# CS 188: Artificial Intelligence
# Fall 2011

Lecture 15: Bayes' Nets III: Inference
10/13/2011

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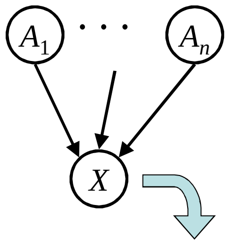
Many slides over this course adapted from Stuart Russell, Andrew Moore

## Bayes' Net Semantics

- A set of nodes, one per variable X
- A directed, acyclic graph
- A conditional distribution for each node
  - A collection of distributions over X, one for each combination of parents' values

$$P(X|a_1 \ldots a_n)$$

  - CPT: conditional probability table
  - Description of a noisy "causal" process




$$P(X|A_1 \ldots A_n)$$

*A Bayes net = Topology (graph) + Local Conditional Probabilities*

## Probabilities in BNs

- For all joint distributions, we have (chain rule):

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | x_1, \ldots, x_{i-1})$$

- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \mathit{parents}(X_i))$$

- This lets us reconstruct any entry of the full joint
- Not every BN can represent every joint distribution
  - The topology enforces certain conditional independencies

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## All Conditional Independences

- Given a Bayes net structure, can run d-separation to build a complete list of conditional independences that are necessarily true of the form

$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, \ldots, X_{k_n}\}$$

- This list determines the set of probability distributions that can be represented

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## Same Assumptions, Different Graphs?

- Can you have two different graphs that encode the same assumptions?
  - Yes!
  - Examples:

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## Example: Independence

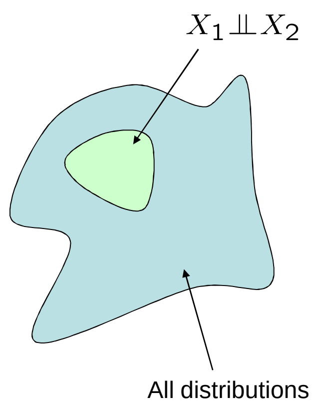
- For this graph, you can fiddle with θ (the CPTs) all you want, but you won't be able to represent any distribution in which the flips are dependent!


$P(X_1)$

| | |
|---|---|
| h | 0.5 |
| t | 0.5 |

$P(X_2)$

| | |
|---|---|
| h | 0.5 |
| t | 0.5 |

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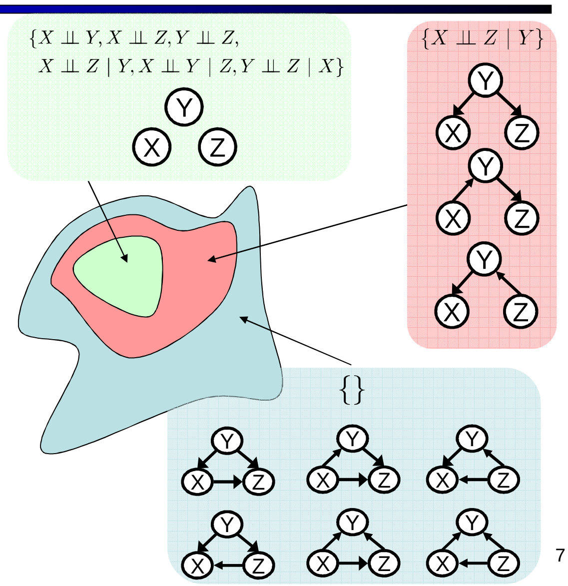
# Topology Limits Distributions

- Given some graph topology G, only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution

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# Causality?

- When Bayes' nets reflect the true causal patterns:
  - Often simpler (nodes have fewer parents)
  - Often easier to think about
  - Often easier to elicit from experts
- BNs need not actually be causal
  - Sometimes no causal net exists over the domain
  - E.g. consider the variables *Traffic* and *Drips*
  - End up with arrows that reflect correlation, not causation
- What do the arrows really mean?
  - Topology may happen to encode causal structure
  - Topology only guaranteed to encode conditional independence
- *More about causality: [Causility – Judea Pearl]

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## Bayes Nets Representation Summary

- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes' net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution

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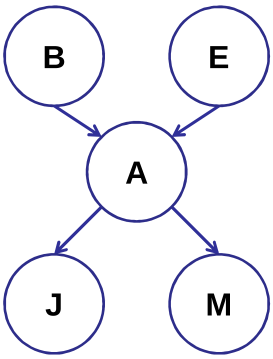
## Inference

- Inference: calculating some useful quantity from a joint probability distribution
- Examples:
  - Posterior probability:

    $$P(Q|E_1 = e_1, \ldots E_k = e_k)$$

  - Most likely explanation:

    $$\text{argmax}_q\ P(Q = q|E_1 = e_1 \ldots)$$

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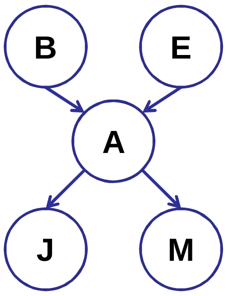
## Inference by Enumeration

- Given unlimited time, inference in BNs is easy
- Recipe:
  - State the marginal probabilities you need
  - Figure out ALL the atomic probabilities you need
  - Calculate and combine them
- Example:

$$P(+b| +j, +m) = \frac{P(+b, +j, +m)}{P(+j, +m)}$$

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## Example: Enumeration

- In this simple method, we only need the BN to synthesize the joint entries

$$\begin{aligned} P(+b,+j,+m) = \\ & P(+b)P(+e)P(+a|+b,+e)P(+j|+a)P(+m|+a)+ \\ & P(+b)P(+e)P(-a|+b,+e)P(+j|-a)P(+m|-a)+ \\ & P(+b)P(-e)P(+a|+b,-e)P(+j|+a)P(+m|+a)+ \\ & P(+b)P(-e)P(-a|+b,-e)P(+j|-a)P(+m|-a) \end{aligned}$$

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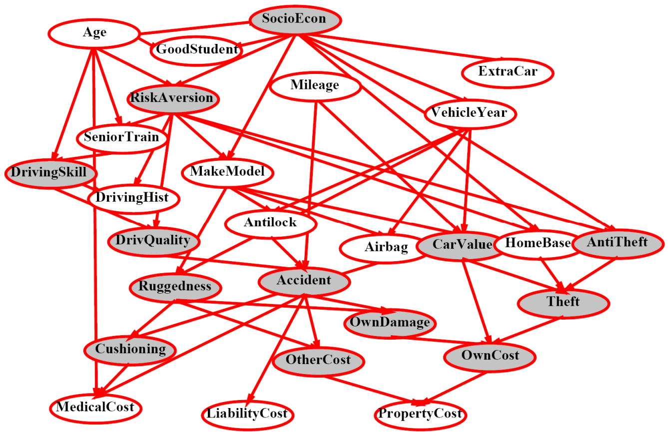
# Inference by Enumeration?






# Variable Elimination

- Why is inference by enumeration so slow?
  - You join up the whole joint distribution before you sum out the hidden variables
  - You end up repeating a lot of work!
- Idea: interleave joining and marginalizing!
  - Called “Variable Elimination”
  - Still NP-hard, but usually much faster than inference by enumeration
- We’ll need some new notation to define VE

# Factor Zoo I

- Joint distribution: P(X,Y)
  - Entries P(x,y) for all x, y
  - Sums to 1

$P(T,W)$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

- Selected joint: P(x,Y)
  - A slice of the joint distribution
  - Entries P(x,y) for fixed x, all y
  - Sums to P(x)

$P(cold,W)$

| T | W | P |
|---|---|---|
| cold | sun | 0.2 |
| cold | rain | 0.3 |

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# Factor Zoo II

- Family of conditionals: P(X |Y)
  - Multiple conditionals
  - Entries P(x | y) for all x, y
  - Sums to |Y|

$P(W|T)$

| T | W | P |
|---|---|---|
| hot | sun | 0.8 |
| hot | rain | 0.2 |
| cold | sun | 0.4 |
| cold | rain | 0.6 |

$P(W|hot)$

$P(W|cold)$

- Single conditional: P(Y | x)
  - Entries P(y | x) for fixed x, all y
  - Sums to 1

$P(W|cold)$

| T | W | P |
|---|---|---|
| cold | sun | 0.4 |
| cold | rain | 0.6 |

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# Factor Zoo III

- Specified family: P(y | X)
  - Entries P(y | x) for fixed y, but for all x
  - Sums to … who knows!

$P(rain|T)$

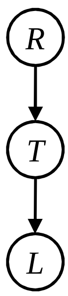
| T | W | P | |
|---|---|---|---|
| hot | rain | 0.2 | $P(rain|hot)$ |
| cold | rain | 0.6 | $P(rain|cold)$ |

- In general, when we write $P(Y_1 \ldots Y_N \mid X_1 \ldots X_M)$
  - It is a "factor," a multi-dimensional array
  - Its values are all $P(y_1 \ldots y_N \mid x_1 \ldots x_M)$
  - Any assigned X or Y is a dimension missing (selected) from the array



# Example: Traffic Domain

- Random Variables
  - R: Raining
  - T: Traffic
  - L: Late for class!
- First query: P(L)




$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

$P(L|R)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

# Variable Elimination Outline

- Track objects called factors
- Initial factors are local CPTs (one per node)

$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

- Any known values are selected
  - E.g. if we know $L = +\ell$ , the initial factors are

$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

$P(+\ell|T)$

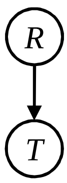
| | | |
|---|---|---|
| +t | +l | 0.3 |
| -t | +l | 0.1 |

- VE: Alternately join factors and eliminate variables



# Operation 1: Join Factors

- First basic operation: joining factors
- Combining factors:
  - Just like a database join
  - Get all factors over the joining variable
  - Build a new factor over the union of the variables involved
- Example: Join on R




$P(R) \times P(T|R) \Rightarrow P(R,T)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

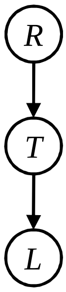
| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

| | | |
|---|---|---|
| +r | +t | 0.08 |
| +r | -t | 0.02 |
| -r | +t | 0.09 |
| -r | -t | 0.81 |


  - Computation for each entry: pointwise products

$$\forall r, t : \quad P(r,t) = P(r) \cdot P(t|r)$$

# Example: Multiple Joins




$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

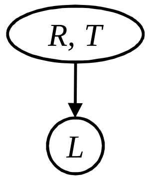
Join R

$P(R,T)$

| | | |
|---|---|---|
| +r | +t | 0.08 |
| +r | -t | 0.02 |
| -r | +t | 0.09 |
| -r | -t | 0.81 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

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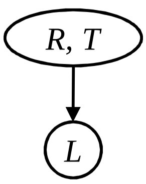
# Example: Multiple Joins




$P(R,T)$

| | | |
|---|---|---|
| +r | +t | 0.08 |
| +r | -t | 0.02 |
| -r | +t | 0.09 |
| -r | -t | 0.81 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

Join T


$P(R,T,L)$

| | | | |
|---|---|---|---|
| +r | +t | +l | 0.024 |
| +r | +t | -l | 0.056 |
| +r | -t | +l | 0.002 |
| +r | -t | -l | 0.018 |
| -r | +t | +l | 0.027 |
| -r | +t | -l | 0.063 |
| -r | -t | +l | 0.081 |
| -r | -t | -l | 0.729 |

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# Operation 2: Eliminate

- Second basic operation: marginalization
- Take a factor and sum out a variable
  - Shrinks a factor to a smaller one
  - A projection operation
- Example:

$P(R,T)$

| | | |
|---|---|---|
| +r | +t | 0.08 |
| +r | -t | 0.02 |
| -r | +t | 0.09 |
| -r | -t | 0.81 |

sum $R$

$P(T)$

| | |
|---|---|
| +t | 0.17 |
| -t | 0.83 |

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# Multiple Elimination

$R, T, L$ $T, L$ $L$

$P(R,T,L)$

| | | | |
|---|---|---|---|
| +r | +t | +l | 0.024 |
| +r | +t | -l | 0.056 |
| +r | -t | +l | 0.002 |
| +r | -t | -l | 0.018 |
| -r | +t | +l | 0.027 |
| -r | +t | -l | 0.063 |
| -r | -t | +l | 0.081 |
| -r | -t | -l | 0.729 |

Sum out R

$P(T,L)$

| | | |
|---|---|---|
| +t | +l | 0.051 |
| +t | -l | 0.119 |
| -t | +l | 0.083 |
| -t | -l | 0.747 |

Sum out T

$P(L)$

| | |
|---|---|
| +l | 0.134 |
| -l | 0.886 |

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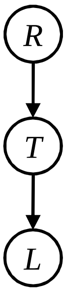
# P(L) : Marginalizing Early!




$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

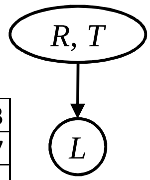
$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

Join R

$P(R, T)$

| | | |
|---|---|---|
| +r | +t | 0.08 |
| +r | -t | 0.02 |
| -r | +t | 0.09 |
| -r | -t | 0.81 |



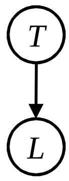

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

Sum out R

$P(T)$

| | |
|---|---|
| +t | 0.17 |
| -t | 0.83 |




$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

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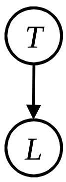
# Marginalizing Early (aka VE*)



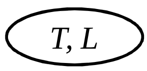

$P(T)$

| | |
|---|---|
| +t | 0.17 |
| -t | 0.83 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

Join T




$P(T, L)$

| | | |
|---|---|---|
| +t | +l | 0.051 |
| +t | -l | 0.119 |
| -t | +l | 0.083 |
| -t | -l | 0.747 |

Sum out T


$P(L)$

| | |
|---|---|
| +l | 0.134 |
| -l | 0.886 |

* VE is variable elimination

# Evidence

- If evidence, start with factors that select that evidence
  - No evidence uses these initial factors:

$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

  - Computing $P(L|+r)$ , the initial factors become:

$P(+r)$

| | |
|---|---|
| +r | 0.1 |

$P(T|+r)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

- We eliminate all vars other than query + evidence

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# Evidence II

- Result will be a selected joint of query and evidence
  - E.g. for P(L | +r), we'd end up with:

$P(+r, L)$

| | | |
|---|---|---|
| +r | +l | 0.026 |
| +r | -l | 0.074 |

Normalize

$P(L|+r)$

| | |
|---|---|
| +l | 0.26 |
| -l | 0.74 |

- To get our answer, just normalize this!

- That's it!

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# General Variable Elimination

- Query: $P(Q|E_1 = e_1, \ldots E_k = e_k)$
- Start with initial factors:
  - Local CPTs (but instantiated by evidence)
- While there are still hidden variables (not Q or evidence):
  - Pick a hidden variable H
  - Join all factors mentioning H
  - Eliminate (sum out) H
- Join all remaining factors and normalize

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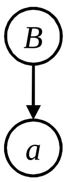
# Variable Elimination Bayes Rule

## Start / Select

$P(B)$

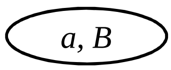
| B | P |
|---|---|
| +b | 0.1 |
| ¬b | 0.9 |




$P(A|B) \rightarrow P(a|B)$

| B | A | P |
|---|---|---|
| +b | +a | 0.8 |
| ~~+b~~ | ~~¬a~~ | ~~0.2~~ |
| ¬b | +a | 0.1 |
| ~~¬b~~ | ~~¬a~~ | ~~0.9~~ |

## Join on B




$P(a, B)$

| A | B | P |
|---|---|---|
| +a | +b | 0.08 |
| +a | ¬b | 0.09 |

## Normalize

$P(B|a)$

| A | B | P |
|---|---|---|
| +a | +b | 8/17 |
| +a | ¬b | 9/17 |

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# Example

$P(B|j,m) \propto P(B,j,m)$

$\boxed{P(B) \quad P(E) \quad P(A|B,E) \quad P(j|A) \quad P(m|A)}$

Choose A

$$\begin{array}{l} P(A|B,E) \\ P(j|A) \\ P(m|A) \end{array} \xrightarrow{\times} P(j,m,A|B,E) \xrightarrow{\Sigma} P(j,m|B,E)$$

$\boxed{P(B) \quad P(E) \quad P(j,m|B,E)}$



# Example

$\boxed{P(B) \quad P(E) \quad P(j,m|B,E)}$

Choose E

$$\begin{array}{l} P(E) \\ P(j,m|B,E) \end{array} \xrightarrow{\times} P(j,m,E|B) \xrightarrow{\Sigma} P(j,m|B)$$

$\boxed{P(B) \quad P(j,m|B)}$

Finish with B

$$\begin{array}{l} P(B) \\ P(j,m|B) \end{array} \xrightarrow{\times} P(j,m,B) \xrightarrow{\text{Normalize}} P(B|j,m)$$

# Variable Elimination

- What you need to know:
  - Should be able to run it on small examples, understand the factor creation / reduction flow
  - Better than enumeration: saves time by marginalizing variables as soon as possible rather than at the end
- We will see special cases of VE later
  - On tree-structured graphs, variable elimination runs in polynomial time, like tree-structured CSPs
  - You'll have to implement a tree-structured special case to track invisible ghosts (Project 4)