CS 188: Artificial Intelligence
Fall 2011

Lecture 14: Bayes' Nets II – Independence

10/11/2011

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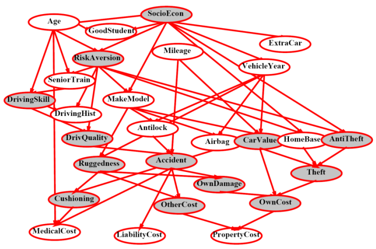
Bayes' Nets

- A Bayes' net is an efficient encoding of a probabilistic model of a domain



- Questions we can ask:
  - Inference: given a fixed BN, what is P(X | e)?
  - Representation: given a BN graph, what kinds of distributions can it encode?
  - Modeling: what BN is most appropriate for a given domain?

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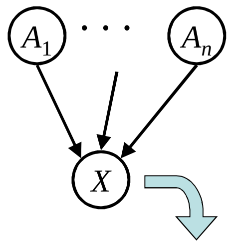
Bayes' Net Semantics

- Let's formalize the semantics of a Bayes' net
- A set of nodes, one per variable X
- A directed, acyclic graph
- A conditional distribution for each node
  - A collection of distributions over X, one for each combination of parents' values

$$P(X|a_1 \ldots a_n)$$

  - CPT: conditional probability table
  - Description of a noisy "causal" process



$$P(X|A_1 \ldots A_n)$$

A Bayes net = Topology (graph) + Local Conditional Probabilities

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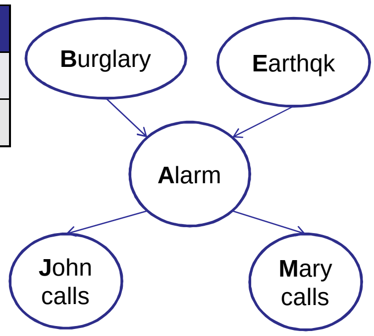
Example: Alarm Network

B	P(B)
+b	0.001
¬b	0.999



E	P(E)
+e	0.002
¬e	0.998

B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	¬a	0.05
+b	¬e	+a	0.94
+b	¬e	¬a	0.06
¬b	+e	+a	0.29
¬b	+e	¬a	0.71
¬b	¬e	+a	0.001
¬b	¬e	¬a	0.999

A	J	P(J|A)
+a	+j	0.9
+a	¬j	0.1
¬a	+j	0.05
¬a	¬j	0.95

A	M	P(M|A)
+a	+m	0.7
+a	¬m	0.3
¬a	+m	0.01
¬a	¬m	0.99

Size of a Bayes' Net

- How big is a joint distribution over N Boolean variables?

  2^N

- How big is an N-node net if nodes have up to k parents?

  $O(N * 2^{k+1})$

- Both give you the power to calculate $P(X_1, X_2, \ldots X_n)$
- BNs: Huge space savings!
- Also easier to elicit local CPTs
- Also turns out to be faster to answer queries (coming)

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Building the (Entire) Joint

- We can take a Bayes' net and build any entry from the full joint distribution it encodes

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \textit{parents}(X_i))$$

  - Typically, there's no reason to build ALL of it
  - We build what we need on the fly

- To emphasize: every BN over a domain implicitly defines a joint distribution over that domain, specified by local probabilities and graph structure

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Bayes' Nets So Far

- We now know:
  - What is a Bayes' net?
  - What joint distribution does a Bayes' net encode?
- Now: properties of that joint distribution (independence)
  - Key idea: conditional independence
  - Last class: assembled BNs using an intuitive notion of conditional independence as causality
  - Today: formalize these ideas
  - Main goal: answer queries about conditional independence and influence
- Next: how to compute posteriors quickly (inference)

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Bayes Nets: Assumptions

- Assumptions we are required to make to define the Bayes net when given the graph:

$$P(x_i|x_1 \cdots x_{i-1}) = P(x_i|parents(X_i))$$

- Probability distributions that satisfy the above ("chain-rule→Bayes net") conditional independence assumptions
  - Often guaranteed to have many more conditional independences
  - Additional conditional independences can be read off the graph
- Important for modeling: understand assumptions made when choosing a Bayes net graph

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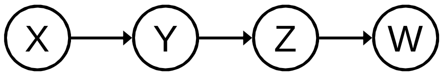
Example



- Conditional independence assumptions directly from simplifications in chain rule:

- Additional implied conditional independence assumptions?

Conditional Independence

- Reminder: independence
  - X and Y are independent if

$$\forall x, y \;\; P(x, y) = P(x)P(y) \dashrightarrow X \perp\!\!\!\perp Y$$

  - X and Y are conditionally independent given Z

$$\forall x, y, z \;\; P(x, y|z) = P(x|z)P(y|z) \dashrightarrow X \perp\!\!\!\perp Y|Z$$

  - (Conditional) independence is a property of a distribution

D-separation: Outline

- Study independence properties for triples
- Any complex example can be analyzed using these three canonical cases

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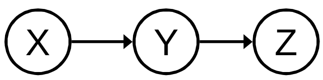
Independence in a BN

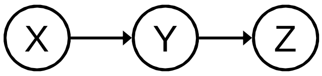
- Important question about a BN:
  - Are two nodes independent given certain evidence?
  - If yes, can prove using algebra (tedious in general)
  - If no, can prove with a counter example
  - Example:



  - Question: are X and Z necessarily independent?
    - Answer: no. Example: low pressure causes rain, which causes traffic.
    - X can influence Z, Z can influence X (via Y)
    - Addendum: they *could* be independent: how?

Causal Chains

- This configuration is a "causal chain"



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Is X independent of Z given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)} = \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)}$$

$$= P(z|y)$$

Yes!

- Evidence along the chain "blocks" the influence

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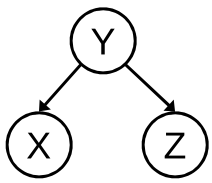
Common Cause

- Another basic configuration: two effects of the same cause
  - Are X and Z independent?
  - Are X and Z independent given Y?



$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)} = \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)}$$

$$= P(z|y)$$

Yes!

Y: Project due

X: Newsgroup busy

Z: Lab full

- Observing the cause blocks influence between effects.

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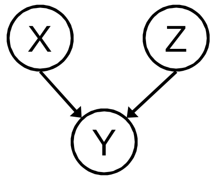
Common Effect

- Last configuration: two causes of one effect (v-structures)
  - Are X and Z independent?
    - Yes: the ballgame and the rain cause traffic, but they are not correlated
    - Still need to prove they must be (try it!)
  - Are X and Z independent given Y?
    - No: seeing traffic puts the rain and the ballgame in competition as explanation?
  - This is backwards from the other cases
    - Observing an effect activates influence between possible causes.



X: Raining

Z: Ballgame

Y: Traffic

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The General Case

- Any complex example can be analyzed using these three canonical cases
- General question: in a given BN, are two variables independent (given evidence)?
- Solution: analyze the graph

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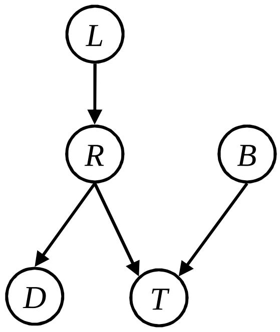
Reachability

- Recipe: shade evidence nodes
- Attempt 1: if two nodes are connected by an undirected path not blocked by a shaded node, they are conditionally independent
- Almost works, but not quite
  - Where does it break?
  - Answer: the v-structure at T doesn't count as a link in a path unless "active"



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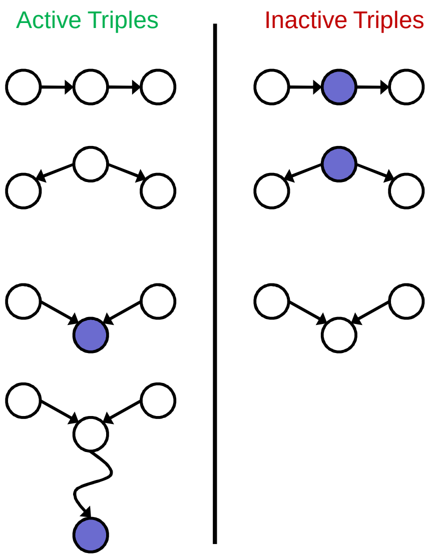
Reachability (D-Separation)

- Question: Are X and Y conditionally independent given evidence vars {Z}?
  - Yes, if X and Y "separated" by Z
  - Look for active paths from X to Y
  - No active paths = independence!
- A path is active if each triple is active:
  - Causal chain $A \rightarrow B \rightarrow C$ where B is unobserved (either direction)
  - Common cause $A \leftarrow B \rightarrow C$ where B is unobserved
  - Common effect (aka v-structure) $A \rightarrow B \leftarrow C$ where B *or one of its descendents* is observed
- All it takes to block a path is a single inactive segment



D-Separation

- Given query $X_i \overset{?}{\perp\!\!\!\perp} X_j | \{X_{k_1}, ..., X_{k_n}\}$
- Shade all evidence nodes
- For all (undirected!) paths between and
  - Check whether path is active
    - If active return $X_i \not\perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$
- (If reaching this point all paths have been checked and shown inactive)
  - Return $X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$

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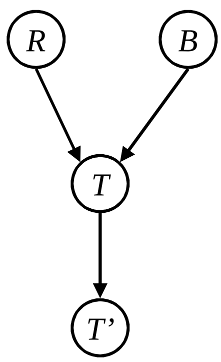
Example

$R \perp\!\!\!\perp B$ *Yes*

$R \perp\!\!\!\perp B | T$

$R \perp\!\!\!\perp B | T'$



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Example

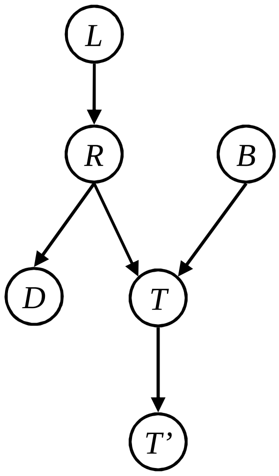
$L \perp\!\!\!\perp T' | T$ *Yes*

$L \perp\!\!\!\perp B$ *Yes*

$L \perp\!\!\!\perp B | T$

$L \perp\!\!\!\perp B | T'$

$L \perp\!\!\!\perp B | T, R$ *Yes*



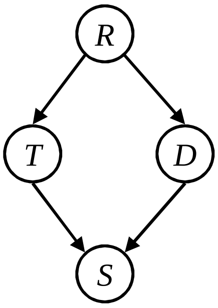
Example

- Variables:
  - R: Raining
  - T: Traffic
  - D: Roof drips
  - S: I'm sad
- Questions:

  $T \perp\!\!\!\perp D$

  $T \perp\!\!\!\perp D | R$ *Yes*

  $T \perp\!\!\!\perp D | R, S$



All Conditional Independences

- Given a Bayes net structure, can run d-separation to build a complete list of conditional independences that are necessarily true of the form

$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

- This list determines the set of probability distributions that can be represented

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Example: Independence

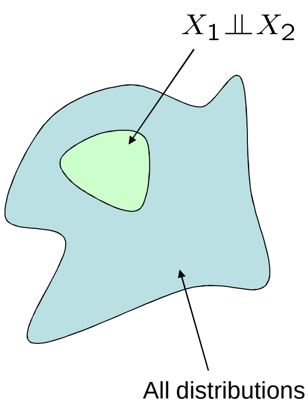
- For this graph, you can fiddle with θ (the CPTs) all you want, but you won't be able to represent any distribution in which the flips are dependent!

$P(X_1)$

h	0.5
t	0.5

$P(X_2)$

h	0.5
t	0.5



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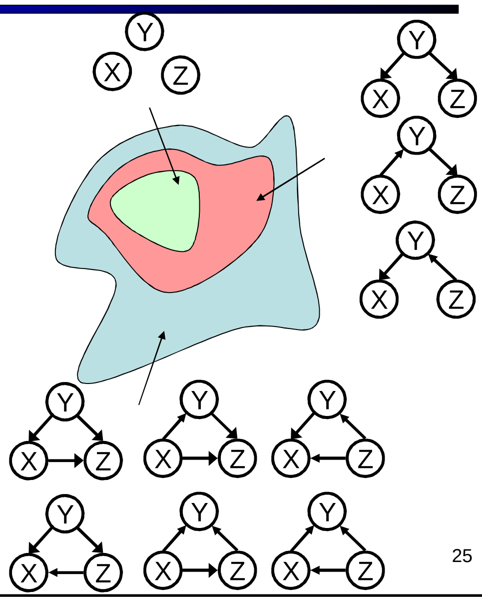
Topology Limits Distributions

- Given some graph topology G, only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution

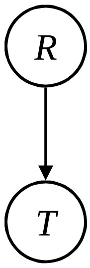


Causality?

- When Bayes' nets reflect the true causal patterns:
  - Often simpler (nodes have fewer parents)
  - Often easier to think about
  - Often easier to elicit from experts
- BNs need not actually be causal
  - Sometimes no causal net exists over the domain
  - E.g. consider the variables *Traffic* and *Drips*
  - End up with arrows that reflect correlation, not causation
- What do the arrows really mean?
  - Topology may happen to encode causal structure
  - Topology only guaranteed to encode conditional independence

Example: Traffic

- Basic traffic net
- Let's multiply out the joint



$P(R)$

r	1/4
¬r	3/4

$P(T|R)$

r	t	3/4
	¬t	1/4
¬r	t	1/2
	¬t	1/2

$P(T, R)$

r	t	3/16
r	¬t	1/16
¬r	t	6/16
¬r	¬t	6/16

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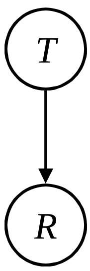
Example: Reverse Traffic

- Reverse causality?



$P(T)$

t	9/16
¬t	7/16

$P(R|T)$

t	r	1/3
	¬r	2/3
¬t	r	1/7
	¬r	6/7

$P(T, R)$

r	t	3/16
r	¬t	1/16
¬r	t	6/16
¬r	¬t	6/16

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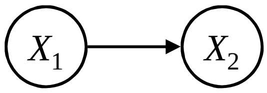
Example: Coins

- Extra arcs don't prevent representing independence, just allow non-independence



$P(X_1)$

h	0.5
t	0.5

$P(X_2)$

h	0.5
t	0.5

$P(X_1)$

h	0.5
t	0.5

$P(X_2|X_1)$

h | h	0.5
t | h	0.5

h | t	0.5
t | t	0.5

- Adding unneeded arcs isn't wrong, it's just inefficient

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Changing Bayes' Net Structure

- The same joint distribution can be encoded in many different Bayes' nets
  - Causal structure tends to be the simplest
- Analysis question: given some edges, what other edges do you need to add?
  - One answer: fully connect the graph
  - Better answer: don't make any false conditional independence assumptions

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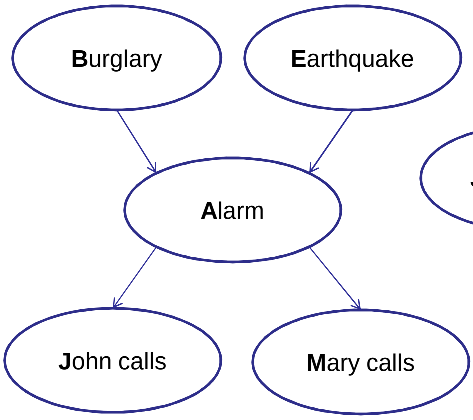
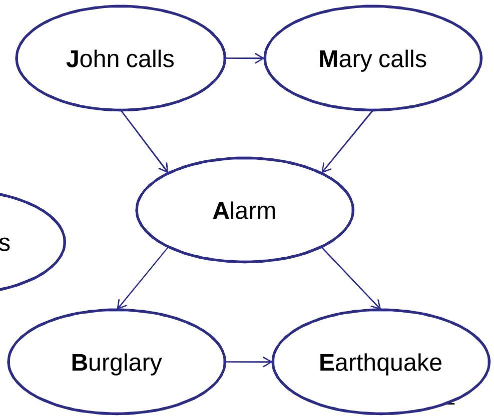
Example: Alternate Alarm



If we reverse the edges, we make different conditional independence assumptions



To capture the same joint distribution, we have to add more edges to the graph

Summary

- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes' net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution

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