CS188 Spring 2012 Section 8: Particle Filters

1 Jabberwock in the wild

Lewis' Jabberwock is in the wild: its position is in a two-dimensional discrete grid, but this time the grid is not bounded. In other words, the position of the Jabberwock is a pair of integers $z = (x, y) \in \mathbb{Z}^2 = \{\dots, -2, -1, 0, 1, 2, \dots\} \times \{\cdots, -2, -1, 0, 1, 2, \cdots\}$. At each time step $t = 1, 2, 3, \dots$, the Jabberwock is in some cell $Z_t = z \in \mathbb{Z}^2$, and it moves to cell Z_{t+1} randomly as follows: with probability $1/2$, it stays where it is, otherwise, it chooses one of the four neighboring cells uniformly at random (fortunately, no teleportation is allowed this week!).

(a) Write the transition probabilities.

$$P(Z_{t+1} = (x', y') | Z_t = (x, y)) = \begin{cases} \frac{1}{2} & \text{if } x' = x, y' = y \\ \frac{1}{8} & \text{if } |x - x'| + |y - y'| = 1 \\ 0 & \text{otherwise} \end{cases}$$

At each time step t you get an observation of the x coordinate R_t in which the Jabberwock sits, but it is a *noisy observation*. Given that the true position is $Z_t = (x, y)$, you observe the correct value $R_t = x$ half of the time, but the other half of the time, the observed value $R_t = r$ is different from x, with the following distribution: the absolute value of the difference between x and r is geometrically distributed with mean 2, and the difference has equal chance of being positive or negative. This means:

$$P(R_t = r | Z_t = (x, y)) = \begin{cases} \frac{1}{2} & \text{if } r = x \\ (\frac{1}{4})(\frac{1}{2})^{(|x-r|-1)} & \text{otherwise} \end{cases}$$

To track the Jabberwock, we will use a particle filter.

In each question, use as many values as needed from the following sequence $\{a_i\}_{1 \le i \le 8}$ of numbers generated independently and uniformly at random from $[0, 1]$ as a source of randomness. Restart at a_1 in each subproblem so that you don't run out.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
0.142	0.522	0.916	0.792	0.036	0.859	0.656	0.149

You can use these values to sample from any discrete distribution of the form $P(X)$ where X takes values in $\{1, 2, \dots, N\}$. Given $a \sim U[0, 1]$, return j such that $\sum_{k=1}^{j-1} P(X = k) \le a < \sum_{k=1}^{j} P(X = k)$. In other words, return the index where the CDF becomes larger than the random a that you drew. (You have to fix an ordering of the elements for this procedure to make sense.)

(b) We now review how to do particle resampling. Suppose that at some time step t there are 4 particles, $p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}$ in our particle filter, with weights $0.25, 0.25, 1.0, 0.5$ say after observation reweighting. Use them to get 4 unweighted particles. Access the particle weights in the order that they appear (other orders are also correct, but this way makes it easier to compare your answer with the solution).

First, the normalizer is 2, so the normalized weights are $0.125, 0.125, 0.5, 0.25$. The first random value a_1 is between $1/8$ and $1/8 + 1/8 = 1/4$, so the first particle sampled is $p^{(2)}$. Continuing in this fashion, (multinomial sampling with replacement) one gets $p^{(2)}, p^{(3)}, p^{(4)}, p^{(4)}$.

Suppose that you know that half of the time, the Jabberwock starts at $z_1 = (0, 0)$, and the other half, at $z_1 = (1, 1)$. Now, you get the following observations: $R_1 = 1, R_2 = 0, R_3 = 1$.

(c) Use a particle filter with 2 particles to compute an approximation to $P(Z_1, Z_2, Z_3|R_1 = 1, R_2 = 0, R_3 = 1)$. Note that we are asking for the joint distribution of these three variables, not the marginal of Z_3. For transitions, use the provided independent random samples distributed as follows: "stay" with probability 1/2, and "up", "down", "left", "right" with probability 1/8 each.

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}
left	stay	left	stay	right	stay	right	left	stay	down	stay	stay	stay	down	down

Hint: Use a_1, a_2 to sample from the prior; to check that you are in the right track, at the end of step $t = 1$, you should have two particles at $(1, 1)$; at the end of step $t = 2$, the first particle should be at $(0, 1)$ and the second at $(1, 1)$.

For $t = 1$:

1. Sampling from the prior, one gets a first particle at $(0, 0)$ and a second at $(1, 1)$. [use a_1, a_2]
2. Observation reweighting gives respective weights of 1/4 and 1/2.
3. Normalization is 3/4, so normalized weights are 1/3 and 2/3.
4. Resampling gives two particles at $(1, 1)$. [use a_3, a_4]

For $t = 2$:

1. Sampling from the transition, one gets a first particle at $(0, 1)$ and a second at $(1, 1)$. [use t_1, t_2]
2. Observation reweighting gives respective weights of 1/2 and 1/4.
3. Normalization is 3/4, so normalized weights are 2/3 and 1/3.
4. Resampling gives one particle at $(0, 1)$ and a second at $(1, 1)$. [use a_5, a_6]

For $t = 3$:

1. Sampling from the transition, one gets a first particle at $(-1, 1)$ and a second at $(1, 1)$. [use t_3, t_4]
2. Observation reweighting gives respective weights of 1/8 and 1/2.
3. Normalization is 5/8, so normalized weights are 1/5 and 4/5.
4. Resampling gives one particle at $(-1, 1)$ and a second at $(1, 1)$. [use a_7, a_8]

So the approximation of the posterior is: probability 1/2 on the path $(1, 1), (0, 1), (-1, 1)$ and probability 1/2 on the path $(1, 1), (1, 1), (1, 1)$.

(d) Use your samples (the unweighted particles in the last step) to evaluate the posterior probability that the x-coordinate of Z_3 is different than the column of Z_3, i.e. $X_3 \neq Y_3$.

Exactly one of the two paths satisfies this property, so the estimate is $\frac{1}{2}$.

(e) Use your samples to evaluate the expected number of distinct x-coordinates visited by the Jabberwock in these 3 time steps.

For the first path, 3 distinct x-coordinates are visited, and only one in the second path, so the estimate is:

$$\frac{1}{2}(3+1) = 2$$

(f) Use your samples to evaluate the posterior probability that the Jabberwock returned to its starting point.

Exactly one of the two paths satisfies this property, so the estimate is $\frac{1}{2}$.

(g) Instead of using the unweighted particles, can you use the weighted particles of the last step to get an estimate of these quantities? How?

Yes, this can be done. This is the same idea as weighted sampling seen earlier. For example, the estimate for the probability that $X_3 \neq Y_3$ becomes $1/5$. As a side note, this estimator generally performs better than using the unweighted particles, and should be preferred in practice. Note that resampling to get unweighted particles is still necessary when executing the filtering, because it is useful to have many samples from the dynamics at high likelihood points of the state space.

(h) Could you use the elimination algorithm to compute these quantities?

No; the state space is infinite, so factors of infinite size (distributions over all points on the plane) would need to be computed and passed around. However, with some cleverness, it could be made to work: given n observations, we know that the Jabberwock is still within a rectangle of size $2n$ centered around the starting point, so all of the probability mass is on a finite subset of the state space. This will still be extremely slow compared to particle filtering, however, because large amounts of computation will be carried out over regions where the Jabberwock cannot possibly be.