Certifying Differential Invariants of Neural Networks using Abstract Duals

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The Setup

Safety Property

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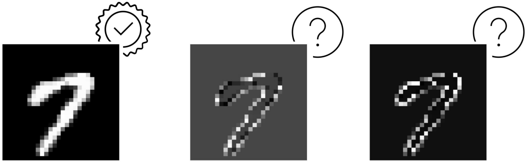

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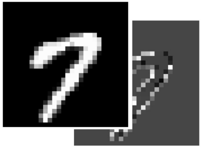

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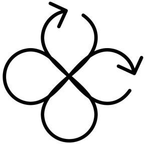

Continuous input spaces

Safety-Critical Domains

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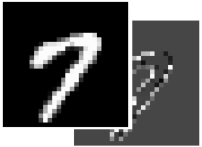

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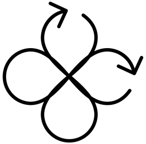

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Safety-Critical Domains

Possible solution

"Check if *global gradient bounds* can provide a simpler yet effective verification domain compared to *layer-wise relational abstractions*"

Why is this Interesting?

The power of global bounds with simplicity of gradient analysis

1. **Simplicity beats complexity**

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   - → Our method detects when the *local linearity assumption collapses*

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The Conceptual Shift

Mathematical Bridge: Mean Value Theorem

$$|f(x) - f(x_0)| \leq \underbrace{\sup_{z \in X} ||\nabla f(z)||}_{\text{Abstract Duals}} \cdot ||x - x_0||$$

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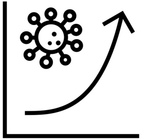
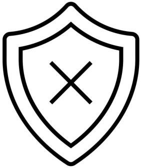
- → Ours: Approximates the derivative $\nabla f(x)$. Coarser, but "jumps" by bypassing intermediate wrapping errors

Contributions

Formalization:
Affine Arithmetic



Gradient Anomaly:
Depth $<$ 3; Sigmoid



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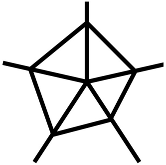
Identifies failures



Evaluation:
Tiny/Small/Std. ϵ

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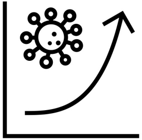

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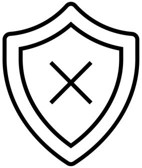

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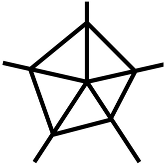

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Statement

"Global gradient bounds provide a simpler verification domain for shallow, smooth networks, while relational domains are essential for deep, non-smooth architectures"

Contribution Idea

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→ **Value Component**: $\hat{x} = \alpha_0 + \sum_{i=1}^{n} \alpha_i \epsilon_i$ (Abstract Input Region)

→ Propogate the interval through the network:

$$\hat{y} = f(\hat{x}) = \underbrace{\left(\alpha_0 + \sum \alpha_i \epsilon_i\right)}_{\text{Value Interval}} + \underbrace{\left(d_0 + \sum d_i \epsilon_i\right)}_{\text{Gradient } \nabla f} \eta$$

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- $\rightarrow$ *Global Lipschitz bound is the supremum*:

$$K = \sup_{x \in X_0} ||\nabla f(x)||_1 = \sum_j \left(|\beta_{0,j}| + \sum_i |\beta_{i,j}| \right)$$

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Implication

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→ Linear approximations (`DeepPoly`) become loose when curvature is high

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In **16.1%** of Tiny Net cases ($\epsilon = 0.01$), the simpler Gradient method certified robustness where `DeepPoly` failed.

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Network	ϵ	**Count**	**%**
Tiny Net	0.01	5	16.1%
Tiny Net	0.02	3	9.7%
Tiny Net	0.03	2	6.5%
Small/Std	All	0	0.0%

Table: Gradient method outperforms `DeepPoly` in shallow, smooth networks; $\{$Tiny: 2×10, Small: 5×20, Standard: $10 \times 20\}$

Experimental Results II

Experiment	Hypothesis	Result
Gradient Instability	Does zero-crossing gradient (instability) predict verification failure?	**100% Correlation** between instability and failure at high perturbation ($\epsilon = 0.12$).
ReLU vs. Sigmoid	Does activation smoothness affect domain performance?	`DeepPoly` dominates on ReLU (100% win rate) not Sigmoid

Table: Summary of structural analysis on MNIST classifiers using `ocaml-nn`[2] with $\epsilon \in \{0.01 \dots 0.12\}$.

[2] https://github.com/ck090/ocaml-nn/tree/main

Thank You!

$\rightarrow$ **Shallow + Smooth** $\rightarrow$ Use **Gradient (Abstract Duals)**

$\rightarrow$ **Deep + Non-smooth** $\rightarrow$ Use **Relational (`DeepPoly`)**

Final Takeaway

"Domain complexity does not dictate robustness, however, alignment with function smoothness does. Expanding to Physics-Informed Neural Networks is an exciting future direction"