A production problem and the Primal Simplex method
Sample and exercises

Gyorgy Dosa

February 17, 2022

Given the next table with the data of a production problem.

	P_1	P_2	P_3	P_4	Cap
Res_1	1	2	1	2	40
Res_2	1	0	1	1	38
Res_3	1	1	3	2	26
Profit	11	15	14	18	

We solve it by the studied method. We do it step by step.
Step 1: we write up the model.

$$\begin{aligned}
x_1 + 2x_2 + x_3 + 2x_4 &\le 40 \\
x_1 + \quad + x_3 + x_4 &\le 38 \\
x_1 + x_2 + 3x_3 + 2x_4 &\le 26 \\
x_j &\ge 0, \quad (1 \le j \le 4) \\
11x_1 + 15x_2 + 14x_3 + 18x_4 &= z \to \max
\end{aligned}$$

Step 2: all constraints are formed to $=$
(by introducing slack variables):

$$\begin{aligned}
x_1 + 2x_2 + x_3 + 2x_4 + s_1 \qquad &\le 40 \\
x_1 + \quad + x_3 + x_4 \quad + s_2 \quad &\le 38 \\
x_1 + x_2 + 3x_3 + 2x_4 \qquad + s_3 &\le 26 \\
\mathbf{x} \ge \mathbf{0}, \mathbf{s} &\ge \mathbf{0} \\
11x_1 + 15x_2 + 14x_3 + 18x_4 &= z \to \max
\end{aligned}$$

Step 3: We build the first simplex tableau.
(In fact this is the fist simplex tableau of the second phase. First Phase: creating a feasible basic solution. Phase 2: Beginning from the result of the

first phase, step by step we reach an optimal basic solution. But, as initially all constraints are $\leq$, we can just start with the second phase. During this course we will not deal with the first phase. It can be performed "similarly" to the second phase.)

B	x_B	a_1	a_2	a_3	a_4	u_1	u_2	u_3
u_1	40	1	2	1	2	1	0	0
u_2	38	1	0	1	1	0	1	0
u_3	26	1	1	3	2	0	0	1
z	0	-11	-15	-14	-18	0	0	0

Step 4: We perform basis transformations.
Recall:

- B: **basis**
- last row: **reduced cost**
- we can choose freely from the last row, where the component of the reduced cost is negative.
- **But**, let us perform this **smart greedy choice** as follows. (It makes possible –not always, but many times – that we need to perform only a few number of transformations.)
- Let us make a **preprocessing** (generally, a very typical way of solving a problem that we start with some kind of preprocessing):
- if we choose column of a_1, who will be the pivot, and what will be the amount of the increment: The minimum is taken in row of u_3, the increment is $\frac{26 \cdot 11}{1} = 286$.
- if we choose column of a_2, the minimum is taken in row of u_1, the increment is $\frac{40 \cdot 15}{2} = \mathbf{300}$.
- if we choose column of a_3, the minimum is taken in row of u_3, the increment is $\frac{26 \cdot 14}{3} \approx 121.33$.
- if we choose column of a_4, the minimum is taken in row of u_3, the increment is $\frac{26 \cdot 18}{2} = 234$.
- So, the increment of the objective is the biggest, if we choose column of a_2. Let us choose this column.
- The transformation:

From this tableau:

B	x_B	a_1	a_2	a_3	a_4	u_1	u_2	u_3
u_1	40	1	**2**	1	2	1	0	0
u_2	38	1	0	1	1	0	1	0
u_3	26	1	1	3	2	0	0	1
z	0	-11	-15	-14	-18	0	0	0

We get this one:

B	x_B	a_1	a_2	a_3	a_4	u_1	u_2	u_3
a_2	20	$1/2$	1	$1/2$	1	$1/2$	0	0
u_2	38	1	0	1	1	0	1	0
u_3	6	$1/2$	0	$5/2$	1	$-1/2$	0	1
z	300	$-7/2$	0	$-13/2$	-3	$15/2$	0	0

Again comes the preprocessing.

- if we choose column of a_1, the minimum is taken in row of u_3, the increment is $\frac{6\cdot(7/2)}{1/2} = \mathbf{42}$.
- if we choose column of a_3, the minimum is taken in row of u_3, the increment is $\frac{6\cdot(13/2)}{5/2} \approx 15.6$.
- if we choose column of a_4, the minimum is taken in row of u_3, the increment is $\frac{6\cdot 3}{1} = 18$.
- So, the increment of the objective is the biggest, if we choose column of a_1. Let us choose this column.
- The transformation:

From this tableau:

B	x_B	a_1	a_2	a_3	a_4	u_1	u_2	u_3
a_2	20	$1/2$	1	$1/2$	1	$1/2$	0	0
u_2	38	1	0	1	1	0	1	0
u_3	6	$\mathbf{1/2}$	0	$5/2$	1	$-1/2$	0	1
z	300	$-7/2$	0	$-13/2$	-3	$15/2$	0	0

We get this one:

B	x_B	a_1	a_2	a_3	a_4	u_1	u_2	u_3
a_2	14	0	1	-2	0	1	0	-1
u_2	26	0	0	-4	-1	1	1	-2
a_1	12	1	0	5	2	-1	0	2
z	342	0	0	11	4	4	0	7

Since all values in the row of reduced cost are non-negative: Optimal solution! Thus we stop.

Step 5: Evaluation

$$x_{OPT} = (12, 14, 0, 0\,|0, 26, 0) \text{ while } z_{OPT} = 342 \text{ which means that:}$$

- we produce 12 and 14 units from the 1th and $2nd$ products, resp. (and nothing from the others)
- remain 26 units from the 2nd resource (and nothing from the others, the other two are consumed)
- the value of the objective function is 342

Exercises for practice:

1.

	P_1	P_2	P_3	P_4	Cap
Res_1	1	2	1	2	40
Res_2	1	0	1	1	10
Res_3	1	1	3	2	26
Profit	11	15	28	18	

The optimal solution is: $x_{OPT} = (0, \frac{94}{5}, \frac{12}{5}, 0\,\big|0, \frac{38}{5}, 0)$ while $z_{OPT} = 349, 2$.

2.

	P_1	P_2	P_3	P_4	Cap
Res_1	1	2	1	2	40
Res_2	1	0	1	1	10
Res_3	1	1	3	2	15
Profit	11	5	28	18	

The optimal solution is: $x_{OPT} = (\frac{15}{2}, 0, \frac{5}{2}, 0\,|30, 0, 0)$ while $z_{OPT} = 152.5$.

3.

	P_1	P_2	P_3	P_4	P_5	Cap
Res_1	1	1	2	1	1	25
Res_2	0	1	0	1	2	42
Res_3	1	0	1	3	1	50
Profit	12	8	15	17	22	

The optimal solution is:

$x_{OPT_1} = (4, 0, 0, 0, 21\,|0, 0, 25)$ and $x_{OPT_2} = (0, 0, 0, 8, 17\,|0, 0, 9)$,

$$x_{OPT} = \lambda \cdot x_{OPT_1} + (1 - \lambda) \cdot x_{OPT_2}, \text{ where } 0 \le \lambda \le 1,$$

while $z_{OPT} = 510$.