Optimization Methods

1. Linear Programming (LP), Simplex Method

Gyorgy Dosa

PE

2022. február 10.

Operations Research

- In fact, OR is some part of "Optimization Methods".
- Here we introduce some basic ideas about decision making and operations research (OR). OR has a very rich literature from the seventies, we propose the following book for the interested readers:
- Wayne L. Winston, Operations Research, Applications and algorithms, Duxbury Press, Belmont, California, 1991 (second Edition), ISBN 0-534-92495-6
- Book of Winston is a very commonly used handbook, considering the main models and algorithms. There are also many other books, as well. Many of them can be found at the following homepage:
- https://www.tankonyvtar.hu/hu/tartalom/

See also: https://en.wikipedia.org/wiki/Operations_research

A short Intro to Linear Programming (LP)

- George Dantzig: Simplex Method, 1947
- In Hungary, Andras Prekopa and others

Linear Programming (LP) is a main technique within OR. There are several solution methods, and within them we must mention the (different versions) of the Simplex Method. The first version called Primal Simplex Method was introduced by George Dantzig in 1947. In the next chapter we briefly introduce it by solving a simple production problem.

A production model and the Primal Simplex Method

Given the next table with the data of a production problem.

	P_1	P_2	P_3	P_4	P_5	Cap
Res_1	1	2	1	3	0	24
Res_2	0	1	1	5	1	43
Res_3	1	0	0	2	2	18
Profit	19	23	15	42	33	

- small factory, several products ($P_1, \ldots, P_5$).
- three resources
- meaning of the columns: e.g. one unit from product P_4, we will use 3 units from resource Res_1, 5 units from resource Res_2, and 2 units from resource Res_3 (and we gain 42 units of money).
- the production will consume the capacities of the resources, and these capacities must not be exceeded
- We will gain certain amount of profit for the production, this is again a linear combination of the *production plan* and the vector of coefficients of the profit. If the production vector is e.g. $x(0,8,0,1,0)$, introducing vector $c(19,23,15,42,33)$ for the profit, the gained profit will be $c \cdot x = (19,23,15,42,33) \cdot (0,8,0,1,0) = 226$.
- Our goal is to find a production plan where all constraints are satisfied and the gained profit is as large as possible.

Let us try!

	P_1	P_2	P_3	P_4	P_5	Cap
Res_1	1	2	1	3	0	24
Res_2	0	1	1	5	1	43
Res_3	1	0	0	2	2	18
Profit	19	23	15	42	33	

a, $x(18, 0, 0, 0, 0)$, $z_1 = 18 \cdot 19 = 342$
b, $x(0, 12, 0, 0, 0)$, $z_2 = 12 \cdot 23 = 276$
c, $x(0, 0, 24, 0, 0)$, $z_3 = 24 \cdot 15 = 360$
d, $x(0, 0, 0, 8, 0)$, $z_4 = 8 \cdot 42 = 336$
e, $x(0, 0, 0, 0, 9)$, $z_5 = 9 \cdot 33 = 297$
f, $x(18, 3, 0, 0, 0)$, also possible!!! $z_6 = 18 \cdot 19 + 3 \cdot 23 = 411$
g, what about $x(0, 0, 24, 0, 9)$, $z_7 = 24 \cdot 15 + 9 \cdot 33 = 657$
h, is it the best, or there exists better?

Definitions

- A vector x is a **solution**, if substituting it, the constraints are fullfilled.
- A vector x is a **feasible solution**, if solution, and moreover $x \geq 0$.
- A vector x is an **optimal solution**, if feasible solution, and "the best one" among all feasible solutions.

	P_1	P_2	P_3	P_4	P_5	Cap
Res_1	1	2	1	3	0	24
Res_2	0	1	1	5	1	43
Res_3	1	0	0	2	2	18
Profit	19	23	15	42	33	

Examples:
$x(-1, 1, 1, 1, 1)$ solution, but not feasible
$x(10, 10, 0, 0, 0)$ feasible, but not solution
$x(5, 1, 1, 0, 0)$ feasible solution
at this point we do not know how to find an optimal solution (if any)

How to find a feasible/optimal solution?

- Is it at all (optimal solution)?
- If yes, how many?
- How to find it?
- There are several methods. One of them is: (Primal) Simplex Method. We introduce it briefly.
- **This will be the first exercise in the mid-test!!!!!**

Let us introduce the following notation:

- Coefficient matrix: A,
- the capacity vector (r.h.s. vector): b,
- profit function (objective function) coefficients c.
- Then the problem is written as follows:

$$Ax \leq b$$
$$x \geq 0$$
$$z = c \cdot x \longrightarrow \max$$

We call it: Linear Program (LP).
Written it in details:

Written it in details:

$$\begin{aligned} x_1 + 2x_2 + x_3 + 3x_4 \qquad\quad &\leq 24 \\ x_2 + x_3 + 5x_4 + x_5 &\leq 43 \\ x_1 \qquad\qquad\qquad + 2x_4 + 2x_5 &\leq 18 \\ x_i &\geq 0, \qquad (1 \leq i \leq 5) \\ 19x_1 + 23x_2 + 15x_3 + 42x_4 + 33x_5 &= z \to \max \end{aligned}$$

recall:

	P_1	P_2	P_3	P_4	P_5	Cap
Res_1	1	2	1	3	0	24
Res_2	0	1	1	5	1	43
Res_3	1	0	0	2	2	18
Profit	19	23	15	42	33	

Let us start to solve it. **First step**: all constraints are formed to $=$ (with the help of introducing so called slack variables):

$$\begin{aligned}
x_1 + 2x_2 + x_3 + 3x_4 \qquad\quad + s_1 \qquad\quad &= 24 \\
x_2 + x_3 + 5x_4 + x_5 \quad + s_2 \quad\; &= 43 \\
x_1 \qquad\qquad\quad + 2x_4 + 2x_5 \qquad + s_3 &= 18 \\
\mathbf{x} &\geq \mathbf{0}, \mathbf{s} \geq \mathbf{0} \\
19x_1 + 23x_2 + 15x_3 + 42x_4 + 33x_5 &= z \to \max
\end{aligned}$$

Instead of the system of equations we can use a brief form to handle the data. This brief form is called **simplex tableau**, which is shown below.

B	x_B	a_1	a_2	a_3	a_4	a_5	u_1	u_2	u_3
u_1	24	1	**2**	1	3	0	1	0	0
u_2	43	0	1	1	5	1	0	1	0
u_3	18	1	0	0	2	2	0	0	1
z	0	-19	-23	-15	-42	-33	0	0	0

- Here B means **basis (**maximal linearly independent set of vectors).
- Now $B = \{u_1, u_2, u_3\}$, the basis is composed (initially) from the three unit vectors.
- The corresponding **feasible basic solution** is $x_B = (0, 0, 0, 0, 0\,|24, 43, 18\,)$. This is a solution of the linear system, moreover each variable is non-negative (i.e. the solution is feasible), and any nonzero component in the vector belongs to a basis vector.
- last row: **reduced cost**

The next theorem is crucial:

Theorem

Exactly one of the following options happens.
a, There is no negative entry in the last row. In this case the corresponding basic solution is optimal.
b, There is a negative value in the last row, such that there is no positive value in its column. In this case there is no optimal solution, as the objective value is not bounded from above.
c, Otherwise (there is a negative value in the last row, but for any such value there exists a positive value in its coloumn) we can perform a basis transformation so that the objective does not decrease.

How can we apply this theorem?

The simplex method (Primal Simplex Method, 2-th phase)

- We perform basis transformations
- We keep the **primal feasibility**: all components of the x vector are non-negative !!!!
- This is ensured by the **minimum rule**!!!
- The objective value will grow only if we **choose a column where the reduced cost is negative**.
- **Dual feasibility**: if the reduced cost is also ≥ 0. Since $x \geq 0$ also holds, this means that the objective value is as big as possible (i.e. optimal). We stop.
- Claim: The simplex method is finite (if we perform it "smartly"). Usually it ends "soon".

- Let us suppose that we choose the coloumn of a_2 for the vector that enters the basis. We are not allowed to choose the vector that will leave the basis, this choice must be made by a rule that is called the **minimum rule**. This is as follows.
- We take the following minimum: $\min\left\{\frac{24}{2}, \frac{43}{1}\right\}$, here the fractions are created so that the numerator is taken from the basic solution, and the denominator is taken from the same row and the chosen coloumn. We cannot divide by zero, and we do not want to divide by negative value.
- The minimum is found as $24/2$, this means that the vector of the row of 24 **must be chosen** as leaving vector.
- So a_2 enters the basis and u_1 leaves the basis. We call the 2 value as **pivot** number, we wrote it by bold letter in the tableau.

How do we perform the basis transformation?

- The row of the pivot is divided by the pivot value (i.e. by 2), and
- 1/2 times of the row of the pivot is subtracted from the second row,
- 0/2 times of the row of the pivot is subtracted from the third row, and
- 23/2 times of the row of the pivot is added to the row of the objective.
- In the above calculations (i.e. 1/2, 0/2, and 23/2) the numerator comes from the coloumn of the pivot value, and the denominator is always the pivot value. After the transformation we get the following tableau.

from this:

B	x_B	a_1	a_2	a_3	a_4	a_5	u_1	u_2	u_3
u_1	24	1	**2**	1	3	0	1	0	0
u_2	43	0	1	1	5	1	0	1	0
u_3	18	1	0	0	2	2	0	0	1
z	0	-19	-23	-15	-42	-33	0	0	0

we get this one:

B	x_B	a_1	a_2	a_3	a_4	a_5	u_1	u_2	u_3
a_2	12	$1/2$	1	$1/2$	$3/2$	0	$1/2$	0	0
u_2	31	$-1/2$	0	$1/2$	$7/2$	1	$-1/2$	1	0
u_3	18	1	0	0	2	**2**	0	0	1
z	276	$-15/2$	0	$-7/2$	$-15/2$	-33	$23/2$	0	0

Now let a_5 enter the basis. By the minimum rule, u_3 leaves the basis. We get the next tableau:

from this:

B	x_B	a_1	a_2	a_3	a_4	a_5	u_1	u_2	u_3
a_2	12	$1/2$	1	$1/2$	$3/2$	0	$1/2$	0	0
u_2	31	$-1/2$	0	$1/2$	$7/2$	1	$-1/2$	1	0
u_3	18	1	0	0	2	**2**	0	0	1
z	276	$-15/2$	0	$-7/2$	$-15/2$	-33	$23/2$	0	0

we get this one:

B	x_B	a_1	a_2	a_3	a_4	a_5	u_1	u_2	u_3
a_2	12	$1/2$	1	**1/2**	$3/2$	0	$1/2$	0	0
u_2	22	-1	0	$1/2$	$5/2$	0	$-1/2$	1	$-1/2$
a_5	9	$1/2$	0	0	1	1	0	0	$1/2$
z	573	9	0	$-7/2$	$51/2$	0	$23/2$	0	$33/2$

Only one negative entry remained in the last row, we must choose this column to enter the basis. According to the minimum rule a_2 leaves the basis.

Summarizing, the rule of the change of the objective function is the following:

- If we choose a column to enter the basis where the value is negative in the row of the objective: the consequence is that the objective value is growing.
- If we choose a column where the value in the bottom is positive: the objective will decrease (but we do not want this as our goal is to maximize the objective)
- If we choose a column where the value in the bottom line is zero: there will be no change in the objective value.

The above works only **if the basic solution is non-degenerate**, which means that if for example the current basis is $B(a_2, u_2, a_5)$, then all out of x_2, s_2, x_5 are positive. In fact, if the value of the variable in the basic solution in the row of the pivot value would be zero then there will be no change in the coloumn of the basic solution, so there is no change in the objective.

Let us see what happened after the transformation:

from this:

B	x_B	a_1	a_2	a_3	a_4	a_5	u_1	u_2	u_3
a_2	12	$1/2$	1	$\mathbf{1/2}$	$3/2$	0	$1/2$	0	0
u_2	22	-1	0	$1/2$	$5/2$	0	$-1/2$	1	$-1/2$
a_5	9	$1/2$	0	0	1	1	0	0	$1/2$
z	573	9	0	$-7/2$	$51/2$	0	$23/2$	0	$33/2$

we get this one:

B	x_B	a_1	a_2	a_3	a_4	a_5	u_1	u_2	u_3
a_3	24	1	2	1	3	0	1	0	0
u_2	10	$-3/2$	-1	0	1	0	-1	1	$-1/2$
a_5	9	$1/2$	0	0	1	1	0	0	$1/2$
z	657	$25/2$	7	0	36	0	15	0	$33/2$

Optimal solution!!!! $x_B = \left(0, 0, 24, 0, 9 \,\middle|\, 0, 10, 0\right)$ while $z_{OPT} = 657$
(as there is no more negative entry in the last row)

Evaluation

B	x_B	a_1	a_2	a_3	a_4	a_5	u_1	u_2	u_3
a_3	24	1	2	1	3	0	1	0	0
u_2	10	$-3/2$	-1	0	1	0	-1	1	$-1/2$
a_5	9	$1/2$	0	0	1	1	0	0	$1/2$
z	657	$25/2$	7	0	36	0	15	0	$33/2$

Optimal solution!!!! $x_B = \left(0, 0, 24, 0, 9 \,\middle|\, 0, 10, 0\right)$ while $z_{OPT} = 657$
which means that:

- we produce 24 and 9 units from the 3-th and 5-th products, resp. (and nothing from the others)
- remain 10 units from the 2-th resource (and nothing from the others, all are consumed)
- the value of the objective function is 657
- Otherwise, we guessed it in advance! We cannot do it "always", but sometimes we can, this is the power of GREEDY!!

Summary of Primal Simplex

- during Primal Simplex we do basis transformations
- we keep the "primal feasibility" i.e. $x \geq 0$, but there is negative in the reduced cost
- we always choose a column where the red. cost is negative. By this, the objective grows.
- if we reached that there is no negative in the last row: also dual feasibility
- these two together: optimality
- **please do calculate several exercises!!!!!**