The **Cutting Stock Problem**, and solving it by the **Gomory-Gilmore algorithm**.

Sample

Let us solve the next problem. C=14, and

size	#
2	30
3	50
5	20
7	80

Phase 0.

We start with the next patterns (column vectors):

$B=\{a^1,a^2,a^3,a^4\}$:

size	a^1	a^2	a^3	a^4
2	7	0	0	0
3	0	4	0	0
5	0	0	2	0
7	0	0	0	2

Then B^{-1} is

1/7	0	0	0
0	1/4	0	0
0	0	1/2	0
0	0	0	1/2

So the pricing vector is $\pi=cB*B^{-1}=[1,1,1,1]*B^{-1}=[1/7; 1/4; 1/2; 1/2]$.

The slave problem looks as follows:

$$1/7\ a1 + 1/4\ a2 + 1/2\ a3 + 1/2\ a4\ -1 \rightarrow \max$$

$$\text{st. } 2a1 + 3a2 + 5a3 + 7a4 \leq 14$$

$$a1, a2, a3, a4 \geq 0, \text{ integer.}$$

The gain/weight ratios:

g/w: 1/14, 1/12, 1/10, 1/14.

Then, the following vector is (possibly) an optimal solution: $\mathbf{a^5(2,0,2,0)}$.

Let us calculate the corresponding objective value, it is z=2/7 + 2/2 -1 =2/7 >0, so it is advantageous to take this new vector a^5 into the basis. It means, $\mathbf{a^5}$ enters the basis.

What vector will leave the basis? $A^5 \rightarrow B \rightarrow ?$

We need to calculate the transformed form of a^5 (denoted by y), and the basis solution, x_B. Recall that $x_B=B^{-1}*b$ and $y= B^{-1}* a^5$. (Recall also so that **b(30,50,20,80)** is the b vector, and the basis inverse B^{-1} is:

1/7	0	0	0
0	1/4	0	0
0	0	1/2	0
0	0	0	1/2

After making the calculation we get x_B(30/7; 50/4; 10; 40) and y(2/7; 0; 1; 0).

Now we apply the minimum rule:

Min {(30/7) / (2/7); 10/1}=10/1 . The corresponding number is chosen to the pivot value in the y vector, that is: y(2/7; 0; **1**; 0). That is, the third number is chosen, so the third row is chosen in the column vector y. This means that vector will leave the basis, which is in the third position.

The current basis is $B=\{a^1,a^2,a^3,a^4\}$, so the leaving vector is a^3.

Phase 1.

Now $B=\{a^1,a^2,a^5,a^4\}$ where the already considered columns are

size	a^1	a^2	a^3	a^4	a^5
2	7	0	0	0	2
3	0	4	0	0	0
5	0	0	2	0	2
7	0	0	0	2	0

Recall that y is at follows: y(2/7; 0; **1**; 0), where the third value is the pivot. It means that the basis transformation should be made so that

- the third row is divided by 1
- the second and fourth rows remain as here we already do have a 0
- we subtract (2/7)-times of the third row from the first row

The old basis inverse is:

1/7	0	0	0
0	1/4	0	0
0	0	1/2	0
0	0	0	1/2

The new basis inverse will be:

1/7	0	-1/7	0
0	1/4	0	0
0	0	1/2	0
0	0	0	1/2

So the new pricing vector is (the numbers in the matrix are added in each column to get):

$\pi(1/7;\ 1/4;\ 5/14;\ 1/2)$.

With this new pricing vector the new slave problem looks like:

$1/7\ a1 + 1/4\ a2 + 5/14\ a3 + 1/2\ a4\ -1 \rightarrow \max$

st. $2a1 + 3a2 + 5a3 + 7a4 \leq 14$

$a1, a2, a3, a4 \geq 0$, integer.

The gain/weight ratios:

g/w: 1/14, 1/12, 1/14, 1/14.

Then, it is easy to see that an optimal solution comes, if we fill completely the knapsack, and as many as possible, from the best item type (i.e. second one, where the g/w ratio is the biggest) is put into the knapsack. So the optimal solution will be (at least it seems so) $\mathbf{a^6(0,3,1,0)}$.

Let us calculate the corresponding objective value, it is $z=3/4 + 5/14 -1 > 0$, so it is advantageous to take this new vector a^6 into the basis. It means, $\mathbf{a^6}$ enters the basis.

What vector will leave the basis? $a^6 \rightarrow B \rightarrow ?$

We need to calculate the transformed form of a^6 (denoted again simply by y), and the basis solution, x_B. Recall that $x_B=B^{-1}*b$ and $y= B^{-1}* a^6$. (Recall also so that **b(30,50,20,80)** is the b vector, and the basis inverse B^{-1} is:

1/7	0	-1/7	0
0	1/4	0	0
0	0	1/2	0
0	0	0	1/2

After making the calculation we get $x_B(10/7; 50/4; 10; 40)$ and $y(-1/7; 3/4; 1/2; 0)$.

Now we apply the minimum rule:

Min {(50/4) / (3/4); 10/(1/2)}=(50/4) / (3/4) . The corresponding number is chosen to the pivot value in the y vector, that is: y(-1/7; **3/4**; 1/2; 0). That is, the second number is chosen, so the second row is chosen in the column vector y. This means that vector will leave the basis, which is in the second position. The current basis is $B=\{a^1,a^2,a^5,a^4\}$, so the leaving vector is a^2.

$$a^6 \rightarrow \ B \ \rightarrow \ a^2$$

Phase 2.

Now $B=\{a^1,a^6,a^5,a^4\}$ where the already considered columns are

size	a^1	a^2	a^3	a^4	a^5	a^6
2	7	0	0	0	2	0
3	0	4	0	0	0	3
5	0	0	2	0	2	1
7	0	0	0	2	0	0

Recall that y is at follows: y(-1/7; **3/4**; 1/2; 0), where the second value is the pivot. It means that the basis transformation should be made so that

- the second row is divided by 3/4
- the fourth row remains as here we already do have a 0
- we add (4/21)-times of the second row to the first row
- we subtract (2/3)-times of the second row from the third row

The old basis inverse is:

1/7	0	-1/7	0
0	1/4	0	0
0	0	1/2	0
0	0	0	1/2

The new basis inverse will be:

1/7	1/21	-1/7	0
0	1/3	0	0
0	-1/6	1/2	0
0	0	0	1/2

So the new pricing vector is $\pi(1/7;\ 3/14;\ 5/14;\ 1/2)$.

With this new pricing vector the new slave problem looks like:

$$1/7\ a1 + 3/14\ a2 + 5/14\ a3 + 1/2\ a4\ -1 \rightarrow \max$$

$$\text{st. } 2a1 + 3a2 + 5a3 + 7a4 \leq 14$$

$$a1, a2, a3, a4 \geq 0, \text{ integer.}$$

The gain/weight ratios:

g/w: 1/14, 1/14, 1/14, 1/14.

Then, it is easy to see that the **maximum of z is 0**. It means that we already **arrived** to an optimal solution of the relaxed model. Still we need to calculate that what is the optimal solution. We get it as $x_B=B^{-1}*b$.

The current basis inverse and the b vector are: **b(30,50,20,80)** and

1/7	1/21	-1/7	0
0	1/3	0	0
0	-1/6	1/2	0
0	0	0	1/2

After making the calculation we get

$x_B(80/21;\ 50/3;\ 5/3;\ 40)$. The optimum value (of the relaxed problem) is so 62.1428.

Since the optimum value of the cutting stock problem is some integer number (the number of stock material that is cut), we know that the optimal solution of the cutting stock problem is at least 63 (62.1428 is rounded up).

Let us try to round up also the components of the x_B vector. We get a new vector as x(4; 17; 2; 40).

Since the current basis is $B=\{a^1,a^6,a^5,a^4\}$, this means that so many stock are cut from the patterns: We cut

- 4 stocks according to column 1,
- 17 according to pattern 6,
- 2 according to pattern 5, and finally we cut
- 40 according to pattern 4.

“We are lucky” in the sense, that the sum of the components is just 63, which means that we do have a “proven” optimal solution of the cutting stock problem. Let us summarize finally that what patterns are used, and how many pieces are cut.

size	a^1	a^2	a^3	a^4	a^5	a^6	produced
2	7	0	0	0	2	0	32
3	0	4	0	0	0	3	51
5	0	0	2	0	2	1	21
7	0	0	0	2	0	0	80
#	4	-	-	40	2	17	

We conclude that all demand is produced (and a bit more).