Transportation problem, **Phase II**.

This was (and still this is) the problem: Fill in the cells so that
- the sum is OK in each row and column
- the total cost is minimum

	D1	D2	D3	D4	
S1	1	5	8	4	18
S2	2	3	5	1	13
S3	1	4	2	5	42
S4	3	7	6	6	58
	34	15	22	60	

We solve it in 2 phases

1th phase: We already determined a feasible basic solution (with 4+4-1=7 cells that we used)

2th phase: Starting from the given feasible basic solution, we will reach the optimal solution (by performing basis transformations)

We got this tableau (by Vogel-Korda),
- which seems very good, but
- we do not know yet whether optimal or not.

	v1	v2	v3	v4
u1	1 18	5	8	4
u2	2	3	5	1 13
u3	1 5	4 15	2 22	5
u4	3 11	7	6	6 47

The next slave algorithm will help:

Find the values of u1,...,u4 and v1,…,v4, so that

- u_i + v_j =c(i,j) for each basis cell (i,j)   !

<u>Claim</u>: If we have a basic solution, then the above system of linear equations does have solution, and the degree of freedom is exactly 1.

So, we can freely choose one variable. Let u1=0.
Then, the further variables will be determined by a <u>path-searching algorithm</u>, as follows:

Repeat
- find a basis cell, for which one variable is already determined, the other one is not
- calculate the value of the other one

End

This is what we get:

	1	4	2	4
0	1 18	5 1	8 6	4 0
-3	2 4	3 2	5 6	1 13
0	1 5	4 15	2 22	5 1
2	3 11	7 1	6 2	6 47

The solution is unique (if u1=0).

Next, let us determine the reduced cost:

For each cell (i,j), the value of the reduced cost is c(i,j) - (u_i + v_j).
We will write them into the right bottom corner of each cell.
(For the basis cells these values are 0, we do not calculate them.)

<u>Theorem</u>: If there is no negative number within the components of the reduced cost, the current solution is optimal.

- So, our solution is currently optimal.
- It does not mean that the result of Vogel-Korda would be always optimal!!!

Let us start a **new example**!

Suppose we are already done with the first phase, and let us suppose that the following is a feasible basic solution (no matter how we got it.)

- Let us check that the slave variables are correct.
- Then let us check also the reduced cost components (if it is positive, we write simply +, as we do not have any further work with it).

	6	2	-1	0
0	1 -5	2 8	3 +	1 +
6	1 -11	1 -7	5 4	6 10
2	1 -7	4 12	1 21	5 +
2	8 15	7 +	6 +	2 2

- If all reduced cost components would be + or 0: optimal solution.
- But now, we have negative entries! It means, we can improve the solution!!!!
- **How?**

We need some definitions.

- **edge**: two cells that are in common row or column
- **path**: a sequence of adjacent edges so that one edge is horizontal, next edge is vertical, next edge is again horizontal, and so on
- **cycle**: a path with the same starting and ending point

Theorem: Starting from any non-basic cell, there is exactly one cycle so that it visits only basic cells (except the stating and ending point).

The theorem holds only if we do have a system of basic cells!
Let us find several such cycles!

Let us find a cycle, starting (and ending) e.g. in cell (2,2), (we put there an X).

	6	2	-1	0
0	1 -5	2 8	3 +	1 +
6	1 -11	1 X ----- I -7	5 ----- 4 I	6 10
2	1 -7	4 I 12 -----	1 I ----- 21	5 +
2	8 15	7 +	6 +	2 2

<u>Claim</u>: If we alternatively reduce/increase the transported volume by 1 unit throughout the cycle, we will gain the value of the reduced cost.

- It means, this change is advantageous.
- Modifying 1 unit, we gain 7 (dollars).
- Modifying 2 units, we gain 2 times 7.
- And so on.
- But we cannot modify more than 4 !
- So, by such a transformation, we can gain 4*7=28 in total!

Let us see some other option:

Let us see some other option, e.g. starting in cell (1,1), (we put there an X).

	6	2	-1	0
0	1 40 X ----- I -5	2 ----- 8 I	3 +	1 +
6	1 I I I -11	1 I 28 I I -7	5 4 ----- I	6 ----- 10 I
2	1 I I I -7	4 I 12 -----	1 I ----- 21	5 I I I +
2	8 I 15 -----	7 ----- ----- ----- +	6 ----- ----- ----- +	2 I ----- 2

- We can modify by at most **8**.
- How do we get: these are the volumes throughout the cycle: X,8,12,21,4,10,2,15,X.
- But any second is growing, the others are decreasing. So, only every second matters!
- These are the important values: -,**8**,-,21,-,10,-,15,-.
- The minimum rule now: **Take the minimum from every second**.
- The first one matters, second one does not matter, third one matters,…
- Every odd step matters only!
- And take the smallest value.
- Now we will gain by the transformation (if we choose this one): **8*5=40**

Let us see one more option, starting in cell (3,1), (we put there an X).

	6	2	-1	0
0	1 40 -5	2 8	3 +	1 +
6	1 110 -11	1 28 -7	5 4 ----- I	6 ----- 10 I
2	1 70 X I -7	4 12	1 I 21	5 I I I +
2	8 I 15 -----	7 ----- ----- ----- +	6 ----- ----- ----- +	2 I ----- 2

The cycle does not contain cell (3,2) !!!
The minimum is min{15,10,21}=10.
We gain (choosing this transformation) so many: 10*7=70.

If we start at (2,1), the gain is 10*11=110.
Let us summarize what we know:

	6	2	-1	0
0	1 40 -5	2 8	3 +	1 +
6	1 110 -11	1 28 -7	5 4	6 10
2	1 105 -7	4 12	1 21	5 +
2	8 15	7 +	6 +	2 2

There are four options to improve.
We can choose as we want.

Let us choose the **best one**. (This is some kind of Greedy choice.)

Possibly choosing the transformation which improves the most, we will have only few transformations. (We do not know in advance, there is no guarantee, as this is a Greedy choice.)

So,

- in cell (2,1) we increase by 10.
- in cell (4,1) we decrease by 10.
- in cell (4,4) we increase by 10.
- in cell (4,1) we decrease by 10, and **this cell leaves the basis**!

We will get this tableau:

We get this tableau after the transformation:

	-5	2	-1	-11
0	1 +	2 8	3 +	1 +
6	1 10	1 28 -7	5 4	6 +
2	1 +	4 12	1 21	5 +
13	8 5	7 32 X -8	6 24 -6	2 12

- We already calculated also the slave variables
- there are several negative components of the reduced cost, we will choose cell (4,2)

We get this tableau after the transformation (adding and reducing 4 alternatively).
We already calculated the slave variables and reduced cost values.

	3	2	-5	-3
0	1 2 -2	2 8	3 +	1 +
-2	1 14	1 +	5 +	6 +
6	1 8 -8	4 8	1 25	5 +
5	8 1	7 4	6 +	2 12

Still there is option to improve (but the improvement is small).
We can choose cell (3,1), and so on….

Summary of the second phase

- we start from a feasible basic solution (4+4-1=7 basis cells)
- Repeat
  - slave variable on the left and up
  - reduced cost: if all are non-negative: optimal solution
  - otherwise: there is negative. Choose a cell with negative reduced cost. (Preferably, where the improvement is the biggest one: this is a greedy choice)
  - transformation