The Canadian and Inter-Provincial Input-Output Models: The Mathematical Framework

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1 Introduction

This paper provides an overview of the Canadian and interprovincial input-output (IO) models and multipliers maintained by the Industry Accounts Division (IAD) of Statistics Canada. The models are classical Leontief-type open IO models, providing detailed information on the impact on the economy of exogenous demands for the outputs of industries. The models are primarily based on the Canadian and provincial IO tables[1] published by the IAD. These tables are published annually with an approximate lag of three years from the reference period.

The paper is an update of previous documentation[2] of the models. It accounts for certain conceptual and presentational changes in the IO tables that have occurred over time. It also provides a more explicit derivation of some of the assumptions that underlie the models.

Input-output models are primarily used for analyzing the propagation of demand throughout the economy. For example, demand for an automobile generates demand in industries supplying the automobile industry, which in turn generates demand for the suppliers of the suppliers. Each industry thus affected requires imports, labour, and other factors. These models can therefore be used to examine the impacts of a large investment project, the impacts of an industry or a final expenditure category on the rest of the economy, or the impact of a change in the demand for certain products.

The models quantify impacts on production, value-added components (including wages and surplus), expenditures, and imports. Other industry-based variables (not available in the IO tables) are integrated into the models to account for employment, resource use (such as energy), or pollutant emissions by industry. Counterfactual analysis can also be applied by introducing changes to the benchmark parameters or coefficients. The interprovincial model extends all the above impacts to the provincial dimension, including detailed impacts on interprovincial trade.

While characterized by a high level of precision due to the detailed information provided by the IO tables, this model also has certain weaknesses. One of the main weaknesses of this model is that it does not incorporate economic behaviour by agents in response to variations in prices. Furthermore, the model does not account for any economies of scale, technological change, or any supply constraints that may cause bottlenecks in the production process.

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[1] Traditionally, the tables published by Statistics Canada are referred to as rectangular or industry-by-commodity IO tables. This differs from the nomenclature for these tables adopted by the international standard “The System of National Accounts 1993”, which refers to them as supply and use tables; the term input-output tables being reserved for the symmetric industry-by-industry or commodity-by-commodity tables.
[2] Hoffman (1980), Mercier *et al*. (1991), and Statistics Canada (1987).

2 The Canadian model

2.1 Organizing the data

The Canadian IO tables are composed of the output, input, and final demand tables. The latter two tables include, for all expenditures, the purchaser and modified basic price valuations, as well as the detailed trade, transport, and tax margins. As a first step, the tables, at the modified basic price valuation, are re-organized into a form more appropriate for the required mathematical manipulations.

The output table is defined in matrix form as $\mathbf{V} = v_{ji}$, where industry *j* produces commodity *i*. The input table is split into two matrices, a matrix for intermediate inputs $\mathbf{U} = u_{ij}$, where commodity *i* is consumed by industry *j* and a matrix $\mathbf{U}^Y = u_{ij}^Y$, of value-added components[3], *i* by industry *j*. The final demand table is also split into a matrix $\mathbf{F} = f_{ij}$, where commodity *i* is purchased by category *j* and a vector $\mathbf{f}^Y = f_j^Y$, of total indirect taxes paid by each category *j*.

Matrix $\mathbf{F}$ is partitioned into the following variables:

c_{ij} = domestic personal expenditures by residents and non-residents
c_i^M = personal expenditures by residents abroad, i.e., travel imports
c_i^X = personal expenditures by non-residents, i.e., travel exports (negative numbers)
c_{ij}^G = current expenditures by level of government
k_{ij} = capital expenditures by industry
s_i = scrap metal (negative numbers)
n_{ij}^A = inventory additions by type of inventory[4]
n_{ij}^W = inventory withdrawals by type of inventory (negative numbers)
x_i = exports
r_i = re-exports
m_i = imports (negative numbers)

The basic accounting identity between total supply and total demand by commodity, valued at modified basic prices is defined as:

$$(1)\quad \sum_j v_{ji} - m_i - \sum_j n_{ij}^W - s_i \equiv \sum_j u_{ij} + \sum_j c_{ij} + c_i^X + c_i^M + \sum_j c_{ij}^G + \sum_j k_{ij} + \sum_j n_{ij}^A + x_i + r_i$$

[3] These value-added components are: indirect taxes and subsidies both on products and production, wages and salaries, supplementary labour income, mixed income, and other operating surplus.
[4] There are two types of inventories in the tables, inventories of "finished goods and goods in process" and inventories of "raw materials and goods purchased for resale".

There are several relationships within this identity which must be made explicit prior to the specification of the model equations. Some of these relationships can be directly extracted from the data, while others are based on conceptual definitions and can only be inferred indirectly.

2.1.1 Exports and domestic supply

All sources of domestic supply, that is, output, inventory withdrawals, and scrap, can, by construction of the IO tables, be split into two basic components of supply for domestic demand and supply for exports.

Sum the output matrix across industries to derive total output for each commodity q_i

(2) $q_i = \sum_j v_{ji}$

Each commodity can then be split into two components, production for domestic demand q_i^D and production for exports q_i^X.

(3) $q_i = q_i^D + q_i^X$

Similarly, inventory withdrawals can be split into supplies for the domestic market $n_i^{W,D}$ [5] and supplies for exports $n_i^{W,X}$. For the purpose of simplification, the distinction between "raw" and "finished goods" inventories is removed in favour of a single aggregate vector of inventory withdrawals.

(4) $n_i^W = n_i^{W,D} + n_i^{W,X}$

Scrap metal is also split into supply for the domestic market s_i^D and supply for exports, s_i^X.

(5) $s_i = s_i^D + s_i^X$

Re-exports can be split into the value of the imported commodities that are to be re-exported, which have no impact on the domestic economy and the transportation margins on those commodities, which are supplied from domestic production. To obtain the value of imports associated with re-exports, a new vector equivalent to the re-export vector is

[5] For the purpose of simplifying notation, wherever the context is clear, the exclusion of an index denotes summation over that index. For example, $n_i^{W,D} \equiv \sum_j n_{ij}^{W,D}$.

defined, $\mathbf{r}^I = r_i^I$; however, the value of the transportation margins commodity in this vector is set to zero. Thus, if as a starting point $r_i^I = r_i$, with $i = 1,\ldots,k,\ldots n$; n being the number of commodities and k, the transportation commodity, then set $r_{i=k}^I = 0$. The total value of transportation services associated with re-exports can then be derived residually as

(6) $r_i^D = r_i - r_i^I$

By definition, domestic production, inventory withdrawals, and scrap metal are the only sources of supply for exports and transportation margins on re-exports.

(7) $q_i^X - n_i^{W,X} - s_i^X = x_i + r_i^D$

2.1.2 Imports and domestic demand

A portion of total imports in the tables are not related to domestic demand. They relate to personal expenditures on international travel and re-exports. Imports that correspond to domestic demand, m_i^D, are derived by removing re-exports and the imports of travel by the personal sector.

(8) $m_i^D = m_i + c_i^M + r_i^I$

Removal of the category of personal expenditures on travel imports makes it obvious that some imports of travel-related services in m_i^D can no longer correspond to personal expenditures. Viewed differently, once personal expenditures on imports of travel are removed, some remaining expenditures on travel-related services by the personal sector must originate solely in domestic production. Thus, it becomes necessary to split personal expenditures into two elements: one element for all non-travel related personal expenditures c_i^O and a second one for travel-related personal expenditures c_i^T [6]. The former will be partially supplied from imports, while the latter vector will only be supplied from domestic production.

(9) $c_i^O = c_i - c_i^T$

Production for the domestic market can be further split into two components, production for personal expenditures on non-travel related services $q_i^{D,T}$ and production for all other domestic demand $q_i^{D,O}$.

[6] These are mostly expenditures on travel, accommodation and other services that cannot be transported across borders.

Domestic supply of travel-related personal expenditures is simply equal to the associated expenditures.

(10) $q_i^{D,T} = c_i^T$

The production for all remaining domestic demand categories is residually derived as

(11) $q_i^{D,O} = q_i^D - q_i^{D,T}$

A clear association can now be derived between total domestic demand (excluding personal expenditures on travel-related services) and supplies of production, imports, inventory withdrawals, and scrap.

(12) $q_i^{D,O} - m_i^D - n_i^{W,D} - s_i^D = u_i + e_i$

where $\mathbf{E}$ is a partitioned matrix of the categories necessary to calculate domestic final demand

$$\mathbf{E} = \left[\mathbf{C}^O \quad \mathbf{c}^X \quad \mathbf{C}^G \quad \mathbf{K} \quad \mathbf{N}^A\right] \text{and therefore } e_i = c_i^O + c_i^X + c_i^G + k_i + n_i^A .$$

2.2 The model assumptions

The model is based on two main assumptions. The first assumption is that industry input coefficients are constant. The second assumption is that all the components of supply: production, imports, inventory withdrawals, and scrap metal, maintain constant relative shares in total supply.

2.2.1 Constant market shares

Define a vector of total industry output $\mathbf{g}$ as

(13) $g_j = \sum_i v_{ji}$

Also, define the market share matrix $\mathbf{D}$ as

(14) $d_{ji} = \dfrac{v_{ji}}{q_i}$

Reorganizing equation (14), summing across index *i*, and inserting equation (13)

(15) $$g_j = \sum_i \left(d_{ji} q_i \right)$$

Gross output by industry can thus be derived as the constant market shares times total commodity output. Through equation (15) it is possible to convert any vector of commodities into their industries of origin.

2.2.2 Constant input coefficients

The input coefficients matrix **B** is defined as the ratio of intermediate inputs to total output by industry.

(16) $$b_{ij} = \frac{u_{ij}}{g_j}$$

Multiplying both sides of equation(16) by g_j and summing the industries across the index j

(17) $$\sum_j u_{ij} = \sum_j \left(b_{ij} g_j \right)$$

Total intermediate inputs are a function of industry output and the constant input coefficients. This is the classical Leontief production function, which implies that there is no substitution between intermediate inputs, or between intermediate inputs and value-added components.

2.2.3 Shares of imports, inventory withdrawals, and scrap

Aside from domestic production, as reflected in industry outputs, there are three other sources of supply, which all appear in the Final Demand table as negative values. These are imports, inventory withdrawals, and scrap metal and used cars. The model is based on the assumption that for each commodity, these sources of supply maintain their relative shares or viewed alternatively, they have a fixed share of the demand for each commodity.

2.2.4 Imports

The IO tables provide an estimate of imports by commodity with limited information regarding the allocation of imports to sources of demand. Thus, in general, average import rates are used to derive an estimate of imports by source of demand.

Define the share of imports in total supply for the domestic uses as μ_i, then

(18) $$\mu_i = \frac{-m_i^D}{q_i^{D,O} - m_i^D - n_i^{W,D} - s_i^D}$$

The derivation of each element in the denominator requires some manipulations. A simpler alternate derivation of μ_i is possible by substituting in the denominator from equation (12),

(19) $$\mu_i = \frac{-m_i^D}{\sum_j u_{ij} + e_i}$$

2.2.5 Inventories and scrap

Define the share of inventories and scrap in total supply as β_i and α_i respectively, then

(20) $$\beta_i = \frac{-n_i^W}{q_i^{D,O} + q_i^X - m_i^D - n_i^W - s_i}$$

(21) $$\alpha_i = \frac{-s_i}{q_i^{D,O} + q_i^X - m_i^D - n_i^W - s_i}$$

Similarly to the treatment of the import shares, by substitution from equations (7) and (10)-(12), scrap and inventory withdrawals can also be transformed into shares of demand.

(22) $$\beta_i = \frac{-n_i^W}{u_i + e_i + c_i^T + x_i}$$

(23) $$\alpha_i = \frac{-s_i}{u_i + e_i + c_i^T + x_i}$$

2.3 The model

2.3.1 Output determination

By substituting equation (16) and reformulating into matrix notation, equations (19), (22), and (23) can be re-stated as

(24) $-\mathbf{m}^{D} = \hat{\boldsymbol{\mu}}(\mathbf{Bg} + \mathbf{e})$[7]

(25) $-\mathbf{n}^{W} = \hat{\boldsymbol{\beta}}(\mathbf{Bg} + \mathbf{e} + \mathbf{x})$

(26) $-\mathbf{s} = \hat{\boldsymbol{\alpha}}(\mathbf{Bg} + \mathbf{e} + \mathbf{x})$

Substituting equations (2), (6), (8), (17), and (24)-(26) into equation (1)

(27) $\mathbf{q} + \hat{\boldsymbol{\mu}}(\mathbf{Bg} + \mathbf{e}) + \mathbf{c}^{M} + \mathbf{r}^{I} + \hat{\boldsymbol{\beta}}(\mathbf{Bg} + \mathbf{e} + \mathbf{x}) + \hat{\boldsymbol{\alpha}}(\mathbf{Bg} + \mathbf{e} + \mathbf{x}) = \mathbf{Bg} + \mathbf{e} + \mathbf{c}^{M} + \mathbf{x} + \mathbf{r}^{D} + \mathbf{r}^{I}$

Multiplying both sides by $\mathbf{D}$ and solving for $\mathbf{g}$ provides the reduced form equation of the model for domestic gross output by industry.

(28) $\mathbf{g} = \left[\mathbf{I} - \mathbf{D}(\mathbf{I} - \hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\alpha}})\mathbf{B}\right]^{-1}\mathbf{D}\left[\left(\mathbf{I} - \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\alpha}}\right)\mathbf{e} + \left(\mathbf{I} - \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\alpha}}\right)\mathbf{x} + \mathbf{r}^{D}\right]$

Equation (28) shows the dependence of domestic output on final expenditures. Post-multiplying the inverse by the observed final expenditures in the Final Demand table, after accounting for any direct leakages from production, i.e., imports, inventory withdrawals, and scrap metal, generates an estimate of total outputs by industry, inclusive of inter-industry transactions, equivalent to the total outputs observed in the benchmark tables.

Alternatively, substituting equations (15) into equation (27) and solving for **q**, we obtain the reduced form equation of the model for domestic gross output by commodity.

(29) $\mathbf{q} = \left[\mathbf{I} - \left(\mathbf{I} - \hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\alpha}}\right)\mathbf{BD}\right]^{-1}\left[\left(\mathbf{I} - \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\alpha}}\right)\mathbf{e} + \left(\mathbf{I} - \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\alpha}}\right)\mathbf{x}\right]$

This equation is very similar to equation (28), except that now, instead of industries, commodities are used to produce other commodities.

The inverse in equation (29) may be used to estimate the impact on gross output by industry (renamed as $\mathbf{g}^{*}$) of any arbitrary change in exogenous demand, i.e., any exogenous shock specified by the model user. Exogenous shocks may be applied to industry outputs $\mathbf{g}^{E}$, as in equation (29.1), or final domestic expenditures $\mathbf{e}^{E}$ and international exports $\mathbf{x}^{E}$, as in equation (29.2).

(29.1) $\mathbf{g}^{*} = \left[\mathbf{I} - \mathbf{D}(\mathbf{I} - \hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\alpha}})\mathbf{B}\right]^{-1}\mathbf{g}^{E}$

(29.2) $\mathbf{g}^{*} = \left[\mathbf{I} - \mathbf{D}(\mathbf{I} - \hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\alpha}})\mathbf{B}\right]^{-1}\mathbf{D}\left[\left(\mathbf{I} - \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\alpha}}\right)\mathbf{e}^{E} + \left(\mathbf{I} - \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\alpha}}\right)\mathbf{x}^{E}\right]$

[7] The hat symbol ^ indicates the diagonalization of a vector.

2.3.2 Detailed impacts

The use of ratios to total industry output provide the means to estimate the impacts of any exogenous shock on a wide range of variables such as GDP components, employment, imports, resource use, and pollution emissions, to list a few. In short, it is possible to study the total direct and indirect impacts on any variable for which a ratio to total gross output can be reasonably determined. That is, for any variable that can be classified according to the IO industry structure and for which an estimate of the complete population is available.

The impact on GDP components of any exogenous shock is equal to the gross output requirements times the GDP-to-output ratios by industry.

$$(30) \qquad u_{ij}^{Y*} = \frac{u_{ij}^{Y}}{g_j} g_j^*$$

Intermediate inputs by industry are equal to the industry output times the input coefficients.

$$(31) \qquad u_{ij}^* = b_{ij} g_j^*$$

Imports per commodity are composed of direct imports related to final domestic expenditures and intermediate inputs[8]. The first term on the right-hand-side of equation (32) shows the value of direct imports, as the import rate times the exogenous final expenditure. The second term on the right-hand-side shows the imports of intermediate inputs summed across industries based on the average import rate by commodity. For industry shocks, only the second term on the right-hand-side is relevant. Of course, the detailed imports by industry may be separately obtained from the bracketed expression in the second term.

$$(32) \qquad m_i^{D*} = \mu_i e_i^E + \sum_j \left(\mu_i u_{ij}^* \right)$$

Other leakages from current period production are calculated in a similar fashion to imports. Again, for industry shocks, only the second and third terms on the right-hand-side of equation (32) are relevant, since e_i^E would be equal to zero.

$$(33) \qquad n_i^{W*} = \beta_i e_i^E + \beta_i x_i^E + \sum_j \left(\beta_i u_{ij}^* \right)$$

$$(34) \qquad s_i^* = \alpha_i e_i^E + \alpha_i x_i^E + \sum_j \left(\alpha_i u_{ij}^* \right)$$

[8] By definition, there are no direct imports associated with exports. These would have been classified as re-exports in the IO tables.

These leakages reflect the rates observed in the benchmark and may of course be set to any other arbitrary ratio (between 0 and 1) at the discretion of the modeller.

Other data sources, compatible with the IO industrial classification and for which it is reasonable to assume a somewhat linear relationship with gross output in the short term, can be integrated into the impacts. For example, let l_j be the number of jobs per industry *j*, then the total impact on jobs l_j^* can be calculated as

$$(35) \qquad l_j^* = \frac{l_j}{g_j} g_j^*$$

Variables measured in volumes, such as pollutant emissions or energy use may also be used in a similar fashion to employment. The mixing of nominal output values and volumes, however, entails some specific risks. This approach can be considered sound if the value and volume measures are for the same year and the analysis is focusing on the structure of the economy for that year. When used for projecting into non-benchmark years, the relationship between values and volumes may be corrupted by the impact of price variations[9]. Of course, the farther the time horizon from the benchmark and the higher the price volatility in the economy, the more important such considerations become. This problem may be attenuated by converting model shocks into constant dollar shocks or by adjusting the ratio of volume to output by output price indices.

2.4 Multipliers

From equation (29), define the industry-by-industry inverse as $\mathbf{INV} = \left[\mathbf{I} - \mathbf{D}(\mathbf{I} - \hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\alpha}})\mathbf{B}\right]^{-1}$. For each industry, its related column in $\mathbf{INV}$ provides an estimate of the production by all other industries used in the production of one unit of output by the given industry. This estimate includes any feedbacks on the industry itself, which appear as the values that exceed unity on the industry row.

Summing across rows provides the gross output multiplier by industry *k*, $GMULT_k$.

$$(36) \qquad GMULT_k = \sum_j INV_{jk}$$

GDP multipliers are obtained by weighting the contribution of each industry by its GDP-to-output ratio.

[9] In fact, other phenomena may impact on this relationship such as technological or regulatory changes or the stage of the business cycle. However, in the short term and barring industry-specific developments, they should be relatively minor.

(37) $$GDPMULT_k = \sum_j \left(INV_{jk} \sum_i \left(\frac{u_{ij}^Y}{g_j} \right) \right)$$

GDP multipliers show the total impact on income-based GDP, including all indirect activities, of any exogenous demand for the outputs of an industry. It is also possible to derive multipliers for each component of GDP by simply isolating the appropriate component *i* in u_{ij}^Y in equation (37).

In a similar fashion to equation (37), employment-to-output ratios can be used to derive employment multipliers. Let l_j be the number of jobs per industry *j*, then the employment multiplier, $LMULT_j$ can be derived as:

(38) $$LMULT_k = \sum_j \left(INV_{jk} \left(\frac{l_j}{g_j} \right) \right)$$

Multipliers for imports, inventory withdrawals, and scrap metal, $MMULT_k$, $NMULT_k$, and $SMULT_k$, respectively, are derived based on the share of each one of these sources of supply in total use by industry.

(39) $$MMULT_k = \sum_j \left(INV_{jk} \sum_i b_{ij} \mu_i \right)$$

(40) $$NMULT_k = \sum_j \left(INV_{jk} \sum_i b_{ij} \beta_i \right)$$

(41) $$SMULT_k = \sum_j \left(INV_{jk} \sum_i b_{ij} \alpha_i \right)$$

When calibrated to the benchmark coefficient and parameter values, the sum of GDP multipliers and all other leakage multipliers must sum to one for each industry.

(42) $$GDPMULT_k + MMULT_k + NMULT_k + SMULT_k = 1$$

This reflects the basic identity that all final expenditures must equal to total incomes generated in the economy, i.e., that expenditure-based GDP must equal income-based GDP, after accounting for all leakages from production. This identity is normally used as a check that all calculations in the model were properly implemented.

3 The Interprovincial Input-Output Model

The Interprovincial Input-Output Model is based on the Provincial Input-Output Accounts. The model is fundamentally similar to the Canadian model, replicating all its linear proportionality assumptions. The interprovincial model, however, accounts for the additional role of interprovincial trade. This leads to a conceptual difference from the national model in the treatment of final expenditures. Thus, although final expenditures are treated as exogenous variables in the national model, interprovincial exports remain endogenous in the interprovincial model. This is necessary to account for the feedbacks generated by economic transactions that cross provincial frontiers. For example, international exports of motor vehicle parts from a province may generate interprovincial imports of steel from other provinces. The interprovincial imports of steel could in turn, however, lead the other provinces to increase their demand for iron ore from the vehicle parts-exporting province.

3.1 Organizing the data

To avoid unnecessary duplication, the provincial model borrows where possible from similar equations in the Canadian model. Where feasible, variables which have the same form in the provincial dimension will simply be distinguished with an additional index denoted as *p*. Some variables or matrices, however, even though they carry the same names as in the Canadian model, will have to be redefined.

Define a new matrix **D** based on replicating equation (14) from the national model for each province[10]. The provincial domestic market share matrix is a block diagonal matrix (of dimensions 14m x 14n, where m is the number of industries and n is the number of commodities), where the sub-matrix $\mathbf{D}_p$ is the domestic market share matrix for each region *p*, for *p* = 1,2,…14.

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D}_{14} \end{bmatrix}$$

The provincial industry technology matrix (14n x 14m) is a block diagonal matrix, where the sub-matrix $\mathbf{B}_p$ is based on equation (16) and represents the domestic industry technology matrix for region *p*.

[10] The provincial IO tables are composed of 10 provinces, 3 territories, and one "government abroad" category. For simplicity, the 14 regions are referred to as provinces and their total number is treated as a constant. It is possible to collapse the number of regions by simply aggregating the IO tables into the desired number of regions, up to 2 regions for example, representing the region of interest and the rest-of-Canada. This would of course be at the cost of a certain loss of precision.

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B}_{14} \end{bmatrix}$$

The provincial gross output matrix, **g**, is a partitioned matrix of the vectors of gross outputs by industry for each province, calculated in a similar manner to equation (13).

$$\mathbf{g} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \vdots \\ \mathbf{g}_{14} \end{bmatrix}$$

As with equation (17) in the Canadian model, intermediate inputs can now, in the provincial dimension, also be defined as a function of gross outputs and the input coefficients:

(43) $\mathbf{u} = \mathbf{Bg}$

The matrix of domestic final demand for all provinces $\mathbf{e}$ is composed of the same subset of final expenditures as in the national model,

$$\mathbf{e} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_{14} \end{bmatrix}$$

The Interprovincial Trade Flow tables show for each commodity all interprovincial and international trade at the modified basic price valuation. Based on these tables, define a variable t_{iod}, which shows for each commodity *i*, imports from all other provinces of origin *o*, in each province of destination *d*. Then, total interprovincial imports m_{ip}^{R} by commodity *i* and province *p*, are the sum of interprovincial imports over the origin dimension

(44) $m_{ip}^{R} = \sum_{o} t_{iod} \qquad (\text{where } d = p)$

While total interprovincial exports x_{ip}^{R} by commodity *i* and province *p*, are simply the sum of the trade flows over the destination dimension

(45) $$x_{ip}^{R} = \sum_{d} t_{iod} \qquad \text{(where } o = p\text{)}$$

Based on the flows tables, which show international imports and re-exports separately, it is possible to assign a variable to international imports m_{ip}^{D}, for commodity *i* in each province *p*. Similarly, a variable for international exports can also be created, x_{ip} for commodity *i* in each province *p*, based on the flows tables.

Two basic identities link supply and demand in the IO tables to the interprovincial trade flow tables. Total domestic supply must equal total interprovincial and international exports in each province

(46) $$q_{ip} - n_{ip}^{W} - s_{ip} \equiv x_{ip}^{R} + x_{ip}$$

While total domestic demand must equal total interprovincial and international imports in each province

(47) $$\sum_{j} u_{ijp} + \sum_{j} e_{ip} \equiv m_{ip}^{R} + m_{ip}^{D}$$

3.2 The model assumptions

As with the national model, market shares as well as technological coefficients remain static. However, in the provincial model, it is necessary to assign demand in each province to the provinces from which production originated. This implies the added assumption that interprovincial and international trade coefficients also remain static.

3.2.1 The trade coefficients

The share of imports from the rest of the world, μ_{ip} can be determined as a share of total imports into each province

(48) $$\mu_{ip} = m_{ip}^{D} / (t_{ip} + m_{ip}^{D})$$

For interprovincial imports, however, things are bit more involved. Based on the interprovincial trade flow tables, define the interprovincial trade coefficients variable r_{iop}, which shows for each commodity *i*, the import shares from all other provinces of origin *o*, in each province of destination *p*. The variable is based on the share of each commodity in total imports. For notational reasons, simply rename the index *d* as *p* in the trade flows variable, i.e., $t_{iop} \equiv t_{iod}$, to obtain

(49) $r_{iop} = t_{iop} / (m_{ip}^R + m_{ip}^D)$

Based on the trade coefficients variable, define a vector of commodities $\mathbf{r}_{op} = r_{iop}$. The provincial commodity share matrix $\mathbf{R}$, is then composed of 14 x 14 sub-matrices formed from the diagonalization of the $\mathbf{r}_{op}$ vectors.

$$\mathbf{R} = \begin{bmatrix} \hat{\mathbf{R}}_{1,1} & \hat{\mathbf{R}}_{1,2} & \cdots & \hat{\mathbf{R}}_{1,14} \\ \hat{\mathbf{R}}_{2,1} & \hat{\mathbf{R}}_{2,2} & & \hat{\mathbf{R}}_{2,14} \\ \vdots & & \ddots & \vdots \\ \hat{\mathbf{R}}_{14,4} & \hat{\mathbf{R}}_{14,2} & \cdots & \hat{\mathbf{R}}_{14,14} \end{bmatrix}$$

Since each interprovincial export is symmetrically an interprovincial import from the perspective of the other provinces and vice versa, the interprovincial trade coefficients can be used to define either interprovincial imports or exports. Thus, when viewed from the angle of the provinces of origin, the trade coefficients also define interprovincial exports associated with domestic demand in each province. This can be seen by reorganizing equation (49) and substituting equation (47) for total domestic demand, to derive interprovincial exports by province based on the demand of all provinces.

(50) $x_{io}^R = \sum_p (r_{iop}(u_{ip} + e_{ip}))$

To convert the above equation into matrix notation, define a vector of inter-provincial exports by commodity for each province as $\mathbf{x}_o^R = x_{io}^R$ and then define a matrix $\mathbf{x}^R$ that groups the interprovincial exports of all provinces

$$\mathbf{x}^R = \begin{bmatrix} \mathbf{x}_1^R \\ \mathbf{x}_2^R \\ \vdots \\ \mathbf{x}_{14}^R \end{bmatrix}$$

With the help of equation (43), equation (50) can then be reformulated as

(51) $\mathbf{x}^R = \mathbf{R}(\mathbf{Bg} + \mathbf{e})$

3.2.2 Inventories and scrap

The inventory and scrap leakage rates are defined as a share of total domestic supply within each province.

(52) $$\beta_{ip} = \frac{-n_{ip}^{W}}{q_{ip} - n_{ip}^{W} - s_{ip}}$$

(53) $$\alpha_{ip} = \frac{-s_{ip}}{q_{ip} - n_{ip}^{W} - s_{ip}}$$

Form the commodity vectors $\boldsymbol{\beta}_p = \beta_{ip}$ and $\boldsymbol{\alpha}_p = \alpha_{ip}$, diagonalize, and reorganize these vectors into the bloc diagonal matrix $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$, respectively.

$$\boldsymbol{\beta} = \begin{bmatrix} \hat{\boldsymbol{\beta}}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \hat{\boldsymbol{\beta}}_2 & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \hat{\boldsymbol{\beta}}_{14} \end{bmatrix} \qquad \boldsymbol{\alpha} = \begin{bmatrix} \hat{\boldsymbol{\alpha}}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \hat{\boldsymbol{\alpha}}_2 & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \hat{\boldsymbol{\alpha}}_{14} \end{bmatrix}$$

3.3 The model

3.3.1 Output determination

By substituting equations (46) and (51) into equation (52), the inventory and scrap leakage rates can be used to reformulate the value of leakages in each province as a function of the domestic demands of all provinces as well as of international exports.

(54) $$\mathbf{n}^{\mathbf{W}} = -\boldsymbol{\beta}(\mathbf{R}(\mathbf{Bg} + \mathbf{e}) + \mathbf{x})$$

By similar substitution for equation (53)

(55) $$\mathbf{s} = -\boldsymbol{\alpha}(\mathbf{R}(\mathbf{Bg} + \mathbf{e}) + \mathbf{x})$$

These leakages reflect the rates observed in the benchmark and may of course be set to any other arbitrary ratio (between 0 and 1) at the discretion of the modeller.

The model is based on the substitution of equations (51), (54), and (55) into the basic supply identity in equation (46), to obtain

(56) $$\mathbf{q} + \boldsymbol{\beta}(\mathbf{R}(\mathbf{Bg} + \mathbf{e}) + \mathbf{x}) + \boldsymbol{\alpha}(\mathbf{R}(\mathbf{Bg} + \mathbf{e}) + \mathbf{x}) = \mathbf{R}(\mathbf{Bg} + \mathbf{e}) + \mathbf{x}$$

Multiplying through by **D** and isolating **g**, provides the reduced form equation

(57) $$\mathbf{g}^{*} = [\mathbf{I} - \mathbf{D}(\mathbf{I} - \boldsymbol{\beta} - \boldsymbol{\alpha})\mathbf{RB}]^{-1}\mathbf{D}(\mathbf{I} - \boldsymbol{\beta} - \boldsymbol{\alpha})(\mathbf{Re} + \mathbf{x})$$

The inverse in equation (57) may be used to estimate the impact by province on gross output by industry $\mathbf{g}^*$, of any exogenous demand in any province.

3.3.2 Detailed impacts

Similarly to equation (31), the impact on GDP of any exogenous shock is equal to the gross output requirements times the GDP-to-output ratios

(58) $$u_{ijp}^{Y*} = \frac{u_{ijp}^{Y}}{g_{jp}} g_{jp}^{*} \qquad (\text{for p} = 1,2,...14)$$

Intermediate inputs by industry are equal to the industry output times the input coefficients

(59) $$u_{ijp}^{*} = \frac{u_{jp}}{g_{jp}} g_{jp}^{*}$$

Imports per commodity m_{ip}^{*} are composed of direct imports related to final domestic expenditures and intermediate inputs. The first term on the right-hand-side of equation (60) shows the value of direct imports, as the import rate times the exogenous final expenditure. The second term on the right-hand-side shows the imports of intermediate inputs summed across industries based on the average import rate by commodity. For industry shocks, only the second term on the right-hand-side is relevant. Of course, the detailed imports by industry may be separately obtained from the bracketed expression in the second term.

(60) $$m_{ip}^{*} = \mu_{ip} e_{ip}^{E} + \sum_{j} \left(\mu_{ip} u_{ijp}^{*} \right)$$

Other leakages from current period production are calculated in a similar fashion to imports. These can be obtained by inserting the g^* variable and the exogenous final expenditure values $\mathbf{e}$ and $\mathbf{x}$, $\mathbf{e}^E$ and $\mathbf{x}^E$ respectively, into equations (54) and (55).

(61) $$\mathbf{n}^{\mathbf{W}*} = -\boldsymbol{\beta}(\mathbf{R}(\mathbf{B}\mathbf{g}^{*} + \mathbf{e}^{E}) + \mathbf{x}^{E})$$

(62) $$\mathbf{s}^{*} = -\boldsymbol{\alpha}(\mathbf{R}(\mathbf{B}\mathbf{g}^{*} + \mathbf{e}^{E}) + \mathbf{x}^{E})$$

The impact on interprovincial trade can be obtained from equations (47) and (49), and the values of u_{ij}^{*} and e_{ij}^{E}

(63) $$t_{iop}^{*} = r_{iop}(u_{ip}^{*} + e_{ip}^{E})$$

3.4 Multipliers

While fundamentally similar to national multipliers, provincial multipliers make it possible to specify the geographic dimension of impacts. From equation (57), extract the provincial inverse as $\mathbf{INV} = [\mathbf{I} - \mathbf{D}(\mathbf{I} - \boldsymbol{\beta} - \boldsymbol{\delta})\mathbf{RB}]^{-1}$, where the $\mathbf{INV}$ matrix has 14 x 14 sub-matrices, each of dimension n x n, where n is the number of industries.

$$\mathbf{INV} = \begin{bmatrix} \mathbf{INV}_{1,1} & \mathbf{INV}_{1,2} & \cdots & \mathbf{INV}_{1,14} \\ \mathbf{INV}_{2,1} & \mathbf{INV}_{2,2} & & \mathbf{INV}_{2,14} \\ \vdots & & \ddots & \vdots \\ \mathbf{INV}_{14,1} & \mathbf{INV}_{14,2} & \cdots & \mathbf{INV}_{14,14} \end{bmatrix}$$

In response to a one unit increase in output for a given industry, its corresponding column in each province *P* of $\mathbf{INV}_{iP}$ provides an estimate of production by all industries in all provinces *i*. For each province *P*, within-province impacts are therefore found in the diagonal sub-matrices of $\mathbf{INV}$, i.e., where $i = P$ in $\mathbf{INV}_{ip}$ and out-of-province impacts are observed in the non-diagonal sub-matrices of $\mathbf{INV}$, i.e., where $i \neq P$ in $\mathbf{INV}_{iP}$.

Adding the matrices in each column of $\mathbf{INV}$ and then summing across rows provides the vector of total gross output multipliers for all industries $\mathbf{gmult}_P$ in each province *P*.

(64) $\mathbf{gmult}_P = \mathbf{i}_n(\mathbf{INV}_{1,P} + \mathbf{INV}_{2,P} + \cdots + \mathbf{INV}_{14,P})$ (for P = 1,2...,14 and n = number of industries)

As with the inverse, or any other multipliers, it is possible to distinguish between within and out-of-province gross output multipliers by separating the on- from the off-diagonal sub-matrices of $\mathbf{INV}$.

GDP multipliers $\mathbf{gdpmult}_p$ are obtained by weighting the contribution of each industry in the inverse by its GDP-to-output ratio. Define a variable of GDP-to-output ratios by industry for each province *p*, as $gdpratio_{ijp}$. Using the same nomenclature for primary inputs as in the national model but adding the index *p* to denote provinces

(65) $$gdpratio_{jp} = \sum_i \left(\frac{u^Y_{ijp}}{g_{jp}} \right)$$

Form an industry-length row vector $\mathbf{o}_P^{GDP}$, based on the variable $gdpratio_{jp}$ for each province *P*. The GDP multipliers $\mathbf{gdpmult}_P$ can then be formulated for each province *P*, as

(66) $$\mathbf{gdpmult}_P = \mathbf{o}_1^{GDP}\mathbf{INV}_{1,P} + \mathbf{o}_2^{GDP}\mathbf{INV}_{2,P} + \cdots + \mathbf{o}_{14}^{GDP}\mathbf{INV}_{14,P} \qquad (\text{for P} = 1,2...,14)$$

GDP multipliers show the total impact on income-based GDP of any exogenous demand for the gross outputs of an industry. It is also possible to derive multipliers for each component of GDP by simply isolating the appropriate component *i* in u_{ijp}^{Y} in equation (65).

Similarly to GDP multipliers, employment multipliers can be calculated by weighting the inverse with ratios of employment to gross output. Define a variable of employment-to-output ratios by industry for each province *P* as $lratio_{jp}$,

(67) $$lratio_{jp} = \frac{l_{jp}}{g_{jp}}$$

Based on $lratio_{jp}$, form an industry-length row vector $\mathbf{o}_P^L$, for each province *p*. Total employment multipliers $\mathbf{lmult}_P$ can then be formulated for each province p, as

(68) $$\mathbf{lmult}_P = \mathbf{o}_1^{L}\mathbf{INV}_{1,P} + \mathbf{o}_2^{L}\mathbf{INV}_{2,P} + \cdots + \mathbf{o}_{14}^{L}\mathbf{INV}_{14,P} \qquad (\text{for P} = 1,2...,14)$$

Employment multipliers show the total impact on employment of any exogenous demand for the gross outputs of an industry.

Multipliers for imports $\mathbf{mmult}_P$, inventory withdrawals $\mathbf{nmult}_P$, and scrap metal $\mathbf{smult}_P$, are based on an estimate by commodity of the average use of these sources of supply in each industry and province.

(69) $$\mathbf{mmult}_P = \boldsymbol{\mu}_1\mathbf{B}_1\mathbf{INV}_{1,P} + \boldsymbol{\mu}_2\mathbf{B}_2\mathbf{INV}_{2,P} + \cdots + \boldsymbol{\mu}_{14}\mathbf{B}_{14}\mathbf{INV}_{14,P} \qquad (\text{for P} = 1,2...,14)$$

(70) $$\mathbf{nmult}_P = \boldsymbol{\beta}_1\mathbf{B}_1\mathbf{INV}_{1,P} + \boldsymbol{\beta}_2\mathbf{B}_2\mathbf{INV}_{2,P} + \cdots + \boldsymbol{\beta}_{14}\mathbf{B}_{14}\mathbf{INV}_{14,P} \qquad (\text{for P} = 1,2...,14)$$

(71) $$\mathbf{smult}_P = \boldsymbol{\alpha}_1\mathbf{B}_1\mathbf{INV}_{1,P} + \boldsymbol{\alpha}_2\mathbf{B}_2\mathbf{INV}_{2,P} + \cdots + \boldsymbol{\alpha}_{14}\mathbf{B}_{14}\mathbf{INV}_{14,P} \qquad (\text{for P} = 1,2...,14)$$

For each industry in each province, the sum of GDP multipliers and all other leakage multipliers must sum to one.

$$\mathbf{gdpmult}_P + \mathbf{mmult}_P + \mathbf{nmult}_P + \mathbf{smult}_P = \mathbf{i} \qquad (\text{for P} = 1,2...,14)$$

4 Other issues

4.1 Conversion of purchaser price final expenditures

The models as presented up to now have been based on IO tables valued in modified basic prices. For final expenditure shocks at purchaser prices, it is necessary to convert to the modified basic price valuation based on the observed margin rates in the IO tables. The geographic aspect is ignored since, generally speaking, the same calculations apply, regardless of the geographic dimension.

Let e_i^P and x_i^P be the purchaser price vectors of domestic final demands and international exports and let e_{ik}^M and x_{ik}^M be the margin matrices associated them, where $k = 1,\ldots,K$, are all vectors of margins for trade and transport but excluding taxes on products. Similarly, let e_{il}^T and x_{il}^T be the tax margin matrices, by detailed type of tax $l = 1,\ldots,L$.

It is then possible to derive the margin rates on domestic expenditures, $\tilde{e}_{ik}^M$ and $\tilde{e}_{ik}^T$, and international exports $\tilde{x}_{ik}^M$ and $\tilde{x}_{il}^T$ as the value of each margin divided by its associated purchaser price expenditure.

(72) $$\tilde{e}_{ik}^M = \frac{e_{ik}^M}{e_i^P}$$

(73) $$\tilde{e}_{il}^T = \frac{e_{il}^T}{e_i^P}$$

(74) $$\tilde{x}_{ik}^M = \frac{x_{ik}^M}{x_i^P}$$

(75) $$\tilde{x}_{il}^T = \frac{x_{il}^T}{x_i^P}$$

Let a_{ik} be a binary mapping from commodity i to margin k. Final domestic expenditures and exports, in basic prices, are derived based on the purchaser price expenditures and the observed margin rates.

(76) $$e_i = e_i^P - \sum_l e_i \tilde{e}_{il}^T - \sum_k e_i \tilde{e}_{ik}^M + \sum_k \left(\sum_i \left(e_i^P \tilde{e}_{ik}^M \right) a_{ik} \right)$$

(77) $$x_i = x_i^P - \sum_l x_i \tilde{x}_{il}^T - \sum_k x_i \tilde{x}_{ik}^M + \sum_k \left(\sum_i \left(x_i^p \tilde{x}_{ik}^M \right) a_{ik} \right)$$

In the right-hand-side of equations (76) and (77), the second and third terms removes the margins associated with each purchaser price commodity in the first term. The sum of the inside bracket in the fourth term aggregates all the individual margins associated with each commodity into a total by margin; a_{ik} then maps the total of each margin to its appropriate margin commodity in the *n* commodities list[11]. Thereafter, the sum of the outside bracket in the fourth term eliminates the margin dimension, which is no longer required.

Total indirect taxes, by type of tax, on final domestic expenditures e^T and exports x^T are based on the observed tax rates by commodity summed across all commodities.

(78) $$e_l^T = \sum_i \left(e_i \tilde{e}_{il} \right)$$

(79) $$x_l^T = \sum_i \left(x_i \tilde{x}_{il} \right)$$

[11] For example, all the retail margins associated with each commodity are summed and assigned to the "retail services" commodity.

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