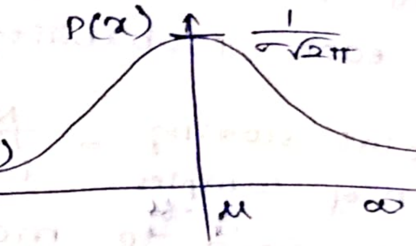
MOT

Gaussian Distribution (Normal Distribution):

$$P(x \mid \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Univariate



$\underbrace{\int_{-\infty}^{\infty} P(x)\,dx = 1 \qquad 0 \le P(x)}$ can be >1 (continuous)

for any fn. to be prob. density fn., these two must satisfy

Multivariate Gaussian:

- Variance tells the spread of the data around μ
- Here variance ~~should~~ also tells about the spread of data over all dimensions (all random variables)

$[x_1, x_2, x_3]$ Here, Σ - covariance matrix does this

It tells how one dimension varies w.r.t other dimension

Consider x to be a vector (matrix - column)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{then} \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

$$P(x \mid \mu, \Sigma) = \frac{1}{(2\pi)^{d/2}\,|\Sigma|} e^{-\frac{1}{2}\underbrace{(x-\mu)^T(\Sigma^{-1})[x-\mu]}_{\text{Mahalanobis distance}}}$$

When all dimension are independent, i.e, do not effect each other then, Σ is a unit vector.

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{then } \Sigma^{-1} \Rightarrow \text{unit vector}$$

Then $(x-\mu)^T \cdot [x-\mu] \rightarrow$ dot product of two vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

then mahalanobis dist. $d_1^2 + d_2^2 + d_3^2$

is large if the r.v is far from μ

is small if r.v is nearer to μ

Consider n values,

$x_1, x_2, \ldots \ldots x_N \rightarrow$ iid

i → independent

i → identically

d → distributed

Joint probability of two independent events is the product of each probability of an event.

$$\underset{\text{fn of samples}}{\text{Joint scoring}} = \prod_{i=1}^{N} P(x_i) \rightarrow \text{likelihood fn.}$$

~~To~~ We need to maximise the likelihood fn, (Joint scoring fn) to find μ, σ (variables) then we get,

$$\mu = \frac{1}{N} \Sigma x_i$$

$$\sigma^2 = \frac{1}{N} \Sigma (x_i - \mu)^2$$

$$\log(\text{Joint scoring fn}) = \log\left(\prod_{i=1}^{N} P(x_i)\right)$$

$$= \sum_{i=1}^{N} \log\left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2}\right)$$

$$= \sum_{i=1}^{N} \left[\log \frac{1}{\sqrt{2\pi}\,\sigma} - \frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right]$$

To maximise, diff w.r.t μ (1st take derivative w.r.t μ bcoz σ depends on μ)

$$-\frac{1}{2} \times 2 \sum_{i=1}^{N} \left(\frac{x_i - \mu}{\sigma}\right) \times \frac{-1}{\sigma} = 0$$

$$\sum_{i=1}^{N} \left(\frac{x_i - \mu}{\sigma}\right) = 0$$

$$\sum_{i=1}^{N} x_i - N\mu = 0 \Rightarrow \boxed{\mu = \frac{1}{N} \sum_{i=1}^{N} x_i}$$

diff wrt σ,

$$+\frac{N}{\cancel{\sigma}} + \frac{-\cancel{2}}{\cancel{2}} \left[\frac{x_i - \mu}{\cancel{\sigma}}\right] \times \left[\frac{x_i - \mu}{\sigma^2}\right] = 0$$

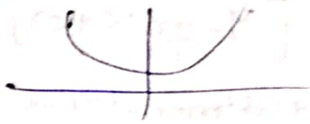
$$\sum_{i=1}^{N} \frac{(x_i - \mu)^2}{\sigma^2} = N$$

$$\boxed{\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2}$$

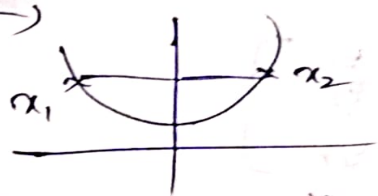
$20x_1 - 30x_2 + 15 = 0$

Convex function: for any ~~line~~ point lieing on a line that is formed by joining any two points on the curve, must be inside the curve. Then, that curve (fn) is said to be convex fn.

$y = ax^2 + bx + c$



Convex optimisation:
(i) Linear Programming
(ii) Quadratic "



$\cancel{\min\ f_0(x)}$

$\cancel{s.t\ \ f_i(x) \le 0}$

$0 \le \theta \le 1$

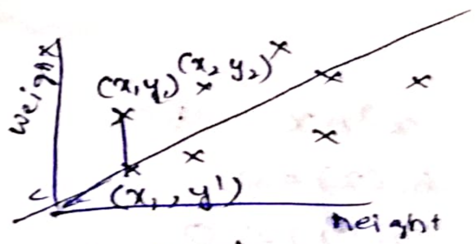
if $f(\theta x_1 + (1-\theta) x_2)$

$\le \theta f(x_1) + (1-\theta) f(x_2)$

then $f(x)$ is said to be a convex fn.

Optimisation:
minimise $f_0(x)$ $\rightarrow x \in R^n$ } Linear Programming
such that $f_i(x) \le 0$

(1) Least squares problem: (2) Linear programming



$K = 2 \rightarrow$ attributes
two - dimensional

$y = mx + c$

$y = mx + c$ should be chosen such that sum of squares of distances (vertical) from line to each point should be minimum.

vertical distance $\Rightarrow y_i - \underbrace{(m x_i + c)}_{y'}$ $\rightarrow$ point on the line (y-coordi)

$(y_1 - (m x_1 + c))^2 + (y_2 - (m x_2 + c))^2 + \cdots (y_n - (m x_n + c))^2 \Rightarrow$ should be minimised

This can be written in vector form as,

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$[Y-(mX+c)]^T [Y-(mX+c)] \Rightarrow (Y-(mX+c))^2$$

X - data ~~set~~ matrix

Multidimensional → Refer TB

$$X = \begin{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} & \end{bmatrix}_{N \times K} \quad \text{→ ~~constants~~} \qquad m = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_K \end{bmatrix} [x_1\, x_2 \cdots x_k]$$

K - attributes (dimensions)

$$(y_1 - (m_1 x_{11} + m_2 x_{12} + m_3 x_{13}))^2 + \cdots$$

$$\theta = \underbrace{[Y - Xm]^T}_{[d_1 \cdots d_N]} \underbrace{[Y - Xm]}_{\text{vector}} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$[d_1 \cdots d_N] \begin{bmatrix} d_1 \\ \vdots \\ d_N \end{bmatrix} = \sum_{i=1}^{N} d_i^2$$

let ~~$Y_i - X_i m$~~ $y_i - x_i m$ be d_i

Linear Programming:

$$\min \; c^T x \quad \cdots \quad x \in \mathbb{R}^K$$

$$a_i^T x \leq 0 \qquad 1 \leq i \leq N$$

linear constraint $\quad Ax \leq 0$

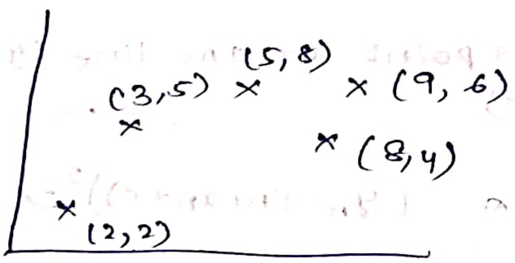
a_i^T → vector matrix

Quadratic Programming: $\quad \min \; x^T A x$

$$h_i(x) = 0$$
$$g_i(x) \leq 0$$

(1) Find if (i) $x \log x$ (ii) e^{ax} (iii) x^α $\quad \alpha \geq 1$, $\alpha \leq 0$ are convex.

(2)



find m, c

(1) $f(x) = x \log x$

$$f(\theta x_1 + (1-\theta) x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$$

$(\theta x_1 + (1-\theta)x_2)\log(\theta x_1 + (1-\theta)x_2) \le \theta x_1 \log x_1 + (1-\theta)(x_2 \log x_2)$

$\log(\theta(x_1 - x_2) + x_2) \le \theta x_1 \log x_1 + x_2 \log x_2 - \theta x_2 \log x_2$

$\le \theta \left(\log \frac{x_1^{x_1}}{x_2^{x_2}}\right) + x_2 \log x_2$

$\log(\theta e^x$

(2) ~~$f(x) = e^{ax}$~~

~~$f(\theta e^{ax_1} + C$~~ e^a

(2) $X = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 8 \\ 9 \end{bmatrix}$ m $\quad Y = \begin{bmatrix} 2 \\ 5 \\ 8 \\ 4 \\ 6 \end{bmatrix}$

$[Y - (mX + c)] \Rightarrow \begin{bmatrix} 2-(2m+c) \\ 5-(3m+c) \\ 8-(5m+c) \\ 4-(8m+c) \\ 6-(9m+c) \end{bmatrix}$

$[Y - (mX+c)]^T [Y - (mX+c)] = \{ (2-(2m+c))^2 + (5-(3m+c))^2 + (8-(5m+c))^2 + (4-(8m+c))^2 + (6-(9m+c))^2$

$4(2-(2m+c))(-2) + 2(5-(3m+c))(-3) + 10(8-(5m+c)) + 16(4-(8m+c)) + 18(6-(9m+c)) = 0$

$2(2-(2m+c)) + 2(5-(3m+c)) + 2(8-(5m+c))$

$+ 2($

(2) $\sum(y_i - (mx_i + c))^2 = \sum(y_i^2 + m^2x_i^2 + c^2 + 2mx_ic - 2mx_iy_i - 2y_ic)$

$= \sum y_i^2 + m^2 \sum x_i^2 + nc^2 + 2mc\sum x_i - 2m\sum x_iy_i - 2c\sum y_i$

$\frac{d}{dm} \Rightarrow \quad 2m\sum x_i^2 + 2c\sum x_i - 2\sum x_iy_i = 0$

$c = 3.54$
$m = 0.26$

$\frac{d}{dc} \Rightarrow \quad 2nc + 2m\sum x_i - 2\sum y_i = 0$

$\sum x_i = 27 \quad \sum y_i = 25 \quad \sum x_i^2 = 183 \quad \sum y_i^2 = 145 \quad \sum x_iy_i = 145$

$366m + 54c - 290 = 0 \quad 10c + 54m - 50 = 0 \quad m = 0.26, \; c = 3.54$

$(\theta x_1 + (1-\theta$ $\log(\theta(x_1$ $\leq \theta($

(2) $f(x) = e^{x^2}$

(2) $x = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 8 \\ 9 \end{bmatrix}$ $y = \begin{bmatrix} 2 \\ 5 \\ 8 \\ 4 \\ 6 \end{bmatrix}$

$[y - (mx + c)] \Rightarrow \begin{bmatrix} 2 - (2m + c) \\ 5 - (3m + c) \\ 8 - (5m + c) \\ 4 - (8m + c) \\ 6 - (9m + c) \end{bmatrix}$

$[y - (mx + c)]^T [y - (mx + c)] = (2 - (2m + c))^2 + (5 - (3m + c))^2 + (8 - (5m + c))^2 + (4 - (8m + c))^2 + (6 - (9m + c))^2$

$4(2 - (2m + c))(-2) - 6[5 - (3m + c)](-3) + 10(8 - (5m + c))$
$+ 16(4 - (8m + c)) + 18(6 - (9m + c)) = 0$

$2(2 - (2m + c)) + 2(5 - (3m + c)) + 2(8 - (5m + c))$

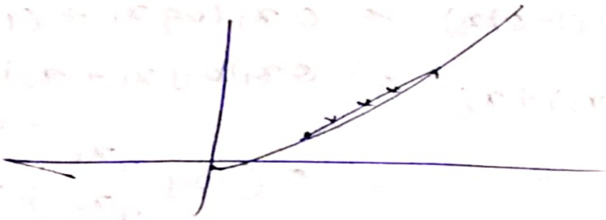
(2) $\Sigma(y_i - (mx_i + c))^2 = \Sigma(y_i^2 + m^2 x_i^2 + c^2 + 2mx_i c - 2mx_i y_i - 2y_i c)$

$\frac{d}{dm} \Rightarrow$

$\frac{d}{dc} \Rightarrow$

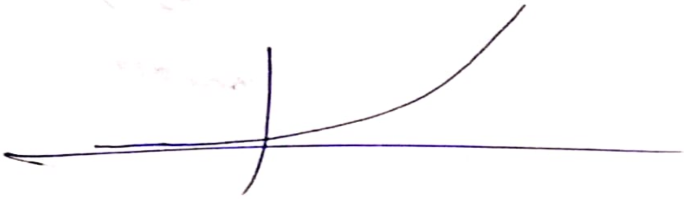
$\Sigma x_i = 27$

① $x \log x$



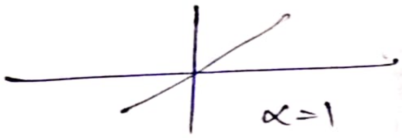
. convex

e^{ax}



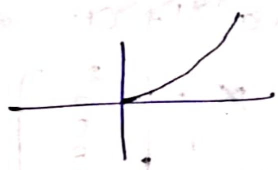
convex.

x^{α}



$\alpha = 1$

. $\alpha = 0$



$0 < \alpha < 1$

. convex

Optimising multi dimensional variate:

$(Y - Xm)^T (Y - Xm)$

$\Rightarrow Y^T Y - \cancel{X m^T} \cancel{X^T m^T Y} \; m^T X^T Y$

$- Y^T X m - \cancel{Y^T m X} + \cancel{m X^T X}$

$+ m^T X^T X m \quad \cancel{+ X^T m^T m X}$

$\Rightarrow \cancel{Y^T Y - 2 Y^T m X + X^T m^T m X}$

$\frac{\partial}{\partial m} \cancel{\Rightarrow} \quad \cancel{-2 Y^T X}$

$\Rightarrow Y^T Y - 2 Y^T X m + m^T X^T X m$

$m = a$ notation

$\frac{\partial}{\partial a} \Rightarrow -\cancel{2} Y^T X + \cancel{2} X^T X a = 0$

$a = (X^T X)^{-1} (Y^T X)$

$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n \times 1}$

$X = \begin{bmatrix} & \\ & \end{bmatrix}_{n \times k}$

$m = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_k \end{bmatrix}_{k \times 1}$

$X^T X \rightarrow$ positive definite matrix

↳ whose eigen values are +ve.

$a^T \quad X \quad a$

$\underset{1\times2}{\begin{bmatrix} a_1 & a_2 \end{bmatrix}} \underset{2\times2}{\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$\begin{bmatrix} a_1 x_{11} + a_2 x_{21} & a_2 x_{12} + a_1 x_{22} \end{bmatrix}$

$\begin{bmatrix} a_1 & a_2 \end{bmatrix}$

$a_1^2 x_{11} + a_1 a_2 x_{21} + a_2^2 x_{22} + a_1 a_2 x_{12}$

$$\frac{\partial(a^T X a)}{\partial a} = \begin{bmatrix} 2a_1 x_{11} + 2a_2 x_{21} \\ 2a_2 x_{22} + 2a_1 x_{21} \end{bmatrix} = 2Xa$$

considering X to symmetric matrix

Affine set: Affine hull

Affine Combination: Linear sum of all data points

$\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$ where $\theta_1 + \theta_2 + \cdots \theta_k = 1$

Convex set:

convex combination → similar to affine comb. except that

Convex hull $0 \le \theta \le 1$

& $\theta_1 + \theta_2 + \cdots + \theta_k = 1$

Affine function:

$x \in$ Affine set

$f(x) = Ax + b$

↳ helps x in transformation

$A \to R^{m\times n}$ $x \in R^{n\times 1}$ $b \in R^m$

$Ax \Rightarrow R^m \to x$ is undergoing transformation.

And b is allowing the vector to move (changes the distance)

fn that allows transformation of a vector with some displacement

Convex fn: Consider two points from convex set. Let them be x, y

$f(\theta x + (1-\theta) y) \le \theta f(x) + (1-\theta) f(y)$ $(x, f(x))$ $(y, f(y))$

convex:

affine: $ax+b$ on R, for any $a, b \in R$

exponential: e^{ax}, for any $a \in R$

powers: x^α on R_{++} for $\alpha \ge 1$ or $\alpha \le 0$

powers of absolute value: $|x|^p$ on R, for $p \ge 1$

negative entropy: $x \log x$ on R_{++}

concave:

powers: x^α on R_{++}, for $0 \le \alpha \le 1$

logarithm: $\log x$ on R_{++}

affine: $ax+b$ on R, for any $a, b \in R$

First-order condition: $f(x): \mathbb{R}^n \to \mathbb{R}$ $x \in \mathbb{R}^n$

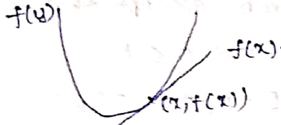
If f is differentiable if dom f is open & gradient. exi:

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \dots \frac{\partial f(x)}{\partial x_n} \right)$$

exists at each $x \in$ dom f

1st order condition: diff. f with convex domain is convex iff

$$f(y) \geq \underbrace{f(x) + \nabla f(x)^T (y-x)}_{\text{1st order approximation around } x} \text{ for all } x, y \in \text{dom } f.$$



1st order approximation of f is global underestimator

Second order condition:

f is twice differentiable if dom f is open & the Hessian $\nabla^2 f(x) \in S^n$

$$\nabla^2 f(x)_{ij} =$$

2nd order condition: for twice diff. f with convex domain

f is convex iff $\nabla^2 f(x) \geq 0$ for all

$a x_1^2 + b x_2^2 + c x_1 x_2$ 2-dim

$$\nabla f(x) = \begin{bmatrix} 2a x_1 + c x_2 \\ 2b x_2 + c x_1 \end{bmatrix} \qquad \nabla^2 f(x) = \begin{bmatrix} 2a & c \\ c & 2b \end{bmatrix}$$

$\hookrightarrow$ size = no. of dimensions $\qquad n \times n \quad 2 \times 2$

least squares:

$$f(x) = \|Ax - b\|_2^2$$

$x : D \times 1$

$A = D \times n$ ($D \downarrow$ no. of data pts, $n \to$ # dimensions.)

$$f(x) = (Ax-b)^T (Ax-b)$$

gradient of $f(x)$: $\nabla(f(x)) = A^T(Ax-b) + (Ax-b)^T A$

$$= 2A^T(Ax-b)$$

$$\nabla^2(f(x)) = 2A^T(A)$$

$\therefore$ convex for any A.

if $\forall x \in \mathbb{R}^k$ $x^T M x \geq 0$ then M is said to be of +ve determinant

quadratic-over-linear: $f(x,y) = x^2/y$

$$\nabla^2 f(x,y) = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^T \succeq 0 \qquad \text{convex for } y > 0$$

$$\nabla f(x,y) = \begin{bmatrix} 2x/y \\ -x^2/y^2 \end{bmatrix} \qquad \nabla^2 f(x,y) = \begin{bmatrix} 2/y & \cancel{-2x/y^2} \\ \cancel{-2x/y^2} & 2x^2/y^3 \end{bmatrix}$$

$$= \begin{bmatrix} 2/y & -2x/y^2 \\ -2x/y^2 & 2x^2/y^3 \end{bmatrix}$$

$$\left(\begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y & -x \end{bmatrix} \quad \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix} \right)$$

$$= 2/y^3 \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix}$$

$$\nabla^2 f(x,y) = 2/y^3 \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^T$$

log sum exp: $f(x) = \log \sum_{k=1}^{n} \exp x_k$ is convex

$$\nabla^2 f(x) =$$

Optimization problem in standard form:

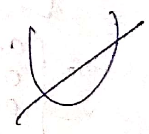
minimize $f_0(x)$

subject to $f_i(x) \leq 0$ $i = 1, \dots, m$

$h_i(x) = 0$ $i = 1, \dots, p$

Eg: min $5x^2 + 10y^2 - 8xy$

s.t $2x - 5y = 6$



min $f(x)$ s.t (x,y) must lie on this line

Ans: $(x,y) = (0, -6/5)$

$2x = 6 + 5y$ $\qquad x = \frac{6+5y}{2}$

$$5\left(\frac{6+5y}{2}\right)^2 + 10y^2 - 8\left(\frac{6+5y}{2}\right)y$$

$f(x) = \log \sum_{k=1}^{n} e^{x_k}$ Proove that this fn. is convex : $x \in R^n$

$$f(x) = \log\left(e^{x_1} + e^{x_2} + e^{x_3} + \dots + e^{x_n}\right)$$

$$\nabla(f(x)) = \begin{bmatrix} \frac{1}{\sum_{k=1} e^{x_k}} \times e^{x_1} \\ \vdots \\ \frac{1}{\sum_{k=1} e^{x_k}} \times e^{x_n} \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{1}{\sum_{k=1}^{n} e^{x_k}} \times e^{x_1} + \left(\frac{-1}{(\sum_{k=1}^{n} e^{x_k})^2}\right) \times e^{x_1} \dots & \dots & \cancel{\frac{1}{\sum e^{x_k}}} - \frac{1}{(\sum_{k=1} e^{x_k})^2} \times e^{x_n} \\ \vdots & & \\ \frac{1}{\sum_{k=1}^{n} e^{x_k}} \times e^{x_n} + \left(\frac{-1}{(\sum_{k=1} e^{x_k})^2}\right) \times e^{x_n} & \dots & \cancel{\frac{-1}{(\sum_{k=1}^{n} e^{x_k})^2}} \times e^{x_1} \end{bmatrix}$$

$$a_i = \frac{e^{x_i}}{\sum_{k=1}^{n} e^{x_k}} \qquad \nabla \Rightarrow a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} \qquad a_1 + a_2 + \dots + a_n = 1$$

$$\frac{\partial a_i}{\partial x_j} = -a_i a_j \qquad \frac{\partial a_i}{\partial x_i} = a_i(1-a_i)$$

$x^T M x$

$$\nabla^2 f(x) = \begin{bmatrix} a_1(1-a_1) & -a_1 a_2 & \dots \\ & a_i(1-a_i) & -a_i a_j \end{bmatrix}$$

a_1
a_2
a_3

$$= \begin{bmatrix} a_1 - a_1^2 & -a_1 a_2 & \dots \\ & a_i - a_i^2 & \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & \\ 0 & & a_3 \\ \vdots & & \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_n \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}$$

To proove $\nabla^2 f(x) \geq 0$ $\quad v^T \cdot \nabla^2 f(x) \cdot v \geq 0$

$$\sum_{i=1}^{n} a_i(1-a_i) v_i^2 - 2\sum_{i,j} a_i a_j v_i v_j$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}_{(n\times 1)} \begin{bmatrix} a_1(1-a_1) & -a_1 a_2 & \dots \\ & a_i(1-a_i) & \end{bmatrix}_{(n\times n)} \begin{bmatrix} v_1 & v_2 & v_3 & \dots & v_n \end{bmatrix}_{(1\times n)}$$

$$\begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}_{(1\times n)} \begin{bmatrix} a_1(1-a_1) & -a_1 a_2 & \dots \\ & a_i(1-a_i) & \end{bmatrix}_{(n\times n)} \begin{bmatrix} v_1 \\ v_2 \\ v_n \end{bmatrix}_{(n\times 1)}$$

$(1\times n)$

$\cancel{\sum_{i,j=1}^{n} a_i v_j (1-a_i)}$

$$\begin{bmatrix} a_1(1-a_1)v_1 + (-a_1 a_2)v_2 + (-a_3 a_3)v_3 \dots & -a_1 a_2 v_1 + a_2(1-a_2)v_2 \dots \end{bmatrix}_{(1\times n)} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{(n\times 1)}$$

$$\sum_{i=1}^{n} a_i(1-a_i)v_i^2 - 2\sum_{i,j=1}^{n} a_i a_j v_i v_j \geq 0 \checkmark$$

$$\sum_{i=1}^{n} a_i v_i^2 - \left(\sum a_i v_i\right)^2 \geq 0$$

$$\sum_{i=1}^{n} a_i v_i^2 - \sum_{i=1}^{n} a_i^2 v_i^2 - 2\sum_{i,j=1}^{n} a_i a_j v_i v_j \geq 0$$

$$\Rightarrow \sum_{i=1}^{n} a_i v_i^2 - \left(\sum_{i=1}^{n} a_i v_i\right)^2 \geq 0$$

$$= (a_1 v_1^2 + a_2 v_2^2 + \dots a_n v_n^2) - (a_1^2 v_1^2 + a_2^2 v_2^2 + \dots + a_n^2 v_n^2)^2 \geq 0$$

Simplex Method:

Model: A program used to represent the real working principle of a system

Simulation - physical system uses model   Model: program

optimisation problems
- category I: set of alternatives - finite & less (scoring/ranking method)
- Set of alternatives is infinite or large.

Solving this:

1. Problem formulation
2. Construction of model
3. Finding soln of model
4. Implementing the soln of model → simulation

Operation reasearch (OR): It is an ~~decision~~ approach to decision-making, which involves a set of methods to operate a system.

ORModels →
- Linear programming
- Integer programming
- Dynamic programming
- Network model, transport model
- Non linear programming

Linear Programming: Technique where we depict complex relationships through linear functions & then find the optimum points

→ Used for obtaining the most optimal soln for problem with given constraints.

→ Here, we formulate our real life prob. into mathematical model.

Decision Variables: Variables which will decide my o/p.

Objective fn.: Objective of making decision (The final obj. we need)

Constraints: Restrictions on decision variables.

Non-negativity restriction: for all linear programs, decision variables ≥ 0

Process:
1. Identify the decision variables
2. Write the objective fn.
3. Mention the constraints
4. Explicitly state the non-negativity restriction.

Solving Linear program →
- Enumeration Method
- Graphical Method
- R language
- opensolver
- Simplex Method
- NorthWest Corner Method & Least Cost Method

Graphical Method: The optimal feasible soln is achieved at the point of int. ~~where~~ of linear inequalities subjected to the constraints.

Feasible region explains the values our model can take.

Simplex Method:

→ Iterative procedure for getting the most feasible soln.

→ Here, we keep transforming the value of basic variables to get maximum value for the objective fn.

* A linear programming fn. is in its standard form if it seeks to maximize the obj. fn

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to constraints, → users < resources → sm resources are unused.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

where $x_i \geq 0$ and $b_i \geq 0$

Variables $s_1, s_2, \dots s_m$ are added to inequalities to remove the inequality. These variables → slack variables.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 = b_1$$

$s_i \geq 0$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1$$ → users > resources → Insufficient resources.

slack variables: $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n - s_1 = b_1$

(surplus variable)

(TBCont)

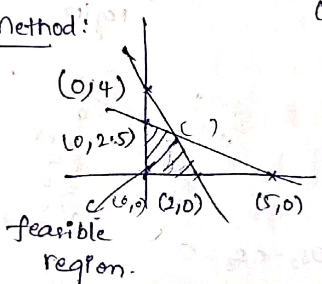
Enumeration Method:

If LPP has [n - variables, m - constants

max. no. of combinations $= {}^nC_m$

Eg: Maximize $\underbrace{Z = 2x_1 + 3x_2}$ s.t. $2x_1 + x_2 \leq 4$ $x_1 + 2x_2 \leq 5$ $x_1, x_2 \geq 0$

Graphical Method:



Considering pts inside the feasible region,

	(0,0)	(0,2.5)	(2,0)	(1,2)	...
Z:	0	7.5	4	8	

max. value

$\therefore x_1 = 1$ $x_2 = 2$

Enumeration Method: $z = 2x_1 + 8x_2$

$2x_1 + x_2 + s_1 = 4$

$x_1 + 2x_2 + s_2 = 5$

$x_1, x_2, s_1, s_2 > 0$

Variables $n=4$, const $m=2$ $\Big]$ ${}^4C_2 = 6 \rightarrow$ max. combinations

↳ 6 corner pts

NOTE: Set $(n-m)$ variables to zero, Solve for other m Variables

Non-basic variable (variables set to zero)	Basic variable (non-zero)	Basic soln	Feasible	objective values (z)
(x_1, x_2)	$(s_1 s_2)$	$(4,5)$	✓	0
$(x_1\ s_1)$	$(x_2\ s_2)$	$(4,-3)$	✗	—
$(x_1\ s_2)$	(x_2, s_1)	$(2.5, 1.5)$	✓	7.5
$(x_2\ s_2)$	$(x_1\ s_1)$	$(5,-6)$	✗	—
$(s_1\ s_2)$	$(x_1\ x_2)$	$(1, 2)$	✓	8
$(x_2\ s_1)$	$(x_1\ s_2)$	$(2,3)$	✓	4

$z = 8$ ✓ $\rightarrow$ Lengthy $\rightarrow$ Not suggested.

Linear Programming Problem:

min $c^T x$ $\quad x \in R^n$

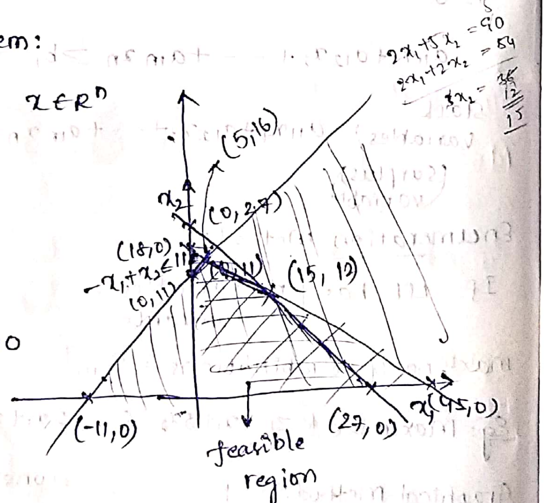
$Ax \geq b \quad x \geq 0$

max $4x_1 + 6x_2$

s.t $\quad -x_1 + x_2 \leq 11$

$x_1 + x_2 \leq 27$

$2x_1 + 5x_2 \leq 90$



$-x_1 + x_2 + s_1 = 11$

$x_1 + x_2 + s_2 = 27$

$2x_1 + 5x_2 + s_3 = 90$

max $Z = 4x_1 + 6x_2 + 0s_1 + 0s_2 + 0s_3$

$Z - 4x_1 - 6x_2 - 0s_1 - 0s_2 - 0s_3 = 0$

$36 \times 4 = 144$ $\quad 36 \times 6 = 216$ $\quad 45 \times 6$ $\quad 45 \times 4 = 180$ $\quad 216$ $\quad 396$

$$\begin{array}{c|ccccc|c|l}
 & x_1 & x_2 & s_1 & s_2 & s_3 & & \\
\hline
s_1 & -1 & 1 & 1 & 0 & 0 & 11 & 11/1 = 11 \\
s_2 & 1 & 1 & 0 & 1 & 0 & 27 & 27/1 = 27 \\
s_3 & 2 & 5 & 0 & 0 & 1 & 90 & 90/5 = 11 \\
\hline
 & -4 & -6 & 0 & 0 & 0 & 0 & \rightarrow \text{objective fn}
\end{array}$$

↳ can be calculated using $(x_1 x_2) = (0, 0)$ (replacing)

(obj. const.)

Consider the variable (column) which has the largest $-$ve value.

$\therefore$ consider $\rightarrow x_2$

Now consider the smallest +ve values from ratio calculated

And make the corresponding column element as pivot (the above considered column)

And make the pivot to '1'

Now, make the remaining column elements to '0' using Row operations.

$$\begin{array}{c|ccccc|c|l}
x_2 & -1 & 1 & 1 & 0 & 0 & 11 & 11/-1 = -11 \\
s_2 & 2 & 0 & -1 & 1 & 0 & 16 & 16/2 = +8 \\
s_3 & 7 & 0 & -5 & 0 & 1 & 35 & 35/7 = 5 \\
\hline
 & -10 & 0 & 6 & 0 & 0 & 66 &
\end{array}$$

$R_2 = R_2 - R_1$

$R_3 = R_3 - 5R_1$

$R_4 = R_4 + 6R_1$

Again follow the same procedure

$$\begin{array}{c|ccccc|c}
x_2 & -1 & 1 & 1 & 0 & 0 & 11 \\
s_2 & 2 & 0 & -1 & 1 & 0 & 16 \\
x_1 & 1 & 0 & -5/7 & 0 & 1/7 & 5 \\
\hline
 & -10 & 0 & 6 & 0 & 0 & 66
\end{array}$$

$R_3 = R_3/7$

$R_1 = R_1 + R_3$

$R_2 = R_2 - 2R_3$

$R_4 = R_4 + 10R_3$

$$\begin{array}{c|ccccc|c}
x_2 & 0 & 1 & 2/7 & 0 & 1/7 & 16 \\
s_1 & 0 & 0 & 3/7 & 1 & -2/7 & 6 \\
x_1 & 1 & 0 & -5/7 & 0 & 1/7 & 5 \\
\hline
 & 0 & 0 & -8/7 & 0 & 10/7 & 116
\end{array}$$

$R_1 = 3R_1 - 2R_2$

$R_3 = 5R_2 + 3R_3$

$R_4 = 8R_2 + 3R_4$

$$\begin{array}{|ccccc|c|c}
0 & 1 & 2/7 & 0 & 1/7 & 16 & 56 \\
0 & 0 & 3/7 & 1 & -2/7 & 6 & 14 \\
1 & 0 & -5/7 & 0 & 1/7 & 5 & -7 \\
\hline
0 & 0 & -8/7 & 0 & 10/7 & 116 &
\end{array}$$

$$\begin{array}{c|ccccc|c}
x_2 & 0 & 1 & 0 & -2/3 & 1/3 & 12 \\
s_1 & 0 & 0 & 1 & 7/3 & -2/3 & 14 \\
x_1 & 1 & 0 & 0 & 5/3 & -1/3 & 15 \\
\hline
 & 0 & 0 & 0 & 26/3 & 2/3 & 132
\end{array}$$

// Start from feasible → Optimality

Computational technique of simplex method:

Conditions: Optimality condition → used for finding entering variables

Feasibility condition

↳ used for finding leaving variables.

Entering variable: Some var. from non-basic var. set to basic var set

Leaving variable: Some var. leaving the basic var. set

1. Maximize $z = 5x_1 + 4x_2$ S.T $6x_1 + 4x_2 \le 24$

$x_1 + 2x_2 \le 6$

$x_1 + x_2 \le 1$

$x_2 \le 2$

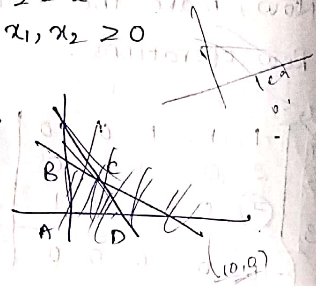
$x_1, x_2 \ge 0$

$6x_1 + 4x_2 + S_1 = 24$

$x_1 + 2x_2 + S_2 = 6$

$x_1 + x_2 + S_3 = 1$

$x_2 + S_4 = 2$



$\left.\begin{matrix} n=6 \\ m=4 \end{matrix}\right]$ (n−m) − set to zero

Non-basic variables: (n−m) = 2

Basic variables: 4

Initial Simplex table:

Basic variables	z	x_1	x_2	S_1	S_2	S_3	S_4	soln
z	1	−5	−4	0	0	0	0	0
S_1	0	6	4	1	0	0	0	24
S_2	0	1	2	0	1	0	0	6
S_3	0	1	1	0	0	1	0	1
S_4	0	0	1	0	0	0	1	2

↳ 4 basic variables (usually slack variables)

With ref to (0,0) $x_1, x_2 = 0 \Rightarrow S_1, S_2, S_3, S_4$ → basic variables

$Z = 5x_1 + 4x_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4 = 0$

∴ Entering variables → (x_1, x_2) for Z

The most negative value for nonbasic variable in the Z-row is considered as entering variable.

$x_1 \Rightarrow$ entering variable (most −ve : since has higher impact) (for min. pro. → opp)

The column containing entering variable – pivot column.

Leaving variable is determined by feasibility condition.

	x_1	soln	Ratio
S_1	6	24	4
S_2	1	6	6
S_3	−1	1	−1
S_4	0	2	∞

Variable having min. +ve ratio is ~~called~~ taken as leaving variable.

S_1 → leaving variable

Pivot row → row containing leaving var.

Pivot element – intersection of pivot row & column.

Next iteration:

Basic	z	x_1	x_2	S_1	S_2	S_3	S_4	soln
z	1	0	−2/3	5/6	0	0	0	20
x_1	0	1	2/3	1/6	0	0	0	4
S_2	0	0	4/3	−1/6	1	0	0	2
S_3	0	0	5/3	1/6	0	1	0	5
S_4	0	0	1	0	0	0	1	2

Gauss Jordan method:

To find out revised entries:

New pivot row = $\dfrac{\text{Old pivot row}}{\text{Pivot element}}$

Remaining rows ⇒ Current row (old) − (Pivot column coeff × New pivot row)

Eg: $Z_{new} = (1, -5, -4, 0, 0, 0, 0, 0) - [(-5) \times (0, 1, 2/3, 1/6, 0, 0, 0, 4)]$

$= (1, 0, -2/3, 5/6, 0, 0, 0, 20)$

Termination criteria: if 'z' row contains all +ve values, we will stop the algorithm.

No. of variables in initial pro. – all should enter the basic row.

Entering variable – x_2 Leaving variable → S_2

	x_2	soln	ratio
x_1			6
S_2			3/2 →
S_3			3
S_4			2

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	soln
z	1	0	0	9/12	1/2	0	0	21 $z = 21$
x_1	0	1	0	1/4	-1/2	0	0	3 $x_1 = 3$
x_2	0	0	1	-1/8	3/4	0	0	3/2 $x_2 = 3/2$
s_3								
s_4								

$\Rightarrow$ max $4x_1 + 6x_2$

s.t $x_1 + x_2 \le 27$ $\qquad$ $x_1 + x_2 + x_3 = 27$

$2x_1 + 5x_2 \le 90$ $\qquad$ $2x_1 + 5x_2 + x_4 = 90$

$-x_1 + x_2 \le 11$ $\qquad$ $-x_1 + x_2 + x_5 = 11$

$x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$, $x_4 \ge 0$, $x_5 \ge 0$

Initial conditions: $x_3 = 0$, $x_4 = 0$, $x_5 = 0$

	$x_2 = 0$: x_1	$x_1 = 0$: x_2
$x_3 = 27 - x_1 - x_2$	27 ✓	27
$x_4 = 90 - 2x_1 - 5x_2$	45	18
$x_5 = 11 + x_1 - x_2$	-11	11 ✓

$z = 4x_1 + 6x_2$

$x_2 = 11 + x_1 - x_5$ $\qquad$ 10

$x_3 = 27 - x_1 - (11 + x_1 - x_5) = 16 - 2x_1 - x_5$

$x_4 = 90 - 2x_1 - 5(11 + x_1 - x_5) = 35 - 7x_1 + 5x_5$ $\qquad$ 30 $\quad 15 = 10 - x_2 + 5$

$z = 4x_1 + 6(11 + x_1 - x_5) = 4x_1 + 66 + 6x_1 - 6x_5$

$10x_1 + 66 - 6x_5$

Q: maximize $z = 2x_1 - x_2 + 2x_3$

s.t $2x_1 + x_2 \le 10$ $\qquad$ $2x_1 + x_2 + s_1 = 10$

$x_1 + 2x_2 - 2x_3 \le 20$ $\qquad$ $x_1 + 2x_2 - 2x_3 + s_2 = 20$

$x_2 + 2x_3 \le 5$ $\qquad$ $x_2 + 2x_3 + s_3 = 5$

$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0$

$z = 2x_1 - x_2 + 2x_3 - 0s_1 - 0s_2 - 0s_3$

$z - 2x_1 + x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3 = 0$

$z =$ maximize $\quad -2x_1 - 3x_2$

$3x_1 + 2x_2 = 14$

$2x_1 - 4x_2 \ge 2$

$4x_1 + 3x_2 \le 19$

$3x_1 + 2x_2 = 14$ $\quad (+a_1)$

$2x_1 - 4x_2 - x_3 = 2$ $\quad (+a_2)$

$4x_1 + 3x_2 + x_4 = 19$

x_3 - surplus variable

x_4 - slack variable.

Taking $(x_1, x_2) = 0$ violates $x_3 \ge 0$ condition ($\because x_3$ becomes -2)

So, we introduce artificial variables $(a_1, a_2 \ge 0)$ ↳ will be added to those eqns which doesn't have slack variables.

slack, surplus - exist in the contraints

artificial variables - doesn't exist.

First phase objective: ~~mini~~ max. $z = -a_1 - a_2$

c_B		0 x_1	0 x_2	0 x_3	0 x_4	-1 a_1	-1 a_2		
-1	a_1	3	2	0	0	1	0	14	14/3
-1	a_2	2	-4	-1	0	0	1	2	2/2 = 1
0	x_4	4	3	0	1	0	0	19	19/4
$c_j - z_j$		5	-2	-1	0	0	0		

1 -2 $\quad$ -1/2 $\quad$ -10

slack

a_2 - leaving variable $\qquad$ x_1 entering variable.

	~~x_1~~	x_2	x_3	~~x_4~~	a_1	a_2
a_1	0	+16/2 -2	+3/2 -1/2	0 0	+7/2 0	-3/2 1/2
x_1	1					
x_4	0					

c_B		0 x_1	0 x_2	0 x_3	0 x_4	-1 a_1	-1 a_2	0	
-1	a_1	0	~~16~~ 8	3/2	0	~~2~~ 1	-3/2	11	11/8
0	x_1	1	-2	-1/2	0	0	1/2	1	1/-2
0	x_4	0	11	2	1	0	-2	15	15/11
		0	8	3/2	0	0	-3/2	-11	

		x_1	x_2	x_3	x_4	a_1	a_2	
-1	a_1	0	0	1/22	-8/11	1	-1/22	$11-\frac{120}{11}=\frac{1}{11}$
0	x_1	1	0	-3/22	2/11	0	1/2+4/11	$1-\frac{30}{11}=\frac{41}{11}$
0	x_2	0	1	2/11	1/11	0	-2/11	15/11
		0	0	+1/22	-8/11	0	- 1/22	1/22

		x_1	x_2	x_3	x_4	a_1	a_2
0	x_3	0	0				
0	x_1	1					
0	x_2	0					

		-2	-3				
		x_1	x_2	x_3	x_4		
0	x_3	0	0	1	-8/11	1/11	
-2	x_1	1	0	0	2/11	-41/11	41/2
-3	x_2	0	1	0	1/11	15/11	15 ←
$C_j - Z_j$		0	0	0	7/11		

		0	0	0	0	-1	-1		
		x_1	x_2	x_3	x_4	a_1	a_2		
-1	a_1	3	2	0	0	1	0	14	14/3
-1	a_2	2	-4	-1	0	0	1	2	1
0	x_4	4	3	0	1	0	0	19	19/4
		5	-2	-1	0	0	0		

		0	0	0	0	-1	-1		
		x_1	x_2	x_3	x_4	a_1	a_2		
-1	a_1	0	8	3/2	0	1	-3/2	11	11/8
0	x_1	1	-2	-1/2	0	0	1/2	1	-1/2
0	x_4	0	11	2	1	0	-2	15	15/11
		0	8	3/2	0	0	-3/2	-11	

		0	0	0	0	-1	-1		
		x_1	x_2	x_3	x_4	a_1	a_2		
-1	a_1	0	0	1/22	-8/11	1	-1/22	1/11	= 2
0	x_1	1	0	-3/22	2/11	0	3/22	41/11	$=\frac{41}{11}\times\frac{22}{-3}$
0	x_2	0	1	2/11	1/11	0	-2/11	15/11	$=\frac{15}{11}\times\frac{11}{2}=7.5$
		0	0	1/22	-8/11	0	-23/22	1/11	

		x_1	x_2	x_3	x_4
0	x_3	0	0	1/22	-8/11
0	x_1				
0	x_2				

		-2	-3	0	0	
		x_1	x_2	x_3	x_4	
0	x_3	0	0	1	-16	2
-2	x_1	1	0	0	-2	4
-3	x_2	0	1	0	3	1
		0	0	0	0	-11

$$\boxed{\therefore\ x_1 = 4,\quad x_2 = 1,\quad Z = -11}$$

Maximize $Z = -5x_1 - 3x_2 + 7x_3$ $7x_3' - 7x_3''$

s.t. $2x_1 + 4x_2 + 6x_3 = 7$

$3x_1 - 5x_2 + 3x_3 \le 5$ → it should be +ve

$-4x_1 - 9x_2 + 4x_3 \le -4$

$x_1 \ge -2$ $0 \le x_2 \le 4$ x_3 - free ↳ $\underbrace{x_3' - x_3''}$, $x_3' \ge 0$, $x_3'' \ge 0$

$(x_1+2) \ge 0$ $x_2 \ge 0$

→ $x_2 \le 4$ another constraint

$x_2 + s_1 = 4$

$2(x_1+2) + 4x_2 + 6x_3 = 7+4$

$3(x_1+2) - 5x_2 + 3x_3 \le 5+6$

$+4(x_1+2) + 9x_2 - 4x_3 \ge 4+8$

$2x' + 4x_2 + 6x_3' - 6x_3'' = 11$

$3x' - 5x_2 + 3x_3' - 3x_3'' + x_4 = 11$

$4x' + 9x_2 - 4x_3' - 4x_3'' - x_5 = 12$

$x_2 + x_6 = 4$

$2x_1 + 4x_2 + 6x_3' - 6x_3'' + a_1 = 11$

$3x_1 - 5x_2 + 3x_3' - 3x_3'' + x_4 = 11$

$4x_1 + 9x_2 - 4x_3' + 4x_3'' - x_5 + a_2 = 12$

$x_2 + x_6 = 4$

$z = -a_1 - a_2$

c_B	c_j	0	0	0	0	0	0	0	-1	-1	0	
		x_1	x_2	x_3'	x_3''	x_4	x_5	x_6	a_1	a_2		
-1	a_1	2	4	6	-6	0	0	0	1	0	11	11/4
0	x_4	3	-5	3	-3	1	0	0	0	0	11	11/-5
-1	a_2	4	9	-4	4	0	-1	0	0	1	12	12/9
0	x_6	0	1	0	0	0	0	1	0	0	4	4/1
	z_j	-6	-13	-2	2	0	1	0	-1	-1	-23	
	$c_j - z_j$	6	13	2	-2	0	-1	0	0	0	23	

c_B	c_j	0	0	0	0	0	0	0	-1	-1	23
		x_1	x_2	x_3'	x_3''	x_4	x_5	x_6	a_1	a_2	
-1	a_1	2/9	0	70/9	-70/9	0	4/9	0	1	-4/9	51/9
0	x_4	47/9	0	7/9	-7/9	1	-5/9	0	0	5/9	159/9
0	x_2	4/9	1	-4/9	4/9	0	-1/9	0	0	1/9	12/9
0	x_6	-4/9	0	4/9	-4/9	0	1/9	1	0	-1/9	-8/9
	$c_j - z_j$	+2/9	0	70/9	-70/9	0	4/9	0	0	~~-5/9~~ -13/9	258/9

c_B	c_j	-5	-3	7	-7	0	0	0	
		x_1	x_2	x_3'	x_3''	x_4	x_5	x_6	
7	x_3'	2/70	0	1	-1	0	4/70	0	51/70
0	x_4	468/90	0	0	0	1	-54/90	0	1539/90
-3	x_2	288/630	1	0	0	0	-6/70	0	116/70
0	x_6	-288/630	0	0	0	0	6/70	1	-764/630
		-ve value	0	0	0	0	-ve value	0	

$x_3' = 51/70 \quad x_2 = 116/70 \quad x_1 = 0 \quad x_3'' = 0$

$z = -5x_1 - 3x_2 + 7x_3' - 7x_3''$

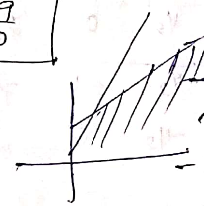
$\boxed{\cancel{z = 9/70}}$ $\boxed{z = \frac{709}{70}}$

$\frac{9}{70} + 10$

maximize $z = x_1 + 2x_2$

$-x_1 + x_2 \le 2$

$-2x_1 + x_2 \le 1$



x_1	x_2	x_3	x_4	
-1	1	1	0	2
-2	1	0	1	1
				1 2

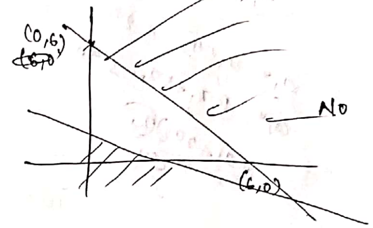
↑ considering this row results negative ratio, making it unbounded.

$x_1 + 2 = 0$

$x_1 = -2$

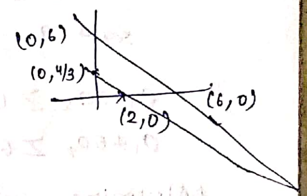
max. $x_1 + x_2 \ge 6$

$2x_1 + 3x_2 \le 4$



No feasible soln in the quadrant

Infeasible



maximize $z = x_1$

$x_1 + x_2 \ge 6$

$2x_1 + 3x_2 \le 4$ $\quad x_1, x_2 \ge 0$

$x_1 + x_2 - s_1 + a_1 = 6$

$2x_1 + 3x_2 + s_2 = 4$

$z' = -a_1$

		0	0	0	0	-1		
		x_1	x_2	s_1	s_2	a_1	soln	ratio
-1	a_1	1	1	-1	0	1	6	6
0	s_2	2	3	0	1	0	4	2
	z_j	-1	-1	1	0	-1	-6	
	$c_j - z_j$	1	1	-1	0	0		

~~x_1~~ a_2 s_1 ~~s_2~~ a_1 ~~soln~~ ratio

C_j		0	0	0	0	-1	soln	ratio
		x_1	x_2	S_1	S_2	a_1		
-1	a_1	0	$-1/2$	-1	$-1/2$	1	4	4
0	x_1	1	$3/2$	0	$1/2$	0	2	∞
Z_j		0	$1/2$	-1	$1/2$	-1	-4	
$C_j - Z_j$		0	$-1/2$	1	$-1/2$	0		

→ unbounded

C_j		10	0	0	0	
		x_1	x_2	S_1	S_2	soln
0	S_1	0	$-1/2$	1	$-1/2$	4
1	x_1	1	$3/2$	0	$17/2$	2
Z_j		01	$3/2$	0	$1/2$	
$C_j - Z_j$		0	$-3/2$	0	$-1/2$	

$\cancel{\boxed{x_1 = 2}}$ → Max.

$4(x_1)+6$

Dual

* Maximise $4x_1 + 6x_2$ → Primary

$x_1 + x_2 \le 27 \quad * a_1$

$2x_1 + 5x_2 \le 90 \quad * a_2$

$a_1 * \quad x_1 + x_2 \le 27$

$a_2 * \quad 2x_1 + 5x_2 \le 90$

$(a_1 + 2a_2) x_1 \ge 4$

$$\left.\begin{array}{l} a_1 + 2a_2 \ge 4 \\ a_1 + 5a_2 \ge 6 \end{array}\right\} \to \text{Constraints}$$

Minimize $27a_1 + 90a_2$

} → Dual L.P.P

* Maximise $z = -5x_1 - 3x_2 + 7x_3$

s.t $2x_1 + 4x_2 + 6x_3 = 7$

$3x_1 - 5x_2 + 3x_3 \le 5$

$-4x_1 - 9x_2 + 4x_3 \le -4$

$4x_1 + 9x_2 - 4x_3 \ge 4$

$x_1 \ge -2$

$0 \le x_2 \le 4$

x_3 - free

$x_3 \to x_3' - x_3''$

$(x_1 + 2) \ge 0$

$x_2 \le 4$

→ $2x_1 + 4x_2 + 6x_3' - 6x_3'' = 11 \quad * a_1$

$3x_1 - 5x_2 + 3x_3' - 3x_3'' \le 11 \quad * a_2$

$4x_1 + 9x_2 - 4x_3' + 4x_3'' \ge 12 \quad * a_3$

$x_2 \le 4 \quad * a_4$

$z = -5x_1 - 3x_2 + 7x_3' - 7x_3''$

$2x_1 + 4x_2 + 6x_3' - 6x_3'' \le 11 \quad * a_1$

$-2x_1 - 4x_2 - 6x_3' + 6x_3'' \le -11 \quad a_2$

$3x_1 - 5x_2 + 3x_3' - 3x_3'' \le 11 \quad a_3$

$-4x_1 - 9x_2 + 4x_3' - 4x_3'' \le -12 \quad a_4$

$x_2 \le 4 \quad a_5$

$x_1 \Rightarrow (2a_1 - 2a_2 + 3a_3 - 4a_4)$

$x_2 \Rightarrow (4a_1 - 4a_2 - 5a_3 - 9a_4 + a_5)$ $\quad a_2, a_5 \le 0$

$x_3' \Rightarrow (6a_1 - 6a_2 + 3a_3 + 4a_4)$ $\quad a_1, a_3, a_4 \ge 0$

$x_3'' \Rightarrow (-6a_1 + 6a_2 - 3a_3 - 4a_4)$

Minimize: $11a_1 - 11a_2 + 11a_3 - 12a_4 + 4a_5$

s.t $2a_1 - 2a_2 + 3a_3 - 4a_4 \ge -5$

$4a_1 - 4a_2 - 5a_3 - 9a_4 + a_5 \ge -3$

$$\left.\begin{array}{l} 6a_1 - 6a_2 + 3a_3 + 4a_4 \ge 7 \\ -6a_1 + 6a_2 - 3a_3 - 4a_4 \ge -7 \end{array}\right\}$$

$6a_1 - 6a_2 + 3a_3 + 4a_4 = 7$

Eg: Minimize $z = \frac{3}{4}x_1 + 150x_2 - \frac{1}{50}x_3 + 6x_4$ $\quad x_1, x_2, x_3, x_4 \ge 0$

s.t $\frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 \le 0$

$\frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 \le 0$

$x_3 \le 1$

		$3/4$	150	$-1/50$	6	0	0			
		x_1	x_2	x_3	x_4	S_1	S_2	S_3	soln	ratio
0	S_1	$1/4$	-60	$-1/25$	9	1	0	0	0	
0	S_2	$1/2$	-90	$-1/50$	3	0	1	0	0	
0	S_3	0	0	1	0	0	0	1		

m - constraints

n - variables

m → basic variables

(n-m) → non basic variables.

Time Com: $(m \times (n-m)) + m + n-m + (m-1) \times (n-m)$

$(m \times (n-m))$ → to calculate $C_j - Z_j$

m → identifying pivot element

$n-m$ → dividing the pivot row by pivot element

$(m-1)$ → m-1 colums

$(n-m)$ → (n-m) rows in each column

→ set to zero.

m - con
n - variables

min $c^T x$ s.t $Ax = b$

$x \geq 0$

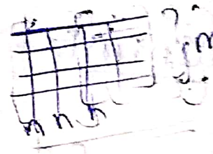
$A_{(m\times n)} \Rightarrow B_{m\times m} \quad N_{m\times(n-m)}$

$A = B||N \qquad \underset{\text{Basic}}{\downarrow} \qquad \underset{\text{Non Basic}}{\downarrow} \qquad (B+N = A)$

$$x_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}_{m\times 1} \qquad x_N = \begin{bmatrix} x_{m+1} \\ \vdots \\ x_{m+(n-m)} \end{bmatrix}_{n-m}$$

$m\times 1$

$AX = b$ can be written as $Bx_B + Nx_N = b$



Standard form:

$$x_B = B^{-1}(b - Nx_n)$$

min $\quad c_B^T x_B + c_N^T x_N$

$$\Rightarrow c_B^T(B^{-1}b - B^{-1}Nx_n) + c_N^T x_n$$

$$\Rightarrow \underset{\text{constant}}{\underset{\downarrow}{c_B^T B^{-1} b}} + \underbrace{(c_N^T - c_B^T B^{-1} N)}_{c_j - z_j} x_N$$