Image & Video Processing

Frequency Domain Filtering

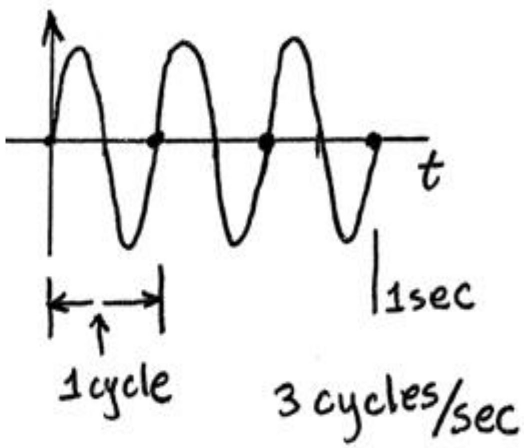
Contents

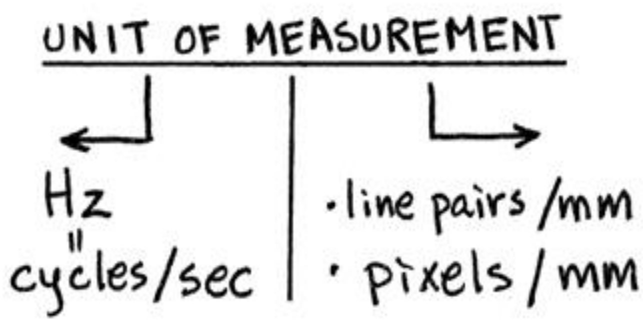
In this lecture we will introduce frequency domain/space and look at mathematical tools to go from time/spatial domain to frequency domain

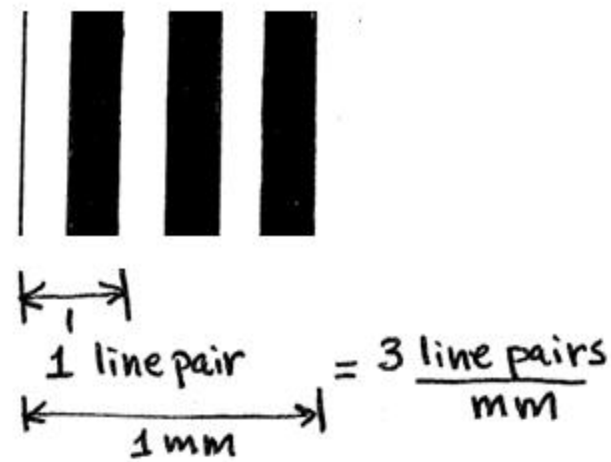
- What is Frequency domain?
- Frequency in Images
- Why Frequency analysis?
- The Fourier series
- The Fourier transform

Frequency Domain/Space

- The term frequency comes up a lot in physics, as some variation in time, describing the characteristics of some periodic motion or behaviour.







Frequency in the Time domain (1D)

Frequency in the Spatial domain (2D)

Frequency Domain/Space

- A signal (wave in time or image in spatial domain) can be decomposed or separated into a sum of sinusoids of different frequencies, amplitude and phase.
- *1D Audio Example:* Consider a complicated sound played on a piano or a guitar. We can describe this sound in two related ways:
  - Temporal Domain: Sample the amplitude of the sound many times a second, which gives an approximation to the sound as a function of time
  - Frequency Domain: Analyse the sound in terms of the pitches of the notes, or the amplitude of each frequency (fundamental plus harmonics) which make up the sound up.

Frequency in Image?

- Image can be represented in two ways:
  - Spatial Domain: Brightness along a line can be recorded as a set of values measured at equally spaced distances apart (represented as 2D array of pixel measurements)
  - Frequency Domain: as a set of spatial frequency values/component (represented as 2D grid of spatial frequencies)
- In short, frequency in image has to do with intensity (brightness or color) variation across the image, i.e. it is a function of spatial coordinates, rather than time.
- Example: If an image represented in frequency space has *high* frequencies then it means that the image has sharp edges or details

Frequency in Image?

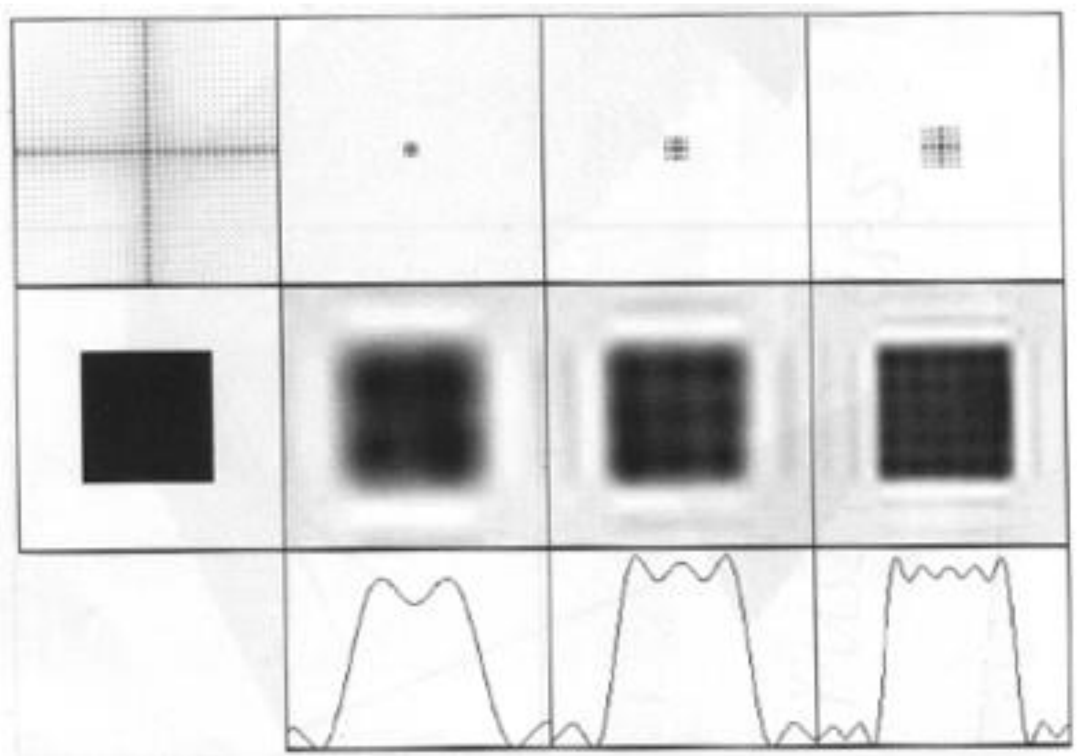


Fig 1. Images in the spatial domain are in the middle row, and their frequency space are shown on the top row. The bottom row shows the varying brightness of the horizontal line through the center of an image.

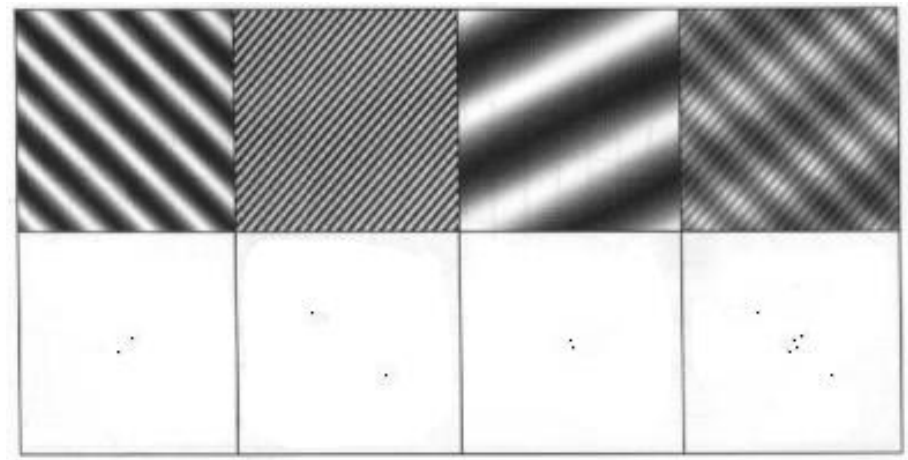
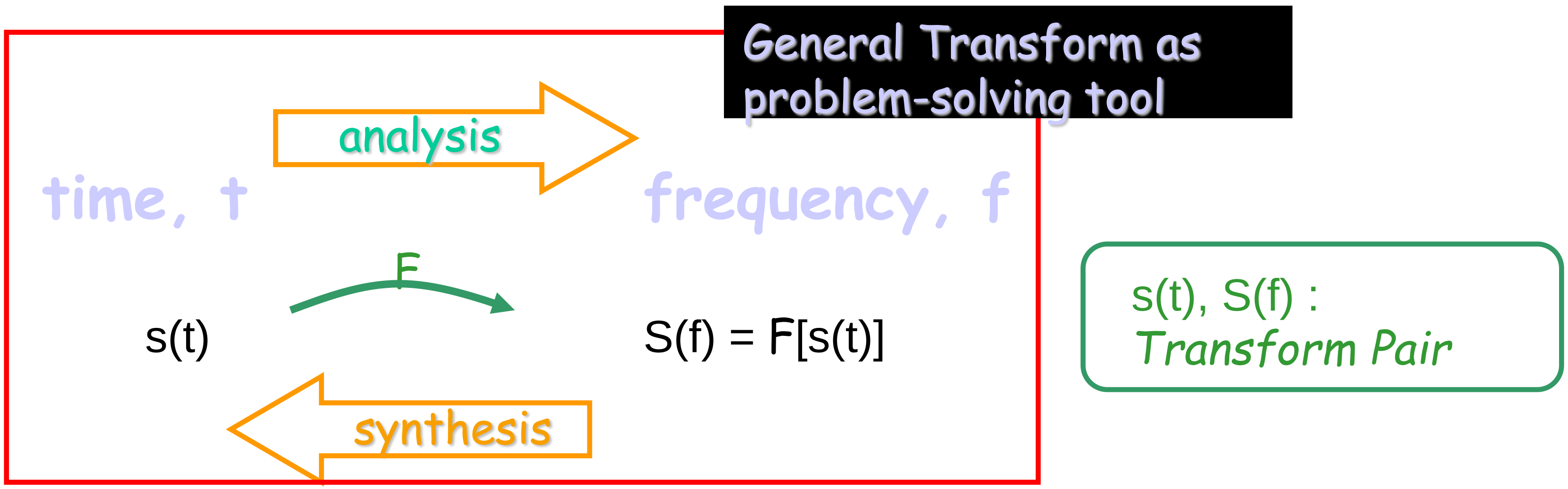


Fig 2. Images with perfectly sinusoidal variations in brightness: The first three images are represented by two dots. You can easily see that the position and orientation of those dots have something to do with what the original image looks like. The 4th image is the sum of the first three.

Why Frequency analysis?

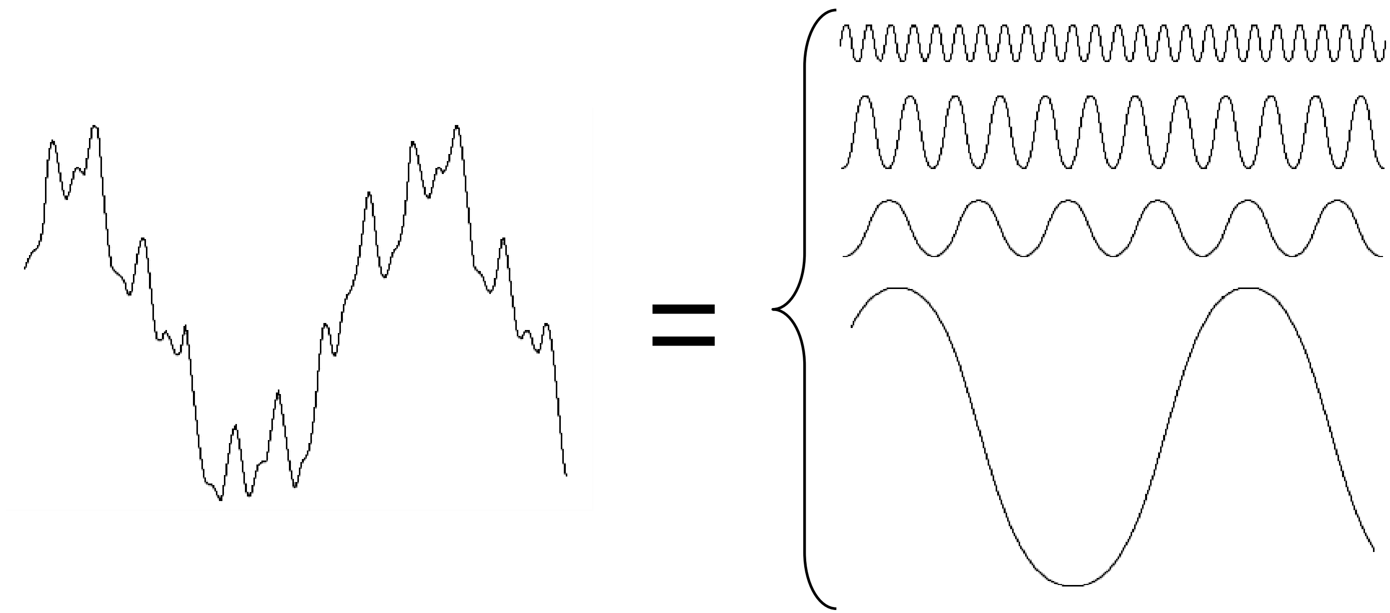
- Makes large filtering operations much faster.
- Powerful & complementary to time/spatial domain analysis techniques.
- Several transforms that makes it easy to go forwards and backwards from the spacial domain to the frequency space: Fourier, Laplace, wavelet, etc.



The Big Idea

Fourier Series and Transforms – given by mathematician **Joseph Fourier**.

One of the most important mathematical theories in modern engineering



Any function that periodically repeats itself can be expressed as a sum of sines and cosines of different frequencies each multiplied by a different coefficient – a *Fourier series*

Images taken from Gonzalez & Woods, Digital Image Processing (2002)

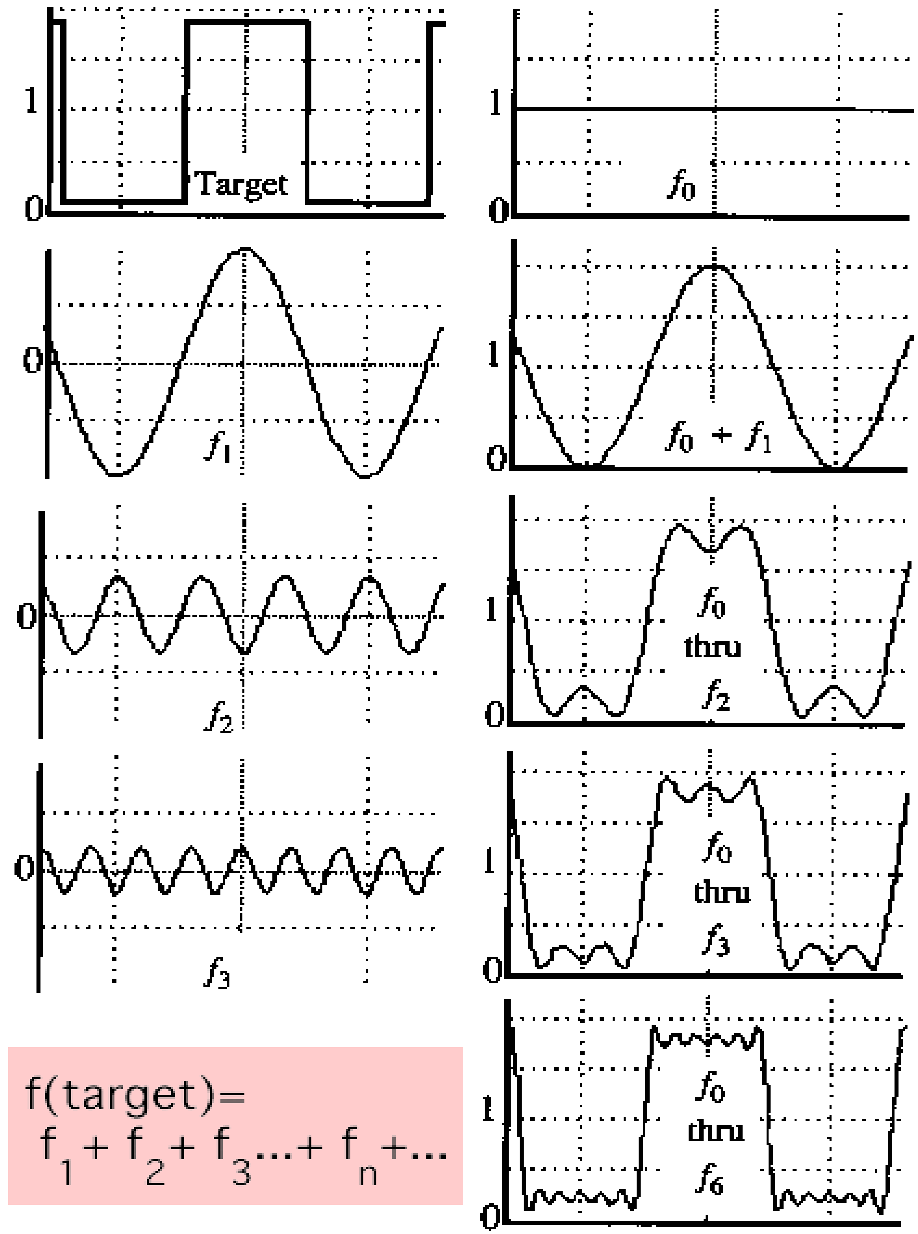
A Sum of Sinusoids

- Our building block:

$$A\sin(\omega x + \phi)$$

- Add enough of them to get any signal *f(x)* you want!

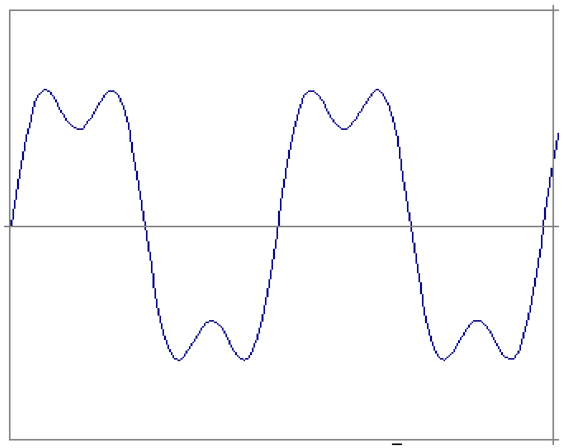
- Which one encodes the coarse vs. fine structure of the signal?



f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

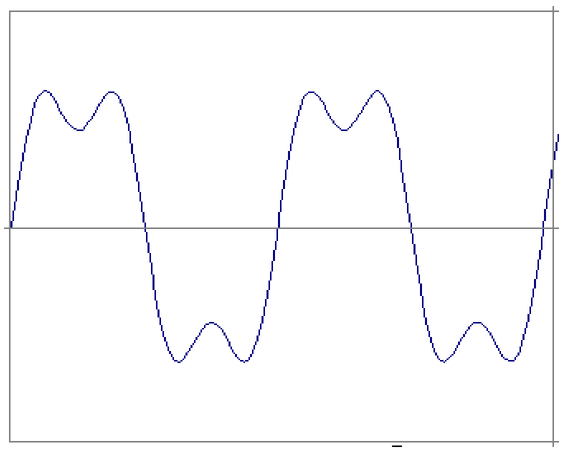
Time and Frequency

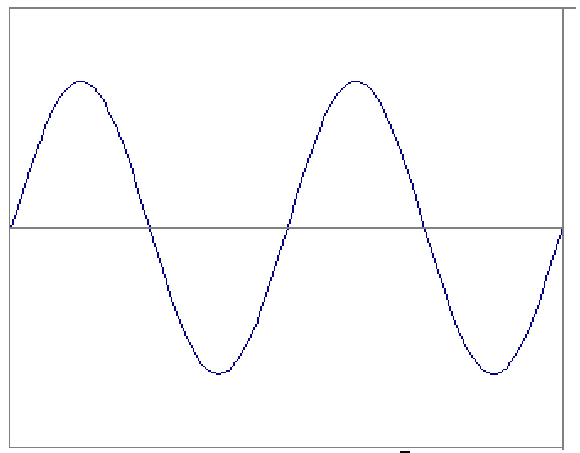
- example : $g(t) = \sin(2pi\, f\, t) + (1/3)\sin(2pi\,(3f)\, t)$

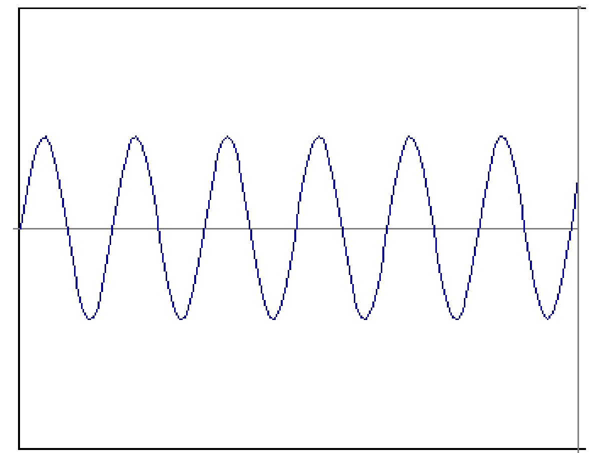


Time and Frequency

- example : $g(t) = \sin(2pi\ f\ t) + (1/3)\sin(2pi\ (3f)\ t)$







Frequency Spectra

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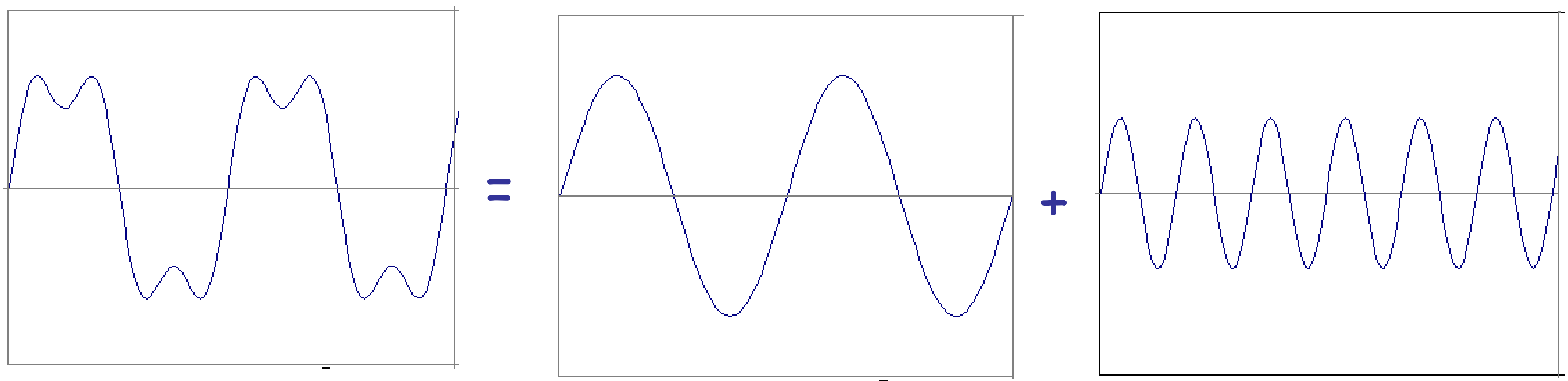
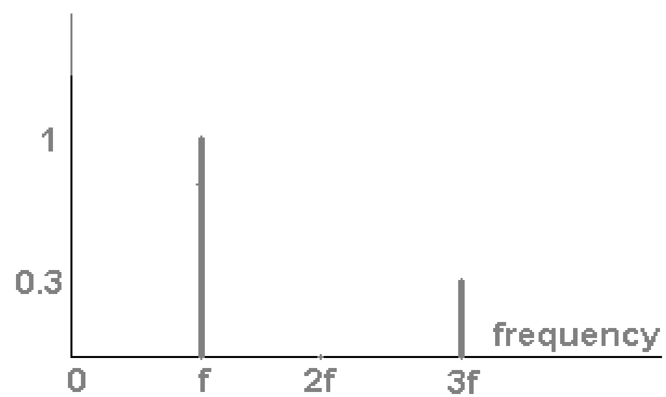
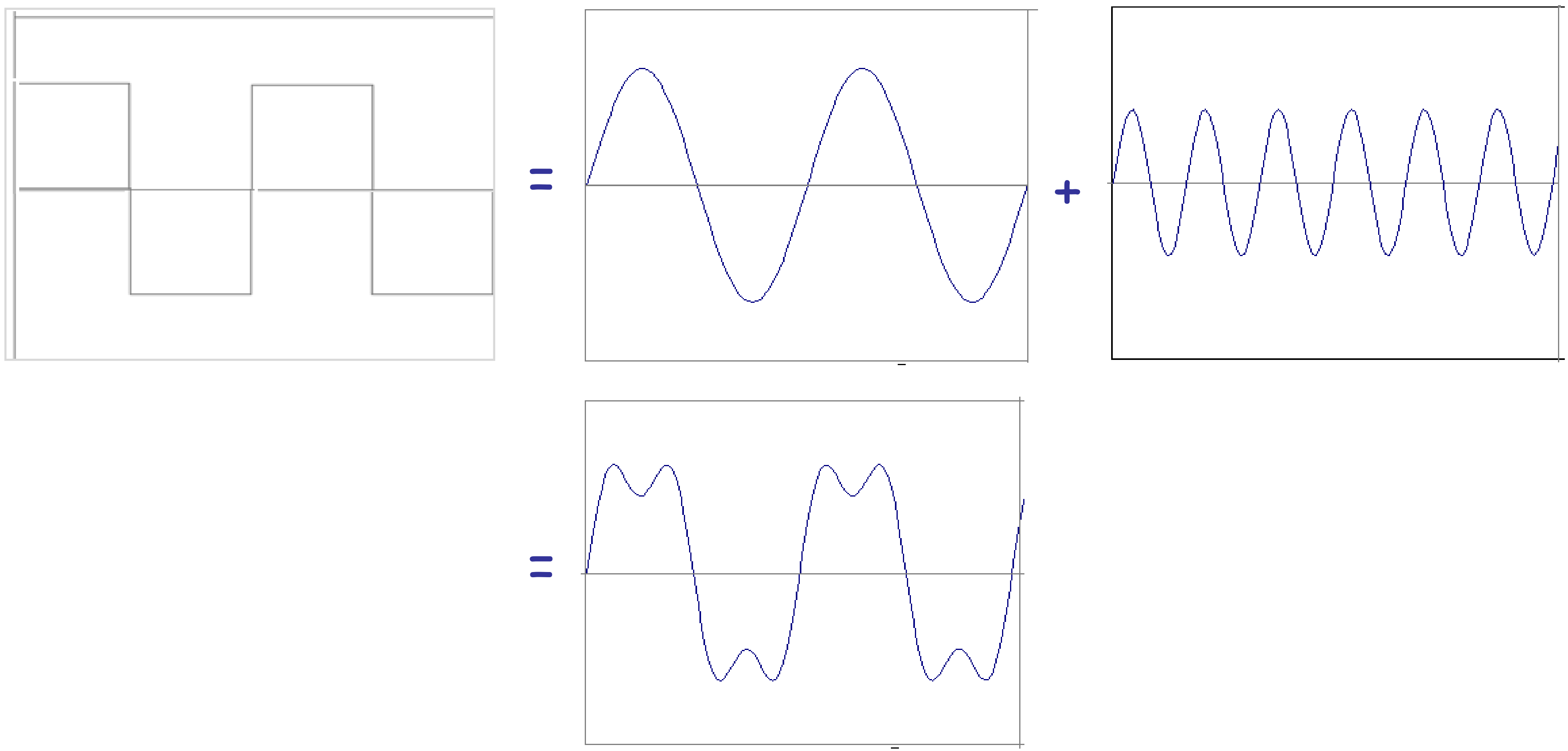
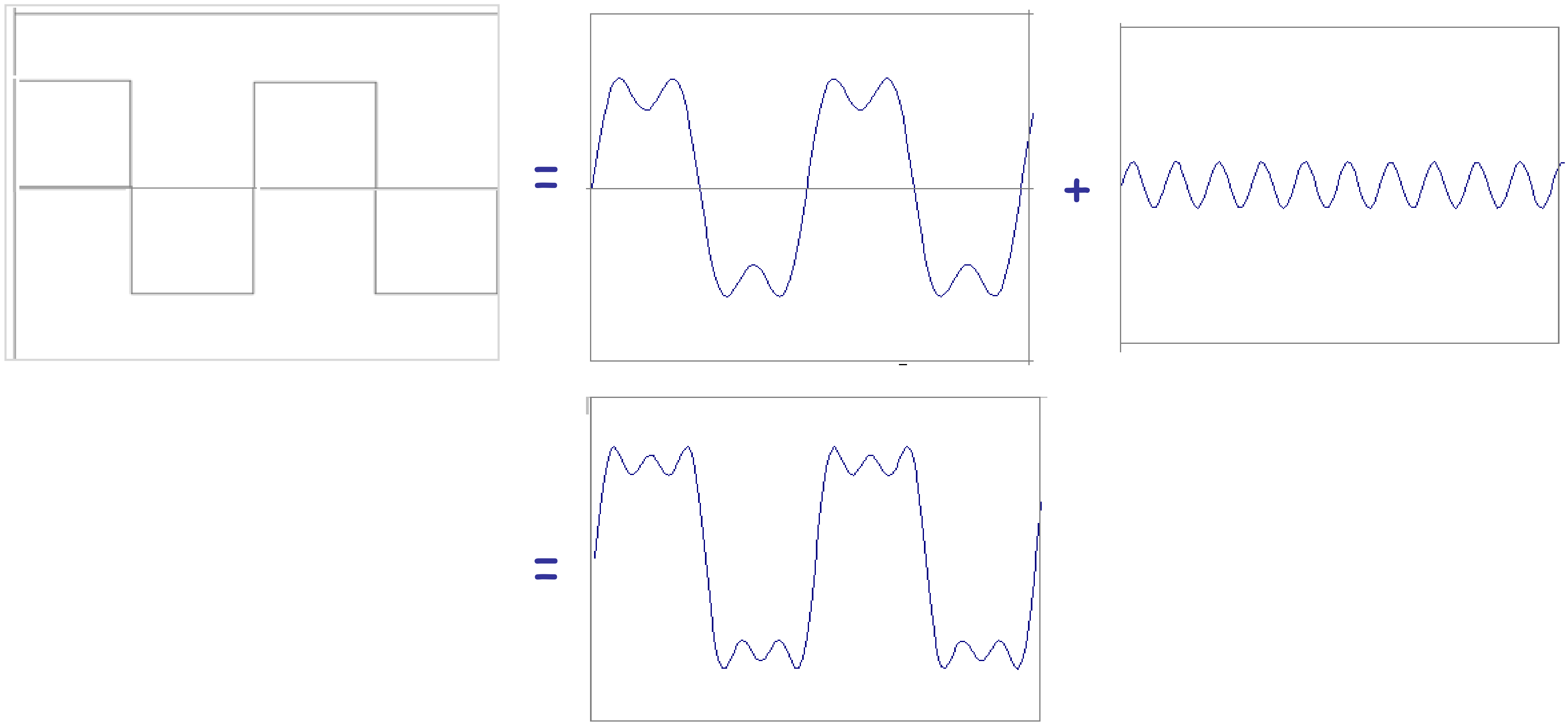




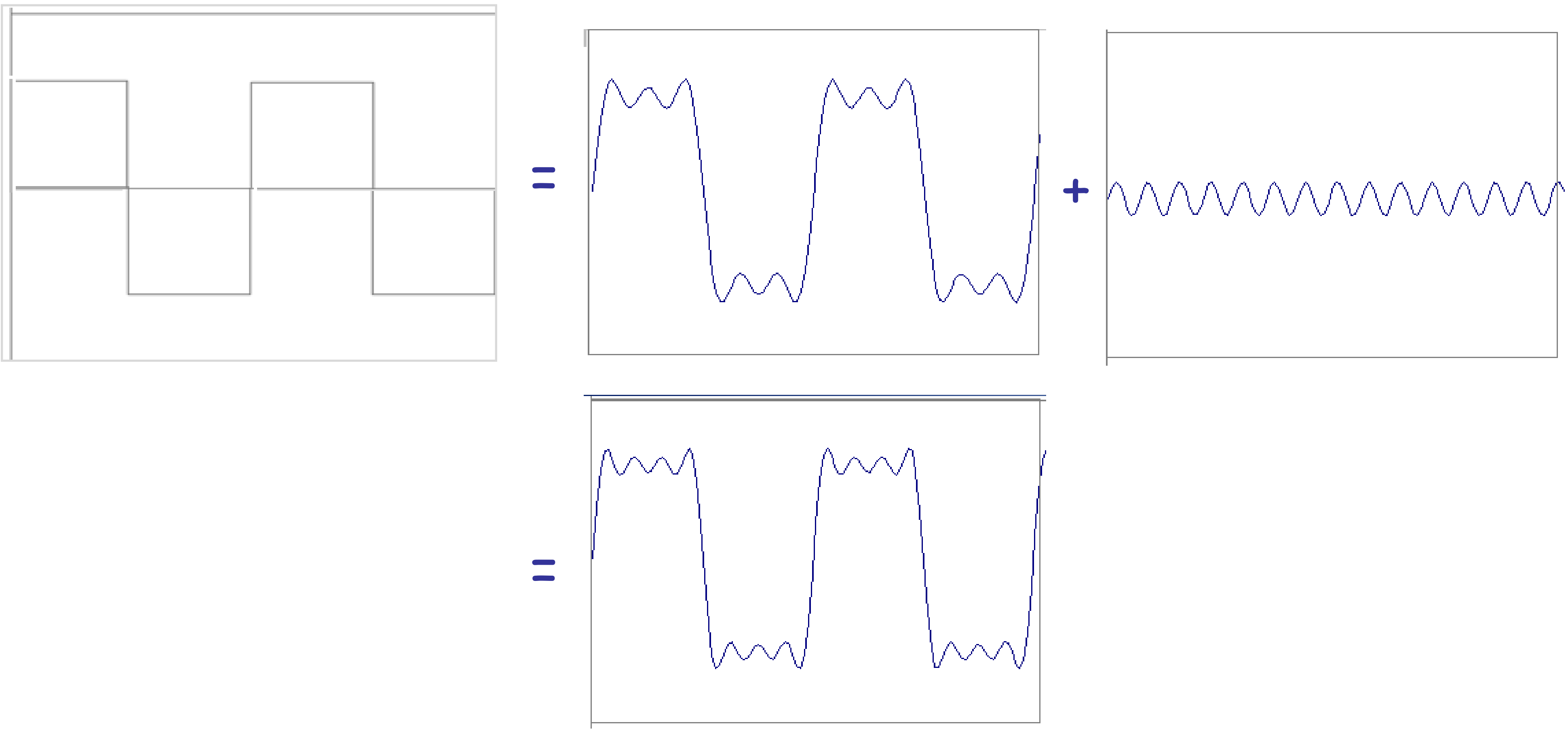
Frequency Spectra



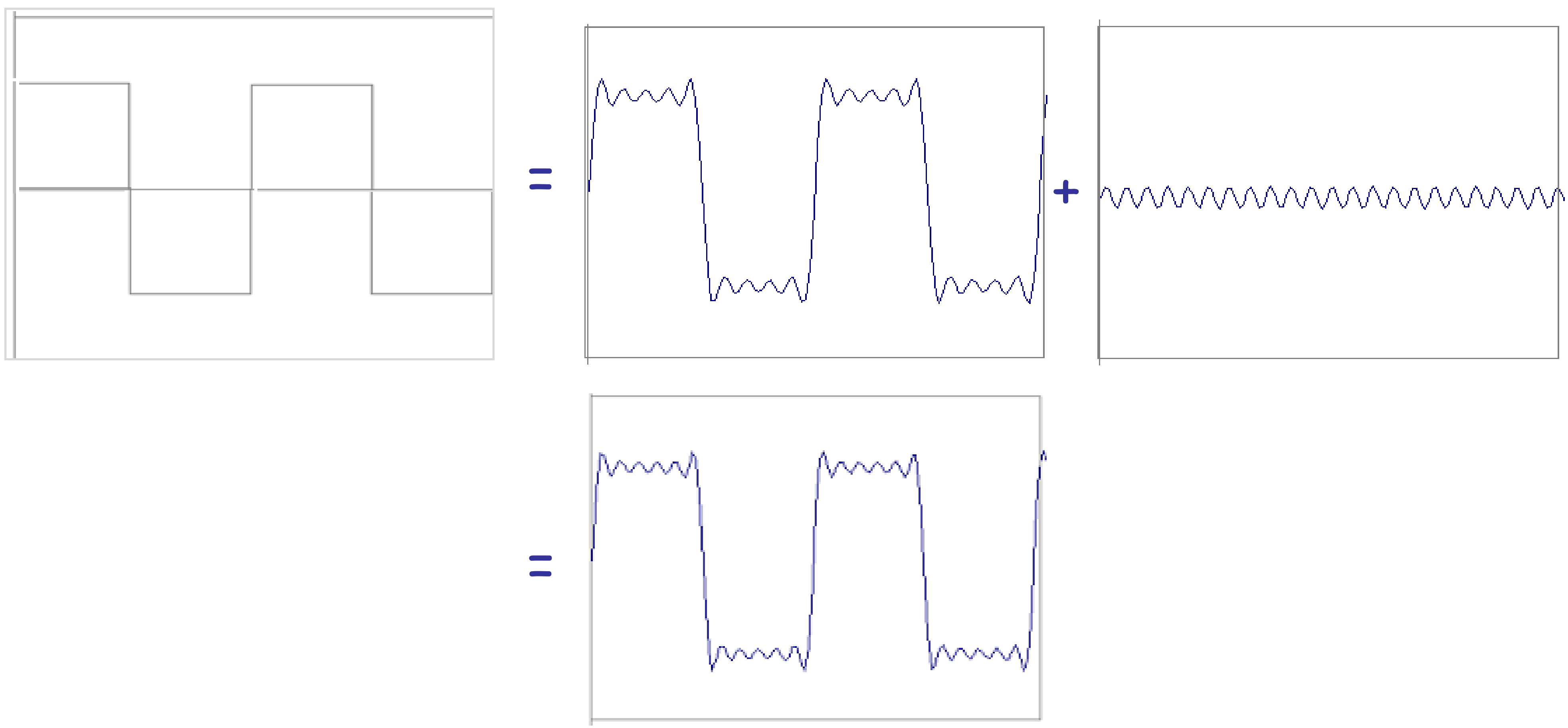
Frequency Spectra



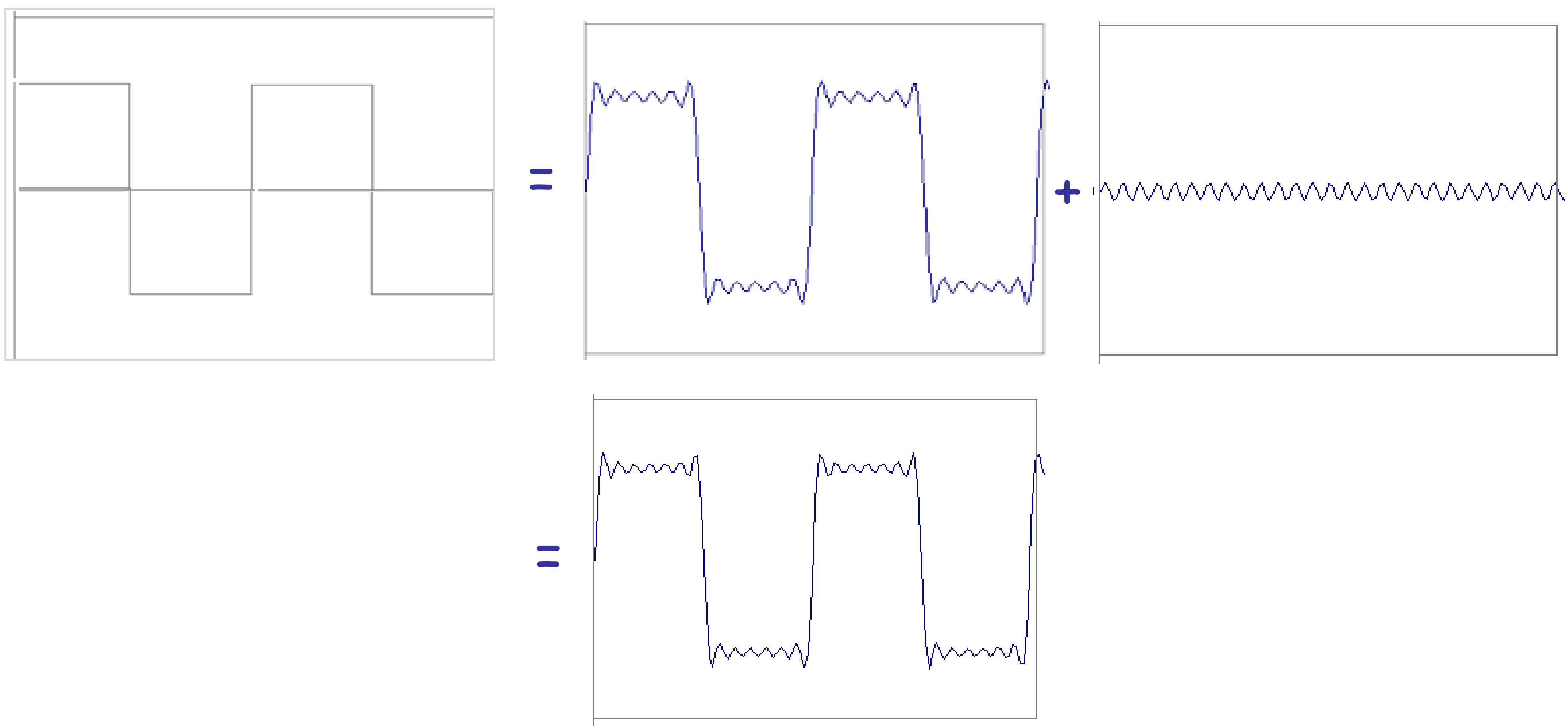
Frequency Spectra



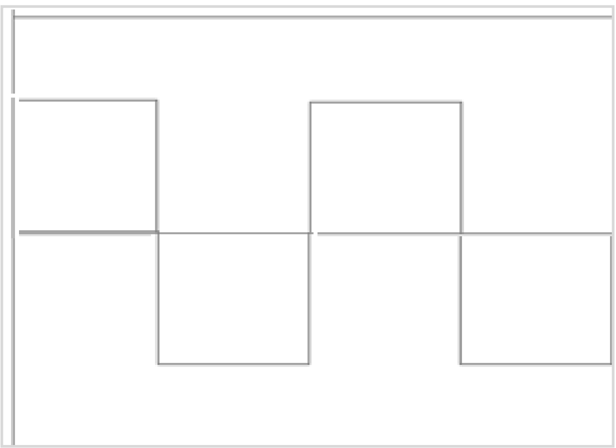
Frequency Spectra



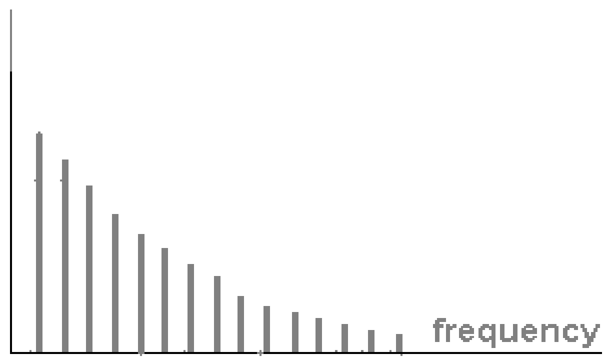
Frequency Spectra



Frequency Spectra



$$= \boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$



Fourier Series

- A general representation of function *f(t)* that is periodic with period T, by Fourier series as:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t dt \qquad n = 1,2,\Lambda$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t dt \qquad n = 1,2,\Lambda$$

Fourier Series

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\omega_0 t)+\sum_{n=1}^{\infty}b_n\sin(n\omega_0 t)$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2}\left(\frac{a_n}{\sqrt{a_n^2+b_n^2}}\cos\omega_n t+\frac{b_n}{\sqrt{a_n^2+b_n^2}}\sin\omega_n t\right)$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2}\left(\cos\theta_n\cos\omega_n t+\sin\theta_n\sin\omega_n t\right)$$

$$=C_0+\sum_{n=1}^{\infty}C_n\cos(\omega_n t-\theta_n)$$

$$C_n=\sqrt{a_n^2+b_n^2}$$

$$f(t)=C_0+\sum_{n=1}^{\infty}C_n\cos(\omega_n t-\theta_n)$$

$$\theta_n=\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

harmonic amplitude

phase angle

Complex Form of the Fourier Series

$$e^{jn\omega_0 t} = \cos n\omega_0 t + j\sin n\omega_0 t \;\;,\;\; e^{-jn\omega_0 t} = \cos n\omega_0 t - j\sin n\omega_0 t$$

$$\cos n\omega_0 t = \frac{1}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) \;,\; \sin n\omega_0 t = \frac{1}{2j}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right) = -\frac{j}{2}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$= \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} a_n\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) - \frac{j}{2}\sum_{n=1}^{\infty} b_n\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - jb_n)e^{jn\omega_0 t} + \frac{1}{2}(a_n + jb_n)e^{-jn\omega_0 t}\right]$$

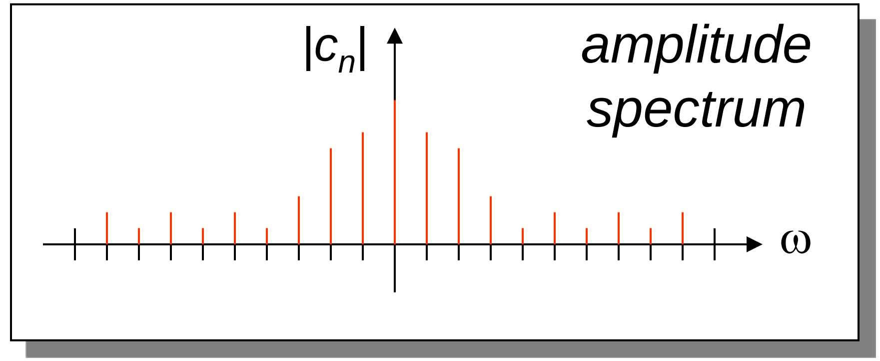
$$= c_0 + \sum_{n=1}^{\infty}\left[c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t}\right]$$

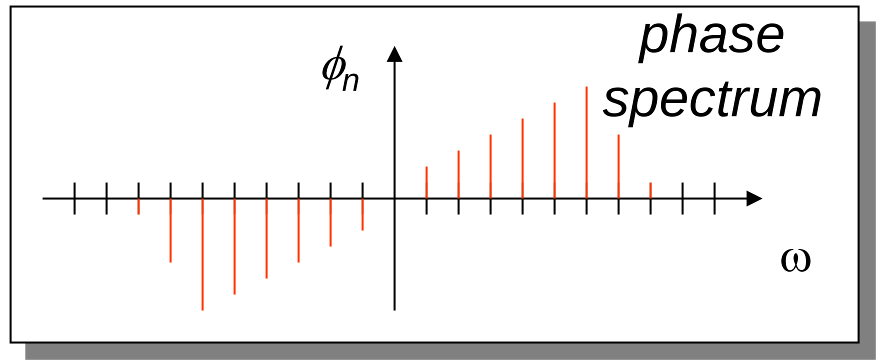
$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Complex Form of the Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt$$





If $f(t)$ is real,

$$c_{-n} = c_n^*$$

$$|c_n| = |c_{-n}| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

Fourier Transform

- We have seen that periodic signals can be represented with the Fourier series
- Can aperiodic signals be analyzed in terms of frequency components?
- Yes, and the Fourier transform provides the tool for this analysis.
- Aperiodic signals can be treated as periodic with period tending to infinity in the limit
- The major difference w.r.t. the discrete line spectra of periodic signals is that the spectra of aperiodic signals are continuous.

Fourier Transform

- We want to understand the frequency ω of our signal. So, let's reparametrize the signal by ω instead of x:

$$f(x) \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow F(\omega)$$

- For every ω from 0 to inf, ***F*(ω)** holds the amplitude *A* and phase ϕ of the corresponding sine

$$A\sin(\omega x + \phi)$$

  - How can *F* hold both? Complex number trick!

$$F(\omega) = R(\omega) + j\, I(\omega)$$

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2} \qquad \phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

$$F(\omega) \longrightarrow \boxed{\text{Inverse Fourier Transform}} \longrightarrow f(x)$$

Fourier Transform – more formally

Represent the signal as an infinite weighted sum of an infinite number of sinusoids

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

Note: $e^{j\theta} = \cos\theta + j\sin\theta \qquad j = \sqrt{-1}$

Arbitrary function $\longrightarrow$ Single Analytic Expression

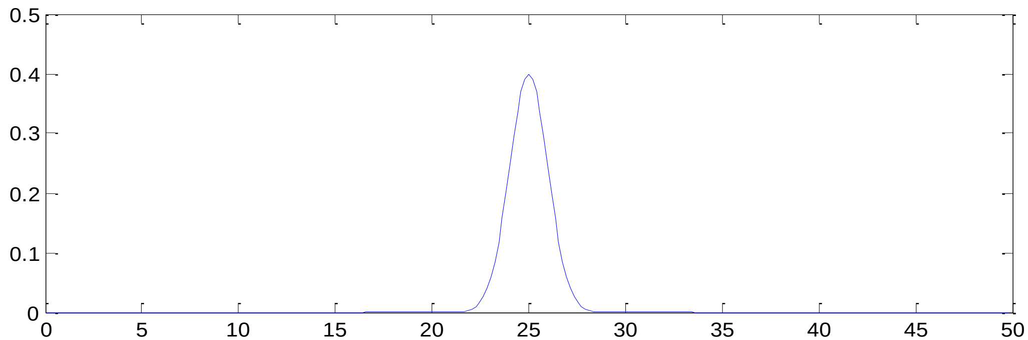
Spatial Domain (*x*) $\longrightarrow$ Frequency Domain (*u*)

(Frequency Spectrum *F(u)*)

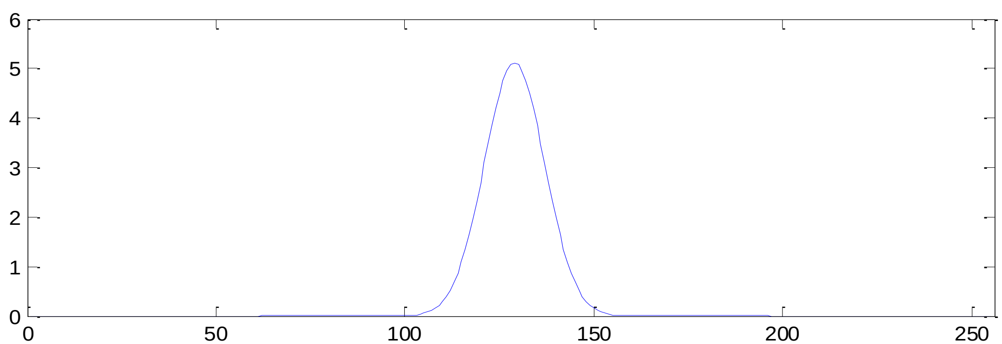
Inverse Fourier Transform (IFT)

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

Famous Fourier Transforms

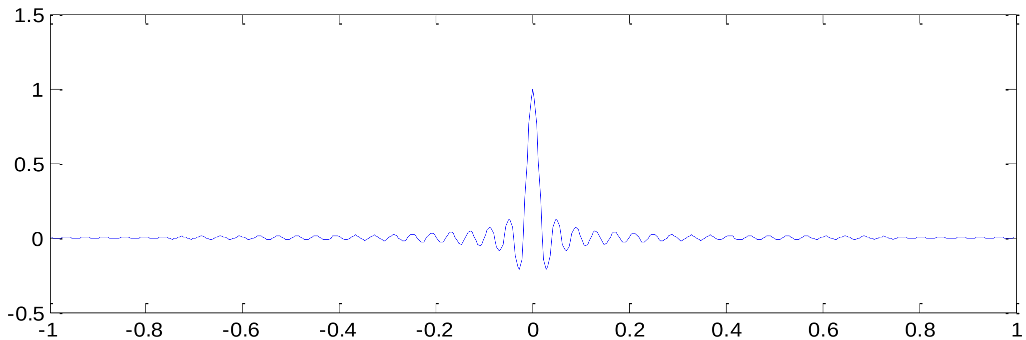


Gaussian

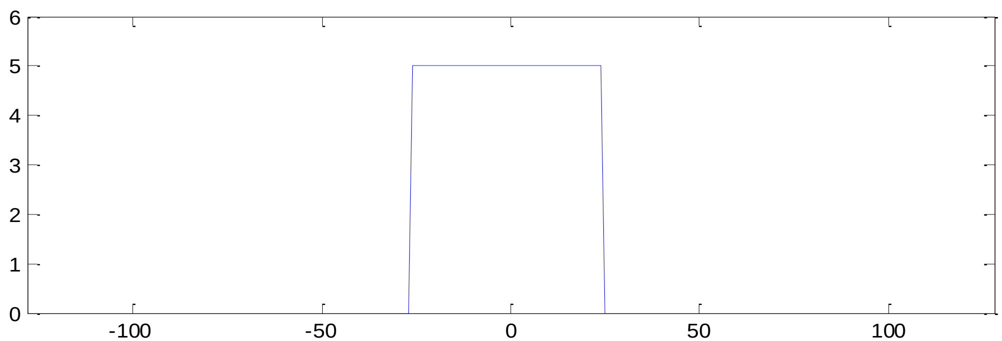


Gaussian

Famous Fourier Transforms

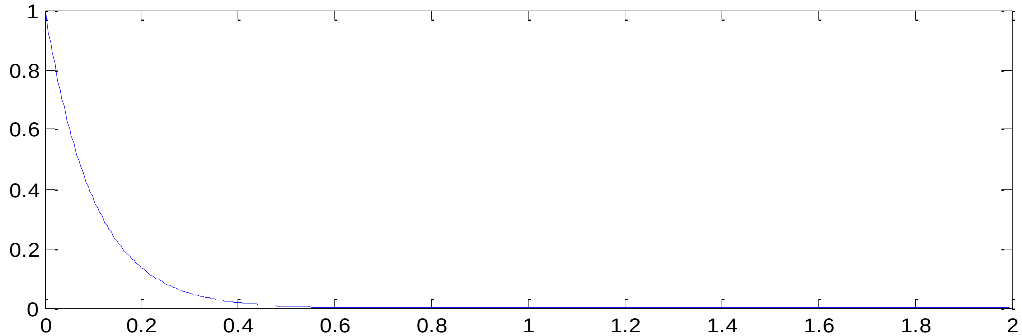


Sinc function

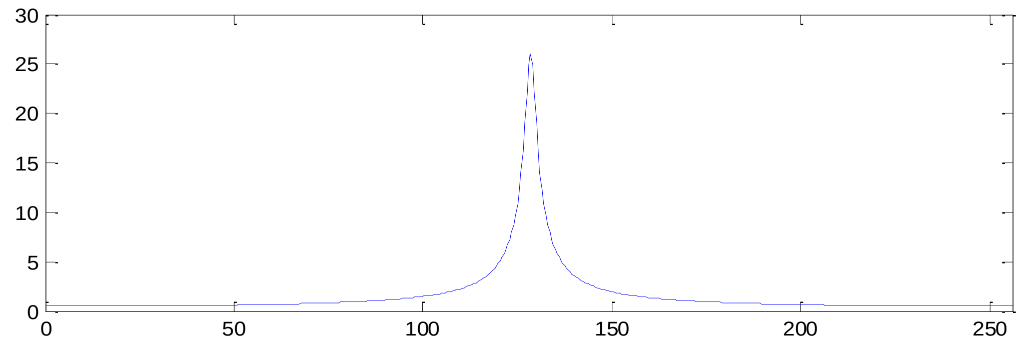


Square wave

Famous Fourier Transforms



Exponential



Lorentzian