# Hypothesis Testing 2

# Light and Heavy Tails

(4 points possible)

## Light and Heavy tails over positive and negative integers

The Random Variable $X$ has a point mass distribution over the integers. In each of the following parts you are given an expression that defines the probability of each integer. The normalizing factor $z$ is a number that ensures that the probability of the whole space is one. Recall that

- when summing over the positive integers $\sum_{i=1}^{\infty} \frac{1}{i^\alpha}$ is infinite if $0 < \alpha \leq 1$ and is finite for $\alpha > 1$.
- When summing over all integers (other than zero) $\sum_{i \neq 0} \frac{1}{i^\alpha}$ the sum is **undefined** if $0 < \alpha \leq 1$ and is finite for $\alpha > 1$.

For each expression, check the true statements. (if the distribution is note well defined, then the expected value and the variance are also undefined.

$$P(X = i) = \begin{cases} \text{if } i \neq 0 & 1/(z|i|^4) \\ \text{if } i = 0 & 0 \end{cases}$$

- [ ] This is a well defined distribution.
- [ ] The expected value of the distribution is finite.
- [ ] The expected value of the distribution is infinite.

# Clinical trials

- Developing a new drug takes 10-15 years and hundreds of millions of dollars.
- After tests on animals, the final stage is a clinical trial – a test on human patients.
- Very expensive: usually limited to a few hundred patients.
- Need to demonstrate **causality:** that taking the medication causes an improvement in the patient's health, and is not just correlated with it. Requires a **controlled study**:
  - **Placebo**
  - **Double blindness**
  - **Fixing the protocol before the start of the trial.**

# Observational Studies

- With access to all electronic medical records, it is becoming possible to measure the effectiveness of a drug on millions of people (contrast with a few hundred in controlled studies).
- **Potential of revolutionizing medical research.**
- **Challenge: hard to control for non-causal correlations.**
- Challenge can be met by controlling for potential causes: wealth, age, race …
- A new trend: when treatment is not critical and resources are limited – treat a randomly selected part of the population. In this case: correlation does imply causation.

# Counting fish

- Suppose we are studying a lake, and we want to estimate how many fish are in it.
- It is not realistic to try and catch all, or even most of them.

- Instead we follow a 3 step process:
  1. Catch $m$ fish, mark them, and release back to the lake.
  2. Let some time pass, so that the marked fish mix with the unmarked.
  3. Catch $l$ fish, count the number of marked fish, call that number the random variable $Y$.

# Sample, mark and return to lake

Catch $m$ fish, mark and release

# Wait

# Sample and count

Catch $l$ fish, $Y$ of them are marked

# Counting Fish continued

- Let $n$ be the number of fish in the lake. The probability that a random fish is marked is
  - m/n
- $Y$ is the sum of $l$ IID Binary random variables whose mean is m/n E(Y)=?
  - $E(Y) = \frac{lm}{n}$
  - $Var(Y) = l\frac{m}{n}\left(1 - \frac{m}{n}\right) \leq$
    - $\leq l\frac{1}{2}\frac{1}{2} = \frac{l}{4};$ $\sigma(Y) = \frac{\sqrt{l}}{2}$
- The 95% confidence interval for $\frac{lm}{n} = \mathrm{E(Y)}$ is
  - $[Y - \sqrt{l}, Y + \sqrt{l}]$
- Therefor the 95% confidence interval on the number of fish is
  - $\left[\frac{lm}{Y+\sqrt{l}}, \frac{lm}{Y-\sqrt{l}}\right]$
- If $l$ is too small then the estimate would be weak.
- If $m$ is too small relative to $n$ then we might not catch any marked fish,

# The structure of statistical tests

1. Define null hypothesis, alternative hypothesis
2. Define the test statistic: $X$ a random variable that is a function of the data (average, difference between averages etc.)
3. Compute (or take from book) the distribution of the test statistic under the null distribution.
   - By convention: 0 – no significance, large values – high significance.
4. Decide on the desired significance level $\alpha$ and, from it, the threshold $X > T$ which is the minimal value of $X$ needed to reject the null hypothesis with significance $\alpha$
5. Run the experiment, compute $X$
   - If $X > T$ reject the null hypothesis
   - Otherwise, the test failed.

# The Z-statistic (1)

- Suppose there is a new serum that is claimed to make people taller. We want to test whether this is true.
- Null Hypothesis: the expected height of 20 years olds is the same whether treated or untreated.
- Alternative Hypothesis: The expected height of treated is larger than of untreated.
- Additional assumptions:
  - The mean $\mu$ of the untreated people is known.
  - The std $\sigma$ of treated people is known.
  - The distribution of the average of $n$ treated people is (close to) a normal distribution (Central Limit Theorem).
- The statistic we use is the normalized average.

# The Z-Statistic (2)

- The statistic we use is the normalized average:
$$Z = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}}$$
- The distribution of $Z$ under the null hypothesis is the standard normal $\mathbb{N}(0,1)$
- Given a desired confidence level $\alpha$ we choose the threshold $T$ so that $Q(T) = \alpha$ or, in other words $T = Q^{-1}(\alpha)$
- The Z-test rejects the null hypothesis if $Z > T$

# The paired t-statistic (1)

- Suppose we want to check which wine people like better, wine1 or wine2.
- To evaluate this we select $n$ people at random give each person a taste of each wine and ask them to rate the two wines on a scale of 1 to 10, where 10 is the best and 1, the worst.
- It is a good idea to have each person rate both wines because rating scales vary from person to person. It is more informative to compare the ratings for wine1 and wine2 given by the same person rather than compare rating of wine1 by person 1 to the rating of wine2 by person 2.
- We thus **pair** the rating and then use the difference: rating-by-person1(wine1) – rating-by-person1(wine2).

# The paired t-statistic (2)

- The Null Hypothesis: the expected difference between the rating is zero = the two wines are equally liked.
- Alternative 1: wine1 is better liked than wine2 = the expected difference is larger than zero
- Alternative 2: wine2 is better liked than wine1 = the expected difference is smaller than zero.
- We define the average of the sample:

$$\bar{X} \doteq \frac{1}{n}\sum_{i=1}^{n} D_i \,, \qquad D_i \doteq R_i^1 - R_i^2$$

- We assume that $n$ is sufficiently large that the distribution of $\bar{X}$ is close to normal.
- It might seem that we are going to end up with the Z-statistic but in this case we don't assume to know the std.

# The paired t-statistic (3)

- Clearly under the null hypothesis $E(R^1) = E(R^2)$ and therefor $E(D) = E(R^1 - R^2) = 0.$
- It might seem that we are going to end up with a Z-statistic but in this case we don't assume to know the std of $R^1, R^2$ or $D$
- We therefor use an estimate of the std:

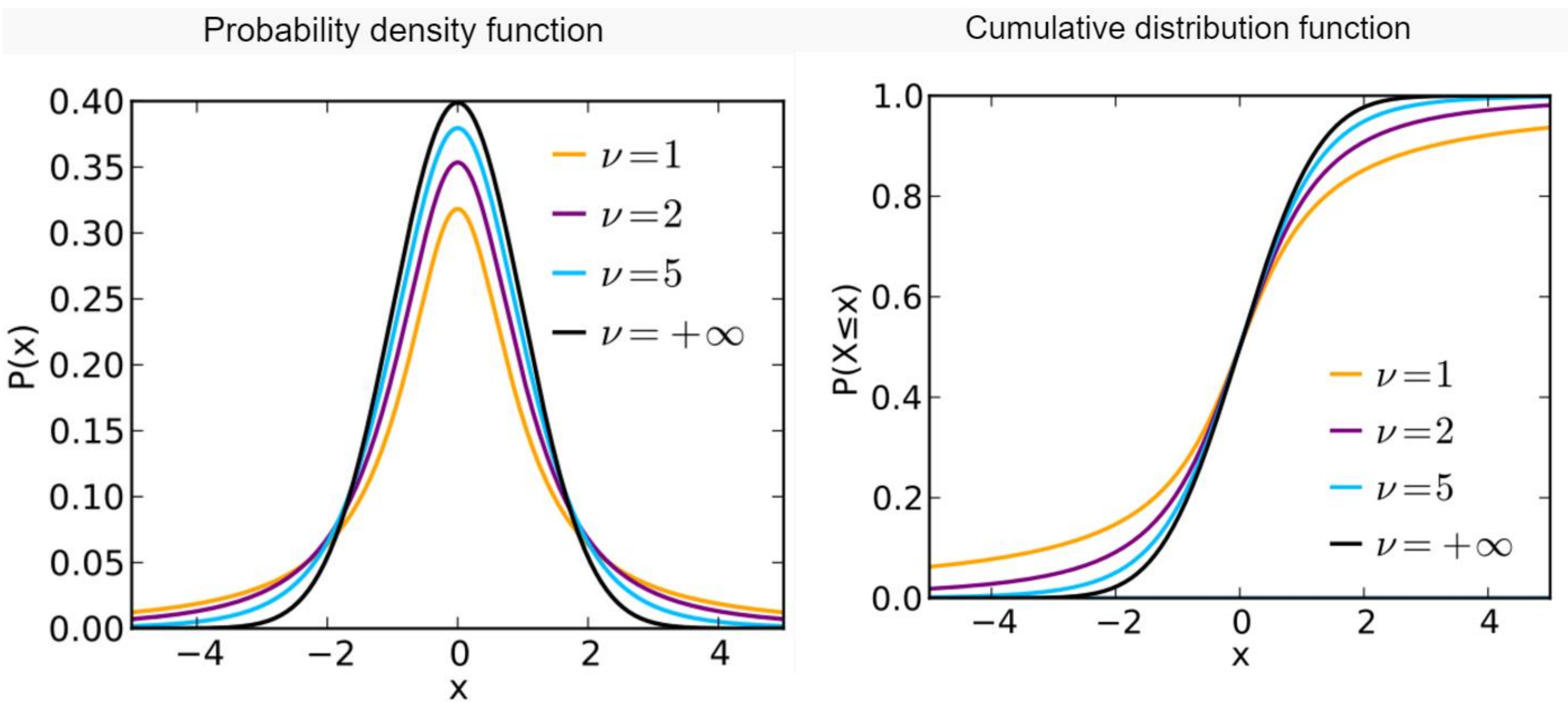
$$\mathrm{s} = \sqrt{\frac{1}{\mathrm{n}-1}\sum_{i=1}^{n} D_i^2}$$

- Using $s$ and $\bar{\mathrm{X}}$ we define the (zero mean) t-statistic to be:

$$t = \frac{\bar{X}}{s/\sqrt{n}}$$

# Paired t-test (4)

- The zero mean t-statistic is $t = \frac{\bar{X}}{s/\sqrt{n}}$
- The distribution of the t-statistic $t$ is the student-t distribution with $\nu = n - 1$ degrees of freedom ($n$ is the number of samples).




As $\nu$ increases, the student t-distribution converges to the standard normal distribution.

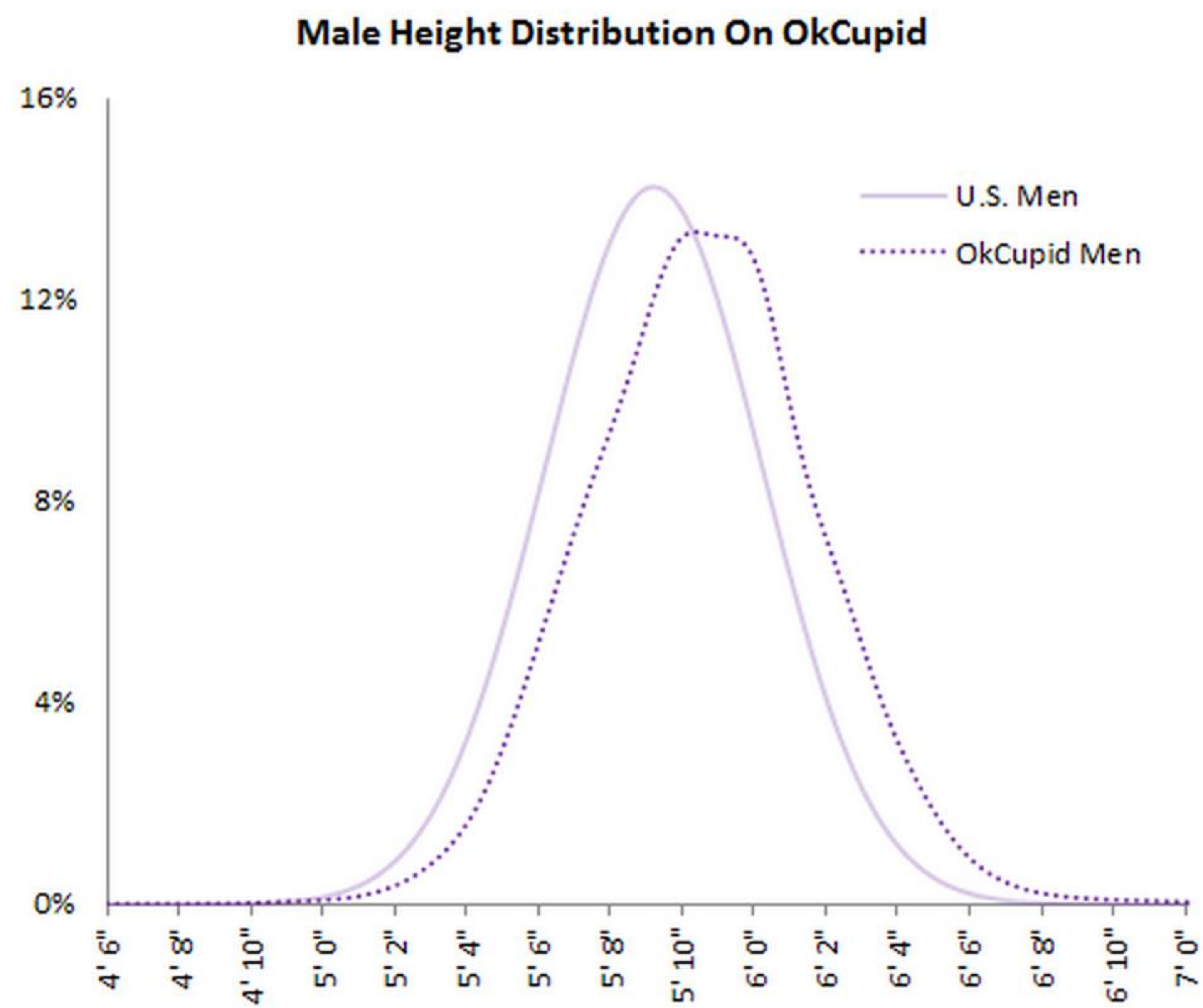
# Other variants of the t-statistic

- Under the null hypothesis the mean of the distribution is $\mu \neq 0$
  - Example: Do men tend to inflate their height when writing match-making ads?
- The two sample t-statistic: compare the expected value of two populations (unpaired)
  - Example: we want to compare the effect of two treatments for a disease and it is not possible to pair – not possible to try both treatments on the same individual.
  - Pooled variance: we assume that the variance of the two populations is the same.
  - Un-pooled variance, we don't assume that the variances are the same.

# Example of single population t-statistic

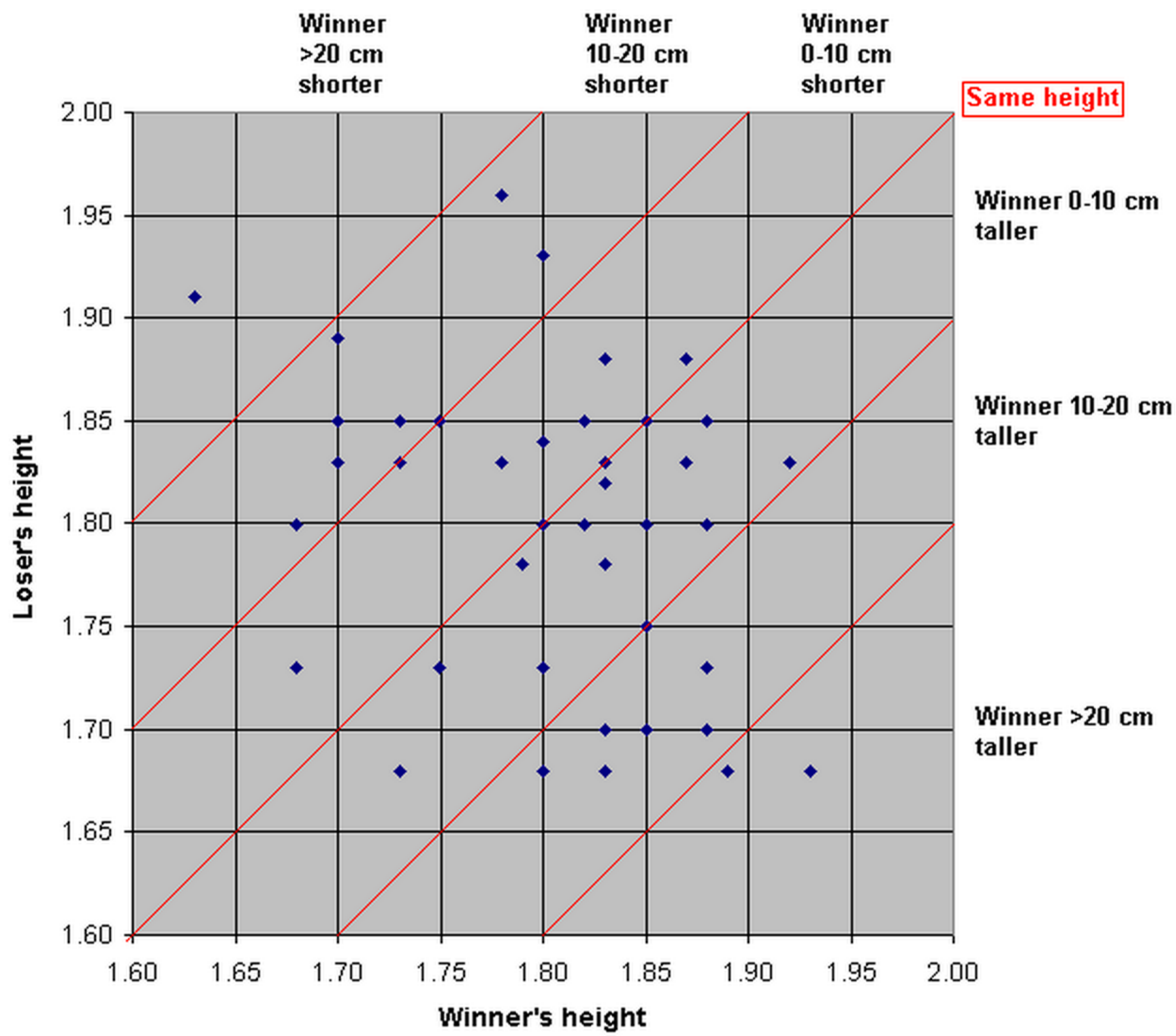
oktrends Dating Research from OkCupid

## The Big Lies People Tell In Online Dating



# Example of a paired t-statistic

Question: are winners in presidential elections taller than their opponent?



# Examples of common statistical tests

## From the matlab Statistics module

| | |
|---|---|
| ranksum | Wilcoxon rank sum test. Tests if two independent samples come from identical continuous distributions with equal medians, against the alternative that they do not have equal medians. |
| runstest | Runs test. Tests if a sequence of values comes in random order, against the alternative that the ordering is not random. |
| signrank | One-sample or paired-sample Wilcoxon signed rank test. Tests if a sample comes from a continuous distribution symmetric about a specified median, against the alternative that it does not have that median. |
| signtest | One-sample or paired-sample sign test. Tests if a sample comes from an arbitrary continuous distribution with a specified median, against the alternative that it does not have that median. |
| ttest | One-sample or paired-sample *t*-test. Tests if a sample comes from a normal distribution with unknown variance and a specified mean, against the alternative that it does not have that mean. |
| ttest2 | Two-sample *t*-test. Tests if two independent samples come from normal distributions with unknown but equal (or, optionally, unequal) variances and the same mean, against the alternative that the means are unequal. |

# Pearson's chi-squared test

From Wikipedia, the free encyclopedia

It tests a null hypothesis stating that the frequency distribution of certain events observed in a sample is consistent with a particular theoretical distribution. The events considered must be mutually exclusive and have total probability 1. A common case for this is where the events each cover an outcome of a categorical variable. A simple example is the hypothesis that an ordinary six-sided die is "fair", i. e., all six outcomes are equally likely to occur.

The value of the test-statistic is

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = N \sum_{i=1}^{n} \frac{(O_i/N - p_i)^2}{p_i}$$

where

$\chi^2$ = Pearson's cumulative test statistic, which asymptotically approaches a $\chi^2$ distribution.

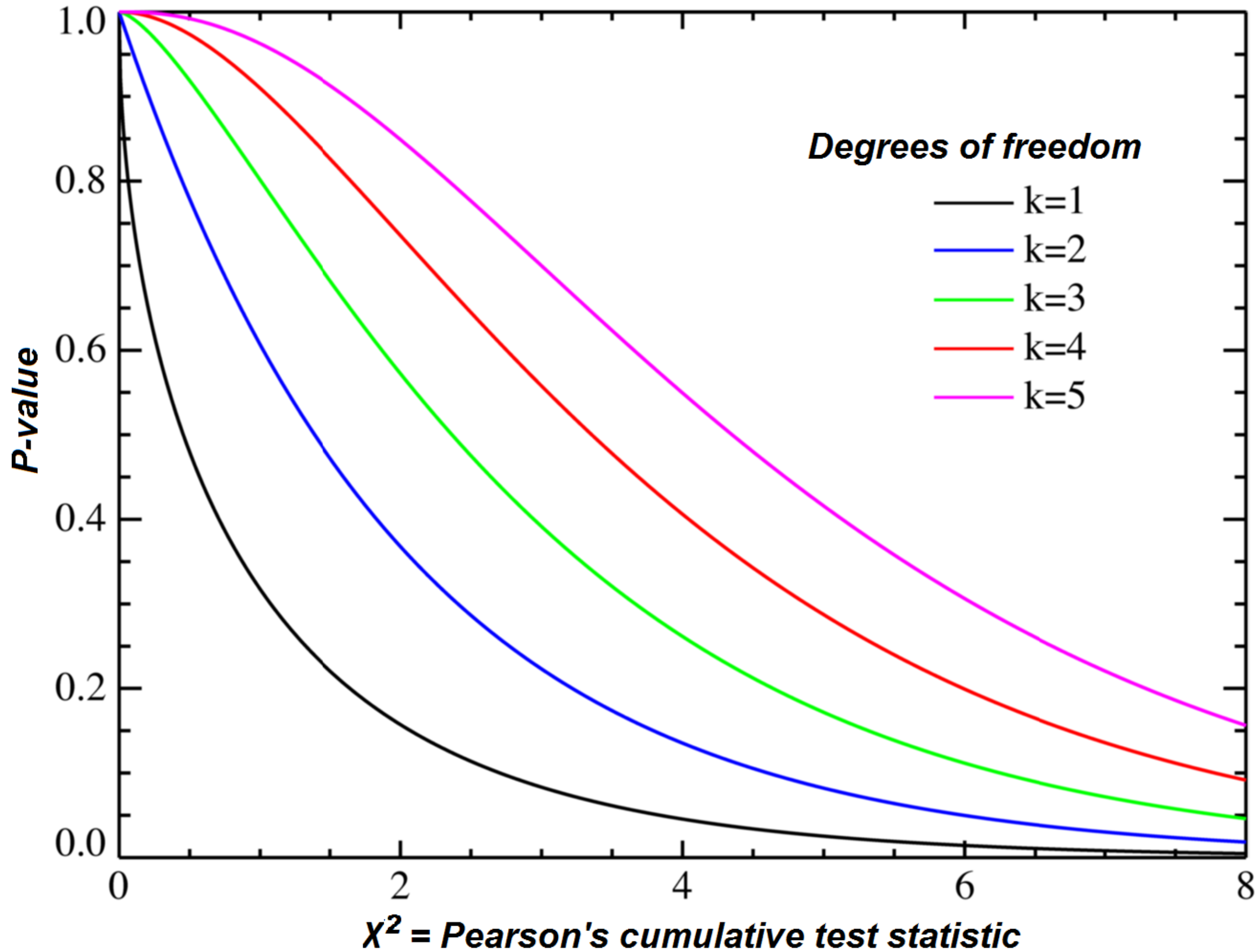
$O_i$ = the number of observations of type *i*.

$N$ = total number of observations

$E_i = Np_i$ = the expected (theoretical) frequency of type *i*, asserted by the null hypothesis that the fraction of type *i* in the population is $p_i$

$n$ = the number of cells in the table.

# The $\chi^2$ distribution




Degrees of freedom:
A distribution over $|\Omega| = d$ possible outcomes has
$k = d - 1$
degrees of freedom.

# Uses of the $\chi^2$ test

- A test of **goodness of fit** establishes whether or not an observed frequency distribution differs from a theoretical distribution.
- A **test of independence** assesses whether paired observations on two variables, expressed in a contingency table, are independent of each other (e.g. polling responses from people of different nationalities to see if one's nationality is related to the response).

# Testing for dependence

- We have two discrete random variables
- True contingency table: the true probability for each cell.
- Empirical contingency table: the fraction of the observations that fall in each in each cell.
- Null Hypothesis: the two variables are independent.
- We can use the chi-square test

# *Empirical contingency tables*

n=100
std~1/10

## independent

true dist

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2500 | 0.5000 | 0.2500 |
| Male | 0.4000 | 0.1000 | 0.2000 | 0.1000 |
| Female | 0.6000 | 0.1500 | 0.3000 | 0.1500 |

empirical dist 1

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2600 | 0.4800 | 0.2600 |
| Male | 0.3900 | 0.1000 | 0.2100 | 0.0800 |
| Female | 0.6100 | 0.1600 | 0.2700 | 0.1800 |

empirical dist 2

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2200 | 0.3900 | 0.3900 |
| Male | 0.4400 | 0.1100 | 0.1400 | 0.1900 |
| Female | 0.5600 | 0.1100 | 0.2500 | 0.2000 |

empirical dist 3

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2200 | 0.5000 | 0.2800 |
| Male | 0.3500 | 0.1000 | 0.1200 | 0.1300 |
| Female | 0.6500 | 0.1200 | 0.3800 | 0.1500 |

## dependent

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1.0000 | 0.3000 | 0.4000 | 0.3000 |
| Male | 0.5000 | 0.1000 | 0.2000 | 0.2000 |
| Female | 0.5000 | 0.2000 | 0.2000 | 0.1000 |

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2700 | 0.5200 | 0.2100 |
| Male | 0.4800 | 0.1200 | 0.2200 | 0.1400 |
| Female | 0.5200 | 0.1500 | 0.3000 | 0.0700 |

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2600 | 0.4300 | 0.3100 |
| Male | 0.4700 | 0.0700 | 0.2100 | 0.1900 |
| Female | 0.5300 | 0.1900 | 0.2200 | 0.1200 |

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.1800 | 0.4000 | 0.4200 |
| Male | 0.5500 | 0.1000 | 0.1600 | 0.2900 |
| Female | 0.4500 | 0.0800 | 0.2400 | 0.1300 |

# *Empirical contingency tables*

n=10,000
std~1/100

## independent

**true dist**

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2500 | 0.5000 | 0.2500 |
| Male | 0.4000 | 0.1000 | 0.2000 | 0.1000 |
| Female | 0.6000 | 0.1500 | 0.3000 | 0.1500 |

**empirical dist 1**

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2473 | 0.5062 | 0.2465 |
| Male | 0.4021 | 0.0977 | 0.2052 | 0.0992 |
| Female | 0.5979 | 0.1496 | 0.3010 | 0.1473 |

**empirical dist 2**

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2530 | 0.4943 | 0.2527 |
| Male | 0.3925 | 0.0982 | 0.1942 | 0.1001 |
| Female | 0.6075 | 0.1548 | 0.3001 | 0.1526 |

**empirical dist 3**

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2457 | 0.5005 | 0.2538 |
| Male | 0.3893 | 0.0936 | 0.1945 | 0.1012 |
| Female | 0.6107 | 0.1521 | 0.3060 | 0.1526 |

## dependent

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1.0000 | 0.3000 | 0.4000 | 0.3000 |
| Male | 0.5000 | 0.1000 | 0.2000 | 0.2000 |
| Female | 0.5000 | 0.2000 | 0.2000 | 0.1000 |

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1.0000 | 0.3000 | 0.4064 | 0.2936 |
| Male | 0.4991 | 0.0958 | 0.2052 | 0.1981 |
| Female | 0.5009 | 0.2042 | 0.2012 | 0.0955 |

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1.0000 | 0.3058 | 0.3895 | 0.3047 |
| Male | 0.5044 | 0.1023 | 0.1989 | 0.2032 |
| Female | 0.4956 | 0.2035 | 0.1906 | 0.1015 |

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2984 | 0.4047 | 0.2969 |
| Male | 0.4970 | 0.0997 | 0.1938 | 0.2035 |
| Female | 0.5030 | 0.1987 | 0.2109 | 0.0934 |

# ***Empirical contingency tables***

n=1,000,000
std~1/1,000

## independent

true dist

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2500 | 0.5000 | 0.2500 |
| Male | 0.4000 | 0.1000 | 0.2000 | 0.1000 |
| Female | 0.6000 | 0.1500 | 0.3000 | 0.1500 |

empirical dist 1

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2500 | 0.4998 | 0.2502 |
| Male | 0.3998 | 0.1001 | 0.2000 | 0.0997 |
| Female | 0.6002 | 0.1499 | 0.2998 | 0.1505 |

empirical dist 2

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2504 | 0.4997 | 0.2499 |
| Male | 0.3991 | 0.1000 | 0.1993 | 0.0999 |
| Female | 0.6009 | 0.1504 | 0.3004 | 0.1500 |

empirical dist 3

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2504 | 0.5000 | 0.2497 |
| Male | 0.4005 | 0.1005 | 0.2003 | 0.0996 |
| Female | 0.5995 | 0.1498 | 0.2997 | 0.1500 |

## dependent

true dist

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1.0000 | 0.3000 | 0.4000 | 0.3000 |
| Male | 0.5000 | 0.1000 | 0.2000 | 0.2000 |
| Female | 0.5000 | 0.2000 | 0.2000 | 0.1000 |

empirical dist 1

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.2997 | 0.4000 | 0.3002 |
| Male | 0.4997 | 0.0994 | 0.2000 | 0.2003 |
| Female | 0.5003 | 0.2003 | 0.2000 | 0.1000 |

empirical dist 2

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.3006 | 0.3996 | 0.2997 |
| Male | 0.5002 | 0.1000 | 0.2001 | 0.2001 |
| Female | 0.4998 | 0.2006 | 0.1995 | 0.0997 |

empirical dist 3

| | marginal | A-average | B-average | C-average |
|---|---|---|---|---|
| marginal | 1 | 0.3001 | 0.3998 | 0.3001 |
| Male | 0.4995 | 0.1000 | 0.1998 | 0.1997 |
| Female | 0.5005 | 0.2001 | 0.2000 | 0.1004 |

# Finding cheaters using the Bedford distribution.

- Suppose we have a large accounting document, with thousands of numbers in it.
- Can we detect if somebody “cooked the books” just by looking at the distribution of the numbers?
- Consider the distribution of the most significant (non zero) digit:

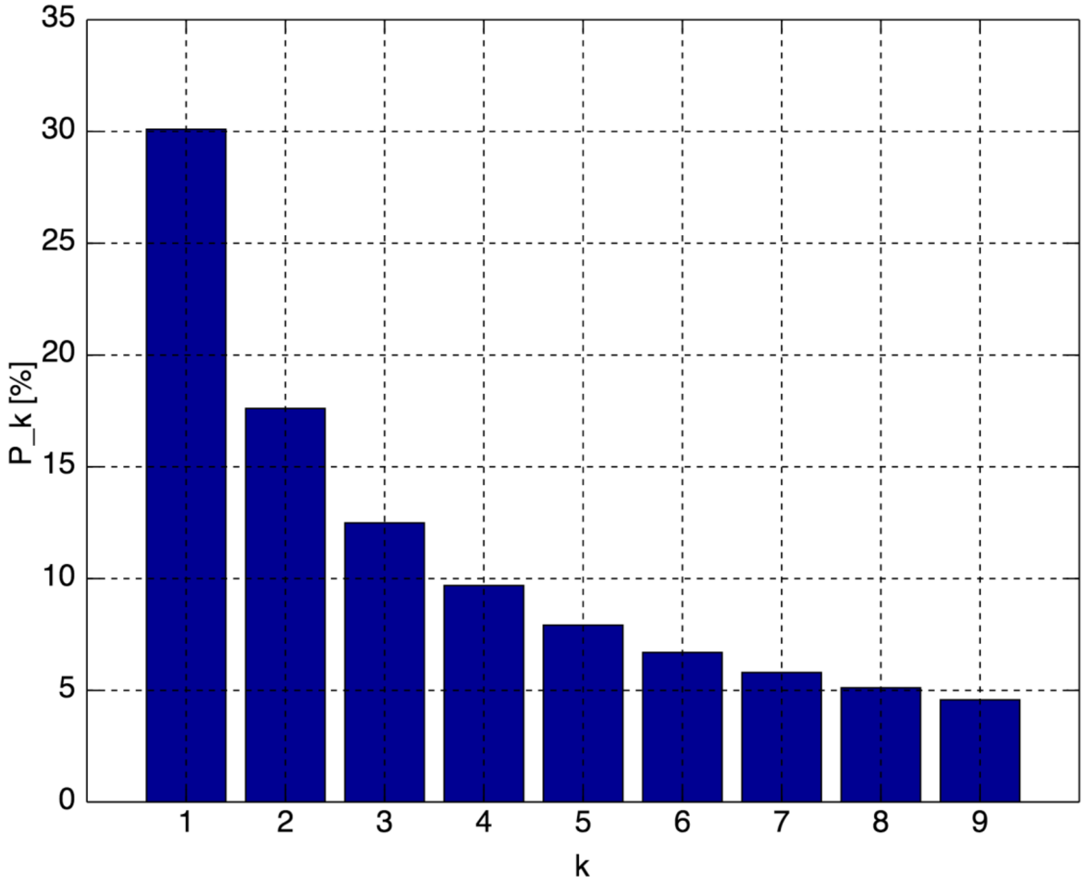
**Expense Report**

| Person | Voucher | Expense | Reimburse | Status | | Pending Approvals | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | Manager | Proj Approver | Customer |
| Admin, Donna M. (donna) | 107 | $1,000.00 | $0.00 | COMPLETED | 9/27/2006 10:01 AM | | | |
| Hayden, Richard (richard) | 102 | $0.00 | $0.00 | COMPLETED | 9/13/2006 1:51 PM | | | |
| Hayden, Richard (richard) | 103 | $20,000.00 | $0.00 | COMPLETED | 9/13/2006 1:54 PM | | | |
| Hayden, Richard (richard) | 104 | ($20,000.00) | $0.00 | COMPLETED | 9/13/2006 1:56 PM | | | |
| Lauer, Matt A. (lauer) | 101 | $622.50 | $122.50 | COMPLETED | 11/7/2006 10:05 AM | | | |
| Lauer, Matt A. (lauer) | 106 | $438.00 | $38.00 | COMPLETED | 10/26/2006 6:11 AM | | | |
| Lauer, Matt A. (lauer) | 108 | $676.00 | $76.00 | COMPLETED | 9/27/2006 10:02 AM | | | |
| Roker, Al (aroker) | 96 | $20.00 | $20.00 | COMPLETED | 9/27/2006 10:02 AM | | | |
| Roker, Al (aroker) | 98 | $790.35 | $290.35 | COMPLETED | 11/7/2006 10:05 AM | | | |
| Roker, Al (aroker) | 99 | $20.00 | $20.00 | COMPLETED | 11/7/2006 10:05 AM | | | |
| Roker, Al (aroker) | 100 | $40.00 | $40.00 | COMPLETED | 11/7/2006 10:05 AM | | | |
| Sawyer, Diane B. (sawyer) | 33 | $560.00 | $560.00 | COMPLETED | 11/7/2006 10:05 AM | | | |
| **Total Report Count:** | | 12 | | | | | | |

* Identifies items that require customer approval first

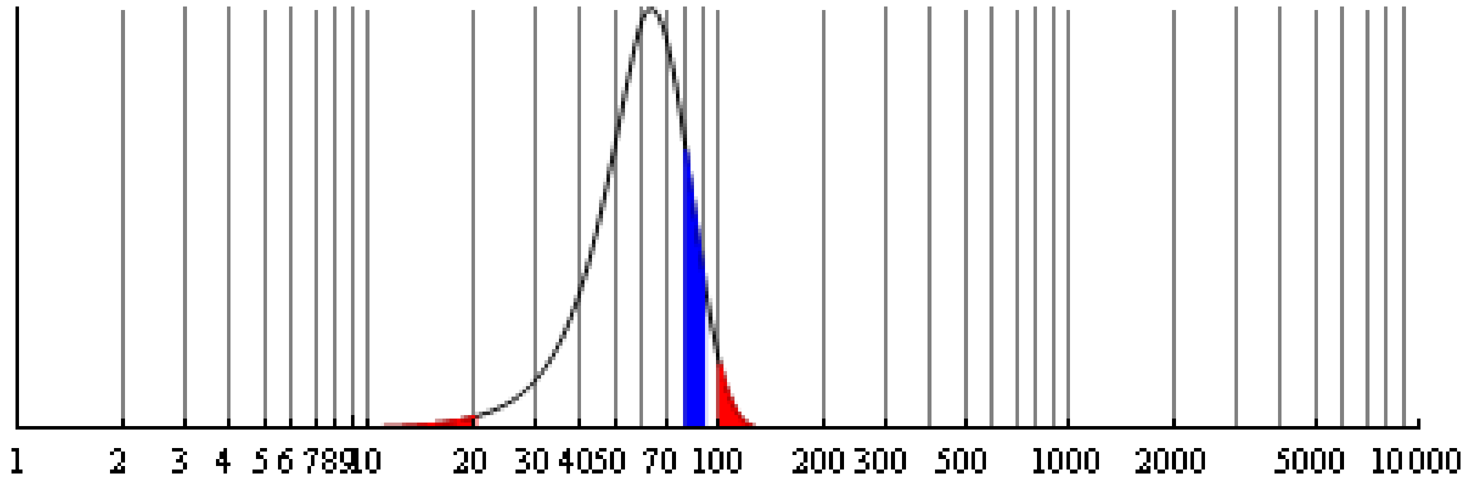
# The Benford distribution

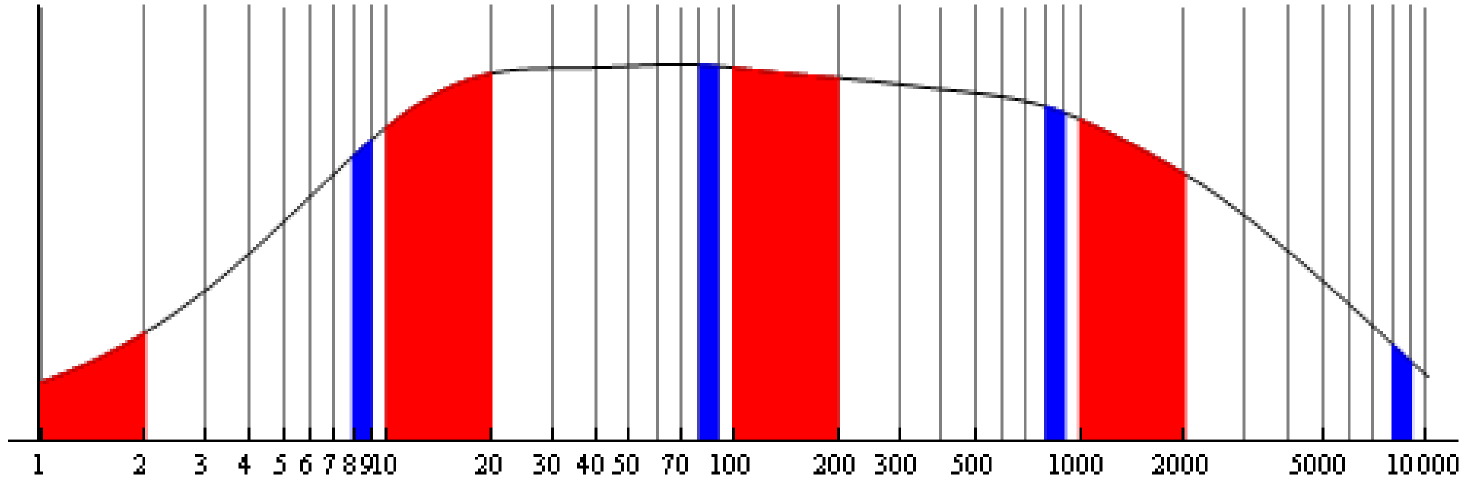
Describes the distribution of the most significant digit in large collection of financial numbers



# How can we explain Benford law?

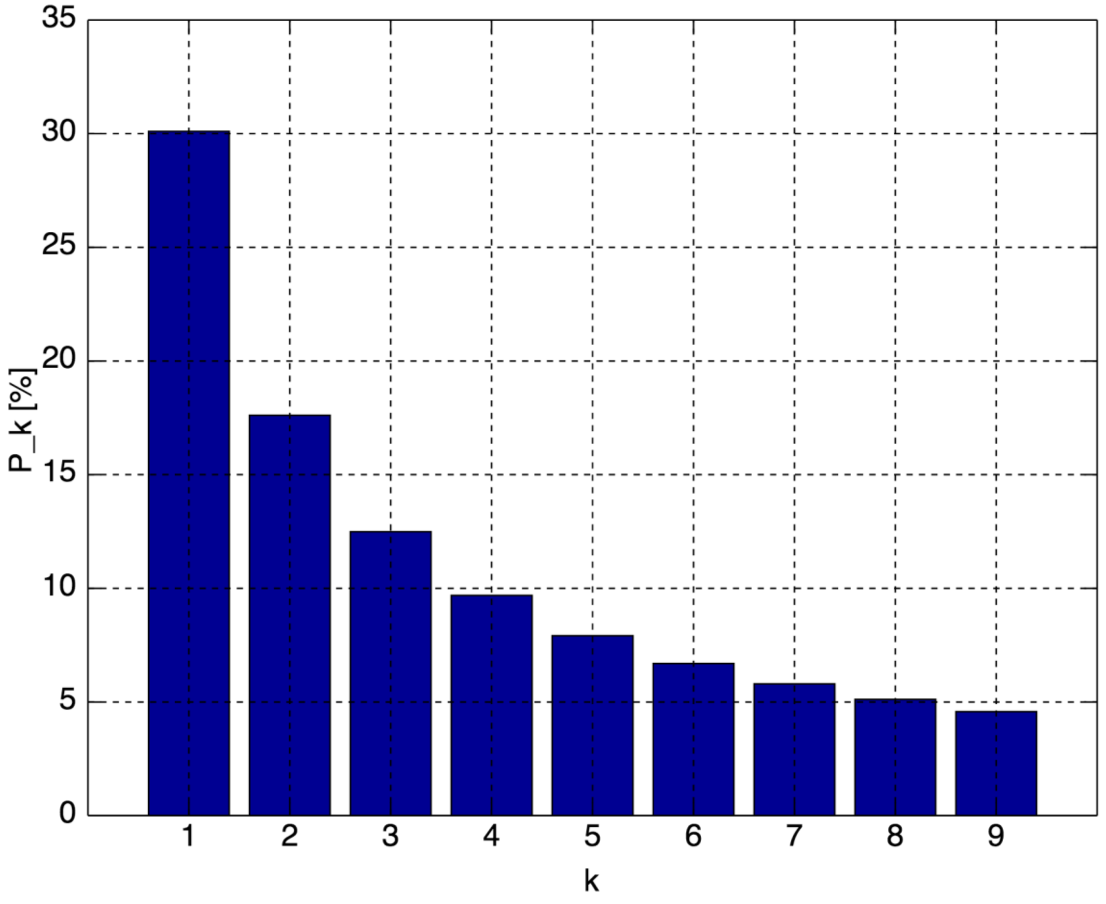
- As currency units are arbitrary, changing the definition of a the currency unit does not fundamentally change the distribution.
- The distribution is approximately constant on a logarithmic scale.
- If the distribution spans several orders of magnitude (from single dollars to thousands of dollars) we get the Benford distribution

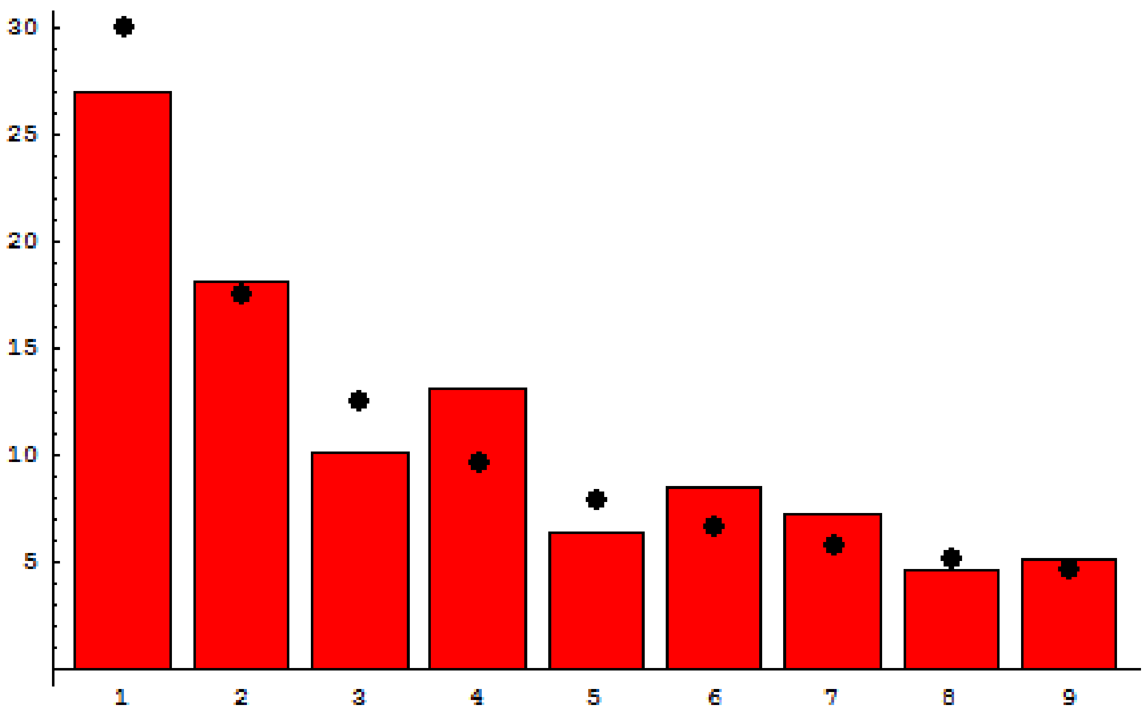


Can we detect accounting fraud using the Benford distribution?

Null Hyp: dist is Benford




Distribution of top digits in a tax return

# Multiple Hypothesis testing

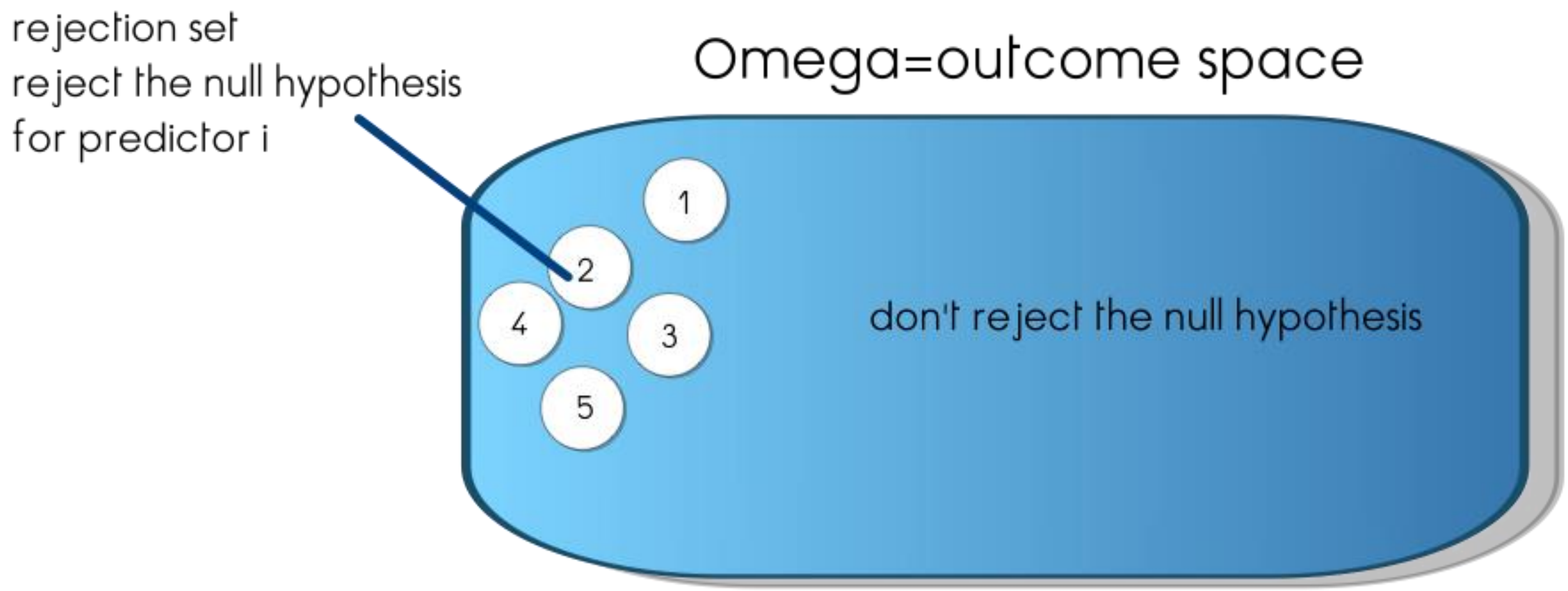
Consider the online ad problem, our goal is to maximize click-through rate. Our null hypothesis is that nothing performs better than picking one of the ads uniformly at random each time.

We have a large number of click-prediction algorithms. Each such algorithm takes as input information about the person, the web page and the ad and predicts the probability that the person will click on the ad.

We can go back in time and compute the expected number of errors each method would have made. We can use a statistical test to quantify the statistical significance of the performance of the method.

Suppose we have 100 methods and use an alpha value of 1%
Suppose for our data we found that one of the 100 methods rejects the null hypothesis at the 1% significance level. How sure can we be that the predictor that we found is better than random?

# The probabilty theory of statistical tests




We don't know what would happen of different samples than the one we observe.
In the worst case the rejection sets are disjoint.

The Bonferroni correction for multiple-hypothesis testing:

If $n$ statistical tests are performed using the same data
and the significance threshold used for all tests is $\alpha$
Then the probability that at least one of the tests will
reject the null hypothesis can be as high as $n\alpha$

# Be a skeptic:

When you read that something has been proven statistically:

1. Ask what was the null hypothesis
2. Ask what was the statistical significance
3. Ask whether similar tests were performed that did not succeed.