Combinatorics 2

Review 1: outcomes, outcome spaces and events

Consider the probability of k heads n tosses of a fair coins.

An outcome: a tuple of length n: HTHHTHH......HHT

Outcome space: Ω the set containing all tuples of length n 2^n

Event: A the set containing all n-tuples with k heads.

$$P(A) = \frac{|A|}{|\Omega|} = \frac{C(n,k)}{2^n}$$

Review 2

Factorial: $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

Permutations: $P(n,k) = \frac{n!}{(n-k)!}$

Combinations: $C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$

The probability of exactly k heads when flipping a fair coin n times:

$$P(A) = \frac{|A|}{|\Omega|} = \frac{C(n,k)}{2^{\cancel{k}\,n}}$$

Binomial Expansion

aa
ab
ba
bb

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

$$(a+b)^3 = (a+b)(a+b)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^3 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} a^3 + \begin{pmatrix} 3 \\ 1 \end{pmatrix} a^2b + \begin{pmatrix} 3 \\ 2 \end{pmatrix} ab^2 + \begin{pmatrix} 3 \\ 3 \end{pmatrix} b^3$$

$$(a+b)^n = \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} a^{n-i} b^i$$

Suppose $a = b = \frac{1}{2}$ then we get:

$$1 = \left(\frac{1}{2} + \frac{1}{2}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{n-i} = \left(\frac{1}{2}\right)^n \sum_{i=0}^{n} \binom{n}{i}$$

Which can also be written as:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Which must be the case because …

Coming back to coin flipping,
How many coin flips do we need to guarantee that there are
at least 60 heads
or
at least 60 tails?

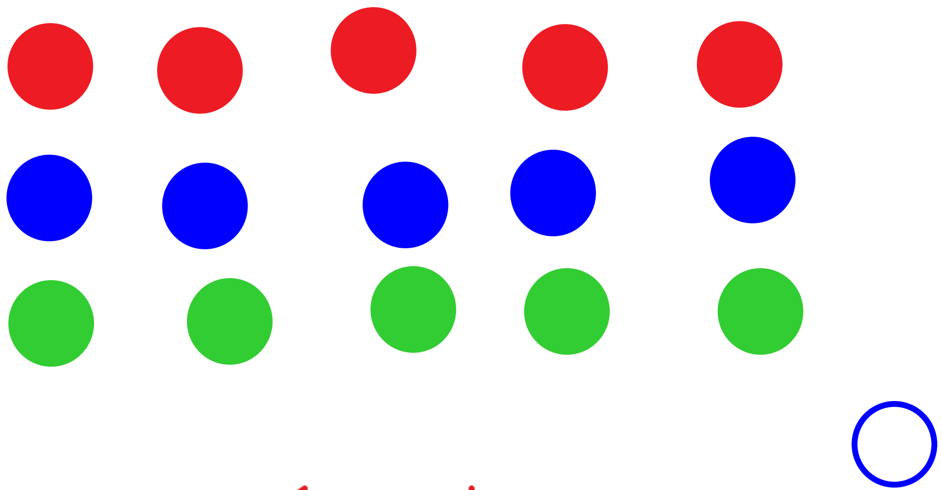
H 59

T 59

$2 \times (60-1)+1$

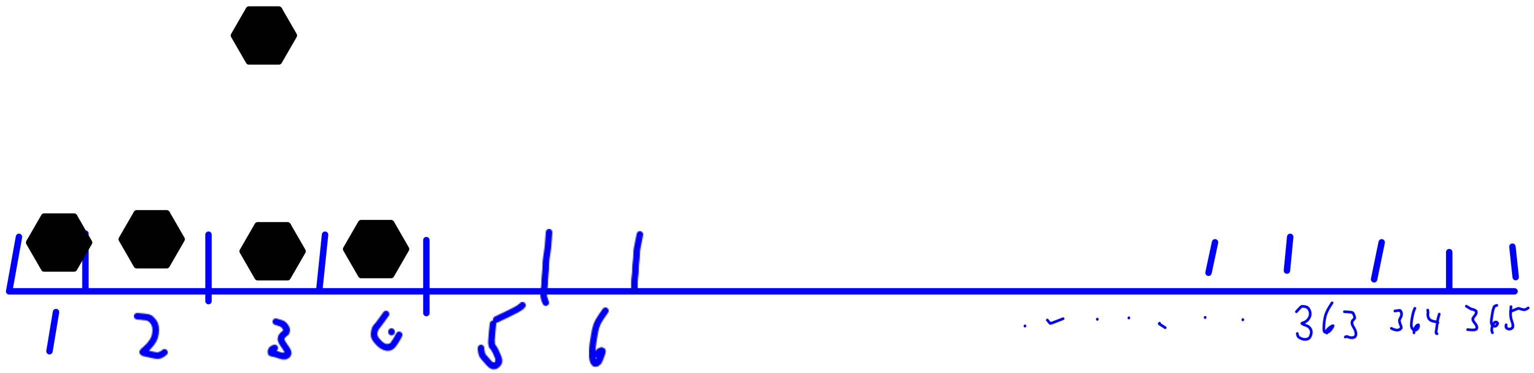
The Pigeon-Hole Principle

There are 3 colors of marbles. How many marbles do we need in order to guarantee that there is at least one color for which we have at least 6 marbles?



$3 \times (6-1) + 1$

How many people need to be in a room so that at least two of them share a birthday? (assume 365 days in a year)



366

The Birthday Paradox

How many people do you need in the room so that at least two of them have the same birthday?

For sure? 366

With probability at least half?

Assume all days have the same probability (1/365)

K = the number of people in the room.

We want to calculate $P(A)$ for the event
A={K birthdays such that at least two are the same}

How many people do you need in the room so that at least two of them have the same birthday?

$$A = \left\{ (i_1, i_2, \ldots, i_K), 1 \le i_j \le 365 \,\middle|\, \exists \;\; 1 \le j_1 < j_2 \le K, i_{j_1} = i_{j_2} \right\}$$

Consider the complement,
No two people have the same birthday

$$A^c = \left\{ (i_1, i_2, \ldots, i_K), 1 \le i_j \le 365 \,\middle|\, \forall \;\; 1 \le j_1 < j_2 \le K, i_{j_1} \ne i_{j_2} \right\}$$

We are using tuples to represent the birthdays, in other words, order is important!

$$A^c \doteq \{x \in \Omega,\; x \notin A\} \qquad A^c = \Omega - A$$

A sequence of K birthdates and no 2 have the same birthday
-> K days out of 365

$$\left|A^c\right| = P(365, K) = \frac{365!}{(365-K)!}$$

$$\Omega = \{1 - 365\}^k \quad |\Omega| = 365^k$$

$A =$ {Tuples where at least one birthday appears more than once}
$A^c =$ {Tuples where no two birthdays are the same}

$$|A^c| = P(365, K) = \frac{365!}{(365-k)!}$$

$$A = \Omega - A^c \quad \Rightarrow \quad |A| = |\Omega| - |A^c|$$

$$Prob(A) = \frac{|A|}{|\Omega|} = \frac{|\Omega| - |A^c|}{|\Omega|} = 1 - \frac{|A^c|}{|\Omega|}$$

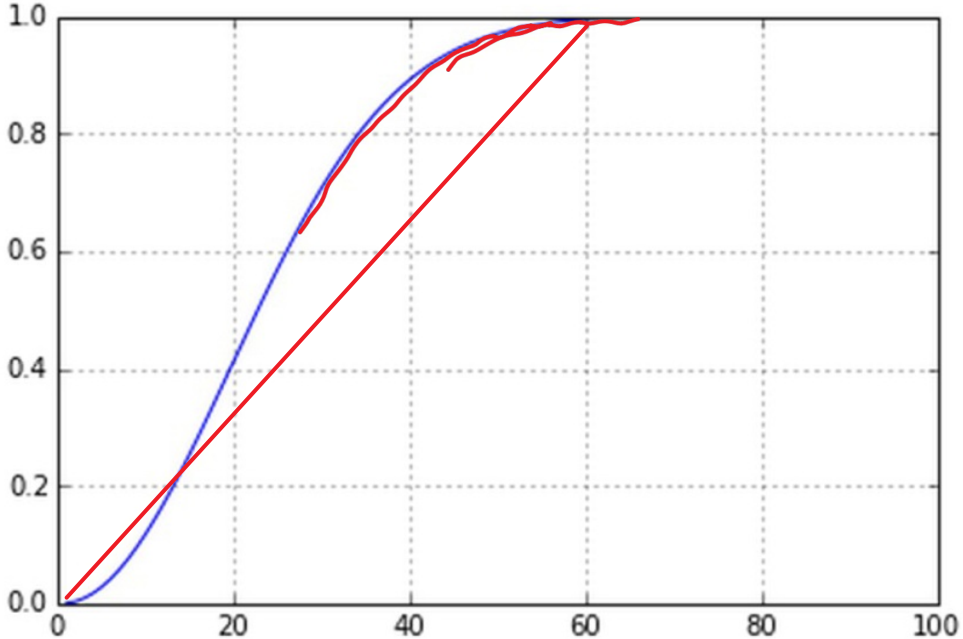
$$Prob(A) = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{P(365,k)}{365^k} = 1 - \left(\frac{365}{365}\right) \times \left(\frac{364}{365}\right) \times \cdots \times \left(\frac{365-k+1}{365}\right)$$

Probability that the first birthday lands on an unoccupied day (certain because this is the first)

Probability that the second birthday lands on an unoccupied day

Probability that the k'th birthday lands on an unoccupied day

1	:	0.00000000
2	:	0.00273973
3	:	0.00820417
4	:	0.01635591
5	:	0.02713557
6	:	0.04046248
7	:	0.05623570
8	:	0.07433529
9	:	0.09462383
10	:	0.11694818
11	:	0.14114138
12	:	0.16702479
13	:	0.19441028
14	:	0.22310251
15	:	0.25290132
16	:	0.28360401
17	:	0.31500767
18	:	0.34691142
19	:	0.37911853
20	:	0.41143838
21	:	0.44368834
22	:	0.47569531
23	:	0.50729723
24	:	0.53834426
50	:	0.97037358
100	:	0.99999969



LLL DD

$C(5,2)\, 26^3\, 10^2$

Excercise 1

How many strings contain 3 letters and two digits?

(digits and letters can repeat and there is no restrictions on their order)

Number of ways to combine 3 letters and 2 digits: C(5,2)

Set of possible 3 letter tuples = {A,...,Z}^3
The size of this set is 26*26*26 = 26^3

$C(5,2)\, 26^3\, 10^2$

Set of 2 digits, size of this set is 10*10=100

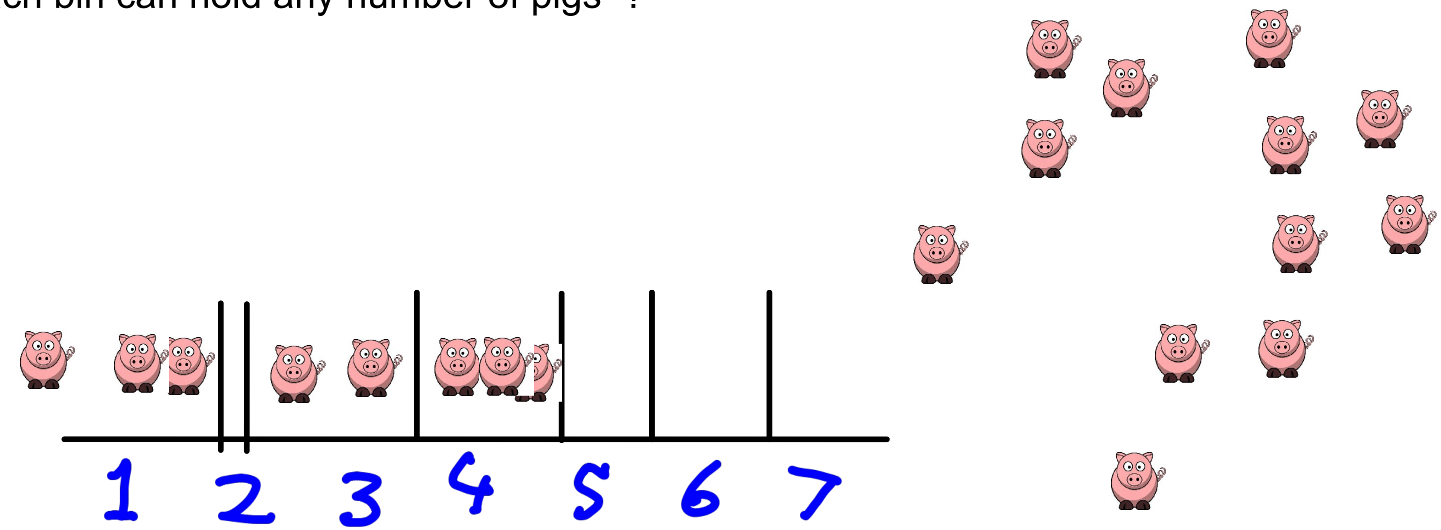
Excercise 2

What is the number of strings that start with a digit followed by 4 letters, followed by 2 digits?

Answer: this is a product set:

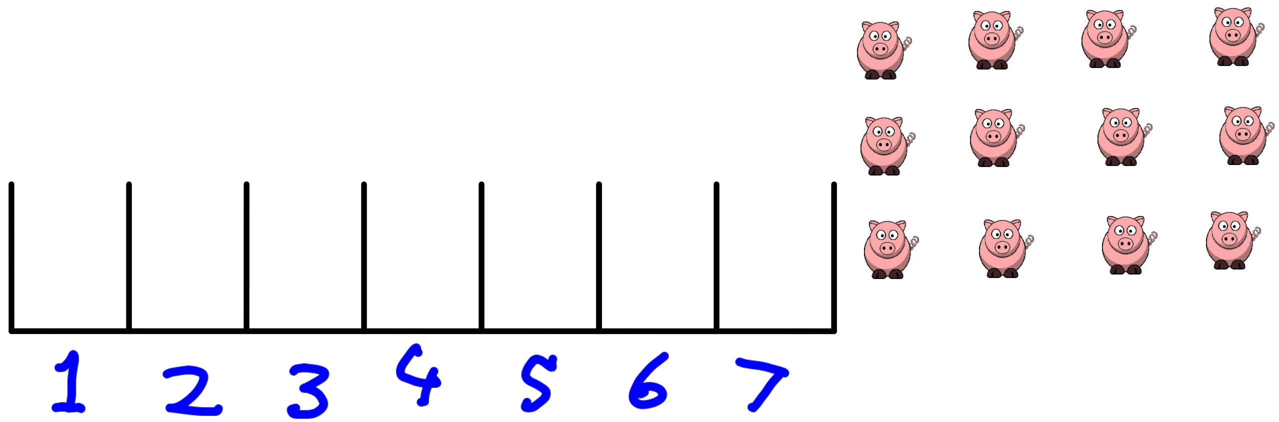
10*26*26*26*26*10*10 = 26^4*10^3

How many different ways to place 12 pigs into 7 pens,
Each bin can hold any number of pigs ?

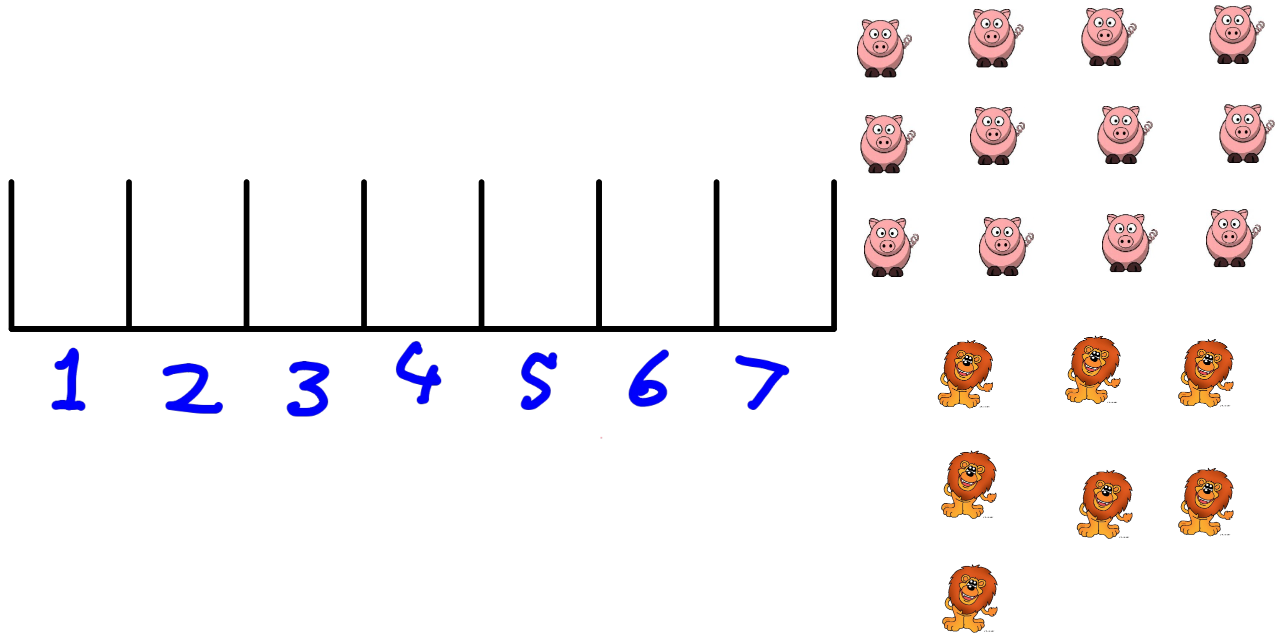


$$\binom{12+(7-1)}{7-1} = \binom{18}{6} = C(18, 6)$$

How many different ways to place 12 pigs into 7 pens,
Each bin can hold any number of pigs ?

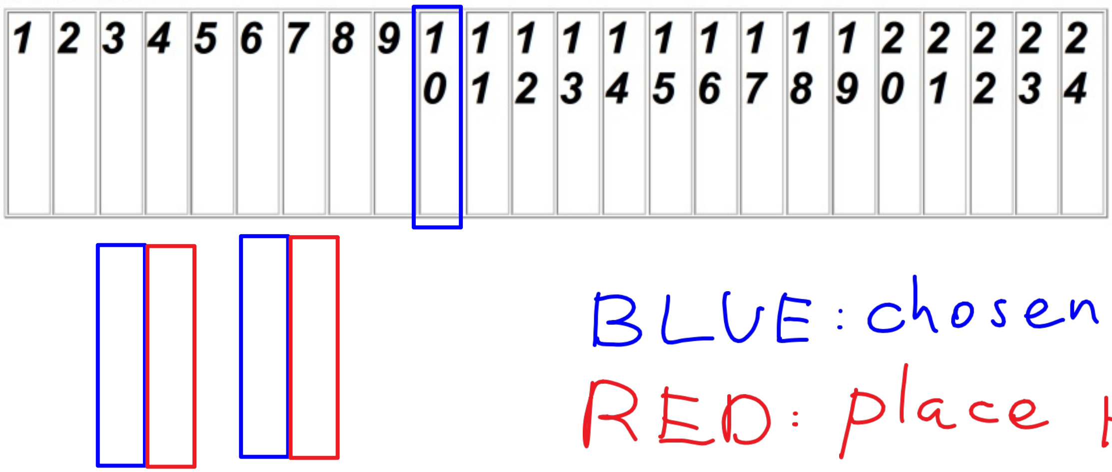


How many different ways to place 12 pigs and 7 lions into into 7 bins, where each bin can contain any number of pigs and lions ?



You have a shelf with 24 books on it. You can pick any 3 books but no two picked books can be right next to each other. How many choices do you have?

24 books:



BLUE: chosen Book

RED: place Holder

If you choose 3 out of 24 books at random, what is the probability that at least 2 would be next to each other?

Size of sample space (Omega):
number of way to choose 3 out of 24 books:
If we care about order of chosen books: P(24,3)
If we don't care about order of chosen books: C(24,3)

$$P(A) = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{P(22,3)}{P(24,3)} = 1 - \frac{C(22,3)}{C(24,3)}$$