Infinite & undefined Expectation

Pareto distributions in Computer Science

Expected value over countably infinite sets

$S =$ a countably infinite subset of R

$S = \{s_1, s_2, \ldots\}$

$X =$ a random variable which gets values in S

$$E(X) \doteq \sum_{i=1}^{\infty} s_i \Pr(X(\omega) = s_i)$$

Recall some facts about series:

1. $$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

2. $$\sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

3. $$\sum_{i=1}^{\infty} \left(\frac{1}{i}\right)^d = \begin{cases} \infty & if\ 0 < d \leq 1 \\ Finite & if\ d > 1 \end{cases}$$

Consider the distribution

distribution is over
the natural numbers = positive integers

$$P(X=i)=\frac{1}{zi^3};\quad z=\sum_{i=1}^{\infty}\frac{1}{i^3}<\infty$$

Distribution is well defined

$$E(X)=\sum_{i=1}^{\infty}\frac{i}{zi^3}=\sum_{i=1}^{\infty}\frac{1}{zi^2}<\infty$$

Expectation is finite

$$E[X^2]=\sum_{i=0}^{\infty}\frac{i^2}{zi^3}=\sum_{i=0}^{\infty}\frac{1}{zi}=\infty$$

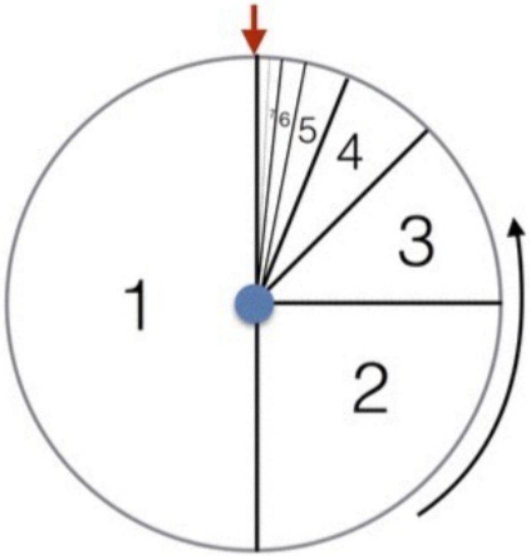
Variance is infinite

Consider next the distribution

$$P(X=i)=\frac{1}{zi^2};\quad z=\sum_{i=1}^{\infty}\frac{1}{i^2}<\infty$$

$$E(X)=\sum_{i=1}^{\infty}\frac{i}{zi^2}=\sum_{i=1}^{\infty}\frac{1}{zi}=\infty$$

Distribution is well defined but
Expectation is infinite



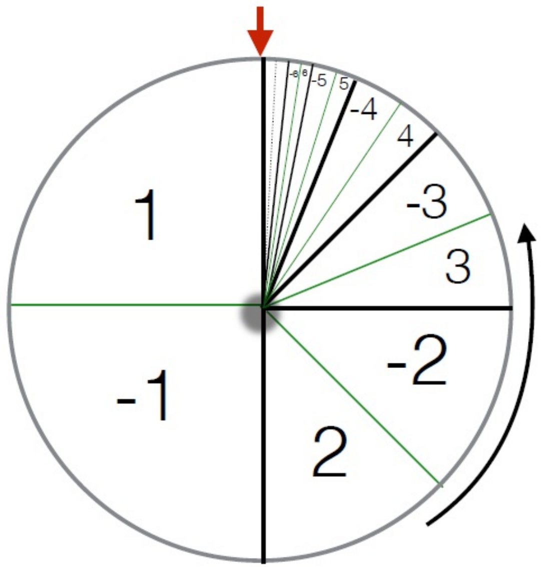
(b) A wheel with
Infinitely many outcomes

$$P(X=i)=\frac{6}{\pi^2 i^2};$$

$$\sum_{i=1}^{\infty} P(X=i)=1$$

$$\sum_{i=1}^{\infty} iP(X=i)=\infty$$

Participation in this game is worth any price
(on the long term)



A wheel with
Infinitely many outcomes
both positive and negative

Consider a game with both wins and losses

$$i \in \{0, -1, +1, -2, +2, \ldots\}$$

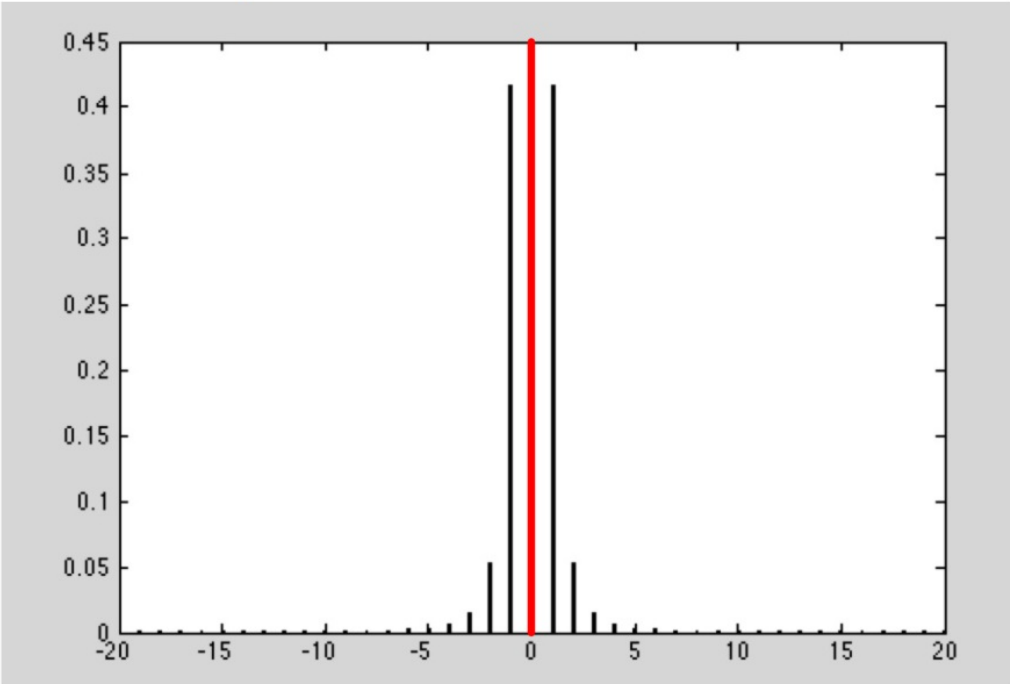
$$P(X=i) = \begin{cases} \dfrac{1}{Z}\dfrac{1}{i^{1.5}} & if \;\; i \neq 0 \\ 0 & if \;\; i = 0 \end{cases} \quad , \quad Z = 2\sum_{i=1}^{\infty}\frac{1}{i^{1.5}}$$

$$\sum_{i=-\infty}^{\infty} P(X=i) = 1$$

$$\sum_{i=\infty}^{\infty} iP(X=i) \;\; is\ undefined$$

Expectation over pos and neg integers: the good case

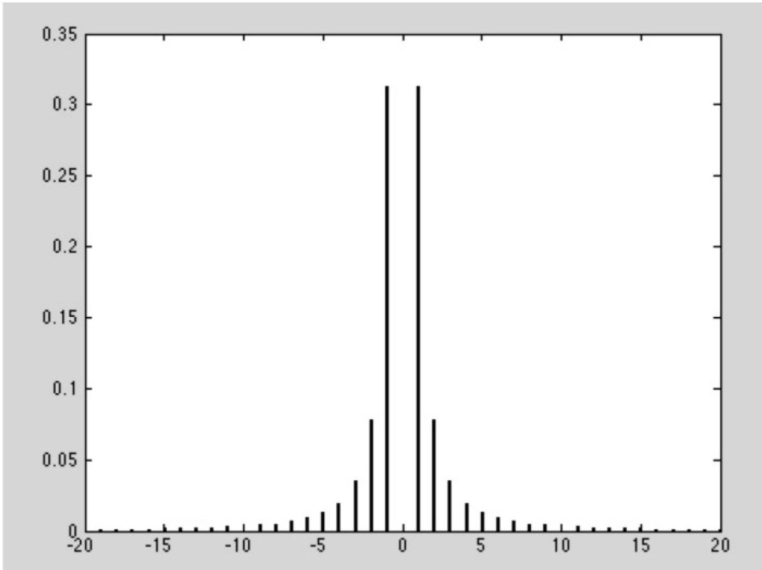
$$P(X=i)=\begin{cases} 0 & \text{if } i=0 \\ \dfrac{1}{Z|i|^3} & \text{if } i\neq 0 \end{cases} \quad ; \quad Z=2\sum_{i=1}^{\infty}\frac{1}{|i|^3}<\infty$$



$$E(X)=\sum_{i=1}^{\infty} iP(X=i)+\sum_{i=-1}^{-\infty} iP(X=i)=\frac{1}{Z}\left(\sum_{i=1}^{\infty}\frac{1}{i^2}-\sum_{i=1}^{\infty}\frac{1}{i^2}\right)=\frac{c-c}{Z}=0$$

A symmetric distribution on pos and neg integers, the bad case

$$P(X=i)=\begin{cases} 0 & \text{if } i=0 \\ \dfrac{1}{Zi^2} & \text{if } i \neq 0 \end{cases} \quad ; \quad Z = 2\sum_{i=1}^{\infty}\frac{1}{i^2} < \infty$$

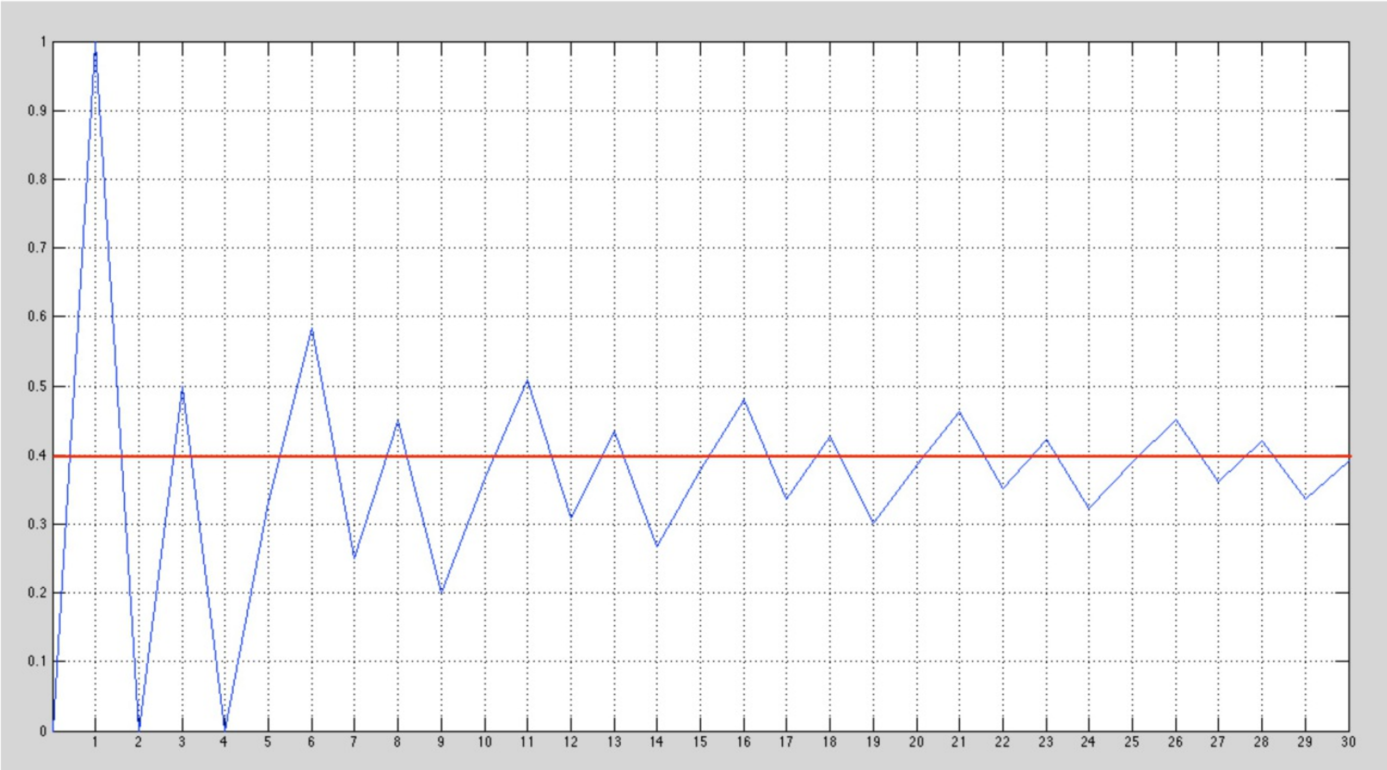


$$E(X) = \sum_{i=1}^{\infty} iP(X=i) + \sum_{i=-1}^{-\infty} iP(X=i) = \frac{1}{Z}\left(\sum_{i=1}^{\infty}\frac{1}{i} - \sum_{i=1}^{\infty}\frac{1}{i}\right) = \frac{\infty - \infty}{Z} = \mathit{undefined}$$

Undefined limit means you can get the limit of your choice by changing the order of summation.

You have at your disposal two infinitely large sums with shrinkingly small pieces:
1/1,1/2,1/3,1/4,....   -1/1,-1/2,-1/3,-1/4,...
Suppose you want the limit to be 0.4, by alternating between positives and negatives you can get arbitrarily close to 0.4 (or to any other number)

1/1-1/1+1/2-1/2+1/3+1/4-1/3+1/5-1/4+1/6+1/7-1/5+1/8-1/6+1/9+1/10-1/7
+1/11-1/8+1/12+1/13-1/9+1/14-1/10+1/15+1/16-1/11+1/17-1/12+1/18 = 0.3919



webwork.cse.ucsd.edu/misc/expected.html

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Let X be a random variable whose probability distribution is defined as:

$$P(X = i) = \begin{cases} \frac{1}{|i|^{\alpha}} & \text{if } i \neq 0 \\ 0 & \text{if } i = 0. \end{cases}$$

$\alpha = 2.5$

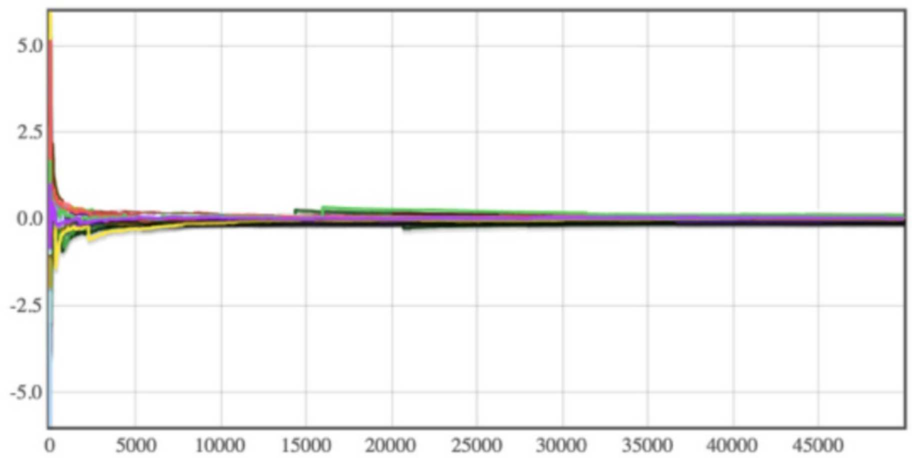
Simulation parameters:

α: 2.5

Number of trajectories: 50

Number of data points: 50000

Run



Finite Expectation

Infinite Variance

Let X be a random variable whose probability distribution is defined as:

$$P(X = i) = \begin{cases} \frac{1}{|i|^{\alpha}} & \text{if } i \neq 0 \\ 0 & \text{if } i = 0. \end{cases}$$

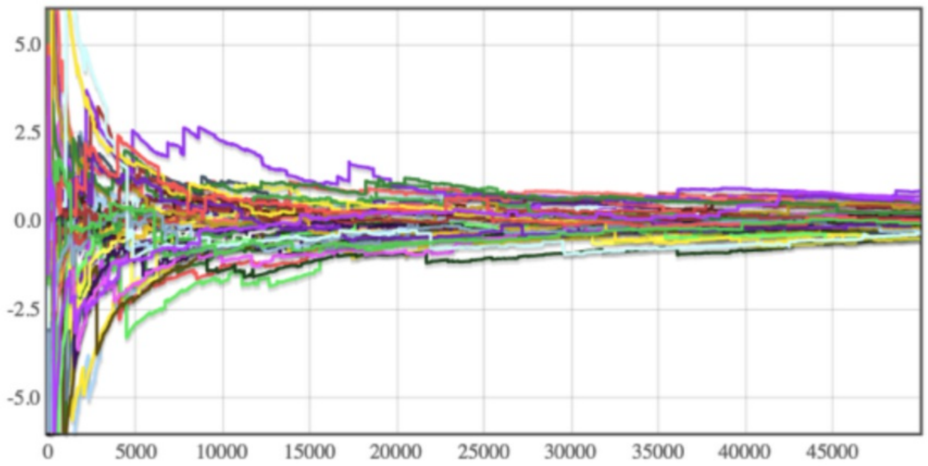
Simulation parameters:

α: 2.0

Number of trajectories: 50

Number of data points: 50000

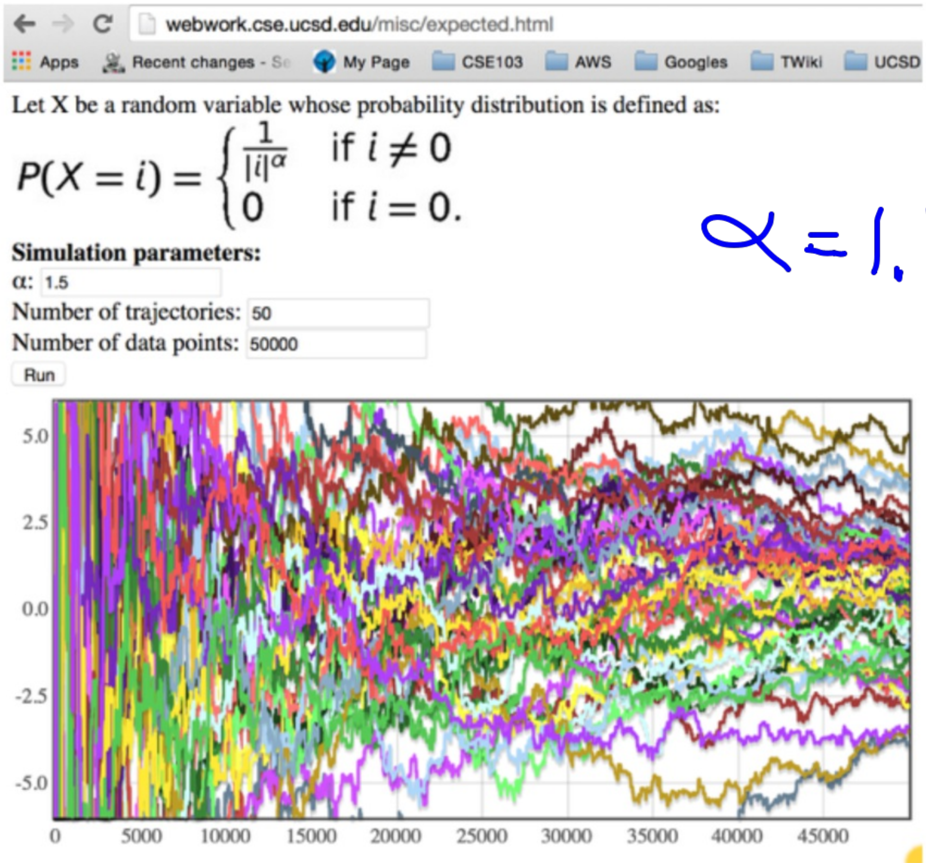
Run



$\alpha = 2$

Undefined expectation

Undefined Variance



$\alpha = 1.5$

Undefined Expectation.

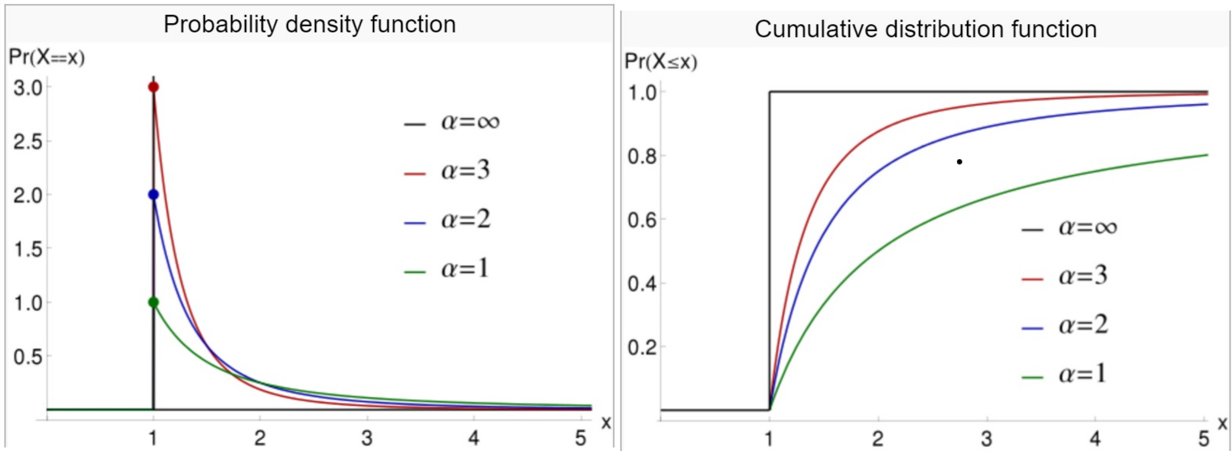
Undefined Variance

Pareto Distribution

Pareto: the contin. version of $\frac{1}{i^k}$

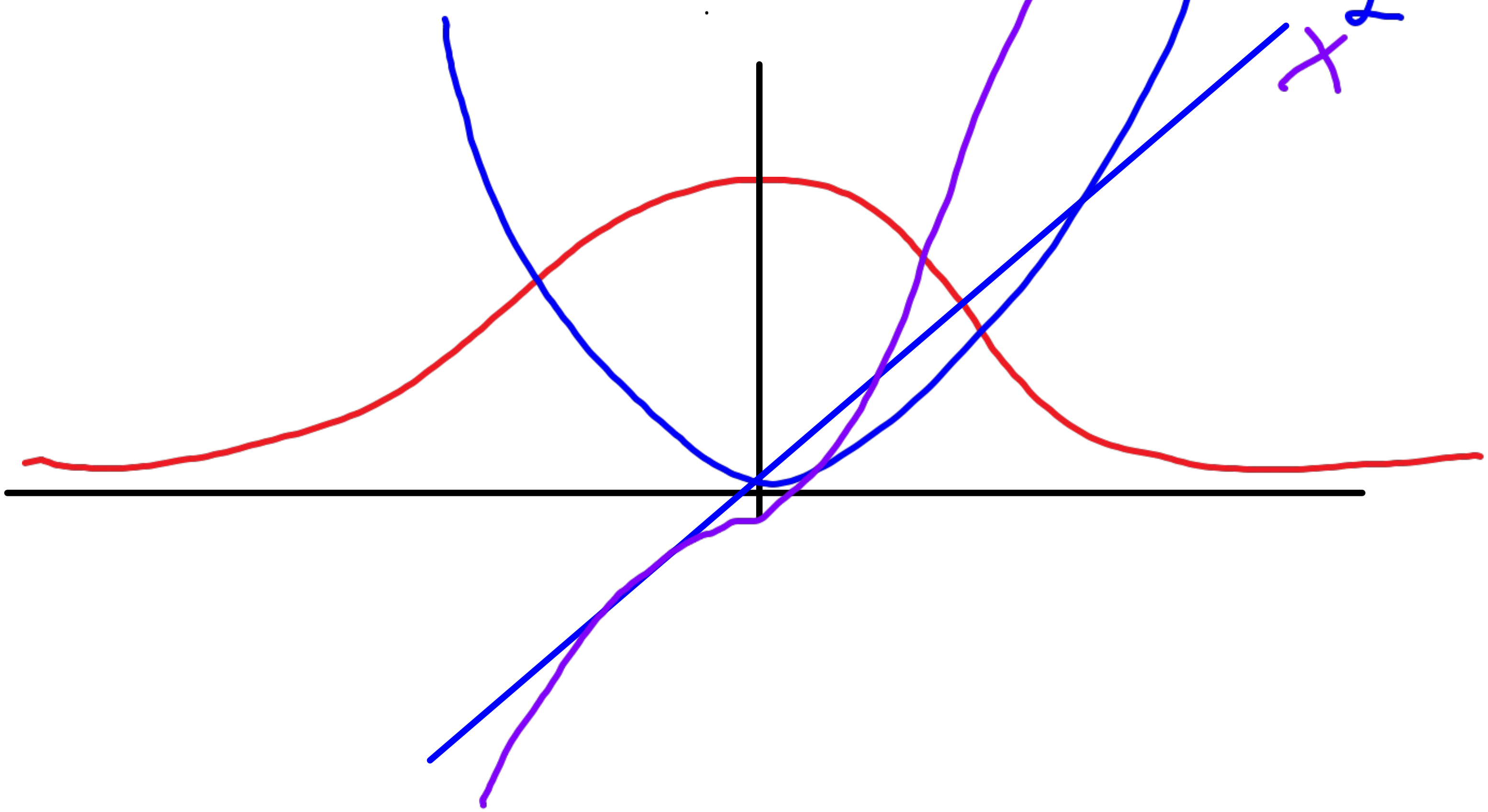
Parameters	$x_m > 0$ scale (real) $\alpha > 0$ shape (real)
Support	$x \in [x_m, +\infty)$
PDF	$\frac{\alpha\, x_m^\alpha}{x^{\alpha+1}}$ for $x \ge x_m$
CDF	$1 - \left(\frac{x_m}{x}\right)^\alpha$ for $x \ge x_m$
Mean	$\begin{cases} \infty & \text{for } \alpha \le 1 \\ \frac{\alpha\, x_m}{\alpha - 1} & \text{for } \alpha > 1 \end{cases}$
Variance	$\begin{cases} \infty & \text{for } \alpha \in (0, 2] \\ \frac{x_m^2 \alpha}{(\alpha-1)^2(\alpha-2)} & \text{for } \alpha > 2 \end{cases}$

Pareto Type I



Moments

Raw	Centered
$E(x)$	$E(x-\mu)=0$
$E(x^2)$	$Var(x)=E((x-\mu)^2)$
$\vdots$	
$E(x^d)$	$E((x-\mu)^d)$



Light and Heavy tail distributions

Light tails

- Exponential, normal
- Exponential tails
- All moments are finite.
- $P(X > 2a | X > a) = P(X > a)$
- Exponential: The time until a program completes does not depend on how long it ran.

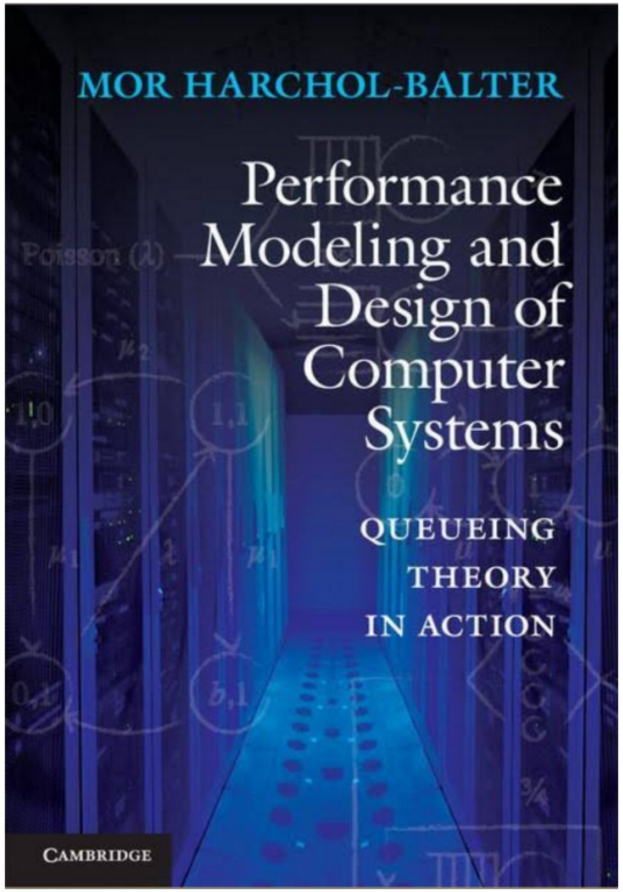
Heavy tails

- Pareto
- Power law.
- Some moments are infinite.
- $P(X > 2a | X > a) = constant$
- Pareto: Decreasing failure rate: the longer a program has run, the longer before it finishes / crashes.
- "Elephant and mice"

Examples of heavy tail distributions

- In the world:
  - Wealth Distribution:
    - Richest 1% of the US population owns 35% of the wealth.
    - Poorest 60% of the US population owns 5% of the wealth.
  - Size of cities
  - Size 0f earthquakes.
  - Frequency of words.
- In computer Science
  - Job run time.
  - Sizes of files in web sites.
  - Internet nodes out-degree
  - Number of packets in an IP flow.

Queuing theory in CS



Queues and light/heavy tails

- Suppose we have K servers.
- If the arrival time of jobs is Poisson and the job size is exponentially distributed (light tailed), then, using a separate queue for each server and random assignment gives good performance.
- If arrival times are Poisson, but distribution of job sizes is heavy tailed, then using a queue for each server and random assignment gives very poor performance (infinite expected waiting time)
  - Why? Small jobs are stuck behind large jobs for a long time, even if another server is free.

Alternatives to FCFS with random assignment

- Round-robin instead of random: a small improvement.
- Join-Shortest-Queue (JSQ): good improvement for light tails job size distribution – not good for heavy tails.
- If job-size known in advance:
  - Size-Interval-Task-Assignment (SITA): small jobs go to server1, larger to server2,… (Express Lane)
  - Least-Work-Left (LWL): Job goes to the server for which the remaining work before job starts to execute is the shortest. (Greedy selfish strategy)

Measuring the weight of the tail

- We consider "bounded Pareto" where there is a maximal job size (S)
- S is a positive random variable
- The ***variance*** of S is: $Var(S) = E(S^2) - E(S)^2$
- We are interested in the relation between the STD and the mean – A unit-less quantity.
  The ***variation coefficient*** of S is:

$$C^2 \doteq \frac{Var(S)}{E(S)^2} = \frac{E(S^2) - E(S^2)}{E(S)^2} = \frac{E(S^2)}{E(S)^2} - 1$$

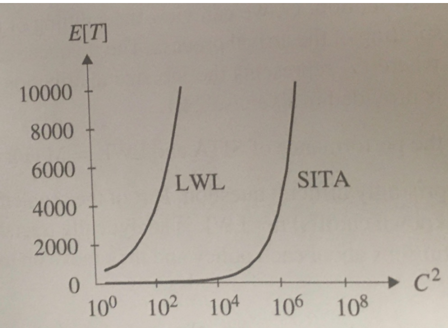


Figure 24.3. Expected response time, $E[T]$, for SITA and LWL versus C^2 in a 2-server system with Bounded Pareto job size distribution with $\alpha = 1.4$ and resource requirement $R = 0.95$.

R = load

α = Pareto parameter.

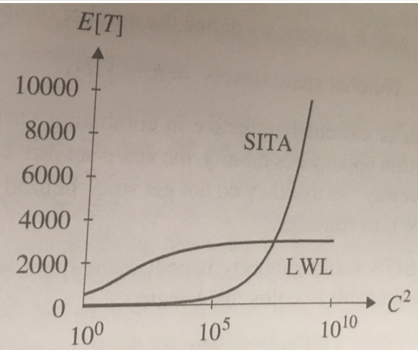


Figure 24.5. Expected response time, $E[T]$, for SITA and LWL versus C^2 in a 2-server system with Bounded Pareto job size distribution with $\alpha = 1.6$ and $R = 0.95$.