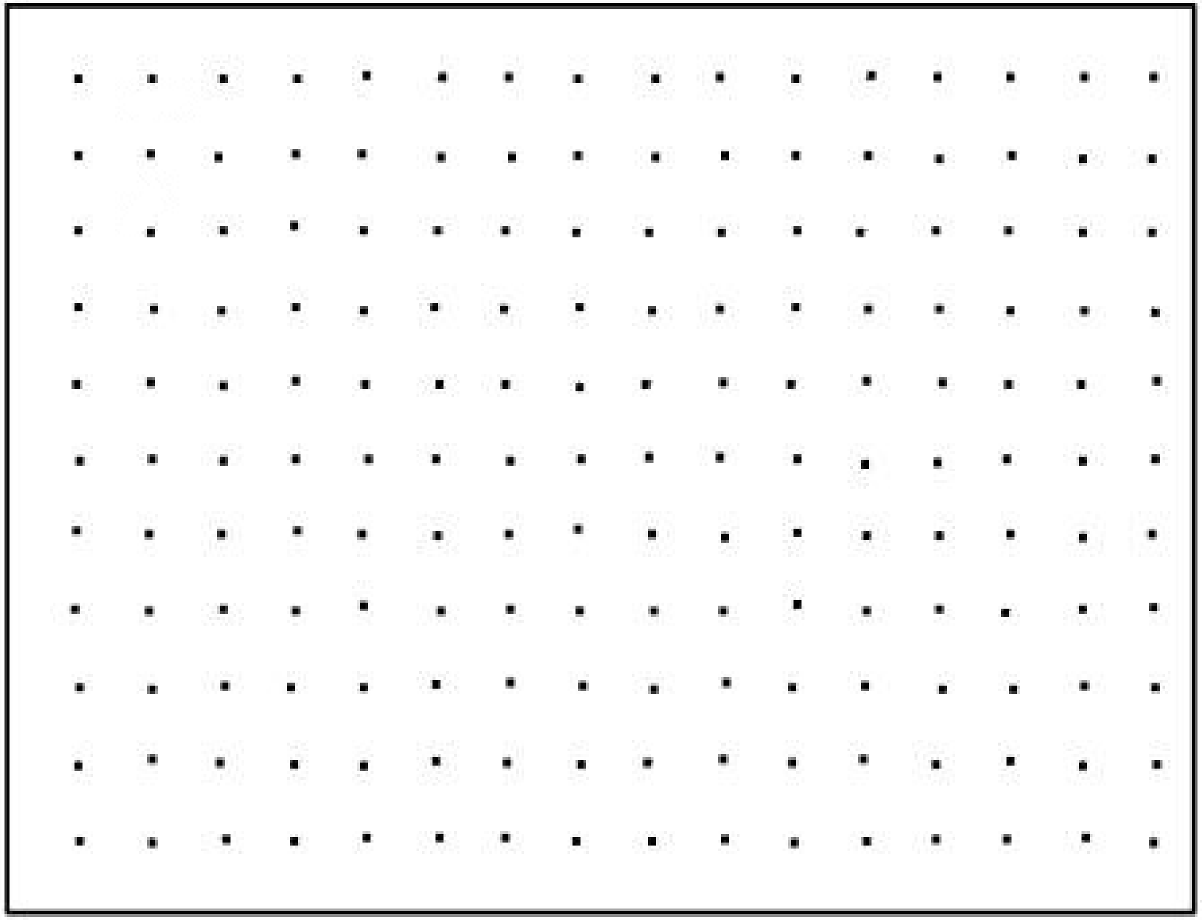
Class 3:
Probability calculations for Uniform distributions over finite spaces.

Ω = outcome space



Motivating Question

* Suppose you flip a fair coin 100 times, with what probability would you get 90 heads and 10 tails?

* The probability of any particular sequence is 1/(2**100)

* To find the answer we need to count the number of sequences of length 100 that contain 90 heads and 10 tails.

* We can write a program to do that. Requires going over

$2^{100} = (2^{10})^{10} \approx (1000)^{10} = 10^{30}$ sequences

* Combinatorics is the science (and art) of counting.

Terminology and notation

1. Outcome: S=HHTHHHT...HH

2. Outcome space: Ω set of all 2^100 possible sequences

$\Omega = \{H, T\}^{100}$

3. Event subset of Ω : the set of sequences with 10 tails

$A = \{ s \in \Omega \mid \#head(s) = 90 \}$

4. Uniform distribution over a finite outcome space =
all sequences of length 100 have the same probability

$P(s) = \frac{1}{2^{100}} = \frac{1}{|\Omega|}$

5. The probability of event A in outcome space Ω

$$P(A) = \frac{|A|}{|\Omega|}$$

Products of sets

$A = \{1,2,3\}$ $B = \{\alpha, \beta\}$

$A \times B = \{1\alpha, 1\beta, 2\alpha, 2\beta, 3\alpha, 3\beta\}$

$|A \times B| = 6 = |A| \times |B|$

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the 2 sides of a coin: $C = \{H, T\}$

the set of 3 coin flips: $\{H,T\} \times \{H,T\} \times \{H,T\} = \{H,T\}^3$

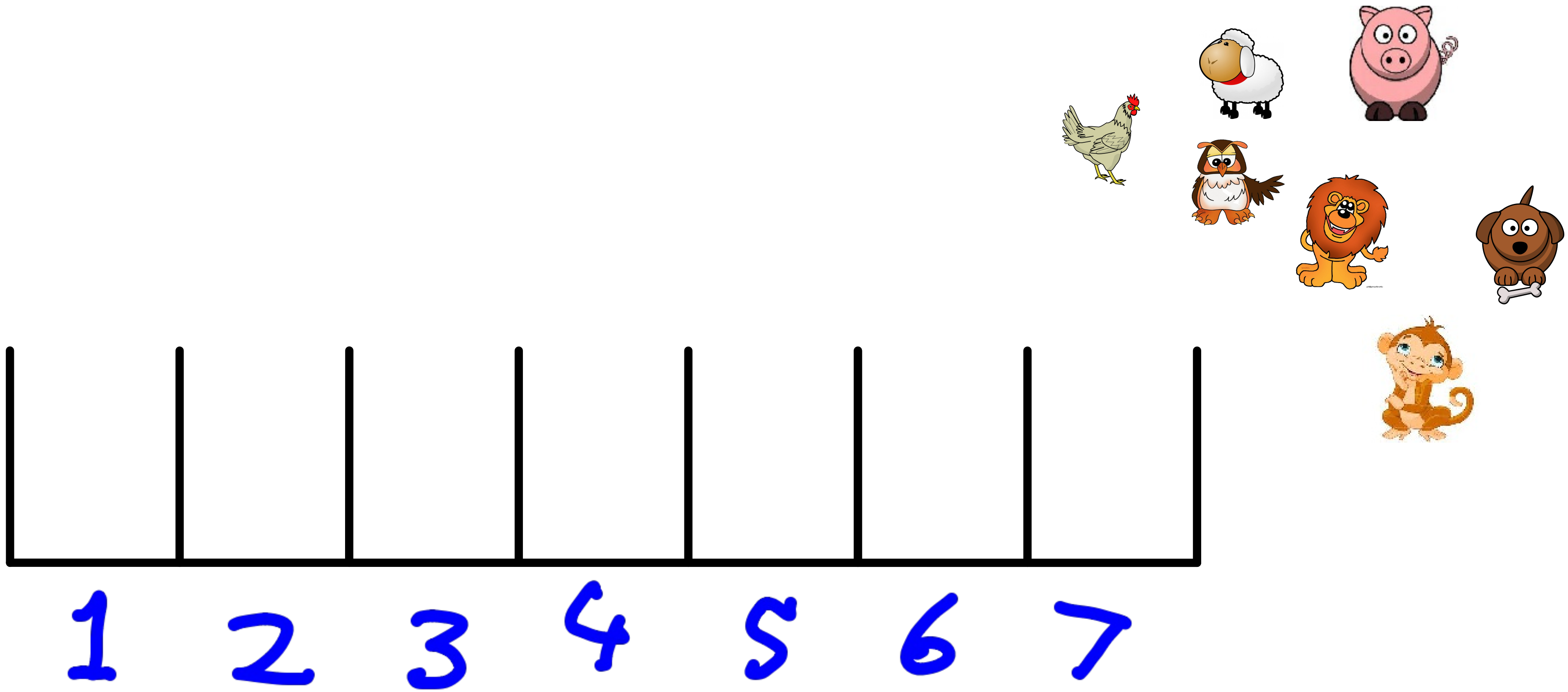
the set of 100 coin flips: $\{H,T\}^{100}$

The size behaves the same way:

$$\left|\{H,T\}^3\right| = 2^3 \qquad \left|\{H,T\}^{100}\right| = 2^{100}$$

The Factorial Function

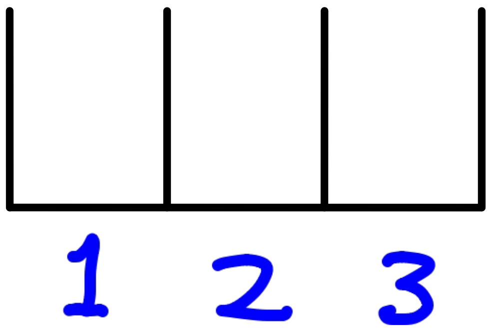
How many different ways are there to place 7 objects in 7 bins?



The Permutation function

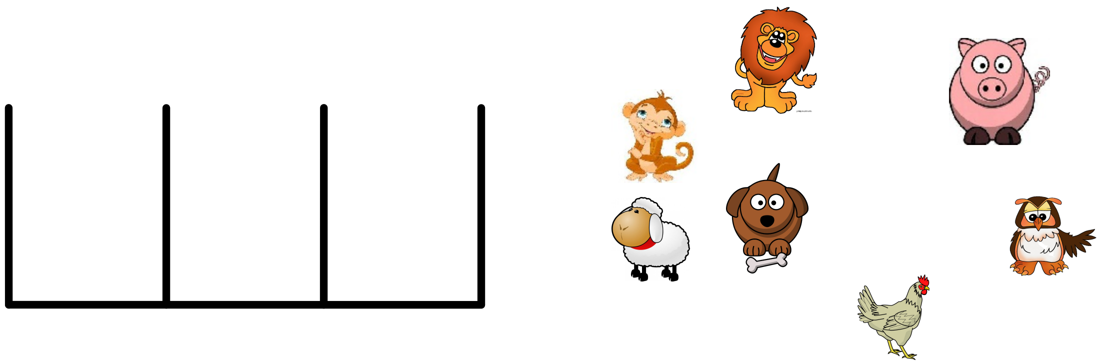
How many different ways are there to place 3 out of 7 objects into 3 bins?

How many different ways are there to select 3 out of 7 objects
when the order matters?



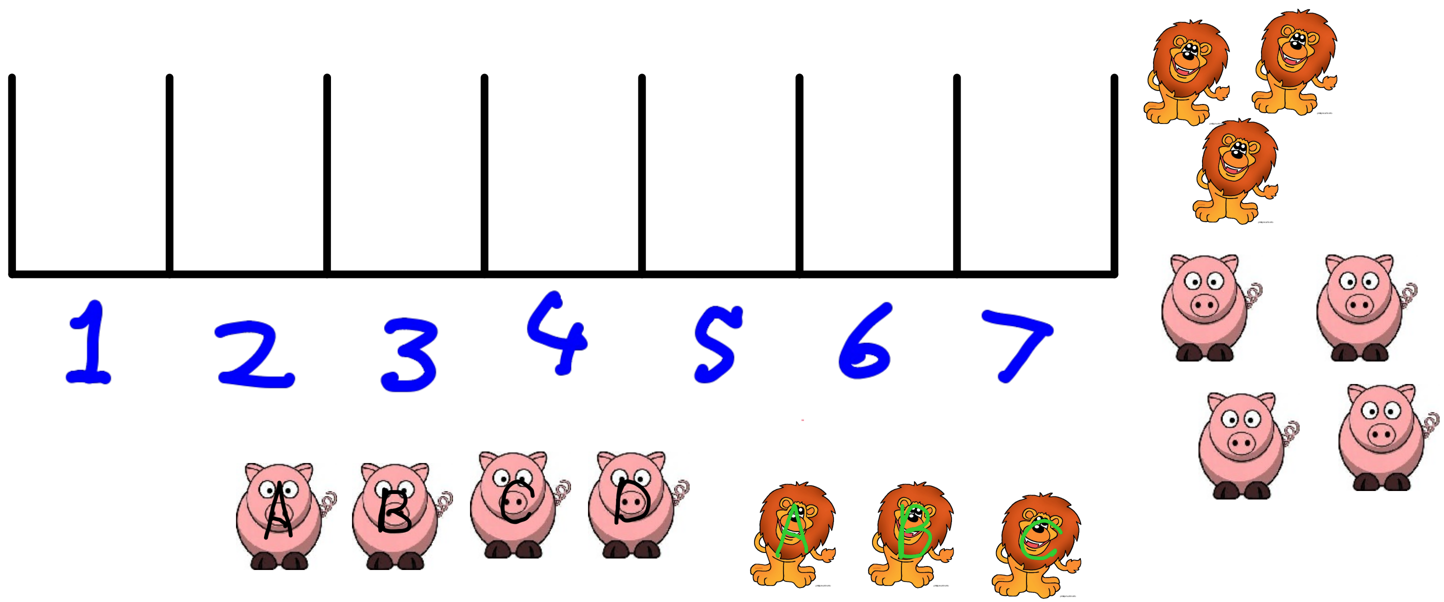
The combination function (selecting a set)

How many different ways are there to select 3 out of 7 objects
when the order does not matter?



The Combination function (counting binary patterns)

How many different ways to place 3 lions and 4 pigs into 7 bins?



* Suppose you flip a fair coin 100 times, with what probability would you get 90 heads and 10 tails?

* The probability of any particular sequence is 1/(2**100)

* To find the answer we need to count the number of sequences of length 100 that contain 90 heads and 10 tails.

100 coin flips

10 tails

90 heads

Total no. of tuples with these properties:

$$\frac{100!}{90!\,10!} = \binom{100}{10} = C(100, 10)$$

$$= \binom{100}{90} = C(100, 90)$$

* Suppose you flip a fair coin 100 times, with what probability would you get 90 heads and 10 tails?

$$\Omega = \{H, T\}^{100} \qquad |\Omega| = 2^{100}$$

$$A = \{s : s \in \Omega \text{ and } s \text{ contain } 10 \text{ tails}\}$$

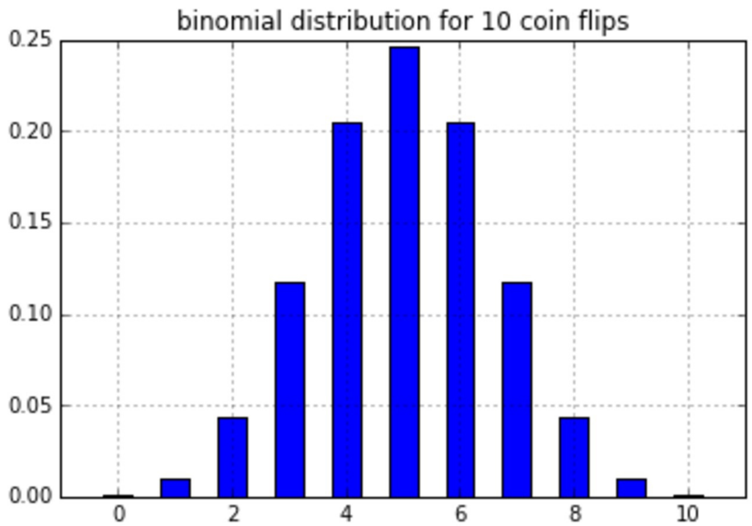
$$|A| = \binom{100}{10}$$

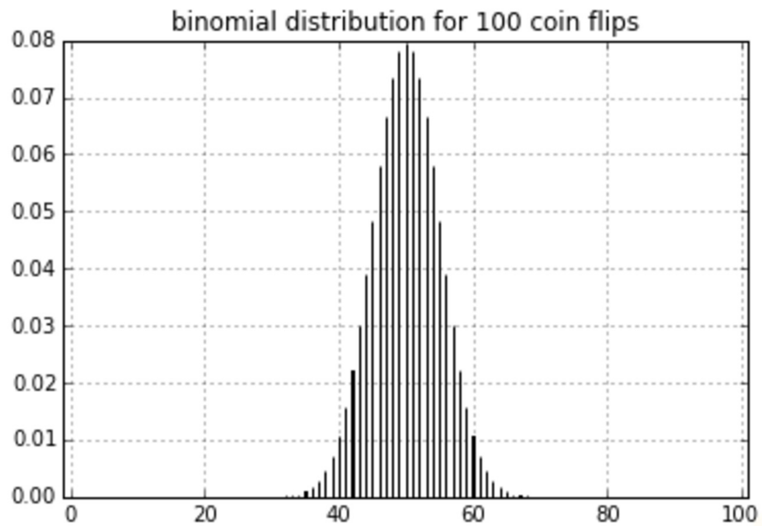
$$P(A) = \frac{|A|}{|\Omega|} = \frac{\binom{100}{10}}{2^{100}} = \frac{100!}{10!\,90!\,2^{100}}$$

$$= \frac{17310309456440}{1267650600228229401496703205376}$$

$$\approx 1.3 \times 10^{-17}$$

binomial distributions





you have cards numbered 1..10,
you pick up card1, then card2, then card3.

What is the probability that the three cards are in increasing order?

Consider the word MISSISSIPPI

Suppose you shuffle the letters and put them in a random order. What is the probability of getting MISSISSIPPI again?
What is the probability of getting IIIIMPPSSSS?

Consider the word BRIGHTLY

Suppose you shuffle the letters and put them in a random order. What is the probability of getting BRIGHTLY again?
What is the probability of getting ?BGHILRTY