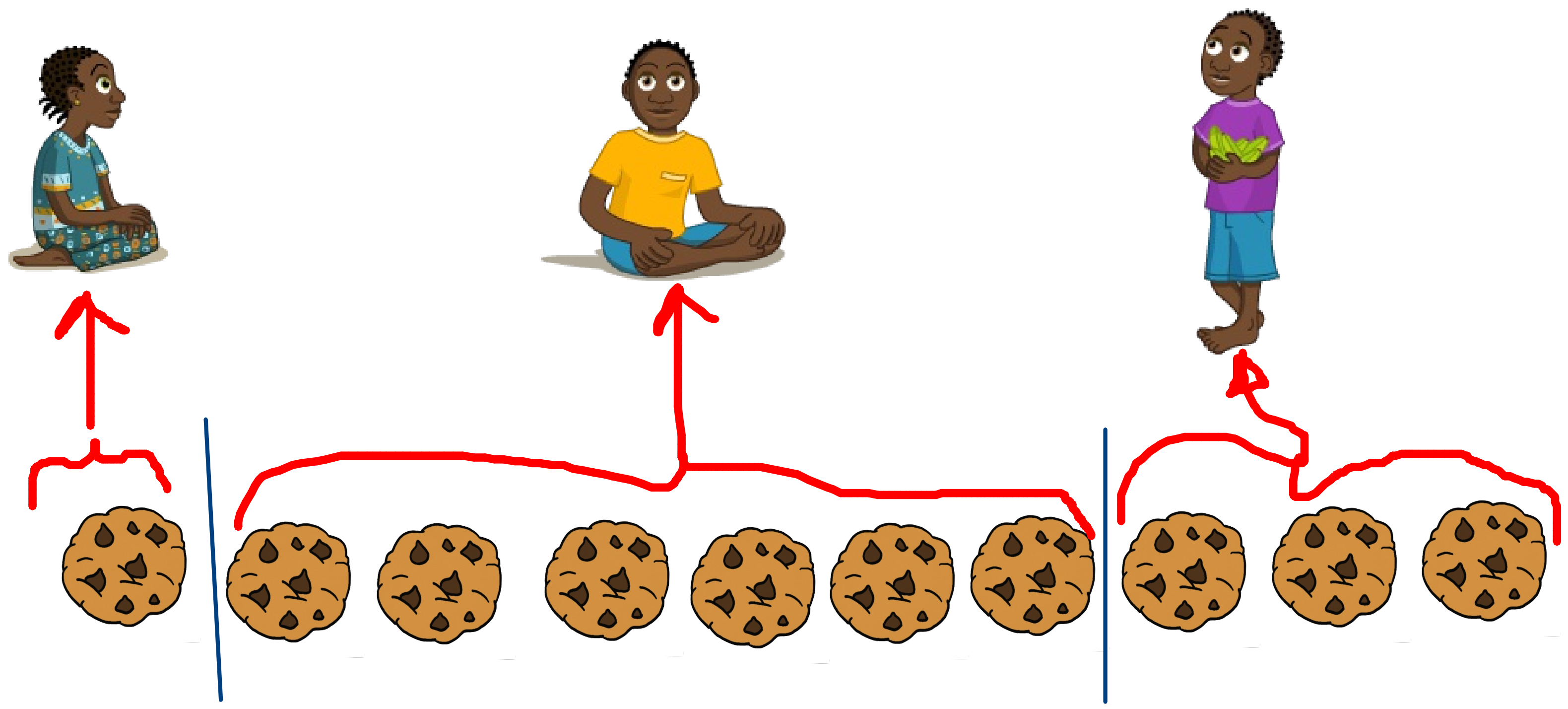
Combinatorics 3
Cookies, Books, Cards
and Some general probability

How many ways to divide 10 cookies among three children? (the cookies are identical and cannot be broken)



We have one fewer vertical lines than children

$C(10+3-1,3-1) = C(10+3-1,10)$

How many ways to split 10 cookies among 3 kids if each kid has to get at least 2 cookies?

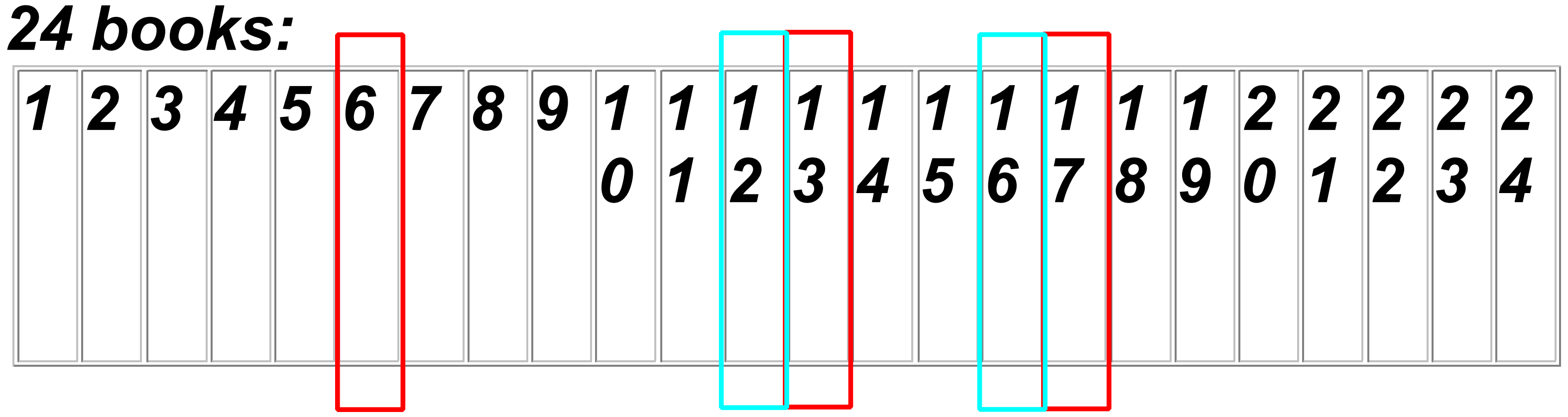
First, give each kid 2 cookies, 4 cookies are left.

Second, divide the remaining cookies among the 3 kids.

$C(4+3-1,3-1) = C(4+3-1,4)$

You have a shelf with 24 books on it. You can pick any 3 books but no two picked books can be right next to each other. How many choices do you have?

24 books:



***Red** rectangle: chosen book*
***Cyan** rectangle: "buffer" book*

Equivalent to choosing 3 out of 24-2=22 books:
If we care about order of chosen books: P(24-2,3)
If we don't care about order of chosen books: C(24-2,3)

If you choose 3 out of 24 books at random, what is the probability that at least 2 would be next to each other?

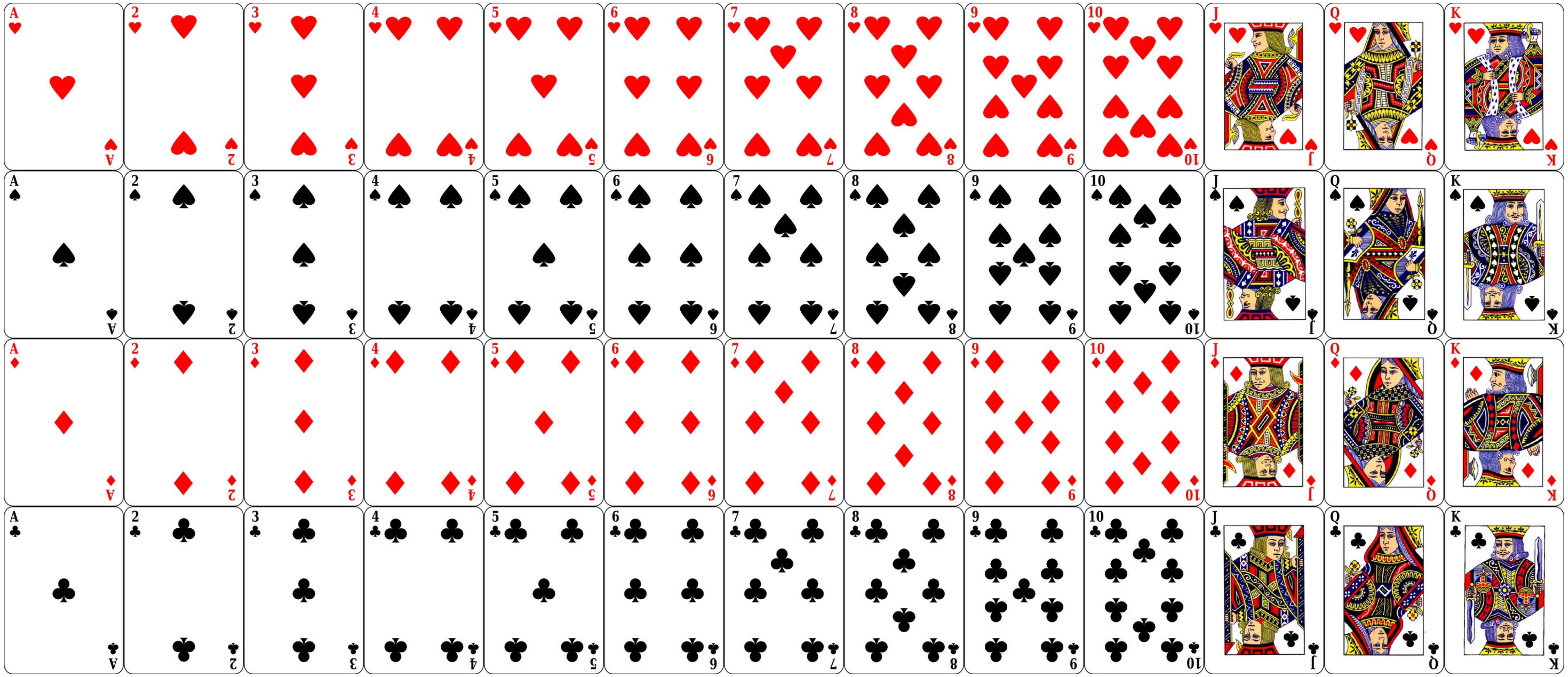
Size of sample space (Omega):
number of way to choose 3 out of 24 books:
If we care about order of chosen books: P(24,3)
If we don't care about order of chosen books: C(24,3)

$$P(A) = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{P(22,3)}{P(24,3)} = 1 - \frac{C(22,3)}{C(24,3)}$$

Play cards

13 ranks

4 Suits



Total: 4X13=52 cards

You pick one card from a shuffled deck.
What is the probability that it is the Ace of Spades?
1/52

You pick one card from a shuffled deck.
What is the probability that it is a spade or a diamond?
2/4 = 1/2

You pick one card from a shuffled deck.
What is the probability that it's rank is higher than 5?

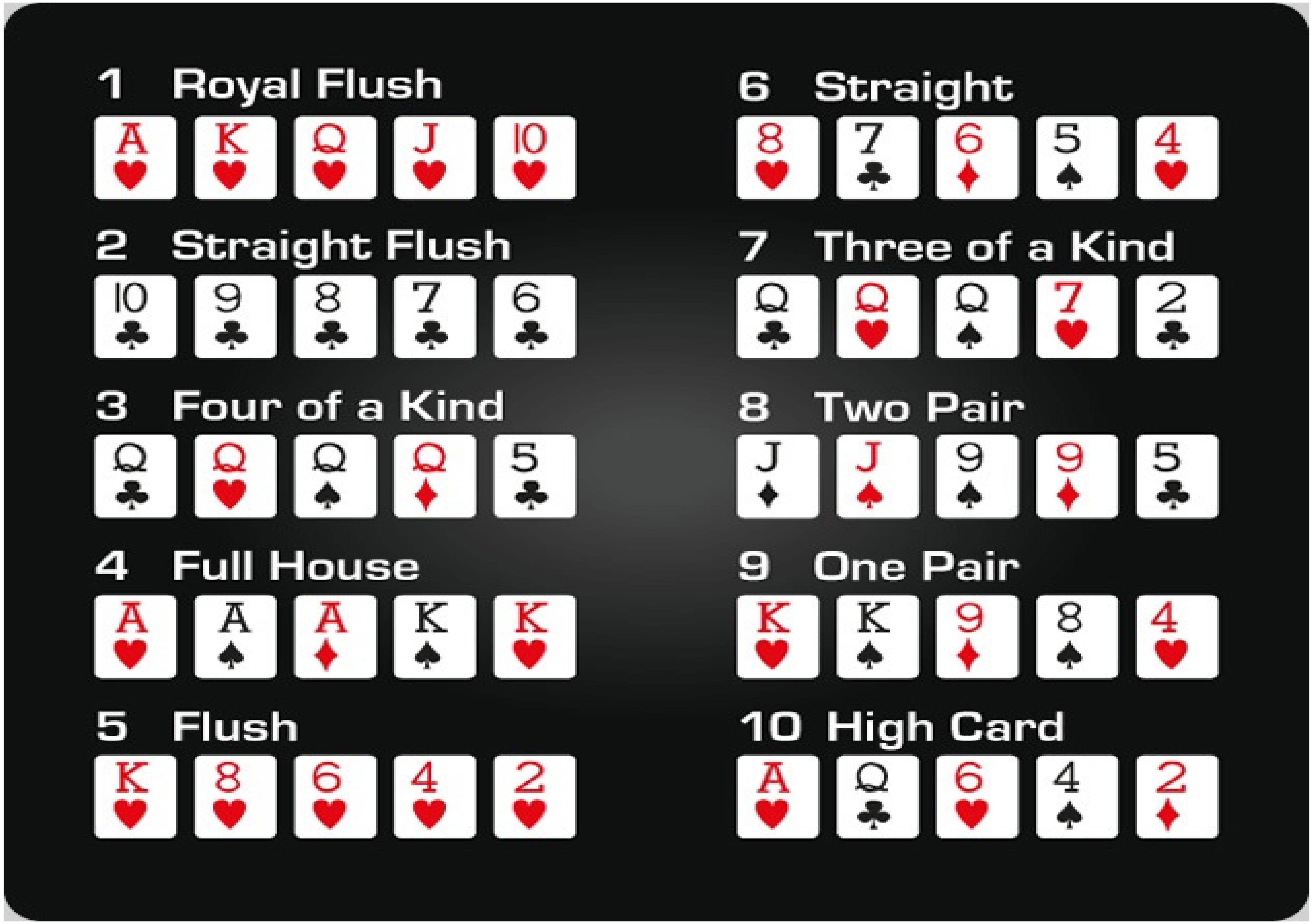
Assuming that Ace is the highest we get 9/13

Basic Poker Rules

1. Each player has two private cards
2. There are 3 shared cards
3. A hand is 2+3=5 cards
4. Hand with highest rank wins

High Rank = Low Probability

The rank of hands in poker

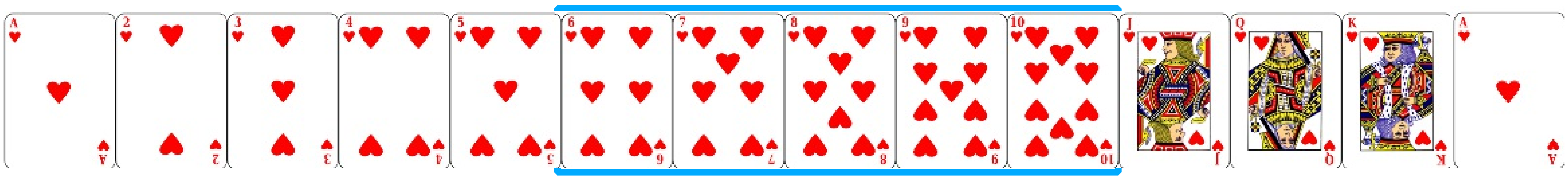


What is the sample space?

The sets of 5 cards out of 52.
Order does not matter
$C(52,5) = 2{,}598{,}960$

1 choice for the card ranks
4 choices for the suit
Prob = 4/C(52,5)

the Ace can be added on either side



9 choices for the card ranks (can't be royal)
4 choices for the suit
Prob = 4*9/C(52,5)=36/C(52,5)

Number of choices for the rank of the 4 cards? 13
choices for the rank of the single? 12
choices for the suit of the single? 4

Prob = (13*12*4)/C(52,5)=624/C(52,5)

Number of choices for the rank of the triple: 13
Number of choices for the rank of the pair: 12
Number of choices for the suits of the triple: $C(4,3)=4$
Number of choices for the suits of the pair : $C(4,2)$

Prob $= (13*12*C(4,3)*C(4,2))/C(52,5) = 3744/C(52,5)$

Number of choices for the ranks of the cards?
(excluding the consecutive ranks from straight flush and royal flush) ***C(13,5)-10***

choices for the suit of the cards? ***4***

Prob = ((C(13,5)-10)*4)/C(52,5)=5148/C(52,5)

How many choices for the card ranks? ***10***

How many choices for the card suits

(cannot be royal flush or straight flush) ? ***4^5-4***

Prob = 10*(4^5-4)/C(52,5)=10,200/C(52,5)

Number of choices for the rank of the triple: ***13***
Number of choices for the suits of the triple: ***C(4,3)=4***
Number of choices for the ranks of the other 2 cards: ***C(12,2)***
Number of choices for the suits of the other 2 cards: ***4*4***

Prob = (13*4*C(12,2)*4*4)/C(52,5) = 54,912/C(52,5)

Unlike full house (2,3) the two pairs are indistinguishable
Number of choices for the ranks of the pairs: C(13,2)
Number of choices for the rank of the single: 11
Number of choices for the suits of the pairs: C(4,2)^2
Number fo hoices for the suit of the single: 4

Prob = (C(13,2)*11*C(4,2)**2*4)/C(52,5) = 123,552/C(52,5)

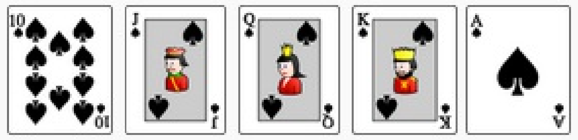
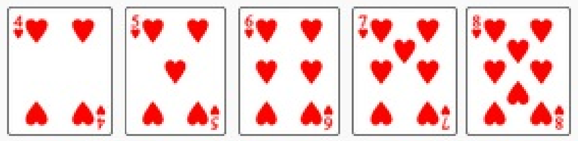
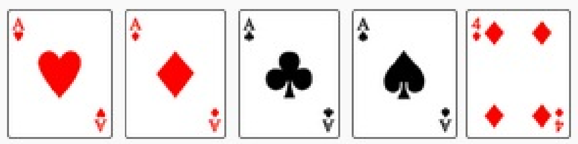
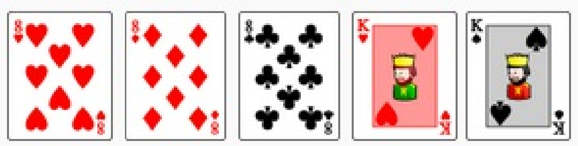
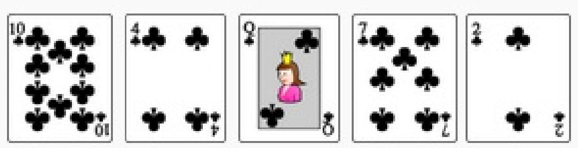
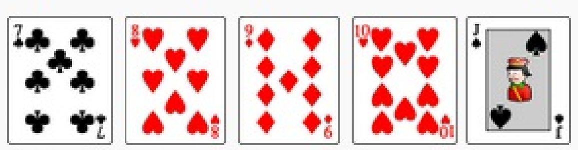
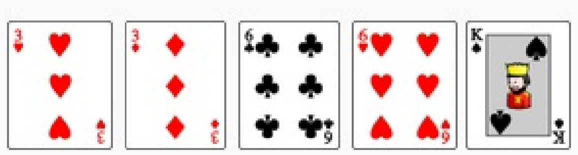
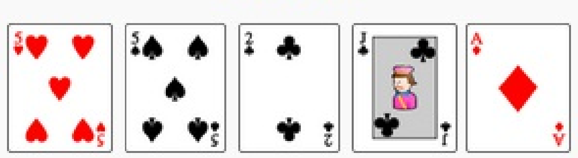
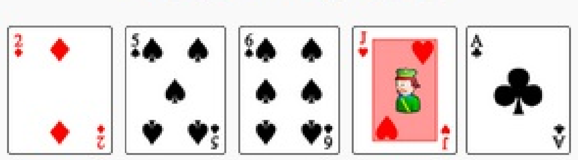
The lowest ranked hand...

*Number of choices for the pair: 13*C(4,2)*

The other cards must not form a pair, else the hand will be two pairs or full house.

Number of possible ranks: C(12,3)
*Number of possible suites: 4**3*

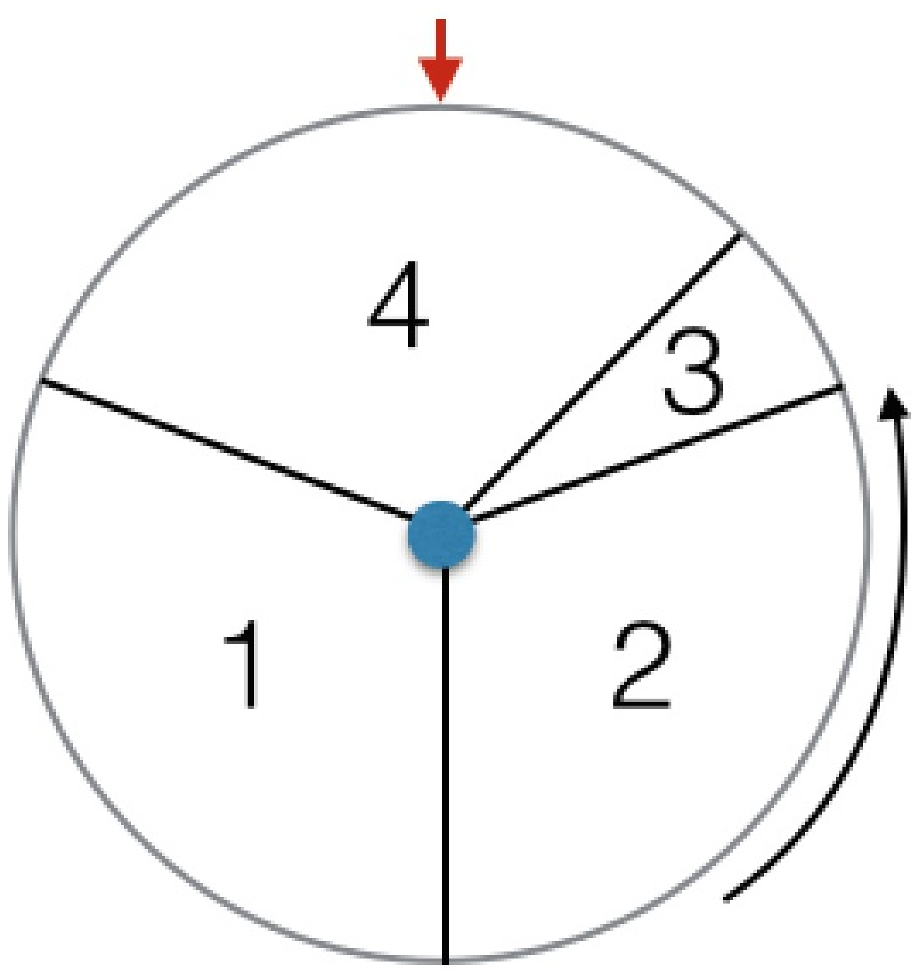
*Prob= (13*C(4,2)*C(12,3)*4**3)/C(52,5) = 1,098,240/C(52,5)*

Hand	Distinct Hands	Frequency	Probability	Cumulative probability	Odds	Mathematical expression of absolute frequency
Royal flush 	1	4	0.000154%	0.000154%	649,739 : 1	$\binom{4}{1}$
Straight flush (excluding royal flush) 	9	36	0.00139%	0.00154%	72,192 : 1	$\binom{10}{1}\binom{4}{1}-\binom{4}{1}$
Four of a kind 	156	624	0.0240%	0.0256%	4,164 : 1	$\binom{13}{1}\binom{12}{1}\binom{4}{1}$
Full house 	156	3,744	0.144%	0.17%	693 : 1	$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$
Flush (excluding royal flush and straight flush) 	1,287	5,148	0.198%	0.367%	508 : 1	$\binom{13}{5}\binom{4}{1}-\binom{10}{1}\binom{4}{1}$
Straight (excluding royal flush and straight flush) 	10	10,200	0.392%	0.76%	254 : 1	$\binom{10}{1}\binom{4}{1}^5-\binom{10}{1}\binom{4}{1}$
Three of a kind 	858	54,912	2.11%	2.87%	46.3 : 1	$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2$
Two pair 	858	123,552	4.75%	7.62%	20.0 : 1	$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$
One pair 	2,860	1,098,240	42.3%	49.9%	1.36 : 1	$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3$
No pair / High card 	1,277	1,302,540	50.1%	100%	0.995 : 1	$\left[\binom{13}{5}-10\right]\left[\binom{4}{1}^5-4\right]$
Total	**7,462**	**2,598,960**	**100%**	**---**	**1 : 1**	$\binom{52}{5}$

General Probabilities

Until now, we considered sample spaces that have a finite number of elements and that the elements have the same probability.

We now consider finite spaces where different outcomes have different probabilities.



Pr(1)=0.30
Pr(2)=0.30
Pr(3)=0.093211...
Pr(4)=0.306789...

Pr(1)+Pr(2)+Pr(3)+Pr(4)=1

Properties of general probability distributions

$$\forall \; A \subseteq \Omega, \; 0 \le P(A) \le 1 \quad P(\Omega) = 1$$

$$\forall \; A, B \subseteq \Omega,$$

$$\forall \; A, B \subseteq \Omega, \; A \cap B = \varnothing$$ ***Disjoint sets***

$$\Rightarrow P(A \cup B) = P(A) + P(B)$$

Implies that: $P(A^c) = 1 - P(A)$

$$A \cup A^c = \Omega, P(\Omega) = 1, A \cap A^c = \varnothing$$

$$\Rightarrow P(A) + P(A^c) = 1$$

The total probability equation

A few simple questions:

If $P(A)=\frac{1}{2},\ P(B)=\frac{2}{3}$, What can be said about $P(A\cap B)$?

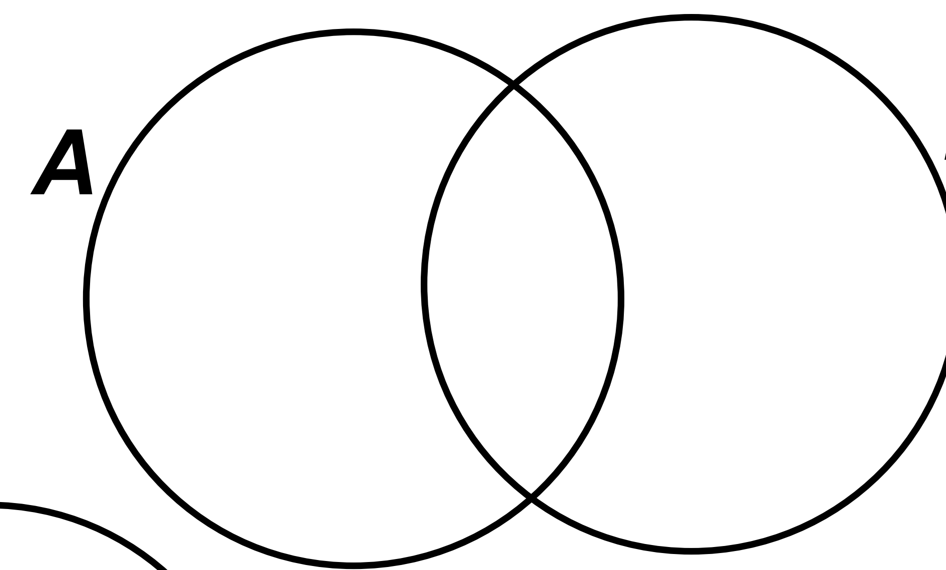
Ω

If $P(A)=\frac{1}{2},\ P(B)=\frac{1}{3},\ P(A\cap B)=\frac{1}{6}$, What is $P(A\cup B)=$?

Ω

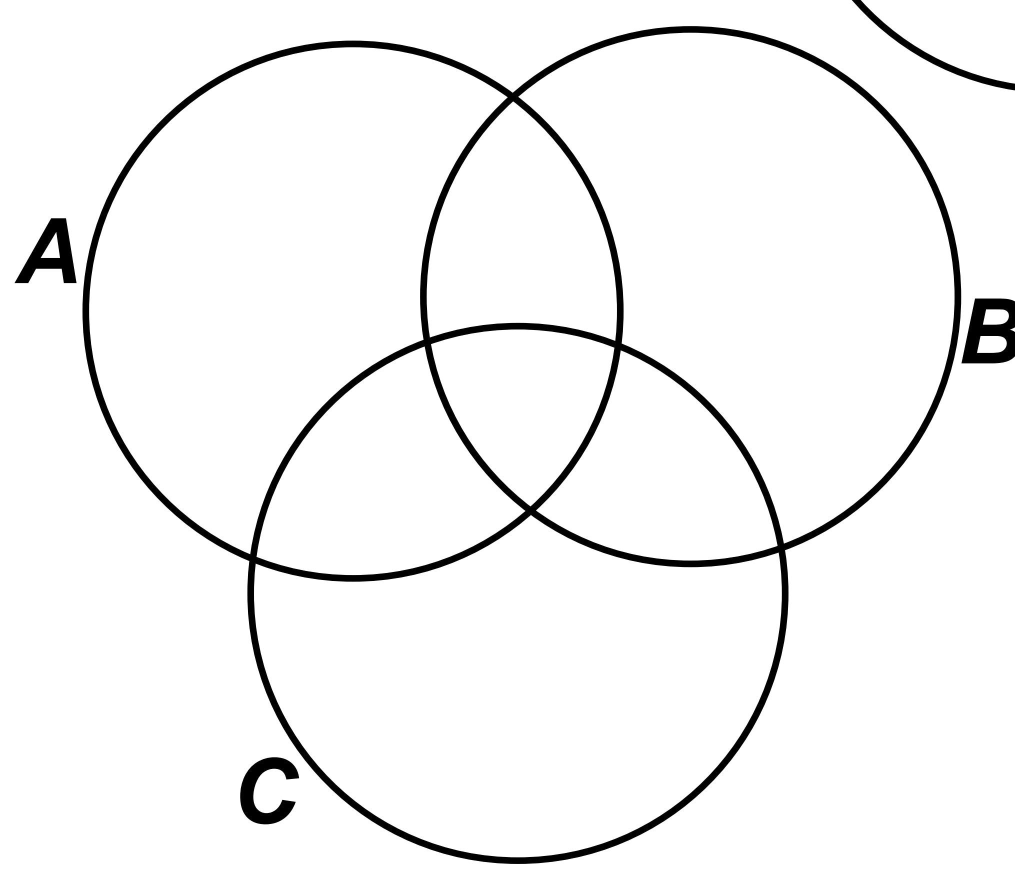
General Formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



How about:

$$P(A \cup B \cup C) = ?$$



For Wed.

1. Finish Week2 homework.
2. Read class notes:
- *Section 4.5 (Poker)*
- *Chapter 5*

The total probability equation for (countably) infinite sets

$$A_1, A_2, A_3, \ldots \subseteq \Omega$$

$$\forall i \neq j,\ A_i \cap A_j = \emptyset$$

$$A_1 \cup A_2 \cup A_3 \cup \cdots \doteq \bigcup_{i=1}^{\infty} A_i = \Omega$$

$$\text{Then } \sum_{i=1}^{\infty} P(A_i) = 1$$

Consider the natural numbers: 1,2,3,...
Is it possible to define a uniform distribution over them?

1st possibility: 0=P(1)=P(2)=... $P(\Omega) =$

2nd possibility: 0<P(1)=P(2)=.... $P(\Omega) =$