- Convergence to the mean Using Variance.

- Covariance & Correlation.

Rules for expected value:

1. If a, b are constants and X is a random variable then

$$E(aX + b) = aE(X) + b$$

2. If X, Y are random variables (dependent or independent)

$$E(X + Y) = E(X) + E(Y)$$

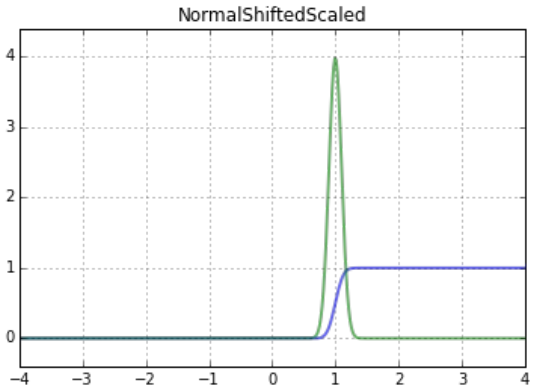
Rules for Variance

1. If a, b are constants and X is a random variable then

$$Var(aX + b) = a^2 Var(X)$$

2. If X, Y are Independent Random Variables, then

$$Var(X + Y) = Var(X) + Var(Y)$$



Shifted and Scaled Normal $N(\mu, \sigma)$
Shift: $\mu = 1$ *scale:* $\sigma = 0.1$

PDF:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

CDF:

$$F(x) = \int_{-\infty}^{x} f(s)ds = 1 - Q\left(\frac{x-\mu}{\sigma}\right)$$

If the RV X is distributed according to $N(\mu, \sigma)$

Then:

$E(X) = \mu$

$Var(X) = \sigma^2$

Mean ≠ Average

Mean =

- $E(X)$ is a property of the distribution, it is not a random variable.
- The average is a random variable:
  - $Average(x_1, x_2, \ldots, x_n) \doteq \frac{1}{n}\sum_{i=1}^{n} x_i$
- When n is large, the average tends to be close to the mean.

The average also called the empirical mean

X_i indicates whethere there was a click-through on the ith presentation of the Ad.

The X_i are Independent and Identically Distributed Binary Random Variables (IID Binary RVs)

$X_i = 1$ indicates there was a click through on the ith presentation of the Ad. Otherwise $X_i = 0$

$\Pr[X_i = 1] = p, \quad \Pr[X_i = 0] = 1 - p, \qquad 0 \le p \le 1, \quad p$ is the click through rate

$E[X_i] = 1 \times p + 0 \times (1 - p) = p \qquad p$ is not a random variable

The average is defined to be $S_n \doteq \frac{1}{n} \sum_{i=1}^{n} X_i \qquad S_n$ is a random variable

From linearity of expectation we know that

$$E[S_n] = E\left[\frac{1}{n} \sum_{i=1}^{n} X_i\right] = \frac{1}{n} \sum_{i=1}^{n} E[X_i] = \frac{1}{n} \sum_{i=1}^{n} p = p$$

Next, we consider $Var(S_n)$

The Binomial Distribution

Exact calculation

Suppose $X_1, X_2, \ldots, X_n$ are

independent identically distributed (IID) random variables

$\Pr[X_i = 1] = p, \quad \Pr[X_i = 0] = 1 - p, \qquad 0 \le p \le 1$

We define the average to be another random variable

$S_n \doteq \frac{1}{n}\sum_{i=1}^{n} X_i$ What is $\Pr\left(S_n = \frac{m}{n}\right), \quad 0 \le m \le n?$

$S_n = \frac{m}{n}$ if and only if for m of the X_i, $X_i = 1$, for $n - m$ of the X_i, $X_i = 0$

The probability of each such sequence is: $p^m (1-p)^{n-m}$

The number of such sequences is: $\binom{n}{m}$

$\Pr\left(S_n = \frac{m}{n}\right) = \binom{n}{m} p^m (1-p)^{n-m}$ The Binomial distribution.

Alternative derivation for the Binomial distribution

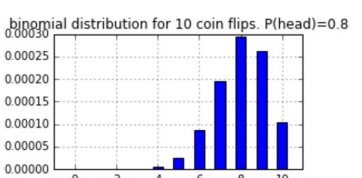
Recall:

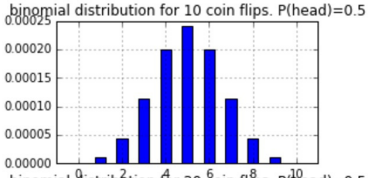
$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

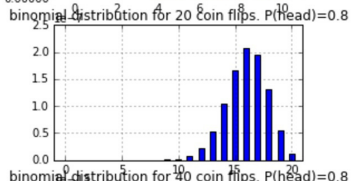
Setting: $a = p, \quad b = (1-p)$

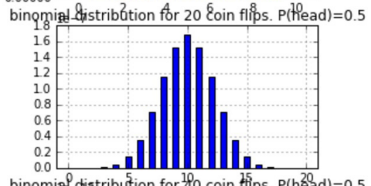
Gives:

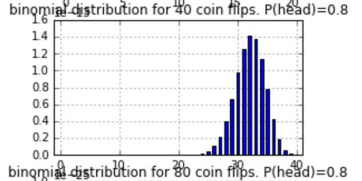
$$1 = \left(p + (1-p)\right)^n = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

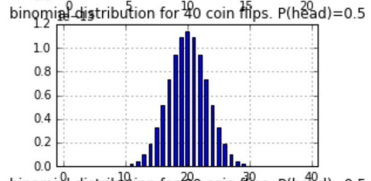

binomial distribution for 10 coin flips. P(head)=0.8
1e−5

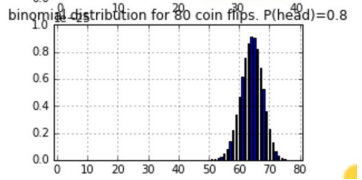

binomial distribution for 10 coin flips. P(head)=0.5

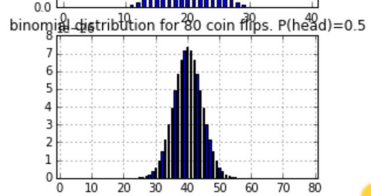

binomial distribution for 20 coin flips. P(head)=0.8
1e−4


binomial distribution for 20 coin flips. P(head)=0.5
1e−5


binomial distribution for 40 coin flips. P(head)=0.8
1e−11


binomial distribution for 40 coin flips. P(head)=0.5
1e−11


binomial distribution for 80 coin flips. P(head)=0.8
1e−23


binomial distribution for 80 coin flips. P(head)=0.5
1e−23

The variance of the average

$S_n \doteq \frac{1}{n}\sum_{i=1}^{n} X_i$ is the number of click-through's in presentations $1,\ldots,n$

$$Var[X_i] = p\times(1-p)^2 + (1-p)\times(0-p)^2 =$$
$$= p\times(1-p)\times(1-p) + p\times p\times(1-p) =$$
$$= p\times(1-p)\times(1-p+p) = p(1-p)$$

As X_i are IID: $Var\left[S_n\right] = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} Var\left[X_i\right] = \frac{1}{n^2} np(1-p) = \frac{p(1-p)}{n}$

$$\sigma\left(S_n\right) = \sqrt{\frac{p(1-p)}{n}}$$

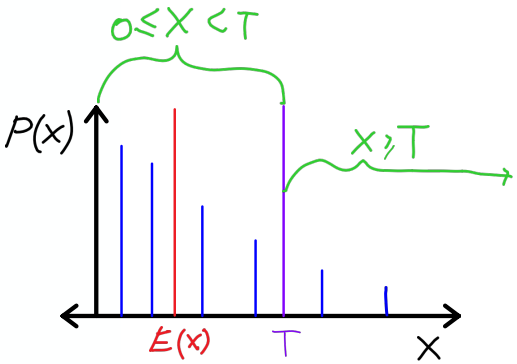
Recall that $\sigma\left(S_n\right)$ is proportional to the width of the distribution of S_n

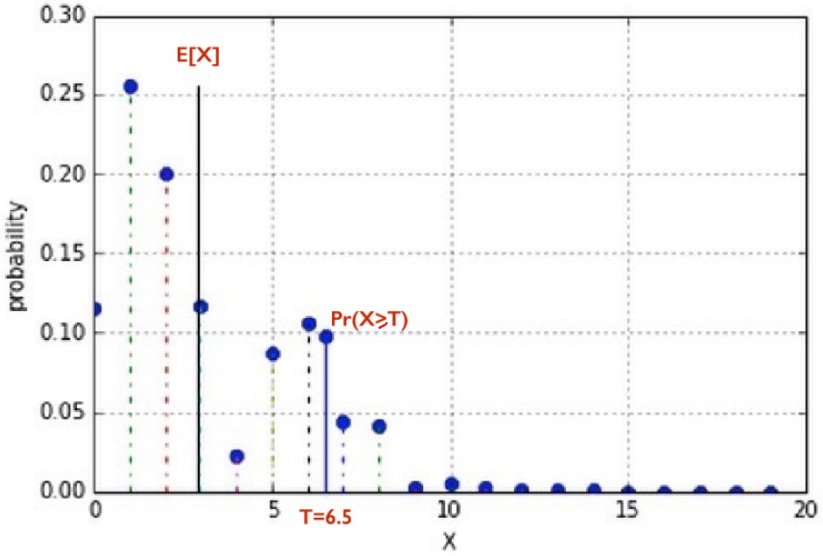
Using the variance to bound the distance from the mean

- Intuition: if the width of the distribution is small then then the probability that the RV is close to the mean is high.
- Proof: to formalize this intuition and make it quantitative we need Chebyshev's bound.
- In order to prove Chebyshev's bound we need Markov bound.
- We'll start with Markov bound.

Markov Bound

- Suppose the RV X is distributed over the **non-negative** integers 0,…,20
- Suppose we know the mean E[X]. Can we bound the probability that X≥T ?

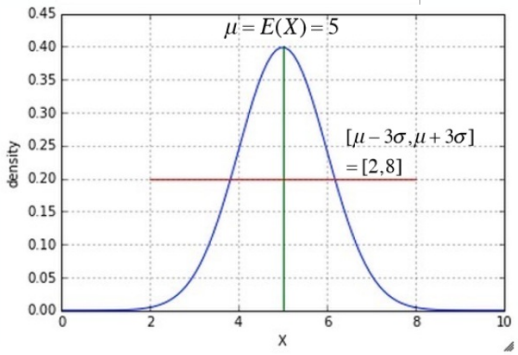


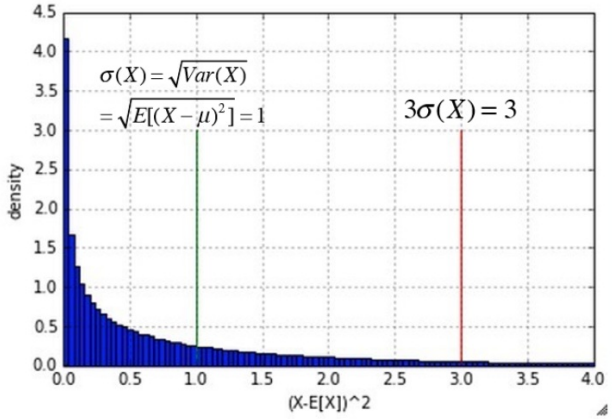


$$E[X] \geq 0 \times \Pr(X < T) + T \times \Pr(X \geq T)$$

$$\Pr(X \geq T) \leq \frac{E(X)}{T}$$

Chebyshev's bound





$$\Pr\left((X-\mu)^2 \geq \lambda^2\right) \leq \frac{E\left[(X-\mu)^2\right]}{\lambda^2} = \frac{Var(X)}{\lambda^2}$$

Plugging in $\lambda = k\sigma(X)$

$$\Pr\left[|X-\mu| \geq k\sigma(X)\right] \leq \frac{\sigma(X)^2}{k^2\sigma(X)^2} = \frac{1}{k^2}$$

In the example shown

$$\mu = E(X) = 5$$

$$\sigma = \sqrt{Var(X)} = 1$$

We choose $k = 3$ to get that

$$\Pr\left(|X-5| \geq 3\right) \leq \frac{1}{k^2} = \frac{1}{9}$$

Applying Chebyshev's bound

$$\Pr\left[\left|X-\mu\right| \geq k\sigma(X)\right] \leq \frac{\sigma(X)^2}{k^2\sigma(X)^2} = \frac{1}{k^2}$$

A few slides ago, we found that

$$\mu(S_n) = p; \quad \sigma(S_n) = \sqrt{\frac{p(1-p)}{n}}$$

$$\Pr\left[\left|S_n - p\right| \geq k\sqrt{\frac{p(1-p)}{n}}\right] \leq \frac{1}{k^2}$$

fixing k and letting n increase we see that doubling n decreases the distance from the mean by $\sqrt{2}$

The Goal of Statistics

True (or underlying) Distribution
Vs
Empirical Distrubution

We have a biased Coin $P(\text{Head}) = \frac{2}{3} = 0.666\ldots$
We flip the Coin 100 times and get $\frac{60 \text{ heads}}{100 \text{ flips}}$
The empirical distribution is $\hat{P}(\text{Heal}) = 0.6$
The emporical distribution is a Random Variable
The true distribution is NOT a Random Variable

Using the empirical distribution We can define:

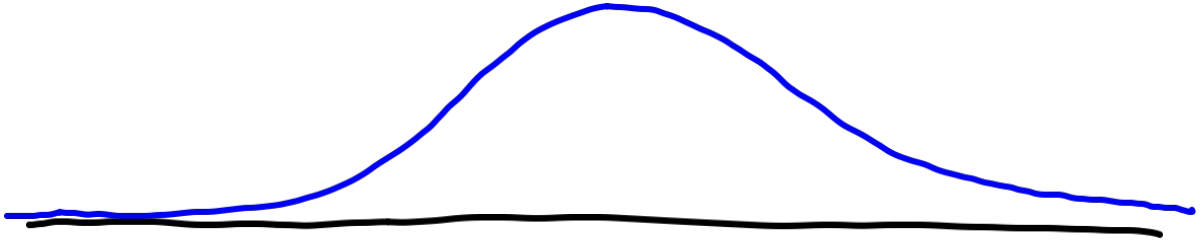
- Empirical mean = average
- Empirical Variance
- Empirical prob. of events.

as the number of Samples $\longrightarrow \infty$
We can estimate the underlying distribution more and more accurately.

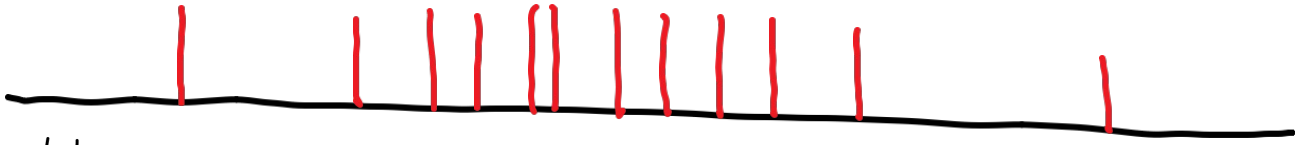
As the number of samples (coin flips) increases
The empirical Converges to the True
The Question is How Fast?

Empirical Dist. Over the Reals

Suppose the true dist. is a density dist over the reals

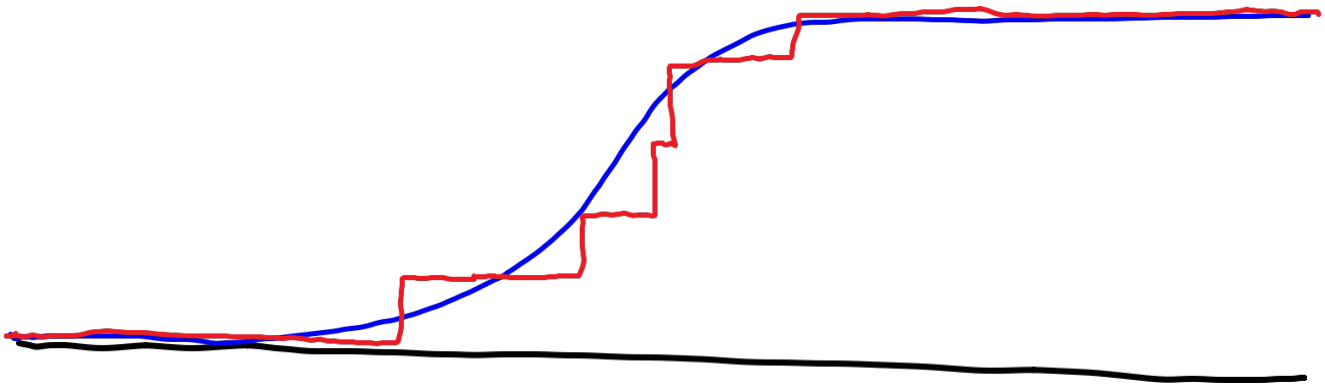


The empirical dist is a point-mass distribution

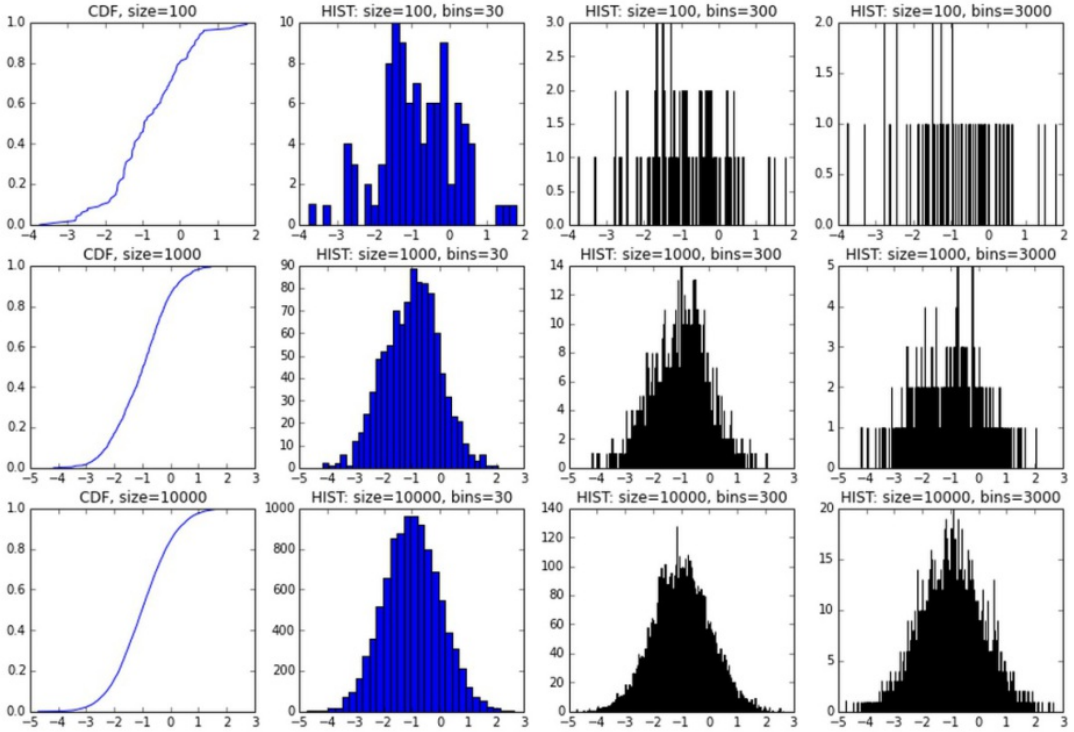


How can the empirical Converge to the density?

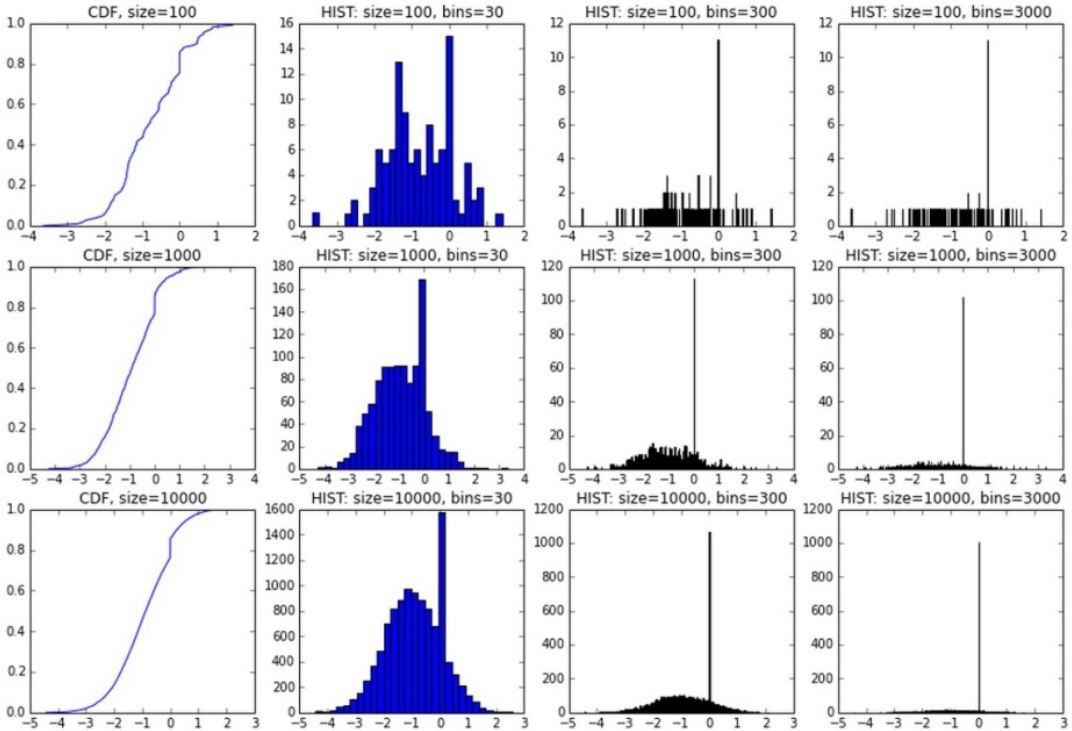
Answer: Consider the CDF



N(-1,1) = A normal distribution centered at -1, with width 1



A mixture of the normal and a point-mass
(10*N(-1,1) + PM(0))



Testing dependence of RVs using

Empirical distribution = DATA

Independent random variables

$$\forall x, y \quad P(X = x \wedge Y = y) = P(X = x)P(Y = y)$$

Independent random variables

$$\forall x, y \quad P(X = x \wedge Y = y) = P(X = x)P(Y = y)$$

	X=0 0.2	X=1 0.1	X=2 0.7
Y=0 0.4	0.08	0.04	0.28
Y=1 0.6	0.12	0.06	0.42

Independent random variables

$$\forall x, y \quad P(X = x \wedge Y = y) = P(X = x)P(Y = y)$$

marginal distributions

		X=0	X=1	X=2
		0.2	0.1	0.7
Y=0	0.4	0.08	0.04	0.28
Y=1	0.6	0.12	0.06	0.42

Do grades depend on gender?

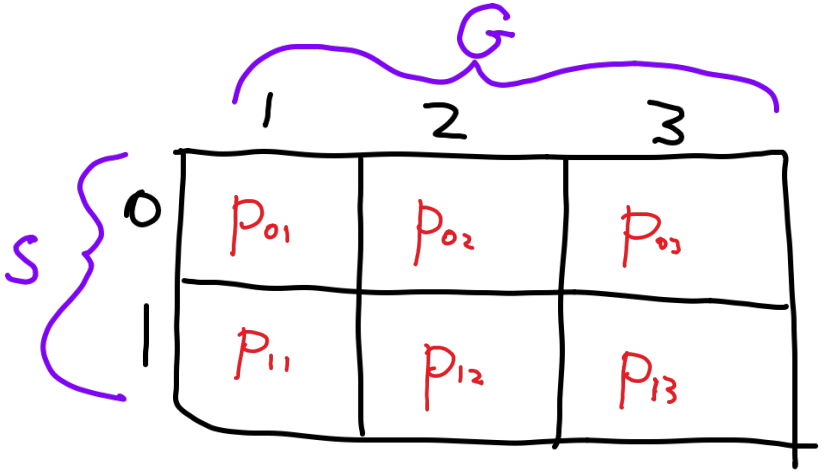
Ω = all students

Define two random Variables

$S : \Omega \to \{0, 1\}$ (male, female)

$G : \Omega \to \{1, 2, 3\}$ (A, B, C) (grade average)

Are S, G dependent or independent?



Empirical contingency tables

n=100
std~1/10

independent

true dist

	marginal	A-average	B-average	C-average
marginal	1	0.2500	0.5000	0.2500
Male	0.4000	0.1000	0.2000	0.1000
Female	0.6000	0.1500	0.3000	0.1500

empirical dist 1

	marginal	A-average	B-average	C-average
marginal	1	0.2600	0.4800	0.2600
Male	0.3900	0.1000	0.2100	0.0800
Female	0.6100	0.1600	0.2700	0.1800

empirical dist 2

	marginal	A-average	B-average	C-average
marginal	1	0.2200	0.3900	0.3900
Male	0.4400	0.1100	0.1400	0.1900
Female	0.5600	0.1100	0.2500	0.2000

empirical dist 3

	marginal	A-average	B-average	C-average
marginal	1	0.2200	0.5000	0.2800
Male	0.3500	0.1000	0.1200	0.1300
Female	0.6500	0.1200	0.3800	0.1500

dependent

	marginal	A-average	B-average	C-average
marginal	1.0000	0.3000	0.4000	0.3000
Male	0.5000	0.1000	0.2000	0.2000
Female	0.5000	0.2000	0.2000	0.1000

	marginal	A-average	B-average	C-average
marginal	1	0.2700	0.5200	0.2100
Male	0.4800	0.1200	0.2200	0.1400
Female	0.5200	0.1500	0.3000	0.0700

	marginal	A-average	B-average	C-average
marginal	1	0.2600	0.4300	0.3100
Male	0.4700	0.0700	0.2100	0.1900
Female	0.5300	0.1900	0.2200	0.1200

	marginal	A-average	B-average	C-average
marginal	1	0.1800	0.4000	0.4200
Male	0.5500	0.1000	0.1600	0.2900
Female	0.4500	0.0800	0.2400	0.1300

Empirical contingency tables

n=10,000
std~1/100

independent

true dist

	marginal	A-average	B-average	C-average
marginal	1	0.2500	0.5000	0.2500
Male	0.4000	0.1000	0.2000	0.1000
Female	0.6000	0.1500	0.3000	0.1500

empirical dist 1

	marginal	A-average	B-average	C-average
marginal	1	0.2473	0.5062	0.2465
Male	0.4021	0.0977	0.2052	0.0992
Female	0.5979	0.1496	0.3010	0.1473

empirical dist 2

	marginal	A-average	B-average	C-average
marginal	1	0.2530	0.4943	0.2527
Male	0.3925	0.0982	0.1942	0.1001
Female	0.6075	0.1548	0.3001	0.1526

empirical dist 3

	marginal	A-average	B-average	C-average
marginal	1	0.2457	0.5005	0.2538
Male	0.3893	0.0936	0.1945	0.1012
Female	0.6107	0.1521	0.3060	0.1526

dependent

	marginal	A-average	B-average	C-average
marginal	1.0000	0.3000	0.4000	0.3000
Male	0.5000	0.1000	0.2000	0.2000
Female	0.5000	0.2000	0.2000	0.1000

	marginal	A-average	B-average	C-average
marginal	1.0000	0.3000	0.4064	0.2936
Male	0.4991	0.0958	0.2052	0.1981
Female	0.5009	0.2042	0.2012	0.0955

	marginal	A-average	B-average	C-average
marginal	1.0000	0.3058	0.3895	0.3047
Male	0.5044	0.1023	0.1989	0.2032
Female	0.4956	0.2035	0.1906	0.1015

	marginal	A-average	B-average	C-average
marginal	1	0.2984	0.4047	0.2969
Male	0.4970	0.0997	0.1938	0.2035
Female	0.5030	0.1987	0.2109	0.0934

Empirical contingency tables

n=1,000,000
std~1/1,000

independent

true dist

	marginal	A-average	B-average	C-average
marginal	1	0.2500	0.5000	0.2500
Male	0.4000	0.1000	0.2000	0.1000
Female	0.6000	0.1500	0.3000	0.1500

empirical dist 1

	marginal	A-average	B-average	C-average
marginal	1	0.2500	0.4998	0.2502
Male	0.3998	0.1001	0.2000	0.0997
Female	0.6002	0.1499	0.2998	0.1505

empirical dist 2

	marginal	A-average	B-average	C-average
marginal	1	0.2504	0.4997	0.2499
Male	0.3991	0.1000	0.1993	0.0999
Female	0.6009	0.1504	0.3004	0.1500

empirical dist 3

	marginal	A-average	B-average	C-average
marginal	1	0.2504	0.5000	0.2497
Male	0.4005	0.1005	0.2003	0.0996
Female	0.5995	0.1498	0.2997	0.1500

dependent

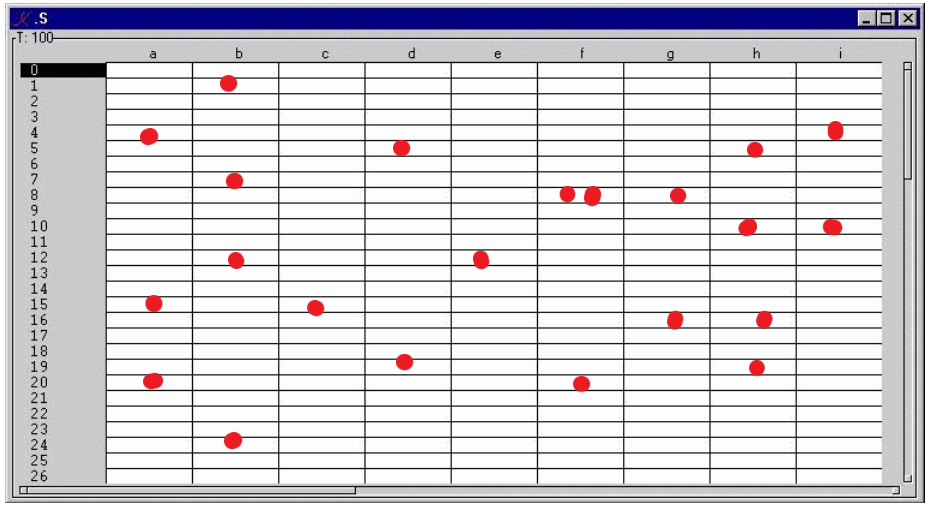
	marginal	A-average	B-average	C-average
marginal	1.0000	0.3000	0.4000	0.3000
Male	0.5000	0.1000	0.2000	0.2000
Female	0.5000	0.2000	0.2000	0.1000

	marginal	A-average	B-average	C-average
marginal	1	0.2997	0.4000	0.3002
Male	0.4997	0.0994	0.2000	0.2003
Female	0.5003	0.2003	0.2000	0.1000

	marginal	A-average	B-average	C-average
marginal	1	0.3006	0.3996	0.2997
Male	0.5002	0.1000	0.2001	0.2001
Female	0.4998	0.2006	0.1995	0.0997

	marginal	A-average	B-average	C-average
marginal	1	0.3001	0.3998	0.3001
Male	0.4995	0.1000	0.1998	0.1997
Female	0.5005	0.2001	0.2000	0.1004

The Data sparcity problem



Suppose $X: \Omega \to \{1\text{-}9\}$ (a-i)

$Y: \Omega \to \{1\text{-}26\}$

Are X,Y dependent?

we will need ALOT of data!

If $X: \Omega \to [0,1]$

$Y: \Omega \to [0,1]$

we will NEVER have enough data

The Covariance approach to finding dependencies

Instead of computing a product for
each entry in the Contingency table
we compute a single number for the whole table.

COV

If **COV** far from zero: X and Y are dependent.
close to zero: We can't say anything

What we gain:

1. faster Convergence.
2. less Computation.

What we lose:

* Some dependencies will not be detected

Sum of two random variables

- The sum of two random variables is a random variable:

$$S(\omega) = X(\omega) + Y(\omega)$$

- The expected value is defined to be:

$$E(X) \doteq \sum_{i=1}^{n} X(\omega_i)P(\omega_i), \quad E(Y) \doteq \sum_{i=1}^{n} Y(\omega_i)P(\omega_i), \qquad E(S) \doteq \sum_{i=1}^{n} S(\omega_i)P(\omega_i),$$

- We can prove the relation $E(X+Y) = E(X) + E(Y)$ in the following way:

- $E(X+Y) = E(S) =$

$$= \sum_{i=1}^{n} S(\omega_i)P(\omega_i) = \sum_{i=1}^{n} \big(X(\omega_i) + Y(\omega_i)\big)P(\omega_i) =$$

$$= \sum_{i=1}^{n} X(\omega_i)P(\omega_i) + \sum_{i=1}^{n} Y(\omega_i)P(\omega_i) = E(X) + E(Y)$$

Product of two random variables

- The product of two random variables is a random variable:

$$M(\omega) = X(\omega) \times Y(\omega)$$

- The expected value is defined to be:

$$E(X) \doteq \sum_{i=1}^{n} X(\ \omega_i) P(\omega_i),\ \ E(Y) \doteq \sum_{i=1}^{n} Y(\omega_i) P(\omega_i), \qquad E(M) \doteq \sum_{i=1}^{n} M(\omega_i) P(\omega_i),$$

- Lets analyze $E(XY)$:

$$E(XY) = E(M) = \sum_{i=1}^{n} M(\omega_i) P(\omega_i) = \sum_{i=1}^{n} \big(X(\omega_i) Y(\omega_i)\big) P(\omega_i) = *$$

- $P(\omega_i)$ can be replaced by $\mathrm{P}(X(\omega) = x \ and \ Y(\omega) = y)$. By summing over the possible values of x, y rather than over the outcomes ω_i . Using these3 observations we e can rewrite the last expression as

$$* = \sum_{x,y} xy\ \mathrm{P}(X(\omega) = x \ and \ Y(\omega) = y) = \#$$

- If $X(\omega),\ Y(\omega)$ are independent random variables then

$$P(X(\omega) = x \ and \ Y(\omega) = y) = P(X(\omega) = x) \times\ P(Y(\omega) = y)$$

- Which implies:

$$\# = \sum_{x,y} xy\ \mathrm{P}(X(\omega) = x \ and \ Y(\omega) = y) = \sum_{x,y} xy\ \mathrm{P}(X(\omega) = x)\ P(Y(\omega) = y) =$$

$$= \sum_{x} x\ \mathrm{P}(X(\omega) = x) \times \sum_{y} y\ \mathrm{P}(Y(\omega) = y) = E(X)E(Y)$$

Expected value for a product of independent RVs

$$E(XY) = \sum_x \sum_y xyP(X = x \wedge Y = y) =$$

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$$= E(X)E(Y)$$

Covariance

Recall $\mu_X \doteq E(X),\ \mu_Y \doteq E(Y)$

$$\mathrm{Cov}(X,Y) \doteq E((X-\mu_X)(Y-\mu_Y)) =$$

$$= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X \mu_Y = E(XY) - E(X)E(Y)$$

Recall $\mathrm{Var}(X) = E(X^2) - E(X)^2 = \mathrm{Cov}(X,X)$

Cov(X,Y)≠0 implies that X and Y are not independent

but Cov(X,Y)=0 **does not imply** that X and Y are independent

Why do we need Corr in addition to Cov ?

- Suppose Cov(X,Y)=3 and Cov(X,Z)=1
- Is X more correlated with Y than with Z?
- Not necessarily, it might be that Y=3Z
- We want to have a measure of correlation that is independent of scaling, i.e.

$$\forall a > 0, b > 0: \;\; Corr(aX, bY) = Corr(X, Y)$$

Correlation coefficient

$$\mathrm{Corr}(X,Y) \doteq \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

- Corr(aX+c,bY+d)=Corr(X,Y) if a,b>0
- Corr(X,Y) varies from -1 to +1
- Corr(X,Y)>0 ⇔ X and Y are "correlated"
- Corr(X,Y)<0 ⇔ X and Y are "anti-correlated"
- Corr(X,Y)=1 ⇔ X=aY, a>0
- Corr(X,Y)=-1 ⇔ X=aY a<0

Correlation coefficient

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{var(X)var(Y)}}$$

- The covariance depends on scaling and units, correlation coefficient does not

$$\forall a > 0, b > 0 \qquad Corr(aX,bY) = Corr(X,Y)$$

- The correlation coefficient varies between -1 and 1.

correlation vs. dependence

- If X, Y are independent then $Cov(X, Y) = 0$ and $Corr(X, Y) = 0$
- If $Corr(X, Y) \neq 0$ then X, Y are dependent.
- No implications in the opposite directions
- If X is a random variable that takes n discrete values, and Y is a random variable that takes m discrete values,
  Then checking for independence requires checking nm values.
- Checking for zero correlation requires calculation of just one value.
- Correlation is the quick and dirty way to detect strong dependencies, but it cannot find them all.

Examples

	X=1	X=2	X=3	X=4
Y=1	1/4	1/4	0	0
Y=2	0	0	0	0
Y=3	0	0	1/4	1/4

$$\mu(X)=2.5,\ \mu(Y)=2$$

$$\text{cov}(X,Y)=\frac{1}{4}(-1.5*-1)+\frac{1}{4}(-.5*-1)+\frac{1}{4}(.5*1)+\frac{1}{4}(1.5*1)=1$$

	X=1	X=2	X=3	X=4
Y=1	0	0	0	1/4
Y=2	0	1/4	1/4	0
Y=3	1/4	0	0	0

$$\mu(X)=2.5,\ \mu(Y)=2$$

$$\text{cov}(X,Y)=\frac{1}{4}(-1.5*1)+\frac{1}{4}(-.5*0)+\frac{1}{4}(.5*9)+\frac{1}{4}(1.5*-1)=-\frac{3}{4}$$

	X=1	X=2	X=3	X=4
Y=1	1/4	0	0	1/4
Y=2	0	0	0	0
Y=3	1/4	0	0	1/4

$$\mu(X) = 2.5,\ \mu(Y) = 2$$

$$\text{cov}(X,Y) = \frac{1}{4}(-1.5*1) + \frac{1}{4}(-1.5*-1) + \frac{1}{4}(1.5*1) + \frac{1}{4}(1.5*-1) = 0$$

P(X=1)=P(X=4)=1/2, P(Y=1)=P(Y=3)=1/2

X and Y are independent because all of the joint probabilities are either 0 or 1/4

	X=1	X=2	X=3	X=4
Y=1	1/8	0	0	1/8
Y=2	0	1/4	1/4	0
Y=3	1/8	0	0	1/8

1. Cov(X,Y)=0
2. X and Y are independent

A. 1 and 2 B. 1 and not 2 C. not 1 and 2 D. not 1 and not 2