

CSE291-C00

Laplacian Operator

Instructor: Hao Su

Famous Motivation

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

"La Physique ne nous donne pas seulement
l'occasion de résoudre des problèmes . . . , elle nous
fait présentir la solution." H. POINCARÉ.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.

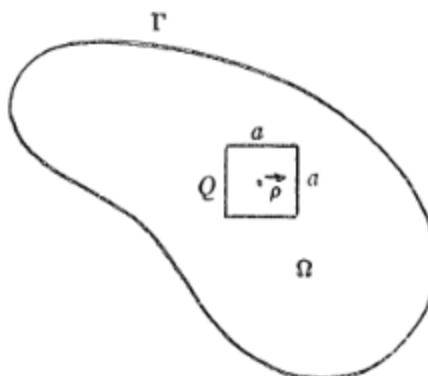
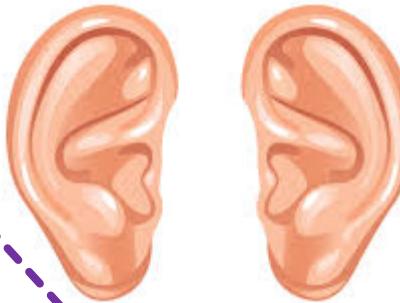


FIG. 1

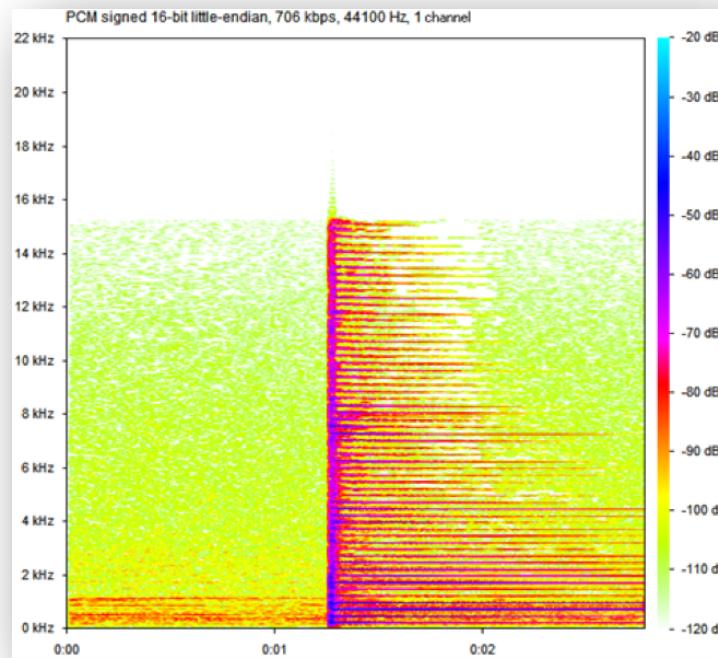
An Experiment



Is this
possible?



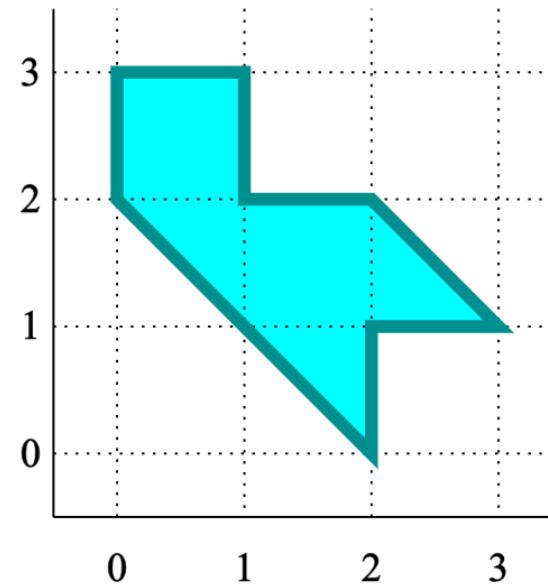
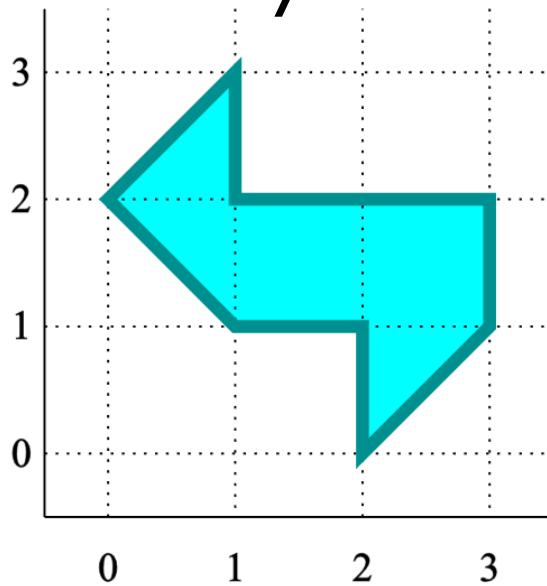
Unreasonable to Ask?



Length
of string

Spoiler Alert

- Has to be a weird drum
- Spectrum tells you a lot!



“No, but...”

Rough Intuition

http://pngimg.com/upload/hammer_PNG3886.png



You can learn a lot
about a shape by
hitting it (lightly)
with a hammer!

Spectral Geometry

What can you learn about its shape from
vibration frequencies and
oscillation patterns?

$$\Delta f = \lambda f$$

Objectives

- Make “vibration modes” more **precise**
- **Progressively more complicated** domains
 - Line segments
 - Regions in
 - Graphs
 - Surfaces/manifolds
- Next time: **Discretization, applications**

Review:

Vector Spaces and Linear Operators

$$\mathcal{L}[\vec{x} + \vec{y}] = \mathcal{L}[\vec{x}] + \mathcal{L}[\vec{y}]$$

$$\mathcal{L}[c\vec{x}] = c\mathcal{L}[\vec{x}]$$

Review:

In Finite Dimensions

A \vec{x}
matrix vector

$\vec{x} \mapsto A\vec{x}$
linear operator

Recall: Spectral Theorems in Linear Algebra

Theorem. Suppose A is **Hermitian**. Then, A has an **orthogonal basis** of eigenvectors. If A is **positive definite**, the corresponding eigenvalues are **nonnegative**.

Minus Second Derivative Operator

“Dirichlet boundary conditions”

$$\{f(\cdot) \in C^\infty([a, b]) : f(0) = f(\ell) = 0\}$$

$$\mathcal{L}[f(\cdot)] := -f''(\cdot)$$

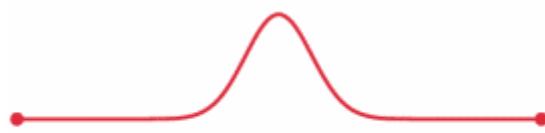
Eigenfunctions:

$$f_k(x) = \sin\left(\frac{\pi k x}{\ell}\right), \quad \lambda_k = \left(\frac{\pi k}{\ell}\right)^2$$

Physical Intuition: Wave Equation

Minus second derivative operator!

$$\frac{\partial^2 u}{\partial t^2} - \underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{Minus second derivative operator!}} = 0$$



Observation

$$\{f(\cdot) \in C^\infty([a, b]) : f(0) = f(\ell) = 0\}$$

$$\begin{aligned}\langle f, \mathcal{L}[f] \rangle &= - \int_0^\ell f(x) f''(x) dx \\ &= -[f(x) f'(x)]_0^\ell + \underbrace{\int_0^\ell f'(x)^2 dx}_{\geq 0}\end{aligned}$$

Hilbert-Schmidt Theorem

Theorem. Let $H \neq 0$ be an infinite-dimensional, separable Hilbert space and let $K \in L(H)$ be compact and self-adjoint. Then, there exists a countable orthonormal basis of H consisting of eigenvectors of K .



Hilbert space: Space with inner product

Separable: Admits countable, dense subset

Compact operator: Bounded sets to relatively compact sets

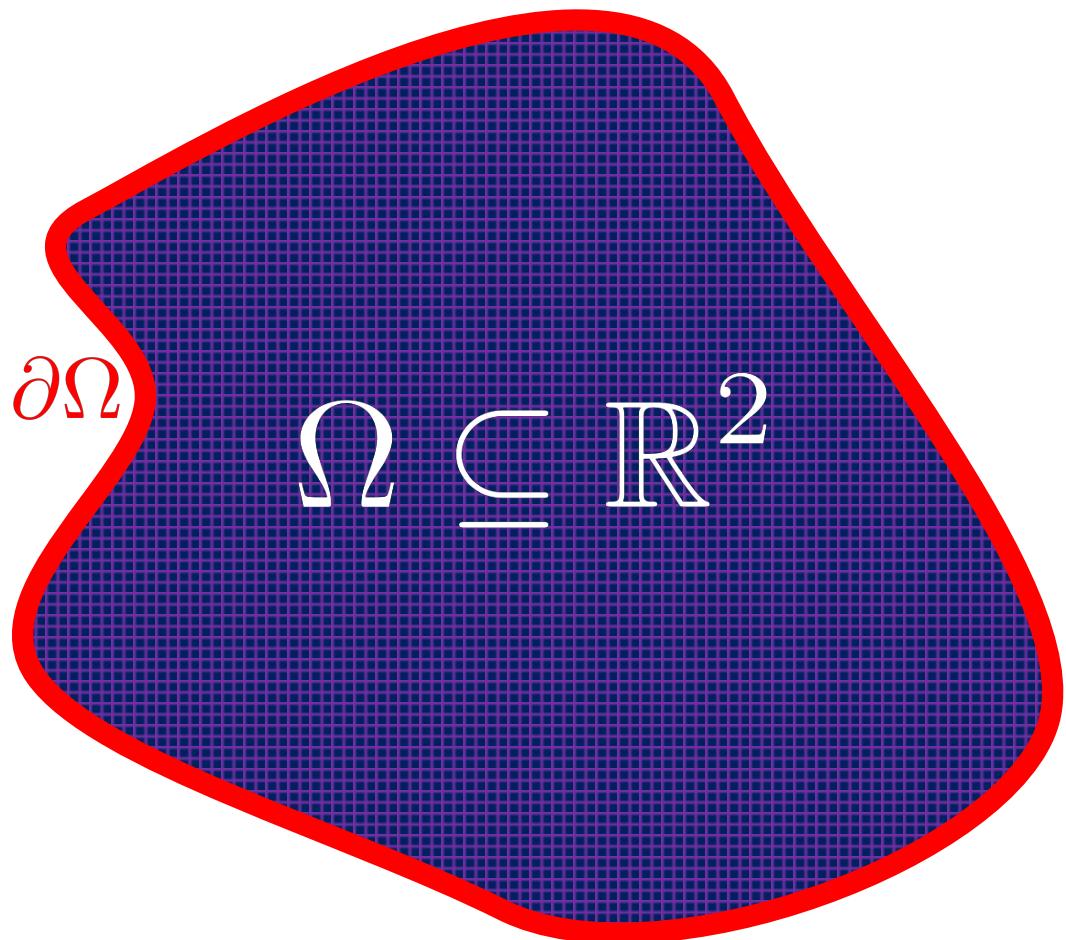
Self-adjoint: $\langle K\nu, w \rangle = \langle \nu, Kw \rangle$

Can you hear the length of an interval?

$$\lambda_k = \left(\frac{\pi k}{\ell} \right)^2$$

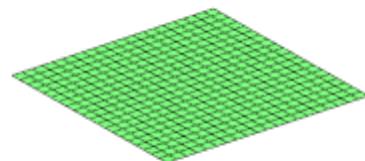
Yes!

Planar Region



Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$
$$\Delta := \sum_i \frac{\partial^2}{\partial x_i^2}$$

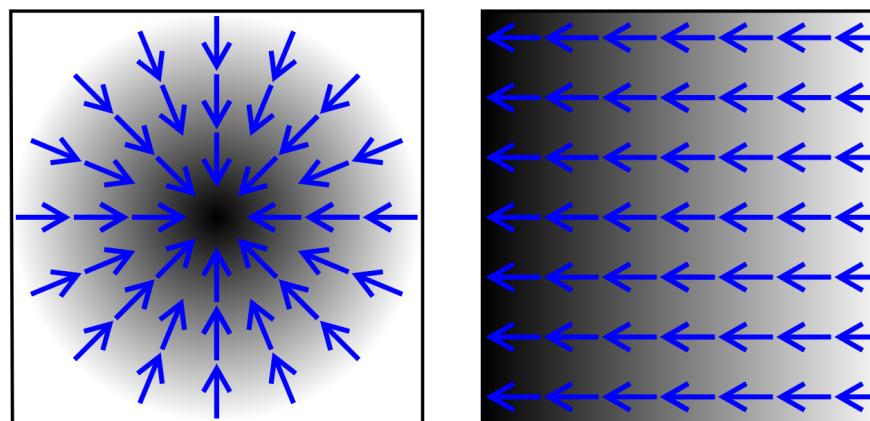


Typical Notation

$$\text{“} \Delta = \nabla \cdot \nabla \text{”}$$

divergence gradient

More
later...



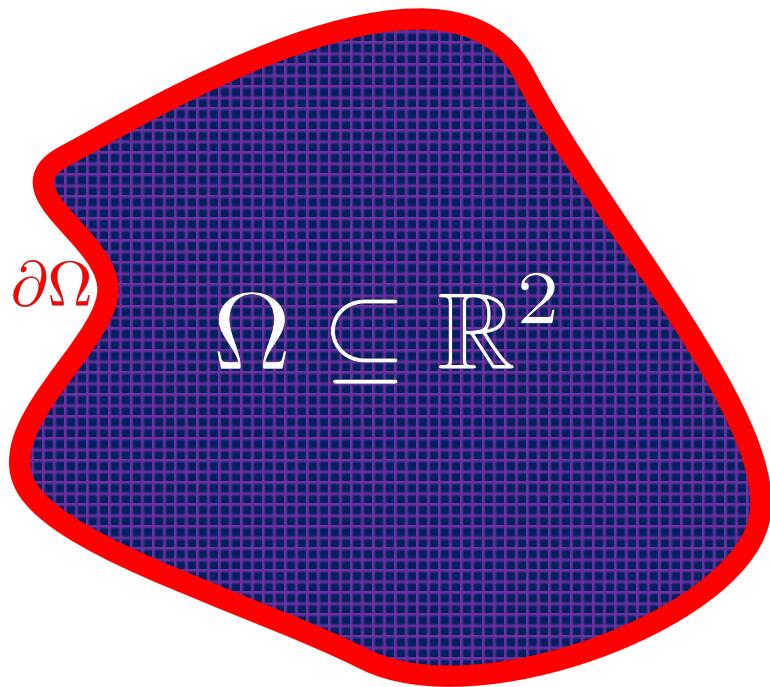
Gradient operator:

$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

Positivity, Self-Adjointness

$$\{f(\cdot) \in C^\infty(\Omega) : f|_{\partial\Omega} \equiv 0\}$$

“Dirichlet boundary conditions”



$$\mathcal{L}[f] := -\Delta f$$

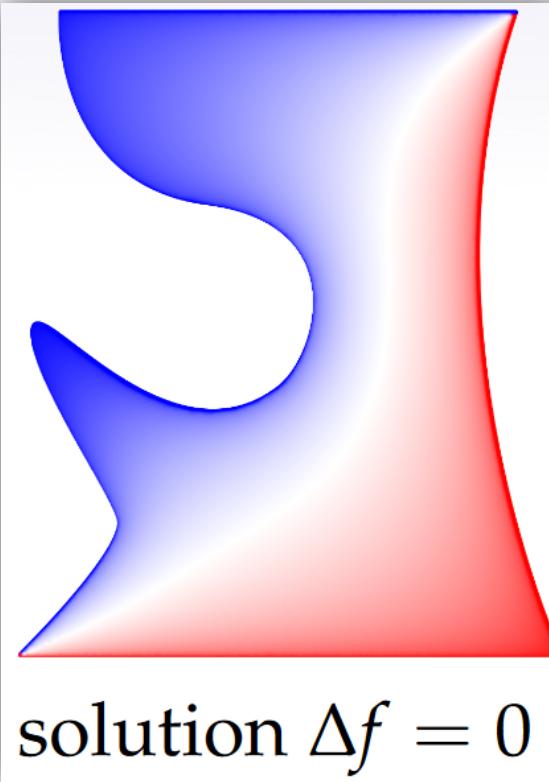
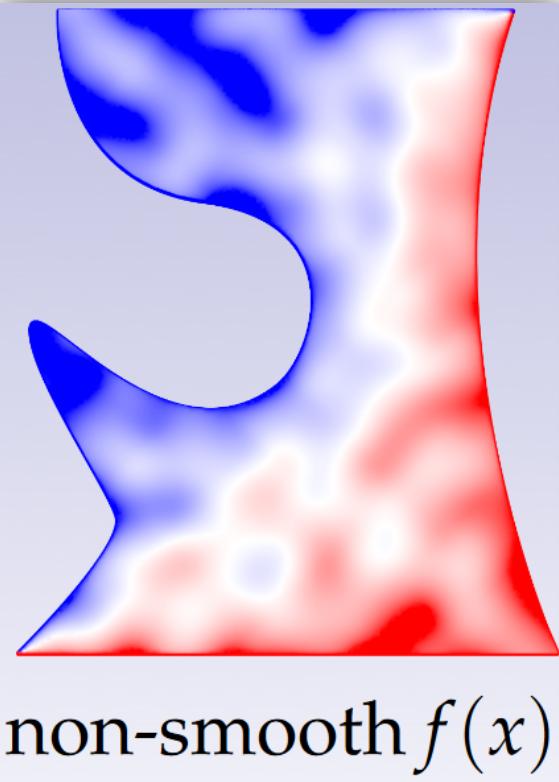
$$\langle f, g \rangle := \int_{\Omega} f(x)g(x) dx$$

On board:

1. Positive: $\langle f, \mathcal{L}[f] \rangle \geq 0$
2. Self-adjoint: $\langle f, \mathcal{L}[g] \rangle = \langle \mathcal{L}[f], g \rangle$

Dirichlet Energy

$$E[f] := \int_{\Omega} \langle \nabla f, \nabla f \rangle dA$$



On board:

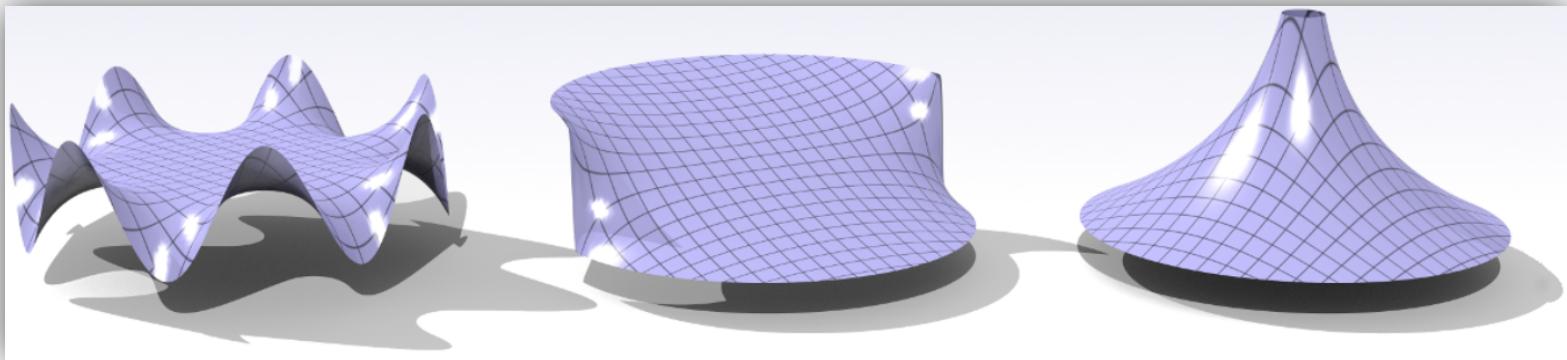
$$\begin{aligned} & \min_f E[f] \\ & \text{s.t. } f|_{\partial\Omega} = g \end{aligned}$$

$$\Delta f \equiv 0$$

"Laplace equation"
"Harmonic function"

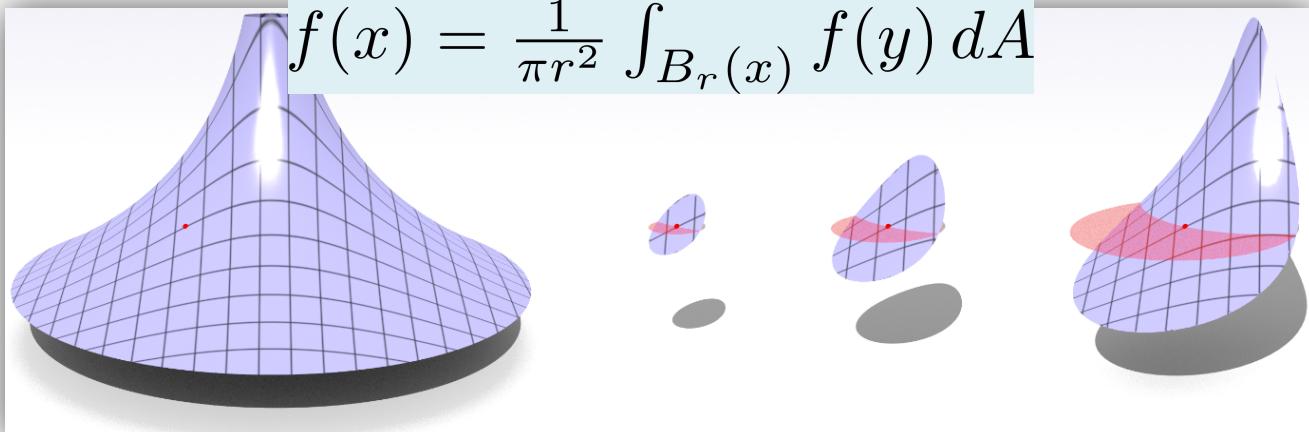
Harmonic Functions

$$\Delta f \equiv 0$$

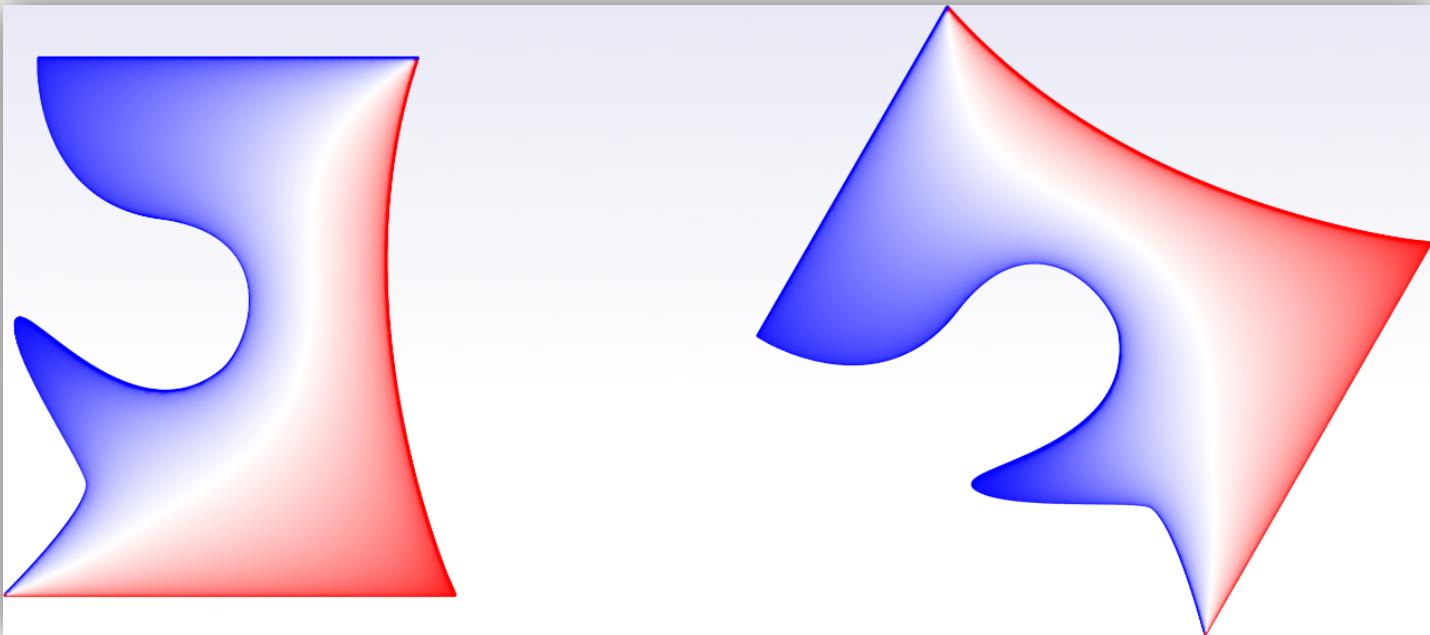


Mean value property:

$$f(x) = \frac{1}{\pi r^2} \int_{B_r(x)} f(y) dA$$



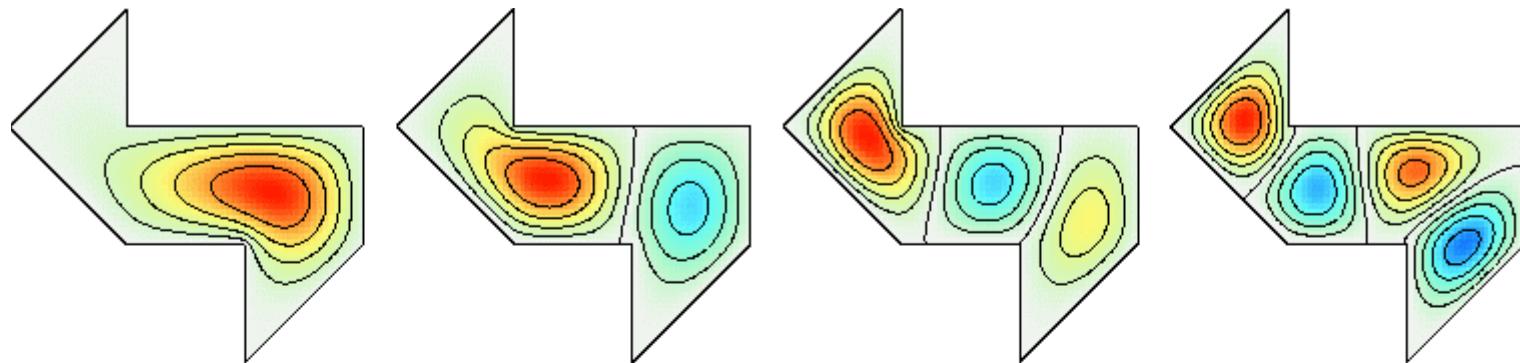
Intrinsic Operator



Images made by E.Vouga

Coordinate-independent (important!)

Another Interpretation of Eigenfunctions



Find critical points of $E[f]$

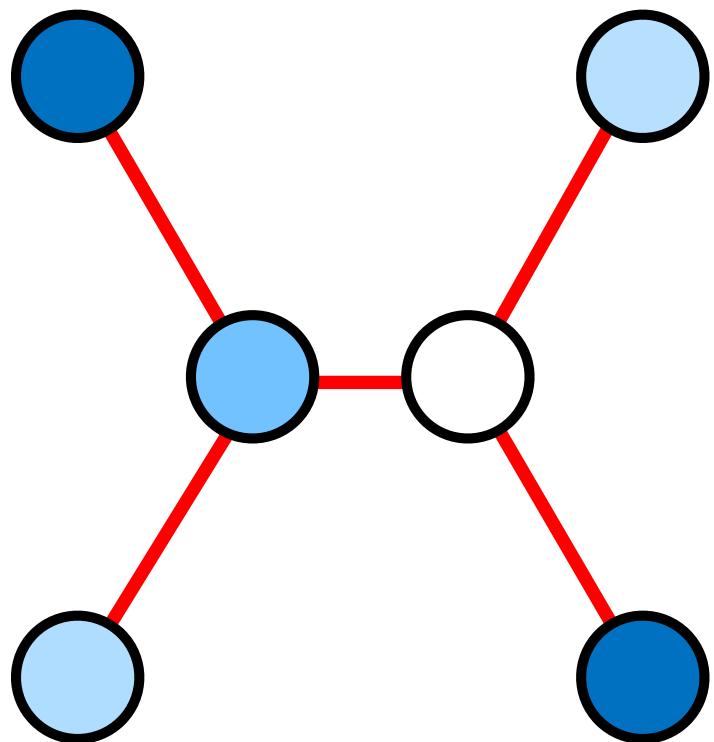
$$\text{s.t. } \int_{\Omega} f^2 = 1$$

<http://www.math.udel.edu/~driscoll/research/gww1-4.gif>

Small eigenvalue: Small Dirichlet Energy

Basic Setup

- **Function:**
One value per vertex



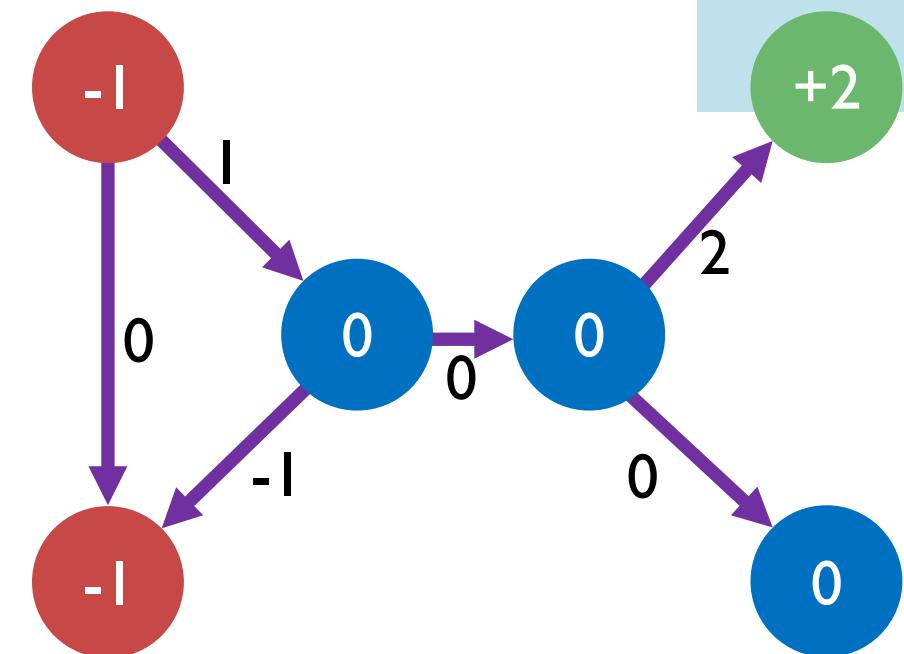


What is the
Dirichlet energy of a
function on a graph?

Differencing Operator

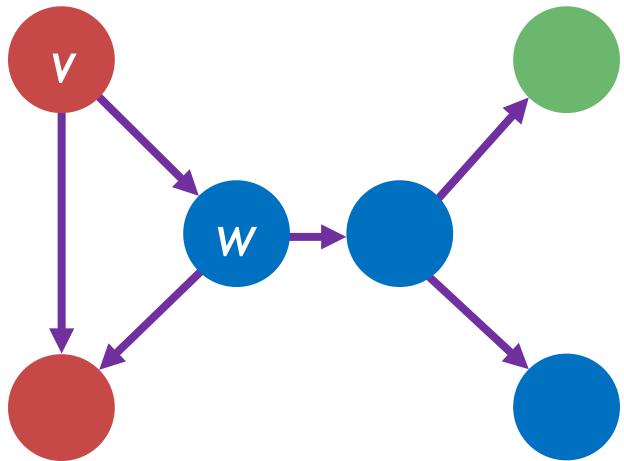
$$D_{ev} := \begin{cases} -1 & \text{if } E_{e1} = v \\ 1 & \text{if } E_{e2} = v \\ 0 & \text{otherwise} \end{cases}$$

$D \in \{-1, 0, 1\}^{|E| \times |V|}$



Orient edges arbitrarily

Dirichlet Energy on a Graph



$$D_{ev} := \begin{cases} -1 & \text{if } E_{e1} = v \\ 1 & \text{if } E_{e2} = v \\ 0 & \text{otherwise} \end{cases}$$

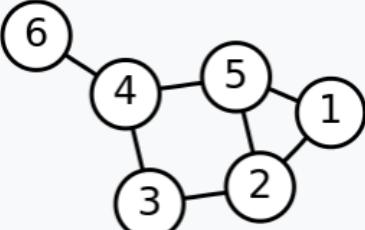
$$E[f] := \|Df\|_2^2 = \sum_{(v,w) \in E} (f_v - f_w)^2$$

(Unweighted) Graph Laplacian

- Symmetric
- Positive definite

$$E[f] = \|Df\|_2^2 = f^\top (D^\top D) f := f^\top L f$$

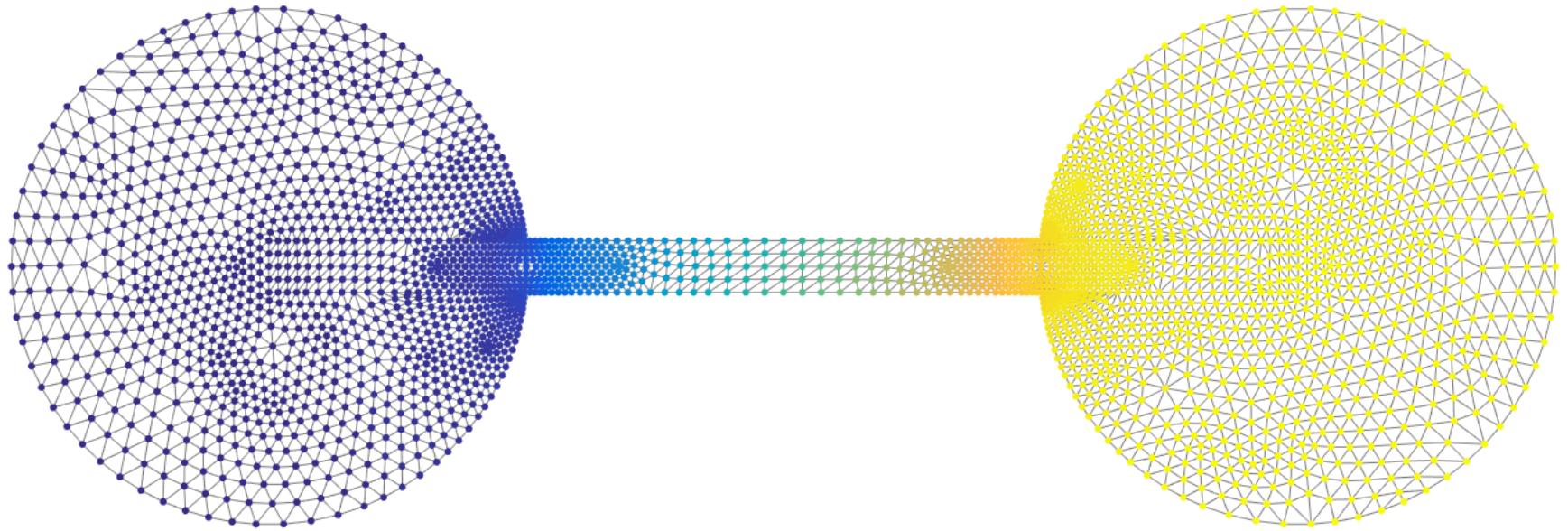
$$L_{vw} = A - D = \begin{cases} 1 & \text{if } v \sim w \\ -\text{degree}(v) & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

Labeled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$



What is the
smallest eigenvalue
of the graph Laplacian?

Second-Smallest Eigenvector



$$Lx = \lambda x$$

Used for graph
partitioning

Fiedler vector (“algebraic connectivity”)

Mean Value Property

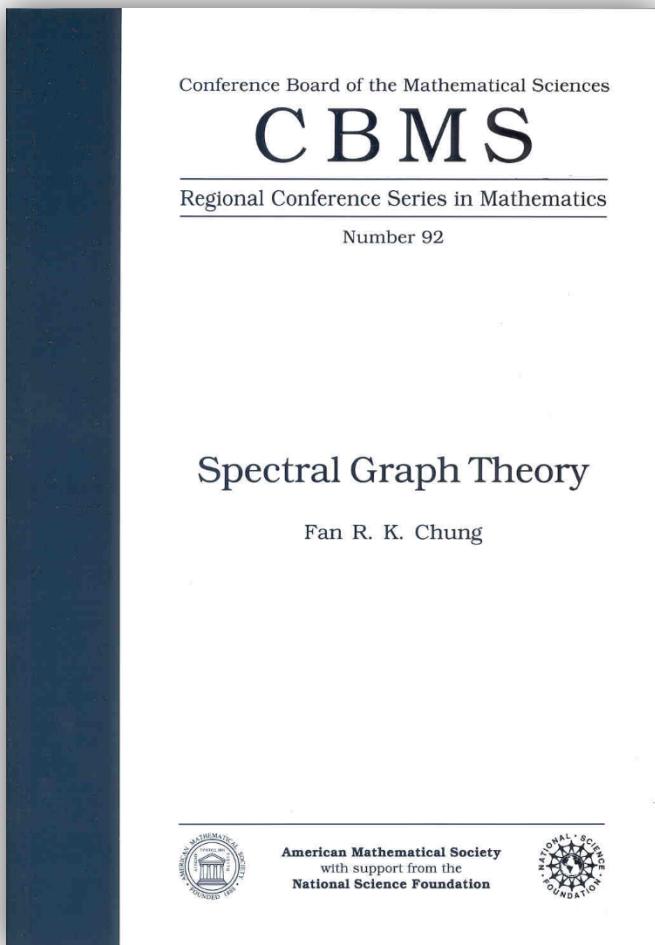
$$L_{vw} = A - D = \begin{cases} 1 & \text{if } v \sim w \\ -\text{degree}(v) & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

$$(Lx)_v = 0$$



Value at v is average of neighboring values

For More Information...

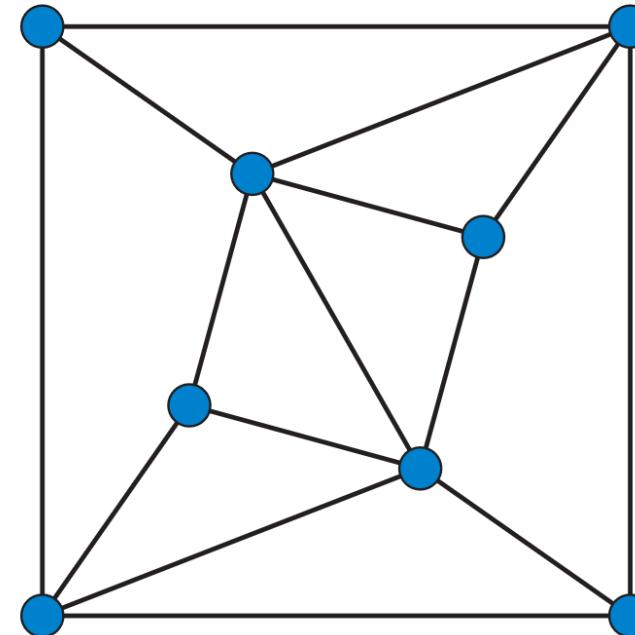
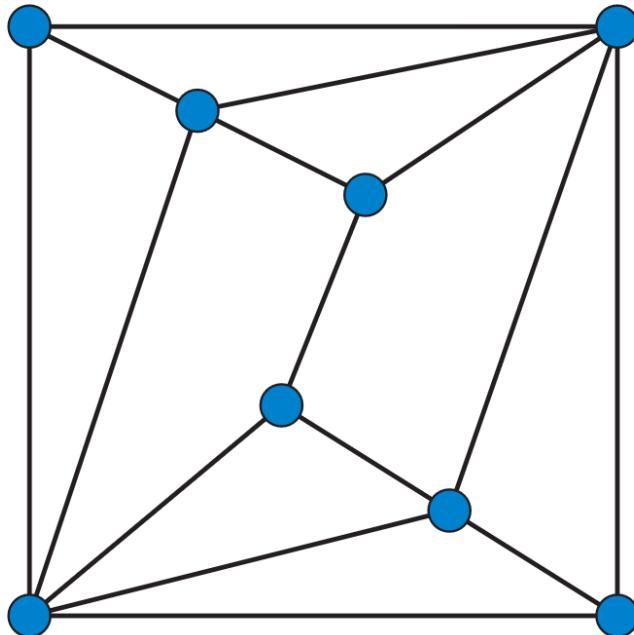


Graph Laplacian encodes lots of information!

Example: Kirchoff's Theorem
Number of spanning trees equals

$$\frac{1}{n} \lambda_2 \lambda_3 \cdots \lambda_n$$

Hear the Shape of a Graph?

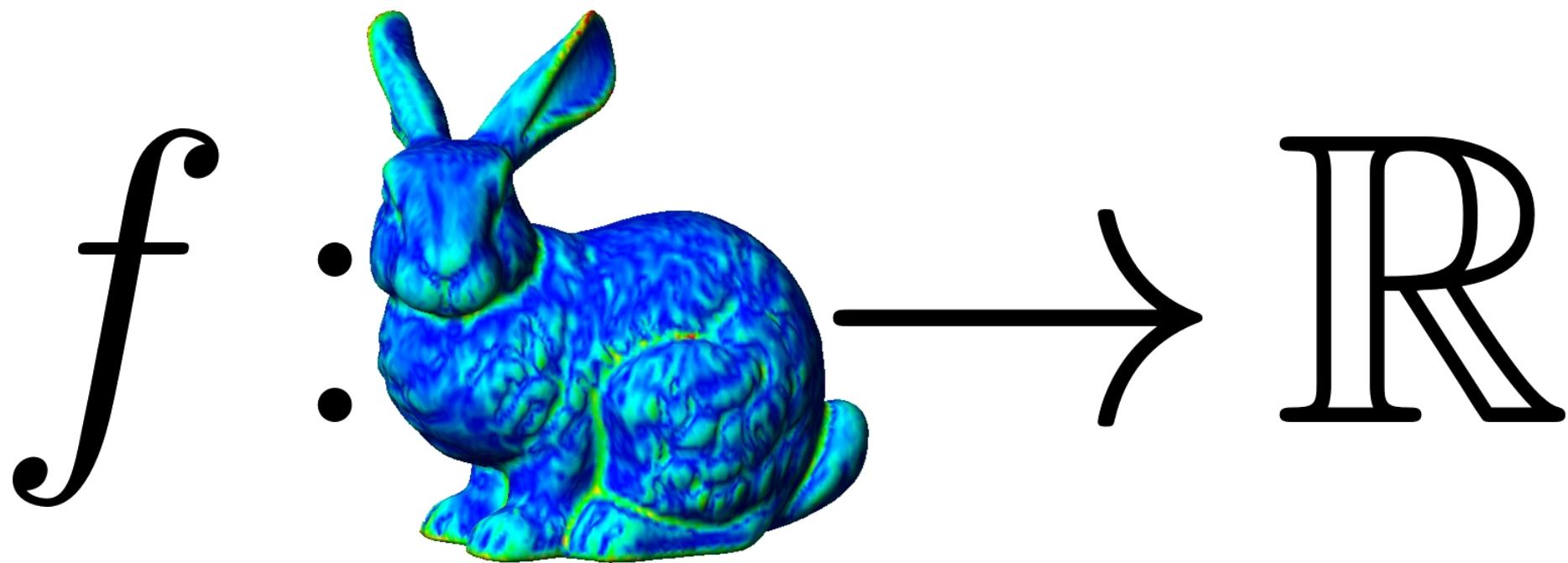


“Enneahedra”

No!

Recall:

Scalar Functions



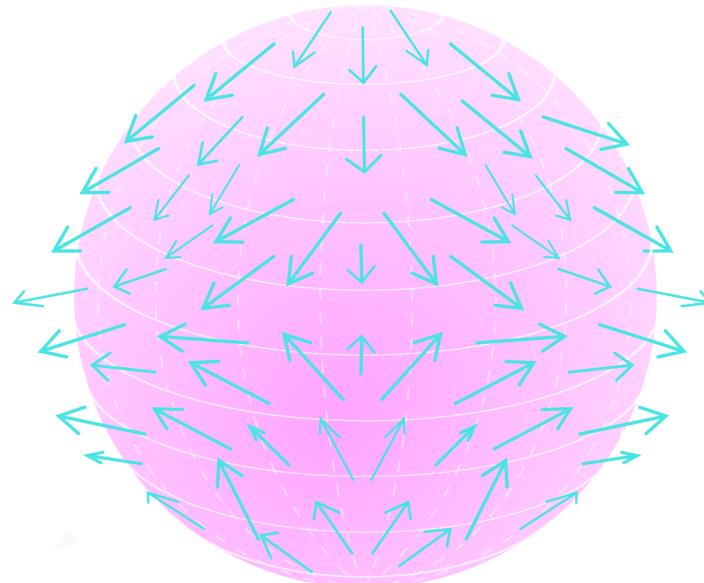
http://www.ieeta.pt/polymero/Screenshots/PolyMeCo_OneView.jpg

Map points to real numbers

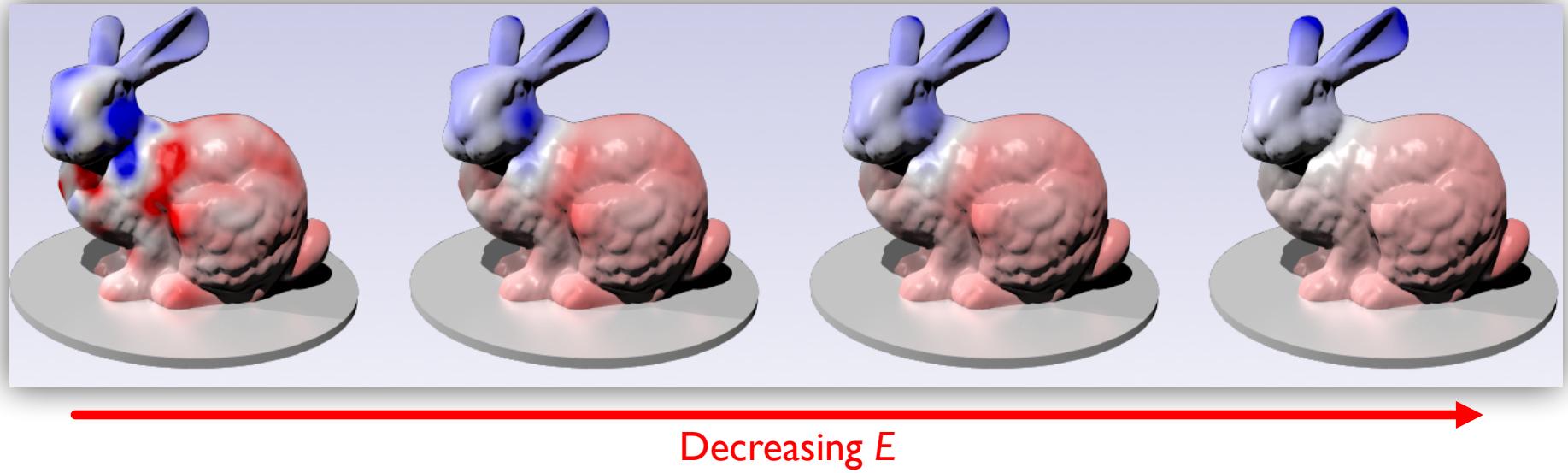
Gradient Vector Field

$\nabla f : S \rightarrow \mathbb{R}^3$ with

$$\begin{cases} \langle (\nabla f)(p), v \rangle = (df)_p(v), v \in T_p S \\ \langle (\nabla f)(p), N(p) \rangle = 0 \end{cases}$$



Dirichlet Energy



$$E[f] := \int_S \|\nabla f\|_2^2 dA$$

From Inner Product to Operator

$$\begin{aligned}\langle f, g \rangle_{\Delta} &:= \int_S \nabla f(x) \cdot \nabla g(x) dA \\ &:= \langle f, \Delta g \rangle\end{aligned}$$

Implies

On the board:
“Motivation” from finite-dimensional linear algebra.

Laplace-Beltrami operator

What is Divergence?

$V : S \rightarrow \mathbb{R}^3$ where $V(p) \in T_p S$

$dV_p : T_p S \rightarrow \mathbb{R}^3$

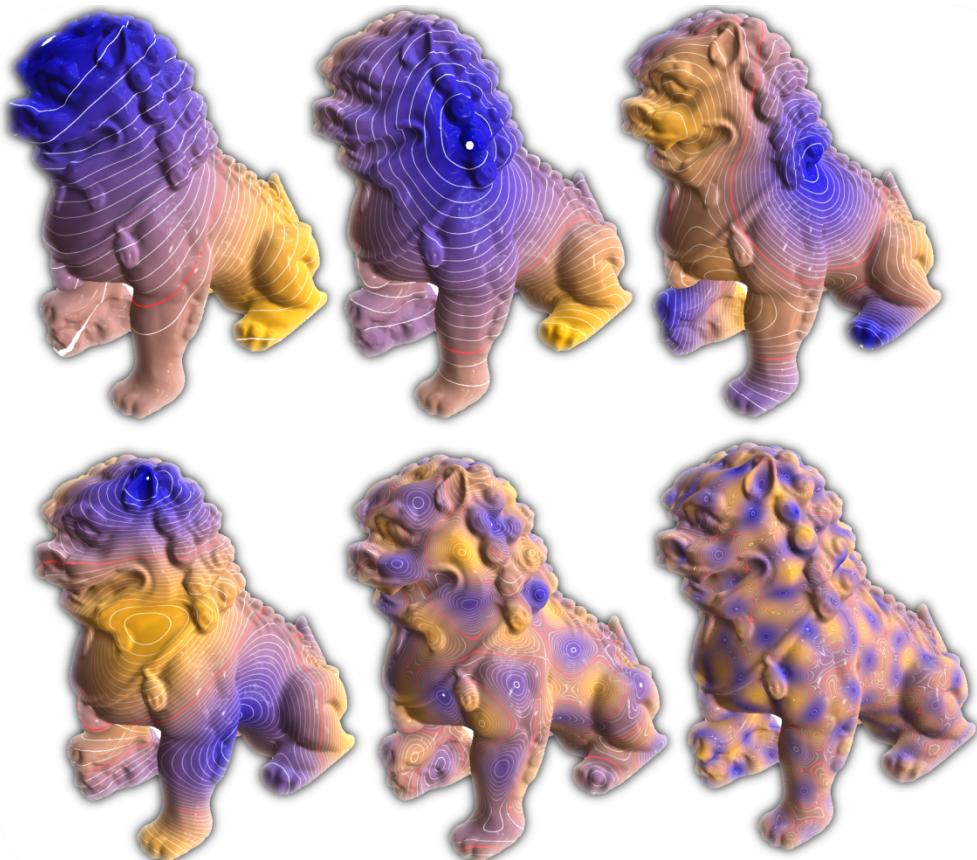
$\{e_1, e_2\} \subset T_p S$ orthonormal basis

$$(\nabla \cdot V)_p := \sum_{i=1}^2 \langle e_i, dV(e_i) \rangle_p$$

Things we **should check** (but probably won't):

- Independent of choice of basis

Eigenfunctions



$$\Delta \psi_i = \lambda_i \psi_i$$

Vibration modes of
surface (not volume!)

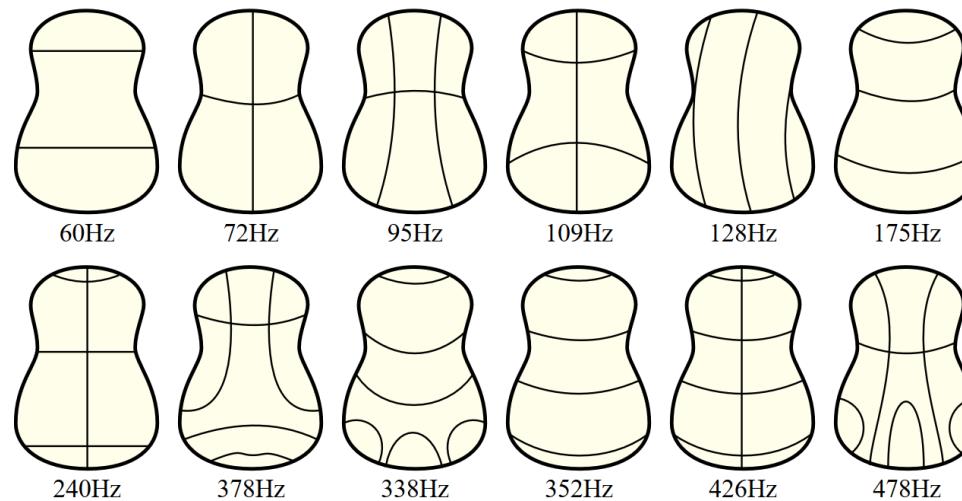
Practical Application



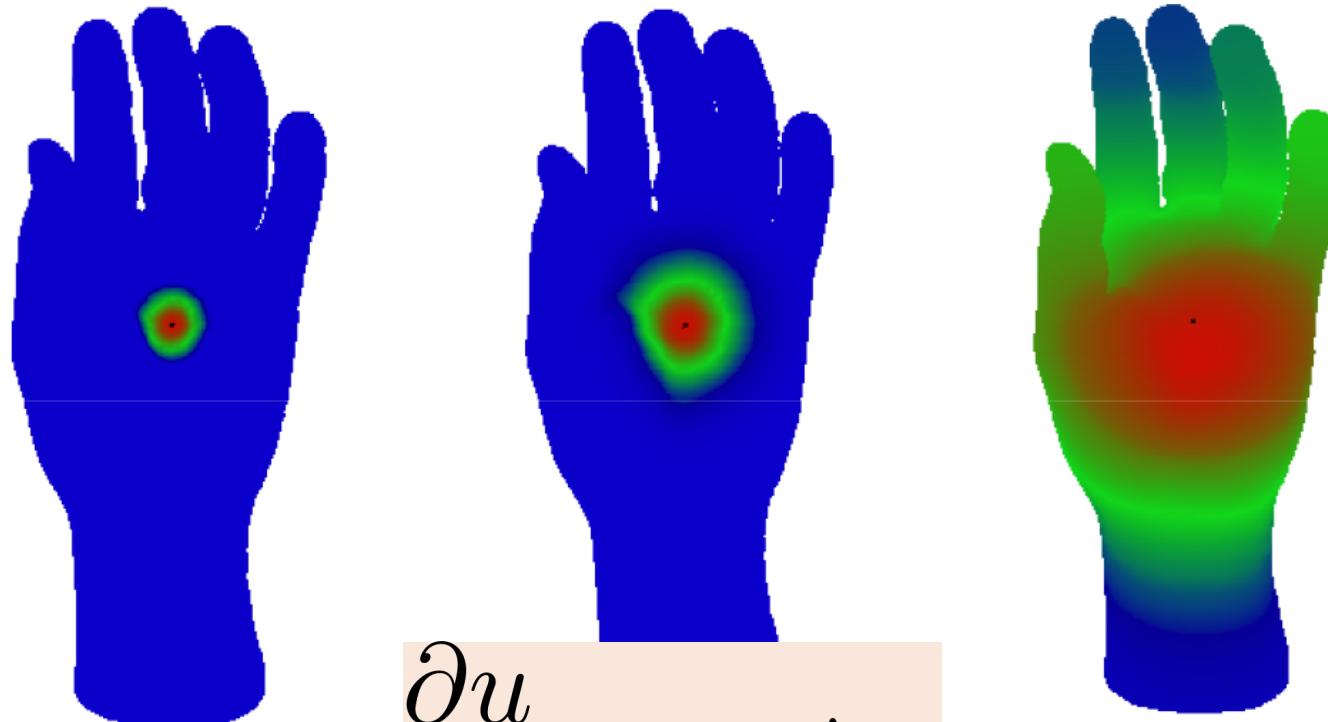
<https://www.youtube.com/watch?v=3uMZZzVvnSiU>

Nodal Domains

Theorem (Courant). The n -th eigenfunction of the Dirichlet boundary value problem has at most n nodal domains.



Additional Connection to Physics

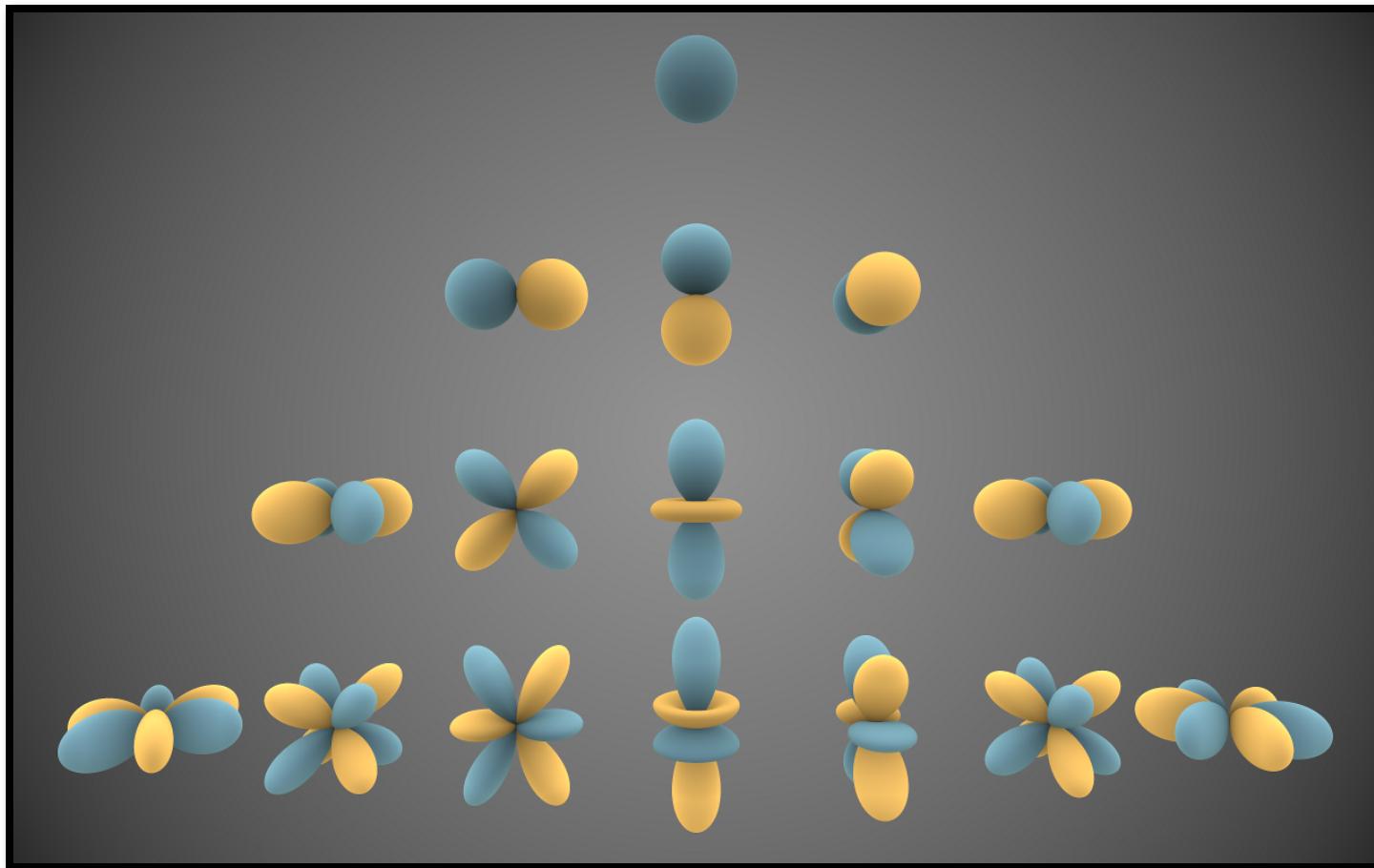


$$\frac{\partial u}{\partial t} = -\Delta u$$

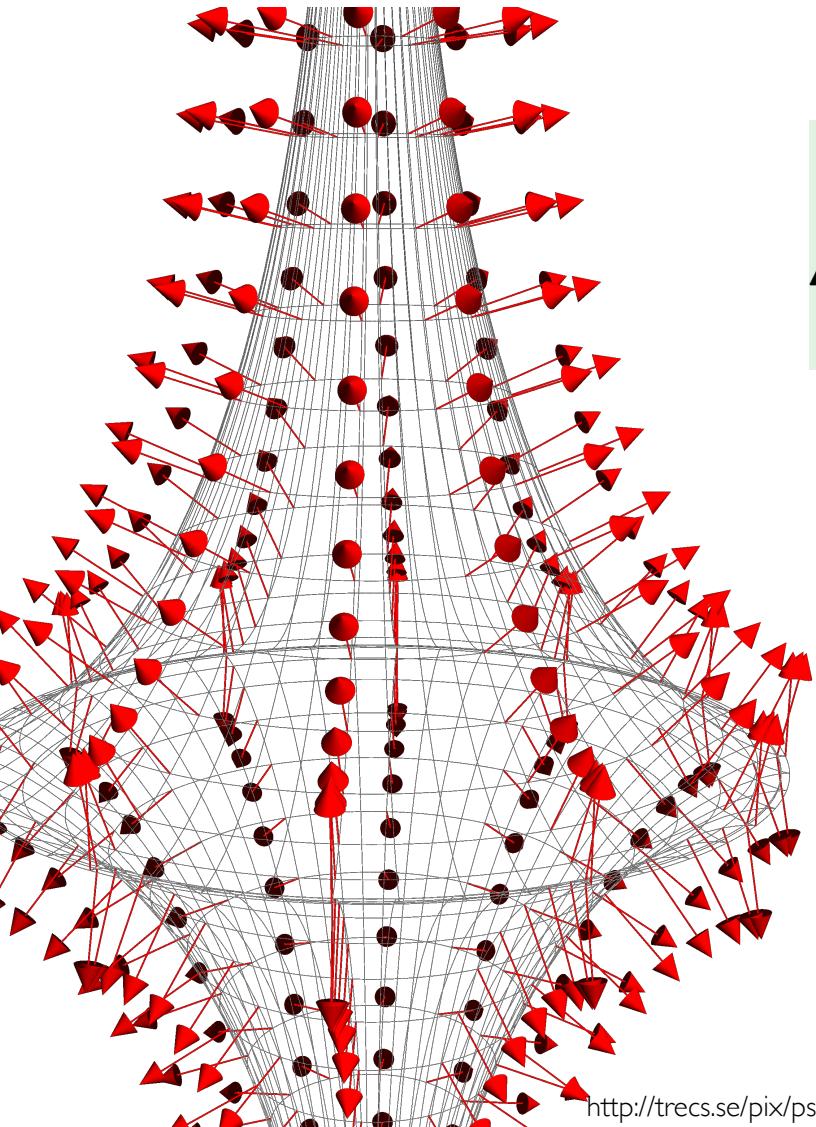
http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf

Heat equation

Spherical Harmonics



Laplacian of xyz function



$$\Delta \vec{x} = \frac{1}{2}(\kappa_1 + \kappa_2)N$$

Intuition:
Laplacian measures difference with neighbors.