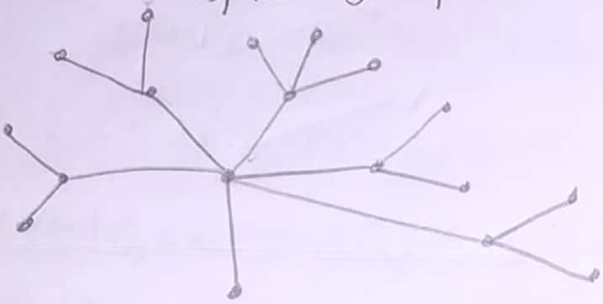
Module : 3

16/09/19

TREE

A Tree is a connected acyclic graph. Tree is a simple graph.



Tree with 1 vertex :

Tree with 2 vertex : 

Tree with 3 vertex : 

Tree with 4 vertex : 

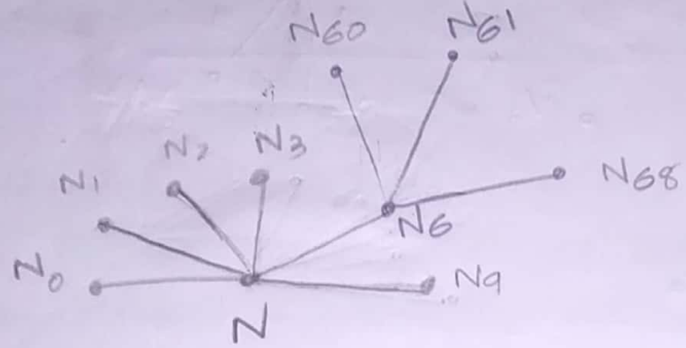
This is not a tree. Because it contains cycle.

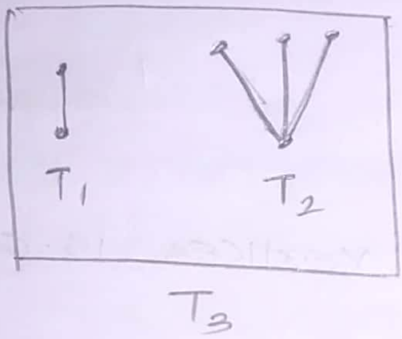
Sorting of mails with pincode.

* 9 digits in pincode

Sorting of mails can be done using decision tree / sorting tree.

~~Root~~





T_3 is not a tree. Because T_1 & T_2 are not connected. But T_1 & T_2 are trees.

Forest: It consist of several disconnected trees.

Spanning Tree

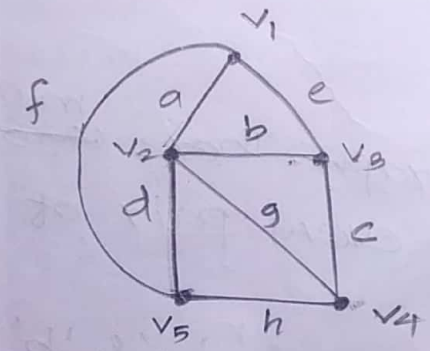
Spanning Tree for a connected graph is a spanning subgraph.

* It is a spanning subgraph.

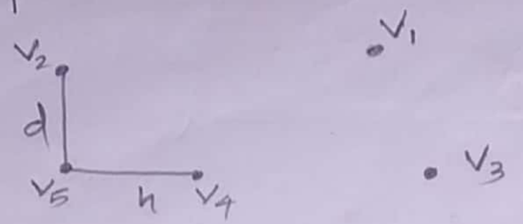
Spanning Subgraph

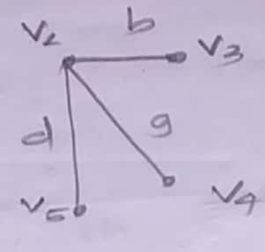
It will have all the vertices of the subgraph but some of the edges.

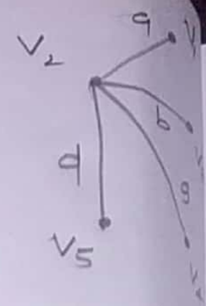
? Draw a spanning subgraph for the given graph.

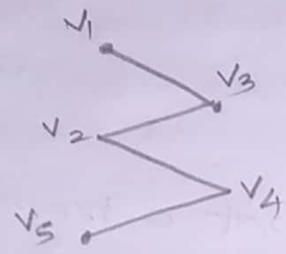


spanning trees are:









Properties of tree

A graph G with 'n' vertices is called a tree if

i) G is connected & has no circuits.

ii) G is connected & has n-1 edges.

iii) It is circuitless & has n-1 edges

iv) There is exactly one path b/w every pair of vertices.

v) G is a minimaly connected graph.

THEOREM:

There is one and only one path b/w every pair of vertices in a tree 'T'.

Proof:

∵ T is a connected graph. There ~~was~~ must exist atleast 1 path b/w every pair of ~~vertex~~ vertices.

Suppose b/w a pair of vertices 'a' & 'b' there exist 2 distinct paths.

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The union of these 2 paths consist of a cycle and a tree cannot have a cycle.

Theorem 2

If in a graph of G there is one and only one path b/w every pair of vertices, G is a tree.

Proof:

Excitence of a path b/w every pair of vertices ensures that G is connected. If there is a circuit in G then there is atleast 1 pair of vertices a & b such that there are 2 distinct path b/w a & b. So G is a tree. Since we are given there is only 1 path b/w every pair of vertices, G is a tree.

Theorem 3

A tree with 'n' vertices has 'n-1' edges.

Proof:

Let us use mathematical induction.

Let n=1, then 

ie, It has 0 edges. (1-1)

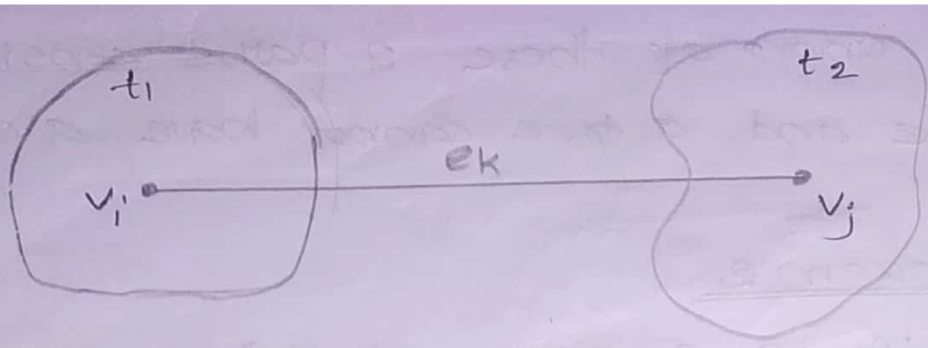
n=2, then 

ie, It has 1 edge (2-1)

n=3, then 

ie, It has 2 edges (3-1)

ie,



Consider a tree with 'n' vertices. In this figure, there exist no other path b/w v_i & v_j except ek according to theorem 1

∴ deletion of ek will disconnect the graph, or T-ek consist of exactly 2 components t_1 & t_2 and each of this components is a tree. Both t_1 & t_2 have fewer than n vertices. Hence T has exactly n-1 edges using theorem 3.

Theorem 4

A ~~The~~ graph is a tree if and only if it is minimally connected.

Proof:

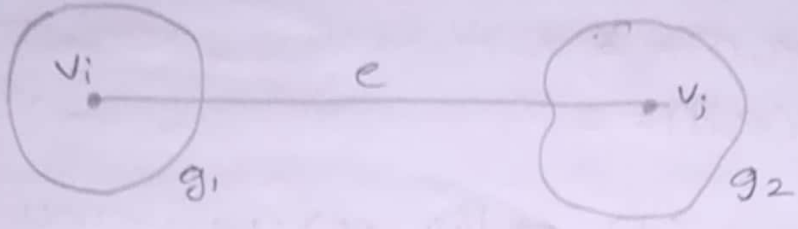
A connected graph said to be minimally connected if removal of any one edge from the graph, disconnects it. So a minimally connected graph cannot have a circuit or it is a tree. Hence proved.

Theorem 5

A graph with 'n' vertices, n-1 edges and no circuit is connected.

Proof :

suppose there exists a circuitless graph 'G' with n vertices and n-1 edges which is disconnected.



~~Assume we~~

Edge e is added such that to graph G is $G = g_1 \cup g_2$

Here, G will consist of two or more circuitless components. Here G consist of 2 components g_1 & g_2.

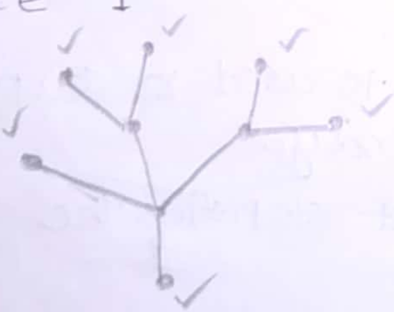
Adding an edge e to g_1 & g_2 will not create a circuit.

$\therefore$ $G \cup e$ is a tree with 'n' vertices & ~~with~~ 'n' edges.

Pendent Vertex in a Tree (leaf nodes)

All leaf nodes are pendent vertices

Pendent vertex is defined as a vertex of degree '1'



Distance & centres in a tree

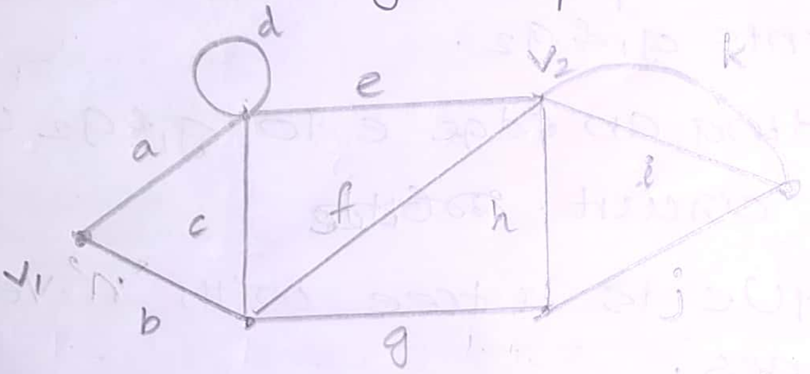
$d(a,b) = 1$

$d(a,c) = 2$

$d(a,d) = 2$

Distance (d), $d(v_i, v_j)$: b/w any 2 vertices v_i & v_j is the length of the shortest path or no. of edges in the shortest path b/w them.

? Identify the shortest path b/w vertices v_1 & v_2 in the following graph.



Paths : (a, e), (b, c, e), (b, f)

Shortest paths : (a, e), (b, f)

length = $d(v_1, v_2) = 2$

Metric

A function $f(x,y)$ is used to compute the distance b/w 2 vertices x & y.

Metric is a fn that statisfies the following 3 conditions:

* Non-negativity : $f(x,y) > 0$ and $f(x,y) = 0$, iff $x = y$

*Symmetry : $f(x,y) = f(y,x)$

*Triangle inequality: $f(x,y) \leq f(x,z) + f(z,y)$ for any z

Theorem 6

The distance b/w the vertices of a connected graph is a metric.

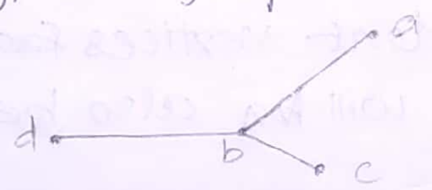
Eccentricity

Eccentricity of a vertex $E(V)$ in a graph G is the distance from v to the vertex furthest from v.

In mathematically,

$$E(V) = \max_{v_i, v} d(v, v_i)$$

? Determine the eccentricity of all the vertices in the given graph.



$E(a) = 2$

$E(b) = 1$

$E(c) = 2$

$E(d) = 2$

Centre of graph

Centre of graph 'G' is a vertex with minimum eccentricity in G.

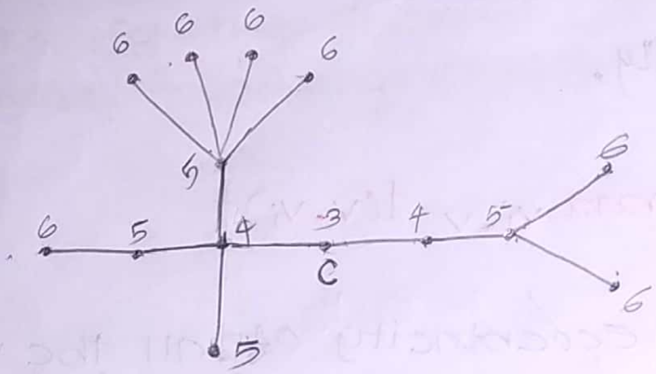
Theorem 7

Every tree has either one or two centres.

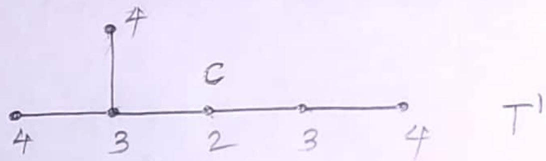
Proof:

The maximum distance max $d(v_i, v)$ from a given vertex v to any another vertex v_i occurs only when v_i is the pendent vertex.

Let us start with a tree 'T' having more than 2 vertices. The tree ~~must~~ must have 2 or more pendent vertex.



Delete all the pendent vertices from 'T' and the resulting graph T' will ~~be~~ also be a tree.



From T', again remove all the pendent vertices and mark the centre.

We continue this process ~~until~~ until we are left with an vertex or an edge.

We continue the process until have a ~~Pe~~ Vertex or Edge.

Property

If a tree has two centers, two centers must be adjacent.

Radius and Diameter of a tree.

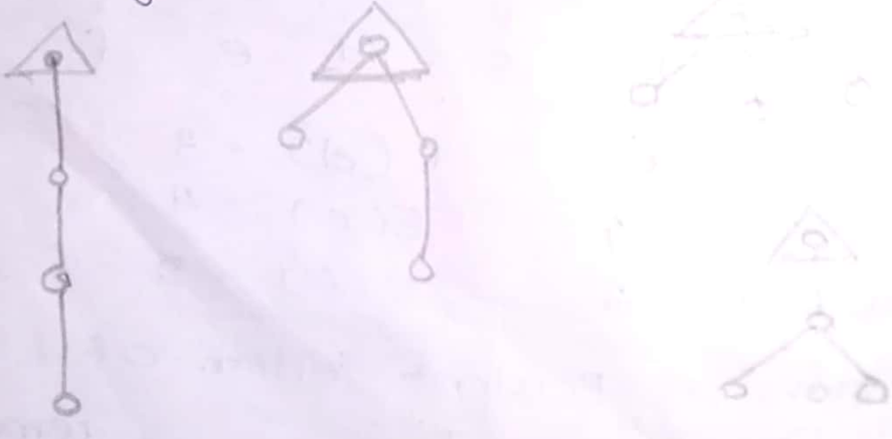
→ Radius : The eccentricity of the centre of a tree is called the radius of the tree

→ Diameter : Diameter of tree is defined as the length of longest path in tree.

Rooted tree :

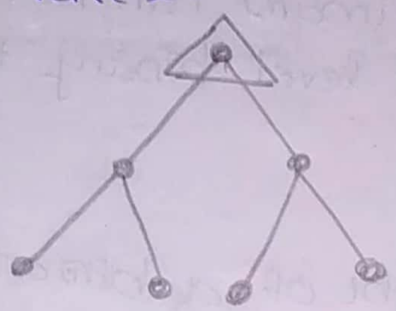
~~Ro~~ A tree in which one vertex called root is distinguished from all other vertex called Rooted tree.

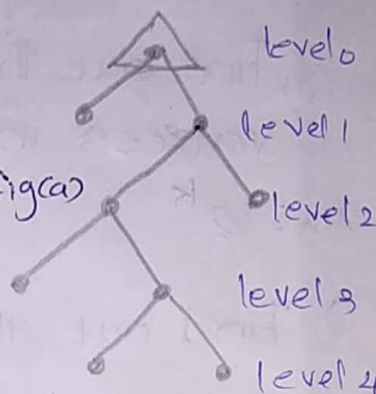
The following figures show different rooted tree having 4 vertices



A binary tree is defined as a tree in which there is exactly one vertex (root) of degree (2) and all remaining vertices are of degree 1 or 2

Draw a binary tree and mark the root vertex.





(fig(a))

Every Binary tree is a rooted tree

<u>Applications of Binary Tree.</u>

→ Searching
→ Sorting.

Internal vertex: A non pendent vertex in a tree called an internal vertex of a tree.

The number of internal vertices in a binary tree ~~num~~ is 1 less than the number of pendent vertices.

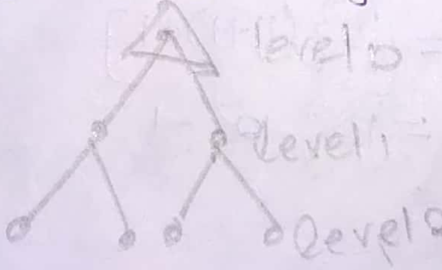
Consider fig (a)

no of pendent vertex = 5
no of internal vertex = 4

level <u>of a ~~tree~~ vertex of tree</u>

In a binary tree a vertex 'v_i' is said to be level 'l_i' if ~~at ve~~ v_i is at a distance l_i from the root

root vertex ~~node~~ is always at level 0.



level to binary tree

figure (a) is 9 vertices & 4 level binary tree

Q; Find out the maximum number of vertices in a k level binary tree.

Ans: 2^k

Q; Find out the height of a binary tree.

maximum level of a binary tree is called the height of a binary tree

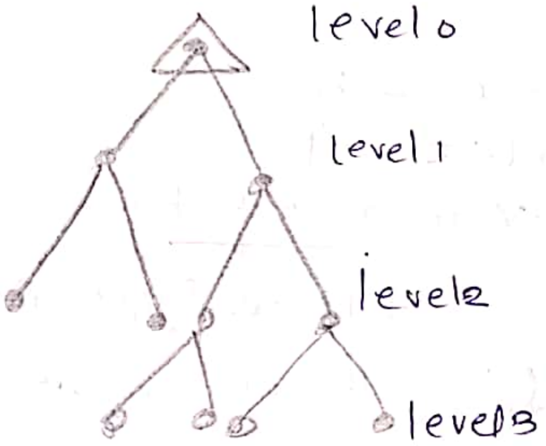
height of tree fig (a) is 4.

Minimum possible height of an n vertex binary tree ~~l_{max}~~

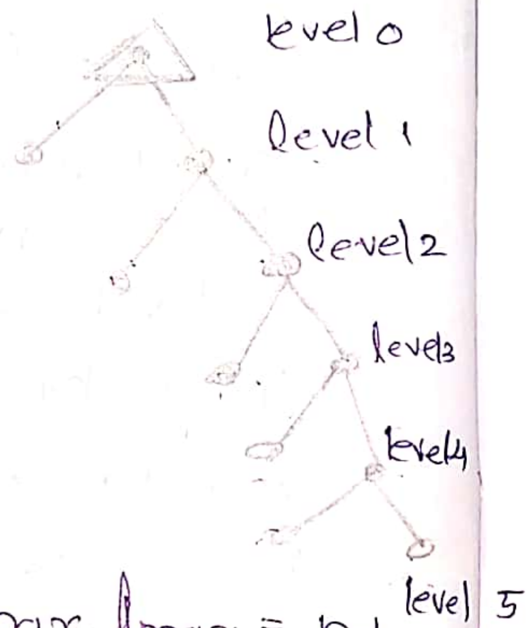
$$\min l_{max} = \lceil \log(n+1) - 1 \rceil$$

$$\max l_{max} = \frac{n-1}{2}$$

balance binary tree.



(skew tree)
Imbalance binary tree.



$$\min l_{max} = \lceil \log(n+1) - 1 \rceil$$
$$= \lceil \log(11+1) - 1 \rceil$$
$$= \log 12 - 1$$
$$=$$

$$\max l_{max} = \frac{n-1}{2}$$
$$= \frac{11-1}{2} = \frac{10}{2}$$
$$= 5$$

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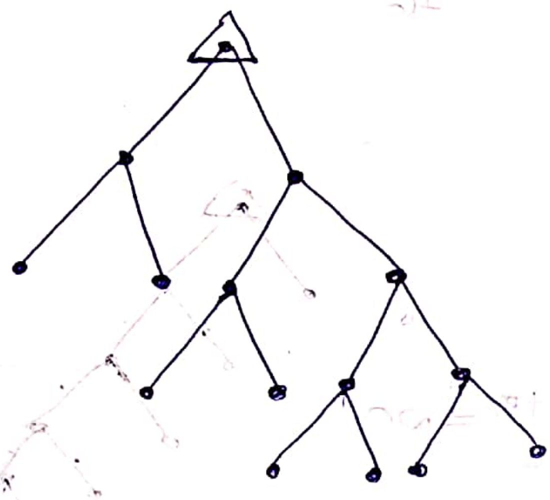
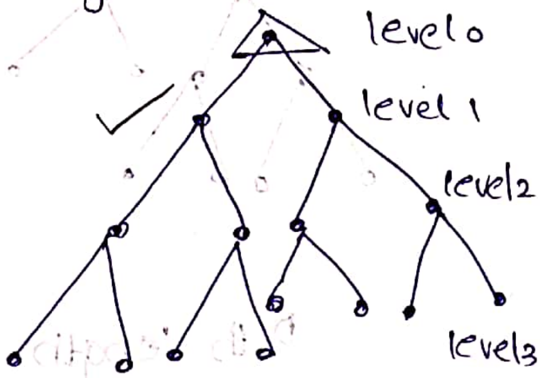
Q; Construct a binary tree with 15 nodes

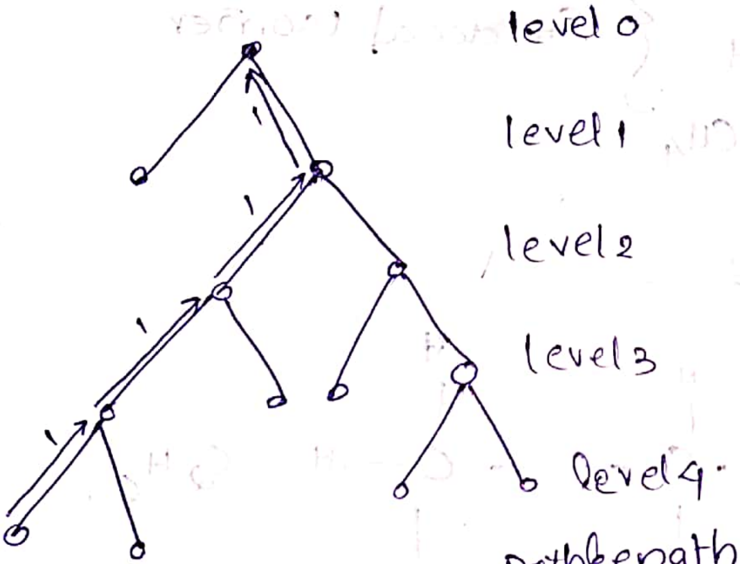
20/8/19 Such that the tree must have minimum height also find the height of the tree?





Path length (External Path length)

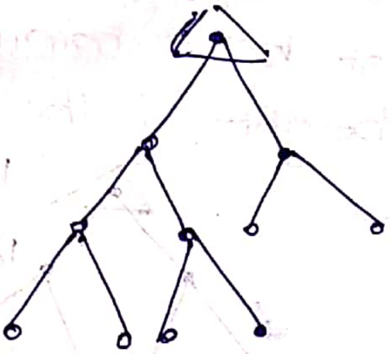
It is defined as the sum of path lengths from root to all pendent vertices or the Sum of level of all the pendent vertices



1+4+4 +3+3 + 4+4 = 23

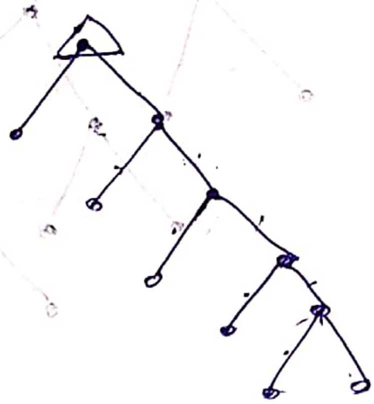
Q; Find the path length of trees ;

The Path length = $\sum$ length of pendent Vertex to the root node



Path length.

$= 3+3+3+3+2+2$
$= 16$

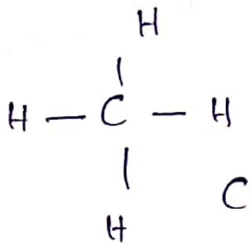


Path length.

$= 1+2+3+4+5+5 = 20$

Counting Trees

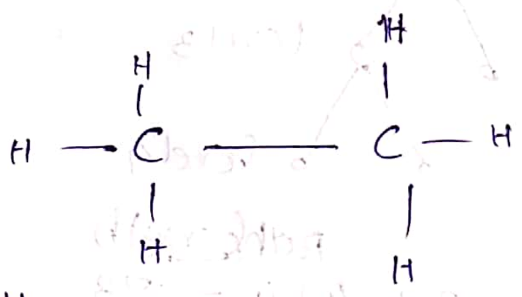
Developed by Arthur Cayley; He discoverd tree by trying to count the structural isomers of saturated hydrocarbons. $C_k H_{2k+2}$



Structural isomer

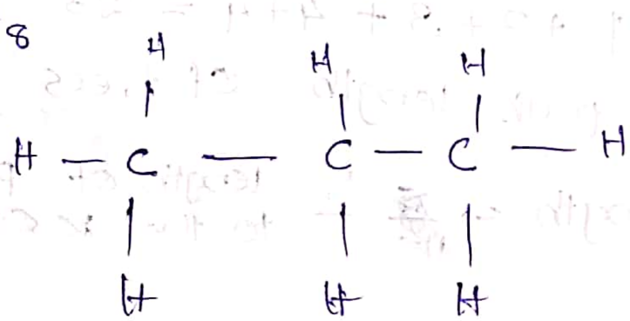
CH_4

C_2H_6



C_2H_6.

C_3H_8



corresponding to their chemical valancy the carbon atom can be represented a vertex ~~hav~~ by a degree 4 and Hydrogen atom by a degree 1.

Total no of vertices in such a graph is;

$n = 3k+2$

And the total number of edges is; $e = \frac{1}{2}$ (sum of degrees)

$= \frac{1}{2}(4k+2k+2)$

$= 3k+1$

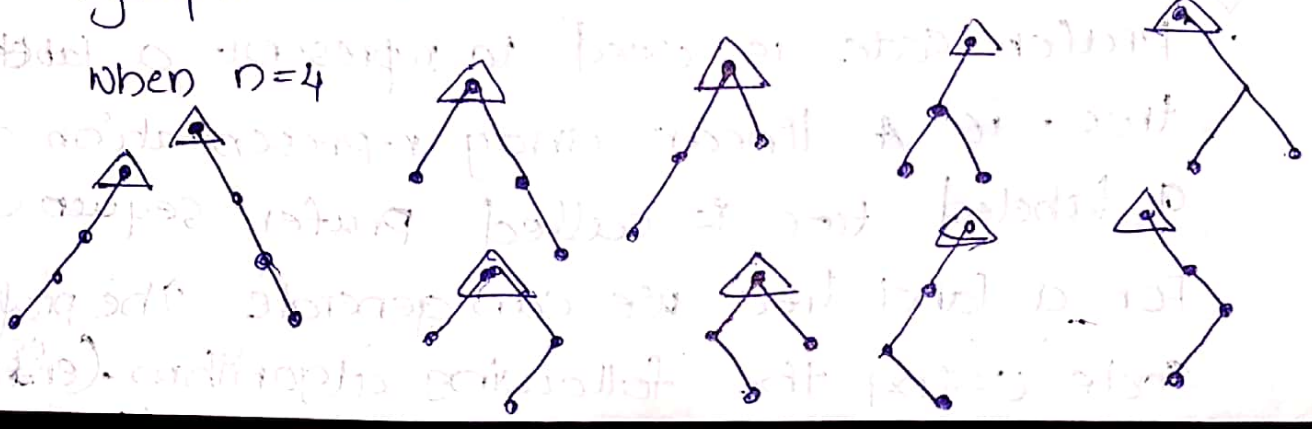
So the number of edges is one less than the number of vertices. And it is a connected graph. ∴ Such a graph is a tree. Thus the problem of counting structural isomers of a given hydro carbon is become a problem of counting trees.

Q; What is the number different trees that one can construct with n distinct vertices.

~~of n~~ labeled graph

A graph in which each vertex is assigned a unique label or a name such that no two vertices have the same label is called the label graph.

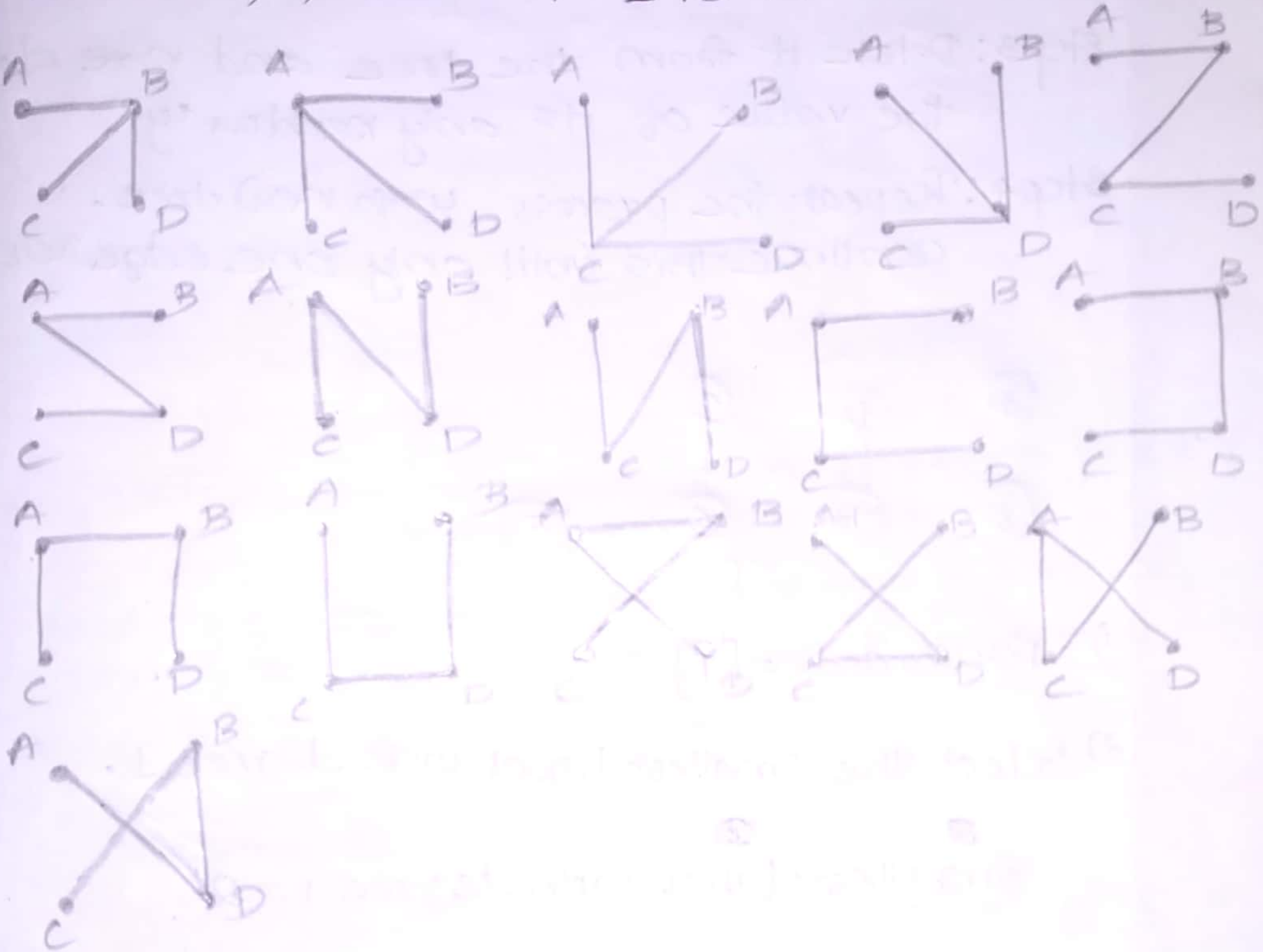
When $n=4$



We have cayley's formula, there are n^{n-2} label trees of order n.

Let us draw different trees for n=4.

we have, $4^{(4-2)} = 4^2 = 16$ label trees.



Prufer Sequence / Prufer Code

It is used to represent a label tree. ie, a linear array representation of a label tree is called prufer sequence. ~~There are~~

2

For a label tree we can generate the Prufer code using the following algorithm. (encoding).

Algorithm

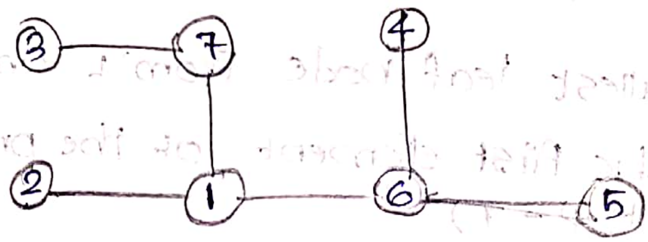
1) Take any tree 'T' whose vertices are labeled from 1 to n.

2. Take the vertex 'x' with a smallest lablel whose degree = 1.

3. Delete it from the tree and note down the value of its only neighbour 'y'

4. Repeat the process with new tree continue this until only one edge remains

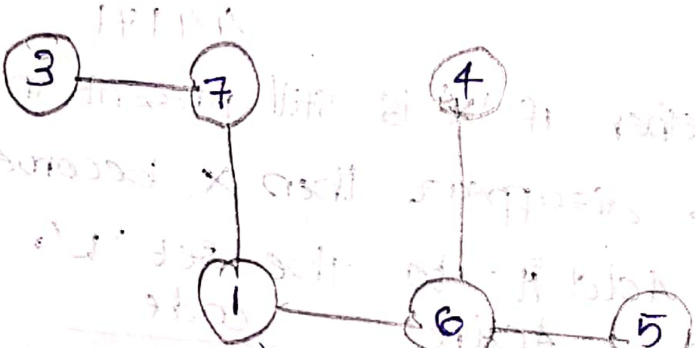
⇒ Consider the tree;



1. Prufer code [ ] /Array

2. Smallest labeled vertex whose degree is 1; which is the vertex '2'.

3.

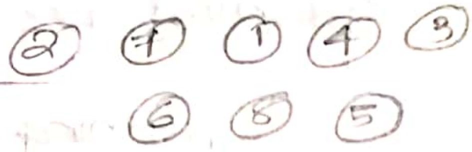


- ALGORITHM FOR CONSTRUCTING A LABELED TREE FROM PRUFER CODE

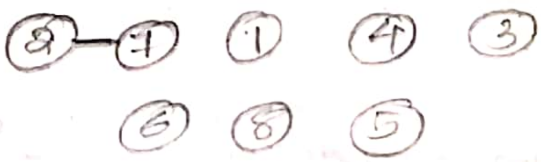
1) Find the node no.s which are missing in the prufer code. These are the leafs of the tree.
Let 'L' be the set of leaf nodes.

Eg: consider the
Prufer code : 744171

Used	Active	Code
	2, 3, 5, 6, 8	744171



2) Select the ~~largest~~ smallest leaf from 'L' and connect it into the 1st elt of the prufer code.

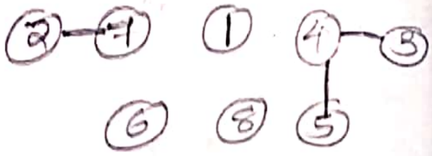


3) Remove 1st elt 'X' from prufercode

Used	Active	Code
	2, 3, 5, 6, 8	744171
2	3, 5, 6, 8	44171

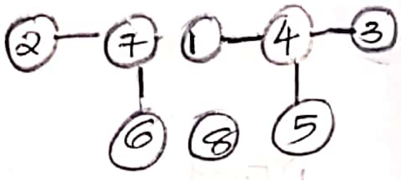
4) Check whether 'x' is till present in the prufer code. If it has disappeared, then 'x' becomes the leaf itself: Add it to the set 'L'

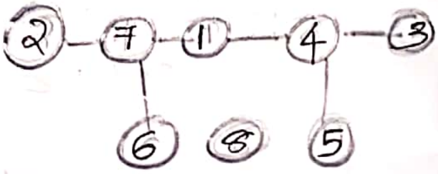
Used	Active	Code
	2,3,5,6,8	744171
2	3,5,6,8	44171
2,3	5,6,8	4171
2,3,5	6,8	171



4 doesnot repeat, so add 4 to the list 'L'

2,3,5	4,6,8	171
2,3,4,5	6,8	71
2,3,4,5,6	7,8	1
2,3,4,5,6,7	1,8	





Spanning Tree

Spanning Tree of a connected ~~to~~ graph 'G', if T is a subgraph of 'G' and 'T' contains

A disconnected graph with 'k' components has a spanning trees.

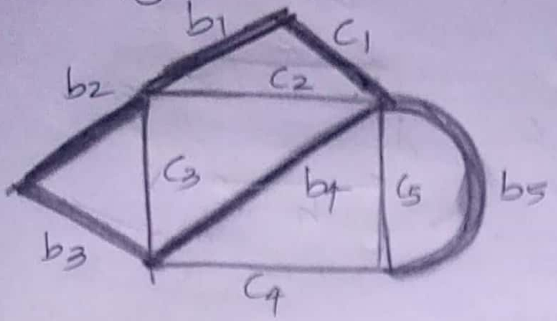
Branch

An edge in a spanning tree 'T' is called the branch of tree.

Chord

An edge of 'G' that is not in the given

spanning tree 'T'



Rank:

Rank of a graph is defined as $r = n - k$

where n = no. of vertices

k = no. of components.

Nullity:

$$\boxed{\mu = e - n + k}$$

* For a spanning tree no. of components $= k$.

* Rank of G = no. of branches in any spanning tree of $G = n-1$

* Nullity of G = no. of chords $= e - n + 1$

* $\boxed{\text{Rank} + \text{nullity} = n - 1 + e - n + 1 = e}$ → no. of edges

25/09 **Fundamental circuit.**

It is a ckt which is form by adding a chord ~~code~~ to a spanning tree.

~~Finding all the Spanning trees of a graph. We can generate one~~

Finding all the spanning trees of a graph.

We can generate one spanning tree from other by the addition of chords and deletion of appropriate branches. This is called cycle interchange or elementary tree transformation.

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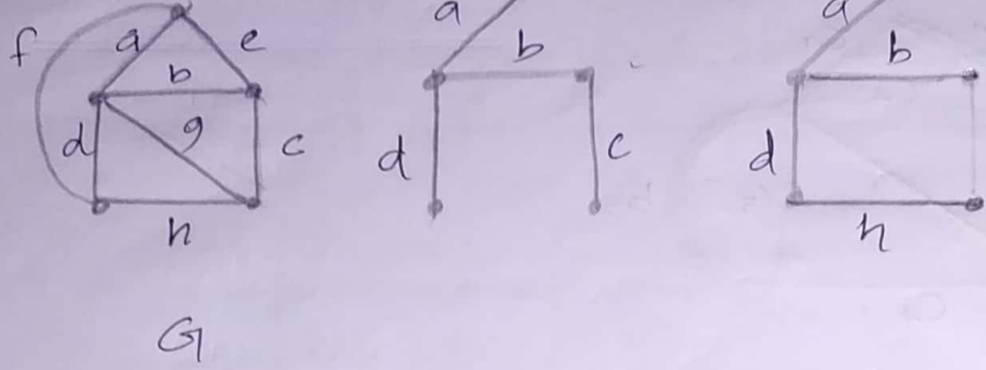

Distance b/w two spanning trees E_i & E_j of a graph G is defined the no. of edges in G present in one tree but not in other.

Ring sum:

Ring sum of 2 spanning trees T_1 and T_2 is the subgraph of G containing all edges of G that are either in T_1 or in T_2 but not in both.

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