4/11/17

5. Matrix Representation of graph.

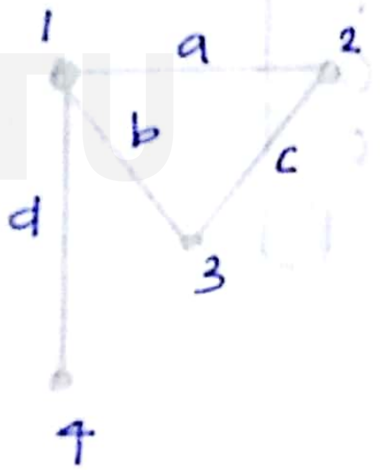
Incidence matrix

G → n vertices m/e - edges, no self loops

$[A_{ij}] = 1$ if e_j is incident v_i
else

n vertices → rows
m edges → columns $(n \times m)$

binary matrix.



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{m \times n} \end{array}$$

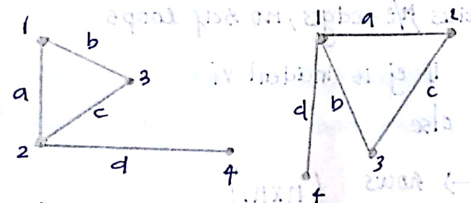
* For each ~~colowmn~~ column there will be two one's in every incidence matric

* ~~Each row~~ No: of ones in each row indicates it's degree.

$$A(G) = \left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array}\right]$$

→ THEOREM:

Two graphs g_1 & g_2 are isomorphic if and only if it incidence matrix $A(g_1)$ and $A(g_2)$ differ only by permutations of rows and columns.



G_1 G_2

$$\begin{array}{c|cccc} & a & b & c & d \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{array} \qquad \begin{array}{c|cccc} & a & b & c & d \\ 1 & 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{array}$$

→ THEOREM

If $A(G)$ is an incidence matrix of a connected graph G with end vertices rank of $\boxed{A(G) = n-1}$

Proof (consider above G_2 graph)

rows of $A(G) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$

sum of each column $= 0$

modulo 2
$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \to A_1 \\ \to A_2 \\ \to A_3 \\ \to A_4 \end{array}$$
$\underline{\underline{0}}$ $\underline{\underline{0}}$ $\underline{\underline{0}}$ $\underline{\underline{0}}$

not ~~Indepen~~ Linearly Independent

$A_4 + A_3 + A_2 = 1$

Sum of vectors $= 0$

Linear ~~Independent~~ dependent: if it can be represented as a Scalar ~~linear~~ multiple / Linear Combination

$A(G)$

$a = [1 \ 2 \ 3]$ $b = [2 \ 4 \ 6]$

$c = [4 \ 5 \ 6]$ $d = [5 \ 7 \ 9]$

$b = 2a$ ∴ linearly ~~independent~~ dependent

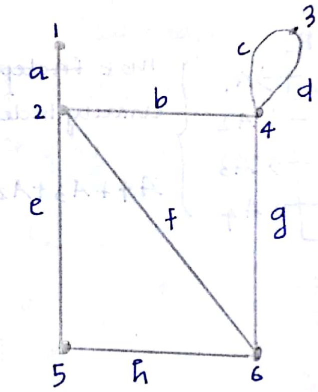
a & c are linearly independent

$a + c = d$

∴ rank of matrix less than n.

$r \leq n-1$; max $(n-1)$

- Incident matrix can be divided into two rows. m rows & $n-m$ rows.



$$
\begin{array}{c|cccccccc}
 & a & b & c & d & e & f & g & h \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
3 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
4 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
6 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}
$$

In a $n \times e$ matrix, ~~it~~ n-1 rows are enough to define the matrix.

$$
\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Reduced}
$$

THEOREM

Reduced incident matrix of a tree is non-singular

Tree → n vertices → (n-1) Edges

$n \times (n-1)$ matrix

redu ↓

$(n-1) \times (n-1)$ square matrix.

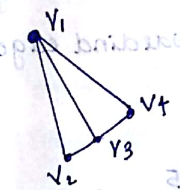
Adjacency Matrix:

A graph G with n vertices and no parallel edge

Adjacency matrix can be defined as.

$X_{ij} = 1$ if $v_i \longleftrightarrow v_j$

$X_{ij} = 0$ else.



$$
\begin{array}{c|cccc}
 & V_1 & V_2 & V_3 & V_4 \\
\hline
V_1 & 0 & 1 & 1 & 1 \\
V_2 & 1 & 0 & 1 & 0 \\
V_3 & 1 & 1 & 0 & 1 \\
V_4 & 1 & 0 & 1 & 0
\end{array}
$$

Properties.

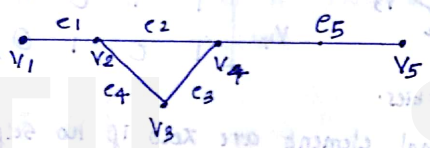
- diagonal elements are zero if no self loop
- diagonal element is one if self loop is present.
- No meaning for parallel edges.
- degree of vertex = no: of one's in row & column.
- Both rows and columns ~~are~~ must be arranged in same order for permutation.
- If G_1 and G_2 are isomorphic then adjacency matrix of G_2 is given by .

$X(G_2) = R^{-1} \cdot X(G_1) \cdot R$

$$X(G) = \left[\begin{array}{c|c} X(g_1) & 0 \\ \hline 0 & X(g_2) \end{array}\right]$$

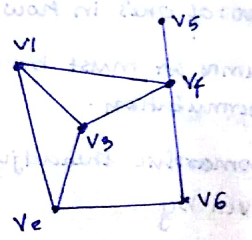
Edge Sequence:-

A sequence of edges for each edge except starting and ending vertex. ~~each edge~~ there is a ~~incident~~ vertex common with a forwarding edges & preceding edge.

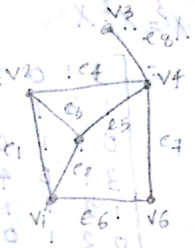


$V_1 e_1 V_2 e_2 V_4 e_3 V_3 e_4 V_2 e_2 V_4 e_5 V_5$

$e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5$



$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$



3/11/2017

Theorem

Let x be a adjuscenty matrix in G. ij^{th} entry X^K is the no: of diff edge sequences of K edges b/w vertices v_i & v_j

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} =$$

$$X^2 = \begin{bmatrix} 3 & 1 & 0 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 1 & 2 & 1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 0 & 2 & 2 \end{bmatrix}$$

- edge sequence of $v_1, v_4 = 3$
- length $= 2$

$X^3 = X^2 \cdot X$

$$= \begin{bmatrix} 3 & 1 & 0 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 1 & 2 & 1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 7 & 3 & 2 & 7 & 6 \\ 7 & 4 & 1 & 8 & 5 & 2 \\ 3 & 1 & 0 & 4 & 1 & 6 \\ 2 & 8 & 4 & 2 & 8 & 7 \\ 7 & 5 & 1 & 8 & 4 & 2 \\ 6 & 2 & 0 & 7 & 2 & 0 \end{bmatrix}$$

4/11/17

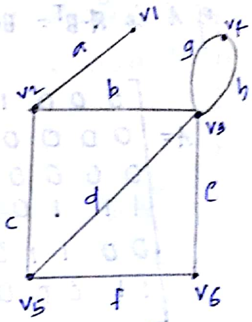
THEOREM:

Let X be the adjusency matrix of a graph G with n vertices $Y = X + X^2 + X^3 + \dots + X^{n-1}$. Then G is connected if and only if for all i,j there exist all distinct i,j where $i \neq j$ & $Y_{ij} \neq 0$.

THEOREM:

In a connected graph distance b/w two vertices v_i & v_j is K ∀ K is a smallest integer for which i,j^{th} the entry is X^K is non-zero

6/11/17 Circuit matrix:-



ckts = $\{c\,b\,e\,f\}$, $\{c\,b\,d\}$, $\{f\,e\,d\}$, $\{g\,h\}$

$b_{ij} = 1$ c_i = include e_j

$= 0$ else.

$$b_{ij} = \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

rows = no: of circuits

columns = edges

consider 'a' to contains no ones – no circuit

column with only one '1' ⇒ self loop.

Permutation of matrix = reliability of rows &

THEOREL

Let B and A be a circuit and incidence matrix ~~respect~~ respectively. Whose.

columns are arranged using same order of edges then every row of B is orthogonal to every row of A ie $A \cdot B^T = B \cdot A^T = 0$

$$A = \begin{bmatrix} 0&0&0&1&0&1&0&0 \\ 0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&1 \\ 1&1&1&0&1&0&0&0 \\ 0&0&1&1&0&0&1&0 \\ 1&1&0&0&0&0&0&0 \end{bmatrix} \quad B = \begin{bmatrix} 0&1&1&0&1&1&0&0 \\ 0&1&1&1&0&0&0&0 \\ 0&0&0&1&1&1&0&0 \\ 0&0&0&0&0&0&1&1 \end{bmatrix}$$

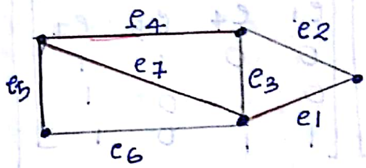
$$B^T = \begin{bmatrix} 0&0&0&0 \\ 1&1&0&0 \\ 1&1&0&0 \\ 0&1&1&0 \\ 1&0&1&0 \\ 1&0&1&0 \\ 0&0&0&1 \\ 0&0&0&1 \end{bmatrix}$$

$$A \cdot B^T = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}$$

$$A \cdot B^T = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$

Consider V_6 and circuit $\{c, d, b\}$ then $\forall$ e and f are not part of circuit. Then consider vertex within the circuit. Each vertex is incident with only two edges.

Fundamental Circuit matrix.



only one chord

fundamental circuit $= \{e_5, e_6, e_7\}\ \{e_4\ e_7\ e_3\}$
$\{e_4, e_7, e_1, e_2\}$

other circuits $= \{e_4\ e_5\ e_6\ e_3\}\ \{e_3\ e_2\ e_1\}$
$\{e_4\ e_5\ e_6\ e_1\ e_2\}$

circuit matrix $B_{ij} =$

$$\begin{array}{c|ccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \hline 1 & 0&0&0&0&1&1&1 \\ 2 & 0&0&1&1&0&0&1 \\ 3 & 1&1&0&1&0&0&1 \\ 4 & 0&0&1&1&1&1&0 \\ 5 & 1&1&1&0&0&0&0 \\ 6 & 1&1&0&1&1&1&0 \end{array}$$

fundamental circuit linearly independent.
Combination of any two fundamental circuit forms other three ~~to~~ circuit which is linearly dependent.

e_2, e_3, e_6 are chords

$$\left[\begin{array}{ccc|cccc} e_6 & e_5 & e_2 & e_1 & e_4 & e_5 & e_7 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

B_f B

9/11/17   A theorem if B is a ckt matrix of a connected graph G with e edges and n vertices rank of B is $e-n+1$

B_f submatrix B.

rank of $B \geq e-n+1$

$A \to$ Incidence matrix

$A \cdot B^T = 0$.

rank of A + rank of $B \leq e$

rank of $B \leq e -$ rank of A

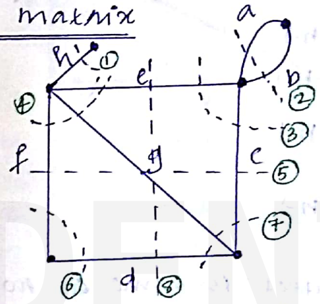
$\leq e-(n-1)$

~~Rank of~~ $\leq e-n+1$ — (2) we know

we know rank of A (incident matrix) $= n-1$.

Rank of $B = e-n+1$

<u>Cut set matrix</u>



$C_{ij} = 1$ i^{th} cut set includes j^{th} edge

$= 0$ else.

cut set $= \{f, g, e\}$

rows - cut set

column - Edges

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \left[\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right] \end{array}$$

properties

Column without 1 is selfloop.

permutation is cinbale to edges.

parallel edge will have identical column.

THEOREM.

Rank of cut set matrix $C(G)$ = rank of incident matrix $A(G)$ = rank of Graph G.

Rank of $G = n-1$

" " $A(G) = n-1$

" " $C(G) = n-1$

Rank of $C(G) \geq n-1$

Consider 6^{th} cutset is same as row of incident matrix but in 4^{th} cut set h is not included but in incident matrix included.

when S & Γ) even

$C \cdot B^T = B \cdot C^T = 0$.

rank of C + rank of $\underbrace{B}_{e-n+1} \leq e$

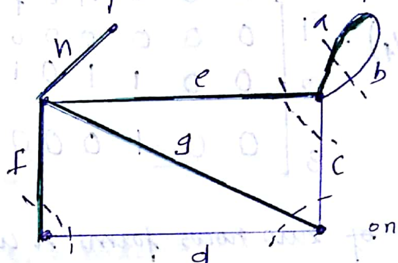
rank of $C \leq e - (e-n+1)$

rank of $C \leq n-1$ — (2)

rank of $C = n-1$

Fundamental cut set matrix

It is represented as $C_f = \left[\underbrace{C_c}_{(n-1)\times\mu} \;\vdots\; I_{n-1} \right]$ chords branches$(n-1)$ $(n-1)\times(n-1)$



green ST.

one branch

Fundamental cut set = {f d} {a, b} {e c} {c g d}

$$
\begin{array}{c}
\begin{array}{ccccccccc} & \multicolumn{3}{c}{\text{chords}} & & \multicolumn{5}{c}{\text{branches}} \end{array} \\
\left[\begin{array}{ccc|ccccc}
b & c & d & a & e & f & g & h \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

$(n-1)\times\mu$: $(n-1)\times(n-1)$: (5×5).

(5×3)

a = 1 0

2^{nd} — {a b}

3^{rd} {c e}

from cutset before

6^{th} {d f}

7^{th} {c g d}

1^{st}. {h}

→ PATH MATRIX (P) : a pair of vertices P(x, y)

$P_{ij} = 1$ i^{th} path include j^{th} edge

$= 0$ else

Path from $V_3 \to V_4$

Paths = { h, e } { h f d e } { h g e }
(1) (2) (3)

$$P(V_3,V_4) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Combining of two rows forms a circuit

THEOREM

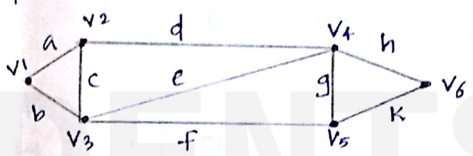
If the edges of a connected graph are arranged in the same order for the columns of the incidence matrix A and path matrix $P(x,y)$. Then the mod 2 product $A \cdot P^T(x,y) = M$ where the matrix M has ones in two rows of x and y. And rest of $n-2$ rows are all zeros

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 8}$$

$$P^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{8\times 3}$$

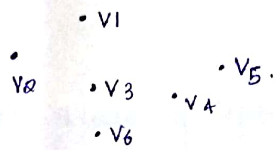
$$A \cdot P^T = \begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

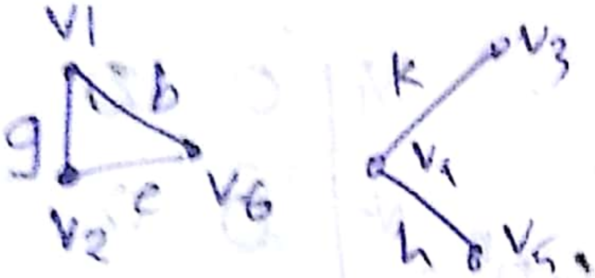
1) Write path matrix



(2) Draw A, X, B, B_f, C & C_f of a complete graph

(3) Draw incidence matrix.





matrix represent as

$$\left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array}\right]$$