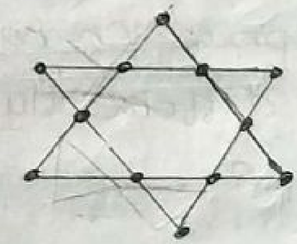
Module 2

Euler Graph

Euler propose that "if some closed walk in a graph contains all the edges of the graph, then the ~~graph~~ walk is called an Euler line: and the graph that contains Euler line is called Euler graph." Euler graph's don't have any isolated vertices and they are always connected.

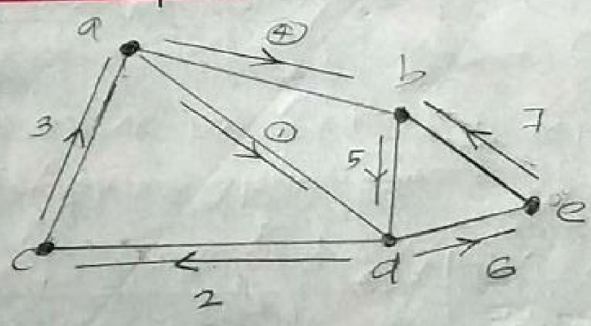
Euler ~~the~~ graph



Unicursal Line

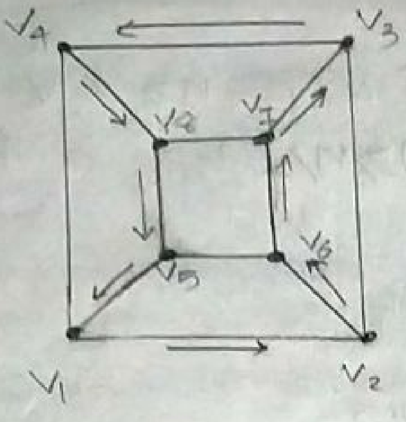
An open walk that includes all edges of a graph without replacing any edge is called open Euler line or Unicursal line.

A graph that has unicursal line is called Unicursal graph.



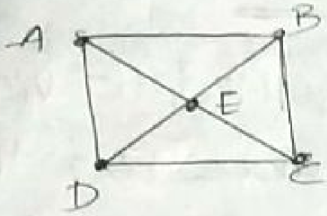
Hamitonian Circuit

A circuit in a connected graph 'G' is said to be Hamiltonian, if it includes every vertex of 'G'



Hamiltonian Path

26/08 Hamiltonian circuits are named after William Rowan Hamiton.



Examples of Hamiltonian Path

A, B, E, C, D

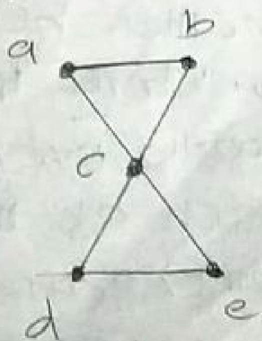
E, B, A, D, C

Example for Hamiltonian circuit:

E, B, A, D, C, E

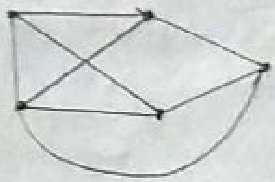
Arbitraarily Traceable Graphs

A vertex 'v' in an Euler graph has a property that an Euler line is always obtain when one follows any walk from vertex such a graph is called an Arbitrarily Traceble graph from vertex v.



Dirac's Theorem for Hamiltonicity.

Every Graph 'G' with $n \geq 3$ vertices and minimum degree $d(G) \geq n/2$ has a hamiltonian cycle.



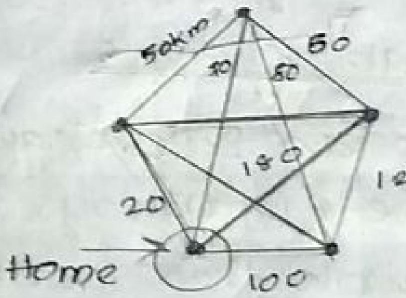
$n=5$

$n/2 = 2.5 < 3$

Minimum degree $= 3$

Travelling Salesman Problem

A salesman is required to visit a no. of cities. given the distance b/w the cities, in what order should he travelled so has to visit every city exactly once and return home. with minimum distance travelled?



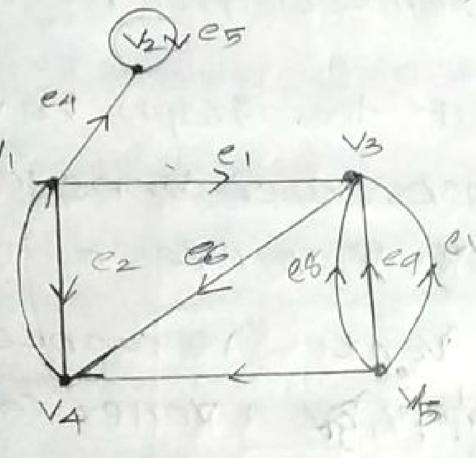
We can represent the cities by vertices & the roads connecting these cities by edges. Here we use a weighted graph. Where the weight or the distance is associated with every edge.

In this problem, if each of the ~~city~~ cities has a road to other cities. We have completed weighted graph. such a complete weighted graph have ~~a~~ several hamiltonian ~~ckt~~ cycle. We only choose the ckt which result in smallest sum of results.

Theorectically, the problem of travelling sale's man can be solved by eliminating all the $\frac{(n-1)!}{2}$ hamiltonian ckt, calculating the distance travelled in each and choosing the shortest one.

Directed Graph (Digraph)

A directed graph 'G' consists of set of vertices $V = \{v_1, v_2, v_3 \dots\}$, ~~and~~ set of edges $E = \{e_1, e_2, e_3 \dots\}$ & a mapping ϕ, that maps every edge on ~~the~~ to some ordered pair vertices v_i, v_j.



? Calculated indegree & outdegree of vertices v_1, v_2, & v_5

If there is an edge e_k from vertex v_i to v_j the vertex v_i is called initial vertex of v_i and the vertex v_j is called terminal vertex of v_j.

? Calculated indegree & outdegree of vertices v_1, v_2 & v_5

H.W

Indegree $(v_1) - 1$ outdegree $(v_1) = 3$

" $(v_2) - 2$ outdegree $(v_2) = 1$

" $(v_5) - 0$ " $(v_5) = 4$

Ore's Theorem

27/08

The graph 'G' is arbitary traceble from a vertex 'v' iff every cycle in 'G' Contains 'V'.

? Prove that the sum of degrees of the Vertices of any finite graph is even.

Either the degree of two vertices is increased by 1 or degree of 1 vertex is $\uparrow$sed by 2.
$\therefore$ Sum of degree of vertices is always increased by 2.

? Show that every simple finite graph has 2 vertices of the same degree.

Assume that the graph have 'n' vertices. Each of those vertices are connected to either 0, 1, 2, ... n-1 other vertices.

If any of vertices is connected to n-1 vertices, there cannot be a vertex which is connected to 0 vertices. It is impossible have a graph with n vertices Such that 1 vertex has degree 0 and another has degree n-1.

- Hand shaking Problem

Show that if n people at in the party and some shake hand with others but not with themselves then at the end there are atleast 2 people who have shaken hands with same no. of peoples.

This is similar to the previous problem. Here, each person attending the party is vertex and every shake hand is an edge.

? Prove that a complete graph with 'n' vertices contains $\frac{n(n-1)}{2}$ edges.

proof by induction.

if $n=1$,

edges $=0$

$\therefore P(1)$ is true.

Assume that it is true for $n=k$.

ie, $P(k)$

ie, A complete graph with k vertices has $\frac{k(k-1)}{2}$ edges.

Now, let $n=k+1$

Then complete graph with $k+1$ vertices has

$$P(k+1) = \frac{(k+1)(k+1-1)}{2} = \frac{(k+1)k}{2} \text{ edges.}$$

Hence proved.

<u>One's theorem proof:</u>

A connected graph is an Euler graph iff every vertex have even degree.

The Euler ckt in the graph must enter and leave vertex. Each visit to a vertex requires both entry and exist edges. So every vertex has even degree.

? Show that any graph in which degree of every vertex is even has an Euler ckt.

? Show that if there are exactly 2 vertices A & B of odd ~~ged~~ degree, there is an euler path from A to B.

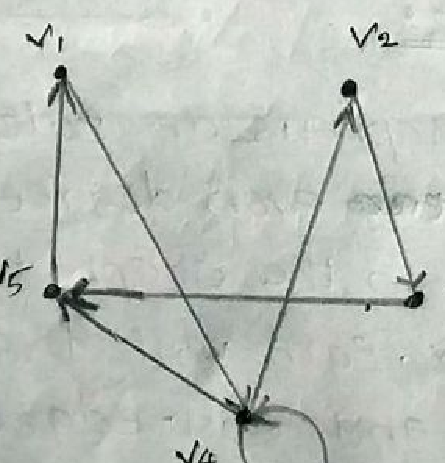
? Show that, if there are more than 2 vertices of odd degree with impossible to construct Euler path.

29/08

? Let 'D' be the digraph whose vertex set is $V = \{V_1, V_2, \ldots V_5\}$ and directed edge set is
$E = \{(V_1, V_4), (V_2, V_3), (V_3, V_5), (V_4, V_2), (V_4, V_4), (V_4, V_5), (V_5, V_1)\}$

Draw the digraph for each of the vertices give indegree and outdegree.

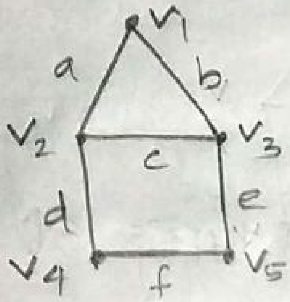
Vertices	V_1	V_2	V_3	V_4	V_5
outdegree $d^+(v)$	1	1	1	2	2
indegree $d^-(v)$	1	1	1	2	2



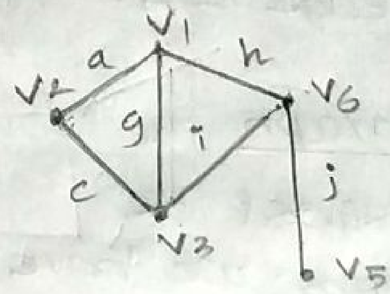
operations of graph

1) Union & Intersection.

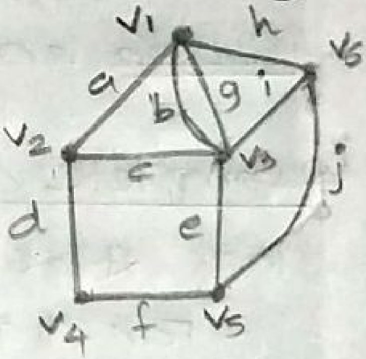
Union of 2 graphs $G_1=(V_1,E_1)$ & $G_2=(V_2,E_2)$. Another graph G_3 such that $G_3 = G_1 \cup G_2$ where $V_3 = V_1 \cup V_2$.



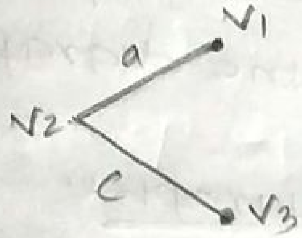
G_1



G_2



$G_3 = G_1 \cup G_2$



$G_4 = G_1 \cap G_2$

Ring sum

It is a graph consisting of the vertex set $V_1 \cup V_2$ of edges that are either in G_1 or in G_2 but not in both.

Types of digraph

1) Simple
2) assymmetric
3) Symmetric
4) Complete

1) Simple digraph

It is a digraph, it has no self loop or parallel edges

2) Assymmetric

digraphs that has atmost one directed edge which means a pair of vertices. But we allowed to have Self loop.

3) Symmetric

Digraphs in which For every edge A-B There is also an edge B-A.

4) Complete digraph

It is defined as

For digraphs, we have 2 types of complete graphs.

1) Complete symmetric digraph
2) complete asymmetric digraph.

1) complete symmetric digraph.

It is a simple digraph in which there is exactly one edge directed from each vertex to every other vertex

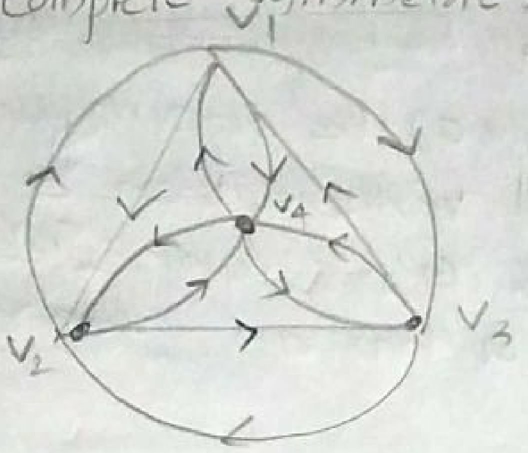
2) Complete assymmetric digraph

It is an asymmetric digraph in which has an edge b/w every pair of vertices.

→ A complete assymmetric digraph of 'n' vertices has $\frac{n(n-1)}{2}$ edges.

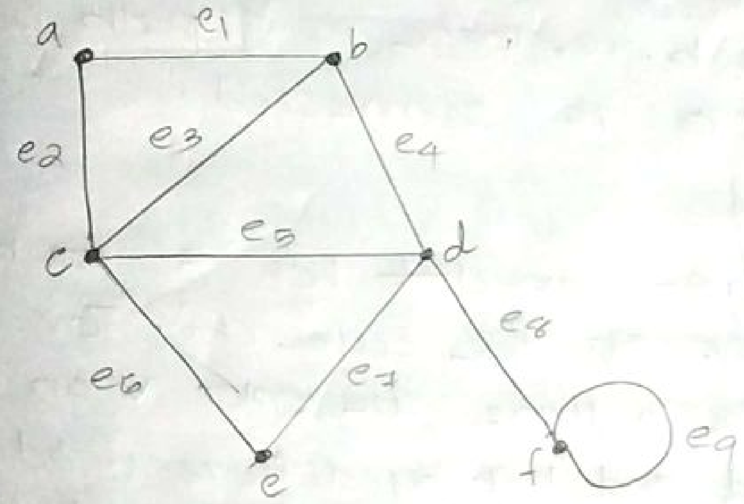
→ A complete symmetric diagraph of 'n' vertices has $n(n-1)$ edges.

Complete Symmetric D



Trail

A trail is a walk with distinct edges.



Trail: a, e_1, b, e_3 c, e_5, d, e_8, f

Check whether the given sequence is trail?

c, e_3, b, e_4, d, e_5, c, e_3, b, e_1, a, e_2, c

No, edges are repeated

Path

It is a walk with distinct vertices.

c, e_2, a, e_1, b, e_4, d

Theorem

Let $G=(V,E)$ be an undirected graph, with no isolated vertices, then G has an Euler circuit iff 'G' is connected and every vertex in G has an even degree.

Proof.

If G has an Euler Ckt, then for all $a, b \in V$ [$\forall$ $a, b \in V$], there is a trail from a and b. That part of a ckt starts at a and terminates at b. ie, there is a path from $a-b$. Hence 'G' is connected.

~~Hence 'G' is connected~~. let 's' be the starting vertex of the Euler Ckt. For any vertex $V(G)$ each time we enter V on the ckt. its ended and left by different edges thus each occurence of b consist of 2 edges or has a degree 2 or has an even degree.

Since the ckt is connected and the degree of the each vertex is even. ~~degree~~

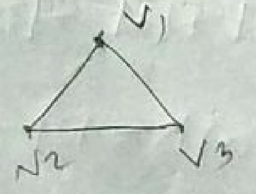
when $n=1$  degree is 2.

$n=2$

degree of $v_1 = 2$

$v_2 = 2$

$n=3$,

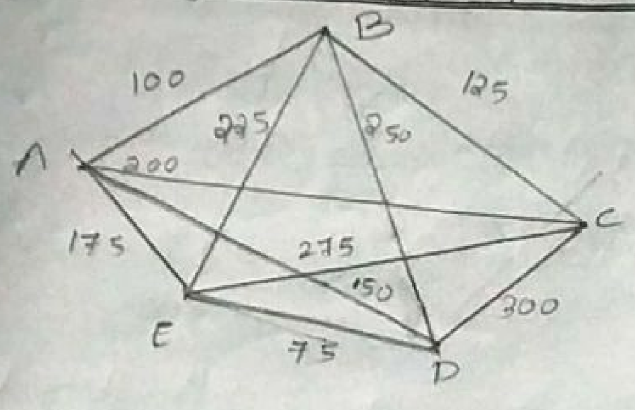


degree of $v_1 = 2$

$v_2 = 2$

$v_3 = 2$

Travelling Salesman Problem



DE – EA – AB – BC – CD Total = 775

75 175 100 125 300

BA – AD – DE – EC – CB Total = 725

100 – 150 75 275 125

CB – BA – AD – ED – DC Total = 750

125 100 150 75 300

ED – DA – AB – BC – CE Total = 725

75 150 100 125 275

DIRAC'S THEOREM

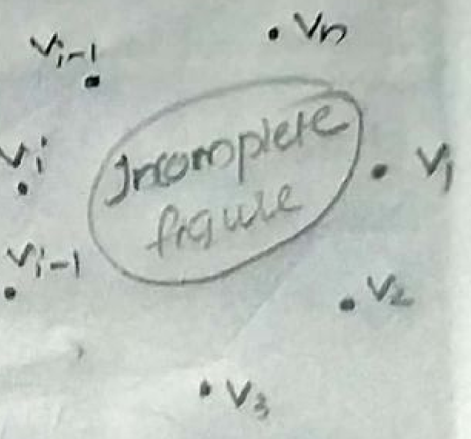
If G is a simple graph with 'n' vertices with $n \geq 3$ and degree of vertex $\geq n/2$. $[d(v) \geq n/2]$ for every vertex v, then G is hamiltonian.

Proof

Suppose that there exist a non-hamiltonian graph. Then there will be a maximal non-hamiltonian graph G with n vertices and degree of V $n/2$ $[d(v) \geq n/2]$ for every $V \in G$.

∴ There exist two non-adjacent vertices V_{i+1} and V_1.

There will be two non-adjacent vertices u & v in G. $\because G$ is maximal and non-hamiltonian. $G+uv$ should be hamiltonian.



Let 'c' be the hamiltonian cycle in $G+uv$.

Let $c = v_1, v_2, v_3, v_{i-1}, v_i, v_{i+1}, \cdots v_{n-1}, v_n, v$

Let $v_n = u$ and $v_1 = v$. Define S as $v_i \in C$.
That means there exist an edge.

$S = \{v_i \in C :$ an edge from u to $v_{i+1}\}$
$T = \{v_j \in C :$ an edge from v to $v_{j+1}\}$

If $v_n \in S$, then there exists an edge from n to v_{n+1} and there exist a loop from u to u.
This happens to be contradiction. So assumption is wrong.

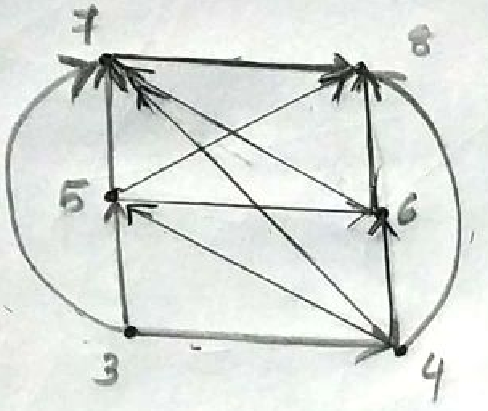
So $v_n \notin S$ ~ $v_n \notin T$

$v_n \notin S \cup T$

$|S \cup T| = |S| + |T| - |S \cap T|$

$|S| = d(n)$

? Draw a digraph that represent the relation is Greater than on the set 3, 4, 7, 5, 8.



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