PH-105 (2012)
Tutorial Sheet 2 (QM Portion)

P46: For an operator $\hat{p}$ defined below, find out $\hat{p}\psi$.

$$\hat{p} \equiv \left(\hat{x} + \frac{\partial}{\partial x}\right)^2$$

P47: Find the eigen function *φ(x)* of the following operator

$$\hat{G} = -i\hbar\frac{d}{dx} + Ax$$

Here *A* is a constant. If this eigen function is subjected to a boundary condition $\varphi(a) = \varphi(-a)$, find out the eigen values.

P48*: Find the angular momentum operator in Cartesian co-ordinate system.

P49*: Show that the expectation value of momentum for any well-behaved function is always real.

P50: If $\psi_1(x,t)$ and $\psi_2(x,t)$ are solutions of time dependent Schrödinger equation, show that $a\psi_1(x,t) + b\psi_2(x,t)$, where *a* and *b* are constants, is also a solution of the same.

P51*: If $\phi_n(x)$ are the solutions of time independent Schrödinger equation, with energies E_n, show that $\psi(x,t) = \sum_n c_n \phi_n(x) e^{\frac{-iE_n t}{\hbar}}$, where C_n are constants, is a solution of time dependent Schrödinger equation. However, show that $\psi(x,0)$ is not a solution of the time independent Schrödinger equation.

P52: There a large number (*N*) of identical experimental set-ups. In each of these set ups, a single particle is described by the wave function $\phi(x) = Ae^{-\frac{x^2}{a^2}}$, where *A* is the normalization constant, and *a* is a constant of the dimension length. If a measurement of position of the particle is carried out at time *t=0* in all these set-ups, it is found that in 100 of these the particle is found within infinitesimal interval of *x=2a* and 2*a*+*dx*. Find out in how many of the measurements the particle would have been found in the infinitesimal interval of *x*=*a* and *a*+*dx*.

P53: Suppose we have 10,000 rigid boxes of same length *L* from *x* = 0 to *x* = *L*. Each box contains one particle of the same mass. All these particles are in the ground state. If a measurement of position of the particle is made in all of these boxes at the same time, in how many of them, the particle is expected to be found between *x* =0 and *L*/*4*. In a particular box, the particle was found to be between x =0 and *L*/*4*. Another measurement of position of the particle is carried out in this box immediately after the first measurement. What is the probability that the particle is again found between *x* =0 and *L*/*4*.

P54: $|\psi_1\rangle$ and $|\psi_2\rangle$ are the normalized eigen functions of an operator $\hat{P}$, with eigen values P_1 and P_2 respectively. If the wave function of the particle is $0.25|\psi_1\rangle + 0.75|\psi_2\rangle$ at *t=0*; find the probability of observing P_1?

P55: Consider a particle confined to a one-dimensional box. Find the probability that the particle in its ground state will be in the central one-third region of the box.

P56*: For a particle in one-dimensional box of side *L*, show that the probability of finding the particle between *x=B* and *x=B+b* approaches the classical value *b/L*, if the energy of the particle is very high.

P57.: Consider a one dimensional infinite square well potential of length L. A particle is in n=3 state of this potential well. Find the probability that this particle will be observed between x = 0 and x = (L/6). Can you guess the answer without solving the integral? Explain how.

P58: Solve the time independent Schrödinger equation for a particle in one dimensional box taking the origin at its mid point and the ends at *±(L/2)*, where *L* is the length of the box.

P59: The wave function of a particle in a one-dimensional box of length *L* and potential zero at time *t=0* is given as follows.

$$\psi(x,0) = \sqrt{\frac{2}{5L}}\sin\frac{\pi x}{L} + \sqrt{\frac{8}{5L}}\sin\frac{4\pi x}{L}$$

(a) Is this wave function normalized?
(b) Is the particle in a stationary state?
(c) If the energy measurement is made on his particle, what are the possible values one might get? What are the corresponding probabilities?
(d) What is the expectation value of the energy of this particle?

P60*. The wave function of a particle in a one-dimensional box of length *L* and potential zero at time *t=0* is given as follows.

$$\psi(x,0) = A\sin\frac{\pi x}{L} + 2A\sin\frac{3\pi x}{L} + \sqrt{11}A\sin\frac{5\pi x}{L}$$

(a) Find the wave function $\Psi(x,t)$, at a later time *t*.

(b) If the measurements of energies are carried out, what are the values that will be found and what are the corresponding probabilities?
(c) If there are a large number of identical systems, each one of them represented by wave functions as above, what would be the average energy found if measurements are done in all of them at the same time
(d) If the measurement yielded the lowest value of energy, what would be the wave function later and what values of energy would be found later.

P61: A particle in a one-dimensional box (V = 0 for 0<x<L , V = ∞ elsewhere) has the following wave function at *t=0*.

$$\phi(x) = A\ \sin\frac{\pi\ x}{L}\cos\frac{2\ \pi\ x}{L} \quad \text{between } 0<x<L \text{ and } 0 \text{ elsewhere}$$

(a) Find A
(b) On making an energy measurement, which are the energy values that the particle may be found to have and with what probability

P62*: Suppose we have 10,000 rigid boxes of same length L, each box containing one particle of the same mass *m*. Let us assume that all particles are described by the same wave function. On performing the energy measurement on all the 10,000 particles at the same time we find only two energy values, one corresponding to *n = 2* and the other corresponding to *n = j*. If in 4,000 measurements, one obtains value corresponding to n=2 and if the average value of the energy found in all the measurements is $\frac{7\hbar^2\pi^2}{2mL^2}$, find the value of *j* and the wave function of the particles.

P63*: A particle in a one-dimensional well (V=0 for 0<x<L, V=∞ elsewhere) has the wave function $\phi(x) = Ax(L-x)$ inside the box and $\phi(x) = 0$ elsewhere at *t=0*. Calculate the expectation value of energy. On making an energy measurement, what is the probability of finding the particle in the ground state?

P64: A particle in a one-dimensional box (V = 0 for 0<x<L , V = ∞ elsewhere) has the following wave function at *t=0*.

$$\phi(x) = A\ \sin^2\left(\frac{\pi x}{L}\right) \text{ between } 0<x<L \text{ and } 0 \text{ elsewhere}$$

(a) Find A
(b) On making an energy measurement, what is the probability that the partle would be found in ground state.
(c) Find the mean value of the energy, if energy measurements aare made in a large number of identiacl boxes representd by this wavefunction.

You could use the following integerals.

$$\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x dx$$
$$\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4}$$

P65: For a particle in a one-dimensional box in n^{th} state, show that the uncertainty product $\Delta x \Delta p_x = \hbar\sqrt{\frac{n^2\pi^2 - 6}{12}}$.

P66: A particle of mass *m* is under the influence of a potential $V = ½\, m\omega^2 x^2$, where *x* is the displacement from origin and ω is a constant. The ground state wave function of this particle is given by $\phi(x) = Ae^{-\left(\frac{m\omega}{2\hbar}\right)x^2}$. Find the mean values $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p_x \rangle$ and $\langle p_x{}^2 \rangle$ correspomding to this wave function and the uncertainty product. For integer *k*, you may use the following integerals.

$$\int_0^\infty u^{2k} e^{-bu^2} du = \frac{1\times 3\times ...(2k-1)}{2^{k+1}}\sqrt{\frac{\pi}{b^{2k+1}}}$$
$$\int_0^\infty u^{2k+1} e^{-bu^2} du = \frac{k!}{2b^{k+1}}$$

P67: A beam of particles coming from left with energy E approaches a potential barrier given below.

$V(x) = 0$ for $x<0$
$V(x) = -V_o$ for $x \geq 0$

Obtain expressions for the transmission and the reflection coefficients and show that their sum is one.

P68: A beam of electrons with energy $E = 4\ eV$ approaches from left hand side a potential barrier defined as $V(x)=0$ for $x<0$ and $V(x)=5\ eV$ for $x>0$. Find the value of *x* inside the barrier for which the probability density is one-fourth the probability density at $x=0$.

P69*: A beam of particles with energy E approaches from left hand side, a potential barrier defined by $V=0$ for $x<0$ and $V=V_o$ for $x>0$, where $V_o>E$.

(a) Find the value of $x=x_o$ ($x_o>0$), for which the probability density is $1/e$ times the probability density at $x=0$.

(b) Take maximum allowed uncertainty Δx for the particle to be localized in the classically forbidden region as x_o. Find the uncertainty this would cause in the energy of the particle. Can then one be sure that its energy E is less than V_o.

P70: (a) A beam of particles of energy E is moving in $+x$ direction is incident on a barrier of height V_o ($V_o<E$) at $x = 0$. Find an expression of the reflection and the transmission coefficient.

(b) If the transmission coefficient in the above problem is 0.36, find E/V_o.

P71*: A beam of particles of energy E and de Broglie wavelength λ, traveling along the positive x-axis in potential free region, encounters a one-dimensional potential barrier of height $V=E$ and width L.

(a) Obtain an expression for the transmission coefficient.

(b) For what value of L (in terms of λ), will the reflection coefficient be half?

P72: A beam of particles (mass 500 keV/c^2) moving along negative x direction of energy 3 eV is incident on a potential given by

$$V=\infty \text{ for } x\le 0$$
$$V=5 \text{ eV for } 0<x\le a$$
$$V=0 \text{ for } x>a$$

The value of 'a' is equal to the de-Broglie wavelength of the particles (in the region x>a).
Write the wave function of the particles in terms of the amplitude of the incident wave. What is the ratio of the probabilities of finding the particle at x = a/2 and at x = a.

P73: A beam of particle of mass m and energy $\frac{4}{3}V_o$ (where V_o is positive constant) is incident from left on the following potential barrier.

$$V = 0 \text{ for } x < -a \text{ and for } x > +a$$
$$V = V_o \text{ for } -a \le x \le +a$$

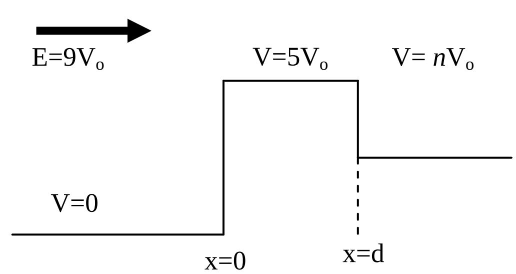
Where $a = \pi\hbar\sqrt{\frac{3}{2mV_o}}$. Write the wave functions in all the three regions and apply boundary conditions, clearly stating them. Find the transmission coefficient of the particles.

P74*: A beam of particles of mass *'m'* and energy 9 V_o (V_o is a positive constant of energy dimension) is incident from left on a barrier given below.

V=0 for x<0

$V=5V_o$ for $x \le d$; where $d = \frac{\pi\hbar}{\sqrt{8mV_o}}$

$V= nV_o$ for $x \ge d$; where n is a number, <u>positive or negative</u>.



It is found that the transmission coefficient from x<0 region to x>d region is 0.75.

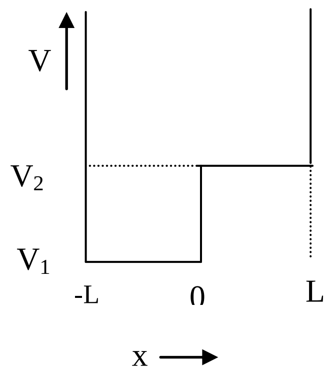
(a) Find *'n'*. Is there more than one possible value for *'n'*?

(b) Find the unnormalized wave function in all the regions in terms of the amplitude of the incident wave for each possible value of *'n'*.

(c) Is there a phase change between the incident and the reflected beam at x=0? If yes determine it for each possible value of *'n'*.
Give your answers by explaining all the steps and clearly writing the boundary conditions used

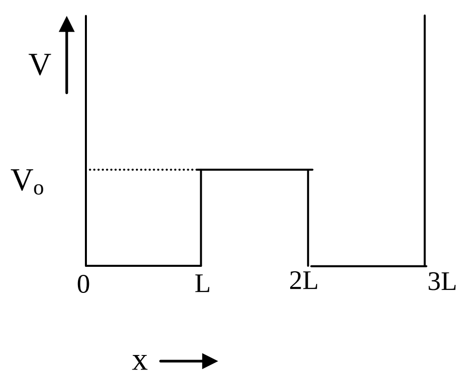
P75: A particle of mass m and energy E ($E>V_2$) is confined to potential well of the following type.

$V=\infty$ for $x<-L$ and for $x>L$
$V=V_1$ for $-L<x<0$
$V=V_2$ for $0<x<L$; $V_2>V_1$



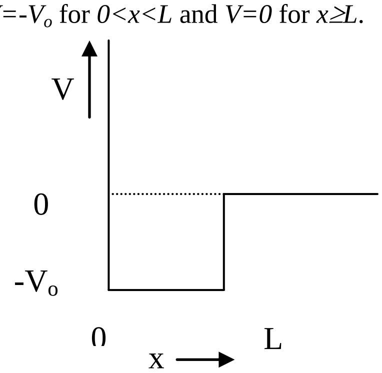
(a) Find the equation governing the energy of the particle.
(b) Find the ratio of the probabilities for the particle to remain in negative x to positive x.
(c) Take the limiting case of $V_1=V_2$ and show that the energies come out to be same as that of a particle in a one-dimensional box of length $2L$.

P76*: A particle of mass m is confined to a one-dimensional box described by $V=0$ for $0<x<L$ and for $2L<x<3L$; $V=V_o$ for $L<x<2L$ and $V=\infty$, everywhereelse.



It is given that the ground state wave function of the particle is independent of x between $L<x<2L$
(a) Find L in terms V_o and m
(b) Find the percentage probabilities of finding the particle in three different regions of different potentials.
(c) Sketch the wave function everywhere in box.

P77*: Consider a one-dimensional potential shown in the figure below. This is described by $V=\infty$ for $x\leq 0$; $V=-V_o$ for $0<x<L$ and $V=0$ for $x\geq L$.



(a) For a beam of particle coming from right with energy $E>0$, what is the reflection coefficient? Can you guess the answer without doing the calculation?
(b) Find the equation that governs the energy for the bound state. Normalize the wave function for this case.
(c) For a given value of L, what should be the value of V_o, so that there is only one bound state.

P78: Consider a potential well given below.

$V = 2V_o$ (where V_o is positive constant) for $x \le 0$

$V = 0$ for $0 < x < L$

$V = V_o$ for $x \ge L$

Find the range of the values of L, for which only one bound state would appear

P79: A particle of mass m is confined to a three dimensional box of sides $L, L/2, L/2$. Find the energy levels of the particle. Find also the wave functions and energy corresponding to lowest energy level that exhibits degeneracy.

P80: A particle of mass m and total energy E is confined to a three-dimensional potential well characterized by the following potentials.

$V = -V_o$ for $0 \le x \le L$ and for $0 \le y \le L/2$ and for $0 \le z \le L/2$

$V = +\infty$ everywhere else

(V_o , L, E are positive)

(a) Starting from three dimensional time independent Schrödinger equation, obtained the quantized energy level.

(b) Deduce the value of the lowest energy level that exhibits degeneracy. Give the corresponding unnormalized wave functions.

(c) Find the maximum value of V_o in terms of L and m, for which no negative energy state exists.

P81: A particle of mass m is under the influence of a potential $V = \frac{1}{2} m\omega^2 x^2$, where x is the displacement from origin and ω is a constant. Show that $\phi(x) = Axe^{-\left(\frac{m\omega}{2\hbar}\right)x^2}$ is a solution of the one-dimensional Schrodinger equation with the above potential. Find the value of energy for this state.

P82*: If the wave function $\psi(r,\theta,\phi,t=0)$ for the case of hydrogen atom is written as a product of three functions as $\psi(r,\theta,\phi,t=0) = R(r)\times\Theta(\theta)\times\Phi(\phi)$ then it can be shown that the radial pert of the Schrödinger equation can be written as follows.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[\frac{2m}{\hbar^2}\left(E+\frac{e^2}{4\pi\varepsilon_o r}\right)-\frac{\ell(\ell+1)}{r^2}\right]R=0$$

where the symbols have their usual meaning. For the ground state of hydrogen atom ℓ=0. For this state

(a) Show that $\psi(r,t=0) = Ae^{-\frac{r}{a}}$ is a solution of this equation. Find the values of A, a and the ground state energy.

(b) Calculate the mean distance, root mean square distance and the most probable distance between the electron and the nucleus in terms of the Bohr radius.

(c) What are the classical and quantum mechanical probabilities of finding the electron ar $r>2a$.

You may use the following standard integral.

$$\int_0^\infty x^p e^{-x}dx = \Gamma(p+1) = p\Gamma(p);\ \ \mathrm{p}+1>0$$

P83: A particle is bound in a potential well of the type V(r) =-Vo for r <a and V(r)=0 for r>a. The wave function $\psi(r,\theta,\phi,t=0)$ for this case is written as a product of three functions as $\psi(r,\theta,\phi,t=0) = R(r)\times\Theta(\theta)\times\Phi(\phi)$ and the radial part of the Schrödinger equation can be written as follows for ℓ=0.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[\frac{2m}{\hbar^2}(E-V)\right]R=0$$

The symbols have their usual meaning. Solve this equation to find the equation which would lead to energy states.

(Hint: Use the substitution $\chi(r)=R(r)\times r$, to reduce the equation into a more familiar differential equation)

P84*: For a particle bound in a particular one-dimensional potential with a property that V(0)=0, the two solutions of Schrödinger Equation are given by $\psi_1(x)$ and $x\psi_1(x)$. The energies corresponding to these solutions are E_1 and E_2 where $E_1 \neq E_2$. Find the unnormalized $\psi_1(x)$, the ratio E_2/E_1 and V(x).

P85*: The Θ and Φ parts of the Schrödinger equation are given as follows.

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\ell(\ell+1)\sin^2\theta = m_l{}^2$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m_l{}^2$$

If we substitute w = cos θ and write P for Θ in the first equation, we get the following equation.

$$\left(1\text{-}w^2\right)\frac{d^2P}{dw^2}-2w\frac{dP}{dw}+\left[\ell(\ell+1)-\frac{m_l{}^2}{1-w^2}\right]P=0$$

It is given that $P = Aw^2 - B$ is a solution of this equation. Find the possible values of $\frac{A}{B}$, ℓ and m_l. Write the form of Θ and Φ for these cases.