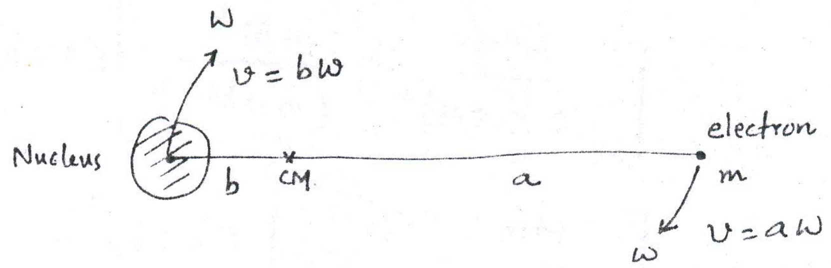
**51. Show that if the nucleus in the Bohr model is assumed to be of finite mass, the angular momentum of the system, allowed radii and energies are all given by identical expressions except for the replacement of $m$ by reduced mass $\mu$.**

Solution:

According to Bohr Model, the $e^-$ is revolving around the nucleus which is at rest. However, in reality the nucleus & electron both are rotating about a common centre of mass.




Take $a + b = r$,

$$ma = Mb \Rightarrow a = \frac{M}{m} b$$

$$a + b = \left(\frac{M}{m} b + b\right)$$

$$a + b = b\left(\frac{M+m}{m}\right)$$

$$\therefore b = \left(\frac{m}{M+m}\right)(a+b)$$

Coulomb force $F = \dfrac{Ze^2}{4\pi\epsilon_0 (a+b)^2}$

This Coulomb force is responsible for $e^-$ and nucleus to rotate about the centre of mass.

For $e^-$ $\quad F = m\omega^2 a$

For nucleus $F = M\omega^2 b = M\omega^2\left(\dfrac{m}{m+M}\right)(a+b)$

$$= \mu\omega^2(a+b)$$

where $\mu = \dfrac{Mm}{m+M}$

$$\Rightarrow \mu r \omega^2 = \frac{Ze^2}{4\pi\epsilon_0 r^2}, \quad r = a+b$$

Angular momentum of nucleus, $L_N = Mb^2\omega$

Angular momentum of electron, $L_e = ma^2\omega$

$\therefore$ Total Angular Momentum $= Mb^2\omega + ma^2\omega$

$$= M\left(\frac{m}{m+M}\right)^2 (a+b)^2\omega + m\left(\frac{M}{m}\right)^2\left(\frac{m}{M+m}\right)^2(a+b)^2\omega$$

$$= \left[\frac{m^2M}{(m+M)^2} + \frac{mM^2}{(m+M)^2}\right](a+b)^2\omega$$

$$= \left[\frac{Mm(m+M)}{(M+M)^2}\right](a+b)^2\omega$$

$$= \left(\frac{mM}{M+m}\right)(a+b)^2\omega$$

$$L = \mu r^2\omega, \quad \boxed{Mb^2 + ma^2 = \mu r^2}$$

Total Energy

$$E = \frac{1}{2}m\omega^2a^2 + \frac{1}{2}M\omega^2b^2 - \frac{Ze^2}{4\pi\epsilon_0(a+b)}$$

$$= \frac{1}{2}\mu r^2\omega^2 - \mu r^2\omega^2$$

$$\boxed{E = -\mu r^2\omega^2}$$