MA 105 D1 Lecture 2

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Monotonic sequences

For the moment we will concentrate on sequences in $\mathbb{R}$.

Definition: A sequence is said to be a monotonically increasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

Definition: A sequence is said to be a monotonically decreasing sequence if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

A monotonic sequence is one that is either monotonically increasing or monotonically decreasing.

Definition: A sequence a_n tends to a limit l, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - l| < \epsilon$$

whenever $n > N$.

A sequence that does not converge is said to diverge, or to be divergent.

Formulæ for limits

If a_n and b_n are two convergent sequences then

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.
3. $\lim_{n\to\infty}(a_n/b_n) = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n$, provided $\lim_{n\to\infty} b_n \neq 0$

Theorem 1: If a_n, b_n and c_n are convergent sequences such that $a_n \leq b_n \leq c_n$ for all n, then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} c_n.$$

Theorem 2: Suppose $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$. If b_n is a sequence satisfying $a_n \leq b_n \leq c_n$ for all n, then b_n converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

Bounded Sequences

Definition: A sequence a_n is said to be bounded if there is a real number $M > 0$ such that $|a_n| \leq M$ for every $n \in \mathbb{N}$. A sequence that is not bounded is called unbounded.

Lemma: Every convergent sequence is bounded.
Proof: Suppose a_n converges to l. Choose $\epsilon = 1$. There exists $N \in \mathbb{N}$ such that $|a_n - l| < 1$ for all $n > N$. In other words, $l - 1 < a_n < l + 1$, for all $n > N$, which gives $|a_n| < |l| + 1$ for all $n > N$. Let

$$M_1 = \max\{|a_1|, |a_2|, \ldots, |a_N|\}$$

and let $M = \max\{M_1, |l| + 1\}$. Then $|a_n| < M$ for all $n \in \mathbb{N}$. $\square$
We will use this Lemma to prove the product rule for limits.

The proof of the product rule

We wish to prove that $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.

Suppose $\lim_{n\to\infty} a_n = l_1$ and $\lim_{n\to\infty} b_n = l_2$. We need to show that $\lim_{n\to\infty} a_n b_n = l_1 l_2$.

Fix $\epsilon > 0$. We need to show that we can find $N \in \mathbb{N}$ such that $|a_n b_n - l_1 l_2| < \epsilon$, whenever $n > N$. Notice that

$$\begin{aligned} |a_n b_n - l_1 l_2| &= |a_n b_n - a_n l_2 + a_n l_2 - l_1 l_2| \\ &= |a_n(b_n - l_2) + (a_n - l_1) l_2| \\ &\leq |a_n||b_n - l_2| + |a_n - l_1||l_2|, \end{aligned}$$

where the last inequality follows from the triangle inequality. So in order to guarantee that the left hand side is small, we must ensure that the two terms on the right hand side together add up to less than ϵ. In fact, we make sure that each term is less than $\epsilon/2$.

The proof of the product rule, continued

Since a_n is convergent, it is bounded by the lemma we have just proved. Hence, there is an M such that $|a_n| < M$ for all $n \in \mathbb{N}$.

Given the quantities $\epsilon/2|l_2|$ and $\epsilon/2M$, there exist N_1 and N_2 such that

$$|a_n - l_1| < \epsilon/2|l_2| \quad \text{and} \quad |b_n - l_2| < \epsilon/2M.$$

Let $N = \max\{N_1, N_2\}$. If $n > N$, then both the inequalities above hold. Hence, we have

$$|a_n||b_n - l_2| \leq M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2} \quad \text{and} \quad |a_n - l_1||l_2| \leq |l_2| \cdot \frac{\epsilon}{2|l_2|} = \frac{\epsilon}{2}.$$

Now it follows that

$$|a_n b_n - l_1 l_2| \leq |a_n||b_n - l_2| + |a_n - l_1||l_2| < \epsilon,$$

for all $n > N$, which is what we needed to prove. $\square$

The proofs of the other rules for limits are similar to the one we proved above. Try them as exercises.

A guarantee for convergence

As we mentioned earlier, proving that a limit exists is hard because we have to guess what its value might be and then prove that it satisfies the definition. The following theorem guarantees the convergence of a sequence without knowing the limit beforehand.

Definition: A sequence a_n is said to be bounded above (resp. bounded below) if $a_n < M$ (resp. $a_n > M$) for some $M \in \mathbb{R}$.

A sequence that is bounded both above and below is obviously bounded.

Theorem 3: A montonically increasing (resp. decreasing) sequence which is bounded above (resp. below) converges.

Remarks on Theorem 3

Theorem 3 clearly makes things very simple in many cases. For instance, if we have a monotonically decreasing sequence of positive numbers, it must have a limit, since 0 is always a lower bound!

Can we guess what the limit of a monotonically increasing sequence a_n bounded above might be?
It will be the supremum or least upper bound (lub) of the sequence. This is the number, say M which has the following properties:

1. $a_n \leq M$ for all n and
2. If M_1 is such that $a_n \leq M_1$ for all n, then $M \leq M_1$.

The point is that a sequence bounded above may not have a maximum but will always have a supremum. As an example, take the sequence $1 - 1/n$. Clearly there is no maximal element in the sequence, but 1 is its supremum.

Another monotonic sequence

Let us look at Exercise 1.5.(i) which considers the sequence

$$a_1 = 3/2 \quad \text{and} \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right).$$

$$\begin{aligned} a_{n+1} < a_n \quad &\iff \quad \tfrac{1}{2}\left(a_n + \tfrac{2}{a_n}\right) < a_n \\ &\iff \quad \sqrt{2} < a_n. \end{aligned}$$

On the other hand,

$$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \geq \sqrt{2}, \quad \text{(Why is this true?)}$$

so $a_{n+1} \geq \sqrt{2}$ for all $n \geq 1$ and $a_1 > \sqrt{2}$ is given.
Hence, $\{a_n\}_{n=1}^{\infty}$ is a monotonically decreasing sequence, bounded below by $\sqrt{2}$. By Theorem 3, it converges.
Exercise 1. What do you think is the limit of the above sequence (Refer to the supplement to Tutorial 1)?

More remarks on limits

Exercise 2. More generally, what is the limit of a monotonically decreasing sequence bounded below? How can you describe it? This number is called the infimum or greatest lower bound (glb) of the sequence.

The proof of Theorem 3 is not so easy and more or less involves understanding what a real number is. It is related to the notion of Cauchy sequences about which I will try to say something a little later (again, refer to the supplement to Tutorial 1).

And finally:

An important remark: If we change finitely many terms of a sequence it does not affect the convergence and boundedness properties of a sequence.

If it is convergent, the limit will not change. If it is bounded, it will remain bounded though the supremum may change. Thus, an eventually monotonically increasing sequence bounded above will converge (formulate the analogue for decreasing sequences).

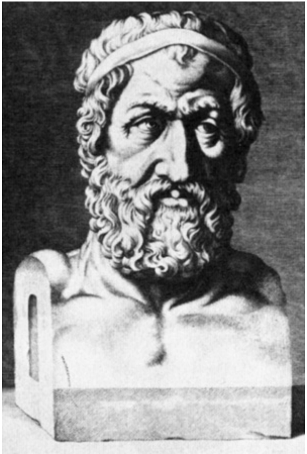
Bottomline: From the point of view of the limit, only what happens for large N matters.

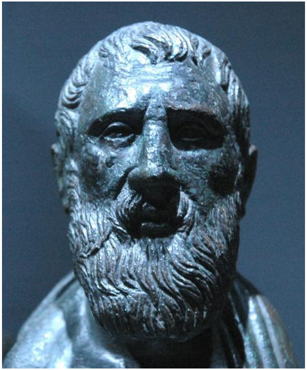
The first man to think about limits?

Zeno of Elea (490 - 460 BCE) was a famous Greek philosopher (source: Wikipedia)

Zeno of Elea

First let us record that we have no idea what Zeno looked like. The picture above was painted in the period 1588 - 1594 CE in Spain, about two thousand years after Zeno's time. Here are two more images of Zeno (also from Wikipedia)





Zeno's Paradoxes

I couldn't find out where the first statue came from and when it was made. The second seems to have come from Herculaneum in Italy (incidentally, Elea (modern Vilia) is a town in Italy). Now Herculaneum was destroyed by a volcanic erruption from the nearby volcano Vesuvius in 79 CE, so it looks like the bust was created within 500 years of Zeno's death. Maybe it was even made during his lifetime and was lying around in some wealthy Roman's house for the next few centuries. Unfortunately, it is not clear whether this statue is one of Zeno of Elea or of another Zeno (of Citium) who lived about 150 years later. So we still really have no clue how he looked.

The important about Zeno is that it would appear that he was the first human to think about limits and limiting processes, at least in recorded history. Most of what we know about him is through his paradoxes, nine of which survive in the works of another famous Greek philosopher Aristotle (384 - 322 CE) , the official guru/tutor of Alexander the Great (aka Sikander in India).

Achilles and the tortoise

One of Zeno's motivations for stating his paradoxes seems to have been to defend his own guru Parmenides' philosophy (whatever that was). Anyway here is his most famous paradox as recorded by Aristotle.

Achilles and the tortoise:
In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.

Aristotle, Physics VI:9, 239b15

General knowledge question: Who was Achilles?

Zeno's paradox animated

Achilles and the Tortoise

A gateway to infinite series

Nowadays, this line of argument does not really bother us, since we understand that an infinite number of terms (in this case consisting of the time travelled in each segment or the distance travelled in each segment) can add up to something finite. Nevertheless there are other philosophical issues that continued to bother mathematicians and physicists for a long time. After all, this kind of discussion does lead us to question whether intervals of time and space can be infinitely subdivided, or if "instantaneous motion" makes sense.

Since we are learning mathematics, we won't speculate on physics or philosophy, but we note that Zeno's argument gives a good way to derive the sum of an infinite geometric series. The geometric series is one of the simplest examples of infinite series, so let us see how this is done.

Geometric series - the formula

Let us suppose that the speed of achilles is v and that the speed of the tortoise is rv for some $0 < r < 1$. We will assume that the tortoise was given a headstart of distance "a".

- The distance covered by Achilles in time t is vt.
- The distance covered by the tortoise in time t is rvt.
- Achilles catches us with the tortoise when $vt = a + rvt$, that is, at time $t = a/(v - rv)$ and when the total distance covered by Achilles is $vt = a/(1 - r)$.

On the other hand,

- Distance covered by the tortoise by the time Achilles has covered distance a is ar.
- Distance covered by the tortoise by the time Achilles has covered distance ar is ar^2 ....
- Total distance covered by Achilles when he has caught up with the tortoise is $a + ar + ar^2 + \dots$.
- Thus we get $a + ar + ar^2 + \cdots = a/(1 - r)$.

Infinite series - a more rigourous treatment

Let us recall what we mean when we write

$$a + ar + ar^2 + \ldots = \frac{a}{1-r}.$$

Another way of writing the same expression is

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

The precise meaning is the following. Form the partial sums

$$s_n = \sum_{k=0}^{n} ar^k.$$

These partial sums $s_1, s_2, \ldots s_n, \ldots$ form a sequence and by $\sum_{k=0}^{\infty} ar^k = a/(1-r)$, we mean $\lim_{n\to\infty} s_n = a/1-r$.
So when we speak of the sum of an infinite series, what we really mean is the limit of its partial sums.

Convergence of the geometric series

So to justify our formula we should show that $\lim_{n\to\infty} s_n = a/1-r$, that is, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| s_n - \frac{a}{1-r} \right| < \epsilon,$$

for all $n > N$.

In other words we need to show that

$$\left| \frac{a(1-r^{n+1})}{(1-r)} - \frac{a}{1-r} \right| = \left| \frac{ar^{n+1}}{1-r} \right| < \epsilon$$

if n is chosen large enough.

But $\lim_{n\to\infty} r^n = 0$, so there exists N such that $r^{n+1} < (1-r)\epsilon/a$ for all $n > N$, so for this N, if $n > N$,

$$\left| s_n - \frac{a}{1-r} \right| < \epsilon.$$

This shows that the geometric series converges to the given expression.

Cauchy sequences

As we saw last time, it is not easy to tell whether a sequence converges or not because we have to first guess what the limit might be and then try and prove that the sequence actually converges to this limit. For a monotonic sequence, things are slightly better since we only need to bound the sequence.

There is another very useful notion which allows us to decide whether the sequence converges by looking only at the elements of the sequence itself. We describe this below.
Definition: A sequence a_n in $\mathbb{R}$ is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon,$$

for all $m, n > N$.

Cauchy sequences: the theorem

Theorem 4: Every Cauchy sequence in $\mathbb{R}$ converges.

Remark 1: One can now check the convergence of a sequence just by looking at the sequence itself!

Remark 2: One can easily check the converse:

Theorem 5: Every convergent sequence is Cauchy.

Remark 3: Remember that when we defined sequences we defined them to be functions from $\mathbb{N}$ to X, for any set X. So far we have only considered $X = \mathbb{R}$, but as we said earlier we can take other sets, for instance, susbets of $\mathbb{R}$. For instance, if we take $X = \mathbb{R} \setminus 0$, Theorem 4 is not valid. The sequence $1/n$ is a Cauchy sequence in this X but obviously does not converge in X. If we take $X = \mathbb{Q}$, the example given in 1.5.(i) $(a_{n+1} = (a_n + 2/a_n)/2)$ is a Cauchy sequence in $\mathbb{Q}$ which does not converge in $\mathbb{Q}$. Thus Theorem 4 is really a theorem about real numbers.