PH-105 Assignment Sheet - 1

Umang Mathur

14. **A particle of mass m is moving with a constant velocity in the $x-y$ plane of an inertial frame S. The observer in S finds that the particle travels a distance of 600 m in a time of $2.5 \times 10^{-6}s$. During this time both the x and y co-ordinates of the particle increase. The increase in the x coordinate is 300 m during the given duration.**

    (a) **Find the proper time difference for the displacement described above.**

    (b) **Find the components of the displacement four vector $\Delta \underset{\sim}{s}$ and the momentum four vector $\underset{\sim}{p}$.**

    (c) **Find the components of the displacement four vector $\Delta \underset{\sim}{s}'$ and the momentum four vector $\underset{\sim}{p}'$ in a frame S, which moves with a speed of $0.5c$ along $+x$-direction of S.**

    Solution :

    (a) In the above problem, $\Delta x = 300\text{m}$, $\Delta y = \sqrt{600^2 - 300^2}m = 300\sqrt{3}\text{m}$ and $\Delta t = 2.5 \times 10^{-6}s$
    Hence,

$$\begin{aligned}
\text{Proper Time} &= \Delta\tau \\
&= \sqrt{(\Delta t)^2 - \frac{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}{c^2}} \\
&= \sqrt{(2.5 \times 10^{-6})^2 - \frac{600^2}{(3 \times 10^8)^2}} \\
&= 1.5 \times 10^{-6}s
\end{aligned}$$

    (b) Displacement four vector $\Delta \underset{\sim}{s} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ ic\Delta t \end{pmatrix} = \begin{pmatrix} 300 \text{ m} \\ 300\sqrt{3} \text{ m} \\ 0 \text{ m} \\ 750i \text{ m} \end{pmatrix}$

    Velocity of particle $= \overrightarrow{u} = \frac{300}{2.5\times10^{-6}}\hat{i} + \frac{300\sqrt{3}}{2.5\times10^{-6}}\hat{j}$ m/s $= 120 \times 10^6\hat{i} + 120\sqrt{3} \times 10^6\hat{j}$ m/s

    Hence, $\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{5}{3}$

    Momentum four vector $\underset{\sim}{p} = \begin{pmatrix} m\gamma_u u_x \\ m\gamma_u u_y \\ m\gamma_u u_z \\ m\gamma_u ic \end{pmatrix} = \begin{pmatrix} \frac{2}{3}mc \text{ m/s} \\ \frac{2\sqrt{3}}{3}mc \text{ m/s} \\ 0 \\ \frac{5}{3} \text{ m/s} \end{pmatrix}$

    (c) The displacement and the momentum four-vectors get transformed as per the Lorentz transformation as follows:

$$\begin{pmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \\ ic\Delta t' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ ic\Delta t \end{pmatrix}$$

$$\begin{pmatrix} m\gamma_{u'} u'_x \\ m\gamma_{u'} u'_y \\ m\gamma_{u'} u'_z \\ m\gamma_{u'} ic \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} m\gamma_u u_x \\ m\gamma_u u_y \\ m\gamma_u u_z \\ m\gamma_u ic \end{pmatrix}$$

Here, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$\beta = \frac{v}{c}$

$u'_x = \frac{u_x - v}{1-\frac{vu_x}{c^2}}$

$u'_y = \frac{u_y}{\gamma(1-\frac{vu_x}{c^2})}$

$u'_z = \frac{u_z}{\gamma(1-\frac{vu_x}{c^2})}$

$\gamma_{u'} = \frac{1}{\sqrt{1-\frac{u'^2_x+u'^2_y+u'^2_z}{c^2}}}$

where, v is the speed of the *primed* frame with respect to the $un-primed$ frame. Here, $v = 0.5c$. Thus, plugging in values and solving the matrix products, we get

Displacement four vector $\Delta \underset{\sim}{s}' = \begin{pmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \\ ic\Delta t' \end{pmatrix} = \begin{pmatrix} -50\sqrt{3} \text{ m} \\ 300\sqrt{3} \text{ m} \\ 0 \text{ m} \\ 400\sqrt{3}i \text{ m} \end{pmatrix}$

Momentum four vector $\underset{\sim}{p}' = \begin{pmatrix} m\gamma_{u'}u'_x \\ m\gamma_{u'}u'_y \\ m\gamma_{u'}u'_z \\ m\gamma_{u'}ic \end{pmatrix} = \begin{pmatrix} \frac{-\sqrt{3}}{9}mc \text{ m/s} \\ \frac{2\sqrt{3}}{3}mc \text{ m/s} \\ 0 \\ \frac{-8\sqrt{3}}{9}imc \text{ m/s} \end{pmatrix}$