QM Tutorial Sheet 3 Q.76

Raghav Gupta

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A particle of mass m is confined to a one-dimensional box described by $V = 0$ for $0 < x < L$ and for $2L < x < 3L, V = Vo$ for $L < x < 2L$ and $V = \infty$, everywhere else.

It is given that the ground state wave function of the particle is independent of x between $L < x < 2L$

(a) Find L in terms Vo and m
(b) Find the percentage probabilities of finding the particle in three different regions of different potentials.
(c) Sketch the wave function everywhere in box.

Take the TISE
$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = (E - V)\Psi$ and let the energy of the particle be E. Now, in region 2, let $\Psi_2(x) = C$ where C is some constant not dependent on x. Plugging Ψ_2 in the TISE, the left hand side becomes zero, hence $E = V_0$ since $\Psi_2 = 0$ is not acceptable.

For region 1, solving the TISE gives
$\Psi_1 = Asin(kx) + Bcos(kx)$
$\Psi_2 = C$
and $\Psi_3 = Dsin(kx) + Ecos(kx)$
Evidently, the wave function would be zero for $x < 0$ and $x > 3L$.

Applying continuity at the boundaries of each region,
$\Psi_1(0) = 0 \Rightarrow B = 0$
$\Psi_1(L) = \Psi_2(L) \Rightarrow Asin(kL) = C$
$\Psi_2(2L) = \Psi_3(2L) \Rightarrow C = Dsin(2kL) + Ecos(2kL)$
$\Psi_3(3L) = 0 \Rightarrow Dsin(3kl) + Ecos(3kL) = 0$
Here $k = \sqrt{\frac{2mV_0}{\hbar^2}}$

As a simplification, we may assume that the wave function in region 3 is a mirror image of the wave function in region 1.

Applying differentiability of wave function at $x = L$, $\Psi_1'(L) = \Psi_2'(L) = Akcos(kL) = 0 \Rightarrow kL = \frac{(2n+1)\pi}{2} \Rightarrow L = \frac{(2n+1)\hbar\pi}{\sqrt{8mV_0}}$.

Now, for ground state, assume that the probability of the particle being in region 1 and in region 3 are equal (symmetry arguments). Also assume that n in the above equation is 0 i.e. its lowest possible value $\Rightarrow A = C$ in ground state. Now, normalizing the wave function, we get $A = \sqrt{\frac{1}{2L}}$

Thus, P{particle exists in region 1} $= \int_0^L A^2 sin^2(kx)\,\mathrm{d}x = 0.25$
P{particle exists in region 2} $= 0.5$
P{particle exists in region 3} $= 0.25$ by assumed symmetry