MA 105 D1 Lecture 18

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Flow lines for vector fields

Recall that a vector field was just a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. For the moment let us suppose that all vector fields under consideration are continuous.

Definition: If $\mathbf{F}$ is a vector field, a flow line or integral curve for $\mathbf{F}$ is a curve $\mathbf{c}(t)$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)).$$

Basically, one is trying to fit a curve so that the tangent vector of the curve at any point is the same as the vector given by the vector field at that point.

If we write $\mathbf{c}(t) = (x(t), y(t), z(t))$, and $F = (F_1, F_2, F_3)$, we see that finding a flow line is equivalent to solving the following system of equations.

$$\begin{aligned} x'(t) &= F_1(x(t), y(t), z(t)) \\ y'(t) &= F_2(x(t), y(t), z(t)) \\ z'(t) &= F_3(x(t), y(t), z(t)) \end{aligned}$$

The del operator on functions

We will assume from now on that our vector fields are smooth.

One important class of vector fields are those that are given by the gradient of a scalar function. We will study these in some detail later.

We define the del operator restricting ourselves to the case $n = 3$:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

The del operator acts on functions $f : \mathbb{R}^3 \to \mathbb{R}$ to give the gradient:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Thus the del operator takes scalar functions to vector fields.

The del operator on vector fields

The del operator can be made to operate on vector fields as follows. For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ we define the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

It is useful to represent it as a determinant:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

Angular velocity

Recall that if a particle P is moving in the three-dimensional space, its position vector $\mathbf{r}$ and velocity vector $\mathbf{v}$ together define a plane at any given instant in time. This plane is called the plane of rotation.

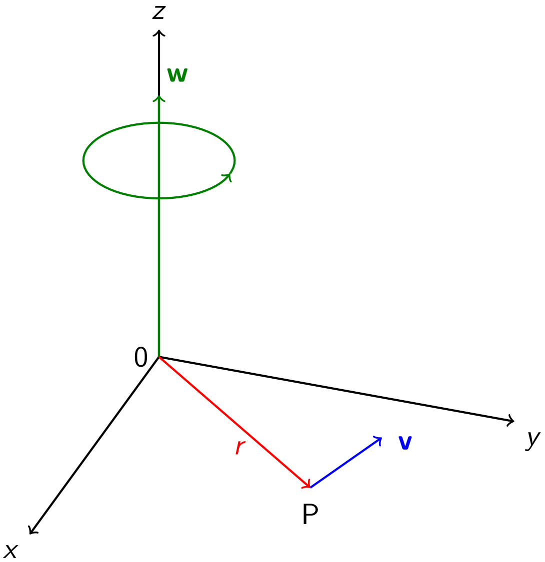
The axis of rotation of the particle is defined as an axis through the origin perpendicular to the plane of rotation. The direction of the axis is obviously given by the direction of the cross product $\mathbf{r} \times \mathbf{v}$.

The angular velocity vector which measures the rate of change of angular displacement is defined as

$$\mathbf{w} = \frac{\mathbf{r} \times \mathbf{v}}{r^2}.$$

Clearly, the direction of $\mathbf{w}$ is the direction of the axis of rotation. In the picture in the next slide we assume that the point P is moving in a plane. In this case we can assume that the axis of rotation is the z-axis, since we can always rotate our picture so that this happens.

Angular velocity for a single particle



The curl and angular velocity

Since the particle moves in the $x - y$ plane, we can write $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. From the definition it follows that

$$\mathbf{v} = \mathbf{w} \times \mathbf{r} = -wy\mathbf{i} + wx\mathbf{j},$$

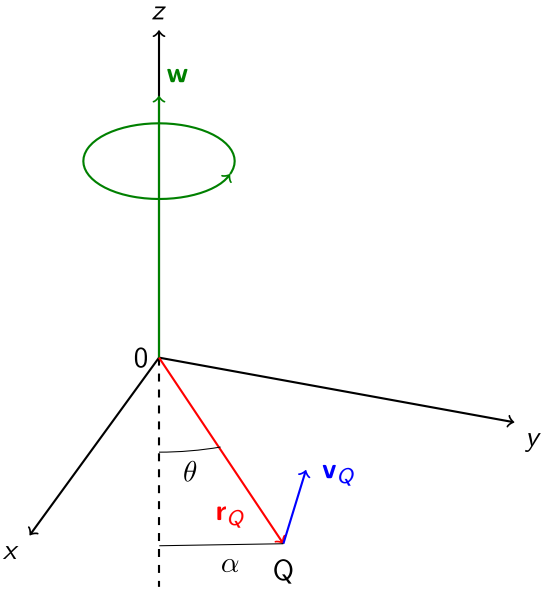
where w is the magnitude of the vector $\mathbf{w}$.
Hence,

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -wy & wx & 0 \end{vmatrix} = 2w\mathbf{k} = 2\mathbf{w}$$

We see that the curl of the velocity is twice the angular velocity.

Suppose we assume that P is part of a rigid body and that Q is another point in the body. In the picture that follows, $\mathbf{r}_Q$ is the position vector of Q and θ is the angle made by $\mathbf{r}_Q$ with the axis of rotation (which we continue to assume is the z axis).

Angular velocity for another particle in the same rigid body



The angular velocity for the point Q

In this picture the velocity vector $\mathbf{v}_Q$ need not be perpendicular to the position vector $\mathbf{r}_Q$. However, a similar calculation goes through.

In the plane formed by $\mathbf{r}_Q$ and $\mathbf{v}_Q$, we note that the component of $\mathbf{v}_Q$ that points in the direction $\mathbf{r}_Q$ contributes nothing to the angular velocity. Hence, the entire contribution to the angular velocity comes from the tangential velocity.

We can easily determine the tangential velocity $\mathbf{v}_T$ of Q. It is directed counterclockwise in along the tangent to a circle parallel to the xy plane with radius α (as in the picture). It follows that

$$\|\mathbf{v}_T\| = w\alpha = wr_Q \sin\theta,$$

whence, we see that $\mathbf{v}_T = \mathbf{w} \times \mathbf{r}_Q$. Now the same computation as before shows that $\nabla \times \mathbf{v}_T = 2\mathbf{w}$.

The angular velocity for a rigid body

What we can conclude from our previous calculations is that for a rigid body, the curl of the velocity vector field is a constant vector field. Its direction at each point is simply the direction of the axis of rotation, while the magnitude is twice the angular speed.

Remark: It is more conventional to denote angular velocity by the letter ω. However, I have used **w** because I am unable to get a good boldface font for ω.

Irrotational flow

Instead of looking at the velocity field of a rigid body we can look at the velocity field $\mathbf{F}$ of the flow of a fluid. In this case, what does it mean if $\nabla \times \mathbf{F} = 0$ at a point P?

It means that the fluid is free from rigid rotations at that point. In physical terms it means that if you put a small paddle wheel face down in the fluid, it will move with the fluid but will not rotate around its axis. In terms of the fluid itself, this translates into there being no whirlpools centred at the point P. In this case, the vector field is called irrotational.

For instance, water draining into sink produces an irrotational field, except at the very centre of the circular drain.

The curl of a gradient

Suppose that $\mathbf{F} = \nabla f$ for some scalar function f. Then

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{k}.$$

Clearly, if f is $\mathcal{C}^2$ (which we will assume), $\nabla \times \mathbf{F} = 0$. In particular, this gives a criterion for deciding whether a vector field arises as the gradient of a function. If its curl is not zero at some point, it cannot arise as a gradient.

Is the condition $\nabla \times F = 0$ sufficient?

Recall that we have previously looked at the vector field

$$\mathbf{F} = \frac{y}{x^2 + y^2} \cdot \mathbf{i} + \frac{-x}{x^2 + y^2} \cdot \mathbf{j},$$

Exercise 1: Check that $\nabla \times \mathbf{F} = 0$. Can you express $\mathbf{F}$ as the gradient of a suitable scalar function?

Finally, as a special case of the curl, we can define the scalar curl. If $\mathbf{F} = (M(x,y), N(x,y), 0)$ is a vector field in the plane, then

$$\nabla \times \mathbf{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}.$$

The function $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$ is called the scalar curl of $\mathbf{F}$.

The divergence of a vector field

The del operator can be made to operate on vector fields to give a scalar function as follows.

Definition: Let $\mathbf{F} = (F_1, F_2, F_2)$ be a vector field. The divergence of $\mathbf{F}$ is the scalar function defined by

$$\operatorname{div} \mathbf{F} := \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

If $\mathbf{F}$ is the velocity field of a fluid, the divergence of $\mathbf{F}$ gives the rate of expansion of the volume of the fluid per unit volume as the volume moves with the flow. In the case of planar vector fields we get the corresponding rate of expansion of area.

Examples

Let us consider the divergence of different vector fields.
Example 1: $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$.

The flow lines of this vector field point radially outward from the origin, so it is clear that the fluid is expanding as it flows. This is reflected in the fact that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2 > 0.$$

If we look at the vector field $\mathbf{F} = -x\mathbf{i} - y\mathbf{j}$, we see that $\nabla \cdot \mathbf{F} = -2$. This is consistent with the fact that the flow lines of the vector field all point towards the origin, and the fact the vectors get smaller in size which means that the fluid is getting compressed.

Example 2: $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$.
In this case the fluid is moving countercolockwise around the origin - so it is neither being compressed, nor is it expanding. One checks easily that $\nabla \cdot \mathbf{F} = 0$.

The change in area in a flow

As before let us assume that our vector field $\mathbf{F} = (F_1, F_2)$ represents the velocity field of a fluid, but this time in just two dimensions. Let us compute the rate of change of a unit area of the fluid as it flows along the integral curve.

We assume that we start at time $t = 0$ at a point $P = (x, y)$ in $\mathbb{R}^3$. We let the point evolve under the flow to a point (X, Y) at time t. Explicitly, we see that

$$X = X(x, y, t) \quad \text{and} \quad Y = Y(x, y, t)$$

The change of variable formula tells us how a unit area changes. One has to simply multiply by the Jacobian determinant of the transformation $\phi(x, y) = (X, Y)$. In this case we have

$$J = \begin{vmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{vmatrix}.$$

The rate of change of area in a flow

We would like to compute the rate at which the unit area is changing. This is simply given by $\frac{\partial J}{\partial t}$. We first write out the function J as

$$J = \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}.$$

It follows that

$$\frac{\partial J}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y}\right] - \frac{\partial}{\partial t}\left[\frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}\right]$$

$$\begin{aligned} &= \frac{\partial}{\partial x}\left(\frac{\partial X}{\partial t}\right)\frac{\partial Y}{\partial y} + \frac{\partial X}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial Y}{\partial t}\right) \\ &- \frac{\partial}{\partial y}\left(\frac{\partial X}{\partial t}\right)\frac{\partial Y}{\partial x} + \frac{\partial X}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial Y}{\partial t}\right). \end{aligned}$$

We keep going by brute force

Now we observe that $(X(t), Y(t))$ describes a flow line for the velocity field (F_1, F_2). Hence, the tangent vector of the curve is the same as the value of $\mathbf{F}$ at any point. Hence,

$$\frac{\partial X}{\partial t} = F_1, \quad \text{and} \quad \frac{\partial Y}{\partial t} = F_2.$$

This is true because (X, Y) describes a flow line of $\mathbf{F}$ as t varies. Hence, the tangent vector of the curve is the same as the value of $\mathbf{F}$ at any point.

Using these equations we

$$\frac{\partial J}{\partial t} = (\nabla \cdot \mathbf{F})J.$$

This shows that if $\nabla \cdot \mathbf{F} = 0$, then $\frac{\partial J}{\partial t} = 0$. Hence, $J(x, y, t)$ is a constant function. But $J(x, y, 0) = 1$, so J must be the constant function 1, that is, there is no change of volume along the flow. This shows that if the divergence is zero the fluid is incompressible.

More exercises

Exercise 2: Try doing the above calculation in three dimensions. By this I mean, take a divergence free vector field in the plane and show that the rate of expansion of volue by unit volume in is 0 along flow lines. This is the same calculation as before but there are now eighteen terms in the Jacobian! Not fun.
Moral: There has got to be an easier way.

Exercise 3: In Lecture 18 we have represented three different vector fields in pictures. Calculate their curls and divergences.

The Laplace operator

Just as the curl of a gradient was 0, we similarly have the divergence of any curl is zero. In other words, if $\mathbf{F}$ is a $\mathcal{C}^2$ vector field,

$$\operatorname{div}(\operatorname{curl}\mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

Exercise 4: If $\nabla \cdot \mathbf{F} = 0$, does it imply that $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$?

Finally, the composition of the gradient followed by the divergence gives one of the most important operators in mathematics and physics. The Laplace operator denoted ∇^2 is defined by

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

One can easily check that the function $f(x, y, z) = \frac{1}{\|\mathbf{r}\|}$ satisfies $\nabla^2 f = 0$.

Vector fields and line integrals

In what follows we only assume that the vector field in question are continuous (not smooth).

We will now define the integral of a vector field along a curve. We will assume that we are given a $\mathcal{C}^1$ curve $\mathbf{c} : [a, b] \to \mathbb{R}^3$ such that $\mathbf{c}'(t) \neq 0$ for any $t \in [a, b]$. Such a curve will be called a regular or non-singular parametrised curve.

We define the line integral of $\mathbf{F}$ over $\mathbf{c}$ as:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt.$$

If we write $\mathbf{c}(t)$ in vector notation, that is

$$\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

we see that

$$\mathbf{c}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

Line integrals - alternate notation

In fact, it is enough to assume that $\mathbf{c}(t)$ is only piecewise $\mathcal{C}^1$ and that $f(\mathbf{c}(t))$ is piecewise continuous. In this case we can break up the interval $[a, b]$ into sub-intervals where $\mathbf{c}'(t)$ and $f(\mathbf{c}(t))$ are continuous.
If $\mathbf{F} = (F_1, F_2, F_3)$ and $\mathbf{c}(t) = (x(t), y(t), z(t))$, we see that

$$\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt.$$

Because of the form of the right hand side the line integral is sometimes written as

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b F_1 dx + F_2 dy + F_3 dz.$$

The expression on the right hand side is just alternate notation for the line integral. It does not have any independent meaning.

An example

Example 1: Evaluate

$$\int_{\mathbf{c}} x^2 dx + xy dy + dz,$$

where $\mathbf{c} : [0, 1] \to \mathbb{R}^3$ is given by $\mathbf{c}(t) = (t, t^2, 1)$.

Solution: We have $\mathbf{c}'(t) = (1, 2t, 0)$, hence

$$(x^2(t, t^2, 1), xy(t, t^2, 1), (t, t^2, 1)) \cdot \mathbf{c}'(t) = t^2 + 2t^4 + 0.$$

$$\int_{\mathbf{c}} x^2 dx + xy dy + dz = \int_0^1 (t^2 + 2t^4) dt = 11/15.$$

Gradient fields

The main observation about line integrals is the following. Suppose the vector field $\mathbf{F}$ can be written as the gradient of a scalar function f, that is, $\nabla f = \mathbf{F}$, then

$$\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt$$

Now the integrand on the right hand side is nothing but the directional derivative of f in the direction of $\mathbf{c}(t)$. Hence, we obtain

$$\int_a^b \frac{d}{dt} f(\mathbf{c}(t))dt = f(c(b)) - f(c(a)).$$

This shows that the value of the line integral depends only on the value of the function at the end points of the curve, not on the curve itself. Such vector fields are called conservative.

Integrating along successive paths

It is easy to see that if $\mathbf{c}_1$ is a path joining two points P_0 and P_1 and $\mathbf{c}_2$ is a path joining P_1 and P_2 and $\mathbf{c}$ is the union of these paths (that is, it is a path from P_0 to P_2 passing through P_1), then

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}.$$

This property follows directly from the corresponding property for Riemann integrals:

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_b^c f(t)dt,$$

where c is a point between a and b. This property will prove useful shortly.

Remark: The union of two $\mathcal{C}^1$ paths may not produce a $\mathcal{C}^1$ path. In fact, line integrals make sense for piecewise $\mathcal{C}^1$ paths, that is for paths for which there may be a finite set of points where the path fails to be differentiable.

Conservative fields

The main observation about line integrals is the following. Suppose the vector field $\mathbf{F}$ can be written as the gradient of a scalar function f, that is, $\nabla f = \mathbf{F}$, then

$$\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

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Connected sets

We have already seen that for many purposes (such as the inverse function theorem) it is best to restrict ourselves to open sets in $\mathbb{R}^n$. For studying conservative vector fields and related questions it is best to restrict our domains further.

Definition: A subset of C of $\mathbb{R}^n$ is called connected if it cannot be written as a disjoint union of two non-empty subsets $C_1 \sqcup C_2$, with $C_1 = C \cap U_1$ and $C_2 = C \cap U_2$, where U_1 and U_2 are open sets.

Connected open domains turn out to be the best settings in which to study many questions about vector fields.

The given definition of connectedness is quite abstract. It turns out that for most situations that we will find ourselves in, we can use a more user-friendly criterion.

Path connectedness

Definition: A subset of C of $\mathbb{R}^n$ is said to be path connected if any two points in the subset can be joined by a path (that is the image of a continuous curve) inside C.

Exercise 1: Try showing that any path connected set is connected.

Thanks to Exercise 1, we see that our intuitive idea of what a connected set should be is consistent with our definition. In any event, all the domains we will come across in the normal course of things will actually be path connected.

Exercise 2: Find a subset in some $\mathbb{R}^n$ that is connected but not path connected (not easy).

Conservative vector fields are gradients

We will now prove the converse to our previous assertion.
Theorem 38: Let $\mathbf{F} : D \to \mathbb{R}^3$ be a conservative vector field on a path connected open domain in $\mathbb{R}^3$. Then $\mathbf{F}$ is the gradient of a scalar function.

Proof: As we discussed in class last time we can easily find a candidate scalar function f. Let $P_0 = (x_0, y_0, z_0)$ be a fixed point in D and let $P = (x, y, z)$ be an arbitrary point in D. We define

$$f(x, y, z) = \int_\gamma \mathbf{F} \cdot d\mathbf{s},$$

where $\gamma : [a, b] \to D$ is any (continuous) path from P_0 to P. Since D is path connected, f is defined on the whole of D. By hypothesis, $f(x, y, z)$ does not depend on which path we took from P to P_0.

It remains to show that $\mathbf{F} = \nabla f$.

The proof of Theorem 38 continued

Let us evaluate $\frac{\partial f}{\partial x}$. We will need to look at the quantity $f(x+h,y,z)$. This is given by the integral

$$\int_{\gamma_2} \mathbf{F} \cdot d\mathbf{s}.$$

where γ_2 is any path joining P_0 and the point $P_1 = (x+h,y,z)$.

We will choose our path as follows. Let γ_2 be the union of the two paths γ and the straight line γ_1 joining P and P_1. This straight line can be described as the set of points

$$\gamma_1(t) = \{(x+th,y,z) \,|\, 0 \leq t \leq 1\}.$$

From the property of line integrals we mentioned earlier

$$\int_{\gamma_2} \mathbf{F} \cdot d\mathbf{s} = \int_{\gamma} \mathbf{F} \cdot d\mathbf{s} + \int_{\gamma_1} \mathbf{F} \cdot d\mathbf{s}.$$

The proof of Theorem 38 continued

Hence we can write $f(x+h,y,z) =$

$$\int_\gamma \mathbf{F}\cdot d\mathbf{s} + \int_0^1 (F_1(x+th,y,z), F_2(x+th,y,z), F_3(x+th,y,z))\cdot(h,0,0)dt$$

Thus

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y,z)-f(x,y,z)}{h} = \lim_{h\to 0}\int_0^1 F_1(x+th,y,z)dt$$
$$= F_1(x,y,z).$$

We can similarly show that

$$\frac{\partial f}{\partial y} = F_2 \quad \text{and} \quad \frac{\partial f}{\partial z} = F_3.$$

This proves what we want.

Where in the proof have we used the fact of path independence of the line integral?