MA 105 D1 Lecture 3

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Autumn 2014, IIT Bombay, Mumbai

Cauchy sequences

As we saw last time, it is not easy to tell whether a sequence converges or not because we have to first guess what the limit might be and then try and prove that the sequence actually converges to this limit. For a monotonic sequence, things are slightly better since we only need to bound the sequence.

There is another very useful notion which allows us to decide whether the sequence converges by looking only at the elements of the sequence itself. We describe this below.

Definition: A sequence a_n in $\mathbb{R}$ is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon,$$

for all $m, n > N$.

Cauchy sequences: the theorem

Theorem 4: Every Cauchy sequence in $\mathbb{R}$ converges.

Remark 1: One can now check the convergence of a sequence just by looking at the sequence itself!

Remark 2: One can easily check the converse:

Theorem 5: Every convergent sequence is Cauchy.

Remark 3: Remember that when we defined sequences we defined them to be functions from $\mathbb{N}$ to X, for any set X. So far we have only considered $X = \mathbb{R}$, but as we said earlier we can take other sets, for instance, susbets of $\mathbb{R}$. For instance, if we take $X = \mathbb{R} \setminus 0$, Theorem 4 is not valid. The sequence $1/n$ is a Cauchy sequence in this X but obviously does not converge in X. If we take $X = \mathbb{Q}$, the example given in 1.5.(i) ($a_{n+1} = (a_n + 2/a_n)/2$) is a Cauchy sequence in $\mathbb{Q}$ which does not converge in $\mathbb{Q}$. Thus Theorem 4 is really a theorem about real numbers.

The completeness of $\mathbb{R}$

A set in which every Cauchy sequence converges is called a a complete set. Thus Theorem 4 is sometimes rewritten as

Theorem 4': The real numbers are complete.

We will see other examples of complete sets, but we can now address (very briefly) the question of what a real number is. More precisely, we can *construct* the set of real numbers starting with the rational numbers.

We let S be the set of all sequences with values in $\mathbb{Q}$. We will put a relation on this set.

The definition of a real number

Two sequence $\{a_n\}$ and $\{b_n\}$ will be related to each other (and we write $a_n \sim b_n$) if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - b_n| < \epsilon$$

for all $n > \mathbb{N}$.

You can check that this is an equivalence relation and it is a fact that it *partitions* the set S into disjoint classes. The set of disjoint classes is denote $S/\sim$.

You can easily see that if two sequences converge to the same limit, they are necessarily in the same class.

Definition: A real number is an equivalence class in $S/\sim$.

So a real number should be thought of as the collection of all rational sequences which converge to it.

Sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$

Most of our definitions for sequences in $\mathbb{R}$ are actually valid for sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$. Indeed, the only thing we really need to define the limit is the notion of distance. Thus if we replace the modulus function $|\ |$ on $\mathbb{R}$ by the distance functions in $\mathbb{R}^2$ and $\mathbb{R}^3$ all the definitions of convergent sequences and Cauchy sequences remain the same.

For instance, a sequence $a(n) = (a(n)_1, a(n)_2)$ in $\mathbb{R}^2$ is said to converge to a point $l = (l_1, l_2)$ (in $\mathbb{R}^2$) if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sqrt{(a(n)_1 - l_1)^2 + a(n)_2 - l_2)^2} < \epsilon$$

whenver $n > N$. A similar defintion can be made in $\mathbb{R}^3$ using the distance function on $\mathbb{R}^3$.

Theorems 2 (the Sandwich Theorem) and 3 (about monotonic sequences) don't really make sense for $\mathbb{R}^2$ or $\mathbb{R}^3$ because there is no ordering on these sets, that is, it doesn't really make sense to ask if one point on the plane or in space is less than the other.

The completeness of other spaces

Theorem 4, however, makes perfect sense - one can define Cauchy in $\mathbb{R}^2$ and $\mathbb{R}^3$ exactly as before, using the distance functions - and indeed, remains valid in $\mathbb{R}^2$ and $\mathbb{R}^3$.

So $\mathbb{R}^2$ and $\mathbb{R}^3$ are complete sets too (but $\mathbb{Q}^2$ and $\mathbb{Q}^3$ are not).

Finally, to emphasise that only the notion of distance matters for such questions we can define a distance function on $X = \mathcal{C}([a, b])$, the set of continuous functions from $[a, b]$ to $\mathbb{R}$, as follows:

$$\mathrm{dist}(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

Then Cauchy and convergent sequences in X can be defined as before, and we can prove (maybe next semester) that X is complete.

The rigourous definition of a limit of a function

Since we have already defined the limit of a sequence rigourously, it will not be hard to define the limit of a real valued function $f : (a, b) \to \mathbb{R}$.

Definition: A function $f : (a, b) \to \mathbb{R}$ is said to tend to (or converge to) a limit l at a point $x_0 \in [a, b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - l| < \epsilon$$

for all $x \in (a, b)$ such that $0 < |x - x_0| < \delta$. In this case, we write

$$\lim_{x \to x_0} f(x) = l,$$

or $f(x) \to l$ as $x \to x_0$ which we read as "$f(x)$" tends to l as x tends to x_0".

This is just the rigourous way of saying that the distance between $f(x)$ and l can be made as small as one pleases by making the distance between x and x_0 sufficiently small.

A subtle point and the rules for limits

Notice that in the definition above, the point x_0 can be one of the end points a or b.

Thus the limit of a function may exist even if the function is not defined at that point.

The rules and formulæ for limits of functions are the same as those for sequence and can be proved in almost exactly the same way. If $\lim_{x \to x_0} f(x) = l_1$ and $\lim_{x \to x_0} g(x) = l_2$, then

1. $\lim_{x \to x_0} f(x) \pm g(x) = l_1 \pm l_2$.
2. $\lim_{x \to x_0} f(x)g(x) = l_1 l_2$.
3. $\lim_{x \to x_0} f(x)/g(x) = l_1/l_2$. provided $l_2 \neq 0$

As before, implicit in the formulæ is the fact that if the limits on the left hand side exist. We prove the first rule below.

The proof of the addition formula for limits

Proof: We first show that $\lim_{x \to x_0} f(x) + g(x) = l_1 + l_2$. Let $\epsilon > 0$ be arbitrary.

Since $\lim_{x \to x_0} f(x) = l_1$ and $\lim_{x \to x_0} g(x) = l_2$, there exist δ_1, δ_2 such that

$$|f(x) - l_1| < \frac{\epsilon}{2} \quad \text{and} \quad |g(x) - l_2| < \frac{\epsilon}{2}$$

whenever $|x - x_0| < \delta_1$ and $|x - x_0| < \delta_2$.

If we choose $\delta = \min\{\delta_1, \delta_2\}$ and if $|x - x_0| < \delta$ then both the above inequalities hold.

Thus, if $|x - x_0| < \delta$, then

$$\begin{aligned} |f(x) + g(x) - (l_1 + l_2)| &= |f(x) - l_1 + g(x) - l_2| \\ &\leq |f(x) - l_1| + |g(x) - l_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

which is what we needed to prove. If we replace $g(x)$ by $-g(x)$ we get the second part of the first rule. $\square$

The Sandwich Theorem(s) for limits of functions

Theorem 5: As $x \to x_0$, if $f(x) \to l_1$, $g(x) \to l_2$ and $h(x) \to l_3$ for functions f, g, h on some interval (a, b) such that $f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b)$, then

$$l_1 \leq l_2 \leq l_3.$$

As before, we have a second version.

Theorem 6: Suppose $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = l$ and If $g(x)$ is a function satisfying $f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b)$, then $g(x)$ converges to a limit as $x \to x_0$ and

$$\lim_{x \to x_0} g(x) = l$$

Once again, note that we do not assume that $g(x)$ converges to a limit in this version of the theorem - we get the convergence of $g(x)$ for free .

Some examples

Let us look at Exercise 1.11. We will use this exercise to explore a few subtle points.
Let $c \in [a, b]$ and $f, g : (a, b) \to \mathbb{R}$ be such that $\lim_{x \to c} f(x) = 0$. Prove or disprove the following statements.

(i) $\lim_{x \to c}[f(x)g(x)] = 0$.

(ii) $\lim_{x \to c}[f(x)g(x)] = 0$, if g is bounded.($g(x)$ is said to be bounded on (a, b) if there exists $M > 0$ such that $|g(x)| < M$ for all $x \in (a, b)$.

(iii) $\lim_{x \to c}[f(x)g(x)] = 0$, if $\lim_{x \to c} g(x)$ exists.

Before getting into proofs, let us guess whether the statements above are true or false.

(i) false

(ii) true

(iii) true.

(i) Notice that $g(x)$ is not given to be bounded - if this was not obvious before, you should suspect that such a condition is needed after looking at part (ii). So the most natural thing to do is to look for a counter-example to this statement by taking $g(x)$ to be an unbounded function. What is the simplest example of an unbounded function $g(x)$ on an open interval?

How about $g(x) = \frac{1}{x}$ on $(0, 1)$?
What would a candidate for $f(x)$ be - what is the simplest example of a function $f(x)$ which tends to 0 for some value of c in $[0, 1]$.

$f(x) = x$, and $c = 0$ is a pretty simple candidate.

Clearly $\lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} 1 = 1 \neq 0$, which shows that (i) is not true in general.

Exercise 1: Can you find a counter-example to (i) with c in (a, b) (that is, c should not be one of the end points)? (Hint: Can you find an unbounded function on a closed interval $[a, b]$?)

Let us move to part (ii).
Suppose $g(x)$ is bounded on (a, b). This means that there is some real number $M > 0$ such that $|g(x)| < M$. Let $\epsilon > 0$. We would like to show that

$$|f(x)g(x) - 0| = |f(x)g(x)| < \epsilon,$$

if $|x - 0| = |x| < \delta$ for some $\delta > 0$.
Since $\lim_{x \to c} f(x) = 0$, there exists $\delta > 0$ such that $|f(x)| < \epsilon/M$ for all $|x| < \delta$. It follows that

$$|f(x)g(x)| = |f(x)||g(x)| < \frac{\epsilon}{M} \cdot M = \epsilon$$

for all $|x| < \delta$, and this is what we had to show.

Part (iii) follows immediately from
the product rule, but can one deduce part (iii) from (ii) instead? Hint: Think back to the lemma on convergent sequences that we proved in Lecture 1: Every convergent sequence is bounded. What is the analogue for functions which converge to a limit at some point? Indeed, you can easily show the following

Lemma 7: Let $f : (a, b) \to \mathbb{R}$ be a function such that $\lim_{x \to c} f(x)$ exists for some $c \in [a, b]$. If $c \in (a, b)$, there exists an (open) interval $I = (c - \eta, c + \eta) \subset (a, b)$ such that $f(x)$ is bounded on I. If $c = a$, then there is a half-open interval $I_1 = (a, a + \eta)$ such that $f(x)$ is bounded on I_1. Similarly if $c = b$, there exists a half-open interval $I_2 = (b - \eta, b)$ such that $f(x)$ is bounded on I_2.

The proof of the lemma above is almost the same as the the lemma for convergent sequences. Basically, replace "N" by "δ" in the proof.

If one applies the Lemma above to $g(x)$, we see that $g(x)$ is bounded in some (possibly) smaller interval $(0, \eta)$. Now apply part (ii) to this interval to deduce that (iii) is true.

Limits at infinity

There is one further case of limits that we need to consider. This occurs when we consider functions defined on open intervals of the form $(-\infty, b)$, (a, ∞) or $(-\infty, \infty) = \mathbb{R}$ and we wish to define limits as the variable goes to plus or minus infinity. The definition here is very similar to the definition we gave for sequences. Let us consider the last case.

Definition: We say that $f : \mathbb{R} \to \mathbb{R}$ tends to a limit l as $x \to \infty$ (resp. $x \to -\infty$) if for all $\epsilon > 0$ there exists $X \in \mathbb{R}$ such that

$$|f(x) - l| < \epsilon,$$

whenever $x > X$ (resp. $x < X$), and we write

$$\lim_{x\to\infty} f(x) = l \quad \text{or} \quad \lim_{x\to-\infty} f(x) = l.$$

or, alternatively, $f(x) \to l$ as $x \to \infty$ or as $x \to -\infty$, depending on which case we are considering.

Limits from the left and right

If $f : (a, b) \to \mathbb{R}$ is a function and $c \in (a, b)$, then it is possible to approach c from either the left or the right on the real line.

We can define the limit of the function $f(x)$ as x approaches c from the left (if it exists) as a number l^- such that for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - l^-| < \epsilon$ whenever $|x - c| < \delta$ and $x \in (a, c)$.

Our notation for this is $\lim_{x \to c-} f(x) = l^-$, and it is also called the left hand (side) limit.

Exercise 2: Write down a definition for the limit of a function from the right. We usually denote the right hand (side) limit by $lim_{x \to c+} f(x)$. Show, using the definitions, that $\lim_{x \to c} f(x)$ exists if and only if the left hand and right hand limits both exist and are equal.

We can also think of the left hand limit as follows. We restrict our attention to the interval (a, c), that is we think of f as a function only on this interval. Call this restricted function f_a. Then, another way of defining the left hand limit is

$$\lim_{x \to c-} f(x) = \lim_{x \to c} f_a(x).$$

It should be easy to see that it is the same as the definition before One can make a similar definition for the right hand limit.

The notions of left and right hand limits are useful because sometimes a function is defined in different ways to the left and right of a particular point. For instance, $|x|$ has different definitions to the left and right of 0.

Calculating limits explicitly

As with sequences, using the rules for limits of functions together with the Sandwich theorem allows one to treat the limits of a large number of expressions once one knows a few basic ones:

(i) $\lim_{x\to 0} x^\alpha = 0$ if $\alpha > 0$, (ii) $\lim_{x\to\infty} x^\alpha = 0$ if $\alpha < 0$,
(iii) $\lim_{x\to 0} \sin x = 0$, (iv) $\lim_{x\to 0} \sin x/x = 1$
(v) $\lim_{x\to 0}(e^x - 1)/x = 1$, (vi) $\lim_{x\to 0} \ln(1+x)/x = 1$

We have not concentrated on trying to find limits of complicated expressions of functions using clever algebraic manipulations or other techniques. However, I can't resist mentioning the following problem.

Exercise 3: Find

$$\lim_{x\to 0} \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}.$$

I will give the solution next time, together with the history of the problem (if I mention the history right away you will be able to get the solution by googling!), but feel free to use any method you like.

Continuity - the definition

Definition: If $f : [a, b] \to \mathbb{R}$ is a function and $c \in [a, b]$, then f is said to be continuous at the point c if and only if

$$\lim_{x \to c} f(x) = f(c).$$

Thus, if c is one of the end points we require only the left or right hand limit to exist.

A function f on (a, b) (resp. $[a, b]$) is said to be continuous if and only if it is continuous at every point c in (a, b) (resp. $[a, b]$).

If f is not continuous at a point c we say that it is dicontinuous at c, or that c is a point of discontinuity for f.

Intuitively, continuous functions are functions whose graphs can be drawn on a sheet of paper without lifting the pencil of the sheet of paper. That is, there should be no "jumps" in the graph of the function.

Continuity of familiar functions: polynomials

What are the functions we really know or understand? What does "knowing" or understanding a function $f(x)$ even mean? Presumably, if we understand a function f, we should be able to calculate the value of the function $f(x)$ at any given point x. But if you think about it, for what functions $f(x)$ can you really do this?

One class of functions is the polynomial functions. More generally we can understand rational functions, that is functions of the form $R(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials, since we can certainly compute the values of $R(x)$ by plugging in the value of x. How do we show that polynomials or rational functions are continuous (on $\mathbb{R}$)?

It is trivial to show from the definition that the constant functions and the function $f(x) = x$ are continuous. Because of the rules for limits of functions, the sum, difference, product and quotient (with non-zero denominator) of continuous functions are continuous. Applying this fact we see easily that $R(x)$ is continuous whenever the denominator is non-zero.

Continuity of other familiar functions

What are the other (continuous) functions we know?
How about the trignometric functions?
Well, here it is less clear how to proceed. After all we can only calculate $\sin x$ for a few special values of x ($x = 0, \pi/6, \pi/4, \ldots$ etc.). How can we show continuity when we don't even know how to compute the function?

Of course, if we define $\sin x$ as the y-coordinate of a point on the unit circle it seems intuitively clear that the y-coordinate varies continuously as the point varies on the unit circle, but knowing the precise definition of continuity this argument should not satisfy you.

We will not prove the continuity of $\sin x$ in this course, though we will given an idea of how this is done next week. So let us assume from now on that $\sin x$ is continuous. How can we show that $\cos x$ is continuous?

The composition of continuous functions

Theorem 8: Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to (e, f)$ be functions such that f is continuous at x_0 in (a, b) and g is continuous at $f(x_0) = y_0$ in (c, d). Then the function $g(f(x))$ (also written as $g \circ f(x)$ sometimes) is continuous at x_0. So the composition of continuous functions is continuous.

Exercise 4: Prove the theorem above starting from the definition of continuity.

Using the theorem above we can show that $\cos x$ is continuous if we show that $\sqrt{x}$ is continuous, since $\cos x = \sqrt{1 - \sin^2 x}$ and we know that $1 - \sin^2 x$ is continuous since it is the product of the sums of two continuous functions ($(1 + \sin x)$ and $(1 - \sin x)$!).

Once we have the continuity of $\cos x$ we get the continuity of all the rational trignometric functions, that is functions of the form $P(x)/Q(x)$, where P and Q are polynomials in $\sin x$ and $\cos x$, provided $Q(x)$ is not zero.