Expected Values

$$\langle x \rangle = \int_{-\infty}^{+\infty} \psi^*(x,t) x \psi(x,t) dx$$

$$= \frac{2}{L}\int_0^L x \sin^2 \frac{n\pi x}{L} dx$$

$$= \frac{2}{L}\int_0^L x \frac{1-\cos\frac{2n\pi x}{L}}{2} dx$$

$$= \frac{1}{L}\int_0^L \left(x - x\cos\frac{2n\pi x}{L} \right) dx$$

$$= \frac{1}{L}\left(\frac{L^2}{2}\right) - \frac{1}{L}\int_0^L x\cos\frac{2n\pi x}{L} dx$$

$$= \frac{L}{2}$$

The last step can be obtained by integrating by parts.

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} \psi^*(x,t) x^2 \psi(x,t) dx$$

$$= \frac{2}{L}\int_0^L x^2 \sin^2 \frac{n\pi x}{L} dx$$

$$= \frac{1}{L}\int_0^L \left(x^2 - x^2\cos\frac{2n\pi x}{L} \right) dx$$

$$= \frac{L^2}{3}\left[1 - \frac{3}{2n^2\pi^2}\right]$$

Verify last step after integrating by parts.

$$\langle p_x \rangle = -i\hbar \int_{-\infty}^{+\infty} \psi^*(x,t) \frac{\partial \psi(x,t)}{\partial x} dx$$

$$= \frac{-2i\hbar}{L} \int_0^L \frac{n\pi}{L} \sin\frac{n\pi x}{L} \cos\frac{n\pi x}{L} dx$$

$$= \frac{-i\hbar n\pi}{L^2} \int_0^L \sin\frac{2n\pi x}{L} dx$$

$$= 0$$

$$\langle p_x^{\,2} \rangle = -\hbar^2 \int_{-\infty}^{+\infty} \psi^*(x,t) \frac{\partial^2 \psi(x,t)}{\partial x^2} dx$$

$$= \frac{2\hbar^2}{L} \left(\frac{n\pi}{L}\right)^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$= \left(\frac{n\pi\hbar}{L}\right)^2$$

$$= 2mE_n$$

Uncertainties

$$\Delta p_x = \sqrt{\langle p_x^{\,2} \rangle - \langle p_x \rangle^2}$$

$$= \frac{n\pi\hbar}{L}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= \sqrt{\frac{L^2}{12 n^2 \pi^2} \left[n^2\pi^2 - 6 \right]}$$

$$\Delta x \Delta p_x = \hbar \sqrt{\frac{n^2\pi^2 - 6}{12}}$$

$= 0.57\hbar$ for n=1

$=1.67\hbar$ for n=2

SOME POSTULATES OF QM

1. System Description and Time Evolution

- A particle under a potential V(x) is described by a wave function $\psi(x)$, which contains the information about all the physical properties of the particle.
- The time evolution of $\psi(x)$ is governed by the time dependent Schrödinger Equation.

- The wave function $\psi(x)$ is single valued, finite and a continuous function of x.

The position derivative $\frac{d\psi}{dx}$ is also continuous , unless V(x) shows infinite jump.

Compare

From Krane (Modern Physics):

When an object moves across the boundary between two regions in which it is subjected to different [forces, potential energies], the basic behavior of the object is found by solving [Newton's second law, the Schrodinger equation].

The [position, wave function] of the object is always continuous across the boundary, and the [velocity, derivative dψ/dx] is also continuous as long as the [force, change in potential energy] remains finite.

2. Operators

- Each dynamical variable that relates to the motion of the particle can be represented by an operator, satisfying certain criteria.
- The only possible result of a measurement of the dynamical variable represented by an operator is one or the other Eigen values of the operator. $\hat{G}\phi_n = g_n\phi_n$

- The Eigen values are real numbers for the operators representing dynamical variables.

Hamiltonian Operator

It is defined as follows

$$\hat{H} \equiv \frac{\hat{p}^2}{2m} + V$$

Eigen Value Equation of Hamiltonian Operator is thus

$$\left(\frac{\hat{p}^2}{2m} + V\right)\phi_n = E_n\phi_n$$

Replacing by their operators we get

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \phi_n}{\partial x^2} + V\phi_n = E_n\phi_n$$

3. Completeness

- The Eigen States of an operator representing a dynamical variable are complete.
- Any admissible wave function can always be expressed in the following way in terms of Eigen functions of any operator. $\psi(x) = \sum_n c_n\phi_n$

- These Eigen functions form the basis.

Example

Any function f(x) that is defined between x=0 and x=L and *zero* everywhere else can be expressed as follows.

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x) \quad \text{where}$$

$$c_n = \int_0^L \phi_n{}^*(x) \;\; f(x)dx$$

The above can be easily visualized using the orthonormality property of the wave function.

4. Probability

- The probability that an Eigen value g_n would be observed as a result of measurement is proportional to the square of the magnitude of the coefficient c_n in the expansion of ψ .

$$P(g_n) \propto |c_n|^2$$

- The proportionality become equality if we have normalized wave function.

5. Collapse of Wave function

- If the measurement gives a particular value of Eigen value g_n, the wave function discontinuously collapses to ϕ_n .