MA 105 D1 Lecture 17

Ravi Raghunathan

Department of Mathematics

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A linear change of coordinates

Suppose we have a change in coordinates given by linear functions composed with translations (such functions are called affine linear functions):

$$x = au + bv + t_1 \quad \text{and} \quad y = cu + dv + t_2.$$

How does the area of the image of a rectangle under this map compare with the area of the original rectangle?

First, let us write down the linear map in more compact notation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

Clearly, a rectangle $[1, 0] \times [0, 1]$ in the $u - v$ plane is mapped to a parallelogram in the $x - y$ plane. The sides of the parallelogram are given by $(a + t_1, c + t_2)$ and $(b + t_1, d + t_2)$.

How does one compute the area of this parallelogram?

The area element for a change of coordinates

It is obviously given by the cross product of the vectors

$$(a, c, 0) \times (b, d, 0) = (ad - bc) \cdot \mathbf{k} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \mathbf{k}.$$

Let us now suppose that we have a general (not linear any more) change of coordinates given by $x = \phi(u, v)$ and $y = \psi(u, v)$. How does the area of a rectangle in the $u - v$ plane change?

Using the chain rule for functions of two variables we see that

$$\Delta x \sim \frac{\partial \phi}{\partial u} \Delta u + \frac{\partial \phi}{\partial v} \Delta v$$

and

$$\Delta y \sim \frac{\partial \psi}{\partial u} \Delta u + \frac{\partial \psi}{\partial v} \Delta v.$$

Using our previous notation, we can write

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \frac{\partial \phi}{\partial u} & \frac{\partial \phi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

The Jacobian

You may recognize the matrix

$$J = \begin{pmatrix} \frac{\partial \phi}{\partial u} & \frac{\partial \phi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{pmatrix}$$

that appears in the preceding formula. It is the derivative matrix for the function $h = (\phi, \psi) : \mathbb{R}^2 \to \mathbb{R}^2$.

We know that the derivative matrix is the linear approximation to the function h, at least in a neighborhood of a point, say (u_0, v_0). Or, to say it slightly differently, in a neighborhood of the point (u_0, v_0), the function h and the function J, behave very similarly.

In particular, it is easy to see how the area of a small rectangle changes under h, since we have already done so in the case of a linear map. It simply changes by the determinant of J! This is how we get the change of variable formulæ.

The change of variables formula

Let D be a region in the xy plane and D^* a region in the uv plane such that $h(D^*) = D$. Then

$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(x(u,v), y(u,v))\frac{\partial(x,y)}{\partial(u,v)}dudv.$$

Let us see what we get in the familiar case of polar coordinates. We have

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix} = r,$$

which is what we have already obtained in this case.

Exercise 6.4: Evaluate the integral

$$\iint_D (x-y)^2 \sin^2(x+y)\,dxdy$$

where D is the parallelogram with vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$.

Solution: Put

$$x = \frac{u-v}{2}, y = \frac{u+v}{2},$$

Then the rectangle

$$R = \{\pi \leq u \leq 3\pi, \;\; -\pi \leq v \leq \pi\}$$

in the uv-plane gets mapped to D, a parallelogram in the xy-plane. Further,

$$J = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

$$\int\int_D (x-y)^2 \sin^2(x+y)\,dxdy = \int\int_R v^2 \sin^2(u)\frac{1}{2}\,dudv$$

$$= \frac{1}{2}\left(\int_{-\pi}^{\pi} v^2 dv\right)\left(\int_{\pi}^{3\pi} \sin^2(u)du\right)$$

$$= \frac{1}{2}\left(2 \times \frac{\pi^3}{3}\right)(\pi) = \frac{\pi^4}{3}.$$

Exercise 6.5: Let D be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1,\ xy = 9$ and the lines $y = x,\ y = 4x$. Find

$$\iint_D dA$$

by transforming it to

$$\iint_E dudv,$$

where $x = \frac{u}{v}, y = uv, v > 0$.

Solution: Put

$$x = \frac{u}{v},\ \ y = uv.$$

Then the rectangle $R = \{1 \le u \le 3, 1 \le v \le 2\}$ in the uv-plane gets mapped to D in the xy-plane. Further,

$$J = \begin{vmatrix} 1/v & -u/v^2 \\ v & u \end{vmatrix} = \frac{2u}{v}$$

Hence,

$$\int\int_D dA) = \mathrm{Area}(D) = \int\int_R \frac{2u}{v} dudv$$

$$= \left(\int_1^3 2udu\right)\left(\int_1^2 \frac{dv}{v}\right) = 8\ln 2.$$

The validity of the change of variables formula

We have pushed many things under the carpet in our hurry to develop a change of variables formula. We must perhaps ask ourselves when and under what conditions the formula might be valid.

The first question to ask is "What does changing coordinates mean?". What kinds of functions h is it reasonable to consider?

For instance, should we allow the following transformation $h(u, v) = (1, 2)$ (a constant map)?

Clearly not, since all information about the region D is lost when we make such a transformation, considering such functions will not lead to anything useful.

Properties to look for in coordinate changes

One of the problems with the preceding example is that it "destroys area", that is, a rectangle with non-zero area is taken to a point with zero area.

It should not be hard to convince yourself that transformations h that do this should not be "allowed". So the function $h(u, v) = (u, 0)$ or the function $h(u, v) = (0, v^2)$ is "disqualified". More generally, when is "area destroyed" by h?

Clearly when $|J| = 0$. So the non-vanishing of $|J|$ is clearly necessary. The remarkable fact is that, locally at least, this guarantees that the function is bijective (just like in the once variable case)!

If we remember the method of substitution in one variable, only substitution changes that were bijective were allowed. We should clearly require this of our function h, not just locally (which is guaranteed if we assume $|J| \neq 0$), but for the whole domain.

Diffeomorphisms

Finally, another requirement comes from the final formula we derived. This includes $|J|$, which, if we want some kind of integrability, forces h to be a $\mathcal{C}^1$ function.

The final condition we must impose is that h must be a diffeomorphism, that is, the inverse function h^{-1} (which exists, since h is assumed bijective) must also be a $\mathcal{C}^1$ function. This will ensure that the image $h(D)$ of a small disc D in the $u - v$ plane will contain a small disc in the $x - y$ plane.

Again, this is automatic locally, if we assume that $|J| \neq 0$. This fact is called the Inverse function theorem.

Coordinate changes

Let us summarize our discussion above. A change of coordinates $h : D \to D' \subset \mathbb{R}^2$ is a function $h(u, v) = (\phi(u, v), \psi(u, v))$ which is $\mathcal{C}^1$ on D, which is a bijection from D to D' and such that h^{-1} is a $\mathcal{C}^1$ function.

There is only one thing we have still been vague about. We have not specified what kind of regions D and D' should be. For our purposes we will assume that D and D' are the interiors of regions of type 3 (the interior of a region of type 3 is simply the part of the region that is not on the boundary - for instance, the open disc is the interior of the closed disc). More generally, we can assume that D and D' are open sets (see next slide).

Open sets

Definition: A subset U of $\mathbb{R}^n$ is called open if around every point $x \in U$ and some $r > 0$, there is a disc $D_r(x)$ of radius r around x contained inside U.

An open interval in $\mathbb{R}$ is an example of an open set in $\mathbb{R}$ ($n = 1$). Finite and countable unions of open intervals are also open sets in $\mathbb{R}$.

Exercise 1: Conversely, prove that any open set in $\mathbb{R}$ is necessarily the (countable) disjoint union of open intervals.

Open discs and rectangles are examples of open sets in $\mathbb{R}^2$ as are unions (finite or infinite) of theses. The n-dimensional open discs and rectangles are open sets in $\mathbb{R}^n$.

The Inverse Function Theorem

Theorem 38: If U is an open set in $\mathbb{R}^n$ and $F : U \to \mathbb{R}^n$ is a $\mathcal{C}^1$ function such that the Jacobian determinant of F at p, $|J_F(p)| \neq 0$, for some point $p \in U$, then the inverse function F^{-1} exists for some open set V containing $F(p)$ and F^{-1} is a $\mathcal{C}^1$ function.

Another way of thinking of the Inverse Function Theorem is the following. If x and y are functions of u and v, the Inverse Function Theorem tells us that (when $|J| \neq 0$) we can write u and v as functions of x and y, at least in some small open set V.

As I mentioned above, in order to check whether the function h gives a coordinate change, it is enough by the Inverse Function Theorem to check that it is bijective and that its Jacobian determinant is everywhere non-zero.

Triple integrals in a box

If we have a function $f : B = [a, b] \times [c, d] \times [e, f] \to \mathbb{R}$ we can integrate it over this rectangular parallelepiped. As in the one and two variable cases, we divide the parallelepiped into smaller ones B_{ijk}, making sure that the length, breadth and height of the small parallelepiped are all small. In particular, we can use the regular partition of order n to obtain the Riemann sum

$$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(t_{ijk}) \Delta B_{ijk},$$

where ΔB_{ijk} is the volume of B_{ijk}, and $t = \{t_{ijk} \in B_{ijk}\}$ is an arbitrary tag.

As before we say that f integrable $\lim_{n\to\infty} S(f, P_n, t)$ exists for any choice of tag t. The value of this limit is denoted by

$$\iiint_B f dV, \iiint_B f(x, y, z) dV \quad \text{or} \quad \iiint_B f(x, y, z) dxdydz.$$

Triple integrals over arbitrary regions

All the theorems for double integrals go through for triple integrals.

First, if f is bounded and continuous in B, except possibly on (a finite union of) graphs of $\mathcal{C}^1$ functions of the form $z = a(x, y)$, $y = b(x, z)$ and $x = c(y, z)$, then it is integrable.

This allows us to define the integral of (say) a continuous function on any bounded region enclosed by a simple $\mathcal{C}^1$ closed curve. As before, simply extend the function by zero on a larger enclosing rectangle.

Once we have defined the triple integral in this way, it remains to evaluate it.

Evaluating triple integrals

Again we have a Fubini Theorem - that is, triple integrals can usually be expressed as iterated integrals, this time by integrating functions of a single variable three times.

Thus, if f integrable on the box B we have

$$\iiint_B f(x,y,z)dxdydz = \int_a^b \int_c^d \int_e^f f(x,y,z)dzdydx.$$

There are, in fact, five other possibilities for the iterated integrals and each of these exists and is equal to the value of the triple integral.

The triple integrals that are easiest to evaluate are those for which the region P in space can be described by bounding z between the graphs of two functions in x and y. This is the analogue of an elementary region in the plane and we will call such regions also elementary regions (but in space). In general, we may be able to express more complicated domains as unions of elementary domains.

Evaluating triple integrals continued

In this case we proceed as follows. Suppose that the region P lies between $z = \gamma_1(x, y)$ and $z = \gamma_2(x, y)$. Suppose that the projection of P on the xy plane is bounded by the curves $y = \phi_1(x)$ and $y = \phi_2(x)$ and the straight lines $x = a$ and $x = b$, then

$$\iiint_P f(x, y, z)dxdydz = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)dzdydx.$$

Example: Let us find the volume of the sphere using the above formula. In other words, let us integrate the function 1 on the region P, where P is the unit sphere.

The sphere can be described as the region lying between $z = -\sqrt{1 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$.

The volume of the unit sphere

The projection of the sphere onto the xy plane gives a disc of unit radius. This can be described as the set of points lying between the curves $-\sqrt{1-x^2}$ and $\sqrt{1-x^2}$ and the lines $x = \pm 1$. Thus our triple integral reduces to the iterated integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dzdydx.$$

This yields

$$2\int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)^{1/2} dy \right] dx.$$

After evaluating the inner integral we obtain

$$2\pi \int_{-1}^{1} \frac{1-x^2}{2} dx = \frac{4}{3}\pi.$$

The change of variables formula in three variables

In three variables, we once again have a formula for a change of variables. The formula has the same form as in the two variable case:

$$\iiint_P f(x,y,z)dxdydz = \iiint_{P^*} g(u,v,w)\frac{\partial(x,y,z)}{\partial(u,v,w)}dudvdw,$$

where $h(P^*) = P$. If the change in coordinates is given by $h = (\phi, \psi, \rho)$, the function g is defined as $g = f(\phi, \psi, \rho)$. The expression

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}$$

is just the Jacobian determinant for a function of three variables.