PH-107 (2014)
Tutorial Sheet 3
(Wave Function, Operators, Particle in a Box)

* Problems to be done in tutorial.

H: Schrödinger Equation and Wave Function:

P46: If we had adopted a convention in which the time dependence of free wave was taken to be of the form $e^{+i\omega t}$, find the momentum and energy operators. Also find out the time dependent Schrödinger Equation.

P47: If $\psi_1(x,t)$ and $\psi_2(x,t)$ are solutions of time dependent Schrödinger equation, show that $a\psi_1(x,t)+b\psi_2(x,t)$, where *a* and *b* are constants, is also a solution of the same.

P48*: If $\phi_n(x)$ are the solutions of time independent Schrödinger equation, with energies E_n, show that $\psi(x,t)=\sum_n c_n\phi_n(x)e^{\frac{-iE_nt}{\hbar}}$, where C_n are constants, is a solution of time dependent Schrödinger equation. However, show that $\psi(x,0)$ is not a solution of the time independent Schrödinger equation.

P49: For an operator $\hat{p}$ defined below, find out $\hat{p}\psi$.

$$\hat{p}\equiv\left(\hat{x}+\frac{\partial}{\partial x}\right)^2$$

P50*: Find the Eigen function $\varphi(x)$ of the following operator

$$\hat{G}=-i\hbar\frac{d}{dx}+Ax$$

Here A is a constant. If this Eigen function is subjected to a boundary condition $\varphi(a)=\varphi(-a)$, find out the Eigen values.

P51*: Find the angular momentum operator in Cartesian co-ordinate system.

P52*: Show that the expectation value of momentum for any well-behaved function is always real.

P53: There are a large number (N) of identical experimental set-ups. In each of these set ups, a single particle is described by the wave function $\phi(x) = Ae^{-\frac{x^2}{a^2}}$ at $t=0$, where A is the normalization constant, and a is a constant of the dimension length. If a measurement of position of the particle is carried out at time $t=0$ in all these set-ups, it is found that in 100 of these the particle is found within infinitesimal interval of $x=2a$ and $2a+dx$. Find out in how many of the measurements the particle would have been found in the infinitesimal interval of $x=a$ and $a+dx$.

P54*: $|\psi_1\rangle$ and $|\psi_2\rangle$ are the normalized eigen functions of an operator $\hat{P}$, with eigen values P_1 and P_2 respectively. If the wave function of the particle is $0.25|\psi_1\rangle + 0.75|\psi_2\rangle$ at $t=0$; find the probability of observing P_1?

P55: An observable '*A*' is represented by an operator $\hat{A}$. Two of its normalized eigenfunctions are given as ϕ_1 and ϕ_2, corresponding to distinct eigenvalues a_1 and a_2 respectively. Another observable '*B*' is represented by an operator $\hat{B}$. Two normalized eigenfunctions of this operator are given as u_1 and u_2 with distinct eigenvalues b_1 and b_2 respectively. The eigenfunctions, ϕ_1 and ϕ_2 can be written in terms of u_1 and u_2 in the following way.

$$\phi_1 = D(3u_1 + 4u_2);\ \phi_2 = F(4u_1 - Pu_2)$$

At time $t=0$, a particle is in a state given as $\left(\frac{2}{3}\phi_1 + \frac{1}{3}\phi_2\right)$.

(a) Find the values of '*D*', '*F*' and '*P*'.

(b) If a measurement of '*A*' is carried out at $t=0$, what are the possible results and what are their probabilities?

(c) Assume that the measurement of '*A*' mentioned above yielded a value a_1. If a measurement of '*B*' is carried out immediately after this, what would be the possible outcomes and what would be their probabilities?

(d) If instead of following the above path, a measurement of '*B*' was carried out initially at $t=0$, what would be the possible outcomes and what would be their probabilities?

Assume that after performing the measurements described in (c), the outcome was b_2.What would be the possible outcomes, if '*A*' was measured immediately after this and what would be the probabilities

I: Particle in a box:

P56: Suppose we have 10,000 rigid boxes of same length L from $x = 0$ to $x = L$. Each box contains one particle of the same mass. All these particles are in the ground state. If a measurement of position of the particle is made in all of these boxes at the same time, in how many of them, the particle is expected to be found between x =0 and $L/4$. In a particular box, the particle was found to be between x =0 and $L/4$. Another measurement of position of the particle is carried out in this box immediately after the first measurement. What is the probability that the particle is again found between x =0 and $L/4$.

P57: Consider a particle confined to a one-dimensional box. Find the probability that the particle in its ground state will be in the central one-third region of the box.

P58*: For a particle in one-dimensional box of side L, show that the probability of finding the particle between $x=B$ and $x=B+b$ approaches the classical value b/L, if the energy of the particle is very high.

P59: Consider a one dimensional infinite square well potential of length L. A particle is in $n=3$ state of this potential well. Find the probability that this particle will be observed between $x = 0$ and $x = (L/6)$. Can you guess the answer without solving the integral? Explain how.

P60: Solve the time independent Schrödinger equation for a particle in one dimensional box taking the origin at its mid-point and the ends at $\pm(L/2)$, where L is the length of the box.

P61: The wave function of a particle in a one-dimensional box of length L and potential zero at time $t=0$ is given as follows.

$$\psi(x,0) = \sqrt{\frac{2}{5L}}\sin\frac{\pi x}{L} + \sqrt{\frac{8}{5L}}\sin\frac{4\pi x}{L}$$

(a) Is this wave function normalized?
(b) Is the particle in a stationary state?
(c) If the energy measurement is made on his particle, what are the possible values one might get? What are the corresponding probabilities?
(d) What is the expectation value of the energy of this particle?

P62*: The wave function of a particle in a one-dimensional box of length *L* and potential zero at time *t=0* is given as follows.

$$\psi(x,0) = A\sin\frac{\pi x}{L} + 2A\sin\frac{3\pi x}{L} + \sqrt{11}A\sin\frac{5\pi x}{L}$$

(a) Find the wave function $\Psi(x,t)$, at a later time *t*.

(b) If the measurements of energies are carried out, what are the values that will be found and what are the corresponding probabilities?

(c) If there are a large number of identical systems, each one of them represented by wave functions as above, what would be the average energy found if measurements are done in all of them at the same time

(d) If the measurement yielded the lowest value of energy, what would be the wave function later and what values of energy would be found later.

P63: A particle in a one-dimensional box (V = 0 for 0<x<L , V = ∞ elsewhere) has the following wave function at *t=0*.

$$\phi(x) = A\ \sin\frac{\pi\ x}{L}\cos\frac{2\ \pi\ x}{L} \quad \text{between } 0<x<L \text{ and } 0 \text{ elsewhere}$$

(a) Find A

(b) On making an energy measurement, which are the energy values that the particle may be found to have and with what probability

P64*: Suppose we have 10,000 rigid boxes of same length L, each box containing one particle of the same mass *m*. Let us assume that all particles are described by the same wave function. On performing the energy measurement on all the 10,000 particles at the same time we find only two energy values, one corresponding to *n = 2* and the other corresponding to *n = j*. If in 4,000 measurements, one obtains value corresponding to n=2 and if the average value of the energy found in all the measurements is $\frac{7\hbar^2\pi^2}{2mL^2}$, find the value of *j* and the wave function of the particles.

P65*: A particle in a one-dimensional well (V=0 for 0<x<L, V=∞ elsewhere) has the wave function $\phi(x) = Ax(L-x)$ inside the box and $\phi(x) = 0$ elsewhere at *t=0*. Calculate the expectation value of energy. On making an energy measurement, what is the probability of finding the particle in the ground state?

P66: A particle in a one-dimensional box (V = 0 for 0<x<L , V = ∞ elsewhere) has the following wave function at *t=0*.

$$\phi(x) = A\ \sin^2\left(\frac{\pi\ x}{L}\right) \text{ between } 0<x<L \text{ and } 0 \text{ elsewhere}$$

(a) Find A

(b) On making an energy measurement, what is the probability that the particle would be found in ground state.

(c) Find the mean value of the energy, if energy measurements are made in a large number of identical boxes represented by this wavefunction.

You could use the following integrals.

$$\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x dx$$

$$\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4}$$

P67: For a particle in a one-dimensional box in n^{th} state, show that the uncertainty product is $\Delta x \Delta p_x = \hbar\sqrt{\frac{n^2\pi^2 - 6}{12}}$.