MA 105 D1/D3 - Review

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Limits

Definition: A function $f : U(\subset \mathbb{R}^2) \to \mathbb{R}$ is said to tend to a limit l as $x = (x_1, x_2)$ approaches $c = (c_1, c_2)$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - l| < \epsilon,$$

whenever $0 < \|x - c\| < \delta$.

Remember, that this means that no matter along which curve one approaches c, the limit must the be same. Keeping this in mind allows one to show in certain examples that a limit does not exist by choosing suitable curves along which it is easy to calculate the limit.

To actually prove the limit exists you must use the definition.

Limits and continuity

When actually evaluating limits you will usually not use the first principles. Instead you will use the rules for limits and the Sandwich theorems.

The definition of limit has been given for functions of two variables. Of course, a similar definition holds for three variables.

Definition: The function $f : U \to \mathbb{R}$ is said to be continuous at a point c if

$$\lim_{x \to c} f(x) = f(c).$$

Again remember that this is a powerful condition since the existence of a limit is implicit in the definition.

Partial derivatives

Definition: The partial derivative of $f : U \to \mathbb{R}$ with respect to x_1 at the point (a, b) is defined by

$$\frac{\partial f}{\partial x_1}(a, b) := \lim_{x_1 \to a} \frac{f((x_1, b)) - f((a, b))}{x_1 - a}.$$

Similarly, one can define the partial derivative with respect to x_2. In this case the variable x_1 is fixed and f is regarded only as a function x_2:

$$\frac{\partial f}{\partial x_2}(a, b) := \lim_{x_2 \to b} \frac{f((a, x_2)) - f((a, b))}{x_2 - b}.$$

The directional derivative

The partial derivatives are special cases of the directional derivative. Let $v = (v_1, v_2)$ be a unit vector. Then v specifies a direction in $\mathbb{R}^2$.

Definition: The directional derivative of f in the direction v at a point $x = (x_1, x_2)$ is defined as

$$\nabla_v = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} = \lim_{t \to 0} \frac{f(x_1 + tv_1, x_2 + tv_2) - f((x, x_2))}{t}.$$

It measures the rate of change of the function f along the path $x + tv$

If we take $v = (1, 0)$ in the above definition, we obtain $\partial f / \partial x_1$, while $v = (0, 1)$ yields $\partial f / \partial x_2$.

Having partial derivatives, or even directional derivatives does not imply a great deal about a function. In fact in Tutorial 4 and in Lectures 9-10 we saw various examples where the partial or directional derivatives exist but the function is not continuous. Likewise we also saw examples where the iterated limits taken in different orders are not equal.

We can also define the derivative of a function in two (or three variables).

Definition: A function $f : U \to \mathbb{R}$ is said to be differentiable at a point (x_0, y_0) if $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ exist and

$$\lim_{(h,k)\to 0} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)h - \frac{\partial f}{\partial y}(x_0, y_0)k}{\|(h, k)\|} \to 0,$$

as $\|(h, k)\| \to 0$.

This is saying that the distance between the tangent plane and the surface is going to zero even after dividing by $\|(h, k)\|$.

A criterion for differentiability

Before we state the criterion, we note that with our definition of differentiability, every differentiable function is continuous.

Theorem 26: Let $f : U \to \mathbb{R}$. If the partial derivatives $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ exist and are continuous in a neighbourhood of a point (x_0, y_0) (that is in a region of the plane of the form $\{(x, y) \mid \|(x, y) - (x_0, y_0)\| < r\}$ for some $r > 0$. Then f is differentiable at x.

In practice this is how we actually check if a function is differentiable. Another way of expressing Theorem 26 is to say that $\mathcal{C}^1$ functions are differentiable.

When viewed as a row vector rather than as a matrix, the Derivative matrix is called the gradient and is denoted $\nabla f(x_0, y_0)$. Thus

$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right).$$

In terms of the coordinate vectors **i** and **j** the gradient can be written as

$$\nabla f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\mathbf{i} + \frac{\partial f}{\partial y}(x_0, y_0)\mathbf{j}.$$

As one sees easily, the gradient is related to the directional derivative in the direction v.

$$\nabla_v f = \nabla f \cdot v.$$

The chain rule

We assume that $x, y : I \to \mathbb{R}$ are differentiable functions from some interval (open or closed) to $\mathbb{R}$ and that $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable.

Theorem 27: With notation as above

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

For a function $w = f(x, y, z)$ in three variables the chain rule takes the form

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

The gradient and the directional derivative

Let $c(t)$ be any curve in $\mathbb{R}^3$. Then, clearly by the chain rule we have

$$\frac{df}{dt} = \nabla f(c(t)) \cdot c'(t).$$

This allows us to see that a function $f(x, y, z)$ increases fastest in the direction of the gradient ∇f.

The total derivative

We now define the derivative for a function $f : U \to \mathbb{R}^n$, where U is a subset of $\mathbb{R}^m$.

The function f is said to be differentiable at a point x if there exists a an $n \times m$ matrix $Df(x)$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(x+h) - f(x) - Df(x) \cdot h\|}{\|h\|} = 0.$$

Here $x = (x_1, x_2, \dots, x_m)$ and $h = (h_1, h_2, \dots h_m)$ are vectors in $\mathbb{R}^m$.

The matrix $Df(x)$ is usually called the total derivative of f. It is also referred to as the Jacobian matrix/Derivative matrix. What are its entries?

The Derivative matrix

Here is the total derivative or the derivative matrix written out fully.

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \cdots & \frac{\partial f_1}{\partial x_m}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \cdots & \frac{\partial f_2}{\partial x_m}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \frac{\partial f_n}{\partial x_1}(x) & \frac{\partial f_n}{\partial x_2}(x) & \cdots & \frac{\partial f_n}{\partial x_m}(x) \end{pmatrix}$$

In the 2×2 case we get

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \end{pmatrix}.$$

Recall that the derivative may be viewed as a linear map, this time from $\mathbb{R}^m$ to $\mathbb{R}^n$ (or, in the case just above, from $\mathbb{R}^2$ to $\mathbb{R}^2$).

Rules for the total derivative

Just like in the one variable case, it is easy to prove that

$$D(f+g)(x) = Df(x) + Dg(x).$$

[Somewhat harder, but only because the notation gets more cumbersome, is the Chain rule:

$$D(f \circ g)(x) = Df(g(x)) \circ Dg(x),$$

where $\circ$ on the right hand side denotes matrix multiplication.

Higher derivatives

Just as we repeatedly differentiated a function of one variable to get higher derivatives, we can also look at higher partial derivatives.

Theorem 28: Let $f : U \to R$ be a function such that the partial derivatives $\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}(f)\right)$ exist and are continuous for every $1 \leq i, j \leq m$. Then,

$$\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}(f)\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}(f)\right).$$

From now on we will use use the following notation. By

$$\frac{\partial^n f}{\partial x_1^{n_1} \partial x_2^{n_2} \ldots, \partial x_k^{n_k}},$$

Because of Theorem 28, for smooth functions it does not matter in what order we take the partial derivatives (actually, we only need $\mathcal{C}^n$ functions if we are taking derivatives of order unto n, that is we only need to assume that all (mixed) partial derivatives of order unto n are continuous.

Maxima and minima

Theorem 29: If (x_0, y_0) is a local extremum point (that is, a minimum or a maximum point) and if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, then (x_0, y_0) is a critical point.

Recall that a A point (x_0, y_0) is called a critical point of $f(x, y)$ if

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$

We want to determine whether a critical point is an extremum and if so whether it is a minimum or maximum. Recall the definition of the Hessian of f. This is the matrix

$$\begin{pmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{pmatrix}.$$

It determinant is discriminant and is sometimes denoted D. Explicitly,

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2.$$

The second derivative test

We give a test for finding local maxima and minima below.

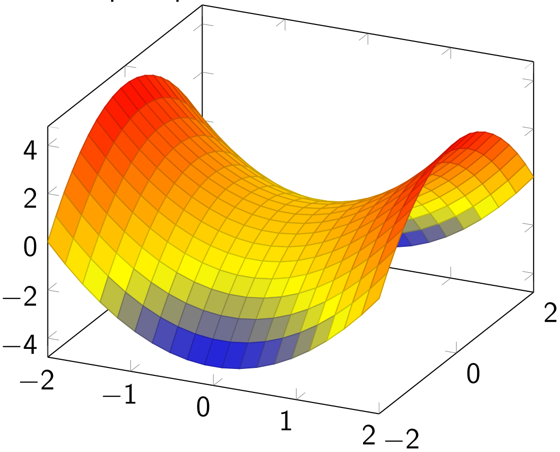
Theorem 30: With notation as above:

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum for f.
2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum for f.
3. If $D < 0$, then (x_0, y_0) is a saddle point for f.
4. If $D = 0$, further examination of the function is necessary.

Near a saddle point the function takes larger or smaller values depending on the direction in which one travels.

Saddle points

Since a picture is worth a thousand words, let me remind you of the one I put up.



The point $(0,0)$ is called a saddle point. This is a picture of the graph of $z = x^2 - y^2$.

Global extrema as local extrema

Definition: A point (x_0, y_0) such that $f(x, y) \leq f(x_0, y_0)$ or $f(x, y) \geq f(x_0, y_0)$ for all (x, y) in the domain being considered is called a global maximum or minimum point respectively.

Theorem 32: A continuous function on a compact set in $\mathbb{R}^2$ will attain its extreme values.

in $\mathbb{R}^n$ compact sets are nothing but closed (complement is open) and bounded sets. In $\mathbb{R}^2$, examples include closed rectangles, closed discs and finite unions of these.

Procedure for finding the global maximum

Assume now that f is a $\mathcal{C}^2$ function on a closed rectangle $\bar{S}$ as above. We can find the global maximum or minimum as follows. We first study all the local extrema which by definition lie in the open rectangle

$$S = \{(x, y) \mid |x - x_0| < a, |y - y_0| < b\}$$

After determining all the local maxima we take the points where the function takes the largest value - say M_1. We compare this with the maximum value of the function on the boundary of the closed rectangle, say M_2. Let M be the maximum of these two values. The points where M is attained are the global maxima.

Sometimes, global extrema may exist even when the domain in $\mathbb{R}^2$ (which is not compact, in fact, not even bounded).

The definition of the Riemann integral

The regular partition of R of order n is is partition defined inductively by $x_0 = a$ and $y_0 = c$ and

$$x_{i+1} = x_i + \frac{b-a}{n} \quad \text{and} \quad y_{j+1} = y_j + \frac{d-c}{n},$$

$1 \leq i, j \leq n-1$. We take $t = \{t_{ij} \in R_{ij}\}$ to be an arbitrary tag.

Definition: We say that the function $f : R \to \mathbb{R}$ is Riemann integrable if the Riemann sum

$$S(f, P_n, t) = \sum_{i,j=0}^{n-1} f(t_{ij})\Delta_{ij}$$

tends to a limit S for any choice of tag t.

The Riemann integral continued

This limit value is usually denoted as

$$\int\int_R f, \quad \int\int_R f(x,y)dA, \quad \text{or} \quad \int\int_R f(x,y)dxdy.$$

The preceding definition is sometimes rewritten as

$$\lim_{n\to\infty} \sum_{i,j=0}^{n-1} f(t_{ij})\Delta_{ij} = \int\int_R f.$$

If $f(x,y) \geq 0$ for all values of x and y, then the Riemann integral has a geometric interpretation. It is obviously the volume of the region under the graph of the function $z = f(x,y)$ and above the rectangle R in xy-plane.

Properties of the Double integral

The Riemann integral for functions of two variables is sometimes called the Double Integral. Its properties may be summed up as follows

1. If R is divided by a (vertical or horizontal) line segment into two rectangles R_1 and R_2 and if f is integrable on R_1 and R_2, then f is integrable on R and

$$\int\int_R f(x,y)dxdy = \int\int_{R_1} f(x,y)dxdy + \int\int_{R_2} f(x,y)dxdy.$$

2. If f_1 and f_2 are integrable, and if $f_1 \leq f_2$ on R, then

$$\int\int_R f_1(x,y)dxdy \leq \int\int_R f_2(x,y)dxdy.$$

3. If $f(x,y) = c$ for all (x,y) in R,

$$\int\int_R f(x,y)dxdy = c \times A(R),$$

where $A(R)$ denotes the area of R.

Further properties

4. If f_1 and f_2 are integrable functions on R, then
$$\int\int_R [f_1 + f_2] dA = \int\int_R f_1 dA + \int\int_R f_2 dA.$$
5. For any constant c,
$$\int\int_R cf dA = c\int\int_R f dA.$$
6. $$\left|\int\int_R f(x,y) dx dy\right| \leq \int\int_R |f(x,y)| dx dy.$$

We have not previously stated the above property in the one variable case where it also holds. All the properties above can be proved quite easily from the definitions, just as in the one variable case.

Reduction to iterated integrals

Theorem 34: If f is integrable on the rectangle $R = [a, b] \times [c, d]$, and if either one of the iterated integrals

$$\int_c^d \left[\int_a^b f(x, y) dx \right] dy \quad \text{or} \quad \int_a^b \left[\int_c^d f(x, y) dy \right] dx$$

exists, then it equals the double integral

$$\int \int_R f(x, y) dxdy.$$

In particular, if both the iterated integrals exist they must be equal.

If f is bounded and continuous on $R \setminus \{\text{a finite union of continuously differentiable curves}\}$, then both iterated integrals will exist and will be equal.

The Slice Method

Cavalieri's basic idea is that we can find the volume of a given solid by slicing it into thin cross sections, calculating the areas of the slices and then adding up these areas.

Let S be a solid and P_x be a family of planes perpendicular to the x-axis with x as x-coordinate such that

1. S lies between P_a and P_b,
2. the area of the slice of S cut by P_x is $A(x)$.

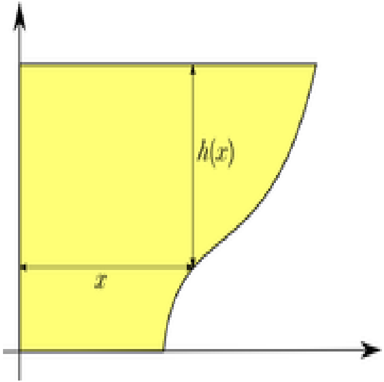
Then the volume of S is given by

$$\int_a^b A(x)dx.$$

Solids of revolution obtained by taking a region B lying between the lines $x = a$ and $x = b$ on the x-axis and the graph of a function $y = f(x)$ and rotating it through an angle 2π around the x-axis: the volume is given by

$$V = \pi \int_a^b [f(x)]^2 dx.$$

The shell method continued



From http://en.wikipedia.org/wiki/Shell_integration

The radius of the cylindrical shell above the point $(x, f(x))$ is x and the height is $f(x)$. Hence its surface area is $2\pi x h(x)$. To get the volume we must integrate, and this yields

$$2\pi \int_a^b x h(x) dx.$$

Elementary regions

We will call the two simple types of regions that we are going to describe elementary regions.

Let $h_1, h_2 : [a, b] \to \mathbb{R}$ be two $\mathcal{C}^1$ functions. Consider the set of points

$$D_1 = \{(x, y) \mid a \leq x \leq b \text{ and } h_1(x) \leq y \leq h_2(x)\}.$$

Such a region is said to be of type 1.

Similarly, if $k_1, k_2 : [c, d] \to \mathbb{R}$ are two $\mathcal{C}^1$ functions, the set of points

$$D_2 = \{(x, y) \mid c \leq y \leq d \text{ and } k_1(y) \leq x \leq k_2(y)\}$$

is called a region of type 2.

If D is a union of regions of types 1 and 2, we call it a region of type 3.

Evaluating integrals on regions of type 1

Let D be a region of type 1 and assume that $f : D \to \mathbb{R}$ is continuous. Let $D \subset R = [\alpha, \beta] \times [\gamma, \delta]$ and let g be the corresponding function on R (obtained by extending f by zero).

The region D is obviously bounded by $\mathcal{C}^1$ curves (the straight lines $x = a$ and $x = b$ and the graphs of the curves $y = h_1(x)$ and $y = h_2(x)$). Hence we can conclude that g is integrable on R.

$$\int_D f(x, y) dxdy = \int \int_R g(x, y) dxdy = \int_\alpha^\beta \left[\int_\gamma^\delta g(x, y) dy \right] dx,$$

where the second equality follows because of Theorem 34. In turn, this gives

$$\int_\alpha^\beta \left[\int_{h_1(x)}^{h_2(x)} g(x, y) dy \right] dx = \int_a^b \left[\int_{h_1(x)}^{h_2(x)} f(x, y) dy \right] dx,$$

since $g(x, y) = 0$ if $y < h_1(x)$ or $y > h_2(x)$.

The change of variables formula

Theorem 36: Let D be a region in the xy plane and D^* a region in the uv plane such that $h(D^*) = D$. Then

$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

In this case we use the notation

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

For polar coordinates:

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$

The mean value theorem for double integrals

Let us first state the mean value theorem for function of one variable.

Theorem 35: Suppose that f is a continuous function on $[a, b]$. There exists a point x_0 in $[a, b]$ such that

$$f(x_0) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Theorem 36: If D is an elementary region in $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ is continuous. There exist (x_0, y_0) in D such that

$$f(x_0, y_0) = \frac{1}{A(D)}\int\int_D f(x, y)dA.$$

For the proof, look at the slides where we derive a formula for the curl as the flux per unit area.

Triple integrals

All the ideas and theorems above extend to triple integrals - the definitions, Fubini's theorem, the Mean Value Theorem, etc. See D1 Lecture 25 or D3 Lectures 27 and 39 of the notes for this material.

Vector fields

Recall that a vector field is just a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Assume that all vector fields under consideration are $\mathcal{C}^1$.

We define the del operator restricting ourselves to the case $n = 3$:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

The del operator acts on functions $f : \mathbb{R}^3 \to \mathbb{R}$ to give the gradient:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Thus the del operator takes scalar functions to vector fields.

It operates as the curl to take vector fields to other vector fields:

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

It operates as the divergence to take vector fields to scalar functions:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

The curl of the velocity vector of a particle in a rigid body gives the angular velocity of the rigid body. If $\mathbf{F}$ is the velocity field of a fluid and it is curl free, then the fluid is irrotational.

The divergence of a fluid measures the rate of expansion of the fluid per unit area/volume,

Finally, for any scalar function f, curl(grad(f) $= 0$ and for any vector field div(curl $\mathbf{F}$) $= 0$.

Line integrals

Given a vector field $\mathbf{F}$ and a non-singular parametrised curve, we can define the line integral of $\mathbf{F}$ over $\mathbf{c}$:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt.$$

It is also sometimes simply written as

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b F_1 dx + F_2 dy + F_3 dz.$$

If $\mathbf{c}$ is the union of successive paths $\mathbf{c}_1$ and $\mathbf{c}_2$, we have

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}.$$

Conservative fields

The main observation about line integrals is the following. Suppose the vector field $\mathbf{F}$ can be written as the gradient of a scalar function f, that is, $\nabla f = \mathbf{F}$, then

$$\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt$$

Now the integrand on the right hand side is nothing but the directional derivative of f in the direction of $\mathbf{c}(t)$. Hence, we obtain

$$\int_a^b \frac{d}{dt} f(c(t))dt = f(c(b)) - f(c(a)).$$

Conservative fields are those that are gradient fields. Because of the remark above, they can sometimes be defined as fields for which the line integrals depend only on the end points, not on the curves themselves.

Conservative vector fields are gradients

We will now prove the converse to our previous assertion.
Theorem 39: Let $\mathbf{F} : D \to \mathbb{R}^3$ be a conservative vector field on a path connected open domain in $\mathbb{R}^3$. Then $\mathbf{F}$ is the gradient of a scalar function.

As long as one is traversing the curve from $\mathbf{c}(a)$ to $\mathbf{c}(b)$, the line integral is independent of the parametrisation.

Parametrised surfaces

Definition: Let D be a domain in $\mathbb{R}^2$. A parametrised surface is a function $\mathbf{\Phi} : D \to \mathbb{R}^3$.

We will assume that the parametrisation is $\mathcal{C}^1$ and non-singular - that is the normal vector is non- zero at every point.

All graphs $z = f(x, y)$ can be thought of as parametrised surfaces as can be solids of revolution around an axis.

We have two tangent vectors we can easily get for a parametrised surface:

$$\frac{\partial \mathbf{\Phi}}{\partial u}(u_0, v_0) := \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

and

$$\frac{\partial \mathbf{\Phi}}{\partial v}(u_0, v_0) := \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

The tangent plane to a parametrised surface

The normal to the surface is given by

$$\mathbf{n} = \mathbf{\Phi}_u \times \mathbf{\Phi}_v = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right).$$

Hence, the tangent plane is given by the equation

$$\mathbf{n} \cdot v = 0.$$

The surface area integral

The surface area of a parametrised surface is given by the double integral

$$\iint_S dS = \iint_D \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\,dudv.$$

The area is nothing but the integral of the constant function 1 on the surface S. We can likewise integrate any scalar function $f : S \to \mathbb{R}$:

$$\iint_S fdS = \iint_D f(x,y,z)\sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\,dudv.$$

If Σ is a union of parametrised surfaces S_i that intersect only along their boundary curves, then we can define

$$\iint_\Sigma fdS = \sum_i \iint_{S_i} fdS.$$

Green's Theorem

Theorem 41 (Green's theorem: Let D be a connected open set in $\mathbb{R}^2$ with a positively oriented boundary consisting of a finite union simple closed curves each of which is piecewise continuously differentiable curve. If $M : D \to \mathbb{R}$ and $N : D \to \mathbb{R}$ are $\mathcal{C}^1$ functions, then

$$\int_{\partial D} Mdx + Ndy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

The importance of Green's theorem is that it converts a surface integral into a line integral. Depending on the situation, one may be easier to evaluate than the other.
Recall that the positive orientation of C is given by the vector field

$$\mathbf{k} \times \mathbf{n}_{\text{out}},$$

where $\mathbf{n}_{\text{out}}$ is the normal vector field pointing outward along the curve.

Physically, this means that if one walks along C in the direction of the positive orientation, the region D is always on one's left.

Orientable and oriented surfaces

Definition: A surface S is said to be orientable if there exists a continuous vector field $\mathbf{F} : S \to \mathbb{R}^3$ such that for each point P in S, $\mathbf{F}(P)$ is a unit vector normal to the surface S at P.

At each point of S there are two possible directions for the normal vector to S. The question is whether the normal vector field be can be chosen so that the resulting vector field is continuous.

An orientable surface together with a specific choice of continuous vector field $\mathbf{F}$ of unit normal vectors is called an oriented surface. The choice of vector field is called an orientation.

The orientation of parametrised surfaces

Let us suppose that we are given an oriented geometric surface S that is described as a $\mathcal{C}^1$ non-singular parametrised surface $\mathbf{\Phi}(u, v)$.

Notice that a parametrised surface $\mathbf{\Phi}$ comes equipped with a natural vector field of unit normal vectors:

$$\hat{\mathbf{n}} = \frac{\mathbf{\Phi}_u \times \mathbf{\Phi}_v}{\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|}.$$

Definiition: If the unit normal vector $\hat{\mathbf{n}}$ agrees with the given orientation of S we say that the parametrisation $\mathbf{\Phi}$ is orientation preserving. Otherwise we say that $\mathbf{\Phi}$ is orientation reversing.

Independence of parametrisation

Let S be an oriented surface. Let $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ be two $\mathcal{C}^1$ non-singular parametrisations of S and let $\mathbf{F}$ be a continuous vector field on S.

- If $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are orientation preserving, then
$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$
- If $\mathbf{\Phi}_1$ is orientation preserving and $\mathbf{\Phi}_2$ is orientation reversing, then
$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

For an oriented surface, the notation

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{n} dS,$$

is unambiguous.

Orienting the boundary of an oriented surface

The boundary of an oriented surface is oriented so that the surface lies to the left of an observer walking along the boundary with his head in the direction of the unit normal vector given by the choice of orientation.

Thus, the boundary of an oriented surface automatically acquires an orientation.

Theorem 42: Let S be a bounded oriented surface (more precisely, an oriented non-singular $\mathcal{C}^1$ surface) and let $\mathbf{F} : D \to \mathbb{R}^3$ be a $\mathcal{C}^1$ vector field, for some region D containing S. Assume further that the boundary ∂S of S is the disjoint union of simple closed curves each of which is a piecewise non-singular parametrised curve. Then

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

The following special case is often of great interest: If S is a closed surface we see that the right hand side is zero, since there is no boundary.

Gauss's divergence theorem

If we have a closed surface (this is a surface without a boundary) then we can speak of an inside and an outside. Hence, such a surface automatically acquires an orientation. As usual, we assume that S is given as a union of piecewise differentiable non-sinuglar surfaces.

Theorem 44: Let $S = \partial W$ be a closed oriented surface enclosing the region W with the outward normal giving the positive orientation. Let $\mathbf{F}$ be a $\mathcal{C}^1$ vector field defined on W. Then

$$\iiint_W (\nabla \cdot \mathbf{F}) dV = \iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}.$$

Clearly, the importance of Gauss's theorem is that it converts surface integrals to volume integrals and vice-versa. Depending on the context one may be easier to evaluate than the other.

The Poincaré Lemma for $\mathbb{R}^3$

The PoincareĹemma for $\mathbb{R}^3$ refers to the following pairs of statements.
Theorem 46: A $\mathcal{C}^1$ vector field $\mathbf{F}$ on $\mathbb{R}^3$ is conservative if and only if $\nabla \times \mathbf{F} = 0$.

Theorem 49: If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ and $\nabla \cdot \mathbf{F} = 0$, then $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$.

Remember that the above statements are only true for $\mathbb{R}^3$. They may not be true for subsets having a more complicated geometry, such as $\mathbb{R}^3 \setminus z-$axis.