MA 105 D1: Tutorial 13

Ravi Raghunathan

Department of Mathematics

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Solutions to problems from Tutorial 13

These slides provide some solutions to the problems assigned in Tutorial 13. Sometimes, I have only indicated the methods to be used or the final answer. I have also commented on some issues that came up during the class.

Exercise 13.2: Verify the Divergence Theorem for

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$$

for the region in the first octant bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Solution: We denote by R the region bounded by the given plane. We first evaluate the volume integral in Gauss' Theorem.

The solution to Exercise 13.2

We have $\operatorname{div}\mathbf{F} = (y+z+x)$.

$$\begin{aligned}\iiint_R (x+y+z)dV &= \iiint_R xdV + \iiint_R ydV + \iiint_R zdV \\ &= \int_0^c \int_0^{b(1-\frac{z}{c})} \int_0^{a(1-\frac{y}{b}-\frac{z}{c})} xdxdydz + (\cdots) + (\cdots) \\ &= \frac{a^2bc}{24} + \frac{ab^2c}{24} + \frac{abc^2}{24} \\ &= \frac{abc}{24}(a+b+c).\end{aligned}$$

We now evaluate the other side of Gauss' Theorem.

The solution to Exercise 13.2, continued

The boundary of the region R consists of four triangular surfaces, three of which lie in the planes formed by the three coordinate planes. For each of these regions we can easily check that $\mathbf{F} \cdot \mathbf{n} = 0$.

For instance, we have

$$S_1 : z = 0; \frac{x}{a} + \frac{y}{b} \leq 1, x, y \geq 0$$

as one of the three boundary pieces, and along S_1

$$\mathbf{n} = -\mathbf{k}, \quad \text{so} \quad \mathbf{F} \cdot \mathbf{n} = -xz = 0 \quad (\text{as } z = 0 \text{ on } S_1),$$

and the other two triangular surfaces are treated similarly. This leaves us only the the triangular surface S_4 defined by $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and $x, y, z \geq 0$.

The solution to Exercise 13.2, continued

Along S_4, the outward normal (to $z = c(1 - \frac{x}{a} - \frac{y}{b}) \equiv f(x, y)$) is $(\frac{c}{a}, \frac{c}{b}, 1)$ so that

$$\begin{aligned}
\iint_{S_4} \mathbf{F} \cdot \mathbf{n} dS &= \iint_{\frac{x}{a}+\frac{y}{b}\le 1; x,y\ge 0} \left(\frac{cxy}{a} + \frac{cyz}{b} + zx\right) dS \\
&= \int_0^a \int_0^{b(1-\frac{x}{a})} \frac{cxy}{a} dxdy + (\cdots) + (\cdots) \\
&= \frac{ab^2c}{24} + \frac{abc^2}{24} + \frac{a^2bc}{24} \\
&= \frac{abc}{24}(a + b + c).
\end{aligned}$$

This proves what we want.

Exercise 13.5

Exercise 13.5: Let V be the volume of a region bounded by a closed surface S and $\mathbf{n} = (n_x, n_y, n_z)$ be its outer unit normal. Prove that

$$V \;=\; \iint_S x\, n_x\, dS = \iint_S y\, n_y\, dS = \iint_S z\, n_z\, dS$$

Solution: Let $\mathbf{F} = x\mathbf{i}$, and apply the divergence theorem. Then

$$V = iiint_R 1\, dV = \iint_S x n_x dS.$$

Similarly, taking $\mathbf{F} = y\mathbf{j}$ and $\mathbf{F} = z\mathbf{k}$, we get the other two integrals.

Exercise 13.7

Exercise 13.7: Compute

$$\iint_S yz\, dydz + zx\, dzdx + xy\, dxdy,$$

where S is the unit sphere.

Solution: In class I made things unnecessarily complicated. Here is the correct calculation. When the parametrisation is given by a graph $z = f(x, y) = \sqrt{1 - x^2 - y^2}$, we know that

$$\|\Phi_x \times \Phi_y\| = \sqrt{1 + f_x^2 + f_y^2}.$$

One sees easily that $f_x = x/z$ and $f_y = y/z$, from which it follows that

$$dS = \frac{(x^2 + y^2 + z^2)dxdy}{z} = \frac{dxdy}{z}.$$

It follows that $\mathbf{F} \cdot \mathbf{n} dS = yzdydz + zxdzdx + xydxdy$ when $\mathbf{F} = (yz, zx, xy)$ and $\mathbf{n} = (x, y, z)$

The solution to Exercise 13.7

Applying Gauss' Theorem to the volume enclosed by the sphere, and observing that $\nabla \cdot \mathbf{F} = 0$, we see that the given integral is identically zero.

Many students pointed out that one can use the ideas of Exercise 11.2 to do the same calculation as above a little more quickly.

Exercise 13.8

Exercise 13.8: Let $\mathbf{u} = -x^3\mathbf{i} + (y^3 + 3z^2 \sin z)\mathbf{j} + (e^y \sin z + x^4)\mathbf{k}$ and S be the portion of the sphere $x^2 + y^2 + z^2 = 1$ with $z \geq \frac{1}{2}$ and $\mathbf{n}$ is the unit normal with positive z-component. Use Divergence theorem to compute

$$\iint_S (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, dS.$$

Solution: The basic idea is to use the divergence theorem to change the surface over which the surface integral is calculated. Let S denote the given surface and S_1 the disc $x^2 + y^2 + z^2 = 1$, $z = 1/2$. Then $S \cup S_1$ is a closed surface (without a boundary!) to which we may apply Gauss's Theorem. Since $\nabla \cdot (\nabla \times u) = 0$ we see that

$$\iint_S = -\int_{S_1} (\nabla \times u) dS.$$

However, it is easy to see that $(\nabla \times u) \cdot \mathbf{n} = 0$ on the surface S_1,

so the desired integral is 0.