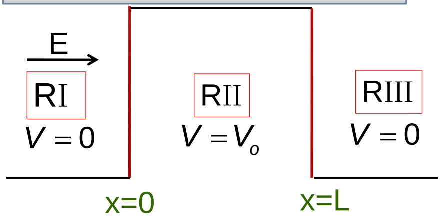
Tunneling

$$V(x) = V_o \quad \text{for } 0 < x < L$$
$$= 0 \quad \text{for } x < 0 \text{ or } x > L$$



Particle approaches from left with $E<V_o$.

The General Solutions

RI $\phi_I(x) = Ae^{ikx} + Be^{-ikx}; \; k^2 = \dfrac{2mE}{\hbar^2}$

RII $\phi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x}; \; \alpha^2 = \dfrac{2m(V_o - E)}{\hbar^2}$

RIII $\phi_{III}(x) = Fe^{ikx} + Ge^{-ikx}; \; k^2 = \dfrac{2mE}{\hbar^2}$

Is any coefficient zero?

Boundary Conditions

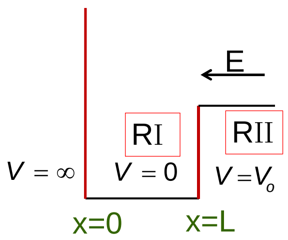
$$G = 0$$
$$A + B = C + D$$
$$ik(A - B) = \alpha(C - D)$$
$$Ce^{\alpha L} + De^{-\alpha L} = Fe^{ikL}$$
$$\alpha(Ce^{\alpha L} - De^{-\alpha L}) = Fike^{ikL}$$

Transmission Coefficient

$$\frac{1}{T}=\left|\frac{A}{F}\right|^2=1+\frac{1}{4}\left(\frac{\alpha}{k}+\frac{k}{\alpha}\right)^2\sinh^2\alpha L$$

$$\boxed{\sinh(x)=\frac{e^x-e^{-x}}{2};\cosh(x)=\frac{e^x+e^{-x}}{2}}$$

Free State



Can the answer for the reflection coefficient be guessed?

RI

$$\phi_I(x)=A\sin k_1x+B\cos k_1x;\ k_1^2=\frac{2mE}{\hbar^2}$$

RII

$$\phi_{II}(x)=Ce^{ik_2x}+De^{-ik_2x};\ k_2^2=\frac{2m(E-V_o)}{\hbar^2}$$

Boundary Conditions

$$\phi_I(0)=0\Rightarrow$$

$$B=0$$

$$\phi_I(L)=\phi_{II}(L)\Rightarrow$$

$$A\sin k_1L=Ce^{ik_2L}+De^{-ik_2L}$$

$$\phi_I'(L)=\phi_{II}'(L)\Rightarrow$$

$$Ak_1\cos k_1L=ik_2\left(Ce^{ik_2L}-De^{-ik_2L}\right)$$

Reflection Coefficient

$$R = \left|\frac{C}{D}\right|^2$$

$$A\sin k_1 L = Ce^{ik_2 L} + De^{-ik_2 L}$$

$$\frac{A}{D}\sin k_1 L = \frac{C}{D}e^{ik_2 L} + e^{-ik_2 L}$$

$$Ak_1\cos k_1 L = ik_2\left(Ce^{ik_2 L} - De^{-ik_2 L}\right)$$

$$\frac{A}{D}\cos k_1 L = i\frac{k_2}{k_1}\left(\frac{C}{D}e^{ik_2 L} - De^{-ik_2 L}\right)$$

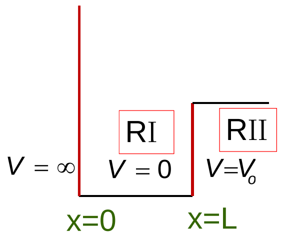
Eliminate $\frac{A}{D}$ from the following equations:

$$\frac{A}{D}\sin k_1 L = \frac{C}{D}e^{ik_2 L} + e^{-ik_2 L}$$

$$\frac{A}{D}\cos k_1 L = i\frac{k_2}{k_1}\left(\frac{C}{D}e^{ik_2 L} - De^{-ik_2 L}\right)$$

$$\frac{C}{D} = e^{-2ik_2 L} \times \frac{\left(\cos k_1 L + i\frac{k_2}{k_1}\sin k_1 L\right)}{\left(-\cos k_1 L + i\frac{k_2}{k_1}\sin k_1 L\right)}$$

Bound State



$$\phi_I(x) = A\sin kx + B\cos kx;$$

$$k^2 = \frac{2mE}{\hbar^2}$$

$$\phi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x};$$

$$\alpha^2 = \frac{2m\left(V_o - E\right)}{\hbar^2}$$

$$B = 0$$

$$C = 0$$

$$\phi_I(L) = \phi_{II}(L) \Rightarrow A\sin kL = De^{-\alpha L}$$

$$\phi_I'(L) = \phi_{II}'(L) \Rightarrow Ak\cos kL = -\alpha De^{-\alpha L}$$

This gives

$$\cot kL = -\frac{\alpha}{k}$$

Solution with E=V_o

$$\alpha = \frac{\sqrt{2m\left(V_o - E\right)}}{\hbar} = 0$$

$$k = \frac{\sqrt{2mE}}{\hbar} = \frac{\sqrt{2mV_o}}{\hbar}$$

The boundary conditions now becomes.

$$A\sin kL = D$$
$$Ak\cos kL = 0$$

Since A can not be zero, hence this implies

$$\cos kL = 0 \text{ or } kL = (2n+1)\frac{\pi}{2}$$
$$\frac{\sqrt{2mV_o}}{\hbar}L = (2n+1)\frac{\pi}{2}$$
$$V_o = \frac{\hbar^2\left[(2n+1)\frac{\pi}{2}\right]^2}{2mL^2}$$

Bound States

There will be no bound state if

$$V_o < \frac{\hbar^2\pi^2}{8mL^2}$$

There will be only one bound state if

$$\frac{\hbar^2\pi^2}{8mL^2} < V_o < \frac{9\hbar^2\pi^2}{8mL^2}$$

Comments

- A finite V_o is needed for a bound state to exist.
- A bound state always exists in the case of a particle in finite square well potential.