MA 105 D1 Lecture 20

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The Jordan curve theorem

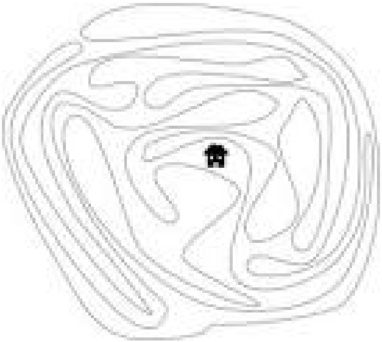
Recall that an oriented curve is a curve together with a choice of orientation. We have shown that the line integral of a vector field along a curve is unchanged when an orientation preserving reparametrisation is applied to the curve (if the orientation is reversed, the sign of the line integral changes).

Suppose we have a simple closed curve C in the plane. Then C divides the plane $\mathbb{R}^2$ into an "inside" and an "outside". More precisely, we have the Jordan curve theorem.

Theorem 40: The complement of a simple curve C in the plane consists of exactly two connected components, one of which is bounded and the other which is not bounded.

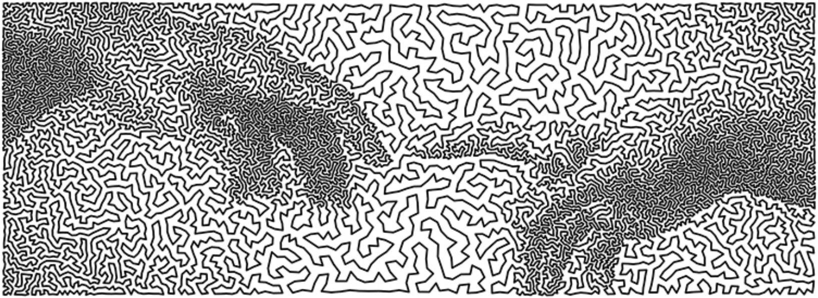
(A connected component consists of all points which can be joined to each other by a path which does not intersect C.)

Is the house inside or outside the wall



http://plus.maths.org/content/winding-numbers-topography-and-topology-ii

There's nothing obvious about the Jordan curve theorem!



from http://www.dominoartwork.com/tspart.html

Orientation for the an enclosed region in the plane

The bounded component of $\mathbb{R}^2 \setminus C$ is obviously what we usually refer to as the interior or inside and the unbounded component is clearly the exterior or “outside”.

The Jordan curve theorem perhaps strikes one as intuitively obvious, but was quite difficult to prove. Although Jordan claimed a proof of this theorem in 1887, it does not seem to have been accepted as complete for many years. More recently, however, some mathematicians indicate that the proof required only a minor bit of polishing and didn't really have a gap.

Once we accept Jordan's theorem, we see that C encloses a bounded region D in the plane which is, after all, a surface. There is a natural notion of positive orientation on the surface D - clearly it is given by the vector field $\mathbf{k}$ - the unit normal vector pointing in the direction of the positive z axis.

Orienting the boundary curve

As in the previous slide, we think of the curve C as the boundary of a region D in the plane, but C may now consists of several components or pieces and D may have "holes".

Definition: The positive orientation of C is given by the vector field

$$\mathbf{k} \times \mathbf{n}_{\text{out}},$$

where $\mathbf{n}_{\text{out}}$ is the normal vector field pointing outward along the curve.

Physically, this means that if one walks along C in the direction of the positive orientation, the region D is always on one's left

As we shall see later, if C is a closed curve in space bounding an oriented surface S, the orientation of S naturally induces an orientation on the boundary C. The above example is a special case of this.

Green's Theorem

With the preliminaries out of the way, we are now in a position to state the first major theorem of vector calculus, namely Green's Theorem.

Theorem 41 (Green's theorem: Let D be a connected open set in $\mathbb{R}^2$ with a positively oriented boundary consisting of a finite union of piecewise continuously differentiable curves. If $M : D \to \mathbb{R}$ and $N : D \to \mathbb{R}$ are $\mathcal{C}^1$ functions, then

$$\int_{\partial D} Mdx + Ndy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

The importance of Green's theorem is that it converts a surface integral into a line integral. Depending on the situation, one may be easier to evaluate than the other.

Example 1: Let C be the circle of radius r oriented in the counterclockwise direction, and let $M(x, y) = -y$ and $N(x, y) = x$. Evaluate

$$\int_C M(x, y)dx + N(x, y)dy.$$

Solution: Let $\mathbf{F} = (-y, x, 0)$ and D denote the disc of radius r. Then $N_x - M_y = 2$. Hence, by Green's theorem

$$\int_C M(x, y)dx + N(x, y)dy = \int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D 2dxdy = 2\pi r^2.$$

Area of a region

The preceding example shows us that the area of a region enclosed can be expressed as a line integral.
If C is a positively oriented curve that bounds a region D , then the area $A(D)$ is given by

$$A(D) = \frac{1}{2}\int_C xdy - ydx.$$

Example 2: Let us use the formula above to find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: We parametrise the curve C by $\mathbf{c}(t) = (a\cos t, b\sin t)$, $0 \leq t \leq 2\pi$. By the formula above, we get

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\int_C xdy - ydx \\
&= \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)dt - (b\sin t)(-a\sin t)dt \\
&= \frac{1}{2}\int_0^{2\pi} abdt = \pi ab.
\end{aligned}$$

Polar coordinates

Suppose we are given a simple positively oriented closed curve $C : (r(t), \theta(t))$ in polar coordinates. Then, by the area formula above, we know that the area enclosed by C is given by

$$\frac{1}{2}\int_C xdy - ydx := \frac{1}{2}\int_C \left(x(r,\theta)\frac{dy}{dt} - y(r,\theta)\frac{dx}{dt} \right) dt$$

$$\frac{1}{2}\int_a^b x(r(t),\theta(t))\frac{\partial y}{\partial r}\frac{dr}{dt}dt + \frac{1}{2}\int_a^b x(r(t),\theta(t))\frac{\partial y}{\partial \theta}\frac{d\theta}{dt}dt$$

$$+\frac{1}{2}\int_a^b y(r(t),\theta(t))\frac{\partial x}{\partial r}\frac{dr}{dt}dt - \frac{1}{2}\int_a^b y(r(t),\theta(t))\frac{\partial x}{\partial \theta}\frac{d\theta}{dt}dt$$

$$= \frac{1}{2}\int_a^b r(t)\cos\theta(t)\sin\theta(t)\frac{dr}{dt}dt - \frac{1}{2}\int_a^b r^2(t)\cos^2\theta(t)\frac{d\theta}{dt}dt$$

$$-\frac{1}{2}\int_a^b r(t)\sin\theta(t)\cos\theta(t)\frac{dr}{dt}dt + \frac{1}{2}\int_a^b r(t)^2\sin^2\theta(t)\frac{d\theta}{dt}dt$$

$$= \frac{1}{2}\int_C r^2 d\theta.$$

Exercise 10.3.(i): Find the area of the cardioid $r = a(1 - \cos\theta)$, $0 \leq \theta \leq 2\pi$.

Solution: Using the formula we have just derived, the desired area is simply

$$\frac{1}{2}\int_0^{2\pi} a^2(1-\cos\theta)^2 d\theta = a^2 \int_0^{2\pi} -2\cos\theta + \frac{\cos 2\theta}{2} + \frac{3}{2} d\theta$$
$$= \frac{3a^2\pi}{2}.$$

Orienting the boundary curve

Let us summarise our results from last time. Let D be a region in the plane (that is, a connected open set) with boundary consisting a finite union of piecewise continuously differentiable curves. Then the outer boundary curve is given the counterclockwise orientation, while the inner boundary curves are oriented in the clockwise direction. More explicitly:

Definition: The positive orientation of C is given by the vector field

$$\mathbf{k} \times \mathbf{n}_{\text{out}},$$

where $\mathbf{n}_{\text{out}}$ is the normal vector field pointing outward along the curve.

Physically, this means that if one walks along C in the direction of the positive orientation, the region D is always on one's left.

Green's Theorem

Here is the first major theorem of vector calculus, namely Green's Theorem.

Theorem 41 (Green's theorem: Let D be a connected open set in $\mathbb{R}^2$ with a positively oriented boundary consisting of a finite union of piecewise continuously differentiable curves. If $M : D \to \mathbb{R}$ and $N : D \to \mathbb{R}$ are $\mathcal{C}^1$ functions, then

$$\int_{\partial D} Mdx + Ndy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

The importance of Green's theorem is that it converts a surface integral into a line integral. Depending on the situation, one may be easier to evaluate than the other.

A proof of Green's theorem for regions of special type

We give a proof of Green's theorem when the region D is both of type 1 and type 2 - the rectangle being the most important such figure.

Thus we will assume that D lies between $x = a$, $x = b$, $y = \phi_1(x)$ and $y = \phi_2(x)$. Similarly, we will assume the that D lies between $y = c$, $y = D$, $x = \psi_1(y)$ and $x = \psi_2(y)$.

Consider the double integral

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_D \frac{\partial N}{\partial x} dxdy - \iint_D \frac{\partial M}{\partial y} dxdy.$$

Since this is a region of type 2, the first double integral on the right hand side can be written as

$$\iint_D \frac{\partial N}{\partial x} dxdy = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial N}{\partial x}(x, y) dxdy.$$

The proof of Green's theorem, continued

Using the Fundamental Theorem of Calculus we get

$$\int_c^d N(\psi_2(y), y) - N(\psi_1(y), y)dy = \int_c^d N(\psi_2(y), y)dy - \int_c^d N(\psi_1(y), y)dy.$$

Now we have to interpret the integrals on the right hand side as line integrals. But this is easy, since the integrand is already essentially in a parametrised form. Indeed, we can parametrise $x = \psi_2(y)$ by $\mathbf{c}(t) = (\psi_2(t), t)$, $c \leq t \leq d$. Hence, we get

$$\int_{\mathbf{c}} N dy = \int_c^d N(\psi_2(t), t)\frac{dy}{dt}dt = \int_c^d N(\psi_2(t), t)dt.$$

But if we change the name "t" to "y", we get the first of our integrals above. Similarly, the second integral becomes the line integral

$$-\int_c^d N(\psi_1(y), y)dy.$$

Completing the proof of Green's theorem

Why is there a minus sign? Because the integral is being taken in the opposite direction (that is, downward in the picture).

We need to do the line integral along the two horizontal lines $y = c$ and $y = d$. Since y is constant along these curves, we have $dy = 0$. As a result, these two line integrals are just zero. Hence, we find that

$$\iint_D \frac{\partial N}{\partial x} dxdy = \int_{\partial D} Ndy.$$

Using the fact that D is a region of type 1, the integral

$$\iint_D \frac{\partial M}{\partial y} dxdy = -\int_{\partial D} Mdx.$$

Where does the minus sign come from? From the fact that $y = \phi_2(x)$ is oriented in the direction of decreasing x.

Subtracting the two equations above, we get

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{\partial D} Mdx + \int_{\partial D} Ndy.$$

A more general case

How does one proceed in general, that is for more general regions which may not be of both type 1 and type 2. We can try proceeding as follows:

- Break up D into smaller regions each of which is of both type 1 and type 2 but so that any two pieces meet only along the boundary.
- Apply Green's theorem to each piece.
- Observe that the line integrals along the interior boundaries cancel, leaving only the line integral around the boundary of D.

Gauss's Divergence Theorem in the plane

Theorem 41: Let D be a region as in Green's theorem and let $\mathbf{n}$ be the outward unit normal vector on the positively oriented boundary ∂D. Let $\mathbf{F} : D \to \mathbb{R}^2$ be a vector field.

$$\int_{\partial D} (\mathbf{F} \cdot \mathbf{n}) ds = \int \int_D \operatorname{div} \mathbf{F} dxdy.$$

Since $\mathbf{k} \times \mathbf{n} = \mathbf{T}$, we have $\mathbf{n} = -\mathbf{k} \times \mathbf{T}$, where $\mathbf{T} = \mathbf{c}'/\|\mathbf{c}'\|$ is the unit tangent vector to the curve $\mathbf{c}(t)$ which parametrises ∂D (we will assume that ∂D can be parametrised by a single curve - otherwise break up the curve into parametrisable pieces...). Hence,

$$\int_{\partial D} (\mathbf{F} \cdot \mathbf{n}) ds = -\int_{\partial D} \mathbf{F} \cdot (\mathbf{k} \times \mathbf{T}) ds.$$

But the integrand on the right hand side is just the scalar triple product which can be rewritten as $(\mathbf{F} \times \mathbf{k}) \cdot \mathbf{T}$, so

$$\int_{\partial D} (\mathbf{F} \cdot \mathbf{n}) ds = -\int_{\partial D} (\mathbf{k} \times \mathbf{F}) \cdot \mathbf{T} ds = \int_{\partial D} (\mathbf{k} \times \mathbf{F}) \cdot d\mathbf{s}.$$

The proof of the divergence theorem, continued

If we write $\mathbf{F} = (A, B)$, then $\mathbf{k} \times \mathbf{F} = A\mathbf{j} - B\mathbf{i}$. If we apply Green's theorem we get

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} ds = \iint_D \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} \right) dxdy = \iint_D \operatorname{div} \mathbf{F} dxdy.$$

We can interpret the above theorem in the context of fluid flow. If $\mathbf{F}$ represents the flow of a fluid, then the left hand side of the divergence theorem represents the net flux of the fluid across the boundary ∂D. On the other hand, the right hand side represents the integral over D of the rate $\nabla \cdot \mathbf{F}$ at which fluid area is being created. In particular if the fluid is incompressible (or, more generally, if the fluid is being neither compressed nor expanded) the net flow across ∂D is zero.

Again, as we shall see next week, this theorem has a three-dimensional analogue.

The definition of a parametrised surface

A curve is a "one-dimensional" object. Intuitively, this means that if we want to describe a curve, it should be possible to do so using just one variable or parameter. To do line integration, we further required some extra properties of the curve - that it should be $\mathcal{C}^1$ and non-singular. In this lecture we are going to develop the analogous theory for surfaces.

In order to describe a surface, which is a two-dimensional object, we clearly need two parameters. And, in order to do we will need some analogue of a non-singular curve.

Definition: Let D be a domain in $\mathbb{R}^2$. A parametrised surface is a function $\mathbf{\Phi} : D \to \mathbb{R}^3$.

Geometric parametrised surfaces

In all examples, D will be a connected open set in $\mathbb{R}^2$. As with curves and paths we will distinguish between the surface $\mathbf{\Phi}$ and its image. Thus the image $S = \mathbf{\Phi}(D)$ will be called the geometric surface corresponding to $\mathbf{\Phi}$.

(In fact, it is customary to call paths geometric curves.)

Note that for a given $(u, v) \in D$, $\mathbf{\Phi}(u, v)$ is a vector in $\mathbb{R}^3$. Each of the coordinates of the vector depends on u and v. Hence we write

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where x, y and z are scalar functions on D.

Examples

Example 1: Graphs of real valued functions are parametrised surfaces.

Indeed, let $f(x, y)$ be a scalar function and let $z = f(x, y)$. If D is the domain of f in $\mathbb{R}^2$, we can define the parametrised surface $\mathbf{\Phi}$ by

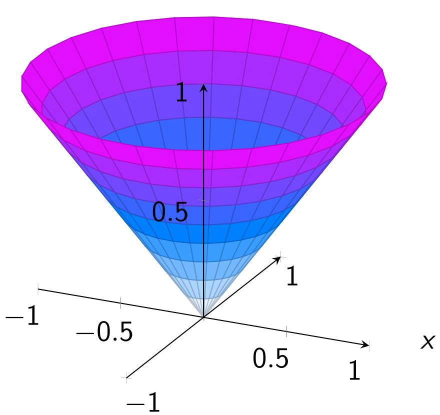
$$\mathbf{\Phi}(u, v) = (u, v, f(u, v)).$$

More specifically, we have $x(u, v) = u$, $y(u, v) = v$ and $z(u, v) = f(u, v)$.
Example 2: Consider $\mathbf{\Phi} : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$x = u \cos v, \quad y = u \sin v, \quad \text{and} \quad z = u,$$

$u \geq 0$. What geometric surface does it describe?

A right-circular cone.



The graph of $z = \sqrt{x^2 + y^2}$, also known as the parametrised surface

$$x = u\cos v, \quad y = u\sin v, \quad \text{and} \quad z = u, \quad u \geq 0.$$

A hyperboloid with one sheet

Example 3: Find a parametrisation for the hyperboloid of one sheet: $x^2 + y^2 - z^2 = 1$

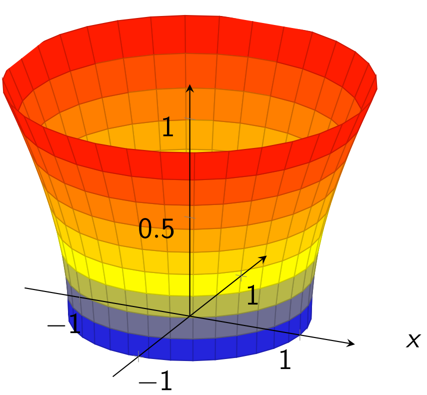
Solution: Note that the figure is obviously invariant under rotations about the z axis, since this would leave the quantities $x^2 + y^2$ and z^2 both unchanged. Whenever one has such invariance, it is logical to use polar coordinates to parametric x and y. Thus, we set

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

The equation of the hyperboloid now becomes $r^2 - z^2 = 1$, so it is reasonable to parametric r and z by hyperbolic functions: $r = \cosh u$, $z = \sinh u$. Hence, we obtain

$$x(u,v) = \cosh u \cos\theta,\ \ y(u,v) = \cosh u \sin\theta \quad \text{and} \quad z = \sinh u$$

as a parametrisation. Here $0 \le \theta \le 2\pi$ and $-\infty < u < \infty$.



The graph of $x^2 + y^2 - z^2 = 1$, also known as the parametrised surface

$$x = \cosh u \cos\theta, \quad y = \cosh u \sin\theta, \quad \text{and} \quad z = \sinh u,$$

Surfaces of revolution around the x-axis

Example 4: The preceding example involved rotations around the z-axis. If we have the graph of a function $y = f(x)$ which we rotate around x-axis, we can parametrise it as follows:

$$x = u, \quad y = f(u)\cos v, \quad \text{and} \quad z = f(u)\sin v$$

Here $a \leq u \leq b$ if $[a, b]$ is the domain of f, and $0 \leq v \leq 2\pi$.

Exercise 1: Find a parametrisation for a torus - that is the outer surface of a doughnut or cycle tire tube.

This shows that parametrised surfaces are more general than graphs of functions.

A picture of the torus