MA 105 D1 Lecture 7

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Review of Taylor's theorem

Darboux integration

Riemann integration

Taylor's Theorem

The Taylor polynomials at a depend on a, so we should really be writing $P_{k,a}(x)$ rather than $P_k(x)$, but we omit the extra subscript a so that our notation does not get too complicated.

The Taylor polynomials are rigged exactly so that the degree n Talyor polynomial has the same first n derivatives at the point a as the function $f(x)$ has, that is, $P_n^{(k)}(a) = f^{(k)}(a)$ for all $0 \leq k \leq n$, where $f^{(0)}(x) = f(x)$ by convention.

Theorem 19: Let $f \in \mathcal{C}^n([a, b])$ and suppose that $f^{(n+1)}$ exists on (a, b). Then there exists $c \in (a, b)$ such that

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

It is customary to denote the function $f(b) - P_n(b)$ by $R_n(b)$. Taylor's Theorem gives us a simple formula for $R_n(b)$. If we can make $R^n(b)$ small, we can approximate our function $f(x)$ by a polynomial.

Calculating the values of functions

As we have also mentioned several times, calculators and computers calculate the values of various common functions like trigonometric polynomials and expressions in $\log x$ and e^x by using Taylor series.

The great advantage of Taylor series is that one can estimate the error since we have a simple formula for the error which can be easily estimated. For instance, for the function $\sin x$, the n-th derivative is either $\pm \sin x$ or $\pm \cos x$, so in either case $|f^{(n}(x)| \leq 1$. Hence,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

If we take $x = 1$, and we want to compute $\sin 1$ to an error of less than 10^{-16}, we need only make sure that $(n+1)! > 10^{16}$, which is achieved when $n \geq 21$.

Supplementary exercises

Remark We have stated Taylor's theorem for $f(b)$ expanding at the the point a with $a < b$. We could have equally well stated the theorem for $b < a$.

Exercise 1: To how many terms must you compute the Taylor series of sin in order to make sure that it approximates the value of sin to within 10^{-32} anywhere on the real line.

Exercise 2: Assume that $f(x)$ is a $\mathcal{C}^\infty$ function on $[a, b]$. Let $x_0 \in (a, b)$. Is it possible that the Taylor series of $f(x)$ about x_0 does not converge to $f(x)$ in any open interval around x_0?

Partitions

Definition: Given a closed interval $[a, b]$, a partition P of $[a, b]$ is simply a collections of points

$$P = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}.$$

We can think of the points of the partition as dividing the original interval $[a, b]$ into sub-intervals $I_j = [x_{j-1}, x_j]$, $1 \leq j \leq n$. Indeed $I = \cup_j I_j$ and if two sub-interval intersect, they have at most one point in common. Hence, the notation "partition".
Definition: A partition $P' = \{a = x_0' < x_1' < \ldots < x_m' = b\}$ is said to be a refinement of the partition P if for each $x_i \in P$, there exists an $x_j' \in P'$ such that $x_i = x_j'$.

Intuitively, a refinement P' of a partition P will break some of the sub-intervals in P into smaller sub-intervals. Any two partitions have a common refinement.

Lower and Upper sums

Given a partition $P = \{a = x_0 < x_1 < \ldots < x_{b-1} < x_n = b\}$ and a function $f : [a, b] \to \mathbb{R}$, we define two associated quantities. First we set:

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad 1 \leq i \leq n$$

Defintion: We define the Lower sum as

$$L(f, P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}).$$

Similarly, we can define the Upper sum as

$$U(f, P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}).$$

In case the words "infimum" and "supremum" bother you, you can think "minimum" and "maximum most of time since we will usually be dealing with continuous functions on $[a, b]$.

One basic example

In order to illustrate what we are saying we will take the following basic example. Let $[a, b] = [0, 1]$ and let $f(x) = x$.

One of the most natural partitions on an interval is a partition that divides the interval into sub-intervals of equal length. For $[0, 1]$, this is

$$P_n = \{0 < 1/n < 2/n < \ldots < (n-1)/n < 1\}.$$

On the interval $I_j = [\frac{j-1}{n}, \frac{j}{n}]$, where does the function $f(x) = x$ take its minimum? its maximum?
Clearly, the minimum $m_j = \frac{j-1}{n}$ is attained at $\frac{j-1}{n}$ and the maximum $M_j = \frac{j}{n}$ at $\frac{j}{n}$. And finally, $\frac{j}{n} - \frac{j-1}{n} = \frac{1}{n}$, for all $1 \leq j \leq n$.

An example of a refinement of P_n is P_{2n}, or, more generally, P_{kn} for any natural number k.

The Darboux integrals

We now define the lower Darboux integral of f by

$$L(f) = \sup\{L(f,P) \colon P \text{ is a partition of } [a,b]\},$$

where the supremum is taken over all partitions of $[a,b]$.
and similarly the upper Darboux integral of f by

$$U(f) = \inf\{U(f,P) \colon P \text{ is a partition of } [a,b]\},$$

and again the infimum is over all partitions of $[a,b]$. (This time there is no escaping inf and sup!)

If $L(f) = U(f)$, then we say that f is Darboux-integrable and define

$$\int_a^b f(t)\,dt := U(f) = L(f).$$

This common value of the two integrals is called the Darboux integral.

Back to the example

Let us calculate $L(f, P_n)$ and $U(f, P_n)$ in the example we gave.

$$L(f, P_n) = \sum_{j=1}^{n} \frac{(j-1)}{n} \cdot \frac{1}{n} = \sum_{j=0}^{n-1} \frac{j}{n^2}.$$

This can be evaluated explicitly:

$$L(f, P_n) = \frac{n(n-1)}{2} \cdot \frac{1}{n^2} = \frac{1}{2} - \frac{1}{2n}.$$

Similary, we can check that

$$U(f, P_n) = \frac{n(n+1)}{2} \cdot \frac{1}{n^2} = \frac{1}{2} + \frac{1}{2n}.$$

Can we conclude that the Darboux integral is $1/2$ by letting $n \to \infty$? Unfortunately, no.

Useful properties of the Darboux sums

Since, for any partition P, $L(f, P) \leq U(f, P)$, we have

$$L(f) \leq U(f).$$

In fact, for any two partitions P_1 and P_2, we have

$$L(f, P_1) \leq U(f, P_2).$$

This is easy to see - the lower sum computes the sum of the areas of rectangles that lie entirely below the curve while the upper sum computes the sum of the areas of rectangles whose "tops" lie above the curve.

One of the most useful properties of the Darboux sums is the following. If P' is a refinement of P then obviously

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P).$$

Riemann Sums

There is another way of getting at the integral due to Riemann which may be a little more intuitive and is better for calculation. This is done via Riemann sums.

To define the notion of a Riemann sum we need one more piece of data. Suppose that for each of the intervals I_j we are given a point $t_j \in I_j$. We will denote the collection of points t_j by t. The pair (P, t) is sometimes called a tagged partition.

Definition: We define the Riemann sum associated to the function f, and the tagged partition (P, t) by

$$R(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}).$$

The norm of a partition

As must be clear, the Lower sum, Upper sum and Riemann sum all give approximations to the area between the lines $x = a$ and $x = b$ and between the curve $y = f(x)$ and the x-axis and

$$L(f, P) \leq R(f, P, t) \leq U(f, P).$$

The point is to make this statement quantitatively precise. We define the norm of a partition P (denoted $\|P\|$) by

$$\|P\| = \max_j \{|x_j - x_{j-1}|\}, \quad 1 \leq j \leq n.$$

The norm gives some measure of the "size" of a partition, in particular, it allows us to say whether a partition is big or small.

When the size of the partition is small, it means that every interval in the partition is small.

The Riemann integral

Definition 1: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|R(f, P, t) - R| < \epsilon,$$

whenever $\|P\| < \delta$. In this case R is called the Riemann integral of the function f on the interval $[a, b]$.

In other words, for all sufficiently "small" or "fine" partitions, the Riemann sums must be within ϵ of R.

Notice, that as long as $\|P\|$ is small, it doesn't matter exactly where the x_j's or the t_j's are in the interval $[a, b]$.

Also notice that if P' is a refinement of P, then $\|P'\| \leq \|P\|$.

The Riemann integral continued

Intuitively, we can see that the smaller or finer the partition, the better the area under the curve is represented by the Riemann sum.

The reason that the Riemann integral is useful is because the definition we have given is actually equivalent to the following apparently weaker definition.

Definition 2: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists a partition P such that for every tagged refinement of (P', t') of (P, t),

$$|R(f, P', t') - R| < \epsilon.$$

The nice thing about the above definition is that one only has to check that $|R(f, P', t') - R|$ is small for for refinements of a fixed partitions, and not all partitions.

Back to our example

Using Definition 2 of the Riemann integral it is easy to see that the function $f(x) = x$ is Riemann integrable.

Let $\epsilon > 0$ be arbitrary. For our fixed partition we take $P = P_n$ where $n > \frac{1}{2\epsilon}$ is some fixed number.

The Riemann sum is trapped between the upper and lower sums:

$$L(f, P_n) = \frac{1}{2} - \frac{1}{2n} \leq R(f, P_n, t) \leq U(f, P_n) = \frac{1}{2} + \frac{1}{2n},$$

for any choice t of points in the intervals I_j, that is,

$$|R(f, P_n, t) - \frac{1}{2}| < \frac{1}{2n} < \epsilon,$$

because of our choice of n. Moreover, if (P', t') is any refinement of (P_n, t) we have

$$L(f, P_n) \leq L(f, P') \leq R(f, P', t') \leq U(f, P') \leq U(f, P_n),$$

whence it follows that

$$|R(f, P', t) - \frac{1}{2}| < \epsilon.$$

The example continued

As the preceding example shows, Definition 2 of the Riemann integral is really easy to work with. Why do we then care about Definition 1 or the Darboux integral?

The reason is that while Definition 2 is good for showing that a given function is Riemann integrable, the other definitions are often better for proving the *abstract properties* of integrals.

In fact, this will be clear in the tutorial exercises. You will see that sometimes the Darboux integral is better than the Riemann integral.

Before going any further we will formally state what we have already been referring to for several slides.

Comparison with the Darboux integral

Theorem 20: The Riemann integral exists if and only if the Darboux integral exists and in this case the two integrals are equal.

With this theorem in hand, we see that the function $f(x) = x$ is also Darboux integrable.

How does one prove Theorem 20? It is not too hard but it takes some work and is roundabout.

The easiest way is to proceed as follows. It is clear that if f is Riemann integrable in the sense of Definition 1, it is Riemann integrable in the sense of Definition 2. Next, one shows that if f is Riemann integrable in the sense of Definition 2, then it is Darboux integrable. And finally, one can show that if the Darboux integral exists, then the Riemann integral exists in the sense of Definition 1. An interested student can try this as an exercise.