MA 105 D1 Lecture 15

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Autumn 2014, IIT Bombay, Mumbai

Regular Partitions

Recall that in Definition 2 for a Riemann integrable function of the one variable integral we saw that it was enough to restrict our attention to a fixed family of partitions. This is what we will do, taking a particularly simple family of partitions.

The regular partition of R of order n is is partition defined inductively by $x_0 = a$ and $y_0 = c$ and

$$x_{i+1} = x_i + \frac{b-a}{n} \quad \text{and} \quad y_{j+1} = y_j + \frac{d-c}{n},$$

$1 \leq i, j \leq n-1$. We take $t = \{t_{ij} \in R_{ij}\}$ to be an arbitrary tag.

Definition: We say that the function $f : R \to \mathbb{R}$ is Riemann integrable if the Riemann sum

$$S(f, P_n, t) = \sum_{i,j=0}^{n-1} f(t_{ij}) \Delta_{ij}$$

tends to a limit S for any choice of tag t.

The Riemann integral continued

This limit value is usually denoted as

$$\int\int_R f, \quad \int\int_R f(x,y)dA, \quad \text{or} \quad \int\int_R f(x,y)dxdy.$$

The preceding definition is sometimes rewritten as

$$\lim_{n\to\infty} \sum_{i,j=0}^{n-1} f(t_{ij})\Delta_{ij} = \int\int_R f.$$

If $f(x,y) \geq 0$ for all values of x and y, then the Riemann integral has a geometric interpretation. It is obviously the volume of the region under the graph of the function $z = f(x,y)$ and above the rectangle R in xy-plane.

The integral may also be interpreted as mass in some physical situations; for example, if we have a rectangular plate and $f(x,y)$ represents the density of the plate at a given point, then the integral above gives the mass of the whole plate.

The main theorem

In the one variable case, we saw that a bounded function with at most a finite number of discontinuities on a closed bounded interval is Riemann integrable. The reason that a finite number of discontinuities do not matter is that points have length zero. What might be the analogous result in two variables?

In two variables the geometry of the set of points of discontinuity can be more complicated. Still, what are the analogues of points in this case? In other words what sets have "zero area"?

Theorem 33: If a function f is bounded and continuous on R except possibly along a finite number of graphs of $\mathcal{C}^1$ functions, then f is integrable on R.

Properties of the Double integral

The Riemann integral for functions of two variables is sometimes called the Double Integral. Its properties may be summed up as follows

1. If R is divided by a (vertical or horizontal) line segment into two rectangles R_1 and R_2 and if f is integrable on R_1 and R_2, then f is integrable on R and

$$\int\int_R f(x,y)dxdy = \int\int_{R_1} f(x,y)dxdy + \int\int_{R_2} f(x,y)dxdy.$$

2. If f_1 and f_2 are integrable, and if $f_1 \leq f_2$ on R, then

$$\int\int_R f_1(x,y)dxdy \leq \int\int_R f_2(x,y)dxdy.$$

3. If $f(x,y) = c$ for all (x,y) in R,

$$\int\int_R f(x,y)dxdy = c \times A(R),$$

where $A(R)$ denotes the area of R.

Further properties

4. If f_1 and f_2 are integrable functions on R, then

$$\int\int_R [f_1 + f_2] dA = \int\int_R f_1 dA + \int\int_R f_2 dA.$$

5. For any constant c,

$$\int\int_R cf dA = c\int\int_R f dA.$$

6.

$$\left|\int\int_R f(x,y)dxdy\right| \leq \int\int_R |f(x,y)|dxdy.$$

We have not previously stated the above property in the one variable case where it also holds. All the properties above can be proved quite easily from the definitions, just as in the one variable case.

Calculating integrals

While we have now given a reasonable definition of the integral for functions of two variables, actually calculating integrals using the definition proves much too complicated. After all, even in the one variable method, integrating any but the simplest functions using the definition of the Riemann integral is more or less impossible. Instead, we proved the fundamental theorem of calculus and used the fact that the integral and the antiderivative were the same in order to evaluate the integrals of various standard functions.

The key idea is to reduce integration in two variables to integrating in one variable (but doing it twice, that is, iteratedly).
In fact, this idea goes back all the way to Archimedes, but was perhaps first extensively used by Cavalieri, a student of Galileo (note that this was before Newton and Liebnitz developed the Fundamental Theorem of Calculus).

Iterated Integrals

If $f : R \to \mathbb{R}$ is a function we can define the iterated integrals as follows. We can first define functions of y and x respectively as follows, provided the integrands below are Riemann integrable as functions of one variable:

$$h(y) = \int_a^b f(x,y)dx \quad \text{and} \quad g(x) = \int_c^d f(x,y)dy.$$

We then consider the integrals

$$\int_c^d h(y)dy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy \quad \text{and}$$

$$\int_a^b g(x)dx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx.$$

These integrals (if they exist) are called iterated integrals

If you think about it, it is not obvious that either of the integrals above should be equal to the double integral, but, in fact, they will be in the most common situations we encounter.

Reduction to iterated integrals

Theorem 34: If f is integrable on the rectangle $R = [a, b] \times [c, d]$, and if either one of the iterated integrals

$$\int_c^d \left[\int_a^b f(x, y) dx \right] dy \quad \text{or} \quad \int_a^b \left[\int_c^d f(x, y) dy \right] dx$$

exists, then it equals the double integral

$$\int \int_R f(x, y) dxdy.$$

In particular, if both the iterated integrals exist they must be equal.

Theorem 34 greatly simplifies the evaluation of double integrals since we now only need to need to do single variable integration repeatedly.

Examples:

Example 1 (Marsden, Tromba and Weinstein page 288): Compute $\int\int_R \sin(x+y)dxdy$, where $R = [0,\pi]\times[0,2\pi]$.

Solution:

$$\begin{aligned}
\int\int_R \sin(x+y)dxdy &= \int_0^{2\pi}\left[\int_0^{\pi}\sin(x+y)dx\right]dy \\
&= \int_0^{2\pi}[-\cos(x+y)|_{x=0}^{\pi}]dy \\
&= \int_0^{2\pi}[\cos y-\cos(y+\pi)]dy \\
&= [\sin y-\sin(y+\pi)]|_{y=0}^{2\pi} = 0
\end{aligned}$$

Example 2 (Marsden, Tromba and Weinstein, page 289): If D is a plate defined by $1 \leq x \leq 2, 0 \leq y \leq 1$ (measured in centimeters), and the mass density $\rho(x, y) = ye^{xy}$ grams per square centimeter. Find the mass of the plate.

Solution: The total mass of the plate is got by integrating over the rectangular region covered by D:

$$\begin{aligned}\int\int_D \rho(x, y)dxdy &= \int_0^1 \int_0^2 ye^{xy}dxdy = \int_0^1 (e^{xy}|_{x=1}^2 dy \\ &= \int_0^1 (e^{2y} - e^y)dy = \tfrac{e^2}{2} - e + \tfrac{1}{2}\end{aligned}$$

Bonaventura Cavalieri (1598 - 1647)

http://en.wikipedia.org/wiki/File:Bonaventura_Cavalieri.jpeg

Cavalieri's Principle

The volumes of two solids are equal if the areas of their corresponding cross sections are equal.

http://en.wikipedia.org/wiki/File:Cavalieri

The Slice Method

Cavalieri's basic idea is that we can find the volume of a given solid by slicing it into thin cross sections, calculating the areas of the slices and then adding up these areas.

Let S be a solid and P_x be a family of planes perpendicular to the x-axis with x as x-coordinate such that
1. S lies between P_a and P_b,
2. the area of the slice of S cut by P_x is $A(x)$.
Then the volume of S is given by

$$\int_a^b A(x)dx.$$

Applying this to the solid graph of $z = f(x, y)$ above a rectangle R in the plane, we see that we get exactly the second of our iterated integrals.

Thus Cavalieri's principle is actually a generalization of the method of iterated integrals. Note that in order to apply the principle we do not require the solid to necessarily lie above a rectangular region in the plane.

Cavalieri's principle is particularly useful in computing the volumes of solids of revolution. These are obtained by taking a region B lying between the lines $x = a$ and $x = b$ on the x-axis and the graph of a function $y = f(x)$ and rotating it through an angle 2π around the x-axis.

Solids of revolution

In this case, we can easily compute the cross-sectional area $A(x)$, since each cross section is nothing but a disc. The radius of the circle is nothing but $f(x)$. Hence, the area $A(x)$ is given by

$$A(x) = \pi[f(x)]^2,$$

and the volume V of the solid is given by

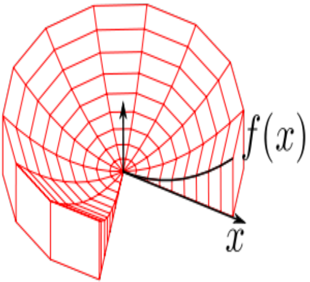
$$V = \pi \int_a^b [f(x)]^2 dx.$$

Solids of revolution may also arise by rotating the graph of a function $f(x)$. around the y-axis. In this case, we can follow the procedure above, replacing x by y and the function $f(x)$ by its inverse.

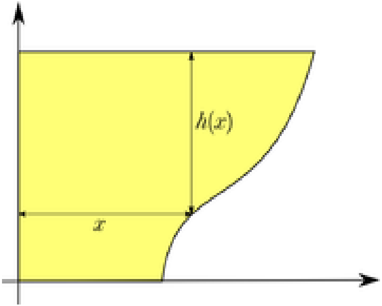
The shell method

There is another way to compute the volume of a solid of revolution obtained by rotating the graph of a function around the y-axis. It is called the shell method.

In this case, rather than slicing the solid by cross sections, we view the solid as being made of cylindrical shells.



The shell method continued



Both pictures from http://en.wikipedia.org/wiki/Shell_integration
As you can see, from the picture above, the radius of the cylindrical shell above the point $(x, f(x))$ is x and the height is $f(x)$. Hence its surface area is $2\pi x h(x)$. To get the volume we must integrate, and this yields

$$2\pi \int_a^b x h(x) dx.$$

The washer method

This is a variant on the previous methods. Sometimes we have have calculate the volume of a solid of revolution which is hollow, where the shape of the hollow part of the solid is also given as a solid of revolution. Thus, we can think of the solid as being obtained by rotating the region that lies between the graphs of two functions $f_1(x)$ and $f_2(x)$ on an interval $[a, b]$ around an axis. If we are rotating around the x-axis, we get

$$\pi \int_a^b [f_2(x)^2 - f_1(x)^2] dx.$$

When we use the shell method, we get the formula

$$2\pi \int_a^b x[f_2(x) - f_1(x)] dx.$$

Above, we assume that $f_2(x)$ lies further away from the axis of rotation than $f_1(x)$. This method of calculating the volumes of hollow solids of revolution is called the washer method.

Exercise 3.15: A round hole of radius $\sqrt{3}$ cms is bored through the center of a solid ball of radius 2 cms. Find the volume cut out.

Solution: We may describe the desired volume as the difference of the volume of the sphere of radius 2 and a certain hollow solid of revolution.

Let us use the slice method first.

The hollow solid of revolution may be described as being obtained by rotating the region between the line $x = \sqrt{3}$ and $x = \sqrt{4 - y^2}$ around the y-axis. The two curves intersect at the points $(\sqrt{3}, \pm 1)$.

The volume of the hollow solid is given by

$$\int_{-1}^{1} \pi x^2 dy - \pi(\sqrt{3})^2 2] = 2\pi[\int_{0}^{1} (4 - y^2) dy - 3] = \frac{4}{3}\pi.$$

The volume of the sphere is $\frac{32}{3}\pi$. Hence the required volume is

$$\frac{32}{3}\pi - \frac{4}{3}\pi = \frac{28}{3}\pi.$$

We could also use the shell method to solve this problem. In the case we will get

$$\begin{aligned} 32\pi/3 - \int_{\sqrt{3}}^{2} 2\pi x(2y)dx \quad &= 32\pi/3 - 4\pi \textstyle\int_{\sqrt{3}}^{2} x\sqrt{4 - x^2}dx \\ &= 32\pi/3 - 4\pi(1/3) = 28\pi/3 \end{aligned}$$

Double integrals over arbitrary regions

We have seen how to compute the volume of a solid that lies above a rectangle in the xy-plane and below the graph of a surface $z = f(x, y)$. In general, however, volumes that we wish to compute may not lie above a rectangle in the plane but over a region which may have quite a complicated shape. In other words, the domain of the function $f(x, y)$ may not be a rectangle but may describe some other shape in the plane. How does one handle this?

There are actually two issues involved here. One is to give a satisfactory definition of the integral over arbitrary regions and the other is to be able to calculate the value of the integrals that we will define.

Defining the double integral over a region

The first issue is easily solved. Assume that $f(x, y)$ is defined in some region D in the plane. We will assume that D is bounded. This means that there is an $r > 0$ such that $D \subset B_r$, where B_r is the ball (or disc) of radius r around the origin. Since any ball can be enclosed by a larger rectangle R, we may assume that $D \subset R$.

Now it is quite easy to define the integral of f on the region D. Define a function

$$g(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \notin D \end{cases}$$

on R. If g is integrable on R, we will say that f is integrable on D and we define

$$\int\int_D = f(x, y)dxdy = \int_R g(x, y)dxdy.$$

Intuitively, it is clear that this is the right definition. After all, since the function g is simply defined to be zero outside the region D, the region outside D contributes no volume to the integral. Thus all the contribution to the integral over R actually comes only from the region D.

You may, however, be bothered by the following question: How can one tell if the function g is integrable or not? Again, when the region D is bounded by a $\mathcal{C}^1$ closed curve (e.g. an ellipse) and f is a continuous function this is not hard to see. In this case the function g on R has the property that g is a bounded function on R which is discontinuous only on the closed curve which is the boundary of D (a $\mathcal{C}^1$ curve can always be realised as a finite union of graphs of $\mathcal{C}^1$ functions!). Hence, by Theorem 33, g is integrable.