MA 105 D1 Lecture 12

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Autumn 2014, IIT Bombay, Mumbai

Review - problems involving the gradient

Exercise 1: Find the points on the hyperboloid $x^2 - y^2 + 2z^2 = 1$ where the normal line is parallel to the line that joins the points $(3, -1, 0)$ and $(5, 3, 6)$.

Solution: The hyperboloid is an implicitly defined surface. A normal vector at a point (x_0, y_0, z_0) on the hyperboloid is given by the gradient of the function $x^2 - y^2 + 2z^2$ at (x_0, y_0, z_0):

$$\nabla f(x_0, y_0, z_0) = (2x_0, -2y_0, 4z_0).$$

We require this vector to be parallel to the line joining the points $(3, -1, 0)$ and $(5, 3, 6)$. This line lies in the same direction as the vector $(5 - 3, 3 + 1, 6 - 0) = (2, 4, 6)$. Thus we need only solve the equations

$$2x_0 = 1, \quad -2y_0 = 4, \quad 4z_0 = 6,$$

which give $x_0 = 1/2$, $y_0 = -2$ and $z_0 = 3/2$. Thus, we need to find λ such that $\lambda(1/2, -2, 3/3)$ lies on the hyperboloid. Substituting in the equation yields $\lambda = \pm\sqrt{2/3}$.

Problems involving the gradient, continued

Exercise: Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin xy$ at the point $(1, 0)$ has the value 1.

Solution: We compute ∇f first:

$$\nabla f(x, y) = (2x + y \cos xy, x \cos xy),$$

so at $(1, 0)$ we get, $\nabla f(1, 0) = (2, 1)$.

To find the directional derivative in the direction $v = (v_1, v_2)$ (where v is a unit vector), we simply take the dot product with the gradient:

$$\nabla_v f(1, 0) = 2v_1 + v_2.$$

This will have value "1" when $2v_1 + v_2 = 1$, subject to $v_1^2 + v_2^2 = 1$, which yields $v_1 = 0, v_2 = 1$ or $v_1 = 4/5, v_2 = -3/5$.

Review of the gradient

Exercise: Find $D_uF(2,2,1)$ where D_u denotes the directional derivative of the function $F(x,y,z) = 3x - 5y + 2z$ and u is the unit vector in the outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at the point $(2,2,1)$.

Solution: The unit outward normal to the sphere $g(x,y,z) = 9$ at $(2,2,1)$ is given by

$$\frac{\nabla g(2,2,1)}{\|\nabla g(2,2,1)\|}.$$

We see that $\nabla g(2,1,1) = (4,4,2)$ so the corresponding unit vector is $(2,2,1)/3$.

To get the directional derivative we simply take the dot product of ∇F with u:

$$(3,5,2) \cdot (2,2,1)/3 = 6.$$

Comments: Also, there is no need to compute the gradient to find the normal vector to the sphere - it is obviously the radial vector at the point $(2,2,1)$!.

Review of the gradient, continued

Exercise: Find the equations of the tangent plane and the normal line to the surface

$$F(x, y, z) := x^2 + 2xy - y^2 + z^2 = 7$$

at $(1, -1, 3)$.

Solution: We first compute the gradient of F to get $\nabla F(x, y, z) = (2x + 2y, 2x - 2y, 4z)$. At $(1, -1, 3)$, this yields the vector $\lambda(0, 4, 6)$ which is normal to the given surface at $(1, -1, 3)$. The point $(1, 3, 9)$ also lies on the normal line so its equation is

$$x = 1, \frac{y+1}{4} = \frac{z-3}{6}.$$

The equation of the tangent plane is given by

$$4(y + 1) + 6(z - 3) = 0,$$

since it consists of all lines orthogonal to the normal and passing through the point $(1, -1, 3)$.

Inf, Sup

Consider the set of points in the interval $[0, 1/2)$. What is its supremum? Clearly $1/2$.

On the other hand consider the subset of number of $[0, 1/2)$ of the form $1/2 - 1/2n$. What is its supremum? Clearly, also $1/2$.

Suppose we have a subset S of $[0, 1/2)$ which has the properly that for every $n \in \mathbb{N}$, there is a $y \in S$ such that $y \geq 1/2 - 1/2n$. What is its supremum? Again, $1/2$.

In general taking the supremum over a subset yields a smaller number than taking the supremum over the whole set. The point I am trying to make is that special subsets of a given set may have the same supremum as the whole set

The Darboux lower integral for $f(x) = x$

Suppose I take any partition P of the interval $[0, 1]$ and take the Darboux lower sum $L(f, P)$ for $f(x) = x$. What does one get? Some number $x \in [0, 1/2)$.

On the other hand, if I take the partition P_n given by interval of equal length $1/n$, what does one get for $L(f, P_n)$? $1/2 - 1/2n$.

Now the set of partitions $P' = P \cup P_n$ $(n \in \mathbb{N})$ is a *subset* of the set of *all* partitions. So *sup* of $L(f, P')$ while only n varies will be less than or equal to the sup of $L(f, P)$ as we vary all P.

On the other hand, for each partition P, the partition $P' = P_n \cup P$ is a refinement of both P_n and P. So $L(f, P') \geq 1/ - 1/2n$.
It follows that for each P, there is a P' such that $L(f, P) \leq 1/2 - 1/2n \leq L(f, P')$. Thus taking the supremum over all P will yield something less than or equal to what we get when taking the supremum only over partitions of the form P'.

The point in the preceding slide is that the set S of all partitions P' as above has the property that we want. Namely

$$\sup_{P'} L(f, P') = \sup_{P} L(f, P).$$

Thus in order to say the lower integral exists, you can look at this special family and take its supremum, which is $1/2$.

A similar analysis of the upper sums show that the upper Darboux integral is also $1/2$.

Small aside: why is $L(f, P) \leq 1/2$ for any P?

Notice that $x_{i-1} \leq \frac{x_{i-1}+x_i}{2}$. So,

$$L(f, P) = \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1}) \leq \sum_{1=1}^{n} \frac{(x_{i-1} + x_i)}{2}(x_i - x_{i-1}).$$

This last sum is the telescoping sum:

$$\sum_{i=1}^{n} \frac{x_i^2}{2} - \frac{x_{i-1}^2}{2} = \frac{x_n^2}{2} - \frac{x_0^2}{2} = 1/2.$$

Riemann Integration

First, note the correction that I issued for the main theorem on Riemann integration which was Theorem 21. The correct statement is as follows.

Theorem 21: Every bounded function f on $[a, b]$ which has at most only a finite number of discontinuities is Riemann integrable.

Now back to discussing the function $f(x) = x$. How does one prove that it is Riemann integrable using Definition 2?

Let $\varepsilon > 0$ be arbitrary. Let $N > 1/2\varepsilon$ and let P_N be the partition $\{0 < 1/N, 2/N, \ldots < 1\}$. Let (P, t) be any tagged refinement of P_N. Then

$$L(f, P_N) \leq R(f, P, t) \leq U(f, P_N),$$

i.e.,

$$1/2 - 1/2N \leq R(f, P, t) \leq 1/2 + 1/2N.$$

But this shows that $|1/2 - R(f, P, t)| \leq 1/2N < \varepsilon$. This shows that the function is Riemann integrable and has Riemann integral $1/2$.