MA 105 D1 Lecture 25

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Triple integrals

Exercises involving triple integrals

Triple integrals in a box

If we have a function $f : B = [a, b] \times [c, d] \times [e, f] \to \mathbb{R}$ we can integrate it over this rectangular parallelepiped. As in the one and two variable cases, we divide the parallelepiped into smaller ones B_{ijk}, making sure that the length, breadth and height of the small parallelepiped are all small. In particular, we can use the regular partition of order n to obtain the Riemann sum

$$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(t_{ijk}) \Delta B_{ijk},$$

where ΔB_{ijk} is the volume of B_{ijk}, and $t = \{t_{ijk} \in B_{ijk}\}$ is an arbitrary tag.

As before we say that f integrable $\lim_{n\to\infty} S(f, P_n, t)$ exists for any choice of tag t. The value of this limit is denoted by

$$\iiint_B f dV, \iiint_B f(x, y, z) dV \quad \text{or} \quad \iiint_B f(x, y, z) dxdydz.$$

Triple integrals over arbitrary regions

All the theorems for double integrals go through for triple integrals.

First, if f is bounded and continuous in B, except possibly on (a finite union of) graphs of $\mathcal{C}^1$ functions of the form $z = a(x, y)$, $y = b(x, z)$ and $x = c(y, z)$, then it is integrable.

This allows us to define the integral of (say) a continuous function on any bounded region enclosed by a simple $\mathcal{C}^1$ closed curve. As before, simply extend the function by zero on a larger enclosing rectangle.

Once we have defined the triple integral in this way, it remains to evaluate it.

Evaluating triple integrals

Again we have a Fubini Theorem - that is, triple integrals can usually be expressed as iterated integrals, this time by integrating functions of a single variable three times.

Thus, if f integrable on the box B we have

$$\iiint_B f(x,y,z)dxdydz = \int_a^b \int_c^d \int_e^f f(x,y,z)dzdydx.$$

There are, in fact, five other possibilities for the iterated integrals and each of these exists and is equal to the value of the triple integral.

The triple integrals that are easiest to evaluate are those for which the region P in space can be described by bounding z between the graphs of two functions in x and y. This is the analogue of an elementary region in the plane and we will call such regions also elementary regions (but in space). In general, we may be able to express more complicated domains as unions of elementary domains.

Evaluating triple integrals continued

In this case we proceed as follows. Suppose that the region P lies between $z = \gamma_1(x, y)$ and $z = \gamma_2(x, y)$. Suppose that the projection of P on the xy plane is bounded by the curves $y = \phi_1(x)$ and $y = \phi_2(x)$ and the straight lines $x = a$ and $x = b$, then

$$\iiint_P f(x, y, z)dxdydz = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)dzdydx.$$

Example: Let us find the volume of the sphere using the above formula. In other words, let us integrate the function 1 on the region P, where P is the unit sphere.

The sphere can be described as the region lying between $z = -\sqrt{1 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$.

The volume of the unit sphere

The projection of the sphere onto the xy plane gives a disc of unit radius. This can be described as the set of points lying between the curves $-\sqrt{1-x^2}$ and $\sqrt{1-x^2}$ and the lines $x = \pm 1$. Thus our triple integral reduces to the iterated integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dzdydx.$$

This yields

$$2\int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)^{1/2} dy \right] dx.$$

After evaluating the inner integral we obtain

$$2\pi \int_{-1}^{1} \frac{1-x^2}{2} dx = \frac{4}{3}\pi.$$

The change of variables formula in three variables

In three variables, we once again have a formula for a change of variables. The formula has the same form as in the two variable case:

$$\iiint_P f(x,y,z)dxdydz = \iiint_{P^*} g(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw,$$

where $h(P^*) = P$. If the change in coordinates is given by $h = (\phi, \psi, \rho)$, the function g is defined as $g = f(\phi, \psi, \rho)$. The expression

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}$$

is just the Jacobian determinant for a function of three variables.

Exercise 7.8

Exercise 7.8: Express the solid $D = \{(x, y, z) | \sqrt{x^2 + y^2} \leq z \leq 1\}$ as a solid region of Type 1, i.e., of the form

$$\{(x, y, z) | a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x), \xi_1(x, y) \leq z \leq \xi_2(x, y)\}.$$

Solution: We are looking at the points of a right circular cone below the plane $z = 1$. To express this as a solid region of Type 1, we need find the functions ξ_1 and ξ_2 as well as the region U in the plane on which these functions are defined. We then need to show that U can be described as a region of Type 2 in the plane.

The solution to Exercise 7.8

The functions ξ_1 and ξ_2 are already given to us:

$$\xi_1(x,y) = \sqrt{x^2+y^2} \quad \text{and} \quad \xi_2(x,y) = 1.$$

To find U we simply take the projection of our solid to the $x-y$ plane. Clearly, U is just the unit disc in the plane which we have to describe as a Type 2 region. This is easy. Take

$$\phi_1(y) = -\sqrt{1-y^2} \quad \text{and} \quad \phi_2(y) = \sqrt{1-y^2},$$

while y ranges from $a=-1$ to $b=1$.

This is the desired description.

Exercise 7.9

Exercise 7.9

$$I = \int_0^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} x dz \right) dy \right) dx.$$

Sketch the region of integration and evaluate the integral by expressing the order of integration as $dxdydz$.

Solution: The solid region W being described is one quarter of the part of the paraboloid of revolution that lies below the plane $z = 2$. More specfically, it is the part of the paraboloid lying in the first quadrant $x \geq 0$ and $y \geq 0$.

The solution to Exercise 7.9

We will describe the solid region as one lying between two graphs given by functions of y andd z. Indeed, we see that $0 < x < \sqrt{z - y^2}$.

We project this region onto the $y - z$ plane. The projection D is given by

$$D = \{(y, z) \,|\, 0 \leq z \leq 2,\, 0 \leq y \leq \sqrt{z}\}.$$

Thus the given triple integral can be rewritten as

$$\int_0^2 \left(\int_0^{\sqrt{z}} \left(\int_0^{\sqrt{z-y^2}} x dx \right) dy \right) dz.$$

$$= \int_0^2 \left(\int_0^{\sqrt{z}} \frac{z - y^2}{2} dy \right) dz = \frac{2}{15} \times z^{\frac{5}{2}} \Big|_0^2 = \frac{8\sqrt{2}}{15}.$$

Exercise 7.10

Exercise 7.10 Using suitable change of variables, evaluate:

(i)

$$I = \iiint_D (z^2x^2 + z^2y^2)dxdydz$$

where D is the cylindrical region $x^2 + y^2 \leq 1$ bounded by $-1 \leq z \leq 1$.

(ii)

$$I = \iiint_D \exp(x^2 + y^2 + z^2)^{3/2} dxdydz$$

over the region enclosed by the unit sphere in $\mathbb{R}^3$.

Solution: (i) In cylindrical coordinates we obtain

$$\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{1} z^2 r^2 r dr dz d\theta = \frac{\pi}{3}.$$

(ii) In spherical coordinates we get:

$$\int_{0}^{2\pi}\int_{0}^{\pi}\int_{0}^{1} e^{-\rho^3}\rho^2 d\rho d\theta d\phi = \frac{4\pi(e-1)}{3}.$$