MA 105 D1 Lecture 8

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Lower and Upper sums

Given a partition $P = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$ and a function $f : [a, b] \to \mathbb{R}$, we define two associated quantities. First we set:

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad 1 \leq i \leq n$$

Defintion: We define the Lower sum as

$$L(f, P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}).$$

Similarly, we can define the Upper sum as

$$U(f, P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}).$$

The Darboux integrals

We now define the lower Darboux integral of f by

$$L(f) = \sup\{L(f, P) \colon P \text{ is a partition of } [a, b]\},$$

where the supremum is taken over all partitions of $[a, b]$.
and similarly the upper Darboux integral of f by

$$U(f) = \inf\{U(f, P) \colon P \text{ is a partition of } [a, b]\},$$

and again the infimum is over all partitions of $[a, b]$.

If $L(f) = U(f)$, then we say that f is Darboux-integrable and define

$$\int_a^b f(t)\, dt := U(f) = L(f).$$

This common value of the two integrals is called the Darboux integral.

An example of a function that is not Darboux integrable

Here is a function that is not Darboux integrable of $[0, 1]$. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

It should be clear that no matter what partition one takes the infimum on any sub-interval in the partition will be 0 and the supremum will be 1.

From this one can see immediately that

$$L(f, P) = 0 \neq 1 = U(f, P),$$

for every P, and hence that $L(f) = 0 \neq 1 = U(f)$.

The Riemann integral

Definition 1: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|R(f, P, t) - R| < \epsilon,$$

whenever $\|P\| < \delta$. In this case R is called the Riemann integral of the function f on the interval $[a, b]$.

In other words, for all sufficiently "small" or "fine" partitions, the Riemann sums must be within ϵ of R.

Notice, that as long as $\|P\|$ is small, it doesn't matter exactly where the x_j's or the t_j's are in the interval $[a, b]$.

The Riemann integral continued

Definition 2: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists a partition P such that for every tagged refinement (P', t') of (P, t)

$$|R(f, P', t') - R| < \epsilon.$$

The nice thing about the above definition is that one only has to check that $|R(f, P', t') - R|$ is small for for refinements of a fixed partitions, and not all partitions.

Theorem 20: The Riemann integral exists if and only if the Darboux integral exists and in this case the two integrals are equal.

Another example

From now on we will use any of the three definitions - the Darboux definition, Definition 1 and Definition 2 for the integral interchangeably and we will often use only the words Riemann integral.

Let us look at Exercise 3.1 (which was assigned for the tutorial). We have to show that the function $f : [0, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 \text{ if } x \in [0, 1] \\ 2 \text{ if } x \in (1, 2] \end{cases}$$

is Riemann integrable from first principles.

The solution

We know from Theorem 20 that we are allowed to use any of the three definitions for the integrability, so let us show that this function is Darboux integrable.

Let $P = \{0 = x_0 < x_1, \dots x_{n-1}, x_n = 2\}$ be an arbitrary partition. The point 1 lies in only one of the partitions, say $[x_{i-1}, x_i]$. for some i. We assume that $1 \neq x_i$ and treat this case first.

$$\begin{aligned} L(f,P) &= \sum_{j=1}^{i}(x_j - x_{j-1}) + \sum_{j=i+1}^{n} 2(x_j - x_{j-1}) && (1) \\ &= x_i + 2(2 - x_i) = 4 - x_i, && (2) \end{aligned}$$

where x_i is a point in $(1, 2]$.

If $x_i = 1$ for some i, then

$$L(f,P) = \sum_{j=1}^{i+1}(x_j - x_{j-1}) + \sum_{j=i+2}^{n} 2(x_j - x_{j-1}) \tag{3}$$

$$= (x_{i+1} - x_0) + 2(2 - x_{i+1}) = 4 - x_{i+1}, \tag{4}$$

where x_{i+1} is a point in $(1,2]$.

In either case $\sup_P L(f,P) = L(f) = 3$.

The Upper sums $U(f,P)$ can be treated in exactly the same way. In either of the cases we have treated above we get $U(f,P) = 4 - x_{i-1}$, for a point $x_{i-1} \in [0,1)$. It follows that $U(f) = \inf_P U(f,P) = 3$.

We have thus shown that $L(f) = U(f) = 3$ which shows that the function is Darboux integrable.

The main theorem for Riemann integration

The main theorem of Riemann integration is the following:

Theorem 21: Let $f : [a, b] \to \mathbb{R}$ be a bounded function that is continuous at all but finitely many points of $[a, b]$. Then f is Riemann integrable on $[a, b]$.

In fact, one can allow even countably many discontinuities and the Theorem will remain true.

Exercise 1: Those of you who have an extra interest in the course should think about trying to prove both Theorem 21 and the extension to countably many discontinuities (Warning: there is one crucial fact about continuous functions that we have not covered that you will have to discover for yourself).

Properties of the Riemann integral

From the definition of the Riemann integral we can easily prove the following properties. We assume that f and g are Riemann integrable. Then

$$\int_a^b [f(t) + g(t)]dt = \int_a^b f(t)dx + \int_a^b g(t)dt,$$

$$\int_a^b cf(t)dt = c\int_a^b f(t)dt,$$

for any constant $c \in \mathbb{R}$, and finally if $f(t) \leq g(t)$ for all $t \in [a, b]$, then

$$\int_a^b f(t)dt \leq \int_a^b g(t)dt.$$

Implicit in the properties above is that fact if f and g are Riemann integrable, then so are $f + g$ and cf.

Properties (1) and (2) say that the Riemann integral is a linear map from the set of all Riemann integrable functions to $\mathbb{R}$.

Proving the properties of the integral

It is not hard to prove either of the properties. One needs only to use the corresponding properties for inf and sup:

$$\inf_P U(f+g,P) = \inf_P U(f,P) + \inf_P U(g,P),$$
$$\sup_P L(f+g,P) = \sup_P L(f,P) + \sup_P L(g,P),$$

and

$$\inf_P U(cf,P) = c \inf_P U(f,P), \quad \sup_P L(cf,P) = c \sup_P L(f,P)$$

These are quite easy to do and I leave this as an exercise to the student (the third property is particularly easy).

More generally, if there are two quantities x and y (not necessarily lower or upper sums) lying in some two subsets of the real numbers, then we only get $\sup(x+y) \leq \sup x + \sup y$ and $\inf(x+y) \geq \inf x + \inf y$ and $\sup cx = c \sup x$ and $\inf cx = c \inf x$ (again this is very easy to see from the definition of inf and sup). When x and y range over sets of non-negative numbers, we get equality.

The Least Upper Bound Axiom

The most interesting fact about inf and sup is, however, the following:

Theorem 22: If a set of real numbers is bounded above, it has a supremum (or least upper bound). If a set of real numbers is bounded below it has an infimum (or greatest lower bound).

Theorem 22 above is equivalent to the fact that the real numbers are complete (Theorem 4 in Lecture 4), a fact that we have not proved (remember that completeness means that every Cauchy sequence in $\mathbb{R}$ converges). It is a very useful form of the completeness and from now on we will use it whenever necessary. It is sometimes referred to as the Least Upper Bound axiom.

Another property of the Riemann Integral

Theorem 23: Suppose f is Riemann integrable on $[a, b]$ and $c \in [a, b]$. Then

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Proof: First we note that if $c = a$ or $c = b$, there is nothing to prove.

Next, if $c \in (a, b)$ we proceed as follows. If P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$, then $P_1 \cup P_2 = P'$ is obviously a partition of $[a, b]$. Thus, partitions of the form $P_1 \cup P_2$ constitute a subset of the set of all partitions of $[a, b]$. For such partitions P' we have

$$L(f, P') = L(f, P_1) + L(f, P_2).$$

Let us denote by $L(f)_{[a,c]}$ (resp. $L(f)_{[c,b]}$) the Darboux lower integral of f on the interval $[a, c]$ (resp. $[c, b]$).

If we take the supremum over all partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ we get

$$\sup_{P'} L(f, P') = L(f)_{[a,c]} + L(f)_{[c,b]}.$$

Now the supremum on the left hand side is taken only over partitions P' having the special form $P_1 \cup P_2$. Hence it is less than or equal to $\sup_P L(f, P)$ where this supremum is taken over all partitions P. We thus obtain

$$L(f)_{[a,c]} + L(f)_{[c,b]} \leq L(f).$$

On the other hand, for any partition $P = \{a < x_1 < \ldots x_{n-1} < b\}$ we can consider the partition $P' = P \cup \{c\}$. This will be a refinement of the partition P and can be written as a union of two partitions P_1 of $[a, c]$ and P_2 of $[c, b]$.

By the property for refinements for Darboux sums we know that $L(f, P) \leq L(f, P')$.

Thus, given any partition P of $[a, b]$, there is a refinement P' which can be written as the union of two partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ respectively, and by the above inequality,

$$\sup_P L(f, P) \leq \sup_{P'} L(f, P'),$$

where the first supremum is taken over all partitions of $[a, b]$ and the second only over those partitions P' which can be written as a union of two partitions as above. This shows that

$$L(f) \leq L(f)_{[a,c]} + L(f)_{[c,b]},$$

so, together with the previous inequality, we get

$$L(f) = L(f)_{[a,c]} + L(f)_{[c,b]}.$$

The same kind of reasoning applies to the Upper sums which allows us to prove the required property.

Motivation

The Fundamental Theorem of calculus allows us to relate the process of Riemann integration to the process of differentiation. Essentially, it tells us that integrating and differentiating are inverse processes. This is a tremendously useful theorem for several reasons.

It turns out that (Riemann) integrating even simple functions is much harder than differentiating them (if you don't believe me, try integrating $(\tan x)^3$ via Riemann sums!). In practice, however, integration is what we need to do to solve physical problems. Usually, when we are studying the motion of a particle or a planet what we find is that the position of a particle, which is a function of time, satisfies some differential equation. Solving the differential equation involves performing the inverse operation of taking some combination of derivatives. The simplest such inverse operation is taking the inverse of the first derivative, which the Fundamental Theorem says, is the same as integrating.

Calculating Integrals

Thus, calculating integrals is one of the basic things one needs to do for solving even the simplest physics and engineering problems. The problem is that this is quite difficult to do.

Once we know the derivatives of some basic functions (polynomials, trigonometric functions, exponentials, logarithms) we can differentiate a wide class of functions using the rules for differentiation , especially the product and chain rules. By contrast, the only rule for Riemann integration that can be proved from the basic definitions is the sum rule.

The Fundamental Theorem solves this problem (partially) because it allows us to deduce formalæ for the integrals of the products and the composition of functions from the corresponding rules for derivatives.

The Fundamental Theorem - Part I

Theorem 24: Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let

$$F(x) = \int_a^x f(t)dt$$

for any $x \in [a, b]$. Then $F(x)$ is continuous on $[a, b]$, differentiable on (a, b) and

$$F'(x) = f(x),$$

for all $x \in (a, b)$.

Proof: We know that $f(t)$ is Riemann integrable for any $x \in [a, b]$ because of Theorem 21 (every continuous function is Riemann integrable).

The proof of Part I continued

By Theorem 23 we know that

$$\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt,$$

for $x + h \in [a, b]$. Hence

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \cdot \int_x^{x+h} f(t)dt.$$

We know that if $f(t) \leq g(t)$ on $[a, b]$, then $\int f(t)dt \leq \int g(t)dt$. We apply this to the three functions $m(h)$, f and $M(h)$, where $m(h)$ and $M(h)$ are the constant functions given by the minimum and maximum of the function f on $[x, x+h]$ to get:

$$m(h) \cdot h \leq \int_x^{x+h} f(t)dt \leq M(h) \cdot h.$$

Dividing by h and taking the limit gives

$$\lim_{h\to 0} m(h) \leq \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \leq \lim_{h\to 0} M(h).$$

But f is a continuous function, so $\lim_{h \to 0} m(h) = \lim_{h \to 0} M(h) = f(x)$. By the Sandwich theorem for limits (use version 2), we see that limit in the middle exists and is equal to $f(x)$, that is $F'(x) = f(x)$. This proves the first part of the Fundamental Theorem of Calculus. $\square$

This first form of the Fundamental Theorem allows us to compute definite integrals. Keeping the notation as in the Theorem we obtain

Corollary:

$$\int_c^d f(t)dt = F(d) - F(c),$$

for any two points $c, d \in [a, b]$.

The Fundamental Theorem of Calculus Part 2

Theorem 24: Let $f : [a, b] \to \mathbb{R}$ be given and suppose there exists a continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b) and which satisfies $g'(x) = f(x)$. Then, if f is Riemann integrable on $[a, b]$,

$$\int_a^b f(t)dt = g(b) - g(a).$$

Note that this statement does not assume that the function $f(t)$ is continuous, and is hence, stronger than the corollary we have just stated.

Proof: We can write:

$$g(b) - g(a) = \sum_{i=1}^{n} [g(x_i) - g(x_{i-1})],$$

where $\{a = x_0, x_1, \ldots, x_n = b\}$ is an arbitrary partition of $[a, b]$. Using the mean value theorem for each of the intervals $I_j = [x_j, x_{j-1}]$, we can write

The proof of the Fundamental Theorem part II continued

$$g(x_i) - g(x_{i-1}) = g'(c_i)(x_i - x_{i-1}).$$

where $c_i \in (x_{i-1}, x_i)$.

Substituting this in the previous expression and using the fact that $g'(c_i) = f(c_i)$, we get

$$g(b) - g(a) = \sum_{i=1}^{n} [f(c_i)(x_i - x_{i-1})].$$

The calculation above is valid for any partition The right hand side obviously represents a Riemann sum. By hypothesis f is Riemann integrable. It follows (using Definition 1, for example) that as $\|P\| \to 0$, the right hand side goes to the Riemann integral. $\square$