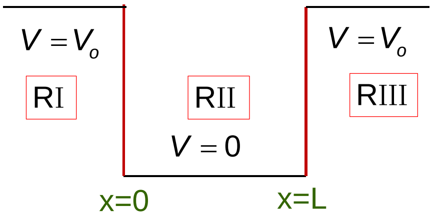
Particle in a Finite Potential Well

$$V(x) = 0 \quad \text{for } 0 < x < L$$
$$\quad = V_o \quad \text{for } x < 0 \text{ or } x > L$$



We take E<V_o for the bound state problem.

RI $$\frac{d^2\phi_I(x)}{dx^2} + \frac{2m}{\hbar^2}[E - V_o]\phi_I(x) = 0$$

RII $$\frac{d^2\phi_{II}(x)}{dx^2} + \frac{2m}{\hbar^2}[E]\phi_{II}(x) = 0$$

RIII $$\frac{d^2\phi_{III}(x)}{dx^2} + \frac{2m}{\hbar^2}[E - V_o]\phi_{III}(x) = 0$$

General Solutions

RI $$\phi_I(x) = Ae^{\alpha x} + Be^{-\alpha x}; \ \alpha^2 = \frac{2m(V_o - E)}{\hbar^2}$$

RII $$\phi_{II}(x) = C\sin(kx) + D\cos(kx);$$
$$k^2 = \frac{2mE}{\hbar^2}$$

RIII $$\phi_{III}(x) = Ge^{\alpha x} + Fe^{-\alpha x}; \ \alpha^2 = \frac{2m(V_o - E)}{\hbar^2}$$

Well-behaved Wave Functions

$$B = 0$$
$$G = 0$$

$$\phi_I(x) = Ae^{\alpha x}$$
$$\phi_{II}(x) = C\sin(kx) + D\cos(kx)$$
$$\phi_{III}(x) = Fe^{-\alpha x}$$

Boundary Conditions Applied

RI- RII

$$\phi_I(0) = \phi_{II}(0)$$

$$Ae^0 = C\sin(0) + D\cos(0)$$

$$A = D$$

$$\frac{d\phi_I}{dx}(0) = \frac{d\phi_{II}}{dx}(0)$$

$$A\alpha e^0 = Ck\cos(0) - Dk\sin(0)$$

$$A\alpha = Ck$$

RII- RIII

$$\phi_{II}(L) = \phi_{III}(L)$$

$$C\sin(kL) + D\cos(kL) = Fe^{-\alpha L}$$

$$\frac{d\phi_{II}}{dx}(L) = \frac{d\phi_{III}}{dx}(L)$$

$$Ck\cos(kL) - Dk\sin(kL) = -F\alpha e^{-\alpha L}$$

$$A = D$$

$$A\alpha = Ck$$

$$C\sin(kL) + D\cos(kL) = Fe^{-\alpha L}$$

$$Ck\cos(kL) - Dk\sin(kL) = -F\alpha e^{-\alpha L}$$

Solving the Equations

Express all constants in terms of A.

$$D = A$$

$$C = \frac{\alpha}{k}A$$

$$\frac{\alpha}{k}A\sin(kL) + A\cos(kL) = Fe^{-\alpha L}$$

$$\frac{\alpha}{k}Ak\cos(kL) - Ak\sin(kL) = -F\alpha e^{-\alpha L}$$

Energy Eigen Values

Divide the last two equations.

$$\frac{\alpha\cos(kL)-k\sin(kL)}{\alpha\sin(kL)+k\cos(kL)}=-\frac{\alpha}{k};$$

$$\hbar\alpha=\sqrt{2m(V_o-E)}$$

$$\hbar k=\sqrt{2mE}$$

The above equation governs the allowed energy levels.

Normalization

$$|A|^2\left[\int_{-\infty}^{0}e^{2\alpha x}dx+\int_{0}^{L}\left(\frac{\alpha}{k}\sin kx+\cos kx\right)^2dx\right]+$$

$$|A|^2e^{2\alpha L}\left(\frac{\alpha}{k}\sin kL+\cos kL)\right)^2\int_{L}^{\infty}e^{-2\alpha x}dx=1$$