Consider Fermions

In how many ways 2 Fermions can occupy 3 degenerate states of energy E_1?

$$\frac{3!}{2!1!} = 3$$

In how many ways 3 Fermions can occupy 5 degenerate states of energy E_2?

$$\frac{5!}{3!2!} = 10$$

In how many ways 2 Fermions can occupy 3 degenerate states of energy E_1 and 3 Fermions can occupy 5 degenerate states of energy E_2?

$$\frac{3!}{2!1!} \times \frac{5!}{3!2!} = 10 \times 3 = 30$$

In how many ways n_i Fermions can occupy g_i degenerate states of energy E_i for all values of i?

$$\prod_i \frac{g_i!}{n_i!(g_i - n_i)!}$$

This was for one set of n_i. There may be many more sets which satisfy the same conditions put on the total number of particles and total energy. The set for which this product would be highest would be most probable.

$$\prod_i \frac{g_i!}{n_i!(g_i - n_i)!}$$

Example

	g_i	n_{i1}	n_{i2}
2E	15	2	3
E	10	4	2
0	5	3	4

$$E = 8E$$
$$N = 9$$

Total Combinations for Set 1

$$\frac{15!}{2!13!} \times \frac{10!}{4!6!} \times \frac{5!}{2!3!}$$
$$= \frac{15 \times 14}{2} \times \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2} \times \frac{5 \times 4}{2}$$
$$= 105 \times 210 \times 10$$
$$= 2{,}20{,}500$$

Total Combinations for Set 2

$$\frac{15!}{3!12!} \times \frac{10!}{2!8!} \times \frac{5!}{4!1!}$$
$$= \frac{15 \times 14 \times 13}{3 \times 2} \times \frac{10 \times 9}{2} \times 5$$
$$= 455 \times 45 \times 5$$
$$= 1{,}02{,}375$$

Prescription

Maximize relative to n_i,

$$ln\left(\prod_i \frac{g_i!}{n_i!(g_i - n_i)!}\right)$$

Subject to the conditions

$$\sum_i n_i = N;\ \sum_i n_i E_i = E$$

First Type: Maxwell-Boltzmann Distribution

$$n_i = g_i e^{-\alpha} e^{-\frac{E_i}{kT}}$$

Second Type: Bose-Einstein Distribution

$$n_i = \frac{g_i}{e^{\alpha} e^{\frac{E_i}{kT}} - 1}$$

BOSONS

Third Type: Fermi-Dirac Distribution

$$n_i = \frac{g_i}{e^{\alpha} e^{\frac{E_i}{kT}} + 1}$$

FERMIONS

Observations

- The distribution does not directly depend upon N.
- The constant α has to be evaluated from the number of particles. Its value is to be put equal to zero, if the number of particles are not constant.

Observations

- For higher values of E_i/kT, the other two distributions approximate to M.B. distribution.

Bose Einstein Statistics

Can be written as

$$n_i = \frac{g_i}{e^{\left(\frac{E_i}{kT}+\alpha\right)} - 1}$$

The value of α is positive, otherwise for some energy values n_i would diverge and even become negative.

Fermi Dirac Statistics

Is normally written as

$$n_i = \frac{g_i}{e^{\frac{E_i - E_F}{kT}} + 1}$$

E_F is called Fermi energy.

Fermi Energy

- n_i is always less or equal to g_i.
- n_i/g_i can be termed as the probability of occupancy of a state with an energy E_i.
- E_F denotes the energy of the highest occupied level at $T=0$ K and the one for which probability is half at $T>0$.

Return to the Problem

- For the energy levels given by particle in a three dimension box, how the states would get filled up when we have a very large number of particles?

$$E = \frac{\pi^2 \hbar^2}{2mL^2}\left(n_x{}^2 + n_y{}^2 + n_z{}^2\right)$$

- We have to evaluate the degeneracies g_i of all the levels, before we can proceed.
- Is it really a simple problem if we are talking of a very large number of particles in a box of macroscopic dimensions?

Density of State

- Considering that we have to handle a large number of particles, we change g_i to a continuous variable, called Density of State.
- The density of states $g(E)dE$ are defined as the number of states between energy E and $E+dE$.