PH-107 (2014)
Tutorial Sheet 4
(Wave Mechanics)

* Problems to be done in tutorial.

J: Free State Problems:

P68: A beam of particles coming from left with energy E approaches a potential barrier given below.

$$V(x) = 0 \quad \text{for } x<0$$
$$V(x) = -V_o \text{ for } x \geq 0$$

Obtain expressions for the transmission and the reflection coefficients and show that their sum is one.

P69: A beam of electrons with energy $E = 4\ eV$ approaches from left hand side a potential barrier defined as $V(x)=0$ for $x<0$ and $V(x)=5\ eV$ for $x>0$. Find the value of x inside the barrier for which the probability density is one-fourth the probability density at $x=0$.

P70*: A beam of particles with energy E approaches from left hand side, a potential barrier defined by $V=0$ for $x<0$ and $V=V_o$ for $x>0$, where $V_o>E$.

(a) Find the value of $x=x_o$ ($x_o>0$), for which the probability density is $1/e$ times the probability density at $x=0$.

(b) Take maximum allowed uncertainty Δx for the particle to be localized in the classically forbidden region as x_o. Find the uncertainty this would cause in the energy of the particle. Can then one be sure that its energy E is less than V_o.

P71: (a) A beam of particles of energy E is moving in $+x$ direction is incident on a barrier of height V_o ($V_o<E$) at $x = 0$. Find an expression of the reflection and the transmission coefficient.

(b) If the transmission coefficient in the above problem is 0.36, find E/V_o.

P72*: A beam of particles of energy E and de Broglie wavelength λ, traveling along the positive x-axis in potential free region, encounters a one-dimensional potential barrier of height $V=E$ and width L.

(a) Obtain an expression for the transmission coefficient.

(b) For what value of L (in terms of λ), will the reflection coefficient be half?

P73: A beam of particles (mass 500 keV/c^2) moving along negative x direction of energy 3 eV is incident on a potential given by

$$V = \infty \text{ for } x \le 0$$
$$V = 5 \text{ eV for } 0 < x \le a$$
$$V = 0 \text{ for } x > a$$

The value of 'a' is equal to the de-Broglie wavelength of the particles (in the region x>a). Write the wave function of the particles in terms of the amplitude of the incident wave. What is the ratio of the probabilities of finding the particle at x = a/2 and at x = a.

P74: A beam of particle of mass m and energy $\frac{4}{3}V_o$ (where V_o is positive constant) is incident from left on the following potential barrier.

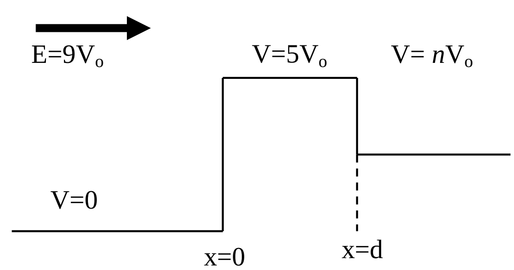
$$V = 0 \text{ for } x < -a \text{ and for } x > +a$$
$$V = V_o \text{ for } -a \le x \le +a$$

Where $a = \pi\hbar\sqrt{\frac{3}{2mV_o}}$. Write the wave functions in all the three regions and apply boundary conditions, clearly stating them. Find the transmission coefficient of the particles.

P75*: A beam of particles of mass *'m'* and energy 9 V_o (V_o is a positive constant of energy dimension) is incident from left on a barrier given below.

$$V=0 \text{ for } x<0$$
$$V=5\,V_o \text{ for } x \le d\text{ ; where } d = \frac{\pi\hbar}{\sqrt{8mV_o}}$$

V= nV_o for x ≥d; where n is a number, positive or negative.



It is found that the transmission coefficient from x<0 region to x>d region is 0.75.

(a) Find *'n'*. Is there more than one possible value for *'n'*?

(b) Find the un-normalized wave function in all the regions in terms of the amplitude of the incident wave for each possible value of *'n'*.

(c) Is there a phase change between the incident and the reflected beam at x=0? If yes determine it for each possible value of *'n'*.

Give your answers by explaining all the steps and clearly writing the boundary conditions used

P76: A beam of particles of energy $9V_o$ is incident from left on a double potential barrier defined as follows:

$$V = 0 \text{ for } x < 0$$
$$V = 5V_o \text{ for } 0 \le x < d$$
$$V = 8V_o \text{ for } x \ge d$$

Here, $d = \frac{\pi\hbar}{\sqrt{2mV_o}}$ and V_o is positive. Find the wave function of the particle in all the regions in terms of the amplitude of the incident wave. Also find the reflection and transmission coefficients and show that their sum is one.

K: Bound State Problems:

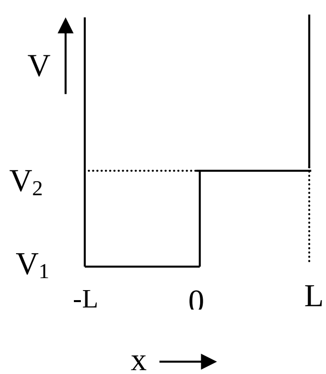
P77: A particle of mass m and energy E ($E>V_2$) is confined to potential well of the following type.

$$V=\infty \text{ for } x<-L \text{ and for } x>L$$
$$V=V_1 \text{ for } -L<x<0$$
$$V=V_2 \text{ for } 0<x<L;\ V_2>V_1$$

(a) Find the equation governing the energy of the particle.

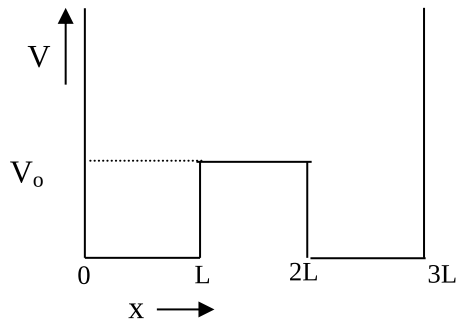
(b) Find the ratio of the probabilities for the particle to remain in negative x to positive x.

(c)Take the limiting case of $V_1=V_2$ and show that the energies come out to be same as that of a particle in a one-dimensional box of length $2L$.



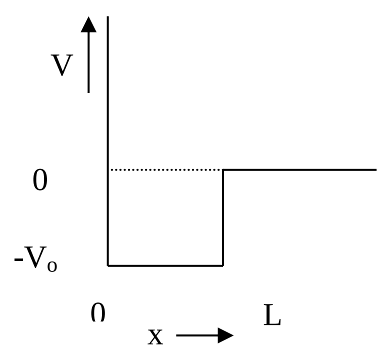
P78*: A particle of mass m is confined to a one-dimensional box described by $V=0$ for $0<x<L$ and for $2L<x<3L$; $V=V_o$ for $L<x<2L$ and $V=\infty$, everywhere else. It is given that the ground state wave function of the particle is independent of x between $L<x<2L$

(a) Find L in terms V_o and m
(b) Find the percentage probabilities of finding the particle in three different regions of different potentials.
(c) Sketch the wave function everywhere in box.



P79: Consider a one-dimensional potential shown in the figure below. This is described by $V=\infty$ for $x\leq 0$; $V=-V_o$ for $0<x<L$ and $V=0$ for $x\geq L$.

(a) For a beam of particle coming from right with energy $E>0$, what is the reflection coefficient? Can you guess the answer without doing the calculation?
(b) Find the equation that governs the energy for the bound state. Normalize the wave function for this case.
(c) For a given value of L, what should be the value of V_o, so that there is only one bound state.



P80: Consider a potential well given below.

$V= 2V_o$ (where V_o is positive constant) for $x \leq 0$

$V= 0$ for $0< x<L$

$V=V_o$ for $x \geq L$

Find the range of the values of L, for which only one bound state would appear.

P81: Consider a potential well as defined below:

$V=V_0$ for $x < 0$; $V=0$ for $0 < x < L$; $V=V_0$ for $x > L$

(a) Find the value of V_0, for which the particle will have only one bound state at $E=V_0/2$.

(b) Find the values of x in the classically forbidden region, for which the probability of finding the particle will be (1/e) of the value at $x = L$.

P82: Consider the following one dimensional potential well (V_o is positive)

$$V = 0 \text{ for } x < 0$$
$$V = -V_o \text{ for } 0 \le x < d$$
$$V = +4V_o \text{ for } x \ge d$$

Find the un-normalized wave function for a particle bound in the well and roughly plot it in your answer-book for the ground state. Find the equation that determines the energy of the quantized levels. Find the least value of 'd' for which a bound state will exist.

P83*: A particle of mass m is in the second excited state (n=3) of an 1-d infinite square well which extends from x=0 to x=L. Suddenly the well expands to double its size (i.e. from x=0 to x=2L), leaving the wave function undisturbed momentarily. If the energy of the particle is now measured, what are the probabilities of finding it in the ground state and in the first excited state (n=2)? At a later time the wave function of the particle is given by the same functional form, but now extends between x=0 and 2L. Find out the probability of finding the particle in the ground state and n=2 state now.

L: Three Dimensional Box:

P84*: A particle of mass m is confined to a three dimensional box of sides L,$L/2$, $L/2$. Find the energy levels of the particle. Find also the wave functions and energy corresponding to lowest energy level that exhibits degeneracy.

P85: A particle of mass *m* and total energy *E* is confined to a three-dimensional potential well characterized by the following potentials.

$$V = -V_o \text{ for } 0 \le x \le L \text{ and for } 0 \le y \le L/2 \text{ and for } 0 \le z \le L/2$$

$$V = +\infty \text{ everywhere else}$$

$$(V_o,\ L,\ E \text{ are positive})$$

(a) Starting from three dimensional time independent Schrödinger equation, obtained the quantized energy level.

(b) Deduce the value of the lowest energy level that exhibits degeneracy. Give the corresponding un-normalized wave functions.

(c) Find the maximum value of V_o in terms of *L* and *m*, for which no negative energy state exists.

M: Harmonic Oscillator:

P86: A particle of mass *m* is under the influence of a potential $V = \frac{1}{2}m\omega^2x^2$, where *x* is the displacement from origin and ω is a constant. The ground state wave function of this particle is given by $\phi(x) = Ae^{-\left(\frac{m\omega}{2\hbar}\right)x^2}$. Find the mean values $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p_x \rangle$ and $\langle p_x{}^2 \rangle$ corresponding to this wave function and the uncertainty product. For integer *k*, you may use the following integrals.

$$\int_0^\infty u^{2k}e^{-bu^2}du = \frac{1\times 3\times ...(2k\text{-}1)}{2^{k+1}}\sqrt{\frac{\pi}{b^{2k+1}}}$$

$$\int_0^\infty u^{2k+1}e^{-bu^2}du = \frac{k!}{2b^{k+1}}$$

P87: A particle of mass *m* is under the influence of a potential $V = \frac{1}{2} m\omega^2x^2$, where *x* is the displacement from origin and ω is a constant. Show that $\phi(x) = Axe^{-\left(\frac{m\omega}{2\hbar}\right)x^2}$ is a solution of the one-dimensional Schrodinger equation with the above potential. Find the value of energy for this state.

P88*: For a particle bound in a particular one-dimensional potential with a property that V(0)=0, the two solutions of Schrödinger Equation are given by $\psi_1(x)$ and $x\psi_1(x)$. The energies corresponding to these solutions are E_1 and E_2 where $E_1 \ne E_2$. Find the un-normalized $\psi_1(x)$, the ratio E_2/E_1 and V(x).

P89*: (a) One of the solutions of the Schrödinger equation for a particle of mass *m* experiencing a potential $V = \frac{1}{2}kx^2$, *k* being constant is $\phi = Axe^{-\alpha x^2}$. Find out the value of α and the energy.

(b) Find the value of $|x| = x_o$, beyond which the region is classically forbidden for the particle.

(c) Consider a particle bound between $-x_o < x < +x_o$ in the following potential

$V = V_o$ for $x < -x_o$

$V = \frac{1}{2}kx^2$ for $-x_o < x < +x_o$

$V = V_o$ for $x > +x_o$

Can $\phi = Axe^{-\alpha x^2}$ still be a valid wave function for the region $-x_o < x < +x_o$? Please give precise reasons.

N: Spherically Symmetrical Potential:

P90*: If the wave function $\psi(r,\theta,\phi,t=0)$ for the case of hydrogen atom is written as a product of three functions as $\psi(r,\theta,\phi,t=0) = R(r) \times \Theta(\theta) \times \Phi(\phi)$ then it can be shown that the radial pert of the Schrödinger equation can be written as follows.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\frac{2m}{\hbar^2}\left(E + \frac{e^2}{4\pi\varepsilon_o r}\right) - \frac{\ell(\ell+1)}{r^2}\right]R = 0$$

where the symbols have their usual meaning. For the ground state of hydrogen atom ℓ=0. For this state

(a) Show that $\psi(r,t=0) = Ae^{-\frac{r}{a}}$ is a solution of this equation. Find the values of *A*, *a* and the ground state energy.

(b) Calculate the mean distance, root mean square distance and the most probable distance between the electron and the nucleus in terms of the Bohr radius.

(c) What are the classical and quantum mechanical probabilities of finding the electron ar *r>2a*.

You may use the following standard integral.

$$\int_0^\infty x^p e^{-x} dx = \Gamma(p+1) = p\Gamma(p); \quad p+1>0$$

P91: A particle is bound in a potential well of the type V(r) =-Vo for r <a and V(r)=0 for r>a. The wave function $\psi(r,\theta,\phi,t=0)$ for this case is written as a product of three functions as $\psi(r,\theta,\phi,t=0)=R(r)\times\Theta(\theta)\times\Phi(\phi)$ and the radial part of the Schrödinger equation can be written as follows for ℓ=0.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[\frac{2m}{\hbar^2}(E-V)\right]R=0$$

The symbols have their usual meaning. Solve this equation to find the equation which would lead to energy states.

(Hint: Use the substitution $\chi(r)=R(r)\times r$, to reduce the equation into a more familiar differential equation)

P92*: The Θ and Φ parts of the Schrödinger equation are given as follows.

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\ell(\ell+1)\sin^2\theta=m_l^{\,2}$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=-m_l^{\,2}$$

If we substitute w = cos θ and write P for Θ in the first equation, we get the following equation.

$$\left(1-w^2\right)\frac{d^2P}{dw^2}-2w\frac{dP}{dw}+\left[\ell(\ell+1)-\frac{m_l^{\,2}}{1-w^2}\right]P=0$$

It is given that $P=Aw^2-B$ is a solution of this equation. Find the possible values of $\frac{A}{B}$, ℓ and m_ℓ. Write the form of Θ and Φ for these cases.