Roll Number:

PH107 (September 11, 2014) Mid-Semester Examination

a. Please write your **Name, Roll Number, Division and Tutorial Batch** on answer sheets and Roll Number on Question Paper.

b. All steps must be shown. All explanations must be clearly given for getting credit. Just the correct final answer does not guarantee the full credit.

c. Possession of mobile phone and exchange of calculators during examination are strictly prohibited.

d. Note that the question paper is printed on both sides

Weightage: 30% **Time: 2 Hours**

1. (a) The absolute temperature of a perfectly black body is changed so that the peak wavelength in its spectrum increases by 10%. What will be the ratio of the total initial and final power radiated by the black body.

Solution:

Let the initial temperature be T_1 and it be changed to T_2. Also let λ be the wavelength corresponding to temperature T_1, for which the intensity is largest. Then as per the question we have:

$$\lambda T_1 = 1.1 \times \lambda T_2$$
$$\Rightarrow \frac{T_1}{T_2} = 1.1$$

(1.5 marks)

Applying Stefan's law, taking P_1 and P_2 is the total power radiated by the black body, we shall get.

$$\frac{P_1}{P_2} = \left(\frac{T_1}{T_2}\right)^4 \approx 1.46$$

(1.5 marks)

(b) Assume a one-dimensional chain of Avogadro number (N_A) of atoms. If all motions are confined only in one dimension, use Einstein's model to find the specific heat at an absolute temperature T for such a chain. Also find the lower and higher temperature limits of the specific heat.

(3+3 marks)

Solution:

As the motion is confined to only one dimension, the total number of degrees of freedom is N_A, which are all vibrational. According to Einstein model, we assume that all the atoms are vibrating with the same frequency ν_E. Hence total energy E of the system is given by:

$$E = N_A \times \frac{h\nu_E}{e^{\frac{h\nu_E}{kT}} - 1}$$

(0.5 marks)

We shall thus get the specific heat as follows:

$$\therefore C_v = \frac{dE}{dT} = N_A \times \frac{h\nu_E}{\left(e^{\frac{h\nu_E}{kT}} - 1\right)^2} \times \frac{h\nu_E}{kT^2}$$

$$= N_A k \frac{\left(\frac{h\nu_E}{kT}\right)^2}{\left(e^{\frac{h\nu_E}{kT}} - 1\right)^2} = \frac{\left(\frac{h\nu_E}{kT}\right)^2}{\left(e^{\frac{h\nu_E}{kT}} - 1\right)^2} R$$

(0.5 marks)

In the low temperature limit, neglect 1 in the denominator, thus getting following expression of specific heat.

$$\therefore C_v \approx \left(\frac{h\nu_E}{kT}\right)^2 e^{-\frac{2h\nu_E}{kT}} R$$

This tends to zero as T tends to zero due to exponential term.

(1 mark)

For the high temperature limit, write the exponential term in series and neglect terms other than the first two. We shall then get:

$$\therefore C_v \approx \frac{\left(\frac{h\nu_E}{kT}\right)^2}{\left(1 - \frac{h\nu_E}{kT} - 1\right)^2} R = R$$

(1 marks)

2. (a) The dispersion relation for electrons in a medium is given by $\omega = A sin\left(\frac{ka}{2}\right)$. Here A is a constant, a is a constant with the dimension of length. ω and k have their usual meaning. Plot the variation of the group velocity of the electron as a function of k in the range $0 \le k \le \frac{\pi}{a}$.

Solution:

The group velocity is given by

$$v_g = \frac{d\omega}{dk} = \frac{Aa}{2}\cos\left(\frac{ka}{2}\right)$$

(1 mark)

The maximum value of group velocity in the given range is $\frac{Aa}{2}$ at k=0.

The minimum value of group velocity is 0 at $k = \frac{\pi}{a}$.

(2 mark for graph. The correct values of maximum and minimum must be shown. No partial marking for the graph.)

(b) Assume that in a hydrogen atom, the life time of an excited state is $\sim 10^{-8}$ s. Taking the uncertainty product as $\hbar$, find the width of the energy levels. For which value of quantum number 'n', the separation between two consecutive levels will be of the same order as the width of the level.

(3+3 marks)

Solution:

The uncertainty principle gives:

$$\Delta E \sim \frac{\hbar}{\Delta t} = 6.56 \times 10^{-8} eV$$

(1 mark)

The energy separation between levels $(n+1)$ and n is given as

$$E_{n+1} - E_n = 13.6\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) eV$$

For large values of n, this difference can be written as

$$E_{n+1} - E_n = \frac{13.6}{n^2}\left(1 - \left(1 + \frac{1}{n}\right)^{-2}\right) eV$$

$$\approx \frac{13.6}{n^2}\left(1 - \left(1 - \frac{2}{n}\right)\right) eV$$

$$= 2 \times \frac{13.6}{n^3}$$

(1 mark)

In order this difference is equal to natural width, we must have

$$2 \times \frac{13.6}{n^3} = 6.56 \times 10^{-8}$$

This gives

$$2 \times \frac{13.6}{n^3} = 6.56 \times 10^{-16}$$

$$n^3 = \frac{2 \times 13.6}{6.56 \times 10^{-8}}$$

This gives $n \sim 746$

(1 mark. Exact value may differ slightly depending upon constants that students have taken. Only correct order is important)

Alternately some student can directly differentiate as follows to get difference between two consecutive energy states in the limit of large n.

$$E = -\frac{13.6}{n^2} eV$$

$$\therefore \Delta E = +\frac{2 \times 13.6}{n^3} \Delta n$$

Then take $\Delta n = 1$.

3. (a) Write an eigen value equation for operator $(\hat{p}_x)^2$ and find its eigen function φ (x). If the eigen function is subjected to a boundary condition, $\varphi(0) = \varphi(L) = 0$, find the eigen values.

Solution:

The eigen value equation gives:

$$-\hbar^2 \frac{d^2\phi}{dx^2} = g\phi$$
$$\Rightarrow \frac{d^2\phi}{dx^2} = -\frac{g}{\hbar^2}\phi$$

(1 mark)

The general solution of this equation is

$$\phi = A\sin\frac{\sqrt{g}}{\hbar}x + B\cos\frac{\sqrt{g}}{\hbar}x$$

(1 mark)

Applying the boundary conditions, we get

$$\phi(0) = 0 \Rightarrow B = 0$$
$$\phi(L) = 0 \Rightarrow A\sin\frac{\sqrt{g}}{\hbar}L = 0$$

(1 mark)

As A cannot be zero, we have

$$\frac{\sqrt{g}}{\hbar}L = n\pi$$
$$\Rightarrow g = \frac{n^2\pi^2\hbar^2}{L}$$

(1 mark)

(b) For a particle in a box of length L, the normalized eigen functions of the particle are given by $\varphi_n(x) = \sqrt{\frac{2}{L}}\,sin\left(\frac{n\pi x}{L}\right)$

Find the probability of finding the particle in the region $0 < x < L/4$. Determine the value of 'n' for which this probability is maximum. Also find the value of the maximum probability.

Solution:

The required probability P is given as:

$$P = \frac{2}{L}\int_0^{L/4} \sin^2\frac{n\pi x}{L}$$

(1 mark)

This gives

$$P = \frac{2}{L}\int_0^{L/4} \sin^2 \frac{n\pi x}{L}$$

$$= \frac{1}{L}\int_0^{L/4}\left(1-\cos\frac{2n\pi x}{L}\right)$$

$$= \frac{1}{L}\left(\frac{L}{4}-\frac{L}{2n\pi}\sin\frac{n\pi}{2}\right)$$

$$= \frac{1}{4}-\frac{1}{2n\pi}\sin\frac{n\pi}{2}$$

(1 mark)

Now $\sin\frac{n\pi}{2}=0$ for even values of n. For $n=(4m+1)$, $\sin\frac{n\pi}{2}=1$ and for $n=(4m-1)$, $\sin\frac{n\pi}{2}=-1$

Hence largest probability is possible only for $n=(4m-1)$. Because the second term is inversely proportional to n, the largest probability would be for n=3.

(1 mark for the explanation, to be given only if proper explanation is given)

The largest value of probability is given by

$$P = \frac{1}{4}+\frac{1}{6\pi}$$

(1 mark)

(c) Let φ_1 and φ_2 be two arbitrary normalized eigen functions, implying that $\int|\varphi_1|^2 dx = \int|\varphi_2|^2 dx = 1$. These functions obey the condition that $\int \varphi_1^* \varphi_2 dx = \alpha$ (where α is a real, non-zero constant). Find a **normalized** linear combination of φ_1 and φ_2, which is orthogonal to φ_1.

(4+4+4 marks)

Solution:

Let the linear combination be given as follows:

$\phi = c_1\phi_1 + c_2\phi_2$

(0.5 mark)

For orthogonality with ϕ_1, we must have

$$\int_{-\infty}^{+\infty} \phi_1 * (c_1\phi_1 + c_2\phi_2)\, dx = 0$$

(1 mark)

Using the fact that that individual functions are normalized and the condition given in question we get

$$c_1 + c_2\alpha = 0$$
$$\Rightarrow \alpha = -\frac{c_1}{c_2}$$

(0.5 mark)

As this linear combination is to be normalized, we shall get

$$\int_{-\infty}^{+\infty} (c_1\phi_1 + c_2\phi_2)^* (c_1\phi_1 + c_2\phi_2)\, dx = 1$$
$$\Rightarrow \int_{-\infty}^{+\infty} |c_1\phi_1|^2\, dx + \int_{-\infty}^{+\infty} |c_2\phi_2|^2\, dx + 2c_1c_2\alpha = 1$$
$$\Rightarrow c_1^2 + c_2^2 + 2c_1c_2\alpha = 1$$

(1 mark)

Writing $c_1 = -\alpha C_2$ and substituting in the above equation, we shall get

$$c_2^2 = \frac{1}{1-\alpha^2}$$

Also we shall get

$$c_1^2 = \frac{\alpha^2}{1-\alpha^2}$$

But the sign of c_1 and c_2 should be opposite. Hence we get the final answer as

$$\phi = \pm\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\phi_1 - \frac{1}{\sqrt{1-\alpha^2}}\phi_2\right)$$

(1 mark)

4. A photon is incident on a hydrogen atom at rest. The hydrogen atom is initially in its ground state. The photon ionizes the atom and as a result an electron is released with certain kinetic energy. Neglect the recoil energy of the hydrogen atom nucleus. Immediately after the release, the electron encounters a positron (a particle with same mass as that of electron but with positive charge), which is at rest. The electron and positron form a positronium atom **in the first excited state**. (A positronium atom is one in which electron and positron revolve around their common centre of mass). As a result of formation of this atom, a photon is released, which moves in the same direction as that of the incident electron. It is found that this photon has a wavelength which is four times the incident (original) photon. (Apply Bohr's model to this problem and assume that all energy and speeds are non-relativistic).

Find

(a) the energy of the original photon and

(b) the speed with which the positronium atom would be moving.

(6 marks)

Solution:

Let E_p be the energy of the original photon and Ee be the energy of the released electron after the ionization.

Conservation of energy during the absorption of photon (neglecting recoil as mentioned in question paper) gives:

E_p - 13.6 eV= E_e ...(1)

(1 mark)

The first excited state energy of the positronium atom (Taking reduced mass into consideration) $= -\frac{1}{2}\frac{13.6}{n^2} = -1.7eV$

(1 mark)

If E_{pos} is the kinetic energy of the centre of mass of positronium atom and $h\nu$ is the energy of released photon, the energy conservation gives:

$E_e + 1.7 = E_{pos} + h\nu$ (in eV) ...(2)

(1 mark)

The momentum conservation gives the following (m is the mass of the electron):

$$\sqrt{2mE_e} = \frac{h\upsilon}{c} + \sqrt{4mE_{pos}}$$

$$\therefore \sqrt{4mE_{pos}} = \sqrt{2mE_e} - \frac{h\upsilon}{c}$$

$$\Rightarrow 4mE_{pos} = 2mE_e + \left(\frac{h\upsilon}{c}\right)^2 - 2 \times \frac{h\upsilon}{c} \times \sqrt{2mE_e}$$

$$\Rightarrow E_{pos} = \frac{E_e}{2} + \left(\frac{h\upsilon}{4mc^2} - \frac{\sqrt{2mE_e}}{2mc}\right)h\upsilon$$

(1 mark)

The first term in the bracket on right hand side is negligible because $h\nu \lll 4mc^2$. The second term in the bracket is also negligible because $\sqrt{2mE_e}$ is the initial momentum of electron and because the problem is non-relativistic it has to be much smaller than $2mc$. This is because momentum is mass times velocity and velocity is much smaller than c. We thus get.

$$E_{pos} \approx \frac{E_e}{2}$$

(1 mark)

If someone has obtained this answer in any other correct way or even by arguing that for same energy photon momentum is negligible in comparison to electron momentum in non-relativistic limit, this mark should be given.

Substituting in (2) we get

$$E_e + 1.7 = h\nu + \frac{E_e}{2}$$
$$\Rightarrow \frac{E_e}{2} = h\nu - 1.7$$
Substituting from (1), and realizing that $E_p = 4h\nu$ we get
$$\frac{1}{2}\left(E_p - 13.6\right) = \frac{E_p}{4} - 1.7$$
$$\Rightarrow \frac{E_p}{4} = 6.8 - 1.7$$
$$\Rightarrow E_p = 20.4\ eV$$
If v_{pos} is the speed of positronium atom then

$$\frac{1}{2}(2m)v_{pos}{}^2 = \frac{\left(20.4 - 13.6\right)}{2} = 3.4\ eV$$
$$\Rightarrow v_{pos} = \sqrt{\frac{3.4 \times 1.6 \times 10^{-19}}{9.1 \times 10^{-31}}} \approx 7.7 \times 10^5 m/s$$

(1 mark)

If the students have substituted the numbers and have not calculated the final velocity or have calculated it wrong, the last mark can be given.

If in this problem some student has missed 1.4 eV in Equation (2) or has given a different sign, at least 3 marks corresponding to step 3,4 and 5 should be deducted.

Useful data

$m_e = 9.1 \times 10^{-31} kg \approx 0.5 MeV/c^2$; $1\,eV = 1.6 \times 10^{-19} J$; $\hbar = 1.05 \times 10^{-34} Js = 6.56 \times 10^{-16} eVs$

$h = 6.63 \times 10^{-34} Js = 4.13 \times 10^{-15} eVs$; $k = 1.38 \times 10^{-23}\ J/K = 8.62 \times 10^{-5}\ eV/K$

$m_p = 1.67 \times 10^{-27} kg \approx 940 MeV/c^2$; $c = 3 \times 10^8\ m/s$; $e = 1.6 \times 10^{-19} C$

$\lambda_m T = 2.9x10^{-3} Km$; Ground State Energy of hydrogen atom = -13.6 eV

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