MA 105 D1 Lecture 22

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Autumn 2014, IIT Bombay, Mumbai

Independence of parametrisation

Let S be an oriented surface. Let $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ be two $\mathcal{C}^1$ non-singular parametrisations of S and let $\mathbf{F}$ be a continuous vector field on S.

- If $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are orientation preserving, then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

- If $\mathbf{\Phi}_1$ is orientation preserving and $\mathbf{\Phi}_2$ is orientation reversing, then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = - \iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

For an oriented surface, the notation

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{n} dS,$$

is unambiguous.

Fluid flow

If S is an oriented surface and $\mathbf{F}$ is the velocity field of a fluid moving in three dimensions, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

is the net rate (units of volume/units of time) at which the fluid is crossing the surface in the outward direction (if the value of the integral is negative, this means that the net flow is inward).

Because of this interpretation of the surface integral, it is sometimes called the flux of the vector field.

Example: Find the flux of the vector field $\mathbf{j}$ across the hemisphere H defined by $x^2 + y^2 + z^2 = 1$, $x \geq 0$, oriented in the direction of increasing x.

An example: flow through a hemisphere

Without actually making the computation, we can easily see that the total net flow will be zero. This is because the amount of fluid entering the left half of the hemisphere is equal to the amount exiting the right half of the hemisphere, by symmetry.

We have already calculated the normal to the sphere for this parametrisation by spherical coordinates. It is

$$-(\sin\phi)\,\mathbf{r} = -\sin\phi(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi).$$

Hence,

$$-(\sin\phi)\,\mathbf{r}\cdot\mathbf{j} = -\sin^2\phi\sin\theta.$$

Note, that since we are dealing only with a hemisphere, we have $-\pi/2 \leq \theta \leq \pi/2$. We know that $\mathbf{\Phi}$ is orientation reversing. Hence,

$$\iint_H \mathbf{j}\cdot d\mathbf{S} = \int_0^{\pi}\int_{-\pi/2}^{\pi/2} \sin^2\phi\sin\theta d\theta d\phi = 0.$$

Gauss's Law

The flux of an electric field $\mathbf{E}$ over a "closed" surface is equal to the net charge Q enclosed by the surface. In terms of surface integrals we get

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = Q.$$

As a special case, let us consider an electric field of the form $\mathbf{E} = E\hat{\mathbf{n}}$, where E is a constant scalar. Then Gauss's law takes the form

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \iint_S E dS = Q.$$

It follows that

$$E = \frac{Q}{A(S)}.$$

Coulomb's law

In the particular case that E arises from a point charge Q_1, symmetry assures us that $\mathbf{E} = E\mathbf{n}$, where $\mathbf{n}$ is the unit normal to any sphere centered at Q_1. From this one easily computes the force on a second point charge Q_2 at a distance R from Q_1:

$$\mathbf{F} = \mathbf{E}Q_2 = EQ_2\mathbf{n} = \frac{Q_1 Q_2}{4\pi R^2}\mathbf{n}.$$

This is nothing but Coulomb's law for the force between two point charges.

(You may ask where the constant of proportionality $1/4\pi\epsilon_0$ has gone. I can give two answers: the first that I was working with cgs units where the proportionality is just 1. The other is that I am a mathematician not a physicist and I don't care about constants and always assume that the units are chosen so that the constant is 1.)

Fourier's Law

Let $T(x, y, z)$ denote the temperature at a point of a region V. The famous law of heat flow due to Fourier says that heat flows from regions of higher temperature to regions of lower temperature. More specifically, the heat flow vector field is proportional to the gradient field ∇T.

We write $\mathbf{F} = -k\nabla T$ for this vector field (why is there a negative sign?). Hence, if S is a surface through which heat is flowing,

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

is the total rate of heat flow or flux across S.

Example: Suppose the scalar field $T(x, y, z) = x^2 + y^2 + z^2$ represents the temperaturre function at each point, and let S be the unit sphere $x^2 + y^2 + z^2 = 1$ oriented with outward normal vector. Find the heat flux across the surface if $k = 1$.

Solution: The heat flow field is given by

$$\mathbf{F} = -\nabla T(x, y, z) = -2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}.$$

The outward unit normal vector on S is simply given by $\hat{\mathbf{n}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. We have

$$\mathbf{F} \cdot \hat{\mathbf{n}} = -2x^2 - 2y^2 - 2z^2 = -2$$

as the normal component of $\mathbf{F}$. Now the surface integral is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -2 \iint_S dS = -8\pi.$$

In what direction is the heat flux flowing?

Homeomorphisms

Recall that a homeomorphism $f : U_1 \to U_2$ from one subset of $\mathbb{R}^n$ to another is a continuous, bijective map such that f^{-1} is also continuous.

The point is that many properties are preserved under homeomorphisms (which may not be preserved under a map which is only bijective and continuous). For instance we have the Invariance of Domain theorem.

Theorem: The space $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic unless $m = n$.

The preceding theorem says that our intuitive notion of dimension can be given mathematical meaning. Incidentally, the previous theorem is closely connected to the Jordan curve theorem (and its generalisation to $\mathbb{R}^n$, the Jordan-Brouwer separation theorem).

Surfaces with boundary

The sphere S^2 clearly does not have a boundary. Neither does the torus. However, the upper hemisphere $z = \sqrt{1 - x^2 + y^2}$ obviously has the unit circle $x^2 + y^2 = 1$ as its boundary. How does one define this more precisely?

Notice that if one is at any point in the sphere, there is a small set around that point that is homeomorphic to an open disc in $\mathbb{R}^2$ (this is just a fancy way of saying that the sphere has no edge). On the other hand, on the hemisphere, if we are at one of the points $(x, \sqrt{1 - x^2}, 0)$, then this is not true any more. This allows us to make precise what we mean by the intuitive notion of boundary.

Definition: Points around which there are no sets in the surface homeomorphic to open discs are called boundary points. The set of all boundary points is called the boundary.

Closed surfaces

A closed surface in $\mathbb{R}^3$ is a surface which is bounded, whose complement is open and which has no boundary points.

For those of you who remember the word, we can define a closed surface as a compact surface that has no boundary.

Examples of closed surfaces are the sphere, the ellipsoid and the torus.

Examples of surfaces that are not closed surfaces include the surface of the paraboloid of revolution, or the one-sheeted hyperboloid (not bounded), the open unit disc in $\mathbb{R}^2$ (the complement is not open!) and the upper hemisphere (has boundary points).

The same definitions go through for higher dimensions. In that case we talk about or, hypersurfaces or, more generally, sub-manifolds without boundary.

Orienting the boundary

Let us assume that we are given an oriented surface S with a boundary that is a simple closed non-singular parametrised curve (or, more generally, a disjoint union of simple closed curves each of who is a piecewise non-singular parametrised curve). Suppose that we have chosen an orientation on S. How is the boundary oriented?

So that the surface lies to the left of an observer walking along the boundary with his head in the direction of the unit normal vector given by the choice of orientation.

Thus, the boundary of an oriented surface automatically acquires an orientation.

Stokes' Theorem

Theorem 42: Let S be a bounded oriented surface (more precisely, an oriented non-singular $\mathcal{C}^1$ surface) and let $\mathbf{F} : D \to \mathbb{R}^3$ be a $\mathcal{C}^1$ vector field, for some region D containing S. Assume further that the boundary ∂S of S is the disjoint union of simple closed curves each of which is a piecewise non-singular parametrised curve. Then

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

The following special case is often of great interest: If S is a closed surface we see that the right hand side is zero, since there is no boundary.

Examples

Example 4 (page 433, Marsden, Tromba and Weinstein): Find the integral of $\mathbf{F}(x, y, z) = z\mathbf{i} - x\mathbf{j} - y\mathbf{k}$ around the triangle with vertices $(0, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$.

Solution: We will use Stokes' theorem for the given triangle T and we will denote by S the surface it bounds. Stokes' theorem says

$$\int_T \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS,$$

where $\mathbf{n}$ is the unit normal vector in the direction of positive orientation.

The given triangle lies in the yz-plane. If the surface is to lie to the left of an observer walking around the triangle in the order described, the surface must be oriented so that the unit normal points in the direction of the positive x-axis. So $\mathbf{n} = \mathbf{i}$.

Examples, continued

Calculating the curl of $\mathbf{F}$, we get

$$\nabla \times \mathbf{F} = -\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Hence,

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = -1,$$

and

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS = -1 \times A(S) = -1 \cdot \frac{1}{2} \cdot 2 \cdot 2 = -2.$$

Examples, continued

Example 5 (page 434, Marsden, Tromba and Weinstein): Let C be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. Let C be oriented so that when it is projected onto the xy-plane the resulting curve is traversed counterclockwise. Evaluate

$$\int_C -y^3 dx + x^3 dy - z^3 dz.$$

Solution: The idea is to use Stokes' theorem. For this we must find a suitable surface bounded by C. Let S be the surface defined by the inequalities

$$z = 1 - x - y \quad \text{and} \quad x^2 + y^2 \leq 1.$$

Clearly the curve C bounds S. The unit normal to S is given by $\pm\frac{1}{\sqrt{3}}(1, 1, 1)$. We must choose $\frac{1}{\sqrt{3}}(1, 1, 1)$ to orient our surface positively so that we traverse C in the counterclockwise direction.

We would like to identify the given line integral as the integral of a vector field along C. For this we set $\mathbf{F} = -y^3\mathbf{i} + x^3\mathbf{j} - z^3\mathbf{k}$.

By Stokes' theorem, the given line integral is equal to the surface integral

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_S (3x^2 + 3y^2)\mathbf{k} \cdot \mathbf{n}\sqrt{3}dxdy,$$

where the equality uses the fact that

$$d\mathbf{S} = \mathbf{n}dS = \mathbf{n}\sqrt{1 + f_x^2 + f_y^2}dxdy$$

for a surface $z = f(x, y)$.
Using polar coordinates we see that the desired integral

$$= 3\int_0^1 \int_0^{2\pi} r^2 r d\theta dr = \frac{3\pi}{2}.$$

Example 6 (page 436, Marsden, Tromba, Weinstein): Evaluate the surface integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS,$$

where S is the portion of the surface of a sphere defined by $x^2 + y^2 + z^2 = 1$ and $x + y + z \geq 1$, $\mathbf{F} = \mathbf{r} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The outward normal is used to orient S.

Solution: How does one proceed? One can do this directly as a surface integral or use Stokes' theorem but in either case the evaluation is quite tedious.

Idea: Change the surface, keeping the boundary (and its orientation) unchanged!

After all, Stokes' theorem doesn't care what surface is being bounded by the curve. The surface integral (no matter what the surface) is equal to the line integral on the boundary.

With this idea in mind, we let C be the curve where the sphere and the plane $x+y+z=1$ intersect, and we let D_1 be the region of this plane enclosed by C (D_1 is just a disc).

We have to make sure that we orient D_1 so that C has the same orientation as in the given problem. The normals to S_1 are given by

$$\mathbf{n}_{S_1} = \pm\frac{1}{\sqrt{3}} \cdot (1,1,1)$$

Which normal should one take for orienting D_1?
Clearly $\frac{1}{\sqrt{3}} \cdot (1,1,1)$.

Now $\nabla \times \mathbf{F} = -2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$. Hence

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n}_{D_1} = -2\sqrt{3}.$$

Hence

$$\iint_{D_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{D_1} -2\sqrt{3}dS = -2\sqrt{3}A(D_1),$$

where $A(D_1)$ is the area of D_1.

We can compute the area of $A(D_1)$ as follows.

The idea is to use the underlying symmetry. The problem as presented is completely symmetric in the variables x, y and z. It is clear that ∂S is a circle and the centre of the circle must thus have all three coordinates equal.

The sum of the coordinates is 1 since the centre lies on the given plane. Hence the it must be the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. It is easy to see that the point $(1, 0, 0)$ lies on the circle ∂S. It follows that the square of the radius is

$$\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{2}{3}.$$

Hence the required surface integral

$$-2\sqrt{3}A(D_1) = -\frac{4}{\sqrt{3}}\pi.$$

Circulation and curl

We said earlier that the curl of the velocity field of a fluid at a given point tells us whether the fluid is rotating around an axis placed at that point perpendicular to the plane of the fluid. We will see that this is an application of Stokes' Theorem.

Let C_ϵ be the circle of radius ϵ centered at a point P_0 and suppose that the enclosed disc D_ϵ has unit normal vector $\mathbf{n}$. Let $\mathbf{F}$ be a $\mathcal{C}^1$ vector field defined in a region containing P_0. Then, we will show that

$$[(\nabla \times \mathbf{F})(P_0)] \cdot \mathbf{n} = \lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} \int_{C_\epsilon} \mathbf{F} \cdot d\mathbf{s}.$$

We will need to combine Stokes' theorem with the mean value theorem for area integrals.

Using the mean value theorem

Applying Stokes' theorem to the disk D_ϵ, we get

$$\iint_{D_\epsilon} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \int_{C_\epsilon} \mathbf{F} \cdot d\mathbf{s}.$$

By the mean value theorem, there is point P_ϵ in each disc D_ϵ such that

$$[(\nabla \times \mathbf{F})(P_\epsilon)] \cdot \mathbf{n} = \frac{1}{A(D_\epsilon)} \iint_{D_\epsilon} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$$

This shows that the normal component of the curl can be interpreted as the circulation per unit area in the plane.

Recalling that $A(D_\epsilon) = \pi\epsilon^2$ and letting $\epsilon \to 0$, we obtain the desired result.