64. The dispersion relation for a lattice wave propogating in a one dimensional chain of atoms of mass m bound together by a force constant β is given by the relation $\omega = \omega_o \sin(ka/2)$, where a is the distance between atoms and $\beta = \omega_o^2 m/4$.

(a) Show that in the long wavelength limit the medium is non-dispersive.

(b) Find the group and phase velocities at $k = \frac{\pi}{a}$.

[Ans: $0, \omega_o a/\pi$]

Solution:

(a.) $\omega = \omega_0 \sin(ka/2)$, $\beta = \frac{\omega_0^2 m}{4}$

Here $k = 2\pi/\lambda$

For the case a, ie long wavelength, k will be small

$\sin(ka/2) \sim ka/2$

$\therefore \omega = \omega_0 \, ka/2$

$\therefore V_p = \omega/k = \omega_0 a/2$

$V_g = \frac{d\omega}{dk} = \frac{\omega_0 a}{2}$

This is independent of k, Hence the medium is non-dispersive.

(b) $\omega = \omega_0 \sin(ka/2)$

$V_p = \omega/k \Big|_{k=\pi/a} = \frac{\omega_0}{k} \sin(\pi/2) = \frac{\omega_0 a}{\pi}$

$V_g = \frac{d\omega}{dk} = \frac{\omega_0 a}{2} \cos(ka/2) \Big|_{k=\pi/a} = 0.$