Look-out for New Laws

Classical Equations:

1. Particle $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$

2. Wave $\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$

Superposition of Waves

- We know that electromagnetic waves even of different wavelengths superimpose.
- If we say that the wave equation is the wave governing the electromagnetic waves it implies that superposition should also be allowed by the wave equation.

- If ψ_1 and ψ_2 are solutions of the wave equation, $\psi_1+\psi_2$ is also a solution of the wave equation.
- Let us verify it as this is key for interference, formation of wave packet etc.

If ψ_1 and ψ_2 are solutions of the wave equation.

$$\frac{\partial^2\psi_1}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi_1}{\partial t^2}$$

$$\frac{\partial^2\psi_2}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi_2}{\partial t^2}$$

$$\frac{\partial^2\psi_1}{\partial x^2}+\frac{\partial^2\psi_2}{\partial x^2}=\frac{1}{c^2}\left(\frac{\partial^2\psi_1}{\partial t^2}+\frac{\partial^2\psi_2}{\partial t^2}\right)$$

$$\frac{\partial^2(\psi_1+\psi_2)}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2(\psi_1+\psi_2)}{\partial t^2}$$

We see that $\psi_1+\psi_2$ is also a solution of the wave equation. But note here c is same for both the waves

Superposition of Particle Wave?

- A particle can travel with any speed.
- In fact the wavelength of the particle depends on speed.

- If we take two particle waves, corresponding to same type of particle (e.g. electron), with different wavelengths, they would not superimpose, if the standard wave equation was governing it.
- So can we form a wave packet?
- We probably need another equation.

How to Solve the Problem?

- If the constant appearing in the wave equation was not dependent on dynamical properties like speed, momentum or energy, the superposition could have been valid.

- Initially consider a non-relativistic free particle. For this particle we have the following relationship.

$$K = \frac{p^2}{2m}$$

- Momentum can be related to wave vector k, while energy to ω.

If second derivative with x was related to the first derivative with t, things could be different. Why not try the following.

$$\frac{\partial^2 \psi}{\partial x^2} = \gamma \frac{\partial \psi}{\partial t}$$

But are the standard displacement equation given by a pure trigonometric equation a solution of it?

Traditionally we take the solution of the following form.

$$\psi = Ae^{i(kx-\omega\, t)}$$

$$\psi = Ae^{i(kx-\omega t)}$$

$$\frac{\partial \psi}{\partial t} = -Ai\omega e^{i(kx-\omega t)}$$

$$\frac{\partial \psi}{\partial x} = Aike^{i(kx-\omega t)}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -Ak^2 e^{i(kx-\omega t)}$$

$$-Ak^2e^{i(kx-\omega t)} = \gamma\left(-Ai\omega e^{i(kx-\omega t)}\right)$$

$$\gamma = \frac{k^2}{i\omega}$$

$$= \frac{2m\hbar^2k^2}{2m\hbar i\hbar\omega}$$

$$= \frac{2m}{i\hbar}$$

Note: We have taken

$$p = \hbar k; \quad K = \hbar\omega$$

One set of argument is to put

$$E = \hbar\omega = K + m_o c^2$$

Then express the constant rest mass energy factor as a phase

$$\psi = Ae^{i(kx-\omega\ t)}$$

$$= Ae^{\frac{i}{\hbar}\left(px-\left(K+m_o c^2\right)t\right)}$$

$$= Ae^{\frac{i}{\hbar}(px-Kt)}e^{-\left(\frac{i}{\hbar}m_o c^2 t\right)}$$

Going back to our old equation, we get the following

$$\frac{\partial^2\psi}{\partial x^2} = \gamma\frac{\partial\psi}{\partial t}$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{2m}{i\hbar}\frac{\partial\psi}{\partial t}$$

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t}}$$

We shall use in QM, K as the kinetic energy and E is the total non-relativistic energy, not including rest mass energy.

Notes

- The value of γ indeed turned out to independent of dynamical variables, but depend on m.
- We could have also used for a wave travelling in +x direction.

$$\psi = Ae^{-i(kx-\omega\, t)}$$

Notes *(Cond.)*

- In that case the value of γ would have changed sign. No Physics would have changed.
- Conventionally we use only the following.

$$\psi = Ae^{i(kx-\omega\, t)}$$

- The wave function turns out to be complex. But that shall not pose any problem so long the observables turn out to be real.
- We shall later see the physical interpretation of the wave function.

Operators

- Abstract but an essential component of QM.
- Are defined to maintain consistency with classical mechanics.

Momentum operator is defined the following way.

$$\hat{p}_x \equiv -i\hbar \frac{\partial}{\partial x}$$

This implies that

$$\hat{p}_x \psi(x,t) \equiv -i\hbar \frac{\partial \psi(x,t)}{\partial x}$$

Energy Operator

Energy operator is defined as

$$\hat{E} \equiv i\hbar \frac{\partial}{\partial t}$$

Like before this implies that

$$\hat{E}\psi(x,t) \equiv i\hbar \frac{\partial \psi(x,t)}{\partial t}$$

Relook at the Old Equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} = i\hbar \frac{\partial \psi(x,t)}{\partial t}$$

$$\frac{1}{2m}\left[-i\hbar \frac{\partial}{\partial x}\right]\left[-i\hbar \frac{\partial}{\partial x}\right]\psi(x,t) = \left[i\hbar \frac{\partial}{\partial t}\right]\psi(x,t)$$

$$\frac{1}{2m}\left[\hat{p}_x\right]\left[\hat{p}_x\right]\psi(x,t) = \left[\hat{E}\right]\psi(x,t)$$

$$\frac{\hat{p}_x^{\,2}}{2m}\psi(x,t) = \hat{E}\psi(x,t)$$