MA 105 D1 Lecture 16

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Defining the double integral over a region

Assume that $f(x, y)$ is defined in some region D in the plane. We will assume that D is bounded. This means that there is an $r > 0$ such that $D \subset B_r$, where B_r is the ball (or disc) of radius r around the origin. Since any ball can be enclosed by a larger rectangle R, we may assume that $D \subset R$.

Now it is quite easy to define the integral of f on the region D. Define a function

$$g(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \notin D \end{cases}$$

on R. If g is integrable on R, we will say that f is integrable on D and we define

$$\int \int_D = f(x, y) dxdy = \int_R g(x, y) dxdy.$$

Intuitively, it is clear that this is the right definition. After all, since the function g is simply defined to be zero outside the region D, the region outside D contributes no volume to the integral. Thus all the contribution to the integral over R actually comes only from the region D.

You may, however, be bothered by the following question: How can one tell if the function g is integrable or not? Again, when the region D is bounded by a $\mathcal{C}^1$ closed curve (e.g. an ellipse) and f is a continuous function this is not hard to see. In this case the function g on R has the property that g is a bounded function on R which is discontinuous only on the closed curve which is the boundary of D (a $\mathcal{C}^1$ curve can always be realised as a finite union of graphs of $\mathcal{C}^1$ functions!). Hence, by Theorem 33, g is integrable.

Evaluating double integrals over non-rectangular regions

Merely defining the integral, as above, does not solve the problem of actually evaluating it. We will be able to evaluate double integrals when the domain D falls into one of three types. These three types will exhaust all the most common situations we will find ourselves in.

Let $\gamma(t)$ be a simple closed curve in $\mathbb{R}^2$, that is, $\gamma : [0,1] \to \mathbb{R}^2$ is a continuous function such that $\gamma(0) = \gamma(1)$. Let us assume that our region D is bounded by the graph of γ. In what follows we will assume that γ is piecewise $\mathcal{C}^1$, that is, it is a $\mathcal{C}^1$ function except possibly at a finite number of points.

Basic idea: We can think of any such region as the union of two different kinds of simpler regions which we will describe below.

Elementary regions

We will call the two simple types of regions that we are going to describe elementary regions.

Let $h_1, h_2 : [a, b] \to \mathbb{R}$ be two continuous functions. Consider the set of points

$$D_1 = \{(x, y) \mid a \leq x \leq b \text{ and } h_1(x) \leq y \leq h_2(x)\}.$$

Such a region is said to be of type 1.

Similarly, if $k_1, k_2 : [c, d] \to \mathbb{R}$ are two continuous functions, the set of points

$$D_2 = \{(x, y) \mid c \leq y \leq d \text{ and } k_1(y) \leq x \leq k_2(y)\}$$

is called a region of type 2.

If D is a union of regions of types 1 and 2, we call it a region of type 3.

One example

Let us start with the simplest example. Take the closed disc D_r of radius r around the origin. What kind of region is it?

If we take $h_1(x) = \sqrt{r^2 - x^2}$ and $h_2(x) = -\sqrt{r^2 - x^2}$, we see that D_r is of type 1.

If we take $k_1(y) = \sqrt{r^2 - y^2}$ and $k_2(y) = -\sqrt{r^2 - y^2}$, we see that D_r is of type 2.

We could also view the disc as a region of type 3, by dividing it into four quadrants.

Evaluating integrals on regions of type 1

Let D be a region of type 1 and assume that $f : D \to \mathbb{R}$ is continuous. Let $D \subset R = [\alpha, \beta] \times [\gamma, \delta]$ and let g be the corresponding function on R (obtained by extending f by zero).

The region D is obviously bounded by continuous curves (the straight lines $x = a$ and $x = b$ and the graphs of the curves $y = h_1(x)$ and $y = h_2(x)$). Hence we can conclude that g is integrable on R.

$$\int\int_D f(x,y)dxdy = \int\int_R g(x,y)dxdy = \int_\alpha^\beta \left[\int_\gamma^\delta g(x,y)dy\right] dx,$$

where the second equality follows because of Theorem 34. In turn, this gives

$$\int_\alpha^\beta \left[\int_{h_1(x)}^{h_2(x)} g(x,y)dy\right] dx = \int_a^b \left[\int_{h_1(x)}^{h_2(x)} f(x,y)dy\right] dx,$$

since $g(x,y) = 0$ if $y < h_1(x)$ or $y > h_2(x)$.

Evaluating integrals on regions of type 2

Using exactly the same reasoning as in the previous case (basically, interchanging the roles of x and y) we can obtain a formula for regions of type 2. We get

$$\int\int_D f(x,y)dxdy = \int_c^d \left[\int_{k_1(y)}^{k_2(y)} f(x,y)dx\right] dy.$$

Both of these formulæ can be viewed as special cases of Cavalieri's principle when $f(x,y) \geq 0$. In the first case we are slicing by planes perpendicular to the x-axis, while in the second case, we are slicing by planes perpendicular to the y-axis.

A final remark about Theorem 34. In fact, it is true that the moment one of the iterated integrals exists, so does the other provided the function is bounded. Hence, we may use either to evaluate a double integral.

Examples

Example 1 (page 298 of Marsden, Tromba and Weinstein but slightly modified): Evaluate the integral

$$\int\int_D \sqrt{1-y^2}\,dxdy,$$

where D is the part of the unit disc that lies in the first quadrant.

Solution: Clearly the region D can be viewed as either a type 1 region or a type 2 region. Let us write down the iterated integrals in each case.

Type 1: Let $h_1(x) = 0$ and $h_2(x) = \sqrt{1-x^2}$. We may view our region as being contained in the square $[0,1] \times [0,1]$. Then, our integral becomes

$$\int_0^1 \left[\int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dy \right] dx.$$

Type 2: Let $k_1(y) = 0$ and $k_2(y) = \sqrt{1 - y^2}$ and R be $[0,1] \times [0,1]$. Then our integral becomes

$$\int_0^1 \left[\int_0^{\sqrt{1-y^2}} \sqrt{1 - y^2} dx \right] dy.$$

One of the iterated integrals above is clearly easier to evaluate than the other. Which one?

Clearly the second one. Let us evaluate it.
We get

$$\int_0^1 \left[\sqrt{1-y^2}x\right]_0^{\sqrt{1-y^2}} dy = \int_0^1 (1-y^2)dy = \left(y - \frac{y^3}{3}\right)\Big|_0^1 = \frac{2}{3}.$$

The point of the above exercise is to show you that it is often easier to evaluate one of the iterated integrals. In particular, if a given iterated integral looks difficult, you can switch the order of integration and try to do the other iterated integral.

The mean value theorem for double integrals

Let us first state the mean value theorem for functions of one variable.

Theorem 35: Suppose that f is a continuous function on $[a, b]$. There exists a point x_0 in $[a, b]$ such that

$$f(x_0) = \frac{1}{b-a}\int_a^b f(x)dx.$$

How does one interpret the above statement geometrically?

Theorem 36: If D is an elementary region in $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ is continuous. There exist (x_0, y_0) in D such that

$$f(x_0, y_0) = \frac{1}{A(D)}\int\int_D f(x, y)dA.$$

Polar Coordinates

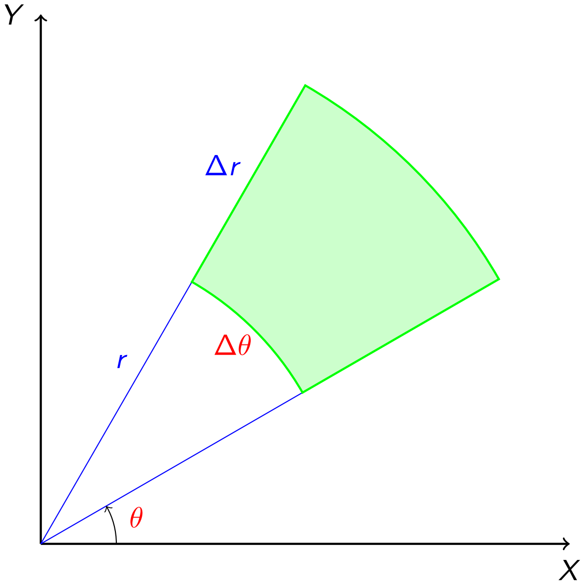
Instead of starting with a completely arbitrary coordinate system, let us see what happens when we use polar coordinates. If we have a function $f : D \to \mathbb{R}$, as a function of the usual (cartesian) $x - y$ coordinates, we can get a function g of the polar coordinates r and θ by composition:

$$g(r, \theta) = f(r \cos \theta, r \sin \theta).$$

To integrate the function g on the domain D we need to cut up D into small rectangles, but these will be rectangles in the $r - \theta$ coordinate system.

What shape does a rectangle $[r, r + \Delta r] \times [\theta, \theta + \Delta\theta]$ represent in the $x - y$ plane?

An area element in polar coordinates



The integral in polar coordinates

Clearly a part of a sector of a circle. What is the area of this part of a sector? It is

$$\frac{1}{2} \cdot [(r + \Delta r)^2 \Delta\theta - r^2 \Delta\theta] \sim r \Delta r \Delta\theta.$$

It follows that the integral we want is approximated by a sum of the form

$$\sum_{i,j} g(t_{ij}) r_i \Delta r_i \Delta\theta_j,$$

where $t = \{t_{ij}\}$ is a tag for the partition of the "rectangle" in polar coordinates. Taking limits, we see that

$$\int\int_D f(x, y) dxdy = \int\int_{D^*} f(r\cos\theta, r\sin\theta) r dr d\theta,$$

where D is the image of the region D^*. This is the change of variable formula for polar coordinates.

The integral of the Gaussian

We would like to evaluate the following integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

What does this integral mean? - so far we have only looked at Riemann integrals inside closed bounded intervals, so the end points were always finite numbers a and b.

An integral like the one above is called an improper integral. We can assign it a meaning as follows. It is defined as

$$\lim_{T\to\infty} \int_{-T}^{T} e^{-x^2}\, dx,$$

provided, of course, this limit exists. We will see how to evaluate this.

The most amazing trick ever

Consider

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy.$$

We view this product as an iterated integral!

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dxdy.$$

But this iterated integral can be viewed as a double integral on the whole plane. Using polar coordinates, we can write this as

$$\int_0^{2\pi} \left[\int_0^{\infty} e^{-r^2} rdr \right] d\theta.$$

But we can now evaluate the inner integral. Hence, we get

$$\int_0^{2\pi} \left[-\frac{1}{2} e^{-r^2} \Big|_0^{\infty} \right] d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \pi$$

The answer

Since $I^2 = \pi$, we see that $I = \sqrt{\pi}$.

Using the above result you can easily conclude that

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}.$$

The integral above arises in a number of places in mathematics - in probability, the study of the heat equation, the study of the Gamma function (next semester) and in many other contexts.

There are many other ways of evaluating the integral I, but the method above is easily the cleverest.

The definition of a mathematician

The preceding trick for evaluating the integral I, was discovered by Joseph Liouville (1809 -1892), a great mathematician in his own right.

Liouville inspired William Thomson, Lord Kelvin, the famous Scotch physicist, to one of the most satisfying definitions of a mathematician that has ever been given, "Do you know what a mathematician is?" Kelvin once asked a class. He stepped to the board and wrote:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Putting his finger on what he had written, he turned to the class and said:

"A mathematician is one to whom that is as obvious as that twice two makes four is to you. Liouville was a mathematician."

Joseph Liouville (1809 - 1892)