Roll Number:

PH107(October 10, 2014) QUIZ 2

a. Please write your **Name, Roll Number, Division and Tutorial Batch on answer sheets and Roll Number on the Question Paper.**
b. All steps must be shown. All explanations must be clearly given for getting credit. Just the correct final answer does not guarantee the full credit.
c. Possession of mobile phone and exchange of calculators during the examination are strictly prohibited.

Weightage: 15% **Time: 50 minutes**

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1. A particle of mass *m* is confined to a one dimensional, infinite square well potential well of width *a*. Its wave function at a certain instant is given by

$$\psi(x) = \frac{i}{2}\sqrt{\frac{2}{a}}\, sin\left(\frac{\pi x}{a}\right) + A\, sin\left(\frac{3\pi x}{a}\right) - \frac{1}{2}\sqrt{\frac{2}{a}}\, sin\left(\frac{4\pi x}{a}\right)$$

(a) Find the value of *A*.
(b) If the energy is measured, what is the most probable result?
(c) What is the expectation value of energy in such a measurement?
(d) If the measurement gives the energy corresponding to the third term (of the given wave function), what is the time dependent wave function after that measurement?

[4 marks]

Solution:

The wave function can be written as $\psi = \frac{i}{2}\psi 1 + A'\psi 3 - \frac{1}{2}\psi 4$

Normalization demands $\left(\frac{-i}{2}\right)\left(\frac{i}{2}\right) + A'^2 + \left(-\frac{1}{2}\right)^2 = 1 \Rightarrow \frac{1}{4} + A'^2 + \frac{1}{4} = 1$

$(a)\ \therefore A' = \sqrt{\frac{1}{2}} \quad and\ since\ A = A'\sqrt{\frac{2}{a}} = \sqrt{\frac{1}{a}}$ ....(1 mark)

probability of getting E_1 as energy is $\left(\frac{-i}{2}\right)\left(\frac{i}{2}\right) = \frac{1}{4}$

probability of getting E_3 as energy is $A'^2 = \frac{1}{2}$

probability of getting E_4 as energy is $\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

(b) Therefore, the most probable energy is $E3 = \frac{9\pi^2\hbar^2}{2ma^2}$ ....(1 mark)

(c) Average energy=$\frac{1}{4}E1 + \frac{1}{2}E3 + \frac{1}{4}E4 = \frac{35\pi^2\hbar^2}{8ma^2}$ ....(1 mark)

(d) Wave function collapses to $\psi 4 = \sqrt{\frac{2}{a}}\, sin\frac{4\pi x}{a}\, e^{-i\frac{E4t}{\hbar}}$ ....(1 mark)

2. A beam of particles of mass *m* is incident on a step potential defined as $V=0$ for $x<0$ and $V=V_o$ for $x>0$. The wave function corresponding to the incident beam is given by $2Ae^{-ik_1x}$, while that for transmitted beam is given by Ae^{-ik_2x}. Find the energy of the particles in terms of V_o. Also find the reflection and transmission coefficients and the un-normalized wave functions in both the regions. (V_o, k_1 and k_2 are positive).

[5 marks]

Solution:

With the given wave function for the incident beam, the particles are incident on the barrier from right hand side. Hence the wave function for $x< 0$ is given as

$$\phi_2 = Ae^{-ik_2x}; \ \ \hbar k_2 = \sqrt{2mE}$$

The same for $x> 0$ is given by

$$\phi_1 = 2Ae^{-ik_1x} + Be^{+ik_1x}; \ \ \hbar k_1 = \sqrt{2m(E-V_o)}$$

1 mark for writing wave functions

Matching wave functions as $x = 0$

$$A = 2A + B \Rightarrow B = -A$$

Matching the derivatives of the wave functions at $x = 0$

$$-ik_2A = ik_1(-2A+B) \Rightarrow k_2 = 3k_1$$

1 mark for applying boundary conditions

This gives

$$\sqrt{2mE} = 3\times\sqrt{2m(E-V_o)}$$
$$\Rightarrow E = 9(E-V_o)$$
$$\Rightarrow E = \frac{9}{8}V_o$$

1 mark for getting energy

$$R = \left|\frac{B}{2A}\right|^2 = 0.25$$
$$T = 1 - R = 0.75$$

Or

$$T = \left|\frac{A}{2A}\right|^2 \frac{k_2}{k_1} = 0.75$$

1 mark for the coefficients

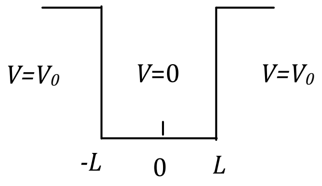
Unnormalized wave functions are given as:

$$\phi_2 = Ae^{-ik_2x};\ \ \hbar k_2 = \sqrt{2mE}$$

$$\phi_1 = 2Ae^{-ik_1x} - Ae^{+ik_1x};\ \ \hbar k_1 = \sqrt{2m(E - V_o)}$$

1 mark for complete wave functions

3. A particle of mass m is bound to a one dimensional square well as shown below.



The wave function of the particle in the region $-L<x<L$ is such that its quarter wave length fits inside the box.

(a) Find the energy of the particle in terms of m and L. Is this the lowest allowed energy value? Give reasons to justify.

(b) Find V_0 in terms of m and L for such a bound state to exist.

(c) Find the un-normalized wave function of the particle in the three regions.

[6 marks]

Solution:

Given $\lambda = 8L \Rightarrow k = \dfrac{2\pi}{\lambda} = \dfrac{\pi}{4L}$

This gives energy of the particle as $E = \dfrac{\hbar^2 k^2}{2m} = \dfrac{\pi^2\hbar^2}{32mL^2}$

1 mark for energy

As the potential is symmetrical about x=0, and it has to match the boundary condition both at x=-L and at x=+L, the only possible solution between $-L \le x \le +L$ is

$\phi_2 = C\cos kx$ with $k = \dfrac{\pi}{4L}$.

Similarly for $x < -L$

$\phi_1 = Ae^{+\alpha x}$ with $\hbar\alpha = \sqrt{2m(V_o - E)}$

And for $x > +L$

$\phi_3 = Ae^{-\alpha x}$

1 mark for writing wave functions

As these wave functions, when plotted would never cross x-axis they are ground state wave function and, therefore, represents the lowest energy of the particle.

1 mark for realizing that this is ground state wave function

Matching wave functions at x=-L

$$Ae^{-\alpha L} = C\cos kL = \frac{C}{\sqrt{2}}$$

Matching derivatives of wave functions at *x*=-*L*

$$A\alpha e^{-\alpha L} = Ck\sin kL = \frac{C}{\sqrt{2}}\frac{\pi}{4L}$$

1 mark for matching boundary conditions

Substituting from earlier boundary condition, we get

$$\alpha\frac{C}{\sqrt{2}} = \frac{C}{\sqrt{2}}\frac{\pi}{4L}$$

$$\Rightarrow \alpha = \frac{\pi}{4L}$$

This gives

$$\alpha = \frac{\sqrt{2m\left(V_o - E\right)}}{\hbar} = \frac{\pi}{4L}$$

$$\Rightarrow V_o - E = \frac{\pi^2\hbar^2}{32mL^2}$$

$$\therefore V_o = \frac{\pi^2\hbar^2}{16mL^2}$$

1 mark for getting potential

We now get

$$Ae^{-\pi/4} = \frac{C}{\sqrt{2}}$$

$$\Rightarrow C = \sqrt{2}Ae^{-\pi/4}$$

Therefore, the wave functions are

$$\phi_1 = Ae^{+\frac{\pi x}{4L}}$$

$$\phi_2 = \sqrt{2}Ae^{-\pi/4}\cos\frac{\pi x}{4L}$$

$$\phi_3 = Ae^{-\frac{\pi x}{4L}}$$

1 mark for complete wave functions

Note: The same result would have been obtained if we matched the boundary conditions at x=+L