MA 105 D1 Lecture 23

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Maxwell's equation

Let $\mathbf{E}$ and $\mathbf{H}$ be time-dependent electric and magnetic fields, respectively. One of Maxwell's equations is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}.$$

Let S be a surface with boundary C. Define

$$\int_C \mathbf{E} \cdot d\mathbf{s} = \text{voltage drop around C}$$

and

$$\iint_S \mathbf{H} \cdot d\mathbf{S} = \text{magnetic flux across S.}$$

We will show that Faraday's Law can be derived from this equation of Maxwell.

Faraday's Law

Faraday's Law: The voltage (drop) around C equals the negative rate of change of magnetic flux through S.

Using Stokes' theorem

$$\int_C \mathbf{E} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S}.$$

Now we use Maxwell's equation to obtain

$$= \iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \iint_S -\frac{\partial \mathbf{H}}{\partial t} \cdot d\mathbf{S}$$

The key observation is that we can move the $\frac{\partial}{\partial t}$ across the integral sign. We can do this because the parameter t is independent of the variables dS occurring in the surface integral. This is a very useful trick called "differentiating under the integral sign".

We will not justify this step of differentiating under the integral sign (although we have all the tools necessary to do so). What we get is

$$\iint_S -\frac{\partial \mathbf{H}}{\partial t} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \iint_S \mathbf{H} \cdot d\mathbf{S}.$$

And this is nothing but Faraday's Law.

Exercise 2: Show that differentiating under the integral sign above is justified.

Stokes' theorem for a graph

We give the proof of this theorem in the special case when the surface S is the graph of a surface $z = f(x, y)$ for a $\mathcal{C}^1$ function f. We think of the graph as a parametrised surface $(x, y, z(x, y))$, where (x, y) lies in the domain D in $\mathbb{R}^2$. Assume further that Green's theorem applies to the domain D.

How does one orient ∂S?

Since D is a planar region, it has a natural positive orientation given by the positive direction of the z axis. Now we orient the boundary ∂D as in Green's theorem, that is, in the counterclockwise direction.

Exercise 2: Once ∂D has been oriented as above, ∂S is automatically oriented. The surface S is oriented by choosing $\mathbf{\Phi}_u \times \mathbf{\Phi}_v$ as the normal with positive orientation. Verify in an example of your choosing that the region S remains to the left of an observer walking on ∂S in the direction of positive orientation.

The left hand side of Stokes' formula

Let $\mathbf{F} = (F_1, F_2, F_3)$. Then

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{j}$$

For the graph of a surface, the normal vector has the form

$$\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right).$$

It follows that

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D &\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\left(-\frac{\partial z}{\partial x}\right) \\ &+ \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\left(-\frac{\partial z}{\partial y}\right) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy\end{aligned}$$

This gives us the left hand side of Stokes' formula.

The right hand side of Stokes' formula

Let $(x(t), y(t))$, $a \leq t \leq b$, be a parametrisation of the boundary of D, oriented using the upward normal. The curve

$$\mathbf{c}(t) = (x(t), y(t), f(x(t), y(t)))$$

gives the boundary ∂S of S and is given the same orientation as ∂D.

We have

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt.$$

Now we use the chain rule to conclude that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Substituting this in the previous expression gives us

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial D} \left(F_1 + F_3 \frac{\partial z}{\partial x} \right) dx + \left(F_2 + F_3 \frac{\partial z}{\partial y} \right) dy.$$

The conclusion of the proof

Applying Green's theorem (and using the chain rule) we get

$$\iint_D \left(\frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F_3}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial F_3}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + F_3 \frac{\partial^2 z}{\partial x \partial y} \right)$$
$$- \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F_3}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial F_3}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + F_3 \frac{\partial^2 z}{\partial x \partial y} \right) dxdy.$$

Now one can compare this expression with the previous one to see that Stokes' theorem is proved in this case.

The basic assumptions

We give a proof of Stokes's theorem for parametrised surfaces (which include spheres, tori and the outer surface of a cuboid/parallelepiped).
The general statement of Stokes' theorem will then follow by decomposing the given surface into a union of parametrised surfaces which intersect only along curves (more specifically, any two parametrised surfaces should intersect in a finite union of graphs of curves).

We will assume that S is a bounded geometric surface in $\mathbb{R}^3$ given by a non-singular $\mathcal{C}^1$ parametrisation $\mathbf{\Phi} : D \to \mathbb{R}^3$, where D is a bounded subset of $\mathbb{R}^2$.
We will further assume that the map $\mathbf{\Phi} : D \to S$ is a bijective map so $\mathbf{\Phi}(D) = S$ and that ∂D consists of a finite disjoint union of simple closed curves each of which is a piecewise $\mathcal{C}^1$ non-singular parametrised curve.
(Note that because of the inverse function theorem $\mathbf{\Phi} : D \to S$ is actually a diffeomorphism, though we will not be using this in the proof).

Orientation of the boundary of the surface

Exercise 1: Check that $\mathbf{\Phi}$ takes the boundary ∂D of D to the boundary ∂S of S.

How does one orient ∂S?

Since D is a planar region, it has a natural positive orientation given by the positive direction of the z axis. Now we orient the boundary ∂D as in Green's theorem, that is, in the counterclockwise direction.

Exercise 2: Once ∂D has been oriented as above, ∂S is automatically oriented. The surface S is oriented by choosing $\mathbf{\Phi}_u \times \mathbf{\Phi}_v$ as the normal with positive orientation. Verify in an example of your choosing that the region S remains to the left of an observer walking on ∂S in the direction of positive orientation.

Reduction of the surface integral to a double integral

Now that we have specified clearly how various regions have been oriented, we can start the proof. We will assume that $\mathbf{F} : W \to \mathbb{R}^3$ is a $\mathcal{C}^1$-vector field on a subset W of $\mathbb{R}^3$ which contains S.

Let us first compute the surface integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

We will write $\mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$, where F_1 and F_2 and F_3 are the three scalar components of $\mathbf{F}$. We will also write

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where x, y and z are the three scalar components of $\mathbf{\Phi}$. With this notation, we have previously seen that

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_D [(\nabla \times \mathbf{F})(\mathbf{\Phi}(u, v))] \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv.$$

Calculating the curl and the normal vector

Once can easily show that

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

One can also easily calculate the normal vector to the surface:

$$\boldsymbol{\Phi}_u \times \boldsymbol{\Phi}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}.$$

$$= \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)\mathbf{i} + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)\mathbf{j} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\mathbf{k}$$

Calculating the dot product of the curl and the normal

We now calculate the dot product

$$(\nabla \times \mathbf{F}) \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v).$$

It is

$$\frac{\partial F_3}{\partial y}\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial F_3}{\partial y}\frac{\partial z}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial F_2}{\partial z}\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} + \frac{\partial F_2}{\partial z}\frac{\partial z}{\partial u}\frac{\partial y}{\partial v}$$
$$\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial F_1}{\partial z}\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial F_3}{\partial x}\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial F_3}{\partial x}\frac{\partial x}{\partial u}\frac{\partial z}{\partial v}$$
$$\frac{\partial F_2}{\partial x}\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial F_2}{\partial x}\frac{\partial y}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial F_1}{\partial y}\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

We thus have an explicit expression for the double integral

$$\iint_D [(\nabla \times \mathbf{F})(\mathbf{\Phi}(u,v))] \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv.$$

Parametrising the boundary of S

Let us assume that ∂D is given by a pieciewise $\mathcal{C}^1$ curve. To prove the more general theorem we have stated when the boundary is in several pieces, we simply parametrise each piece and take the resulting sum of the line integrals.

Let $(u(t), v(t))$, $a \leq t \leq b$, be a parametrisation of ∂D. Then

$$\mathbf{c}(t) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t))),$$

$a \leq t \leq b$ is a parametrisation of ∂S. Hence, the tangent vector is given by

$$\mathbf{c}'(t) = \left(\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}, \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt}, \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} \right).$$

Our aim is to evaluate the line integral that appears in Stokes's theorem

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Using the parametrisation we have given we can write this integral out as

Reduction to a line integral in the plane

$$\int_a^b F_1(x(u(t),v(t)),y(u(t),v(t)),z(u(t),v(t)))\left(\frac{\partial x}{\partial u}\frac{du}{dt}+\frac{\partial x}{\partial v}\frac{dv}{dt}\right)dt$$
$$+F_2(x(u(t),v(t)),y(u(t),v(t)),z(u(t),v(t)))\left(\frac{\partial y}{\partial u}\frac{du}{dt}+\frac{\partial y}{\partial v}\frac{dv}{dt}\right)dt$$
$$+F_3(x(u(t),v(t)),y(u(t),v(t)),z(u(t),v(t)))\left(\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}\right)dt.$$

After changing variables this line integral has the less cumbersome form:

$$\int_{\partial D} F_1(x(u,v),y(u,v),z(u,v))\left[\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right]$$
$$+F_2(x(u,v),y(u,v),z(u,v))\left[\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right]$$
$$+F_3(x(u,v),y(u,v),z(u,v))\left[\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right].$$

Using Green's theorem

The previous integral can be rewritten as

$$\int_{\partial D} \left(F_1 \frac{\partial x}{\partial u} + F_2 \frac{\partial y}{\partial u} + F_3 \frac{\partial z}{\partial u} \right) du + \left(F_1 \frac{\partial x}{\partial v} + F_2 \frac{\partial y}{\partial v} + F_3 \frac{\partial z}{\partial v} \right) dv.$$

This integral is of the form

$$\int_{\partial D} M(u,v)du + N(u,v)dv,$$

so we may apply Green's Theorem to it. Thus, we obtain

$$\int_{\partial D} M(u,v)du + N(u,v)dv = \iint_D (N_u(u,v) - M_v(u,v))dudv.$$

Recall that we have already expressed the surface integral in Stokes' theorem as a double integral over D. So all that is required now is to show that the integrand in the integral above is the same as the one we obtained before.

Evaluating the integrand

We proceed to calculate N_v and M_u.

$$M_v = \frac{\partial}{\partial v}\left(F_1(x,y,z)\frac{\partial x}{\partial u} + F_2(x,y,z)\frac{\partial y}{\partial u} + F_3(x,y,z)\frac{\partial z}{\partial u}\right)$$

$$= \left(\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial u} + F_1(x,y,z)\frac{\partial^2 x}{\partial v \partial u} + \dots$$

Similarly

$$N_u = \frac{\partial}{\partial u}\left(F_1(x,y,z)\frac{\partial x}{\partial v} + F_2(x,y,z)\frac{\partial y}{\partial v} + F_3(x,y,z)\frac{\partial z}{\partial v}\right)$$

$$= \left(\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial v} + F_1(x,y,z)\frac{\partial^2 x}{\partial u \partial v} + \dots$$

The end of the proof

Recall that we need to calculate $N_u - M_v$. By looking at the two expressions in the previous slide, we see that first and fourth terms cancel, leaving four terms behind. There will similarly be four surviving terms coming from the terms involving F_2 and F_3 giving a total of twelve such terms. One can then easily see that these are exactly the twelve terms that occur in the double integral that came from the surface integral. This completes the proof.

As I mentioned earlier, the proof above works when ∂D has just one component. If there are several we need to parametrise each one and write down the corresponding line integrals. Green's theorem will then say that the sum of these line integrals is the double integral that we obtained above.

The Jordan-Brouwer Separation Theorem

Recall that a closed surface S is a compact (bounded and with open complement) surface which has no boundary. It divides $\mathbb{R}^3 \setminus S$ into two components - the bounded component or inside, and the unbounded component or outside. This last fact is not at all easy to prove. It is the analogue of the Jordan curve theorem for $\mathbb{R}^3$.

Denote by S^n the set of points $(x_1, \ldots, x_n n+1)$ in $\mathbb{R}^{n+1}$ such that $x_1^2 + x_2^2 + \ldots x_{n+1}^2 = 1$. This set is called the unit n-sphere. When $n = 1$ we get the unit circle. When $n = 2$ the sphere in $\mathbb{R}^3$.

Theorem: Let S be homeomorphic to the sphere S^n. Then $\mathbb{R}^{n+1} \setminus S$ consists of exactly two connected components - one bounded and the other unbounded.

The theorem above is known as the Jordan-Brouwer separation theorem. When $n = 1$, it reduces to the Jordan curve theorem.

Closed surfaces and enclosed volumes

We will take for granted that a closed surface S in $\mathbb{R}^3$ has an inside (the bounded component) and outside (the unbounded component). We denote the region enclosed by S and including the points of S as well, by W. Then $\partial W = S$ and S is the boundary of W.

(The boundary of W can be defined as was done for surfaces. However, we give a different (but, of course, equivalent) definition instead. A point P in $\mathbb{R}^3$ is a boundary point of W if every open ball in $\mathbb{R}^3$ containing P intersects both W and W^c, the complement of W.)

Gauss's theorem states that the flux of a vector field out of a closed surface is equal to the divergence of that vector field over the volume enclosed by the surface.

Gauss's divergence theorem

Theorem 44: Let $S = \partial W$ be a closed oriented surface enclosing the region W with the outward normal giving the positive orientation. Let $\mathbf{F}$ be a $\mathcal{C}^1$ vector field defined on W. Then

$$\iiint_W (\nabla \cdot \mathbf{F}) dV = \iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}.$$

Clearly, the importance of Gauss's theorem is that it converts surface integrals to volume integrals and vice-versa. Depending on the context one may be easier to evaluate than the other.

Examples

Example 1 (page 446 of Marsden, Tromba and Weinstein): Let $\mathbf{F} = 2x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, and let S be the unit sphere. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Solution: Using Gauss' theorem we see that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W (\nabla \cdot \mathbf{F}) dV,$$

where W is the unit ball bounded by the sphere. Since $\nabla \cdot \mathbf{F} = 2(1 + y + z)$ we get

$$2 \iiint_W (1+y+z) dV = 2 \iiint_W dV + 2 \iiint_W y dV + 2 \iiint_W z dV.$$

Notice that the last two integrals above are 0, by symmetry. Hence, the flux is simply

$$2 \iiint_W dV = \frac{8\pi}{3}.$$

Example 2 (page 448 of Marsden, Tromba and Weinstein)

Calculate the flux of $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the unit sphere.

Solution: Again if we use Gauss's theorem we see that we need only evaluate

$$\iiint_W (\nabla \cdot \mathbf{F}) dV = \iiint_W 3(x^2 + y^2 + z^2) dxdydz,$$

where W is the unit ball.

This problem is clearly ideally suited to the use of spherical coordinates. Making a change of variables, we get

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{1} 3\rho^4 \sin\phi \, d\rho d\phi d\theta = \frac{12\pi}{5}$$

Divergence and flux

Just as we were able to relate the circulation and curl using Stokes' theorem, we can relate the flux and the divergence using the divergence theorem.

Theorem 45: Let B_ϵ be the solid ball of radius ϵ centered at a point P in space and let S_ϵ be its boundary sphere. Let $\mathbf{F}$ be a vector field defined in an open set around P. Then

$$\nabla \cdot \mathbf{F}(P) = \lim_{\epsilon \to 0} \frac{1}{V(B_\epsilon)} \iint_{S_\epsilon} \mathbf{F} \cdot d\mathbf{S},$$

where $V(B_\epsilon)$ is the volume of B_ϵ.

Proof: Use the divergence theorem and the mean value theorem for triple integrals. The proof is virtually the same as the one relating the circulation of a vector field to its curl.

Incompressibility

Note that there is nothing special about solid balls - any class of regions for which the divergence theorem holds and whose volumes shrink to 0 will give the same result.

Theorem 45 gives us another way of showing that the divergence at a point P is the net rate per unit volume at which the fluid is flowing outwards at P.

In particular, if the divergence is 0, this means that the net outward flow of the fluid across any surface is zero. This clearly corresponds to our notion of what an incompressible fluid ought to be. Because of the theorem above, incompressible fluids are also called divergence free fluids.

A formula for the divergence in spherical coordinates

As a rather nice application of the preceding circle of ideas we can compute the formula for the divergence in spherical coordinates.

As I just said in the previous slide, we can apply Theorem 45 to any shape enclosed by a surface for which Gauss' theorem holds. In this case, we will apply it to the infinitessimal volume element W in spherical coordinates at a point (ρ, θ, ϕ). The volume of this region is given by $\rho^2 \sin \phi d\rho d\phi d\theta$.

Let us first calculate the net flux through the two faces which are orthogonal to the radial vector. If S_1 is the outer surface and S_2 in the inner surface the net flux is given by

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

We can easily calculate this net flux as follows. We denote by F_ρ the radial component of the vector field $\mathbf{F}$, and similarly define F_ρ and F_θ.

Caution: these are not to be confused with the notation for the partial derivatives of F with respect to ρ, θ and ϕ. With this notation in hand, the net flux through S_1 and S_2 becomes

$$F_\rho(\rho + d\rho, \phi, \theta)(\rho + d\rho)^2 \sin\phi d\phi d\theta - F_\rho(\rho, \phi, \theta)\rho^2 \sin\phi d\phi d\theta.$$

Using the Mean Value Theorem for the function $F_\rho \rho^2$, the above expression yields

$$\approx \frac{\partial}{\partial\rho}(F_\rho(\rho, \phi, \theta)\rho^2 \sin\phi) d\rho d\phi d\theta.$$

Dividing by the volume element we obtain

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 F_\rho).$$

In exactly the same way we can compute the contribution to the net flux through the surfaces orthogonal to the ϕ and θ directions. These contributions are

$$\frac{1}{\rho \sin\phi}\frac{\partial}{\partial\phi}(\sin\phi F_\phi) \quad \text{and} \quad \frac{1}{\rho\sin\phi}\frac{\partial F_\theta}{\partial\theta}.$$

Putting these terms together we see that

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 F_\rho) + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi F_\phi) + \frac{1}{\rho\sin\phi}\frac{\partial F_\theta}{\partial\theta}.$$

Of course, one could have done this computation by brute force directly, but this is perhaps a more elegant way of doing it.

Gauss' Law of Electrostatics

Gauss' Law Let W be a region in $\mathbb{R}^3$ (we will assume that W is enclosed by a closed surface). and suppose that the origin $(0,0,0)$ in not on the boundary ∂W. Then

$$\iint_{\partial W} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \begin{cases} 4\pi, & \text{if } (0,0,0) \in W \\ 0, & \text{if } (0,0,0) \notin W. \end{cases}$$

Proof: We have previously computed the divergence of the vector field $\mathbf{F}/r^3$. Indeed,

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) = \frac{1}{r^3}\nabla\cdot\mathbf{r} + \nabla\left(\frac{1}{r^3}\right)\cdot\mathbf{r} = 0,$$

provided $\mathbf{r} \neq (0,0,0)$. Hence, if $(0,0,0) \notin W$, the assertion follows immediately from the divergence theorem:

$$\iint_{\partial W} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iiint_{\partial W} \nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right) dV = 0.$$

If $(0,0,0) \in W$, we proceed as follows. Take a solid ball centered at $(0,0,0)$ and contained in W. Consider the region $W \setminus B$. Then the origin is not contained in this region. Hence, applying Gauss's theorem to this region we see that

$$\iint_{\partial W} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS - \iint_{\partial B} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = 0$$

Why is there a minus sign in the left hand side?
Because the inner boundary ∂B of $W \setminus B$ is oriented opposite to the outer boundary ∂W. But

$$\iint_{\partial B} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \iint_{\partial B} \frac{1}{r^2} dS = 4\pi.$$

Note that this calculation shows that the flux through a sphere of a vector field satisfying the inverse square law (such as the electric field given by Coulomb's Law) is independent of the radius of the sphere. In particular we can easily recover Gauss' Law in electrostatics: The flux of an electric field out of a surface is equal to the total charge inside.