Normalization

We must have

$$|A|^2 \int_0^L \sin^2(kx)dx = 1$$

Realizing that *k* can take only discrete values, we replace *k*.

$$|A|^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx = 1$$

$$\frac{|A|^2}{2} \int_0^L \left[1 - \cos\left(\frac{2n\pi x}{L}\right)\right]dx = 1$$

$$\frac{|A|^2}{2}\left[[x]_0^L - \frac{L}{2n\pi}\left[\sin\left(\frac{2n\pi x}{L}\right)\right]_0^L\right] = 1$$

$$\frac{|A|^2}{2}L = 1 \Rightarrow A = e^{i\theta}\sqrt{\frac{2}{L}}$$

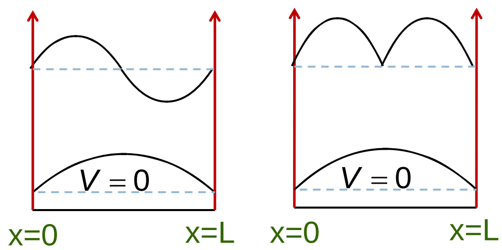
We take the phase angle to be zero. This gives us

$$A = \sqrt{\frac{2}{L}} \text{ and}$$

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \text{ for } 0 < x < L$$

$$= 0 \text{ elsewhere}$$

Wave function and Probability



Ortho-normality of Wave function

We can show that

$$\int_0^L \phi_n^*(x) \ \ \phi_m(x)dx = \delta_{nm}$$

Here the Kronecker-Delta function is defined as follows.

$$\delta_{nm} = \begin{cases} 1 & \text{for } n=m \\ 0 & \text{for } n \neq m \end{cases}$$

Proof

$$\int_0^L \phi_n^*(x) \ \ \phi_m(x)dx$$

$$= \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\text{dx}$$

$$= \frac{1}{\text{L}}\int_0^L \left[\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right]\text{dx}$$

=0 for $n \neq m$

Superposition of wave Function

The wave functions must be able to superpose. We can see that following is a solution of Time Dependent Schrödinger Equation and is general solution.

$$\psi(x,t) = \sum_n c_n \phi_n(x) e^{-i\frac{E_n}{\hbar}t}$$

Here $\phi_n(x)$ are the solutions of the time independent Schrödinger Equation corresponding to energies E_n.

Show that such a state is not a stationary state in general.

Further following is also a valid solution at *t=0*.

$$\psi(x,0) = \sum_n c_n \phi_n(x)$$

But is this a solution of time independent Schrödinger Equation? What would be the value of energy we shall get if a particle is in the above state.

Starting with normalized $\psi(x,0)$, implying

$$\sum_{n=1}^{\infty} |c_n|^2 = 1$$

$|c_n|^2$ is equal to the probability of finding the particle energy to be E_n.

Example

Following are the two normalized wave function corresponding to n=1 and n=4 states.

$$\phi_1(x) = \sqrt{\frac{2}{L}} \sin\frac{\pi x}{L}$$

$$\phi_4(x) = \sqrt{\frac{2}{L}} \sin\frac{4\pi x}{L}$$

Following wave function is not a normalized wave function.

$$\psi(x,0) = \sqrt{\frac{2}{L}} \sin\frac{\pi x}{L} + \sqrt{\frac{2}{L}} \sin\frac{4\pi x}{L}$$

The above can be seen by evaluating

$$\int_0^L \psi^*(x,0)\ \psi(x,0) dx$$

The normalized wave function would be

$$\psi(x,0)=\frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}+\frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}}\sin\frac{4\pi x}{L}$$
$$=\sqrt{\frac{1}{L}}\sin\frac{\pi x}{L}+\sqrt{\frac{1}{L}}\sin\frac{4\pi x}{L}$$

The physical interpretation implies imagining a large number of boxes where the wave function of the particle is given by above. If a measurement of energy is done, in half of them we shall find the particle to be in n=1 and in another half in n=4 state.

Question 1: Is the following wave function a normalized one?

$$\psi(x,0)=\sqrt{\frac{2}{5L}}\sin\frac{\pi x}{L}+\sqrt{\frac{8}{5L}}\sin\frac{4\pi x}{L}$$

We can write the above as follows.

$$\psi(x,0)=\frac{1}{\sqrt{5}}\phi_1(x)+\frac{2}{\sqrt{5}}\phi_4(x)$$

In this case in 20% of the boxes will give an energy corresponding to n=1 and 80% corresponding to n=4.

Question 2: What would be the expected value of energy in such a case?

$$\langle E \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n = 0.2E_1 + 0.8E_4$$

Question 3: If no measurement was done what would be the wave function at a time t.

$$\psi(x,t) = \sqrt{\frac{2}{5L}} \sin\frac{\pi x}{L} e^{-\frac{iE_1 t}{\hbar}} + \sqrt{\frac{8}{5L}} \sin\frac{4\pi x}{L} e^{-\frac{iE_4 t}{\hbar}}$$

We can check that this would not be a stationary state and the probability of finding the particle at a location would be a function of time.

Question 4: If a measurement is done in one of the boxes at t=0 and the energy is found to be E_4, what would be the wave function at a later time t.

The wave function now collapses and the time dependence would be given by.

$$\psi(x,t)=\sqrt{\frac{2}{L}}\sin\frac{4\pi x}{L}e^{-\frac{iE_4 t}{\hbar}}$$

Question 5: What would the measurement of energy yield on this box at a later time?

The particle is now in stationary state. Hence the measurement would lead to E_4.