MA 105 D1 Lecture 24

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The proof of Gauss' Divergence Theorem

Further consequences of the theorems of Gauss, Green and Stokes

Preliminaries

As with Stokes' theorem we will not prove the most general case of Gauss' theorem. Instead we will prove it for the so called elementary solids or regions. These are the analogues of the regions in the plane which are of type 1 as well as type 2.

Recall that in $\mathbb{R}^3$ we have three possible types. A solid W is said to be of type 1 if there is a region D of type 1 or 2 in the plane $\mathbb{R}^2$ and functions $f_1, f_2 : D \to \mathbb{R}$ such that

$$W = \{(x, y, z) : (x, y) \in D, f_1(x, y) \leq z \leq f_2(x, y)\}.$$

In words, W is the solid region that lies between the graphs of two surfaces.

By changing the roles of x, y and z we can make the obvious definitions of regions of types 2 and 3.

The hypotheses

Before embarking on the proof, I must add that I am following Marsden, Weinstein and Tromba's exposition very closely. I would also like to thank my colleague Professor Garge for lending me some of his slides which I have used with only slight modifications.

We will prove Gauss' theorem for regions that are of all three types, that is, they can be realized as the region between two graphs no matter which coordinate we single out.

The most important examples of such regions are the parallelepipeds (or cuboids). Other examples include tetrahedra, pyramids, more generally, convex polyhedra and also spheres.

Recall that we have a $\mathcal{C}^1$-vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ defined and on W. We will further assume that surfaces that bound W (that is, the graphs that appear in the description of W as a region of one type or another) are defined by $\mathcal{C}^1$ functions.

The left hand side

We start with the left hand side (the volume integral) of the theorem. Since,

$$\mathrm{div}(\mathbf{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

the volume integral becomes

$$\iiint_W \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV$$

which breaks up as a sum

$$\iiint_W \frac{\partial P}{\partial x} dV + \iiint_W \frac{\partial Q}{\partial y} dV + \iiint_W \frac{\partial R}{\partial z} dV.$$

The right hand side

Similarly the right hand side breaks up as a sum as follows:

$$\begin{aligned}\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} &= \iint_{\partial W} (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot d\mathbf{S} \\ &= \iint_{\partial W} P\mathbf{i} \cdot d\mathbf{S} + \iint_{\partial W} Q\mathbf{j} \cdot d\mathbf{S} + \iint_{\partial W} R\mathbf{k} \cdot d\mathbf{S}\end{aligned}$$

The theorem will follow if we establish the equalities:

$$\iint_{\partial W} P\mathbf{i} \cdot d\mathbf{S} = \iiint_W \frac{\partial P}{\partial x} dV, \qquad \iint_{\partial W} Q\mathbf{j} \cdot d\mathbf{S} = \iiint_W \frac{\partial Q}{\partial y} dV$$

and

$$\iint_{\partial W} R\mathbf{k} \cdot d\mathbf{S} = \iiint_W \frac{\partial R}{\partial z} dV.$$

Reduction to a single equality

It is clearly enough to prove the theorem for vector fields with only one non-zero component, since any vector field can be written as a sum of three such vector fields as above.

Thus, we can (and will) assume that the vector field $\mathbf{F}$ has only one nonzero component, and we take it to be in the direction of the unit vector $\mathbf{k}$, $\mathbf{F} = R\mathbf{k}$. We will show that the third of the equalities above holds. So we need

$$\iiint_W \frac{\partial R}{\partial z} dV = \iint_{\partial W} R\mathbf{k} \cdot d\mathbf{S},$$

where the solid W is given by a region D in the xy-plane and functions $f_1, f_2 : D \to \mathbb{R}$:

$$W = \{(x, y, z) : (x, y) \in D, f_1(x, y) \leq z \leq f_2(x, y)\},$$

since the region is of type 1.

Reduction to a double integral

Using Fubini's theorem, we write the triple integral as an iterated integral:

$$\iiint_W \frac{\partial R}{\partial z} dV = \iint_D \left(\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial R}{\partial z} dz \right) dxdy.$$

Using the fundamental theorem of calculus, we get

$$\iiint_W \frac{\partial R}{\partial z} dV = \iint_D \left[R\left(x, y, f_2(x, y)\right) - R\left(x, y, f_1(x, y)\right) \right] dxdy.$$

We will not simplify the left hand side any further, but will turn our attention to the right hand side instead.

Computing the integral over the boundary

The boundary of W is a union of six faces: the graph of f_1 is the bottom face S_1, the graph of f_2 is the top face S_2 and the other four faces are S_3, S_4, S_5, S_6 (in any order).

Not all of these four faces may actually occur. If W happens to be the solid cylinder it is possible to use just two faces to describe ∂W. A similar situation presents itself for a solid sphere. In any event we can write

$$\iint_{\partial W} R\mathbf{k} \cdot d\mathbf{S} = \iint_{S_1} R\mathbf{k} \cdot d\mathbf{S} + \iint_{S_2} R\mathbf{k} \cdot d\mathbf{S} + \sum_{i=3}^{6} \iint_{S_i} R\mathbf{k} \cdot d\mathbf{S}.$$

Reduction to surface integrals over two surfaces

Note that the normals to the surfaces S_3, S_4, S_5 and S_6 are normal to the z-axis, because at every point on these surfaces the z-axis is contained in the tangent plane. It follows that

$$\iint_{S_i} R\mathbf{k} \cdot d\mathbf{S} = \iint_{S_i} (R\mathbf{k} \cdot \mathbf{n_i}) dS = 0 \quad 3 \leq i \leq 6.$$

Hence,

$$\iint_{\partial W} R\mathbf{k} \cdot d\mathbf{S} = \iint_{S_1} R\mathbf{k} \cdot d\mathbf{S} + \iint_{S_2} R\mathbf{k} \cdot d\mathbf{S}.$$

It remains to compute the two surface integrals above explicitly. We will use the fact that they are both parametrised surfaces (being graphs).

Computing the surface integrals over the remaining surfaces

The surfaces S_1 and S_2 are graphs of functions f_1 and f_2 respectively, so we parametrise them by $\Phi : D \to \mathbb{R}^3$ and $\Psi : D \to \mathbb{R}^3$ given by

$$\Phi(x,y) = (x, y, f_1(x,y)) \quad \text{and} \quad \Psi(x,y) = (x, y, f_2(x,y)).$$

To compute the surface integrals we must compute the normals. They are

$$\begin{aligned} \Phi_x \times \Phi_y &= \left(\mathbf{i} + \frac{\partial f_1}{\partial x}\mathbf{k}\right) \times \left(\mathbf{j} + \frac{\partial f_1}{\partial y}\mathbf{k}\right) \\ &= -\frac{\partial f_1}{\partial x}\mathbf{i} - \frac{\partial f_1}{\partial y}\mathbf{j} + \mathbf{k} \end{aligned}$$

and

$$\Psi_x \times \Psi_y = -\frac{\partial f_2}{\partial x}\mathbf{i} - \frac{\partial f_2}{\partial y}\mathbf{j} + \mathbf{k}.$$

Note that the normal given by our parametrisation of S_1 is "upward" in the direction of $\mathbf{k}$, that is, it goes inside the solid W, so while computing the surface integral we multiply by -1:

$$\begin{aligned}
\iint_{S_1} R\mathbf{k} \cdot d\mathbf{S} &= -\iint_D R\big(\Phi(x,y)\big)\mathbf{k} \cdot (\Phi_x \times \Phi_y)dxdy \\
&= \iint_D R\big(\Phi(x,y)\big)\mathbf{k} \cdot \left(\frac{\partial f_1}{\partial x}\mathbf{i} + \frac{\partial f_1}{\partial y}\mathbf{j} - \mathbf{k}\right) dxdy \\
&= -\iint_D R\big(x, y, f_1(x,y)\big)dxdy.
\end{aligned}$$

and similarly we get

$$\iint_{S_2} R\mathbf{k} \cdot d\mathbf{S} = \iint_D R\big(x, y, f_2(x,y)\big)dxdy.$$

Adding the two integrals above, we see that Gauss' theorem follows immediately in this case.

Gauss' divergence theorem for more general regions

While we have proved Gauss' theorem only for regions which are simultaneously of types 1, 2 and 3, we stated the theorem for a much more general class of regions - those bounded by closed surfaces.

It turns out that any region bounded by a closed surface can be approximated from within by parallelepipeds which intersect only along planar surfaces. As in the proof of Stokes' theorem for general surfaces the surface integrals on the common boundary of two parallelepipeds cancels because they are oppositely oriented.

It follows that only the surface integral on the outer boundary of the union of two parallelepipeds survives. By decreasing the size of the parallelepipeds we can eventually approximate the region more and more efficiently and prove the general case.

Back to conservative fields

Let $\mathbf{F}$ be a $\mathcal{C}^1$ vector field defined on $\mathbb{R}^3$. We have previously defined a conservative vector field to be one for which the path integrals depend only on the initial and final points of the path and not the path itself. Equivalently, the line integral of $\mathbf{F}$ around any simple closed curve should be 0.

Theorem 46: A vector field $\mathbf{F}$ is conservative if and only if $\nabla \times \mathbf{F} = 0$.

Proof: Let C be a simple closed curve in $\mathbb{R}^3$ and let S be the surface in bounds. Then using Stokes' theorem we see that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0.$$

Conservative fields and gradient fields

Corollary 47: The vector field $\mathbf{F}$ is a gradient field, that is, there exists a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.

This is obvious since we have already shown that conservative fields are gradient fields. We have thus answered one of the basic questions we raised some time ago: When is a curl free vector field a gradient field?

On $\mathbb{R}^3$, always.

I should mention here that in many books, a conservative field is defined as one which is a gradient field.

Subtleties we have ignored so far

What is wrong/not clear about the proof of Theorem 46?

Is it so obvious that a simple closed curve necessarily bounds a surface in $\mathbb{R}^3$?

Let us assume instead that we are on the plane. To assert that a simple closed curve C is the boundary of a compact region in the plane we need a very non trivial result: the Jordan curve theorem. Because of this theorem, we can take the bounded component of $\mathbb{R}^2 \setminus C$ and C will be its boundary.

In $\mathbb{R}^3$ the question is even more subtle.

Back to the plane

We have just invoked the Jordan curve theorem to say that simple closed curves in the plane are necessarily boundaries of surfaces. What happens if we take $\mathbb{R}^2 \setminus \{(0,0)\}$?

It should be clear that in this case the unit circle $x^2 + y^2 = 1$ is not the boundary of a compact region in $\mathbb{R}^2 \setminus \{(0,0)\}$. The region enclosed by the unit circle is the unit disc minus the origin and this set is not closed.

This is why one can find curl free vector fields on this set which are not gradients.

It should thus be clear that the failure of a curl free vector field to be a gradient field is something that depends on the geometry of the region we are considering. In particular, we need to know (at least to apply Stokes' theorem) whether a simple closed curve bounds a compact surface or not.

Homotopy

Definition: Let $\gamma_i : I = [0,1] \to X$, $i = 1,2$, be continuous maps into a subset X of $\mathbb{R}^n$ with $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$. We will say that γ_1 and γ_2 are homotopic if there exists a continuous function

$$F : I \times I \to X$$

such that $F(t,0) = \gamma_0(t)$ and $F(t,1) = \gamma_1(t)$ and $F(0,s) = P$ and $F(1,s) = Q$ for all $0 \leq s \leq 1$.
What this means is that one can continuously deform the curve γ_0 to γ_1 while keeping the end points fixed.

Definition: A space $X \subset \mathbb{R}^n$ in which any two paths are homotopic is called simply connected.

Fact: In a simply connected space, every simple closed curve bounds a (compact) surface.

Curl free vector fields in simply connected regions

Thus, the proof of Theorem 46 is valid. It is just that we are assuming a not-so-easy fact when using Stokes' Theorem.

An alternate way to define a simply connected space is to say that every closed path (that is $\gamma(0) = \gamma(1)$) is homotopic to the constant map.

Theorem 48: Suppose that X is a connected, simply connected surface (non-singular, continuously differentiable) in $\mathbb{R}^3$ and $\mathbf{F}$ is a $\mathcal{C}^1$-vector field defined on X. Then $\nabla \times \mathbf{F} = 0$ if and only if $\mathbf{F}$ is conservative, that is, there exists a function $f : X \to R$ such that $\mathbf{F} = \nabla f$.

There is a version of this theorem for X in $\mathbb{R}^n$. It will work the moment we can make sense of the gradient and the curl in n-dimensions.

Examples

Example 1: The space $\mathbb{R}^n$ is simply connected.

Example 2: The set $\mathbb{R}^2 \setminus \{(0,0)\}$ is not simply connected.

Example 3: The unit circle in $\mathbb{R}^2$ is not simply connected.

Example 4: The unit sphere in $\mathbb{R}^3$.

Example 5: The torus is not simply connected.

Example 6: $\mathbb{R}^3 \setminus z$-axis is not simply connected.

Example 7: All convex sets are simply connected.

Example 8: All star-shaped sets are simply connected.

The divergence and curl

Similar to Corollary 47, we have the following theorem showing that divergence free vector fields necessarily arise as the curls of (other) vector fields. This time, however, the proof is straightforward and does not involve any subtleties.

Theorem 49: If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ and $\nabla \cdot \mathbf{F} = 0$, then $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$.

Proof: In fact, we can find a vector field $\mathbf{G}$ of the special form $\mathbf{G} = G_1(x, y, z)\mathbf{i} + G_2(x, y, z)\mathbf{j}$.
The curl of a vector field $\mathbf{G}$ of this form is

$$-\frac{\partial G_2}{\partial z}\mathbf{i} + \frac{\partial G_1}{\partial z}\mathbf{j} + \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\mathbf{k}.$$

The Poincaré lemma

Thus, we will obtain the required **G** if we solve the equations

$$-\frac{\partial G_2}{\partial z} = F_1, \quad \frac{\partial G_1}{\partial z} = F_2 \quad \text{and} \quad \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = F_3$$

But we can solve these equations simply by integrating:

$$G_1(x, y, z) = \int_0^z F_2(x, y, t)dt - \int_0^y F_3(x, y, 0)dt$$

(Note that there was a mistake in the second term on the RHS displayed in class. "t" has now been replaced by "0" in F_3.) and

$$G_2(x, y, z) = -\int_0^z F_1(x, y, t)dt.$$

This proves our theorem. □

As mentioned earlier one can formulate versions of Corollary 47 and Theorem 49 for $\mathbb{R}^n$ and combine to get just one statement known as the Poincaré lemma.

Theorem 50: Every closed form on $\mathbb{R}^n$ is exact.

The cross derivative test

Corollary 47 tells us that curl free vector fields on $\mathbb{R}^3$ are conservative (or gradient fields). For a vector field $\mathbf{F}$ on $\mathbb{R}^n$ we have the cross-derivative test as one of the special cases of the Poincaré lemma.

Cross derivative test A vector field $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ is conservative if and only if

$$\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} = 0$$

for all $1 \leq i, j \leq n$.

Note that if $n = 3$ this is exactly the condition that the curl is zero.

Carl Friedrich Gauss: The prince of mathematicians (1777 C.E. - 1855 C.E.)

(http://commons.wikimedia.org/wiki/File:Carl_Friedrich_Gauss.jpg)

Gauss - the early years

From "Men of Mathematics" by Eric Temple Bell:

Shortly after his seventh birthday Gauss entered his first school, a squalid relic of the Middle Ages run by a virile brute, one Büttner, whose idea of teaching the hundred or so boys in his charge was to thrash them into such a state of terrified stupidity that they forgot their own names. More of the good old days for which sentimental reactionaries long. It was in this hell-hole that Gauss found his fortune.

Nothing extraordinary happened during the first two years. Then, in his tenth year, Gauss was admitted to the class in arithmetic.

As it was the beginning class none of the boys had ever heard of an arithmetic progression. It was easy then for the heroic Büttner to give out a long problem in addition whose answer he could find by a formula in a few seconds.

The problem was of the following sort, $81297 + 81495 + 81693 + \ldots + 100899$, where the step from one number to the next is the same all along (here 198), and a given number of terms (here 100) are to be added.

It was the custom of the school for the boy who first got the answer to lay his slate on the table; the next laid his slate on top of the first, and so on.

Büttner had barely finished stating the problem when Gauss flung his slate on the table:

"There it lies," he said"Ligget se'" in his peasant dialect.

Then, for the ensuing hour, while the other boys toiled, he sat with his hands folded, favored now and then by a sarcastic glance from Büttner, who imagined the youngest pupil in the class was just another blockhead.

At the end of the period Büttner looked over the slates. On Gauss' slate there appeared but a single number.

To the end of his days Gauss loved to tell how the one number he had written down was the correct answer and how all the others were wrong.

Gauss had not been shown the trick for doing such problems rapidly. It is very ordinary once it is known, but for a boy of ten to find it instantaneously by himself is not so ordinary.

What was Gauss' trick for summing an arithmetic progression?

Philology versus mathematics

At seventeen, Gauss had to decide what he wanted to do. He was a talented student of both philology and mathematics.

What is philology? The study of languages.

At this point he made a remarkable mathematical discovery. He was so excited by this discovery that he decided to do mathematics.

What was this discovery? Gauss discovered that you can construct a regular 17-sided polygon using a straight edge (ruler) and compass only.