MA 105 D1 Lecture 4

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Limits of functions

Definition: A function $f : (a, b) \to \mathbb{R}$ is said to tend to (or converge to) a limit l at a point $x_0 \in [a, b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - l| < \epsilon$$

for all $x \in (a, b)$ such that $0 < |x - x_0| < \delta$. In this case, we write

$$\lim_{x \to x_0} f(x) = l,$$

or $f(x) \to l$ as $x \to x_0$ which we read as "$f(x)$" tends to l as x tends to x_0".

This is just the rigourous way of saying that the distance between $f(x)$ and l can be made as small as one pleases by making the distance between x and x_0 sufficiently small.

Thus the limit of a function may exist even if the function is not defined at that point.

Limits at infinity

There is one further case of limits that we need to consider. This occurs when we consider functions defined on open intervals of the form $(-\infty, b)$, (a, ∞) or $(-\infty, \infty) = \mathbb{R}$ and we wish to define limits as the variable goes to plus or minus infinity. The definition here is very similar to the definition we gave for sequences. Let us consider the last case.

Definition: We say that $f : \mathbb{R} \to \mathbb{R}$ tends to a limit l as $x \to \infty$ (resp. $x \to -\infty$) if for all $\epsilon > 0$ there exists $X \in \mathbb{R}$ such that

$$|f(x) - l| < \epsilon,$$

whenever $x > X$ (resp. $x < X$), and we write

$$\lim_{x \to \infty} f(x) = l \quad \text{or} \quad \lim_{x \to -\infty} f(x) = l.$$

or, alternatively, $f(x) \to l$ as $x \to \infty$ or as $x \to -\infty$, depending on which case we are considering.

Calculating limits explicitly

As with sequences, using the rules for limits of functions together with the Sandwich theorem allows one to treat the limits of a large number of expressions once one knows a few basic ones:

(i) $\lim_{x\to 0} x^\alpha = 0$ if $\alpha > 0$, (ii) $\lim_{x\to\infty} x^\alpha = 0$ if $\alpha < 0$,
(iii) $\lim_{x\to 0} \sin x = 0$, (iv) $\lim_{x\to 0} \sin x/x = 1$
(v) $\lim_{x\to 0}(e^x - 1)/x = 1$, (vi) $\lim_{x\to 0} \ln(1+x)/x = 1$

We have not concentrated on trying to find limits of complicated expressions of functions using clever algebraic manipulations or other techniques. However, I can't resist mentioning the following problem.

Exercise 3: Find

$$\lim x_{\rightarrow} 0 \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}.$$

I will give the solution next time, together with the history of the problem (if I mention the history right away you will be able to get the solution by googling!), but feel free to use any method you like.

Continuity - the definition

Definition: If $f : [a, b] \to \mathbb{R}$ is a function and $c \in [a, b]$, then f is said to be continuous at the point c if and only if

$$\lim_{x \to c} f(x) = f(c).$$

Thus, if c is one of the end points we require only the left or right hand limit to exist.

A function f on (a, b) (resp. $[a, b]$) is said to be continuous if and only if it is continuous at every point c in (a, b) (resp. $[a, b]$).

If f is not continuous at a point c we say that it is dicontinuous at c, or that c is a point of discontinuity for f.

Intuitively, continuous functions are functions whose graphs can be drawn on a sheet of paper without lifting the pencil of the sheet of paper. That is, there should be no "jumps" in the graph of the function.

Continuity of familiar functions: polynomials

What are the functions we really know or understand? What does "knowing" or understanding a function $f(x)$ even mean? Presumably, if we understand a function f, we should be able to calculate the value of the function $f(x)$ at any given point x. But if you think about it, for what functions $f(x)$ can you really do this?

One class of functions is the polynomial functions. More generally we can understand rational functions, that is functions of the form $R(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials, since we can certainly compute the values of $R(x)$ by plugging in the value of x. How do we show that polynomials or rational functions are continuous (on $\mathbb{R}$)?

It is trivial to show from the definition that the constant functions and the function $f(x) = x$ are continuous. Because of the rules for limits of functions, the sum, difference, product and quotient (with non-zero denominator) of continuous functions are continuous. Applying this fact we see easily that $R(x)$ is continuous whenever the denominator is non-zero.

Continuity of other familiar functions

What are the other (continuous) functions we know? How about the trignometric functions? Well, here it is less clear how to proceed. After all we can only calculate $\sin x$ for a few special values of x ($x = 0, \pi/6, \pi/4, \ldots$ etc.). How can we show continuity when we don't even know how to compute the function?

Of course, if we define $\sin x$ as the y-coordinate of a point on the unit circle it seems intuitively clear that the y-coordinate varies continuously as the point varies on the unit circle, but knowing the precise definition of continuity this argument should not satisfy you.

We will not prove the continuity of $\sin x$ in this course, though we will given an idea of how this is done next week. So let us assume from now on that $\sin x$ is continuous. How can we show that $\cos x$ is continuous?

The composition of continuous functions

Theorem 8: Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to (e, f)$ be functions such that f is continuous at x_0 in (a, b) and g is continuous at $f(x_0) = y_0$ in (c, d). Then the function $g(f(x))$ (also written as $g \circ f(x)$ sometimes) is continuous at x_0. So the composition of continuous functions is continuous.

Exercise 4: Prove the theorem above starting from the definition of continuity.

Using the theorem above we can show that $\cos x$ is continuous if we show that $\sqrt{x}$ is continuous, since $\cos x = \sqrt{1 - \sin^2 x}$ and we know that $1 - \sin^2 x$ is continuous since it is the product of the sums of two continuous functions ($(1 + \sin x)$ and $(1 - \sin x)$!).

Once we have the continuity of $\cos x$ we get the continuity of all the rational trignometric functions, that is functions of the form $P(x)/Q(x)$, where P and Q are polynomials in $\sin x$ and $\cos x$, provided $Q(x)$ is not zero.

The continuity of the square root function

Thus in order to prove the continuity of $\cos x$ (assuming the continuity of $\sin x$) we need only prove the continuity of the square root function.

The main observation is that continuity is a local property, that is, only the behaviour of the function near the point being investigated is important.

Let $x_0 \in [0, \infty)$. To show that the square root function is continuous at x_0 we need to show that $\lim_{y \to x_0} \sqrt{y} = \sqrt{x_0}$, that is we need to show that $|\sqrt{y} - \sqrt{x_0}| < \epsilon$ whenever $|y - x_0| < \delta$ for some δ. First assume that $x_0 \neq 0$. Then

$$|\sqrt{y} - \sqrt{x_0}| = \left| \frac{y - x_0}{\sqrt{y} + \sqrt{x_0}} \right| < \frac{|y - x_0|}{\sqrt{x_0}}.$$

If we choose $\delta = \epsilon\sqrt{x_0}$, we see that

$$|\sqrt{y} - \sqrt{x_0}| < \epsilon,$$

which is what we needed to prove. When $x_0 = 0$, I leave the proof as an exercise.

The intermediate value theorem

One of the most important properties of continuous functions is the Intermediate Value Property (IVP). We will use this property repeatedly to prove other results.

Theorem 9: Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. For every u between $f(a)$ and $f(b)$ there exists $c \in [a, b]$ there such that $f(c) = u$.

Functions which have this property are said to have the Intermediate Value Property. Theorem 9 can thus be restated as saying that continuous functions have the IVP.

We will not be proving this property - it is a consequence of the completeness of the real numbers. Intuitively, this is clear. Since one can draw the graph of the function without lifting one's pencil off the sheet of paper, the pencil must cut evey line $y = e$ with e between $f(a)$ and $f(b)$.

The IVT in a picture

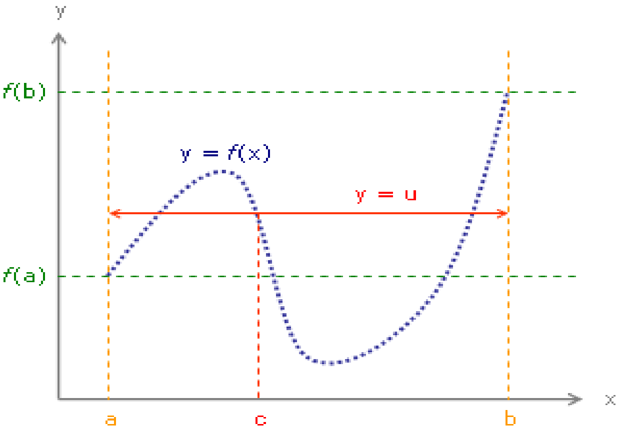

Image created by Enoch Lau see
http://en.wikipedia.org/wiki/File:Intermediatevaluetheorem.png
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Zeros of functions

One of the most useful applications of the intermediate value property is to find roots of polynomials, or, more generally, to find zeros of continuous functions, that is to find points $x \in \mathbb{R}$ such that $f(x) = 0$.

Theorem 10: Every polynomial of odd degree has at least one real root.

Proof: Let $P(x) = a_n x^n + \ldots + a_0$ be a polynomial of odd degree. We can assume without loss of generality that $a_n > 0$. It is easy to see that if we take $x = b > 0$ large enough, $P(b)$ will be positive. On the other hand, by taking $x = a < 0$ small enough, we can ensure that $P(b) < 0$. Since $P(x)$ is continuous, it has the IVP, so there must be a point $x_0 \in (a, b)$ such that $f(x_0) = 0$. $\square$

The IVP can often be used to get more specific information. For instance, it is not hard to see that the polynomial $x^4 - 2x^3 + x^2 + x - 3$ has a root that lies between 1 and 2.

Continuous functions on closed, bounded intervals

The other major result on continuous functions that we need is the following. A closed bounded interval is one of the form $[a, b]$, where $-\infty < a$ and $b < \infty$.

Theorem 11: A continuous function on a closed bounded interval $[a, b]$ is bounded and attains its infimum and supremum, that is, there are points x_1 and x_2 in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$, where m and M denote the infimum and supremum respectively.

Again, we will not prove this, but will use it quite often. Note the contrast with open intervals. The function $1/x$ on $(0, 1)$ does not attain a maximum - in fact it is unbounded. Similarly the function $1/x$ on $(1, \infty)$ does not attain its minimum, although, it is bounded below.

Exercise 5: In light of the above theorem, can you find a continuous function $g : (a, b) \to \mathbb{R}$ for part (i) of Exercise 1.11, with $c \in (a, b)$?

The function $\sin \frac{1}{x}$

Let us look at Exercise 1.13 part (i).
Consider the function defined as $f(x) = \sin \frac{1}{x}$ when $x \neq 0$, and $f(0) = 0$. The question asks if this function is continuous at $x = 0$.
How about $x \neq 0$? Why is $f(x)$ continuous?
Because it is a compostion of the sin function and a rational function $1/x$. Since both of these are continuous when $x \neq 0$, so is $f(x)$.

Let us look at the sequence of points $x_n = 2/n\pi$. Clearly $x_n \to 0$ as $n \to \infty$.
For these points $f(x_n) = \pm 1$. This means that no matter how small I take my δ, there will be a point $x_n \in (0, \delta)$, such that $|f(x_n)| = 1$. But this means that $|f(x) - f(0)| = |f(x)|$ cannot be made smaller than 1 no matter how small δ may be. Hence, f is not continuous at 0.
The same kind of argument will show that there is no value that we can assign $f(0)$ to make the function $f(x)$ continuous at 0.

You can easily check that $f(x)$ has the IVP. However, we have proved that it is not continuous. So IVP $\not\Rightarrow$ continuity.

Sequential continuity

The preceeding example showed that in order to demonstrate that a function, say $f(x)$, is not continuous at a point x_0 it is enough to find a sequence x_n tending to x_0 such that the value of the function $|f(x_n) - f(x_0)|$ remains large. Suppose it is not possible to find such a sequence. Does that mean the function is continuous at x_0? The following theorem answers the question affirmatively.

Theorem 12: A function $f(x)$ is continuous at a point a if and only if for every sequence $x_n \to a$, $\lim_{x_n \to a} f(x_n) = f(a)$.

A function that satisfies the property that for every sequence $x_n \to a$, $\lim_{x_n \to a} f(x_n) = f(a)$ is said to be sequentially continuous. The theorem says that sequential continuity and continuity are the same thing. Indeed, it is clear that a continuous function is necessarily sequentially continuous. It is the reverse that is harder to prove.

Limits of functions of several variables

Just like we did for sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$, we can define the notion of the limit of a function for functions from $\mathbb{R}^2$ to $\mathbb{R}$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. The function $f(x_1, x_2)$ is said to tend to a limit l as $(x_1, x_2) \to (a_1, a_2)$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x_1, x_2) - l| < \epsilon$$

whenever $\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} < \delta$. Notice, that one can now approach the point (a_1, a_2) from any direction in the plane. Our definition requires that the limits from the different directions all exist and be equal. This is quite a powerful condition.

If we have functions from $\mathbb{R}^2 \to \mathbb{R}^2$ we can make exactly the same definition. But this time $l = (l_1, l_2)$ will be in $\mathbb{R}^2$ and so will $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$, so we will have to replace the modulus function by the distance between these two quantities:

$$\sqrt{[f_1(x_1, x_2) - l_1]^2 + [f_2(x_1, x_2) - l_2]^2}.$$

Limits of functions of several variables

The definitions we have made go through for functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, where m and n may be different. For instance, we have considered the case when $m = 2$ and $n = 1$ and also the case $m = 2$ and $n = 2$ above. But we could allow m and n to take any of the values 1, 2 or 3 (in fact, we can allow values greater than 3 as well!).

Exercise 1: Show that

$$\lim_{y \to x} f(y) = l$$

for $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2 \iff$

$$\lim_{y \to x} f_1(y) = l_1 \quad \text{and} \quad \lim_{y \to x} f_2(y) = l_2,$$

where $l = (l_1, l_2)$ In other words, when dealing with limits of functions which are vector-valued, it is enough to study the limits of the coordinate functions.

Continuous functions of several variables

Once the definition of the limit is clear it makes sense to talk of continuity as well. All the definitions remain the same, only the definition of the distance function changes depending on the domain and the range.

For instance, provided we know what “closed and bounded sets” are in $\mathbb{R}^2$ or $\mathbb{R}^3$, Theorem 11 goes through for continuous functions $f : \mathbb{R}^m \to \mathbb{R}$. $(m = 2, 3)$. For functions with more than one variable in the range the first part of Theorem 11 still works, but for the second part things are more compacted (again there is no “ordering” in $\mathbb{R}^2$ or $\mathbb{R}^3$).

While it is easy to see what a bounded set in $\mathbb{R}^m$ should be, closed is a little more complicated and we will not give the definition here. However, a rectangle of the form $[a, b] \times [c, d]$ in $\mathbb{R}^2$ is an example of a closed and bounded set (also called “compact sets” of this form).

Theorem 12 goes through without any problems even when the range is in $\mathbb{R}^2$ or $\mathbb{R}^3$.

The defintion

For now, if you did not understand the rigourous definition of the limit, forget about it. You will be able to understand what follows as long as you remember your 11th standard treatment of limits. Recall that $f : (a, b) \to \mathbb{R}$ is said to be differentiable at a point $c \in (a, b)$ if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. In this case the value of the limit is denoted $f'(c)$ and is called the derivative of f at c. The derivative may also be denoted by $\frac{df}{dx}(c)$ or by $\frac{dy}{dx}|_c$, where $y = f(x)$.

In general, the derivative measures the rate of change of a function at a given point. Thus, if the function we are studying is the position of a particle on the x-coordinate, then $x'(t)$ is the velocity of the particle. If the function we are studying is the velocity $v(t)$ of the particle, then the derivative $v'(t)$ is the acceleration of the particle. If the function we are studying is the population of India, then the derivative measures the rate of change of the population.

The slope of the tangent

From the point of view of geometry, the derivative $f'(c)$ gives us the slope of the curve, that is, the slope of the tangent to the curve $y = f(x)$ at $(c, f(c))$. This becomes particularly clear if we rewrite the derivative as the following limit:

$$\lim_{y \to c} \frac{f(y) - f(c)}{y - c}.$$

The expression inside the limit obviously represents the slope of a line passing through $(c, f(c))$ and $(y, f(y))$, and as y approaches c this line obviously becomes tangent to $y = f(x)$ at the point $(c, f(c))$.

Another way of thinking of the derivative

Another way of thinking of the derivative of the function f at the point x_0 is as follows. If f is differentiable at x_0 we know that

$$\frac{f(x_0+h)-f(x_0)}{h} - f'(x_0) \to 0$$

as $h \to 0$.

Since we are keeping x_0 fixed, we can treat the above quantity as a function of h. Thus we can write

$$\frac{f(x_0+h)-f(x)}{h} - f'(x_0) = o(h)$$

for some function $o(h)$ with the property that $o(h) \to 0$ as $h \to 0$.

Taking a common denominator, we see that

$$\frac{f(x_0+h)-f(x_0)-f'(x_0)h}{h} = o(h) \tag{1}$$

We can use the above equality to give an equivalent definition for the derivative. A function f is said to be differentiable at the point x_0 if there exists a real number (denoted $f'(x_0)$) such that (1) holds for some function $o(h)$ such that $o(h) \to 0$ as $h \to 0$.

The derivative as a linear map

We can rewrite equation (1) as

$$f(x_0 + h) = f(x_0) + f'(x_0)h + o(h)h$$

Thus, the derivative of $f(x)$ at a point x_0 can be viewed as that real number (if it exists) by which you have to multiply h so that the resulting remainder goes to 0 faster than h (that is, the remainder divided by h goes to 0 as h goes to 0).

A function $f : \mathbb{R} \to \mathbb{R}$ which has the property that $f(x + y) = f(x) + f(y)$ is called a linear function (or linear map). All such functions are given by multiplication by a real number, that is, every linear function has the form $f(x) = \lambda x$, for some real number λ. Thus the derivative may be regarded as a linear function (in the variable h). This point of view will be particularly useful in multivariable calculus.