PH-105 Assignment Sheet - 3 (Quantum Mechanics - 2)

Umang Mathur

51. If $\phi_n(x)$ are the solutions of time independent Schrdinger equation, with energies E_n, show that $\psi(x,t) = \sum\limits_n C_n \phi_n(x) e^{\frac{-iEnt}{\hbar}}$, where C_n are constants, is a solution of time dependent Schrdinger equation. However, show that $\psi(x,0)$ is not a solution of the time independent Schrdinger equation

<u>**Solution**</u> :
The Time Independent Schrodinger Equation for one-dimensional space is:

$$\hat{H}\psi = \hat{E}\psi$$

where, $\hat{H} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ and $\hat{E} = i\hbar\frac{\partial}{\partial t}$.

Also, since $\phi_n(x)$ are the solutions of time independent Schrdinger equation, with energies E_n,

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\Big(\phi_n(x)\Big) = E_n\phi_n(x) \tag{1}$$

Thus, we have,

$$\begin{aligned}
\hat{H}\psi(x,t) &= \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\sum_n C_n\phi_n(x)e^{\frac{-iE_nt}{\hbar}} \\
&= \sum_n C_n e^{\frac{-iE_nt}{\hbar}}\left(\frac{-\hbar^2}{2m}\frac{d^2\phi_n(x)}{dx^2}\right) \\
&= \sum_n C_n e^{\frac{-iE_nt}{\hbar}}\Big(E_n\phi_n(x)\Big) \text{ using (1)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\hat{E}\psi(x,t) &= i\hbar\frac{\partial}{\partial t}\sum_n C_n\phi_n(x)e^{\frac{-iE_nt}{\hbar}} \\
&= \sum_n C_n\phi_n(x)\left(i\hbar\frac{\partial}{\partial t}\left(e^{\frac{-iEnt}{\hbar}}\right)\right) \\
&= \sum_n C_n e^{\frac{-iE_nt}{\hbar}}\Big(E_n\phi_n(x)\Big)
\end{aligned}$$

Thus,

$$\hat{H}\psi(x,t) = \hat{E}\psi(x,t)$$

However,

$$\hat{H}\psi(x,0) = \sum_n C_n E_n \phi_n(x)$$

. Thus, for $\psi(x,0)$ to be a solution of TISE, we must have $\hat{H}\psi(x,0) = E\psi(x,0)$ for some real constant E, i.e.,

$$\begin{aligned}
&\sum_n C_n E_n\phi_n(x) = E\big(\sum_n C_n\phi_n(x)\big) \\
&\sum_n C_n\phi_n(x)\big(E - E_n\big) = 0
\end{aligned}$$

However, since, $\hat{H}$ is a Hermitian operator, the eigenvalues $\phi_n(x)$ must be orthogonal (and therefore, linearly independent).
Thus, this is possible only if $E = E_n \forall n$.
But, since all E_ns are distinct (assuming non-degenerate levels in one-dimensional space), this is not possible unless $\psi(x, t)$ is not a linear combination but only a single eigen-function.