

MA 105 D1 Lecture 14

September 19, 2014

Global extrema

Lagrange multipliers

The Riemann integral in several variables

Boundedness of continuous functions of two variables

In one variable we saw that continuous functions are bounded in closed bounded intervals. More generally, we can take a finite union of such intervals and the function will remain bounded. Such sets are called compact sets. What is the analogue for \mathbb{R}^2 ?

It is a little harder to define compact sets in \mathbb{R}^2 , but we can give examples. The **closed** disc

$$\bar{D}_r = \{(x, y) \mid \|(x, y) - (x_0, y_0)\| \leq r\}$$

of radius r around a point $(x_0, y_0) \in \mathbb{R}^2$ is an example. Another example is the **closed** rectangle:

$$\bar{S} = \{(x, y) \mid |x - x_0| \leq a, |y - y_0| \leq b\}$$

Finite unions of such sets will also be compact sets. As before, we have

Theorem 32: A continuous function on a compact set in \mathbb{R}^2 will attain its extreme values.

Global extrema as local extrema

Definition: A point (x_0, y_0) such that $f(x, y) \leq f(x_0, y_0)$ or $f(x, y) \geq f(x_0, y_0)$ for all (x, y) in the domain being considered is called a **global maximum or minimum point** respectively.

In light of Theorem 32, these always exist for continuous functions on closed rectangles or discs.

Assume now that f is a \mathcal{C}^2 function on a closed rectangle \bar{S} as above. We can find the global maximum or minimum as follows. We first study all the local extrema which by definition lie in the **open** rectangle

$$S = \{(x, y) \mid |x - x_0| < a, |y - y_0| < b\}$$

(This is because, in order to have a local extremum we need to have a whole disc around the point. A point on the boundary of the closed rectangle does not have such a disc around it where the function is defined.)

After determining all the local maxima we take the points where the function takes the largest value - say M_1 . We compare this with the maximum value of the function on the boundary of the closed rectangle, say M_2 . Let M be the maximum of these two values. The points where M is attained are the global maxima. We can treat a function defined on the closed disc in the same way. Once again, the local extrema will have to lie in the open disc and we will have to consider these values as well as the values of the function on the boundary circle.

Finding global extrema in \mathbb{R}^2

Sometimes, global extrema may exist even when the domain in \mathbb{R}^2 (which is not compact, in fact, not even bounded).

For instance, in the example $z = 2(x^2 - y^2)e^{-\frac{x^2+y^2}{2}}$, we can check that when $x^2 + y^2 > 16$, $z \leq 1/2$, that is, outside the disc $\bar{D}_4(0,0)$ (try proving this - I have not chosen this disc optimally).

In the closed disc $\bar{D}_4(0,0)$ we have already found the critical points and the local maxima and minima. We see that $f(\sqrt{2}, 0) = f(-\sqrt{2}, 0) = 2/e > 1/2$. There cannot be other local maxima since we have checked all the other critical points, and local maxima can occur only at the critical points. Hence, these points are global maxima in $\bar{D}_4(0,0)$.

Now, outside the disc $\bar{D}_4(0,0)$, we know that $z = f(x, y) < 1/2$. Hence, we see that the value $2/e > 1/2$ is the maximum value taken on all of \mathbb{R}^2 . Thus, this particular function actually has a global maximum (in fact, two global maxima) on \mathbb{R}^2 .

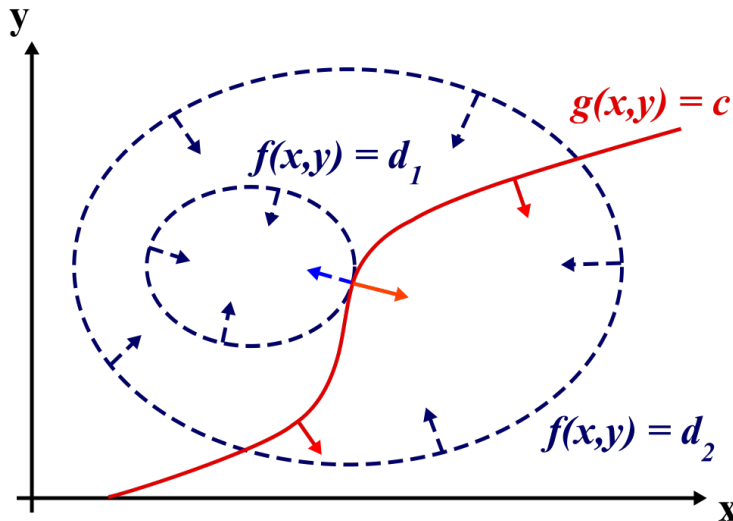
The two variable problem

Suppose we are given a function $f(x, y)$ in two variables. We would like maximize or minimize it **subject to the constraint that $g(x, y) = c$** . In geometric terms, we want to find the maximum or minimum values of f while staying on the curve $g(x, y) = c$.

Where might we find these maxima or minima?

One way of doing it would be to walk along the curve $g(x, y) = c$ meeting the level curves $f(x, y) = d$ as d increases. The smallest value of d (if it exists!) such that $g(x, y) = c$ and $f(x, y) = d$ meet will surely be the minimum value for the function $f(x, y)$ on the curve $g(x, y) = c$. What special property holds at this point?

In pictures



[http://en.wikipedia.org/wiki/Lagrange_multiplier#mediaviewer/
File:LagrangeMultipliers2D.svg](http://en.wikipedia.org/wiki/Lagrange_multiplier#mediaviewer/File:LagrangeMultipliers2D.svg)

The condition on the normals

From the picture one might guess that at the value d_1 the curves are tangent to each other, or, equivalently, their **normals are parallel** to each other. Why is this?

Suppose that this were not the case. Recall that the normals to the level curves of f are the directions in which f is decreasing on increasing the fastest. If this normal is not perpendicular to the curve $g(x, y) = c$, then it will have a component tangent to this curve.

Hence, the function $f(x, y)$ will be either increasing or decreasing at the point of intersection of the two curves as one goes along $g(x, y) = c$. This shows that the point of intersection cannot be an extreme point.

The gradient condition

From the discussion in the previous slide, it follows that the normals of the curves $f(x, y) = d$ and $g(x, y) = c$ must be parallel to each other at the extrema. We are thus looking for points (x_0, y_0) such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0),$$

subject to the constraint condition.

$$g(x_0, y_0) = c.$$

The λ that occurs above is called the **Lagrange multiplier**. The points (x_0, y_0) satisfying the above conditions are like the critical points we encounter in the unconstrained problem. We have to actually check whether they are extrema or not.

Checking for the extrema

To check for extrema all we have to do is to evaluate the function $f(x, y)$ at each of points (x_0, y_0) that we find after solving the equation above. This will enable us to decide whether a given point is a maximum or minimum.

Exercise: Find the maximum and minimum values of the function $f(x, y) = x^2 + 2y^2$ that lie on the circle $x^2 + y^2 = 1$.

Solution: We know that we must solve the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad x^2 + y^2 = 1,$$

where $g(x, y) = x^2 + y^2$. The first equation yields the pair of equations

$$2x = \lambda \cdot 2x \quad \text{and} \quad 4y = \lambda \cdot 2y.$$

The first equation above yields $x = 0$ or $\lambda = 1$. If $x = 0$, we must have $y = \pm 1$. If $\lambda = 1$, then $4y = 2y$ in the second equation, so $y = 0$ and $x = \pm 1$.

Thus we must evaluate $f(x, y)$ at the points $(\pm 1, 0)$ and $(0, \pm 1)$.
We obtain $f(1, 0) = 1$, $f(-1, 0) = 1$, $f(0, 1) = 2$, $f(0, -1) = 2$.
Thus the maximum value is 2 and the minimum 1.

The three variable problem

The same reasoning as before applies to the three variable constrained problem, that is, to find the maxima and minima of a function $f(x, y, z)$ subject to $g(x, y, z) = c$. If we are moving on the surface of g , this means that at any point, we are in the tangent plane of g . On the other hand, the directions in which f is stationary, are those perpendicular to its gradient.

It follows that we must once again solve the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = c,$$

to obtain the extreme points, and then evaluate the function $f(x, y, z)$ at these points.

Exercise: Find the dimensions of the box with the largest volume having a surface area of 150cm^2 .

Solution: The volume of the box is given by $f(x, y, z) = xyz$. This has to be maximized subject to the constraint that $g(x, y, z) = 2xy + 2yz + 2zx = 150$.

The condition $\nabla f = \lambda \nabla g$ yields

$$yz = \lambda(2y + 2z), \quad zx = \lambda(2z + 2x), \quad xy = \lambda(2x + 2y).$$

while the constraint condition give $xy + yx + zx = 75$.

Multiplying the previous equations by x , y and z respectively we get

$$xyz = 2\lambda x(y + z), \quad xyz = 2\lambda y(z + x), \quad xyz = 2\lambda z(x + y).$$

Equating the first two gives $\lambda = 0$ or $x = y$. But if $\lambda = 0$ we get $y = 0$ or $z = 0$, which is not possible. The equations above are symmetric in x , y and z , so if $x = y$ we must have $x = y = z$. This yields $3x^2 = 75$ or $x = y = z = 5$.

Rectangles

Any **rectangle** R in the plane can be described as the set of points in the cartesian product $[a, b] \times [c, d]$ of two closed intervals.

We can define the notion of a partition of a rectangle. How can this be done?

The easiest way is to take a partition P_1 of $[a, b]$ and a partition P_2 of $[c, d]$ and take the **product of the two partitions**. Thus if

$$P_1 = \{a = x_0, x_1, \dots, x_m = b\} \quad \text{and} \quad P_2 = \{c = y_0, y_1, \dots, y_n = d\},$$

we take the collection of points

$$P = \{(x_i, y_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

The point (x_i, y_j) is the left bottom corner of the rectangle $R_{ij} = (x_i, x_{i+1}) \times (y_j, y_{j+1})$. As i and j vary, we get a family of rectangles R_{ij} , $0 \leq i \leq m-1$, $0 \leq j \leq n-1$. By identifying each rectangle with its left bottom corner we can think of P as the collection of these rectangles R_{ij} . Clearly, $R = \cup_{i,j} R_{ij}$, and the collection of rectangles P is called a partition of R .

Partitions of rectangles and the integral

You may recall that we have several equivalent definitions of Riemann integration in one variable. We can model the notion of Riemann integration in two variables on any one of those. Clearly, the notion of upper and lower sums associated to a partition makes sense so we could define the Darboux integral as before.

We set $\Delta_{ij} = (x_{i+1} - x_i) \times (y_{j+1} - y_j)$, the area of the rectangle R_{ij} .

If $f : R \rightarrow \mathbb{R}$ is a function of two variables, define

$$m_{ij} = \inf_{(x,y) \in R_{ij}} f(x, y)$$

and the lower sum of f associated to the partition P by.

$$L(f, P) = \sum_{i,j}^{m-1,n-1} m_{ij} \Delta_{ij}.$$

We can similarly define the upper sums, and now it is clear that the lower and upper integrals, and hence also the Darboux integral can be defined.

The Riemann integral

A second approach arises by defining the norm of a partition of a rectangle. Again, this is easy. In the one variable case the norm of a partition was simply the length of the largest of the sub-intervals. What is the analogue for rectangles? Clearly, we could define

$$\|P\|_1 = \max_{0 \leq i \leq m-1, 0 \leq j \leq n-1} (x_{i+1} - x_i) \times (y_{j+1} - y_j).$$

As before, we can define a tagged partition (P, t) , where $t = \{t_{ij}\}_{i,j}$ is a collection of points such that $t_{ij} \in R_{ij}$.

The Riemann sum S associated to (P, t) is defined by

$$S(f, P, t) = \sum_{i,j}^{m-1,n-1} f(t_{ij}) \Delta_{ij}.$$

Now it should be clear how to define the Riemann integral: it is a number S such that for any $\epsilon > 0$, there is a δ such that

$$|S(f, P, t) - S| < \epsilon,$$

for every tagged partition (P, t) with the property that $\|P\|_1 \leq \delta$.

One problem with the previous definition

There is one problem with the definition made in the last slide. The norm that we have selected for partitions is not such a good one.

For instance, one could take $P_1 = \{0 < 1/2 < 1\}$ and $P_{2,n} = \{0 < 1/n, 2/n, \dots < 1\}$ and take $P_n = P_1 \times P_{2,n}$ as a partition of the unit square. In this case, clearly $\|P\| = 1/2n$ goes to zero as $n \rightarrow \infty$.

Exercise 1: Find a function $f(x, y)$ on the unit square and an $\epsilon > 0$ such that

$$|S(f, P_n, t) - V| > \epsilon,$$

for any tag t of P_n , where V is the volume of the solid region lying above the unit square and below the graph of $z = f(x, y)$ (note, there are very simple functions with the property).

Remedy

How to fix the problem? The problem is that the rectangles in our partition may be very thin in one direction but remain fat in the other one.

In order to avoid such situations, we have to change the definition of the norm that we have given. We have to select a norm so that when the norm of P is small, both sides of our rectangles are guaranteed to be small.

We define $\|P\| = \max_{i,j} \{(x_{i+1} - x_i), (y_{j+1} - y_j)\}$. Clearly this norm has the desired property and is the correct analogue of the norm of a partition in one variable.

Thus, to get the correct definition of a Riemann integral for a two variable function one must replace $\|P\|_1$ in the definition given above by $\|P\|$.

Regular Partitions

We will not use either of the above approaches, preferring instead the third approach. Recall that in Definition 2 of the one variable integral we saw that it was enough to restrict our attention to a fixed family of partitions. This is what we will do, taking a particularly simple family of partitions.

The **regular partition of R of order n** is a partition defined inductively by $x_0 = a$ and $y_0 = c$ and

$$x_{i+1} = x_i + \frac{b-a}{n} \quad \text{and} \quad y_{j+1} = y_j + \frac{d-c}{n},$$

$1 \leq i, j \leq n-1$. We take $t = \{t_{ij} \in R_{ij}\}$ to be an arbitrary tag.

Definition: We say that the function $f : R \rightarrow \mathbb{R}$ is **Riemann integrable** if the Riemann sum

$$S(f, P_n, t) = \sum_{i,j=0}^{n-1} f(t_{ij}) \Delta_{ij}$$

tends to a limit S for any choice of tag t .

The Riemann integral continued

This limit value is usually denoted as

$$\int \int_R f, \quad \int \int_R f(x, y) dA, \quad \text{or} \quad \int \int_R f(x, y) dx dy.$$

The preceding definition is sometimes rewritten as

$$\lim_{n \rightarrow \infty} \sum_{i,j=0}^{n-1} f(t_{ij}) \Delta_{ij} = \int \int_R f.$$

If $f(x, y) \geq 0$ for all values of x and y , then the Riemann integral has a geometric interpretation. It is obviously the volume of the region under the graph of the function $z = f(x, y)$ and above the rectangle R in xy -plane.

The integral may also be interpreted as mass in some physical situations; for example, if we have a rectangular plate and $f(x, y)$ represents the density of the plate at a given point, then the integral above gives the mass of the whole plate.

The main theorem

In the one variable case, we saw that a bounded function with at most a finite number of discontinuities on a closed bounded interval is Riemann integrable. The reason that a finite number of discontinuities do not matter is that points have length zero. What might be the analogous result in two variables?

In two variables the geometry of the set of points of discontinuity can be more complicated. Still, what are the analogues of points in this case? In other words what sets have “zero area”?

Theorem 33: If a function f is bounded and continuous on R except possibly along a finite number of graphs of \mathcal{C}^1 functions, then f is integrable on R .