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MA 105

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Syllabus

- Review of limits, continuity, differentiability.
- Mean value theorem, Taylor's theorem, maxima and minima.
- Riemann integrals, fundamental theorem of calculus, improper integrals, applications to area, volume.
- Convergence of sequences and series, power series.
- Partial derivatives, gradient and directional derivatives, chain rule, maxima and minima, Lagrange multipliers.
- Double and triple integration, Jacobians and change of variables formula.
- Parametrization of curves and surfaces, vector fields, line and surface integrals.
- Divergence and curl, theorems of Green, Gauss, and Stokes.

Texts/References

[Apo80] T.M. Apostol, Calculus, Volumes 1 and 2, 2nd ed., Wiley Eastern (1980).

[HGM03] Hughes-Hallett, Gleason, Mccallum et al., Calculus Single And Multivariable, 4th ed., John-Wiley and Sons (2003).

[Ste03] James Stewart, Calculus, 5th ed., Thomson (2003).

[TF98] G.B. Thomas and R.L. Finney, Calculus and Analytic Geometry, 9th ed., ISE Reprint, Addison-Wesley (1998).

Policy on Attendance

Students are expected to be present for all lectures and tutorial sessions. If you are absent due to medical reasons, you will need a certificate from the IIT hospital confirming that you were unwell on the relevant day(s). Surprise quizes will be held four times during the semester during the class or tutorial hours. These quizes will together amount to 6% (six percent) of the total marks for the semester.

Evaluation Plan

Four surprise quizes (6 marks)
First Quiz (12 marks) to be held before the mid-semester break.
Mid-semster exam (30 marks)
Second quiz (12 marks) to be held after the mid-semester break.
Final exam (40 marks)
The dates and times of the quizes and exams will be announced in class.

Tutorial sheet 1: Sequences, limits, continuity, differentiability

1. Using the (ϵ-N) definition of a limit, prove the following:

   (i) $\lim\limits_{n\to\infty} \dfrac{10}{n} = 0$

   (ii) $\lim\limits_{n\to\infty} \dfrac{5}{3n+1} = 0$

   (iii) $\lim\limits_{n\to\infty} \dfrac{n^{2/3}\sin(n!)}{n+1} = 0$

   (iv) $\lim\limits_{n\to\infty} \left(\dfrac{n}{n+1} - \dfrac{n+1}{n}\right) = 0$

2. Show that the following limits exist and find them:

   (i) $\lim\limits_{n\to\infty} \left(\dfrac{n}{n^2+1} + \dfrac{n}{n^2+2} + \cdots + \dfrac{n}{n^2+n}\right)$

   (ii) $\lim\limits_{n\to\infty} \left(\dfrac{n!}{n^n}\right)$

   (iii) $\lim\limits_{n\to\infty} \left(\dfrac{n^3+3n^2+1}{n^4+8n^2+2}\right)$

   (iv) $\lim\limits_{n\to\infty} (n)^{1/n}$

   (v) $\lim\limits_{n\to\infty} \left(\dfrac{\cos \pi\sqrt{n}}{n^2}\right)$

   (vi) $\lim\limits_{n\to\infty} \left(\sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)\right)$

3. Show that the following sequences are not convergent:

   (i) $\left\{\dfrac{n^2}{n+1}\right\}_{n\geq 1}$ (ii) $\left\{(-1)^n\left(\dfrac{1}{2} - \dfrac{1}{n}\right)\right\}_{n\geq 1}$

4. Determine whether the sequences are increasing or decreasing:

   (i) $\left\{\dfrac{n}{n^2+1}\right\}_{n\geq 1}$

   (ii) $\left\{\dfrac{2^n 3^n}{5^{n+1}}\right\}_{n\geq 1}$

   (iii) $\left\{\dfrac{1-n}{n^2}\right\}_{n\geq 2}$

5. Prove that the following sequences are convergent by showing that they are monotone and bounded. Also find their limits:

   (i) $a_1 = \dfrac{3}{2}, a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{2}{a_n}\right) \quad \forall\, n \geq 1$

   (ii) $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2+a_n}\ \forall\ n \geq 1$

   (iii) $a_1 = 2,\ a_{n+1} = 3 + \dfrac{a_n}{2}\ \forall\, n \geq 1$

6. If $\lim\limits_{n\to\infty} a_n = L$, find the following: $\lim\limits_{n\to\infty} a_{n+1}$, $\lim\limits_{n\to\infty} |a_n|$.

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2}, \ \forall\, n \geq n_0.$$

8. If $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, show that $\lim_{n\to\infty} a_n^{1/2} = 0$. State and prove a corresponding result if $a_n \to L > 0$.

9. For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following:

   (i) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.

   (ii) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

10. Show that a sequence $\{a_n\}_{n\geq 1}$ is convergent iff both the subsequences $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are convergent to the same limit.

11. Let $f, g : (a, b) \to \mathbb{R}$ be functions and suppose that $\lim_{x\to c} f(x) = 0$ for $c \in [a, b]$. Prove or disprove the following statements.

   (i) $\lim_{x\to c}[f(x)g(x)] = 0$.

   (ii) $\lim_{x\to c}[f(x)g(x)] = 0$, if g is bounded.

   (iii) $\lim_{x\to c}[f(x)g(x)] = 0$, if $\lim_{x\to c} g(x)$ exists.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to\alpha} f(x)$ exists for some $\alpha \in \mathbb{R}$. Show that

$$\lim_{h\to 0}[f(\alpha + h) - f(\alpha - h)] = 0.$$

   Analyze the converse.

13. Discuss the continuity of the following functions:

   (i) $f(x) = \sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$

   (ii) $f(x) = x\sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$

   (iii) $f(x) = \begin{cases} \dfrac{x}{[x]} & \text{if } \ 1 \leq x < 2, \\ 1 & \text{if } \ x = 2, \\ \sqrt{6-x} & \text{if } \ 2 < x \leq 3. \end{cases}$

14. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every $c \in \mathbb{R}$.

15. Let $f(x) = x^2\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?

16. Let $f : (a, b) \to \mathbb{R}$ be a function such that

$$|f(x+h) - f(x)| \leq C|h|^{\alpha}$$

for all $x, x+h \in (a, b)$, where C is a constant and $\alpha > 1$. Show that f is differentiable on (a, b) and compute $f^{'}(x)$ for $x \in (a, b)$.

17. If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then show that

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c-h)}{2h}$$

exists and equals $f^{'}(c)$. Is the converse true ? [Hint: Consider $f(x) = |x|$.]

18. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+y) = f(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

If f is differentiable at 0, then show that f is differentiable at every $c \in \mathbb{R}$ and $f^{'}(c) = f^{'}(0)f(c)$.

19. Using the theorem on derivative of inverse function, compute the derivative of (i) $\cos^{-1} x, \ -1 < x < 1$. (ii) $\operatorname{cosec}^{-1} x, \ |x| > 1$.

20. Compute $\dfrac{dy}{dx}$, given

$$y = f\left(\frac{2x-1}{x+1}\right) \text{ and } f^{'}(x) = \sin(x^2).$$

Supplement

1. A sequence $\{a_n\}_{n \geq 1}$ is said to be Cauchy if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon, \ \forall \, m, n \geq n_0$. In other words, if we choose n_0 large enough, we can make sure that the elements of a Cauchy sequence become as close to each other as want beyond n_0. One can show that a sequence in $\mathbb{R}$ is convergent if and only if it is Cauchy. To show that a convergent sequence is Cauchy is easy. To show that every Cauchy sequence converges is harder and, moreover, involves making a precise definition of the set of real numbers. The fact that every Cauchy sequence in $\mathbb{R}$ converges is called the completeness property of the real numbers.

2. To prove that a sequence $\{a_n\}_{n \geq 1}$ is convergent to a limit L, one needs to first guess what this limit L might be and then verify the required property. However the concept of 'Cauchyness' of a sequence is an intrinsic property, that is, we can decide whether a sequence is Cauchy by examining the sequence itself. There is no need to guess what the limit might be.

3. In problem 5(i), we defined

$$a_0 = 1, \ \ a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n}) \ \ \forall \, n \geq 1.$$

The sequence $\{a_n\}_{n \geq 1}$ is a monotonically decreasing sequence of rational numbers which is bounded below. However, it cannot converge to a rational (why?). This exhibits the need to enlarge the concept of numbers beyond rational numbers. The sequence $\{a_n\}_{n \geq 1}$ converges to $\sqrt{2}$ and its elements a_n's are used to find a rational approximation (in computing machines) of $\sqrt{2}$.

Optional Exercises:

1. Show that the function f in Question 14 satisfies $f(kx) = kf(x)$, for all $k \in \mathbb{R}$.

2. Show that in Question 18, f has a derivative of every order on $\mathbb{R}$.

3. Construct an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous everywhere and is differentiable everywhere except at 2 points.

4. Let $f(x) = \begin{cases} 1, & \text{if } \ x \text{ is rational,} \\ 0, & \text{if } \ x \text{ is irrational.} \end{cases}$ Show that f is discontinuous at every $c \in \mathbb{R}$.

5. Let $g(x) = \begin{cases} x, & \text{if } \ x \text{ is rational,} \\ 1-x, & \text{if } \ x \text{ is irrational.} \end{cases}$ Show that g is continuous only at $c = 1/2$.

6. Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$ be such that $\lim\limits_{x \to c} f(x) > \alpha$. Prove that there exists some $\delta > 0$ such that
$$f(c+h) > \alpha \ \text{ for all } \ 0 < |h| < \delta.$$

7. Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. Show that the following are equivalent:

   (i) f is differentiable at c.

   (ii) There exist $\delta > 0$ and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{h \to 0} \epsilon_1(h) = 0$ and
   $$f(c+h) = f(c) + \alpha h + h\epsilon_1(h) \text{ for all } h \in (-\delta, \delta).$$

   (iii) There exists $\alpha \in \mathbb{R}$ such that
   $$\lim_{h \to 0} \left(\frac{|f(c+h) - f(c) - \alpha h|}{|h|} \right) = 0.$$

8. Suppose f is a function that satisfies the equation $f(x+y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. Suppose also that
$$\lim_{x \to 0} \frac{f(x)}{x} = 1.$$
Find $f(0), f'(0), f'(x)$.

9. Suppose f is a function with the property that $|f(x)| \leq x^2$ for all $x \in \mathbb{R}$. Show that $f(0) = 0$ and $f'(0) = 0$.

10. Show that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point.

Tutorial sheet 2: Rolle's theorem, MVT, maxima/minima

1. Show that all the roots of the cubic $x^3 - 6x + 3$ are real.

2. Let p and q be two real numbers with $p > 0$. Show that the cubic $x^3 + px + q$ has exactly one real root.

3. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ are of different signs and $f^{'}(x) \neq 0$ for all $x \in (a, b)$, show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.

4. Consider the cubic $f(x) = x^3 + px + q$, where p and q are real numbers. If $f(x)$ has three distinct real roots, show that $4p^3 + 27q^2 < 0$ by proving the following:

   (i) $p < 0$.

   (ii) f has a local maximum/minimum at $\pm\sqrt{-p/3}$.

   (iii) The maximum/minimum values are of opposite signs.

5. Use the MVT to prove that $|\sin a - \sin b| \leq |a - b|$, for all $a, b \in \mathbb{R}$.

6. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1, c_2 in (a, b) such that $f^{'}(c_1) + f^{'}(c_2) = 2$.

7. Let $a > 0$ and f be continuous on $[-a, a]$. Suppose that $f^{'}(x)$ exists and $f^{'}(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$. Is it true that $f(x) = x$ for every x?

8. In each case, find a function f which satisfies all the given conditions, or else show that no such function exists.

   (i) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f^{'}(1) = 1$

   (ii) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f^{'}(1) = 2$

   (iii) $f^{''}(x) \geq 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f(x) \leq 100$ for all $x > 0$

   (iv) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f(x) \leq 1$ for all $x < 0$

9. Let $f(x) = 1 + 12|x| - 3x^2$. Find the global maximum and the global minimum of f on $[-2, 5]$. Verify it from the sketch of the curve $y = f(x)$ on $[-2, 5]$.

10. Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local maxima/minima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?

    (i) $y = 2x^3 + 2x^2 - 2x - 1$

    (ii) $y = 1 + 12|x| - 3x^2, \ x \in [-2, 5]$

11. Sketch a continuous curve $y = f(x)$ having all the following properties:

    $f(-2) = 8, \ f(0) = 4, \ f(2) = 0; \ \ f^{'}(2) = f^{'}(-2) = 0;$

    $f^{'}(x) > 0$ for $|x| > 2, \ f^{'}(x) < 0$ for $|x| < 2;$

    $f^{''}(x) < 0$ for $x < 0$ and $f^{''}(x) > 0$ for $x > 0$.

12. Give an example of $f : (0, 1) \to \mathbb{R}$ such that f is

    (i) strictly increasing and convex.

(ii) strictly increasing and concave.

(iii) strictly decreasing and convex.

(iv) strictly decreasing and concave.

13. Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in \mathbb{R}$. Define $h(x) = f(x)g(x)$ for $x \in \mathbb{R}$. Which of the following statements are true? Why?

    (i) If f and g have a local maximum at $x = c$, then so does h.

    (ii) If f and g have a point of inflection at $x = c$, then so does h.

    Additional Exercise

(14) Sketch the curve following the template of exercise 10 $y = \dfrac{x^2}{x^2+1}$

Tutorial sheet 3: Supplement on Taylor series

In this tutorial sheet, we will intersperse the exercises with the text, so you will have to read through the sheet somewhat carefully.

The Kerala School of Mathematics

In the fourteenth century CE, mathematicians in Kerala made a number of mathematical discoveries. Sangamagrāma Mādhavan (1350-1425 CE) appears to have been one of the founders of what is now known as the Kerala School of Mathematics, anticipating many of the later European discoveries. The following is an extract from Wikipedia (http://en.wikipedia.org/wiki/Madhava_of_Sangamagrama):
Among his many contributions, he discovered the infinite series for the trigonometric functions of sine, cosine, tangent and arctangent, and many methods for calculating the circumference of a circle. One of Madhava's series is known from the text Yuktibhāṣā, which contains the derivation and proof of the power series for inverse tangent, discovered by Madhava. In the text, Jyeṣṭhadeva describes the series in the following manner:
"The first term is the product of the given sine and radius of the desired arc divided by the cosine of the arc. The succeeding terms are obtained by a process of iteration when the first term is repeatedly multiplied by the square of the sine and divided by the square of the cosine. All the terms are then divided by the odd numbers $1, 3, 5, \ldots$. The arc is obtained by adding and subtracting respectively the terms of odd rank and those of even rank. It is laid down that the sine of the arc or that of its complement whichever is the smaller should be taken here as the given sine. Otherwise the terms obtained by this above iteration will not tend to the vanishing magnitude."

Exercise 1. *Write down the Taylor series for (i)* $\cos x$*, (ii)* $\arctan x$ *about the point* 0*. Write down a precise remainder term* $R_n(x)$ *in each case.*

Exercise 2. *Our examples of Taylor's series have usually been series about the point* 0*. Write down the Taylor series of the polynomial* $x^3 - 3x^2 + 3x - 1$ *about the point* 1*.*

Exercise 3. *What is the Taylor series of the function* $1729x^{1729} + 1728x^{1728} + 28x^{28} + 6x^6 + 1729$ *about the point* 0*?*

Power series

Exercise 4. *Consider the series* $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ *for a fixed* x*. Prove that it converges as follows. Choose* $N > 2x$*. We see that for all* $n > N$*,*

$$\frac{x^{n+1}}{(n+1)!} < \frac{1}{2} \cdot \frac{x^n}{n!}.$$

It should now be relatively easy to show that the given series is Cauchy, and hence (by the completeness of $\mathbb{R}$*), convergent.*

Taylor series (or more generally "power series") can be differentiated and integrated "term by term". That is if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{then} \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

And similarly,

$$\int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} a_n \int_a^b x^n dx.$$

We will not be proving these facts but you can use them below.

Exercise 5. *Using Taylor series write down a series for the integral*

$$\int \frac{e^x}{x} dx.$$

Optional Exercises

Exercise 6. *Use series to approximate* $\int_0^1 \sqrt{1+x^4}\, dx$ *correct to two decimal places.*

Exercise 7. *Show that the Taylor series of the function* $f(x) = \frac{x}{1-x-x^2}$ *is* $\sum_{n=1}^{\infty} f_n x^n$ *where* f_n *is the* n*th Fibonacci number, that is,* $f_1 = 1, f_2 = 1$, *and* $f_n = f_{n-1} + f_{n-2}$ *for* $n \geq 3$. *By writing* $f(x)$ *as a sum of partial fractions and thereby obtaining the Taylor series in a different way, find an explicit formula for the* n*th Fibonacci number.*

Exercise 8. *Write down the Taylor series for* $\tan x$ *about the point* 0 *(this is much harder than the examples in Exercise 1).*

Exercise 9. *Can you construct a smooth (infinitely differentiable) function which takes the constant value* 0 *outside the interval* $[-1, 2]$ *and the constant value* 1 *on the interval* $[0, 1]$.

Exercise 10. *Prove the irrationality of the number* $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ *as follows. First show that* $e < 3$ *by comparing with a suitable geometric series. By Taylor's theorem (applied to* $a = 0$ *and* $b = 1$*) we know that*

$$e - \sum_{k=0}^{n} \frac{1}{n!} =: R_n = e^{\alpha} \frac{1}{(n+1)!}$$

for some α *between* 0 *and* 1. *Since* $e < 3$, $R_n < \frac{3}{(n+1)!}$. *Now suppose* e *is a rational number* c/d, *where* c *and* d *have no common factors. For* $n = d$, *we see that* $d!R_d$ *is an integer. On the other hand, using the estimate for* R_d *that we have obtained using Taylor's Theorem,* $d!R_d < \frac{d! \times 3}{(d+1)!} < 1$, *if* $d > 2$. *If* $d = 2$, *then one can obviously verify directly that* e *cannot be the fraction* $5/2$.

Tutorial sheet 4: Riemann integration

1. Let $f(x) = 1$ if $x \in [0,1]$ and $f(x) = 2$ if $x \in (1,2]$. Show from the first principles that f is Riemann integrable on $[0,2]$ and find $\displaystyle\int_0^2 f(x)dx$.

2. (a) Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a,b]$. Show that $\displaystyle\int_a^b f(x)dx \geq 0$. Further, if f is continuous and $\displaystyle\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a,b]$.

   (b) Give an example of a Riemann integrable function on $[a,b]$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and $\displaystyle\int_a^b f(x)dx = 0$, but $f(x) \neq 0$ for some $x \in [a,b]$.

3. Evaluate $\lim\limits_{n\to\infty} S_n$ by showing that S_n is an approximate Riemann sum for a suitable function over a suitable interval:

   (i) $S_n = \dfrac{1}{n^{5/2}} \displaystyle\sum_{i=1}^{n} i^{3/2}$

   (ii) $S_n = \displaystyle\sum_{i=1}^{n} \frac{n}{i^2 + n^2}$

   (iii) $S_n = \displaystyle\sum_{i=1}^{n} \frac{1}{\sqrt{in + n^2}}$

   (iv) $S_n = \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} \cos \frac{i\pi}{n}$

   (v) $S_n = \dfrac{1}{n} \left\{ \displaystyle\sum_{i=1}^{n} \left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n} \left(\frac{i}{n}\right)^{3/2} + \sum_{i=2n+1}^{3n} \left(\frac{i}{n}\right)^{2} \right\}$

4. Compute

   (a) $\dfrac{d^2y}{dx^2}$, if $x = \displaystyle\int_0^y \frac{dt}{\sqrt{1+t^2}}$

   (b) $\dfrac{dF}{dx}$, if for $x \in \mathbb{R}$ (i) $F(x) = \displaystyle\int_1^{2x} \cos(t^2)dt$ (ii) $F(x) = \displaystyle\int_0^{x^2} \cos(t)dt$.

5. Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral $\displaystyle\int_a^{a+p} f(t)dt$ has the same value for every real number a. (Hint : Consider $F(a) = \displaystyle\int_a^{a+p} f(t)dt,\ a \in \mathbb{R}$.)

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R},\ \lambda \neq 0$. For $x \in \mathbb{R}$, let
$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x-t)dt.$$
   Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.

7. Find the area of the region bounded by the given curves in each of the following cases.

(i) $\sqrt{x} + \sqrt{y} = 1$, $x = 0$ and $y = 0$.

(ii) $y = x^4 - 2x^2$ and $y = 2x^2$.

(iii) $x = 3y - y^2$ and $x + y = 3$.

8. Let $f(x) = x - x^2$ and $g(x) = ax$. Determine a so that the region above the graph of g and below the graph of f has area 4.5.

9. Find the area of the region inside the circle $r = 6a\cos\theta$ and outside the cardioid $r = 2a(1 + \cos\theta)$.

10. Find the arc length of the each of the curves described below.

    (i) the cycloid $x = t - \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi$.

    (ii) $y = \displaystyle\int_0^x \sqrt{\cos 2t}\, dt,\ 0 \le x \le \pi/4$.

11. For the following curve
$$y = \frac{x^3}{3} + \frac{1}{4x},\ 1 \le x \le 3,$$
find the arc length as well as the the area of the surface generated by revolving it about the line $y = -1$.

12. The cross sections of a certain solid by planes perpendicular to the x-axis are circles with diameters extending from the curve $y = x^2$ to the curve $y = 8 - x^2$. The solid lies between the points of intersection of these two curves. Find its volume.

13. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$.

14. A fixed line L in 3-space and a square of side r in a plane perpendicular to L are given. One vertex of the square is on L. As this vertex moves a distance h along L, the square turns through a full revolution with L as the axis. Find the volume of the solid generated by this motion.

15. A round hole of radius $\sqrt{3}$ cms is bored through the center of a solid ball of radius 2 cms. Find the volume cut out.

Tutorial sheet 5: Functions of two variables, limits, continuity, partial derivatives

1. Find the natural domains of the following functions of two variables:

$$(i)\ \frac{xy}{x^2-y^2} \qquad (ii)\ \ln(x^2+y^2)$$

2. Describe the level curves and the contour lines for the following functions corresponding to the values $c = -3, -2, -1, 0, 1, 2, 3, 4$:
(i) $f(x,y) = x - y$ (ii) $f(x,y) = x^2 + y^2$ (iii) $f(x,y) = xy$

3. Using definition, examine the following functions for continuity at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:

$$(i)\ \frac{x^3y}{x^6+y^2} \quad (ii)\ xy\frac{x^2-y^2}{x^2+y^2} \quad (iii)\ ||x|-|y||-|x|-|y|.$$

4. Suppose $f,\ g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Show that each of the following functions of $(x,y) \in \mathbb{R}^2$ are continuous:
(i) $f(x) \pm g(y)$ (ii) $f(x)g(y)$ (iii) $\max\{f(x),\, g(y)\}$ (iv) $\min\{f(x),\, g(y)\}$.

5. Let

$$f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2} \text{ for } (x,y) \neq (0,0).$$

Show that the iterated limits

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] \ \&\ \lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right]$$

exist and both are equal to 0, but $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

6. Examine the following functions for the existence of partial derivatives at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero.

(i) $xy\dfrac{x^2-y^2}{x^2+y^2}$

(ii) $\dfrac{\sin^2(x+y)}{|x|+|y|}$

7. Let $f(0,0) = 0$ and

$$f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2} \text{ for } (x,y) \neq (0,0).$$

Show that f is continuous at $(0,0)$, and the partial derivatives of f exist but are not bounded in any disc (howsoever small) around $(0,0)$.

8. Let $f(0,0) = 0$ and

$$f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y), & \text{if } x \neq 0,\ y \neq 0 \\ x\sin 1/x, & \text{if } x \neq 0,\ y = 0 \\ y\sin 1/y, & \text{if } y \neq 0,\ x = 0. \end{cases}$$

Show that none of the partial derivatives of f exist at $(0,0)$ although f is continuous at $(0,0)$.

9. Examine the following functions for the existence of directional derivatives and differentiability at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:

$$(i)\;\; xy\frac{x^2-y^2}{x^2+y^2} \quad (ii)\;\; \frac{x^3}{x^2+y^2} \quad (iii)\;\; (x^2+y^2)\sin\frac{1}{x^2+y^2}$$

10. Let $f(x,y) = 0$ if $y = 0$ and

$$f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2} \text{ if } y \neq 0.$$

Show that f is continuous at $(0,0)$, $D_{\underline{u}}f(0,0)$ exists for every vector $\underline{u}$, yet f is not differentiable at $(0,0)$.

11. Show that the function $f(x,y) = \sqrt[3]{xy}$ is continuous and the partial derivatives f_x and f_y exist at the origin but the directional derivatives in all other directions do not exist.

Tutorial Sheet 6: Tangent Planes, Maxima/minima, saddle points, Lagrange multipliers

1. Find the points on the hyperboloid $x^2 - y^2 + 2z^2 = 1$ where the normal line is parallel to the line that joins the points $(3, -1, 0)$ and $(5, 3, 6)$.

2. Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin xy$ at the point $(1, 0)$ has the value 1.

3. Let $F(x, y, z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of F at $(1, -1, 3)$ and the equations of the tangent plane and the normal line to the surface $F(x, y, z) = 7$ at $(1, -1, 3)$.

4. Find $D_{\underline{u}}F(2, 2, 1)$, where $F(x, y, z) = 3x - 5y + 2z$, and $\underline{u}$ is the unit vector in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2, 2, 1)$.

5. Given $\sin(x + y) + \sin(y + z) = 1$, find $\dfrac{\partial^2 z}{\partial x \partial y}$, provided $\cos(y + z) \neq 0$.

6. If $f(0, 0) = 0$ and
$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2} \text{ for } (x, y) \neq (0, 0),$$
show that both f_{xy} and f_{yx} exist at $(0, 0)$, but they are not equal. Are f_{xy} and f_{yx} continuous at $(0, 0)$?

7. Show that the following functions have local minima at the indicated points.
(i) $f(x, y) = x^4 + y^4 + 4x - 32y - 7, \quad (x_0, y_0) = (-1, 2)$

(ii) $f(x, y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3, \quad (x_0, y_0) = (0, 0)$

8. Analyze the following functions for local maxima, local minima and saddle points:
(i) $f(x, y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$ (ii) $f(x, y) = x^3 - 3xy^2$

9. Find the absolute maximum and the absolute minimum of
$$f(x, y) = (x^2 - 4x)\cos y \text{ for } 1 \leq x \leq 3,\ -\pi/4 \leq y \leq \pi/4.$$

10. The temperature at a point (x, y, z) in 3-space is given by $T(x, y, z) = 400xyz$. Find the highest temperature on the unit sphere $x^2 + y^2 + z^2 = 1$.

11. Maximize the $f(x, y, z) = xyz$ subject to the constraints
$$x + y + z = 40 \text{ and } x + y = z.$$

12. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints
$$x + 2y + 3z = 6 \text{ and } x + 3y + 4z = 9.$$

Tutorial Sheet 7: Multiple integrals

1. For the following, write an equivalent iterated integral with the order of integration reversed:

   (i) $\displaystyle\int_0^1 \left[\int_1^{e^x} dy\right] dx$

   (ii) $\displaystyle\int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)dx\right] dy$

2. Evaluate the following integrals

   $\displaystyle\int_0^\pi \left[\int_x^\pi \frac{\sin y}{y} dy\right] dx$

   (ii) $\displaystyle\int_0^1 \left[\int_y^1 x^2 e^{xy} dx\right] dy$

   (iii) $\displaystyle\int_0^2 (\tan^{-1} \pi x - \tan^{-1} x) dx.$

3. Find $\displaystyle\iint_D f(x,y)d(x,y)$, where $f(x,y) = e^{x^2}$ and D is the region bounded by the lines $y = 0,\ x = 1$ and $y = 2x$.

4. Evaluate the integral
$$\iint_D (x-y)^2 \sin^2(x+y) d(x,y),$$
where D is the parallelogram with vertices at $(\pi, 0),\ (2\pi, \pi),\ (\pi, 2\pi)$ and $(0, \pi)$.

5. Let D be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1,\ xy = 9$ and the lines $y = x,\ y = 4x$. Find $\displaystyle\iint_D d(x,y)$ by transforming it to $\displaystyle\iint_E d(u,v)$, where $x = \dfrac{u}{v},\ y = uv,\ v > 0$.

6. Find
$$\lim_{r\to\infty} \iint_{D(r)} e^{-(x^2+y^2)} d(x,y),$$
where $D(r)$ equals:

   (i) $\{(x,y) : x^2 + y^2 \leq r^2\}$.

   (ii) $\{(x,y) : x^2 + y^2 \leq r^2,\ x \geq 0,\ y \geq 0\}$.

   (iii) $\{(x,y) : |x| \leq r,\ |y| \leq r\}$.

   (iv) $\{(x,y) : 0 \leq x \leq r,\ 0 \leq y \leq r\}$.

7. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ using double integral over a region in the plane. (Hint: Consider the part in the first octant.)

8. Express the solid $D = \{(x,y,z) | \sqrt{x^2+y^2} \leq z \leq 1\}$ as
$$\{(x,y,z) | a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x), \xi_1(x,y) \leq z \leq \xi_2(x,y)\}.$$

9. Evaluate
$$I = \int_0^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} x dz \right) dy \right) dx.$$
Sketch the region of integration and evaluate the integral by expressing the order of integration as $dxdydz$.

10. Using suitable change of variables, evaluate the following:

(i)
$$I = \iiint_D (z^2x^2 + z^2y^2) dxdydz$$
where D is the cylindrical region $x^2 + y^2 \leq 1$ bounded by $-1 \leq z \leq 1$.

(ii)
$$I = \iiint_D \exp(x^2 + y^2 + z^2)^{3/2} dxdydz$$
over the region enclosed by the unit sphere in $\mathbb{R}^3$.

Tutorial Sheet 8: Vector fields, parametrized curves

1. Let f, g be differentiable functions on $\mathbb{R}^2$. Show that

   (i) $\nabla(fg) = f\nabla g + g\nabla f$;

   (ii) $\nabla f^n = nf^{n-1}\nabla f$;

   (iii) $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ whenever $g \neq 0$.

2. Let $\mathbf{a}, \mathbf{b}$ be two fixed vectors, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r^2 = x^2 + y^2 + z^2$. Prove the following:

   (i) $\nabla(r^n) = nr^{n-2}\mathbf{r}$ for any integer n.

   (ii) $\mathbf{a} \cdot \nabla\left(\frac{1}{r}\right) = -\left(\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}\right)$.

   (iii) $\mathbf{b} \cdot \nabla\left(\mathbf{a} \cdot \nabla\left(\frac{1}{r}\right)\right) = \frac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{a} \cdot \mathbf{b}}{r^3}$.

3. Prove the following:

   (i) $\nabla \cdot (f\mathbf{v}) = f\nabla \cdot \mathbf{v} + (\nabla f) \cdot \mathbf{v}$

   (ii) $\nabla \times (f\mathbf{v}) = f(\nabla \times \mathbf{v}) + \nabla f \times \mathbf{v}$

   (iii) $\nabla \times \nabla \times \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - (\nabla \cdot \nabla)\mathbf{v}$, where $\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operator.

   (iv) $\nabla \cdot (f\nabla g) - \nabla \cdot (g\nabla f) = f\nabla^2 g - g\nabla^2 f$

   (v) $\nabla \cdot (\nabla \times \mathbf{v}) = 0$

   (vi) $\nabla \times (\nabla f) = 0$

   (vii) $\nabla \cdot (g\nabla f \times f\nabla g) = 0$.

4. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$. Show that

   (i) $\nabla^2 f = \text{div }(\nabla f(r)) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$

   (ii) $\text{div }(r^n\mathbf{r}) = (n+3)r^n$

   (iii) $\text{curl}(r^n\mathbf{r}) = 0$

   (iv) $\text{div }(\nabla\frac{1}{r}) = 0$ for $r \neq 0$.

5. Prove that

   (i) $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$
   Hence, if $\mathbf{u}, \mathbf{v}$ are irrotational, $\mathbf{u} \times \mathbf{v}$ is solenoidal.

   (ii) $\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v}$.

   (iii) $\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{v})$.

   Hint: Write $\nabla = \sum \mathbf{i}\frac{\partial}{\partial x}$, $\nabla \times \mathbf{v} = \sum \mathbf{i}\frac{\partial}{\partial x} \times \mathbf{v}$ and $\nabla \cdot \mathbf{v} = \sum \mathbf{i}\frac{\partial}{\partial x} \cdot \mathbf{v}$

6. (i) If $\mathbf{w}$ is a vector field of constant direction and $\nabla \times \mathbf{w} \neq 0$, prove that $\nabla \times \mathbf{w}$ is always orthogonal to $\mathbf{w}$.

   (ii) If $\mathbf{v} = \mathbf{w} \times \mathbf{r}$ for a constant vector $\mathbf{w}$, prove that $\nabla \times \mathbf{v} = 2\mathbf{w}$.

(iii) If $\rho\mathbf{v} = \nabla p$ where $\rho(\neq 0)$ and p are continuously differentiable scalar functions, prove that $\mathbf{v} \cdot (\nabla \times \mathbf{v}) = 0$.

7. Calculate the line integral of the vector field

$$f(x, y) = (x^2 - 2xy)\mathbf{i} + (y^2 - 2xy)\mathbf{j}$$

from $(-1, 1)$ to $(1, 1)$ along $y = x^2$.

8. Calculate the line integral of

$$f(x, y) = (x^2 + y^2)\mathbf{i} + (x - y)\mathbf{j}$$

once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in the counter clockwise direction.

9. Calculate the value of the line integral

$$\oint_C \frac{(x + y)dx - (x - y)dy}{x^2 + y^2}$$

where C is the curve $x^2 + y^2 = a^2$ traversed once in the counter clockwise direction.

10. Calculate

$$\oint_C ydx + zdy + xdz$$

where C is the intersection of two surfaces $z = xy$ and $x^2+y^2 = 1$ traversed once in a direction that appears counter clockwise when viewed from high above the xy-plane.

Tutorial Sheet 9: Line integrals

1. Consider the helix
$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\,\mathbf{k} \text{ lying on } x^2 + y^2 = a^2,$$
where $a > 0$. Parametrize this in terms of arc length.

2. Evaluate the line integral
$$\oint_C \frac{x^2y\,dx - x^3\,dy}{(x^2+y^2)^2},$$
where C is the square with vertices $(\pm 1, \pm 1)$ oriented in the counterclockwise direction.

3. Let $C : x^2 + y^2 = 1$. Find
$$\oint_C \text{grad } (x^2 - y^2) \cdot d\underline{s}.$$

4. Evaluate
$$\int_{(0,0)}^{(2,8)} \text{grad } (x^2 - y^2) \cdot d\underline{s},$$
where C is $y = x^3$.

5. Compute the line integral
$$\oint_C \frac{dx + dy}{|x| + |y|}$$
where C is the square with vertices $(1,0), (0,1)$, $(-1,0)$ and $(0,-1)$ traversed once in the counter clockwise direction.

6. A force $F = xy\mathbf{i} + x^6y^2\mathbf{j}$ moves a particle from $(0,0)$ onto the line $x = 1$ along $y = ax^b$ where $a, b > 0$. If the work done is independent of b find the value of a.

7. Calculate the work done by the force field $F(x,y,z) = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$ along the curve C of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$ where $z \geq 0, a > 0$ (specify the orientation of C that you use.)

8. Determine whether or not the vector field $f(x,y) = 3xy\mathbf{i} + x^3y\mathbf{j}$ is a gradient on any open subset of $\mathbb{R}^2$.

9. Let $S = \mathbb{R}^2 \setminus \{(0,0)\}$. Let
$$\mathbf{F}(x,y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} := f_1(x,y)\mathbf{i} + f_2(x,y)\mathbf{j}.$$
Show that $\dfrac{\partial}{\partial y} f_1(x,y) = \dfrac{\partial}{\partial x} f_2(x,y)$ on S while $\mathbf{F}$ is not the gradient of a scalar field on S.

10. For $\mathbf{v} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$, show that $\nabla\phi = \mathbf{v}$ for some ϕ and hence calculate $\displaystyle\oint_C \mathbf{v} \cdot d\underline{s}$ where C is any arbitrary smooth closed curve.

11. A radial force field is one which can be expressed as $\mathbf{F} = f(r)\mathbf{r}$ where $\mathbf{r}$ is the position vector and $r = \|\mathbf{r}\|$. Show that $\mathbf{F}$ is conservative if f is continuous.

Tutorial Sheet 10: Green's theorem

1. Verify Green's theorem in each of the following cases:

   (i) $f(x,y) = -xy^2$; $g(x,y) = x^2y$; $R:\ x \geq 0,\ 0 \leq y \leq 1 - x^2$;

   (ii) $f(x,y) = 2xy$; $g(x,y) = e^x + x^2$; where R is the triangle with vertices $(0,0), (1,0)$, and $(1,1)$.

2. Use Green's theorem to evaluate the integral $\oint_{\partial R} y^2\, dx + x\, dy$, where

   (i) R is the square with vertices $(0,0), (2,0), (2,2), (0,2)$.

   (ii) R is the square with vertices $(\pm 1, \pm 1)$.

   (iii) R is the disc of radius 2 and center $(0,0)$ (specify the orientation you use for the curve.)

3. For a simple closed curve given in polar coordinates show using Green's theorem that the area enclosed is given by
$$A = \frac{1}{2} \oint_C r^2 d\theta.$$
   Use this to compute the area enclosed by the following curves:

   (i) The cardioid: $r = a(1 - \cos\theta)$, $0 \leq \theta \leq 2\pi$;.

   (ii) The lemniscate: $r^2 = a^2 \cos 2\theta$,; $-\pi/4 \leq \theta \leq \pi/4$.

4. Find the area of the following regions:

   (i) The area lying in the first quadrant of the cardioid $r = a(1 - \cos\theta)$.

   (ii) The region under one arch of the cycloid
$$\mathbf{r} = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j},\ 0 \leq t \leq 2\pi.$$

   (iii) The region bounded by the limaçon
$$r = 1 - 2\cos\theta,\ \ 0 \leq \theta \leq \pi/2$$
   and the two axes.

5. Evaluate
$$\oint_C xe^{-y^2} dx + [-x^2ye^{-y^2} + 1/(x^2 + y^2)]dy$$
   around the square determined by $|x| \leq a$, $|y| \leq a$ traced in the counter clockwise direction.

6. Let C be a simple closed curve in the xy-plane. Show that

$$3I_0 = \oint_C x^3 dy - y^3 dx,$$

where I_0 is the polar moment of inertia of the region R enclosed by C.

7. Consider $a = a(x, y)$, $b = b(x, y)$ having continuous partial derivatives on the unit disc D. If

$$a(x, y) \equiv 1,\ b(x, y) \equiv y$$

on the boundary circle C, and

$$\mathbf{u} = a\mathbf{i} + b\mathbf{j};\ \mathbf{v} = (a_x - a_y)\mathbf{i} + (b_x - b_y)\mathbf{j}, \mathbf{w} = (b_x - b_y)\mathbf{i} + (a_x - a_y)\mathbf{j},$$

find

$$\iint_D \mathbf{u}\cdot\mathbf{v}\, d(x, y) \text{ and } \iint_D \mathbf{u}\cdot\mathbf{w}\, d(x, y).$$

8. Let C be any counterclockwise closed curve in the plane. Compute $\oint_C \nabla(x^2 - y^2)\cdot\mathbf{n} ds$.

9. Recall the Green's Identities:

   (i) $\displaystyle\iint_R \nabla^2 w\, d(x, y) = \oint_{\partial R} \frac{\partial w}{\partial \mathbf{n}}\, ds.$

   (ii) $\displaystyle\iint_R [w\nabla^2 w + \nabla w\cdot\nabla w]\, d(x, y) = \oint_{\partial R} w\frac{\partial w}{\partial \mathbf{n}}\, ds.$

   (iii) $\displaystyle\oint_{\partial R} \left(v\frac{\partial w}{\partial \mathbf{n}} - w\frac{\partial v}{\partial \mathbf{n}}\right) ds = \iint_R (v\nabla^2 w - w\nabla^2 v)\, d(x, y).$

   (a) Use (i) to compute

   $$\oint_C \frac{\partial w}{\partial \mathbf{n}}\, ds$$

   for $w = e^x \sin y$, and R the triangle with vertices $(0, 0), (4, 2), (0, 2)$.

   (b) Let D be a plane region bounded by a simple closed curve C and let $\mathbf{F}, \mathbf{G} : U \longrightarrow \mathbb{R}^2$ be smooth functions where U is a region containing $D \cup C$ such that

   $$\operatorname{curl}\mathbf{F} = \operatorname{curl}\mathbf{G},\ \operatorname{div}\mathbf{F} = \operatorname{div}\mathbf{G} \text{ on } D\cup C$$

   and

   $$\mathbf{F}\cdot\mathbf{n} = \mathbf{G}\cdot\mathbf{n} \text{ on } C,$$

   where $\mathbf{n}$ is the unit normal to the curve. Show that $\mathbf{F} = \mathbf{G}$ on D.

10. Evaluate the following line integrals where the loops are traced in the counter clockwise sense

(i)

$$\oint_C \frac{y\,dx - x\,dy}{x^2 + y^2}$$

where C is any simple closed curve not passing through the origin.

(ii)

$$\oint_C \frac{x^2 y dx - x^3 dy}{(x^2 + y^2)^2},$$

where C is the square with vertices $(\pm 1, \pm 1)$.

(iii) Let C be a smooth simple closed curve lying in the annulus $1 < x^2 + y^2 < 2$. Find

$$\oint_C \frac{\partial(\ln r)}{\partial y} dx - \frac{\partial(\ln r)}{\partial x} dy.$$

Tutorial Sheet 11: Parametrized surfaces

1. Find a suitable parametrization $\Phi(u, v)$ and the normal vector $\Phi_u \times \Phi_v$ for the following surface:

   (i) The plane $x - y + 2z + 4 = 0$.

   (ii) The right circular cylinder $y^2 + z^2 = a^2$.

   (iii) The right circular cylinder of radius 1 whose axis is along the line $x = y = z$.

2. (a) For a surface S let the unit normal $\mathbf{n}$ at every point make the same acute angle α with z-axis. Let SA_{xy} denote the area of the projection of S onto the xy plane. Show that SA, the area of the surface S satisfies the relation: $SA_{xy} = SA \cos\alpha$.
   (b) Let S be a parallelogram not parallel to any of the coordinate planes. Let S_1, S_2, and S_3 denote the areas of the projections of S on the three coordinate planes. Show that the ares of S is $\sqrt{S_1^2 + S_2^2 + S_3^2}$.

3. Compute the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ which lies within the cylinder $x^2 + y^2 = ay$, where $a > 0$.

4. A parametric surface S is described by the vector equation

$$\Phi(u, v) = u \cos v \, \mathbf{i} + u \sin v \, \mathbf{j} + u^2 \mathbf{k},$$

   where $0 \leq u \leq 4$ and $0 \leq v \leq 2\pi$.

   (i) Show that S is a portion of a surface of revolution. Make a sketch and indicate the geometric meanings of the parameters u and v on the surface.

   (ii) Compute the vector $\Phi_u \times \Phi_v$ in terms of u and v.

   (iii) The area of S is $\frac{\pi}{n}(65\sqrt{65} - 1)$ where n is an integer. Compute the value of n.

5. Compute the area of that portion of the paraboloid $x^2 + z^2 = 2ay$ which is between the planes $y = 0$ and $y = a$.

6. A sphere is inscribed in a right circular cylinder. The sphere is sliced by two parallel planes perpendicular the axis of the cylinder. Show that the portions of the sphere and the cylinder lying between these planes have equal surface areas.

7. Let S denote the plane surface whose boundary is the triangle with vertices at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Let $\mathbf{n}$ denote the unit normal to S having a nonnegative z-component. Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$, using

   (i) The vector representation $\Phi(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + (1 - 2u)\mathbf{k}$.

   –

   (ii) An explicit representation of the form $z = f(x, y)$.

8. If S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$, compute the value of the surface integral (with the choice of outward unit normal)

$$\iint_S xzdy \wedge dz + yzdz \wedge dx + x^2 dx \wedge dy.$$

Choose a representation in which the fundamental vector product points in the direction of the outward normal.

9. A fluid flow has flux density vector

$$\mathbf{F}(x, y, z) = x\mathbf{i} - (2x + y)\mathbf{j} + z\mathbf{k}.$$

Let S denote the hemisphere $x^2+y^2+z^2 = 1$, $z \geq 0$, and let $\mathbf{n}$ denote the unit normal that points out of the sphere. Calculate the mass of the fluid flowing through S in unit time in the direction of $\mathbf{n}$.

10. Solve the previous exercise when S includes the planar base of the hemisphere also with the outward unit normal on the base being $-\mathbf{k}$.

Tutorial Sheet 12: Stokes' theorem

1. Consider the vector field $\mathbf{F} = (x-y)\mathbf{i} + (x+z)\mathbf{j} + (y+z)\mathbf{k}$. Verify Stokes theorem for $\mathbf{F}$ where S is the surface of the cone: $z^2 = x^2 + y^2$ intercepted by
   (a) $x^2 + (y-a)^2 + z^2 = a^2 : z \geq 0$ (b) $x^2 + (y-a)^2 = a^2$

2. Evaluate using Stokes Theorem, the line integral
$$\oint_C yz\,dx + xz\,dy + xy\,dz$$
   where C is the curve of intersection of $x^2 + 9y^2 = 9$ and $z = y^2 + 1$ with clockwise orientation when viewed from the origin.

3. Compute
$$\iint_S (\mathrm{curl}\,\mathbf{v}) \cdot \mathbf{n} dS$$
   where $\mathbf{v} = y\mathbf{i} + xz^3\mathbf{j} - zy^3\mathbf{k}$ and $\mathbf{n}$ is the outward unit normal to S, the surface of the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = -3$.

4. Compute $\oint_C \mathbf{v} \cdot d\mathbf{r}$ for
$$\mathbf{v} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2},$$
   where C is the circle of unit radius in the xy plane centered at the origin and oriented clockwise. Can the above line integral be computed using Stokes Theorem?

5. Compute
$$\oint_C (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz,$$
   where C is the curve cut out of the boundary of the cube
$$0 \leq x \leq a,\ 0 \leq y \leq a,\ 0 \leq z \leq a$$
   by the plane $x + y + z = \frac{3}{2}a$ (specify the orientation of C.)

6. Calculate
$$\oint_C ydx + zdy + xdz,$$
   where C is the intersection of the surface $bz = xy$ and the cylinder $x^2 + y^2 = a^2$, oriented counter clockwise as viewed from a point high upon the positive z-axis.

7. Consider a plane with unit normal $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. For a closed curve C lying in this plane, show that the area enclosed by C is given by
$$A(C) = \frac{1}{2}\oint_C (bz - cy)dx + (cx - az)dy + (ay - bx)dz,$$
   where C is given the anti-clockwise orientation. Compute $A(C)$ for the curve C given by
$$\mathbf{u}\cos t + \mathbf{v}\sin t,\ 0 \leq t \leq 2\pi.$$

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Tutorial Sheet 13: Divergence theorem

1. Verify the Divergence Theorem for

$$\mathbf{F}(x,y,z) = xy^2\mathbf{i} + yz^2\mathbf{j} + zx^2\mathbf{k}$$

for the region

$$R\,:\,y^2+z^2 \leq x^2; 0 \leq x \leq 4.$$

2. Verify the Divergence Theorem for

$$\mathbf{F}(x,y,z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$$

for the region in the first octant bounded by the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

3. Let R be a region bounded by a piecewise smooth closed surface S with outward unit normal

$$\mathbf{n} = n_x\,\mathbf{i} + n_y\,\mathbf{j} + n_z\,\mathbf{k}.$$

Let $u, v\,:\,R \to \mathbb{R}$ be continuously differentiable. Show that

$$\iiint_R u\,\frac{\partial v}{\partial x} dV = -\iiint_R v\,\frac{\partial u}{\partial x} dV + \iint_{\partial R} u\,v\,n_x\,dS.$$

[Hint: Consider $\mathbf{F} = u\,v\,\mathbf{i}$.]

4. Suppose a scalar field ϕ, which is never zero has the properties

$$\|\nabla\phi\|^2 = 4\phi \text{ and } \nabla\cdot(\phi\nabla\phi) = 10\phi.$$

Evaluate $\displaystyle\iint_S \frac{\partial\phi}{\partial\mathbf{n}}\,dS$, where S is the surface of the unit sphere.

5. Let V be the volume of a region bounded by a closed surface S and $\mathbf{n} = (n_x, n_y, n_z)$ be its outer unit normal. Prove that

$$V = \iint_S x\,n_x\,dS = \iint_S y\,n_y\,dS = \iint_S z\,n_z\,dS$$

6. Compute $\displaystyle\iint_S (x^2 dy\wedge dz + y^2\,dz\wedge dx + z^2 dx\wedge dy)$, where S is the surface of the cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.

7. Compute $\displaystyle\iint_S yz\,dy\wedge dz + zx\,dz\wedge dx + xy\,dx\wedge dy$, where S is the unit sphere.

8. Let $\mathbf{u} = -x^3\mathbf{i} + (y^3 + 3z^2\sin z)\mathbf{j} + (e^y\sin z + x^4)\mathbf{k}$ and S be the portion of the sphere $x^2+y^2+z^2 = 1$ with $z \geq \frac{1}{2}$ and $\mathbf{n}$ is the unit normal with positive z-component. Use Divergence theorem to compute $\displaystyle\iint_S (\nabla\times\mathbf{u})\cdot\mathbf{n}\,dS$.

9. Let p denote the distance from the origin to the tangent plane at the point (x, y, z) to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Prove that

   (a) $\iint_S p\, dS = 4\pi abc.$

   (b) $\iint_S \frac{1}{p}\, dS = \frac{4\pi}{3abc}(b^2c^2 + c^2a^2 + a^2b^2)$

10. Interpret Green's theorem as a divergence theorem in the plane.