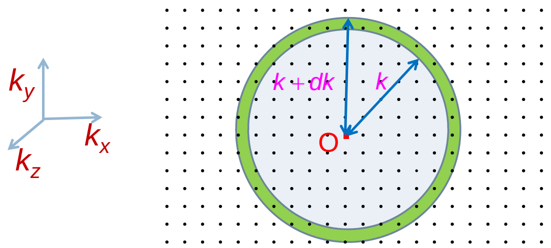
Density of State

- Considering that we have to handle a large number of particles, we change g_i to a continuous variable, called Density of State.
- The density of states $g(E)dE$ are defined as the number of states between energy E and $E+dE$.

We define a variable k as follows

$$E = \frac{\hbar^2}{2m}\left(k_x{}^2 + k_y{}^2 + k_z{}^2\right) = \frac{\hbar^2 k^2}{2m}$$

$$k_i = \frac{n_i \pi}{L};\ i = x, y \text{ or } z$$



Simple number of points

$$g(k)dk = \frac{1}{8}\left(\frac{L^3}{\pi}\right) \times 4\pi k^2 dk$$

Consider other degeneracy factor f.

$$g(k)dk = \frac{f}{8} \times \left(\frac{L^3}{\pi}\right) 4\pi k^2 dk$$

$$g(k)dk = f \times \left(\frac{L}{\pi}\right)^3 \frac{\pi k^2}{2} dk$$

$$E = \frac{\hbar^2 k^2}{2m}$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$dk = \frac{\sqrt{2m}}{\hbar}\frac{1}{2}E^{-\frac{1}{2}}dE$$

Substitute in the first equation.

$$g(k)dk = f \times \left(\frac{L}{\pi}\right)^3 4\pi k^2 dk$$

$$g(E) = \frac{f}{8} \times \left(\frac{L}{\pi}\right)^3 4\pi \frac{2mE}{\hbar^2}\sqrt{\frac{2m}{\hbar^2}}\left(\frac{1}{2}E^{-\frac{1}{2}}\right)$$

$$= f\left(\frac{V}{4\pi^2}\right)\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} E^{\frac{1}{2}}$$

^{4}He, A Boson

Spin Angular Momentum for ^{4}He is Zero and it is a Boson. The factor *f* in D.O.S. is *one*. Consider *N* molecules of such a gas, the mass of each is *m*. Let us imagine that the states of particle in a box are getting occupied by these molecules.

Total number of molecules between energy *E* and *E+dE* is given as follows.

$$n(E)dE = \frac{g(E)dE}{e^{\left(\frac{E}{kT}+\alpha\right)} - 1}$$

We have to determine the value of α, now.

$$n(E)dE = \frac{\left(\frac{V}{4\pi^2}\right)\times\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}dE}{e^{\left(\frac{E}{kT}+\alpha\right)}-1}$$

$$N = \int_0^\infty n(E)dE$$

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$$\frac{N}{V} = \left(\frac{1}{4\pi^2}\right)\times\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{E^{1/2}dE}{e^{\left(\frac{E}{kT}+\alpha\right)}-1}$$

The above integral would give the value of α at a particular temperature.

$$\frac{N}{V} = \left(\frac{1}{4\pi^2}\right)\times\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{E^{1/2}dE}{e^{\left(\frac{E}{kT}+\alpha\right)}-1}$$

The left side of this integral is constant, while right side can depend on temperature, unless temperature dependence of α cancels that out.

As temperature is lowered, α should also become lower. However, the least value of α is zero. Let this happen at a temperature $T = T_c$.

Let us calculate the value of T_c.

Let us first put α=0 in this equation and try to solve it.

$$\frac{N}{V}=\left(\frac{1}{4\pi^2}\right)\times\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty\frac{E^{1/2}dE}{e^{\left(\frac{E}{kT_c}\right)}-1}$$

Let us try the following substitution.

$$\frac{E}{kT_c}=x$$

$$E^{1/2}=(kT_c)^{1/2}x^{1/2}$$

$$dE=(kT_c)dx$$

$$E=0\Rightarrow x=0$$

$$E\to\infty\Rightarrow x\to\infty$$

The integral now becomes as follows. As this is valid only for α=0, we put $T=T_c$.

$$\frac{N}{V}=\left(\frac{1}{4\pi^2}\right)\times\left(\frac{2mkT_c}{\hbar^2}\right)^{3/2}\int_0^\infty\frac{x^{1/2}dx}{e^x-1}$$

The integral value can be obtained from standard Tables and is given as follows.

$$\int_0^\infty\frac{x^{1/2}dx}{e^x-1}=2.612\times\frac{\sqrt{\pi}}{2}$$

We thus get the following expression, which gives us a value of T_c.

$$\frac{N}{V} = \frac{2.612}{8} \times \left(\frac{2mkT_c}{\pi\hbar^2}\right)^{3/2}$$
$$= 0.3265 \times \left(\frac{2mkT_c}{\pi\hbar^2}\right)^{3/2}$$

The density of liquid He is around 140 kg/m^3.

$$\frac{N}{V} \approx \frac{6.02 \times 10^{26}}{4} \times 140$$
$$= 2.1 \times 10^{28}\,\mathrm{m}^{-3}$$

Substituting this number we get $T_c \approx 3.1\,\mathrm{K}$.