MA 105 D1 Lecture 6

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Towards Taylor's Theorem - higher derivatives

We will now introduce some notation. The space $\mathcal{C}^k(I)$, will denote the space of k times continuously differentiable functions on an interval I, for some fixed $k \in \mathbb{N}$, that is, the space of functions for which k derivatives exist and such that the k-th derivative is a continuous function.

The space $\mathcal{C}^\infty(I)$ will consist of functions that lie in $\mathcal{C}^k(I)$ for every $k \in \mathbb{N}$. Such functions are called smooth or infinitely differentiable functions.

From now on we will denote the k-th derivative of a function $f(x)$ by $f^{(k)}(x)$.

Our aim will be to enlarge the class of functions we understand using the polynomials as stepping stones.

The Taylor polynomials

Given a function $f(x)$ which is n times differentiable at some point "a" in an interval I, we can associate to it a family of polynomials $P_0(x), P_1(x), \ldots P_n(x)$ called the Taylor polynomials of degrees $0, 1, \ldots n$ at "a" as follows. We let

$$P_0(x) = f(a),$$

$$P_1(x) = f(a) + f^{(1)}(a)(x-a),$$

$$P_2(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2,$$

We can continue in this way to define

$$P_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor's Theorem

The Taylor polynomials at a depend on a, so we should really be writing $P_{k,a}(x)$ rather than $P_k(x)$, but we omit the extra subscript a so that our notation does not get too complicated.

The Taylor polynomials are rigged exactly so that the degree n Talyor polynomial has the same first n derivatives at the point a as the function $f(x)$ has, that is, $P_n^{(k)}(a) = f^{(k)}(a)$ for all $0 \leq k \leq n$, where $f^{(0)}(x) = f(x)$ by convention.

Theorem 19: Let $f \in \mathcal{C}^n([a,b])$ and suppose that $f^{(n+1)}$ exists on (a,b). Then there exists $c \in (a,b)$ such that

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

It is customary to denote the function $f(b) - P_n(b)$ by $R_n(b)$. Taylor's Theorem gives us a simple formula for $R_n(b)$. If we can make $R^n(b)$ small, we can approximate our function $f(x)$ by a polynomial.

The proof of Taylor's theorem

Proof: Consider the function

$$F(x) = f(b) - f(x) - f^{(1)}(x)(b-x) - \frac{f^{(2)}(x)}{2!}(b-x)^2 - \ldots - \frac{f^{(n)}(x)}{n!}(b-x)^n.$$

Clearly $F(b) = 0$, and

$$F^{(1)}(x) = -\frac{f^{(n+1)}(x)(b-x)^n}{n!}. \tag{1}$$

We would like to apply Rolle's Theorem here, but $F(a) \neq 0$. So consider

$$g(x) = F(x) - \left(\frac{b-x}{b-a}\right)^{n+1} F(a)$$

(this exactly the same method by which we reduced the MVT to Rolle's Theorem), and we see that $g(a) = 0$. Applying Rolle's Theorem we see that there is a $c \in (a, b)$ such that $g'(c) = 0$.

This yields

$$F^{(1)}(c) = -(n+1)\left(\frac{(b-c)^n}{(b-a)^{n+1}}\right)F(a). \tag{2}$$

We can eliminate $F^{(1)}(c)$ using (1). This gives

$$-(n+1)\left(\frac{(b-c)^n}{(b-a)^{n+1}}\right)F(a) = -\frac{f^{(n+1)}(c)(b-c)^n}{n!},$$

from which we obtain

$$F(a) = \frac{(b-a)^{n+1}}{(n+1)!}f^{(n+1)}(c).$$

This proves what we want. $\square$

Remarks on Taylor's Theorem and some examples

Remark 1: When $n = 0$ in Taylor's Theorem we get the MVT. When $n = 1$, Taylor's Theorem is called the Extended Mean Value Theorem.

Remark 2: The Taylor polynomials are nothing but the partial sums of the Taylor Series associated to a $\mathcal{C}^\infty$ function about (or at) the point a:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(b-a)^k.$$

We can show that this series converges provided we know that the difference $f(x) - P_n(x) = R_n(x)$ can be made less than any $\epsilon > 0$ when n is sufficiently large. We will see how to do this for certain simple functions like e^x or $\sin x$.

The Taylor series for e^x

Let us show that the Taylor series for the function e^x about the point 0 is a convergent series for any value of $x = b \geq 0$ and that it converges to the value e^b (a similar proof works for $b < 0$).

In this case, at any point a, $f^{(n)}(a) = e^a$, so at $a = 0$ we obtain $f^{(n)}(0) = 1$. Hence the series about 0 is

$$\sum_{k=0}^{\infty} \frac{b^k}{k!}.$$

If we look at $R_n(b) = e^b - s_n(b)$ we obtain

$$|R_n(b)| = \frac{e^c b^{n+1}}{(n+1)!} \leq \frac{e^b b^{n+1}}{(n+1)!},$$

since $c \leq b$. As $n \to \infty$ this clearly goes to 0. This shows that the Taylor series of e^b converges to the value of the function at each real number b.

Defining functions using Taylor series

Instead of finding the Taylor series of a given function we can reverse the process and define functions using convergent series. Thus, one can define the function e^x as

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

In this case, we have to first show that the series on the right hand side converges for a given value of x, in which case the definition above makes sense.

We show the convergence of such series by showing that they are Cauchy series. This means that we do not have to guess at a value of the limit.

Power series

As we have explained in the previous slide the “correct” (both from the point of view of proofs and of computation) way to define a function like e^x is via convergent series involving non-negative integer powers of x. Such series are called power series and such functions should be viewed as the natural generalizations of polynomials.

The nice thing about power series is that once we know that they converge in some interval $(a - r, a + r)$ around a, it is not hard to show that the functions that they define are continuous functions. In fact, it is not too hard to show that they are smooth functions (that is, that all their derivatives exist). Thus when functions are given by convergent power series, we can automatically conclude they are smooth. This is the advantage of defining functions in this way.

Calculating the values of functions

As we have also mentioned several times, calculators and computers calculate the values of various common functions like trigonometric polynomials and expressions in $\log x$ and e^x by using Taylor series.

The great advantage of Taylor series is that one can estimate the error since we have a simple formula for the error which can be easily estimated. For instance, for the function $\sin x$, the n-th derivative is either $\pm \sin x$ or $\pm \cos x$, so in either case $|f^{(n}(x)| \leq 1$. Hence,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

If we take $x = 1$, and we want to compute $\sin 1$ to an error of less than 10^{-16}, we need only make sure that $(n+1)! > 10^{16}$, which is achieved when $n \geq 21$.

Supplementary exercises

Exercise 1: To how many terms must you compute the Taylor series of sin in order to make sure that it approximates the value of sin to within 10^{-32} anywhere on the real line.

Exercise 2: Assume that $f(x)$ is a $\mathcal{C}^\infty$ function on $[a, b]$. Let $x_0 \in (a, b)$. Is it possible that the Taylor series of $f(x)$ about x_0 does not converge to $f(x)$ in any open interval around x_0?