MA 105 D1 Lecture 19

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Curves and paths

So far I have been using the words "path" and "curve" interchangeably and somewhat carelessly. We will need to be more careful from now on.

We recall that a curve is just a function $\gamma : [a, b] \to \mathbb{R}^3$ while the corresponding path denotes its range or image as a subset of $\mathbb{R}^3$. Thus, two different curves may have the same path. For instance the curves $\gamma_1 : [0, 1/2] \to \mathbb{R}^3$ given by $\gamma_1(t) = (2tx, 2ty, 2tz)$ and $\gamma_2 : [0, 1] \to \mathbb{R}^3$ given by $\gamma_1(t) = (tx, ty, tz)$ describe the same path: namely a straight line from the origin to the point $P = (x, y, z)$.

The point is curves may have the same initial and final points but describe the same trajectory at different speeds. In this case, the path doesn't change.

Reparametrisation

Let $\mathbf{c}(t) : [t_1, t_2]$ be a regular parametrised curve.

Suppose we now make change of variables $t = h(u)$, where h is $\mathcal{C}^1$ diffeomorphism (recall that this means that h is bijective, $\mathcal{C}^1$ and so is its inverse) from $[u_1, u_2]$ to $[t_1, t_2]$. We let $\gamma(u) = \mathbf{c}(h(u))$. Then γ is called a reparametrisation of $\mathbf{c}$. We will assume that $h(u_i) = t_i$ for $i = 1, 2$.

Because h is a diffeomorphism, γ is also a regular parametrised curve.

The line integral of a vector field $\mathbf{F}$ along γ is given by

$$\int_\gamma \mathbf{F}\cdot d\mathbf{s} = \int_{u_1}^{u_2} \mathbf{F}(\gamma(u))\cdot\gamma'(u)du = \int_{u_1}^{u_2} \mathbf{F}(c(h(u))\cdot\mathbf{c}'(h(u))h'(u)du,$$

where the last equality follows from the chain rule. Using the fact that $h'(u)du = dt$, we can change variables from u to t to get

$$\int_\gamma \mathbf{F}\cdot d\mathbf{s} = \int_{t_1}^{t_2} \mathbf{F}(\mathbf{c}(t))\cdot\mathbf{c}'(t)dt = \int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s}.$$

Different parametrisations of the same path

In the one example we gave so far of a path with two different parametrisations the domains of the two parametrised curves were different. Here is an example of a simple path with three different parametrisations with the same domain.

Take the straight line segment between $(0,0,0)$ and $(1,0,0)$.

Here are three different ways of parametrising it:

$$\{t,0,0)\}, \quad \{(t^2,0,0)\} \quad \text{and} \quad \{(t^3,0,0)\},$$

where $0 \le t \le 1$.

In the next slide we will see that there is one privileged choice of parametrisation that is most useful.

The arc length parametrisation

The argument just made shows that the line integral is independent of the choice of parametrisation - it depends only on the path of the non-singular parametrised curve.

There is a natural choice of parametrisation we can make which is useful in many situations. This is the parametrisation by arc length. To this end we set

$$s(t) = \int_a^t \|\mathbf{c}'(u)\| du,$$

where $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is a non-singular curve, from which it follows that $ds = \|\mathbf{c}'(t)\| dt$. Hence,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{T}(t) ds,$$

where $\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}$ is the unit tangent vector along the curve. In other words, the line integral is nothing but the (Riemann) integral of the tangential component of $\mathbf{F}$ with respect to arc length.

Non-singular curves

The preceding example tells us why we assumed that $\mathbf{c}'(t) \neq 0$: notice that this quantity occurs in the denominator of $\mathbf{T}(t)$. The assumption, however, is not essential for much of our theory to work.

For instance, if we take the curve $y = x^{1/3}$, it is certainly not regular at $x = 0$, nor is its parametrisation $(t, t^{1/3})$. However, it does admit a smooth parametrisation - namely (t^3, t)! Notice that in this case $\mathbf{c}'(t) \neq 0$ for all t.

On the other hand the semi-cubical parabola $y = x^{2/3}$ does not admit a smooth parametrisation, so we cannot do away with our assumption altogether, and it is best to just assume it throughout.

Integration of scalar functions long paths

The arc length parametrisation allows us to define the integral of scalar functions along paths. If $f(x, y, z)$ is a continuous function on some domain D in $\mathbb{R}^3$ and $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is a non-singular parametrisation as before, we can define the path integral of f along $\mathbf{c}$ as

$$\int_{\mathbf{c}} f ds := \int_a^b f(x(t), y(t), z(t)) \|\mathbf{c}'(t)\| dt,$$

where $\mathbf{c}(t) = (x(t), y(t), z(t))$.

An example

Example (Marsden, Weinstein and Tromba page 371): Let $\mathbf{c}$ be the helix $(\cos t, \sin t, t)$, $0 \leq t \leq 2\pi$, and let $f(x, y, z) = x^2 + y^2 + z^2$. Evaluate

$$\int_{\mathbf{c}} f(x, y, z) ds.$$

Solution: We must first find $\|\mathbf{c}'(t)\|$:

$$\|\mathbf{c}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

On the curve $\mathbf{c}(t)$, we see that

$$f(x, y, z) = \cos^2 t + \sin^2 t + t^2 = 1 + t^2.$$

Hence,

$$\int_{\mathbf{c}} f(x, y, z) ds = \int_0^{2\pi} (1 + t^2)\sqrt{2} dt = \frac{2\sqrt{2}\pi}{3}(3 + 4\pi^2).$$

Curvature

The arc length parametrisation also allows us to introduce another very useful quantity.

Definition: The curvature κ of a non-singular parametrised curve $\mathbf{c}(t)$ is the rate of change of arc length parametrisation:

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|.$$

Using the chain rule we see that

$$\kappa = \frac{\|d\mathbf{T}/dt\|}{|ds/dt|} = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{c}'(t)\|}.$$

What is the curvature of a circle of radius r?

Note that the reciprocal $R(s) = 1/\kappa(s)$ is called the radius of curvature at s.

Another formula for curvature

Here is an alternate formula for curvature for plane curves. First note that

$$\mathbf{T} = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|},$$

implies that

$$\mathbf{c}'(t) = \frac{ds}{dt}\mathbf{T}.$$

Further

$$\mathbf{c}''(t) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\mathbf{N}\left(\frac{ds}{dt}\right)^2,$$

where $\mathbf{N}(s)$ is the unit normal vector. Taking cross products, we see that

$$\kappa = \frac{\|\mathbf{c}'(t) \times \mathbf{c}''(t)\|}{\|\mathbf{c}'(t)^3\|}.$$

Orientation

There is one further piece of information that a curve carries and that the path (its image) does not. Given two points P and Q in $\mathbb{R}^3$ and a path connecting them, we can ask whether the path is traversed from P to Q or from Q to P.

Since a curve, say $\mathbf{c} : [a, b] \to \mathbb{R}^3$ comes with the information that $\mathbf{c}(a) = P$ while $\mathbf{c}(b) = Q$, (or vice-versa) it allows us to determine the direction in which the path is being traversed.

Note that by composing $\mathbf{c}$ with the function $b + a - t$, we get a new curve $\mathbf{d}(t) = \mathbf{c}(b + a - t)$ which traverses the path in the opposite direction. In this case, the line integral for $\mathbf{F}$ along $\mathbf{d}$ is

$$\int_a^b \mathbf{F}(\mathbf{d}(t))\cdot\mathbf{d}'(t)dt = \int_a^b \mathbf{F}(\mathbf{c}(b+a-t))\cdot\mathbf{d}'(t)dt = \int_b^a \mathbf{F}(\mathbf{c}(u))\cdot\mathbf{c}'(u)du,$$

since we know that $\mathbf{d}'(t) = -\mathbf{c}'(b + a - t)$. Hence,

$$\int_{\mathbf{d}} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$

Oriented curves

Thus, for a conservative field, the line integral along a closed curve is 0.

In view of the discussion above, we see that for a simple closed path lying in a plane, we get a choice of direction - clockwise or anti-clockwise. Making such a choice gives rise to an oriented curve. By convention, the positive orientation corresponds to the anti-clockwise direction and the negative orientation to the clockwise direction.

If the curve does not necessarily lie on a plane, the usual convention is the following:
We will say that the curve is positively oriented if the surface bounded by the curve always lies to the left of an observer walking along the curve in the chosen direction. Otherwise, we will say that the curve is negatively oriented. Note, that this agrees with our convention for planar curves above.

Reparametrisation and orientation

When we calculated the effect of reparametrisation on the line integral a few slides ago, we assumed that $h(u_1) = t_1$ and $h(u_2) = t_2$ (go back and check!) In other words, we assumed that the parametrisation was orientation preserving.

If a reparametrisation does not change the orientation, we say that it is orientation preserving. Otherwise, we say that it is orientation reversing. When we calculated the effect of parametrisation on the line integral a few slides ago, we assumed that the parametrisation was orientation preserving (go back and check!).

With our calculation above, we can make a more refined statement. Orientation preserving parametrisations do not change the line integral. Orientation reversing ones, change the sign of the line integral.

Ampère's Law

We have noted that the line integral can be interpreted as the work done when a particle moves along a path in a force field.

Suppose $\mathbf{H}$ denotes the magnetic field in $\mathbb{R}^3$ and γ is a closed (oriented) curve in $\mathbb{R}^3$. Then Ampère's Law states that

$$\int_\gamma \mathbf{H} \cdot d\mathbf{s} = I,$$

where I is the current passing through the surface bounded by γ.

If $\mathbf{V}$ denotes the velocity field of a fluid, the quantity

$$\int_\gamma \mathbf{V} \cdot d\mathbf{s}$$

is an important quantity in fluid mechanics which is called the circulation of $\mathbf{V}$ around γ.