PH-105 Assignment Sheet - 2 (Quantum Mechanics)

Umang Mathur

1. **Show that the Einstein's expression of specific heat gives a value of 3R at high temperatures and tends to zero as temperature tends to zero.**

   Solution :

   Einsteins' expression of energy is given as follows:

   $$\varepsilon = 3N_A \frac{h\nu_E}{e^{\frac{h\nu_E}{kT}} - 1}$$

   Hence, Specific heat (C_v), is given by

   $$\begin{aligned} C_v &= \frac{\partial \varepsilon}{\partial T} \\ &= 3R(\frac{h\nu_E}{kT})^2 \frac{e^{\frac{h\nu_E}{kT}}}{(e^{\frac{h\nu_E}{kT}} - 1)^2} \end{aligned}$$

   Let $\frac{h\nu_E}{kT} = \alpha$, then we have

   $$C_v = 3R\frac{\alpha^2 e^\alpha}{(e^\alpha - 1)^2}$$

   Now, as $T \to \infty$, $\alpha \to 0$, hence,

   $$\begin{aligned} \lim_{T\to\infty} C_v &= 3R \lim_{\alpha\to 0} \frac{\alpha^2 e^\alpha}{(e^\alpha - 1)^2} \\ &= 3R \lim_{\alpha\to 0} \frac{2\alpha e^\alpha + \alpha^2 e^\alpha}{2e^\alpha(e^\alpha - 1)} \\ &= 3R \lim_{\alpha\to 0} \frac{2\alpha + \alpha^2}{2e^\alpha - 2} \\ &= 3R \lim_{\alpha\to 0} \frac{1+\alpha}{e^\alpha} \\ \therefore C_v &= 3R \end{aligned}$$

   Again, as $T \to 0$, $\alpha \to \infty$, hence,

   $$\begin{aligned} \lim_{T\to 0} C_v &= 3R \lim_{\alpha\to\infty} \frac{\alpha^2 e^\alpha}{(e^\alpha - 1)^2} \\ &= 3R \lim_{\alpha\to\infty} \frac{2\alpha e^\alpha + \alpha^2 e^\alpha}{2e^\alpha(e^\alpha - 1)} \\ &= 3R \lim_{\alpha\to\infty} \frac{2\alpha + \alpha^2}{2e^\alpha - 2} \\ &= 3R \lim_{\alpha\to\infty} \frac{1+\alpha}{e^\alpha} \\ &= 3R \lim_{\alpha\to\infty} e^{-\alpha} \\ \therefore C_v &= 0 \end{aligned}$$