PH-105 Assignment Sheet - 3 (Quantum Mechanics - 2)

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63. **A particle in a one-dimensional well ($V = 0$ for $0 < x < L$, $V = \infty$ elsewhere) has the wave function $\phi(x) = Ax(L-x)$ inside the box and $\phi(x) = 0$ elsewhere at t=0. Calculate the expectation value of energy. On making an energy measurement, what is the probability of finding the particle in the ground state ?**

<u>**Solution**</u> :
The expectation value of energy can be calculated as:

$$\langle E\rangle = \langle \phi^*(x)|\hat{H}|\phi(x)\rangle = \int_0^L \phi^*(x)\Big(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\big(Ax(L-x)\big)\Big)dx = \frac{A^2\hbar^2L^3}{6m}$$

Also, normalizing $\phi(x)$, we have,

$$\langle \phi^*(x)|\phi(x)\rangle = \int_0^L A^2x^2\big(L-x\big)^2 dx = 1, \text{ which gives } A = \sqrt{\frac{30}{L^5}}$$

Thus, $\langle \mathbf{E}\rangle = \mathbf{\frac{5\hbar^2}{mL^2}}$.
For a particle in an infinite well, the eigenvalues of energy operator are given by $\psi_n(x) = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$.
Hence, the probability of finding a particle in ground state is given by

$$\big(\langle\psi_1(x)|\phi(x)\rangle\big)^2 = \Big(\int_0^L \sqrt{\frac{2}{L}}\sin(\frac{\pi x}{L})\big(Ax(L-x)\big)\Big)^2$$

Noting that $A = \sqrt{\frac{30}{L^5}}$, $\int z\sin(az)dz = -\frac{z\cos(az)}{a} + \frac{\sin(az)}{a^2}$, and that $\int z^2\sin(az)dz = \frac{2z\sin(az)}{a^2} - \frac{(2+a^2z^2)\cos(az)}{a^3}$, we have

$$\mathbf{P(ground\ state)} = \big(\langle\psi_1(x)|\phi(x)\rangle\big)^2 = \Big(\sqrt{60}\frac{4}{\pi^3}\Big)^2 = \frac{960}{\pi^6} = 0.9985$$