MA 105 D1 Lecture 9

Ravi Raghunathan

Department of Mathematics

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Tutorial Problem 4.4

Exercise 4.4 Compute

(a) $\frac{d^2y}{dx^2}$, if

$$x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$$

(b) $\frac{dF}{dx}$, if for $x \in \mathbb{R}$

$$(i) F(x) = \int_1^{2x} \cos(t^2) dt$$

and

$$(ii) F(x) = \int_0^{x^2} \cos(t) dt.$$

Problem 4.5

Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral

$$\int_a^{a+p} f(t)dt$$

has the same value for every real number a.

(Hint: Consider $F(a) = \int_a^{a+p} f(t)dt$.)

Problem 4.6

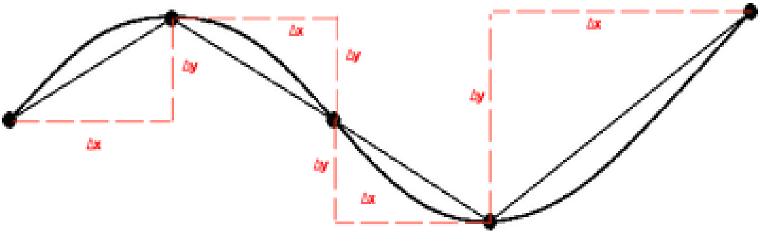
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R},\ \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x - t) dt.$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.

Arc length

The picture below and the discussion on the next slide are from from Wikipedia (http://en.wikipedia.org/wiki/Arc_length).



See: http://en.wikipedia.org/wiki/File:Arclength-2.png

In the picture above, the curve $y = f(x)$ is being approximated by straight line segments which form the hypotenuses of the right angled triangles shown in the picture.

The formula for arc length

Let us denote the arc length of the curve $y = f(x)$ by S. The length of any given hypotenuse in the previous slide is given by the Pythagorean Theorem: $\sqrt{\Delta x^2 + \Delta y^2}$.
Intuitively, the sum of the lengths of the n hypotenuses appears to approximate S:

$$S \sim \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i,$$

where "$\sim$" means approximately equal. We can use this idea to define the arc length as

$$S := \lim_{\Delta x_i \to 0} \sum_{i=1}^{\infty} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

provided this limit exists (in particular, we demand that the limit is a finite number).

Exercise 4.10.(ii) Find the length of the curve

$$y(x) = \int_0^x \sqrt{\cos 2t}\, dt, \; 0 \le x \le \pi/4.$$

Solution: The formula for the arc length of a curve $y = f(x)$ between the points $x = a$ and $x = b$ is given by

$$\int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

For the problem at hand this gives

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 2x} dx = \sqrt{2} \int_0^{\pi/4} \cos(x) dx = 1.$$

Rectifiable curves

Not all curves have finite arc length! Here is an example of a curve with infinite arc length.

Example: Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the curve given by $\gamma(t) = (t, f(t))$, where

$$f(t) = \begin{cases} t\cos\left(\frac{\pi}{2t}\right), & \text{if } t \neq 0 \\ 0 \text{ if } t = 0 \end{cases}$$

If
http://math.stackexchange.com/questions/296397/nonrectifiable-curve
is correct, you should be able to check that this curve has infinite arc length. Try it as an exercise.

Notice that the curve above is given by a continuous function. Curves for which the arc length S is finite are called rectifiable curves. You can easily check that the graphs of piecewise $\mathcal{C}^1$ functions are rectifiable.

Things can get even stranger

In fact, there exist space filling curves, that is curves $\gamma : [0,1] \to [0,1] \times [0,1]$ which are continuous and surjective. Obviously the graph of this curve "fills up" the entire square. Such curves are not rectifiable (can you prove this?)

The existence of such curves should make you question whether your intuitive notion of dimension actually has any mathematical basis. If a line segment can be mapped continuously onto a square, is it reasonable to say that they have different dimensions? After all, this means we can describe any point on the square using just one number.

We will answer this question (without a proof) later in this course. We will also come back to arc length of a curve when studying multivariable calculus.

Functions with range contained in $\mathbb{R}$

We will be interested in studying functions $f : \mathbb{R}^m \to \mathbb{R}$, when $m = 2, 3$. We have already mentioned how limits of such functions can be studied in the first few lectures. Before doing this in detail, however, we will study certain other features of functions in two and three variables.

The most basic thing one needs to understand about a function is the domain on which it is defined. Very often a function is given by a formula which makes sense only on some subset of $\mathbb{R}^m$ and not on the whole of $\mathbb{R}^m$. When studying functions of two or more variables given by formulae it makses sense to first identify this subset, which is sometimes call the natural domain of the function, and to describe it geometrically if possible.

Exercise 5.1: Find the natural domains of the following functions:
(i) $\frac{xy}{x^2-y^2}$

Clearly this function is defined whenever the denominator is not zero, in other words when $x^2 - y^2 \neq 0$.
The natural domain is thus

$$\mathbb{R}^2 \setminus \{(x, y) \mid x^2 - y^2 = 0\},$$

that is, $\mathbb{R}^2$ minus the pair of straight lines with slopes ± 1.
(ii) $f(x, y) = \log(x^2 + y^2)$
This function is defined whenever $x^2 + y^2 \neq 0$, in other words, in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Level curves and contour lines

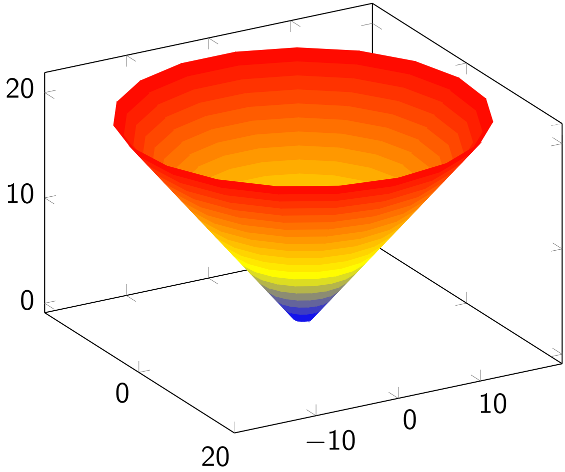
The second thing one should do with a function from $\mathbb{R}^2 \to \mathbb{R}$ is to study its range. This is done in different ways.

One way is to study the level sets of the functions. These are the sets of the form $f(x, y) = c$, where c is a constant. The level set "lives" in the xy-plane.

One can also plot (in three dimensions) the surface $z = f(x, y)$. By varying the value of c in the level curves one can get a good idea of what the surface looks like.

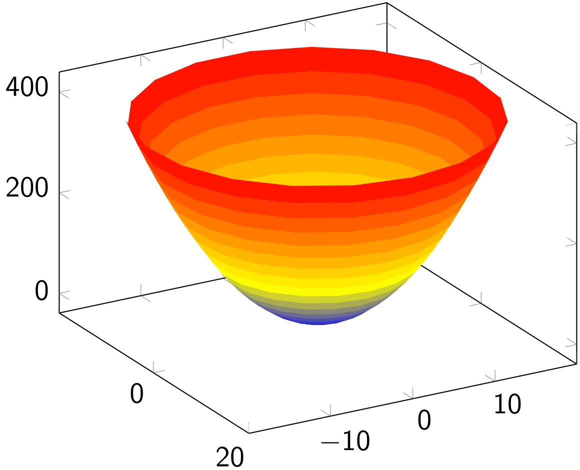
When one plots the $f(x, y) = c$ for some constant c one gets a curve. Such a curve is usually called a contour line (the contour "lives" in the $z = c$ plane).

I have a couple of pictures in the next two slides to illustrate the point.



This is the graph of the function $z = \sqrt{x^2 + y^2}$ lying above the xy-plane. It is a right circular cone.

The contour lines $z = c$ give circles lying on planes parallel to the xy-plane. The curves given by $z = f(x, 0)$ and $z = f(0, y)$ give pairs of straight lines in the planes $y = 0$ and $x = 0$.



This is the graph of the function $z = x^2 + y^2$ lying above the xy-plane. It is a paraboloid of revolution.
The contour lines $z = c$ give circles lying on planes parallel to the xy-plane. The curves $z = f(x, 0)$ or $z = f(y, 0)$ give parabolæ lying in the planes $y = 0$ and $x = 0$. Exercise 5.2.(ii).

Limits

We have already said what it means for a function of two or more variables to approach a limit. We simply have to replace the absolute value function on $\mathbb{R}$ by the distance function on $\mathbb{R}^m$. We will do this in two variables. The three variable definition is entirely analogous. We will denote by U a set in $\mathbb{R}^2$.

Definition: A function $f : U \to \mathbb{R}$ is said to tend to a limit l as $x = (x_1, x_2)$ approaches $c = (c_1, c_2)$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - l| < \epsilon,$$

whenever $0 < \|x - c\| < \delta$.

We recall that

$$\|x\| = \sqrt{x_1^2 + x_2^2}.$$

Continuity

Before talking about continuity we remark the following. In the plane $\mathbb{R}^2$ it is possible to approach the point c from infinitely many different directions - not just from the right and from the left. In fact, one may not even be approaching the point c along a straight line! Hence, to say that a function from $\mathbb{R}^2$ to $\mathbb{R}$ possesses a limit is actually imposing a strong condition - for instance, the limits along all possible curves leading to the point must exist and all these (infinitely many) limits must be equal.

Once we have the notion of a limit, the definition of continuity is just the same as for functions of one variable.

Definition: The function $f : U \to \mathbb{R}$ is said to be continuous at a point c if

$$\lim_{x \to c} f(x) = f(c).$$

The rules for limits and continuity

The rules for addition, subtraction, multiplication and division of limits remain valid for functions of two variables (or three variables for that matter). Nothing really changes in the statements or the proofs.

Using these rules, we can conclude, as before, that the sum, difference, product and quotient of continuous functions are continuous (as usual we must assume that the denominator of the quotient is non zero).

Continuity through examples

Once again, we emphasise that continuity at a point c is a very powerful condition (since the existence of a limit is implicit).

Exercise 5.3.(i) asks whether the function

$$f(x,y) = \begin{cases} \frac{x^3y}{x^6+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Solution: Let us look at the sequence of points $z_n = (\frac{1}{n}, \frac{1}{n^3})$, which goes to 0 as $n \to \infty$. Clearly $f(z_n) = \frac{1}{2}$ for all n, so

$$\lim_{n\to\infty} f(z_n) = \frac{1}{2} \neq 0.$$

This shows that f is not continuous at 0.
But does the limit exist?

Iterated limits

When evaluating a limit of the form $\lim_{(x_1,x_2)\to(c_1,c_2)} f(x_1,x_2)$ one may naturally be tempted to let x_1 go to c_1 first, and then let x_2 go to c_2. Does this give the limit in the previous sense?

Exercise 5.5: Let

$$f(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2}.$$

we have

$$\lim_{x\to 0}\lim_{y\to 0} \frac{x^2y^2}{x^2y^2 + (x-y)^2} = \lim_{x\to 0} 0 = 0$$

Similarly, one has $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0$.

However, choosing $z_n = (\frac{1}{n}, \frac{1}{n})$, shows that $f(z_n) = 1$ for all $n \in \mathbb{N}$. Now choose $z_n = (\frac{1}{n}, \frac{1}{2n})$ to see that the limit cannot exist.

Partial Derivatives

As before, U will denote a subset of $\mathbb{R}^2$. Given a function $f : U \to \mathbb{R}$, we can fix one of the variables and view the function f as a function of the other variable alone. We can then take the derivative of this one variable function.

To make things precise, fix x_2.
Definition: The partial derivative of $f : U \to \mathbb{R}$ with respect to x_1 at the point (a, b) is defined by

$$\frac{\partial f}{\partial x_1}(a, b) := \lim_{x_1 \to a} \frac{f((x_1, b)) - f((a, b))}{x_1 - a}.$$

Similarly, one can define the partial derivative with respect to x_2. In this case the variable x_1 is fixed and f is regarded only as a function x_2:

$$\frac{\partial f}{\partial x_2}(a, b) := \lim_{x_2 \to b} \frac{f((a, x_2)) - f((a, b))}{x_2 - b}.$$

General comments on the marking scheme

If the True/False component of the question was answered correctly, you have received one mark.

The other mark(s) were given for giving a proper reason/counter-example.

The correct answers were FFFTT.

I have not had enough time to go through the data, but I think Q4. was the hardest, followed by Q5. and then Q3.

Once I have all the data, I will post the mean and median scores on moodle/my home page.

Question 1

If the sequence a_n is convergent and b_n is monotonically increasing and bounded, then there exists some $N \in \mathbb{N}$ such that $a_n b_n < a_{n+1} b_{n+1}$ for all $n > N$.

FALSE. Consider the sequence $a_n = (-1)^n/n$ and $b_n = 1$. The sequence $\{a_n\}$ is convergent and $\{b_n\}$ is monotone and bounded, but there is no $N \in \mathbb{N}$ such that $a_n b_n < a_{n+1} b_{n+1}$ for all $n > N$.

Marking scheme:

- No marks if answer was given to be TRUE.
- If answer is FALSE, 1 mark was awarded
- Further, if the student gives a correct counterexample, full marks were awarded.

Some students have given very complicated counter-examples without any justification. In almost all cases they made a mistake and the counter-example was not a counter-example at all. In any event, if it was not easily verifiable by your TA that the counter-example was indeed one, you may not have received a mark.

Question 2

The function $x^6 + 3x + 1$ has exactly two roots in the interval $[-2, -1]$.

FALSE. If the polynomial has two roots in the interval $[-2, -1]$, by Rolle's theorem, the derivative of the polynomial would have a root in the interval $(-2, -1)$. But the derivative is $6x^5 + 3$, the only real root of which lies outside $(-2, -1)$.
Marking scheme:

- No marks if answer was given to be TRUE.
- If answer is FALSE, 1 mark was awarded.
- Further, if the proof was correct (possibly by another method), full marks were awarded.

Many students simply argued that since $f(-2)f(-1) < 0$, the function must cross the x-axis an odd number of times in the interval. Others showed that the function was a monotonically decreasing function in the interval, and hence, its graph could cut the x-axis only once etc. No marks were awarded if students only said that the function was convex/$f''(x) > 0$.

Question 3

The largest value of δ such that the inequality $|x^2 - 1| < 10^{-100}$ holds for all $x \in (1 - \delta, 1 + \delta)$ is $\delta = 10^{-101}$.

FALSE. For any $h \in (-\delta, \delta)$, we have $|(1+h)^2 - 1| = |2h + h^2| \leq |h|(2 + |h|) < \delta(2+\delta)$. Even if we take $\delta = (1/3)10^{-100} > 10^{-101}$, then $\delta(2+\delta) < 10^{-100}$. Hence, 10^{-101} is not the largest such δ

- No marks if answer was given to be TRUE.
- If answer is FALSE, 1 mark was awarded.
- If the student gave an explicit $\delta > 10^{-101}$ (like $(1/3)10^{-100}$ in the solution above) for which the inequality holds, 1/2 mark was awarded.
- If the student also proves that this explicit δ is correct, award full marks.

Many students gave $\delta = 5 \times 10^{-101}$ as the largest δ. This is incorrect (the actual value is just less than this). Many other attempted to use the binomial theorem but in almost all cases their argument were wrong/incomplete.

Question 4

$$\left(\frac{\pi}{22}\right)\cos\left(\frac{\pi}{22}\right)+\left(\frac{2\pi}{11}\right)\cos\left(\frac{5\pi}{22}\right)+\left(\frac{2\pi}{11}\right)\cos\left(\frac{9\pi}{22}\right)+\left(\frac{\pi}{22}\right)\cos\left(\frac{5\pi}{11}\right.$$

$$<\left(\frac{\pi}{26}\right)+\left(\frac{3\pi}{13}\right)\cos\left(\frac{\pi}{26}\right)+\left(\frac{3\pi}{13}\right)\cos\left(\frac{7\pi}{26}\right)$$

[1+2]

TRUE. Let $f : [0, \pi/2] \to \mathbb{R}$ be the cosine function. Consider the partitions $P_1 = \{0 < \pi/22 < 5\pi/22 < 9\pi/22 < 5\pi/11 < \pi/2\}$ and $P_2 = \{0 < \pi/26 < 7\pi/26 < \pi/2\}$. Then, the LHS is just the lower sum $L(f, P_1)$ and the RHS is the upper sum $U(f, P_2)$. Since every lower sum is at most every upper sum, we have the inequality.

Marking scheme:

- No marks if the answer was given to be FALSE.
- If answer is TRUE, 1 mark was awarded
- If the proof attempted to use upper and lower sums to prove the bound, 1 additional mark was awarded.
- If the proof was correct, full marks were awarded.

Question 4, continued

Very few students did this problem correctly. One or two succeeded without using upper and lower sums, by sheer brute force, but most attempts without lower and upper sums were incorrect.

Question 5

Let $P_2(x)$ denote the Taylor polynomial of degree 2 about the point $a = 0$ for the function $\log(1 + x)$. The inequality $|f(x) - P_2(x)| < 0.05$ holds for all x in $[0, \frac{1}{2}]$.

TRUE. Fix any $x \in [0, 1/2]$. By Taylor's theorem, we have $|f(x) - P_2(x)| = |f^{(3)}(y)x^3/3!| = |x^3/3(1 + y)^3|$ for some $y \in [0, x]$. Since $x \in [0, 1/2]$, we see that the error term is at most $1/(3 \cdot 2^3) = 1/24 < 0.05$. Hence, the inequality is true for all $x \in [0, 1/2]$.

Marking scheme:

- No marks if the answer was FALSE.
- If the answer is TRUE, 1 mark was awarded.
- If Taylor's theorem was applied correctly (i.e. the remainder term was given as $f^{(3)}(y)x^3/3!$), then 1 mark was awarded.
- If the rest of the calculation is correct, full marks were awarded.

Question 5

Many students wrote the remainder term as $f^{(3)}(x)x^3/3!$. This is incorrect - the point y lies in $(0, x)$. One or two student used a different method showing that $f(x) - P_2(x)$ was a monotonically decreasing function and proceeding further.