PH108

Lecture 09:

Electrostatic energy

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Recall our simple definition of electrostatic energy

When a test charge is moved in an electric field from *a* to *b*

Work done = Force * distance:

$$W = \int_a^b q_2 \vec{E} \cdot \overrightarrow{dl} = q_2[\Phi(b) - \Phi(a)]$$

Work done in moving per unit charge ≡ Energy

If $\Phi(r)$ is due to a point charge q_1 and *a* is ∞

$$Energy = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}$$

For *n* point charges the expression is simple

For *n point* charges:

$$Work\ done = \frac{1}{4\pi\epsilon_0}\frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{r_{ij}}$$

$$Work\ done = \frac{1}{2}\sum_{i=1}^{n} q_i \Phi(r_i)$$

Potential at r_i due to all charges *except* q_i

Generalize to continuous charge $\rho(r)$:

$$\boxed{Energy = \frac{1}{2}\int \rho d\tau \Phi}$$

Creation of energy: pulling unlike sign charges apart

Video credit MIT 8.02T TEAL http://goo.gl/1173Bc

5 + and 5 – charges are located at $x=0$. Some force pulls the + charges away to $x=a$. Work done? Energy storage?

Generalize to a continuous charge distribution ρ

$$Energy = \frac{1}{2}\int \rho \Phi d\tau$$

$$= \frac{1}{2}\int \epsilon_0 (\vec{\nabla} \cdot \vec{E}) \Phi d\tau$$ Poisson's equation

$$= \frac{\epsilon_0}{2}\left[\int_\tau \left(-\vec{E} \cdot \vec{\nabla}\Phi\right) d\tau + \oint \Phi \vec{E} \cdot d\vec{\sigma} \right]$$ Integration by parts

Any surface that bounds V

$$= \frac{\epsilon_0}{2}\left[\int_\tau E^2 d\tau + \oint \Phi \vec{E} \cdot d\vec{\sigma} \right]$$

Any volume V that encloses ρ

Energy of a charged sphere of radius R

Take volume = sphere of radius $a > R$

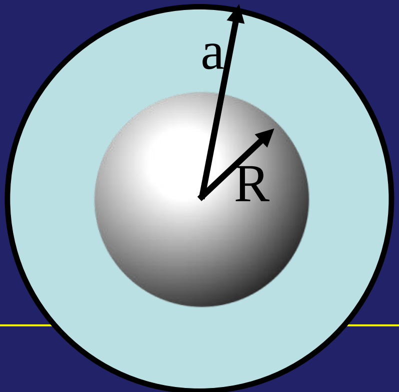
$$Energy = \frac{\epsilon_0}{2}\left[\int_\tau E^2 d\tau + \oint \Phi\vec{E}\cdot d\vec{\sigma}\right]$$

volume *surface*

$$\frac{\epsilon_0}{2}\int_0^R \left(\frac{\rho r}{3\epsilon_0}\right)^2 4\pi r^2 dr + \frac{\epsilon_0}{2}\int_R^a \left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 4\pi r^2 dr \qquad \epsilon_0 \frac{q}{4\pi\epsilon_0 a}\frac{q}{4\pi\epsilon_0\, a^2}4\pi a^2$$

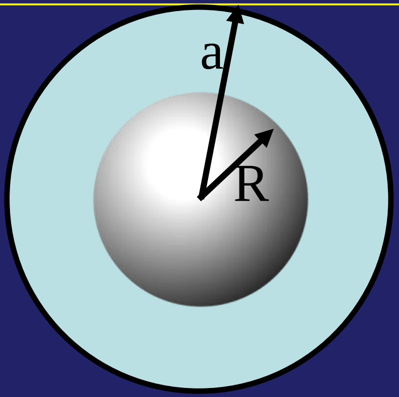
$$\frac{1}{2}\frac{q^2}{4\pi\epsilon_0\, 5R} + \frac{1}{2}\frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{R} - \cancel{\frac{1}{a}}\right) \qquad \cancel{\frac{1}{2}\frac{q^2}{4\pi\epsilon_0\, a}}$$

$$\boxed{\frac{3}{5}\frac{q^2}{4\pi\epsilon_0 R}}$$



There is a problem with energy of a point charge

$$\boxed{Energy = \frac{3}{5}\frac{q^2}{4\pi\epsilon_0 R}}$$



Recall: energy is stored in the *Field*

As R *decreases*, Energy *increases*

In the limit $R \to 0$ (i.e. point charge), Energy $\to \infty$!

Can R be allowed to $\to 0$? NO

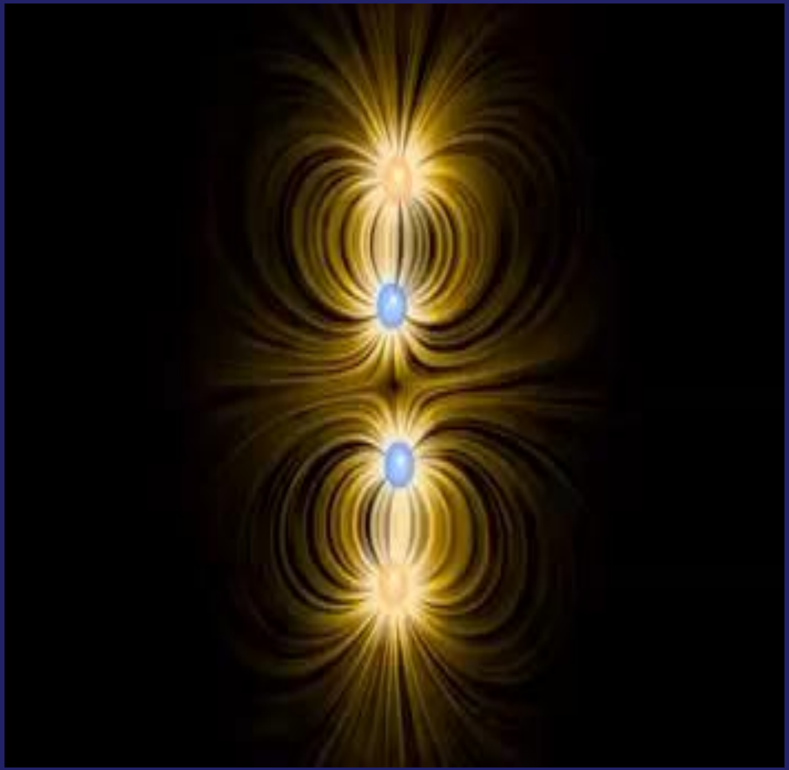
When $E \geq 2m_ec^2$, Energy is sufficient to 'create' e^+e^- (QM)

$$Classical\ radius\ of\ electron\ R_e = \frac{3}{5}\frac{q^2}{4\pi\epsilon_0 2m_ec^2}$$

Electric force (classical) + quantum effects hold matter together

Hydrogen atom: 1 +, 1 – charge.

What keeps the charges apart?



Video credit MIT 8.02T TEAL http://goo.gl/TWSdxW

Hydrogen molecule: H_2

Both atoms have net charge = 0. What keeps them together?

What are the practical features of electrostatics?

We have seen that in a *conductor* in static/steady state all charge resides on the surface

If charges were free in ‘empty’ space they would collapse to a point, with no net charge anywhere.
But they don’t – field energy & quantum mechanics keeps them a minimum distance apart.

Apart from tutorial problems and transients,
in practice $\rho = 0$

Laplace's equation becomes the most commonly used tool to find Φ

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$ Poisson equation

For ρ=0 reduces to $\boxed{\nabla^2 \Phi = 0}$ Laplace's equation

$\Phi(\vec{r})$ is a scalar field : to be expressed in suitable coordinates

$\Phi(x, y, z)$ ↓ $(Ae^{ik_x x} + Be^{-ik_y x})$

$\Phi(r, \theta, \phi)$ ↓ $(A_l r^l + \frac{B_l}{r^{l+1}}) Y^{lm}(\theta, \phi)$

$\Phi(r, \phi, z)$ ↓ $\alpha ln(r) + (A_l r^l + \frac{B_l}{r^l})(C_m \cos m\phi + D_m \sin m\phi)$

General form of solutions

Constants are determined by boundary conditions