MA-106 Linear Algebra

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D1 - Lecture 3

Recall: we have seen matrix multiplication, elementary matrix and permutation matrix.

Let us review their action by 2×2 case.

- $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c + 2a & d + 2b \end{pmatrix}$
$(E_{21}(2) : \; R_2 \mapsto R_2 + 2R_1)$.
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} a + 2b & b \\ c + 2d & d \end{pmatrix}$
$(E_{21}(2) : \; C_1 \mapsto C_1 + 2C_2)$.
- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \qquad (P_{12} : \; R_1 \leftrightarrow R_2)$.
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & d \\ a & c \end{pmatrix} \qquad (P_{12} : \; C_1 \leftrightarrow C_2)$.

$$
\begin{aligned}
&E_{21}(2)\,E_{31}(3)\,E_{32}(4)\\
&= E_{21}(2)\,E_{31}(3)\,\begin{pmatrix} e_1 & e_2 + 4e_3 & e_3 \end{pmatrix}\\
&= E_{21}(2)\,\begin{pmatrix} e_1 + 3e_3 & e_2 + 4e_3 & e_3 \end{pmatrix}\\
&= \begin{pmatrix} e_1 + 3e_3 + 2e_2 & e_2 + 4e_3 & e_3 \end{pmatrix}\\
&= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}
\end{aligned}
$$

There seems to be some doubts about this slide, so I am adding one more slide clarifying this. I should have done it by row operations only which is the next slide.

added after class

Recall $E_{ij}(a)A$ corresponds to row operation $A_i \mapsto A_i + aA_j$.

$$E_{21}(2)\,E_{31}(3)\,E_{32}(4) = E_{21}(2)\,E_{31}(3)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 4 & 1\end{pmatrix}$$

$$= E_{21}(2)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 4 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 4 & 1\end{pmatrix}.$$

$$E_{32}(4)\,E_{31}(3)\,E_{21}(2) = E_{32}(4)\,E_{31}(3)\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$= E_{32}(4)\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 11 & 4 & 1\end{pmatrix}$$

Triangular Factors and Row Exchange

Consider $\begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} \quad (Ax = b)$

In Gauss Elimination, the row operations

$$R_2 \mapsto R_2 - 2R_1, \quad R_3 \mapsto R_3 + R_1, \quad R_3 \mapsto R_3 + R_2$$

changes A to an upper triangular matrix.

$$U := E_{32}(1)\ E_{31}(1)\ E_{21}(-2)\ A =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$b' = E_{32}(1)\ E_{31}(1)\ E_{21}(-2)\ b = \begin{pmatrix} 5 \\ -12 \\ 2 \end{pmatrix},$$

Triangular . . . continued

Previous system $Ax = b$ is equivalent to $\boxed{Ux = b'}$.

Example: The row operations

$$R_2 \mapsto R_2 - 2R_1, \quad R_2 \mapsto R_2 + 2R_1$$

leaves the matrix unchanged. Hence they are inverse to each other. In matrix terms,

$$E_{21}(-2)\ E_{21}(2) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore the inverse of $E_{ij}(a)$ is $E_{ij}(-a)$.

Remark: In the previous example,

$$E_{32}(1)\ E_{31}(1)\ E_{21}(-2)\ A = U$$

Hence $$A = E_{21}(2)\ E_{31}(-1)\ E_{32}(-1)\ U$$

Triangular Factorization

Product of lower triangular matrices is lower triangular. For example

$$L := E_{21}(2)\ E_{31}(-1)\ E_{32}(-1) =$$

$$\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ \mathbf{2} & 1 & 0\\ \mathbf{-1} & \mathbf{-1} & 1\end{pmatrix}$$

• In the Gaussian elimination of A, if no exchanges of rows are required, then $\boxed{A = L\,U}$, where

• L is a lower triangular matrix with diagonal entries 1 and *below the diagonals are* multipliers $(2, -1, -1$ in above example)

• U is an upper triangular matrix, which is got after forward elimination, with *diagonal entries as pivots*.

Example: $A = \begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix}$.

A can not be factored as LU (check directly), since the Gaussian elimination of A needs a row exchange.

$$P_{12}\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix} = I\ U$$

• If A is $n \times n$ non-singular, then there is a matrix P which is a product of permutation matrices (needed to take care of row exchanges in elimination process) such that $\boxed{PA = LU}$, where L and U are defined earlier.

• What happens when A is $n \times m$ (rectangular). We will come to this case next week.

"If A is non-singular then $PA = LU$, where P is a product of permutation matrices".

To see this, you start with Gaussian elimination on A. If you do not need row exchange then $A = LU$.

Suppose after eliminating x, $(2, 2)$ entry is zero. Then we need to interchange R_2 with R_3 (if $(3, 2)$ entry is non-zero).

Now if we replace A by $P_{23}A$ and then start my elimination, then we do not need to interchange R_2 with R_3. for second pivot.

Proceed. If we need to interchange say R_3 with R_4, then you start your Gaussian elimination with $P_{34}P_{23}A$. Now you do not need to interchange R_3 with R_3, and so on.

Inverse of a Matrix

Let A be $n \times n$ matrix. The inverse of A is a $n \times n$ matrix B (denoted by A^{-1}) such that $AB = I$ and $BA = I$.

- $\boxed{A^{-1} \text{ is unique, if it exists.}}$ To see this, if B and C are inverse of A, then $AB = I$ and $CA = I$. Hence
$$B = I.B = (CA).B = C(AB) = C.I = C.$$
- If A is invertible (i.e. A^{-1} exist), then system $Ax = b$ has a unique solution, namely $x = A^{-1}b$.
- $Ax = 0$ has a non-zero solution $\implies$ A is not invertible. Note $x = 0$ is also a solution of $Ax = 0$, hence solution is not unique.
- A is invertible if and only if A has n pivots.
- An upper triangular matrix A is invertible if and only if its diagonal entries are non-zero.

- Assume A and B are invertible. Is AB invertible? Yes

$$(AB)^{-1} = B^{-1}A^{-1}.$$

- Assume A, B are invertible. Is $A + B$ invertible? No (in general).

  Example: $I + (-I) = (0)$.

• Compute inverse of invertible matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$.

If $A^{-1} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$, where x_i is the i-th column of A^{-1}, then $AA^{-1} = I$ gives three systems of linear equations

$$Ax_1 = e_1, \quad Ax_2 = e_2, \quad Ax_3 = e_3$$

where e_i is the i-th column of I. Since the coefficient matrix A is same, we can solve them simultaneously as follows: