MA-108 Ordinary Differential Equations

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D1 - Lecture 6

Examples

Describe the method to solve the following differential equation and find solution.

- $y' = \dfrac{x^2+3x+2}{y-2}, y(1) = 4$ non-linear, Separable
- $(x-2)(x-1)y' - (4x-3)y = (x-2)^3$ Linear non-homogeneous
- $(1+x^2)y' + 2xy = \dfrac{1}{(1+x^2)y}$ Bernoulli Equation
- $y' = \dfrac{2x+y+1}{x+2y-4}$ Can be converted to a separable equation, use substitution $X = x+2$, $Y = y-3$.
- $3x^2y^2 + 2x^3y\dfrac{dy}{dx} = 0.$ Exact equation

Exact Equation

Example. Solve $$3x^2y^2 + 2x^3y\frac{dy}{dx} = 0 \qquad (*)$$

Note $3x^2y^2 = \dfrac{\partial}{\partial x}(x^3y^2)$ and $2x^3y = \dfrac{\partial}{\partial y}(x^3y^2)$.

Let $G(x, y) = x^3y^2$. Then

$$3x^2y^2 + 2x^3y\frac{dy}{dx} = 0$$

$$\implies \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} = 0$$

$$\implies \frac{d}{dx}G(x, y(x)) = 0$$

Therefore,

$$G(x, y) = C$$

is a solution of given ODE (*).

Definition. A first order ODE written in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is said to be **exact** if there exists a function G such that

$$\frac{\partial G}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial G}{\partial y} = N(x,y).$$

If ODE is exact, then

$$G(x,y) = C$$

is an implicit solution of ODE.

When is an ODE exact?

Theorem

Consider ODE $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0 \quad (*)$.

Assume functions M, N, $\dfrac{\partial M}{\partial y}$ *and* $\dfrac{\partial N}{\partial x}$ *be continuous in an open rectangle* $R := \{a < x < b,\ c < y < d\}$.

Then (*) is an exact ODE on R *if and only if* M *and* N *satisfies the condition* $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ *on* R.

In other words, there exists a function $G : R \to \mathbb{R}$ *such that* $\dfrac{\partial G}{\partial x} = M$ *and* $\dfrac{\partial G}{\partial y} = N$ *if and only if* $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ *on* R.

Exact Equations

Which of the following ODE's are exact?

1. $(2x+3)+(2y-2)y' = 0$ Exact
2. $\dfrac{dy}{dx} = \dfrac{ax+by}{bx+cy}.$ Not Exact
3. $(y/x+6x)dx + (\ln x - 2)dy = 0 \quad x, y > 0.$ Exact
4. $(3x^2y+2xy+y^3)dx + (x^2+y^2)dy = 0.$ Not Exact

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Example. Solve $(2x+3)+(2y-2)y' = 0.$

The ODE is exact, so we need to find $\phi(x,y)$ such that

$$\frac{\partial \phi}{\partial x} = 2x+3 \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 2y-2$$

Integrating first equation gives

$$\phi(x,y) = x^2 + 3x + h(y)$$

This gives

$$\frac{\partial \phi}{\partial y} = \frac{dh}{dy} = 2y-2 \implies h(y) = y^2 - 2y + C_1$$

Therefore, an implicit solution to ODE is

$$\phi(x,y) = x^2 + 3x + y^2 - 2y = C$$

Example.

Solve $(y/x + 6x)dx + (\ln x - 2)dy = 0 \quad x, y > 0.$

This is exact, so we need to find $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = \frac{y}{x} + 6x \text{ and } \frac{\partial \phi}{\partial y} = \ln x - 2$$

Integrating the first equation gives

$$\phi(x, y) = y \ln |x| + 3x^2 + h(y)$$

This gives

$$\frac{\partial \phi}{\partial y} = \ln |x| + \frac{dh}{dy} = \ln x - 2 \implies h(y) = -2y$$

Therefore the solution is given by

$$\phi(x, y) = y \ln |x| + 3x^2 - 2y = C$$

Method of integrating factors

Example. Solve $(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0.$

It is not exact. **Q.** Can it be converted to an exact equation?

The idea is to multiply the equation by a function $\mu(x, y)$ so that it becomes exact. There is no algorithm for choosing μ.

Assume $\mu(3x^2y + 2xy + y^3)dx + \mu(x^2 + y^2)dy = 0$ is exact.

Then exactness condition $\partial M/\partial y = \partial N/\partial x$ for $M + Ny' = 0$

$$\Rightarrow \mu(3x^2+2x+3y^2)+\frac{\partial\mu}{\partial y}(3x^2y+2xy+y^3) = 2x\mu+\frac{\partial\mu}{\partial x}(x^2+y^2)\ (*)$$

From observation, we choose μ to be independent of y.

Then $\partial\mu/\partial y = 0$ and equation (*) becomes

$$3\mu(x^2 + y^2) = \frac{d\mu}{dx}(x^2 + y^2) \implies \frac{d\mu}{dx} = 3\mu \implies \mu = Ce^{3x}$$

The ODE now becomes

$$e^{3x}(3x^2y + 2xy + y^3)dx + e^{3x}(x^2 + y^2)dy = 0.$$

Verify this is in fact exact. Hence there exists $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = e^{3x}(3x^2y + 2xy + y^3) \quad \text{and} \quad \frac{\partial \phi}{\partial y} = e^{3x}(x^2 + y^2)$$

Integrating the first equation gives

$$\phi(x, y) = e^{3x}x^2y + \frac{1}{3}\,e^{3x}y^3 + h(y)$$

$$\frac{\partial \phi}{\partial y} = e^{3x}x^2 + e^{3x}y^2 + \frac{dh}{dy} = e^{3x}(x^2 + y^2)$$

This gives $\dfrac{dh}{dy} = 0 \implies h(y) = C$. The solution of ODE is

$$\phi(x, y) = e^{3x}(x^2y + \frac{1}{3}\,y^3) = C$$

Q. Is $\phi = e^{3x}(x^2y + \frac{1}{3}\,y^3) = C$ the solution to our original ODE?

In general, how will the solutions to the two equations be related?

$$\phi'(x,y) = e^{3x}2xy + 3e^{3x}x^2y + e^{3x}x^2y' + 3e^{3x}\,\frac{y^3}{3} + e^{3x}y^2y' = 0.$$

Then $e^{3x}(2xy + 3x^2y + x^2y' + y^3 + y^2y') = 0$

Since e^{3x} is non-zero for all $x \in \mathbb{R}$.

We have $2xy + 3x^2y + x^2y' + y^3 + y^2y' = 0$.

Thus every $y(x)$ which is a solution to the new exact equation is a solution to the original equation and vice versa.

Q. In general, if μ is an integrating factor, are the solutions to $\mu y' = \mu f(x,y)$ same as the solutions to $y' = f(x,y)$ and vice versa?

Finding the integrating factor

Definition. We say $\mu(x,y)$ is a integrating factor of ODE

$$M(x,y) + N(x,y)y' = 0 \text{ if } \mu M + \mu N y' = 0 \text{ is exact}$$

i.e. $$\frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}$$

i.e. $$\mu \left(M_y - N_x \right) = \frac{\partial \mu}{\partial x} N - \frac{\partial \mu}{\partial y} M$$

If the original equation was exact, then $\mu \equiv 1$ is an integrating factor. In general, there is no clear way to determine μ.

However, if we assume that $\mu = \mu(x)$ is independent of y, then

$$\frac{1}{\mu} \frac{d\mu}{dx} = \frac{M_y - N_x}{N} \implies \frac{M_y - N_x}{N} := p(x)$$

is a function of x only. In this case,

$$\mu = e^{\int p(x) \ dx}$$

Finding the integrating factors

Similarly, if we assume that $\mu = \mu(y)$ is independent of x, then

$$\frac{1}{\mu}\frac{d\mu}{dy} = \frac{N_x - M_y}{M} \implies \frac{N_x - M_y}{M} := q(y)$$

is a function of y. In this case, $\mu = e^{\int q(y)\ dy}$.

THEOREM. Consider $\quad M(x,y) + N(x,y)y' = 0 \quad (*)$.

Assume that M, N, M_y, N_x are continuous on an open rectangle R. Then μ is an integrating factor of (*), where

$$\mu = \begin{cases} e^{\int p(x)\ dx} & \text{if } \dfrac{M_y - N_x}{N} := p(x) \quad \text{on } R \\ e^{\int q(y)\ dy} & \text{if } \dfrac{N_x - M_y}{M} := q(y) \quad \text{on } R \end{cases}$$

Combining both cases, if $M_y - N_x = p(x)N - q(y)M$ on R,

then $\quad \mu = e^{\int p(x)\ dx} e^{\int q(y)\ dy}$

Ex. $\cos x \cos y \; dx + (\sin x \cos y - \sin x \sin y + y) \; dy = 0.$

Verify that this is a non-linear, non-separable and non-exact. We check if it can be made exact.

$$M(x,y) = \cos x \cos y, \quad N(x,y) = \sin x \cos y - \sin x \sin y + y$$

Then, $M_y - N_x = \; - \cos x \sin y - \cos x \cos y + \cos x \sin y.$
Note, $N_x - M_y/M = 1.$
The integrating factor will be e^y. Then

$e^y \cos x \cos y \; dx + e^y(\sin x \cos y - \sin x \sin y + y) \; dy = 0$ is exact.

$$\frac{\partial \phi}{\partial x} = e^y \cos x \cos y, \quad \frac{\partial \phi}{\partial y} = e^y(\sin x \cos y - \sin x \sin y + y)$$

Integrating first equation $\dfrac{\partial \phi}{\partial x} = e^y \cos x \cos y$, we get $\phi(x, y) = e^y \sin x \cos y + h(y)$.

This gives $\dfrac{d\phi}{\partial y} = e^y \sin x \cos y - e^y \sin x \sin y + \dfrac{dh}{dy} = e^y(\sin x \cos y - \sin x \sin y + y)$.

This implies $\dfrac{dh}{dy} = ye^y$, so $h(y) = e^y y + e^y + C$.

Therefore an implicit solution of ODE is

$$\phi(x, y) = e^y(\sin x \cos y + y + 1) = C$$

Example. Solve $(3x^2y^3 - y^2 + y)dx + (-xy + 2x)dy = 0$.

This equation is non-linear, not separable, non-homogeneous and non-exact.

We will try to see if it can be converted to an exact equation using an integrating factor.

We have

$$M(x,y) = 3x^2y^3 - y^2 + y, \quad N(x,y) = -xy + 2x$$

Then

$$M_y - N_x = 3x^2 3y^2 - 2y + 1 + y - 2 = 9x^2y^2 - y - 1$$

Clearly $-M_y + N_x/M$ is not independent of y and $M_y - N_x/N$ is not independent of x.

Can we write $M_y - N_x = p(x)N - q(y)M$ for some p and q in some open rectangle?

We have $M(x,y) = 3x^2y^3 - y^2 + y, \quad N(x,y) = -xy + 2x$

$M_y - N_x = 9x^2y^2 - y - 1$

Want $M_y - N_x = p(x)N - q(y)M$.

Choose $p(x) = -2/x$ and $q(y) = -3/y$. Then

$$p(x)N - q(y)M = M_y - N_x$$

The integrating factor is then given by

$$e^{\int -2/x \ dx} e^{\int -3/y \ dy} = \frac{1}{x^2y^3}$$

$$\frac{1}{y^3x^2}[(3x^2y^3 - y^2 + y) \ dx + (-xy + 2x)] = 0$$

is exact. Solve it.

Is an integrating factor unique? In general, this need not be the case.
As observed, if μ is an integrating factor, then so is $c\mu$ for $c \neq 0$. What about upto constant multiple? No.

For example,

$$3xy + y^2 + (x^2 + xy)y' = 0$$

is not exact. Show that $\mu(x, y) = \dfrac{1}{xy(2x + y)}$ is an integrating factor of the ODE.

Find another integrating factor of the same ODE. We can show that $\mu(x) = x$ is also an integrating factor.

However one integrating factor may give a simpler ODE than the other.

Classify the given ODE and give a method to solve it. Explain in what region does the solution exist.

- $(2t - 5y)y' = t - y$ Homogeneous non-linear
- $y = ye^y y' - 2xy'$ Integrating factor
- $\sin 2x + \cos 3yy' = 0$ separable
- $y' + (2/t)y - (\cos t)/t^2 = 0$ Linear
- $y/x + 6x + (lnx - 2)y' = 0$ Exact

Picard's Iteration Method

Picard's iteration method gives a rough guide to solving a given IVP. It is useful in proving the existence and uniqueness theorem of the IVP $y' = f(t, y), \quad y(0) = 0$.

We will now give a rough sketch of the idea of the proof using this method. Note it is sufficient to assume the IVP is $y(0) = 0$, since the solution can obtained for any other initial condition by making appropriate substitution.

Suppose $y = \phi(t)$ is a solution to the IVP. Then,

$$\frac{d\phi}{dt} = f(t, \phi(t)), \quad \phi(0) = 0.$$

That is,

$$\phi(t) = \int_0^t f(s, \phi(s))ds, \quad \phi(0) = 0.$$

Picard's Iteration Method

The previous equation is called an integral equation in the unknown function ϕ.

Conversely, if the integral equation holds, then by the Fundamental Theorem of Calculus,

$$y' = \frac{d\phi}{dt} = f(t, \phi(t)) = f(t, y).$$

Thus, solving the integral equation is equivalent to solving the IVP.

We define, iteratively, a sequence of functions $\phi_n(t)$ for every integer $n \geq 0$ as follows: Let

$$\begin{aligned} \phi_0(t) &\equiv 0 \\ \phi_1(t) &= \int_0^t f(s, \phi_0(s))ds \end{aligned}$$

Picard's Iteration Method

More generally,

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds.$$

Note: Each ϕ_n satisfies the initial condition $\phi_n(0) = 0$. None of the ϕ_n may satisfy $y' = f(t, y)$. Suppose for some n, $\phi_{n+1} = \phi_n$. Then,

$$\phi_{n+1} = \phi_n = \int_0^t f(s, \phi_n(s))ds,$$

and this implies

$$\frac{d}{dt}(\phi_n(t)) = f(t, \phi_n(t))$$

is a solution of the given IVP.

Picard's Iteration Method

In general, the sequence $\{\phi_n\}$ may not terminate. In fact, all the ϕ_n may not even be defined outside a small region in the domain.

However, it is possible to show that, if $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ is continuous in some open rectangle (hence continuous and bounded in a smaller closed rectangle), the sequence converges to a function

$$\phi(t) = \lim_{n \to \infty} \phi_n(t)$$

which will be the unique solution to the given IVP.

Picard's Iteration Method

Example: Solve the IVP:

$$y^1 = 2t(1+y);\ \ y(0) = 0.$$

The corresponding integral equation is

$$\phi(t) = \int_0^t 2s(1+\phi(s))ds.$$

Let $\phi_0(t) \equiv 0$. Then,

$$\begin{aligned}
\phi_1(t) &= \int_0^t 2sds = t^2, \\
\phi_2(t) &= \int_0^t 2s(1+s^2)ds = t^2 + \frac{t^4}{2}, \\
\phi_3(t) &= \int_0^t 2s(1+s^2+\frac{s^4}{2})ds = t^2 + \frac{t^4}{2} + \frac{t^6}{6}.
\end{aligned}$$

Picard's Iteration Method

We claim:

$$\phi_n(t) = t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \ldots + \frac{t^{2n}}{n!}.$$

Use induction to prove this:

$$\begin{aligned}
\phi_{n+1}(t) &= \int_0^t 2s(1+\phi_n(s))ds \\
&= \int_0^t 2s\left(1 + s^2 + \frac{s^4}{2} + \ldots + \frac{s^{2n}}{n!}\right) ds \\
&= t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \ldots + \frac{t^{2n}}{n!} + \frac{t^{2n+2}}{(n+1)!}.
\end{aligned}$$

Picard's Iteration Method

Hence $\phi_n(t)$ is the n-th partial sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$.

Applying the ratio test, we get:

$$\left| \frac{t^{2k+2}}{(k+1)!} \cdot \frac{k!}{t^{2k}} \right| = \frac{t^2}{k+1} \to 0$$

for all t as $k \to \infty$. Thus,

$$\lim_{n\to\infty} \phi_n(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = e^{t^2} - 1.$$

Uniqueness

Let's quickly see how to get uniqueness. Suppose ϕ and ψ are solutions of $y^1 = f(x, y), y(0) = 0$. Thus, both these satisfy the integral equation as well. Then,

$$\phi(t) - \psi(t) = \int_0^t (f(s, \phi(s)) - f(s, \psi(s)))ds.$$

Thus,

$$|\phi(t) - \psi(t)| \leq \int_0^t |f(s, \phi(s) - f(s, \psi(s)|ds.$$

The crucial point is that there is a constant, say K, such that

$$|f(s, \phi(s)) - f(s, \psi(s))| \leq K|\phi(s) - \psi(s)|,$$

and this is since we assume $\frac{\partial f}{\partial y}$ is continuous.

Uniqueness

Let

$$U(t) = \int_0^t |\phi(s) - \psi(s)| ds.$$

Clearly, $U(0) = 0, U(t) \geq 0$. Also, $U'(t) = |\phi(t) - \psi(t)|$. So,

$$U'(t) - KU(t) \leq 0.$$

Thus:

$$[e^{-Kt}U(t)]' \leq 0.$$

Integrate from 0 to t and use $U(0) = 0$ to conclude $U(t) \leq 0$. Thus,

$$U(t) \equiv 0,$$

and so

$$U'(t) \equiv 0.$$

Thus, $\phi(t) \equiv \psi(t)$.