MA-108 Ordinary Differential Equations

M.K. Keshari

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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D1 - Lecture 3

Recall

- The graph of a particular solution of an ODE : solution curve.
- An implicit solution of an ODE $F(x, y(x), \ldots, y^{(n)}(x)) = 0$ is an equation $g(x, y) = 0$ which satisfies the differential equation and gives an explicit solution $y(x)$ of ODE on some interval.
- The graph of an implicit solution : integral curve.
- We studied the direction fields of ODE $y' = f(x, y)$.
- Curves $f(x, y) = c$ are called direction fields or the slope fields of $y' = f(x, y)$.

Separable ODE's

Example. Solve $y' = 2xy^2$.
We assume that y is not identically zero and apply separation of variables method.
Rewrite ODE as $\dfrac{1}{y^2}y' = 2x$ and integrate it.
The solution is given by $\dfrac{-1}{y} = x^2 + C$, or $y = \dfrac{-1}{x^2 + C}$.
Note that the function $y \equiv 0$ is also a solution to this equation. This solution cannot be obtained for any choice of C in the previous solution.

Find the interval of validity for this solution, given that $y(0) = y_0$.
If $y_0 \neq 0$ then, $C = -\dfrac{1}{y_0}$. Hence $y = \dfrac{-y_0}{y_0x^2 - 1}$

Separable ODE's

Recall $y = \dfrac{-1}{x^2 + C}$ and $y = \dfrac{-y_0}{y_0 x^2 - 1}$ if $y_0 \neq 0$.

What happens when $y_0 < 0$? or $y_0 = 0$? or $y_0 > 0$?

When $y_0 < 0$, the solution is defined for all x.

When $y_0 = 0$, we get the solution $y \equiv 0$.

When $y_0 > 0$, the solution is valid when $x \in \mathbb{R} - \{\pm 1/\sqrt{y_o}\}$.
$\square$

Separable ODE's

Example: Find the solution to the initial value problem:

$$\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}; \quad y(0) = 1.$$

Assume $y \neq 0$. Then,

$$\frac{1+2y^2}{y}dy = \cos x \; dx.$$

Integrating,

$$\ln|y| + y^2 = \sin x + c.$$

As $y(0) = 1$, we get $c = 1$. Hence a particular solution to the IVP is

$$\ln|y| + y^2 = \sin x + 1.$$

Note: $y \equiv 0$ is a solution to the DE, but it is not a solution to the given IVP.

Existence and Uniqueness for 1st order linear ODE

Theorem

Let

$$y' + p(x)y = f(x)$$

be linear 1st order ODE. Assume functions p *and* f *are continuous on an open interval* $I = (a, b)$*. Let* $x_0 \in I$*. If we fix the initial condition*

$$y(x_0) = y_0,$$

then the IVP has a unique solution $y = \phi(x)$ *on the interval* I*.*

- For example, the solution $R(t) = -200e^{t}$ (here $k = 1$) for the IVP $R' = R - 300; \quad R(0) = 100$ is unique over the interval $(-\infty, \infty)$. Why? Note in this problem both $p(x)$ and $f(x)$ are constants and hence continuous over $\mathbb{R}$.

- The interval of existence and uniqueness of the solution is independent of y_0.

- The uniqueness condition implies that the one parameter family of solutions, (for 1st order linear ODE) obtained using variation of parameters method, is in fact a general solution on the interval I.

Example. Solve

$$ty' + y = 3t^4.$$

Write the ODE in standard form

$$y' + (1/t)y = 3t^3.$$

The functions $p(t) = 1/t$ and $f(t) = 3t^3$ are continuous over $I = \mathbb{R} - \{0\}$. By existence and uniqueness theorem, every IVP has a unique solution on the interval $\mathbb{R} - \{0\}$.

Solve the homogeneous part

$$y' + (1/t)y = 0$$

by separation of variables method. We have

$$\ln|y| = -\ln|t| + K, \quad \text{i.e.} \quad y = \frac{C}{t}$$

Take $y_1 = 1/t$ as solution of homogeneous part.

Applying variation of parameters, the solution of given ODE is $y = uy_1$, where u is the solution of

$$u' = \frac{3t^3}{y_1} = 3t^4$$

Note that $y_1 = 1/t$ is not zero on $I = \mathbb{R} - \{0\}$.

• In general, the solution for the 1st order linear homogeneous equation $y' + p(x)y = 0$ will be non-zero on the given I, Why? since the solution is always the exponential of a function.

• Solving $u' = \dfrac{3t^3}{y_1} = 3t^4$, we get $u = \frac{3}{5}t^5 + C$. Hence $y = (1/t)[(3/5)t^5 + C]$, is a solution to $y' + (1/t)y = 3t^3$ on $\mathbb{R} - \{0\}$.

• For different initial values $t_0 \in \mathbb{R} - \{0\}$ with $y(t_0) = y_0 \in \mathbb{R}$, the constant C will get determined.

• By Uniqueness theorem, solution $y = (3/5)t^4 + C(1/t)$ is a general solution. There are no other solutions.

Linear vs Non-Linear Differential Equations

Theorem (Existence of solution for $y' = f(x, y)$)

Consider $y' = f(x, y)$. Assume the functions f and $\partial f/\partial y$ are continuous in some rectangle $a < x < b,\ c < y < d$ containing the point (x_0, y_0). Then, in some interval $(t_0 - h, t_0 + h) \subset (a, b)$, there is a unique solution $y = \phi(t)$ of the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$.

- Note the theorem says that the solution and the interval where the solution is valid, of a non-linear ODE depends on the choice of our initial condition.

Linear vs Non-Linear Differential Equations

- The solution of a non-linear ODE obtained using a particular method may not be a general solution.
- For example, for the non-linear ODE $y' = 2xy^2$, our solution $y = -1/(x^2 + C)$ does not give the solution $y \equiv 0$ for any value of C.
- Also, not every value of C will give an actual solution, unlike in the case of the solution to linear ODEs.

  We showed that the solution obtained using separation of variables method,

$$\int M(x)\, dx + \int N(y)\, dy = C$$

  is a solution to an initial value problem for some values of C, not necessarily for all C.

Bernoulli Equation

Let us now discuss other methods to solve non-linear differential equations.

Example: Consider $y' + y = xy^2$. This is clearly not linear and not separable. The solution to homogeneous part $y' + y = 0$ is given by $y = Ce^{-x}$. We imitate the variation of parameters method and set $y = ue^{-x}$.

$$\begin{aligned} u'e^{-x} - ue^{-x} + ue^{-x} &= u^2e^{-2x}x \\ \implies u'e^{-x} &= u^2e^{-2x}x \\ \implies \frac{u'}{u^2} &= xe^{-x} \end{aligned}$$

We can now apply separation of variables method

$$\begin{aligned} -1/u &= -(1+x)e^{-x} + C \\ u &= 1/((1+x)e^{-x} - C) \\ \implies y &= e^{-x}/((1+x)e^{-x} - C) = 1/(1 + x - Ce^{x}) \end{aligned}$$

Epidemics

The math. model of an epidemic gives a non-linear ODE. Suppose that a given population can be divided into two parts: those who have a given disease and can infect others, and those who do not have it but are susceptible.

Let x be the proportion of susceptible individuals and y be the proportion of infectious individuals; then $x + y = 1$. Assume that the disease spreads by contact between sick and wellat the rate dy/dt proportional to number of such contacts. Further, assume that members of both groups move about freely among each other, so the number of contacts is proportional to the product of x and y. Since $x = 1 - y$, we obtain the initial value problem

$$dy/dt = \alpha y(1 - y), \; y(0) = y_0,$$

where α is a positive proportionality factor, and y_0 is the initial proportion of infectious individuals.

Bernoulli Equation

A non-linear differential equation of the form $y' + p(x)y = f(x)y^r$ is said to be a **Bernoulli Equation**.

Example: $xy' - 2y = \dfrac{x^2}{y^6}$

Let us rewrite it as $y' - \dfrac{2}{x}y = \dfrac{x}{y^6}$, and consider solutions over intervals in $\mathbb{R} - \{0\}$.

The solution to homogeneous part is $y_1 = x^2$.
The solution to non-homogeneous part is $y = uy_1$, where u satisfies $u'y_1 = x(uy_1)^{-6}$. Rewrite it as $u^6u' = x(x^2)^5 = x^{11}$.
By integrating, we get $(1/7)u^7 = (1/12)x^{12} + C$.
Solution: $(1/7)y^7 = [(1/12)x^{12} + C]y_1^7 = [(1/12)x^{12} + C]x^{14}$.

Notice we no longer have an explicit solution here .

Converting to Separable Equation

Solve $xy' = y + x$. Rewrite it as $y' = y/x + 1$.
Clearly, this is not separable, but we can make it separable by changing our variables.

Let $v = y/x$ or $y = vx$. Then $y' = v'x + v$
Given ODE is $v'x + v = v + 1$ or $v'x = 1$.

Apply separation of variables to get $y = (\ln|x| + C)x$.

Definition

A differential equation $y' = f(x, y)$ is said to be **homogeneous** if if it can be written as $y' = q(y/x)$.

For example, $y' = \dfrac{y^2 + xy - x^2}{x^2} = (\dfrac{y}{x})^2 + \dfrac{y}{x} - 1$.
Any non linear ODE $y' = q(y/x)$ can be converted to a separable equation by substituting $y = vx$.