Tutorial-5, MA 108 (ODE)
Spring 2015, IIT Bombay

1. Determine if the following improper integrals exist.

   (a) $\int_0^\infty (t^2+1)^{-1}\, dt$

   (b) $\int_1^\infty t^{-2}e^t\, dt$

2. Find the Laplace transform of following functions.

   (a) $\cosh t \sin t$, (b) $\cosh^2 t$, (c) $t \sinh 2t$, (d) $\sin(t + \dfrac{\pi}{4})$.

   (e) $f(t) = \begin{cases} e^{-t}, & 0 \le t < 1 \\ e^{-2t}, & t \ge 1 \end{cases}$, (f) $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$

3. (a) Prove that if $L(f(t)) = F(s)$, then $L(t^k f(t)) = (-1)^k F^{(k)}(s)$.

   [Hint. Assume that we can differentiate the integral $\int_0^\infty e^{-st} f(t)dt$ with respect to s under the integral sign.]

   (b) Using $L(1) = 1/s$, show that $L(t^n) = \dfrac{n!}{s^{n+1}}$, n an integer.

4. Show that if f is piecewise continuous and of exponential order, then $\lim_{s\to\infty} F(s) = 0$.

5. Show that if f is continuous on $[0, \infty)$ and of exponential order $s_0 > 0$, then

$$L\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}L(f), \quad s > s_0.$$

6. Suppose f is piecewise continuous and of exponential order, and $\lim_{t\to 0+} f(t)$ exists. Show that

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(r)dr.$$

7. Suppose s is piecewise continuous on $[0, \infty)$.

   (a) Prove: If the integral $g(t) = \int_0^t e^{-s_0\tau} f(\tau)d\tau$ satisfies the inequality $|g(t)| \le M$, $t \ge 0$, then f has a Laplace transform $F(s)$ defined for $s > s_0$.

   [Hint. Use integration by parts to show that

$$\int_0^T e^{-st} f(t)dt = e^{-(s-s_0)T} g(T) + (s - s_0)\int_0^T e^{-(s-s_0)t} g(t)dt$$

   (b) Show that if $L(f)$ exists for $s = s_0$, then it exists for $s > s_0$.

   (c) Show that the function $f(t) = te^{t^2}\cos(e^{t^2})$ has a Laplace transform defined for $s > 0$, even though f is not of exponential order.

8. Find the Laplace transform of the following functions.

(a) $\dfrac{\sin \omega t}{t}$, $\omega > 0$, (b) $\dfrac{e^{at} - e^{bt}}{t}$, (c) $\dfrac{\cosh t - 1}{t}$, (d) $\dfrac{\sinh^2 t}{t}$.

9. The **Gamma function** defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

can be shown to converge, if $\alpha > 0$.

(a) Use integration by parts to show that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha),\ \alpha > 0$.

(b) Show that $\Gamma(n+1) = n!$, if $n = 1, 2, \ldots$.

(c) $L(t^\alpha) = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $s > 0$ holds if α is a non-negative integer. Show that this formula is valid for any $\alpha > -1$.

10. Suppose f is continuous on $[0, T]$ and $f(t+T) = f(t)$ for all $t \geq 0$. We say f is periodic with period T.

(a) Show that the Laplace transform $L(f)$ is defined for $s > 0$.

(b) Show that

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt, \quad s > 0$$

11. Find the Laplace transform of the following periodic functions.

(a) $f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 2 - t, & 1 \leq t < 2 \end{cases}, \quad f(t+2) = f(t),\ t \geq 0.$

(b) $f(t) = \begin{cases} 1, & 0 \leq t < 1/2 \\ -1, & 1/2 \leq t < 1 \end{cases}, \quad f(t+1) = f(t),\ t \geq 0.$

(c) $f(t) = \begin{cases} \sin t, & 0 \leq t < \pi \\ 0, & \pi \leq t < 2\pi \end{cases}, \quad f(t+2\pi) = f(t),\ t \geq 0.$

(d) $f(t) = |\sin t|$.

12. Find the inverse Laplace transform of the following functions.

(a) $\dfrac{3}{(s-7)^4}$, (b) $\dfrac{2s-4}{s^2-4s+13}$, (c) $\dfrac{s^2-1}{(s^2+1)^2}$, (d) $\dfrac{s^2-4s+3}{(s^2-4s+5)^2}$, (e) $\dfrac{s^3+2s^2-s-3}{(s+1)^4}$,

(f) $\dfrac{3-(s+1)(s-2)}{(s+1)(s+2)(s-2)}$, (g) $\dfrac{3+(s-2)(10-2s-s^2)}{(s-2)(s+2)(s-1)(s+3)}$, (h) $\dfrac{2+3s}{(s^2+1)(s+2)(s+1)}$,

(i) $\dfrac{3s+2}{(s^2+4)(s^2+9)}$, (j) $\dfrac{17s-15}{(s^2-2s+5)(s^2+2s+10)}$, (k) $\dfrac{2s+1}{(s^2+1)(s-1)(s-3)}$.

13. Solve the following IVP's using Laplace transforms.

    (a) $y'' + 3y' + 2y = e^t,\ y(0) = 1,\ y'(0) = -6,$

    (b) $y'' - 3y' + 2y = 2e^{3t},\ y(0) = 1,\ y'(0) = -1.$

    (c) $y'' + y = \sin 2t,\ y(0) = 0,\ y'(0) = 1,$

    (d) $y'' + 4y = 3\sin t,\ y(0) = 1,\ y'(0) = -1.$

    (e) $y'' + y = t,\ y(0) = 0,\ y'(0) = 2,$

    (f) $y'' + 2y' + y = 6\sin t - 4\cos t,\ y(0) = -1,\ y'(0) = 1.$

    (g) $y'' - 5y' + 6y = 10e^t \cos t,\ y(0) = 2,\ y'(0) = 1,$

    (h) $y'' + 4y' + 5y = e^{-t}(\cos t + 3\sin t),\ y(0) = 0,\ y'(0) = 4.$

14. Suppose that

$$g(t) = \int_0^t f(r)\ dr.$$

If $G(s)$ and $F(s)$ are Laplace transforms of g and f respectively, show that

$$G(s) = F(s)/s.$$