MA-106 Linear Algebra

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Eigenvalues and Eigenvectors: Motivation

- Solve for u: $du/dt = 3u$.
  The solution is $u(t) = c\, e^{3t}$, $c \in \mathbb{R}$
  With initial condition $u(0) = 2$, the solution is $u(t) = 2e^{3t}$.
- Consider the system of Differential Equations (ODE) with initial conditions (IC):

$$\begin{aligned} dv/dt &= 4v - 5w, \quad v(0) = 8, \\ dw/dt &= 2v - 3w, \quad w(0) = 5. \end{aligned}$$

  How does one find the solution?
- Write the system in matrix form $\boxed{d\mathbf{u}/dt = A\mathbf{u},\ \mathbf{u}(0) = \mathbf{u}_0}$
  where $\mathbf{u} = \begin{bmatrix} v \\ w \end{bmatrix}$, $\mathbf{u}_0 = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$, $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$.
  This is a system of linear 1$^{\text{st}}$ order ODE with constant coefficients.
- Assuming the solution is $\mathbf{u}(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} = e^{\lambda t}\,\mathbf{x} = \begin{bmatrix} e^{\lambda t}\, y \\ e^{\lambda t}\, z \end{bmatrix}$, where
  $\mathbf{x} = \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^2$, we need to find λ and x.

Eigenvalues and Eigenvectors: Definition

We have $v' = 4v - 5w$, $w' = 2v - 3w$, where $v(t) = e^{\lambda t}\, y, w(t) = e^{\lambda t}\, z$

$$\lambda\, e^{\lambda t} y = 4e^{\lambda t} y - 5e^{\lambda t} z,$$
$$\lambda\, e^{\lambda t} z = 2e^{\lambda t} y - 3e^{\lambda t} z.$$

Cancelling $e^{\lambda t}$, we get

Eigenvalue problem: Find λ and $\mathbf{x} = (y, z)^T$ satisfying

$$4y - 5z = \lambda y,$$

$$2y - 3z = \lambda z.$$

In the matrix form, it is $\boxed{A\mathbf{x} = \lambda \mathbf{x}}$.
This equation has two unknowns, λ and $\mathbf{x}$.

If there exists a λ such that $A\mathbf{x} = \lambda \mathbf{x}$ has a non-zero solution $\mathbf{x}$, then λ is called an eigenvalue of A and all *nonzero* $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda \mathbf{x}$ are called eigenvectors of A associated to λ.

Q: Given A $n \times n$, how does one find its eigenvalues and eigenvectors?

Eigenvalues and Eigenvectors: Solving $A\mathbf{x} = \lambda\mathbf{x}$

- Write $Ax = \lambda x$ as $\boxed{(A - \lambda I)x = 0.}$
- λ is an eigenvalue of A

$$\begin{aligned}
&\Leftrightarrow \text{there is a nonzero } x \text{ in the nullspace of } A - \lambda I \\
&\Leftrightarrow N(A - \lambda I) \neq 0, \text{ i.e., } \dim\,(N(A - \lambda I)) \geq 1, \\
&\Leftrightarrow A - \lambda I \text{ is singular} \\
&\Leftrightarrow \det(A - \lambda I) = 0.
\end{aligned}$$

- $\det(A - \lambda I)$ is a polynomial in the variable λ of degree n. Hence it has at most n roots $\Rightarrow$ A has atmost n eigenvalues.
- $\det(A - \lambda I)$ is called the characteristic polynomial of A.
- If λ is an eigenvalue of A, then the nullspace of $A - \lambda I$ is called the eigenspace of A associated to eigenvalue λ.
- $\boxed{\lambda = 0 \text{ is an eigenvalue of } A \Leftrightarrow \det(A) = 0 \Leftrightarrow A \text{ is singular.}}$

Eigenvalues and Eigenvectors: Example

To summarise: An eigenvalue of A is a root of its characteristic polynomial, and any non-zero vector
in the corresponding eigenspace is an associated eigenvector.
Recall: The ODE system we want to solve is

$$v' = 4v - 5w, \quad v(0) = 8, \qquad\qquad w' = 2v - 3w, \quad w(0) = 5.$$

The solutions are $v(t) = e^{\lambda t}\, y, w(t) = e^{\lambda t}\, z$, where $(y, z)^T$ is a solution of:

$$\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \lambda \begin{bmatrix} y \\ z \end{bmatrix} \qquad (A\mathbf{x} = \lambda \mathbf{x})$$

The characteristic polynomial of A is

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{vmatrix} \\ &= (4-\lambda)(-3-\lambda) + 10 \\ &= \lambda^2 - \lambda - 2 \qquad = (\lambda+1)(\lambda-2) \end{aligned}$$

The eigenvalues of A are $\boxed{\lambda_1 = -1,\ \lambda_2 = 2.}$

Eigenvalues and Eigenvectors: Example

To find the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ associated to $\lambda_1 = -1$ and $\lambda_2 = 2$ respectively, find $N(A - \lambda_1 I) = N(A + I)$, and $N(A - \lambda_2 I) = N(A - 2I)$.

$(A + I)\mathbf{x}_1 = \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = 0 \Rightarrow$

The eigenspace of A corresponding to $\lambda_1 = -1$ is $N(A + I) = \left\{ \begin{bmatrix} z \\ z \end{bmatrix} \text{ where } z \in \mathbb{R} \right\}$ and $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an associated eigenvector.

Similarly, $(A - 2I)\mathbf{x}_2 = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = 0$

$\Rightarrow N(A - 2I) = \left\{ \begin{bmatrix} \frac{5z}{2} \\ z \end{bmatrix} \text{ where } z \in \mathbb{R} \right\}$.

In particular, $\mathbf{x}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ is an eigenvector associated to $\lambda_2 = 2$.

Thus, the system $d\mathbf{u}/dt = A\mathbf{u}$ has two special solutions

$$\mathbf{u}_1(t) = e^{-t}\mathbf{x}_1 \text{ and } \mathbf{u}_2(t) = e^{2t}\mathbf{x}_2.$$

Complete Solution to ODE

Note: When $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfy $d\mathbf{u}/dt = A\mathbf{u}$, then so does $c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ for scalars c_1 and c_2.

Complete solution: $\quad \mathbf{u}(t) = c_1 e^{-t}\mathbf{x}_1 + c_2 e^{2t}\mathbf{x}_2.$

$$\text{i.e. } \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

$$\text{i.e. } v(t) = c_1 e^{-t} + 5c_2 e^{2t}, \qquad w(t) = c_1 e^{-t} + 2c_2 e^{2t}.$$

If we put the initial conditions $v(0) = 8$ and $w(0) = 5$, then

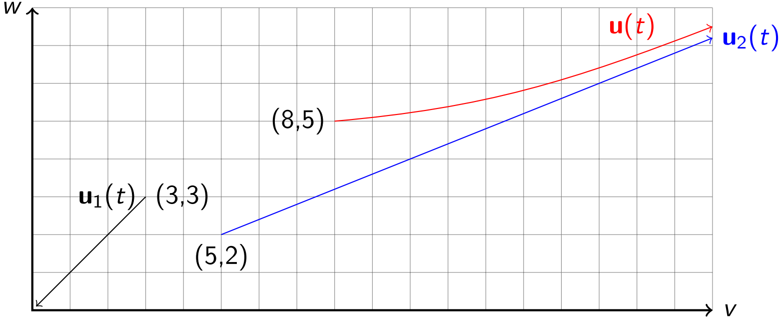
$$c_1 + 5c_2 = 8, \quad c_1 + 2c_2 = 5 \Rightarrow c_1 = 3,\ c_2 = 1.$$

Hence the solution of the original ODE system with the given IC is

$$v(t) = 3e^{-t} + 5e^{2t}, \quad w(t) = 3e^{-t} + 2e^{2t}.$$

Other Initial Conditions

- If ICs are $v(0) = 3, w(0) = 3$, then $c_1 = 3$ and $c_2 = 0$. The solutions are $v_1(t) = 3e^{-t}$ and $w_1(t) = 3e^{-t}$ which decays with time.
- If ICs are $v(0) = 5, w(0) = 2$, then $c_1 = 0$ and $c_2 = 1$. The solutions are $v_2(t) = 5e^{2t}$, $w_2(t) = 2e^{2t}$ which grows with time.
- $$\frac{w(t)}{v(t)} = \frac{3e^{-t} + 2e^{2t}}{3e^{-t} + 5e^{2t}} = 1 - \frac{3e^{2t}}{3e^{-t} + 5e^{2t}} = 1 - \frac{3}{3e^{-3t} + 5} < \frac{2}{5}$$
  $$\frac{w_1(t)}{v_1(t)} = 1, \qquad \frac{w_2(t)}{v_2(t)} = \frac{2}{5}, \qquad \frac{w(t)}{v(t)} \to \frac{2}{5} \quad \text{as} \quad t \to \infty$$



Summary

To solve $Ax = \lambda x$,

1. Find characteristic polynomial $\det(A - \lambda I)$ (of degree n).
2. Find roots of characteristic polynomial.
   [For $n \geq 5$, no formula exist for roots. (Abel, Galois)
   For $n = 3, 4$, formulae for root exist, but not easy to use.]
3. For each eigenvalue λ, to find associated eigenspace, solve $(A - \lambda I)x = 0$.
4. Finding roots of characteristic polynomial is difficult in general. However we know the sum and product of eigenvalues: Write $\det(A - \lambda I) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$ and expand both sides. Comparing the coefficients of λ^{n-1} and λ^0, we get

$$\begin{aligned} \text{Trace of } A &:= a_{11} + \ldots + a_{nn} && \text{(sum of diagonal entries)} \\ &= \lambda_1 + \ldots + \lambda_n && \text{(sum of eigenvalues)} \\ \det(A) &= \lambda_1 \ldots \lambda_n && \text{(product of eigenvalues)} \end{aligned}$$

Examples

In some cases it is easy to find the eigenvalues.

Ex: $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ is diagonal. Characteristic polynomial $(\lambda - 3)(\lambda - 2)$.

Eigenvalues: $\lambda_1 = 3$, $\lambda_2 = 2$.

Eigenvectors: $(A - 3I)x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Similarly, eigenvector associated to λ_2 is $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Further, $\mathbb{R}^2$ has a basis consisting of eigenvectors of A: $\{x_1, x_2\}$.

Ex: If A is a diagonal matrix with diagonal entries $\lambda_1, \cdots, \lambda_n$, then

Eigenvalues: $\lambda_1, \cdots, \lambda_n$

Eigenvectors: $e_1, \cdots, e_n$,

which form a basis for $\mathbb{R}^n$.

Examples

Ex: Projection onto the line $x = y$: $P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ projects onto itself $\Rightarrow \lambda_1 = 1$ with eigenvector x_1.

$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ projects onto $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \lambda_2 = 0$ with eigenvector x_2.

Further, $\{x_1, x_2\}$ is a basis of $\mathbb{R}^2$.

Q: Do a collection of eigenvectors always form a basis of $\mathbb{R}^n$? **A:** No.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Characteristic Polynomial: $\det(A - \lambda I) = (\lambda - 1)^2$.

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 1$.

Eigenvectors: $(A - I)x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Eigenspace of A is 1 dimensional $\Rightarrow \mathbb{R}^2$ has no basis of eigenvectors of A.

Q: What is the advantage of a basis of $\mathbb{R}^n$ consisting of eigenvectors?

Similarity and Eigenvalues

Theorem: If A and B are similar, i.e., $S^{-1}AS = B$ for an invertible matrix S, then they have the same characteristic polynomial.

In particular, they have the same eigenvalues, $\det(A) = \det(B)$ and $\text{Trace}(A) = \text{Trace}(B)$.

Proof. Given: $B = S^{-1}AS$. Want to prove: $\det(A - \lambda I) = \det(B - \lambda I)$.
Note: $\det(B - \lambda I) = \det(S^{-1}AS - \lambda S^{-1}S)$
$$= \det(S^{-1}(A - \lambda I)S) = \det(A - \lambda I). \qquad \square$$

Observe: $A - \lambda I$ and $B - \lambda I$ are similar.

Definition: An $n \times n$ matrix A is diagonalizable if A is similar to a diagonal matrix Λ, i.e., there is an invertible matrix S and a diagonal matrix Λ such that $S^{-1}AS = \Lambda$.

Note: If this happens, the eigenvalues of A are the diagonal entries of Λ.