MA-108 Ordinary Differential Equations

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D1 - Lecture 10

Recall: Abel's Formula

$$W(f, g; x) = W(f, g; a)\, e^{-\int_a^x p(t)dt},$$

when f, g are solution of an ODE $y'' + p(x)y' + q(x)y = 0$, where p, q are continuous on I and $a \in I$.

If f, g are linearly dependent, then $W(f, g; x) = 0$.

Consider $y'' + p(x)y' + q(x)y = 0$, where $p(x)$ and $q(x)$ are continuous on $I = (a, b)$. Suppose f and g are solutions on I. Then f and g are linearly independent on I if and only if $W(f, g; x)$ has no zeros in I.

Consider $y'' + p(x)y' + q(x)y = 0$. If $y_1(x)$ is one solution, then other solution is

$$y_2 = y_1(x) \int \frac{e^{-\int p dx}}{y_1^2} dx.$$

2nd Order Linear ODE's with constant coeff.

For $a, b, c \in \mathbb{R}$ with $a \neq 0$, consider

$$ay'' + by' + cy = 0.$$

Suppose e^{mx} is a solution, where m is a constant. Then, $p(m) = am^2 + bm + c = 0$. It's roots are given by

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We considered solutions when

Case 1: $b^2 - 4ac > 0$. Then characteristic equation has two distinct real roots.

Case 2: $b^2 - 4ac = 0$. Then characteristic equation has two repeated real roots.

two distinct complex conjugate roots case

Ex. Find general solution of $y'' + 4y' + 13y = 0 \quad (1)$.
Characteristic polynomial is $m^2 + 4m + 13 = (m+2)^2 + 9$.
Roots of characteristic equation are $-2 + 3i$ and $-2 - 3i$.
It is reasonable to expect that $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ are solutions of (1). Infact that is true. But they are complex valued solutions and we want real solutions. So let us write

$$e^{(-2+3i)x} = e^{-2x}(\cos 3x + i\sin 3x) \text{ and}$$
$$e^{(-2-3i)x} = e^{-2x}(\cos 3x - i\sin 3x)$$

Sum and difference gives $y_1 = e^{-2x}\cos 3x$ and $y_2 = e^{-2x}\sin 3x$ are fundamental solutions. Hence the general solution is

$$y(x) = e^{-2x}[c_1 \cos 3x + c_2 \sin 3x]$$

Ex. Solve IVP $y'' + 4y' + 13y = 0, \quad y(0) = 3, \; y'(0) = 1$.
$c_1 = 3, \; 1 = \; -2(3) + 3c_2$ gives $c_2 = 7/3$.

THEOREM.

Let $p(m) = am^2 + bm + c$ be the characteristic polynomial of

$$ay'' + by' + cy = 0, \quad \text{where } a, b, c \in \mathbb{R}, \ a \neq 0. \text{ Then}$$

1. If $p(m) = 0$ has distinct real roots m_1, m_2, then the general solution is $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
2. If $p(m) = 0$ has repeated real roots m_1, m_1, then the general solution is $y(x) = e^{m_1 x}(c_1 + c_2 x)$
3. If $p(m) = 0$ has complex conjugate roots $m_1 = \lambda + i\omega$ and $m_2 = \lambda - i\omega$, where $\omega > 0$, then the general solution is
$$y(x) = e^{\lambda x}[c_1 \cos(\omega x) + c_2 \sin(\omega x)]$$

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Examples

Solve the equation $x^2y'' + xy' + y = 0$.

This equation can be transformed into one with constant coefficients by change of variables on the interval $(0, \infty)$. Let $t = \ln x$. Then $x = e^t$. Hence

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt} = \frac{dy}{dx}\,e^t.$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left[\frac{dy}{dx}e^t\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right]e^t + \frac{dy}{dx}\,e^t$$

$$= \frac{d}{dx}\left[\frac{dy}{dx}\right]e^{2t} + \frac{dy}{dt}$$

$$\Longrightarrow \frac{d^2y}{dx^2} = \frac{1}{e^{2t}}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Substituting in the ODE $x^2y'' + xy' + y = 0$, we get

$$e^{2t}\frac{1}{e^{2t}}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + e^t\frac{1}{e^t}\frac{dy}{dt} + y = 0.$$

Equivalently, we have

$$y''(t) + y(t) = 0 \quad (1)$$

The general solution of (1) is

$$y(t) = c_1\cos(t) + c_2\sin(t)$$

Therefore, the general solutions to our original ODE is

$$y(x) = c_1\cos(\ln x) + c_2\sin(\ln x)$$

Cauchy-Euler Equations

In general, the equation

$$x^2y'' + axy' + by = 0$$

where $a, b \in \mathbb{R}$, is called a Cauchy-Euler equation. Assume $x > 0$. Then making the substituting $t = \ln x$, we get a second order ODE with constant coefficients

$$y''(t) + (a-1)y'(t) + by(t) = 0$$

If $y_1(t)$ and $y_2(t)$ are linearly independent solutions to this equation, then the solutions to the Cauchy-Euler equation is given by

$$y(x) = c_1y_1(\ln x) + c_2y_2(\ln x)$$

Cauchy-Euler Equation: Solutions.

THEOREM.

Consider $x^2y'' + axy' + by = 0 \quad (1)$.
Substituting $t = \ln x$ or $x = e^t$ for $x > 0$, (1) becomes
$y''(t) + (a-1)y'(t) + by(t) = 0$ (2).
Let m_1 and m_2 be the roots of the characteristic equation
$p(m) = m^2 + (a-1)m + b = 0$. Then

1. If m_1 and m_2 are real and distinct, then the general solution of (1) is given by $\quad y(x) = c_1x^{m_1} + c_2x^{m_2}$.
2. If $m_1 = m_2$ are real, then the general solution to (1) is given by $\quad y(x) = c_1x^m + c_2x^m \ln x$.
3. When $m_1 = \lambda + i\omega$ and $m_2 = \lambda - i\omega$ are complex conjugates, where $\omega > 0$, then the general solution to (1) is given by

$$y(x) = c_1x^\lambda \cos(\omega \ln x) + c_2x^\lambda \sin(\omega \ln x).$$

Ex. Solve. $x^2y'' + 7xy' + 5y = 0$.

Putting $t = \ln x$, we get $y''(t) + (7-1)y'(t) + 5y(t) = 0$. It's char equation $= m^2 + 6m + 5 = (m+1)(m+5)$.

Hence the general solution is $y(x) = c_1\dfrac{1}{x} + c_2\dfrac{1}{x^5}$.

Ex. Solve $x^2y'' + 7xy' + 9y = 0$.

The characteristic equation of associated const. coeff. ODE is $p(m) = m^2 + (7-1)m + 9 = (m+3)^2$.

Hence the general solution is $y(x) = c_1\dfrac{1}{x^3} + c_2\dfrac{1}{x^3}\ln x$.

Ex. Solve $x^2y'' + 5xy' + 13y = 0$.

The characteristic equation of associated const. coeff. ODE is $p(m) = m^2 + (5-1)m + 13 = (m+2)^2 + 9$.

Hence the general solution is

$$y(x) = \frac{1}{x^2}[c_1\cos(3\ln x) + c_2\sin(3\ln x)].$$

Non-homogeneous Second Order Linear ODE's

Theorem

Let $f(t)$ be any solution of

$$y'' + p(t)y' + q(t)y = r(t)$$

and $y_1(t), y_2(t)$ be a basis of the solution space of the corresponding homogeneous ODE. Then the set of solutions of the non-homogeneous ODE is

$$\{f(t) + c_1 y_1(t) + c_2 y_2(t) \mid c_1, c_2 \in \mathbb{R}\}.$$

Therefore, to solve non-homogeneous ODE, (i) get one particular solution of the non-homogeneous ODE, and (ii) get the general solution of the corresponding homogeneous ODE.

Method of Variation of Parameters

If we can find two linearly independent solutions y_1 and y_2 of

$$y'' + p(x)y' + q(x)y = 0 \quad (1).$$

then we can find a particular solution of

$$y'' + p(x)y' + q(x)y = r(x) \quad (2)$$

where p, q, r are continuous of an open interval I. This method of finding a particular solution, using the solutions of corresponding homogeneous part, is called the method of variation of parameters.

Here, we try to find a partcular solution of (2) of the form

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

Method of Variation of Parameters

Now

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

gives

$$y' = v_1 y_1' + v_1' y_1 + v_2 y_2' + v_2' y_2.$$

Let's assume that v_1 and v_2 satisfy

$$v_1' y_1 + v_2' y_2 = 0.$$

Then

$$y' = v_1 y_1' + v_2 y_2'$$

Thus,

$$y'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'.$$

Method of Variation of Parameters

Substituting y, y', y'' in the given non-homogeneous ODE, we get:

$$(v_1y_1''+v_1'y_1'+v_2y_2''+v_2'y_2')+p(v_1y_1'+v_2y_2')+q(v_1y_1+v_2y_2) = r(x)$$

$$\implies v_1(y_1''+py_1'+qy_1)+v_2(y_2''+py_2'+qy_2)+v_1'y_1'+v_2'y_2' = r(x).$$

$$\implies v_1'y_1' + v_2'y_2' = r(x).$$

Recall that we also have

$$v_1'y_1 + v_2'y_2 = 0.$$

Thus, we have:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}.$$

Method of Variation of Parameters

Therefore,

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{W(y_1, y_2)}, \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W(y_1, y_2)}.$$

Thus,

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx, \;\; v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx.$$

Hence,

$$y = v_1 y_1 + v_2 y_2$$

$$= y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)}\, dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)}\, dx.$$

Ex. Solve $y'' + 6y' + 5y = e^x$.

$y_1 = e^{-x}$ and $y_2 = e^{-5x}$ are two linearly independent solutions of homogeneous part. Wronskian of y_1 and y_2 is $W(e^{-x}, e^{-5x}) = e^{-x}(-5e^{-5x}) - (-e^{-x})e^{-5x} = -4e^{-6x}$.

A particular solution y_p is given by variation of parameter:

$$y_p = y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)}\, dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)}\, dx.$$

$$= e^{-5x} \int \frac{e^{-x}e^{x}}{-4e^{-6x}}\, dx - e^{-x} \int \frac{e^{-5x}e^{x}}{-4e^{-6x}}\, dx$$

$$= -\frac{1}{4}\left[e^{-5x} \int e^{6x}\, dx - e^{-x} \int e^{2x}\, dx\right]$$

$$= -\frac{1}{4}\left[\frac{1}{6}e^{x} - \frac{1}{2}e^{x}\right] = \frac{1}{12}e^{x}$$

Thus the general solution is $y(x) = \dfrac{1}{12}e^{x} + c_1 e^{-x} + c_2 e^{-5x}$

Ex. Solve $y'' + 6y' + 5y = e^{-x}$.

$y_1 = e^{-x}$ and $y_2 = e^{-5x}$ are two LI solutions of homogeneous part. Wronskian $W(e^{-x}, e^{-5x}) = -4e^{-6x}$. A particular solution y_p is given by

$$y_p = y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)}\, dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)}\, dx.$$

$$= e^{-5x} \int \frac{e^{-x} e^{-x}}{-4e^{-6x}}\, dx - e^{-x} \int \frac{e^{-5x} e^{-x}}{-4e^{-6x}}\, dx$$

$$= -\frac{1}{4}\left[e^{-5x} \int e^{4x}\, dx - e^{-x} \int dx\right]$$

$$= -\frac{1}{4}\left[e^{-x}\left(\frac{1}{4} - x\right)\right] = -\frac{1}{16} e^{-x}(1 - 4x)$$

Thus the general solution is given by

$$y(x) = -\frac{1}{16} e^{-x}(1-4x) + c_1 e^{-x} + c_2 e^{-5x} = \frac{1}{4} x e^{-x} + c_1 e^{-x} + c_2 e^{-5x}$$

Ex. Find a particular solution of $y'' + 4y = 3\cos 2t$.

$y_1 = \cos 2t$, $y_2 = \sin 2t$ are solutions of homogeneous part. Wronskian $W(y_1, y_2) = 2$. A particular solution y_p is given

$$y_p = y_2 \int \frac{y_1 r}{W(y_1, y_2)}\, dt - y_1 \int \frac{y_2 r}{W(y_1, y_2)}\, dt.$$

$$= \sin 2t \int \frac{\cos 2t \cdot 3\cos 2t}{2} dt - \cos 2t \int \frac{\sin 2t \cdot 3\cos 2t}{2} dt$$

$$= \sin 2t \int \frac{3}{4}(1 + \cos 4t)\, dt - \cos 2t \int \frac{3}{4} \sin 4t\, dt$$

$$= \frac{3}{4} \sin 2t \left[t + \frac{1}{4} \sin 4t \right] - \frac{3}{4} \cos 2t (-\frac{1}{4} \cos 4t)$$

$$= \frac{3}{4} t \sin 2t + \frac{3}{16} [\sin 2t \sin 4t + \cos 2t \cos 4t]$$

$$= \frac{3}{4} t \sin 2t + \frac{3}{16} \cos 2t.$$

Ex. Find a particular solution of $y'' + y = \csc x$.

$y_1 = \sin x$ and $y_2 = \cos x$ are solutions of homogeneous part.
Wronskian $W(y_1, y_2) = -1$.
A particular solution y_p is given by variation of parameter:

$$y_p = y_2 \int \frac{y_1 r}{W(y_1, y_2)}\, dx - y_1 \int \frac{y_2 r}{W(y_1, y_2)}\, dx.$$

$$= \cos x \int \frac{\sin x \csc x}{-1} dx - \sin x \int \frac{\cos x \csc x}{-1} dx$$

$$= \cos x(-x) + \sin x \ln|\sin x|$$

Hence, a particular solution is given by

$$y(x) = -x \cos x + \sin x \ln|\sin x|.$$

Exercise. Solve the IVP

$$(x^2 - 1)y'' + 4xy' + 2y = \frac{2}{x+1}, \; y(0) = -1, \; y'(0) = -5$$

Given that

$$y_1 = \frac{1}{x-1}, \quad \text{and} \quad y_2 = \frac{1}{x+1}$$

are solutions of homogeneous part.

The solution is given by

$$y(x) = \frac{2\ln|x+1|}{x-1} + \frac{3x+1}{x^2-1}$$