# PH108

## Lecture 10:

## Laplace’s equation: Techniques to solve for $\Phi$ – separation of variables

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Supplementary reading: *Griffiths 3.3.2* -

# Laplace's equation

Start with Poisson equation:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Gauss's law in differential fom

$$\oiint \vec{E} \cdot d\vec{\sigma} = \frac{q_{encl}}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{E} \equiv \lim_{\Delta v \to 0} \frac{1}{\Delta v} \oint \vec{E} \cdot \overrightarrow{d\sigma} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \frac{\bar{\rho} \Delta v}{\epsilon_0}$$

$$\vec{E} = -\vec{\nabla}\Phi$$

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

$$For\ \rho = 0 \qquad \boxed{\nabla^2 \Phi = 0}$$

# Solution to Laplace's equation is unique

Two distinct solutions cannot satisfy Laplace equation with the *same* boundary condition

Suppose $\Phi_1$ is a solution: $\nabla^2\Phi_1 = 0$

Suppose $\Phi_2$ is a second solution: $\nabla^2\Phi_2 = 0$

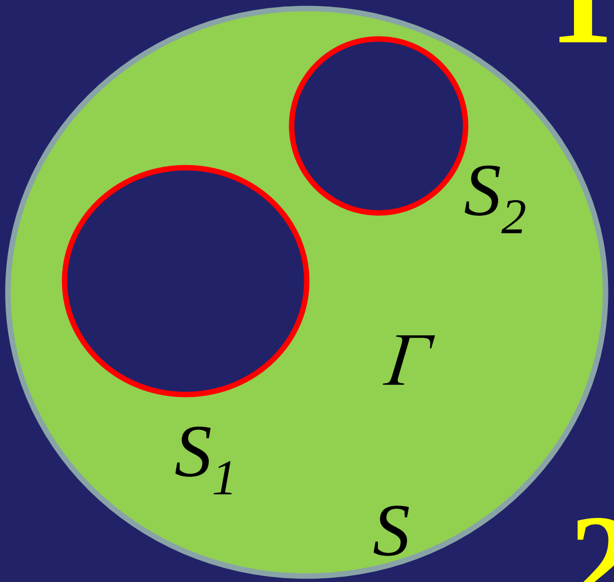
Let $\Phi_3 = (\Phi_2 - \Phi_1) \rightarrow \nabla^2\Phi_3 = \nabla^2\Phi_2 - \nabla^2\Phi_1 = 0$

$\Phi_1 = \Phi_2$ at boundary, so $\Phi_3 = 0$ at boundary and everywhere*

**More rigorous proof in the appendix*

# Boundary conditions are of two types

Need to determine Φ *in the volume Γ*




**1** Value of Φ is specified at the boundary surfaces $S_1 \ldots S_2, S$

<u>Dirichlet</u> boundary condition

**2** Value of $\frac{\partial \Phi}{\partial n}$ is specified at the boundary surfaces $S_1 \ldots S_2, S$

<u>Neumann</u> boundary condition

Uniqueness theorem works when *either* Dirichlet *or* Neumann b.c. specified, *not both*

*(see appendix)*

# Any solution that works is good

Consequence of Uniqueness theorem:

**Problem:** You need to find $\Phi$

You are given boundary conditions that $\Phi$ must satisfy (Dirichlet or Neumann)

You arrive at a solution for $\Phi$ that satisfies $\nabla^2\Phi=0$ and the boundary conditions by *any* method

(intuition, appeal to a higher power, appeal to your neighbor etc)

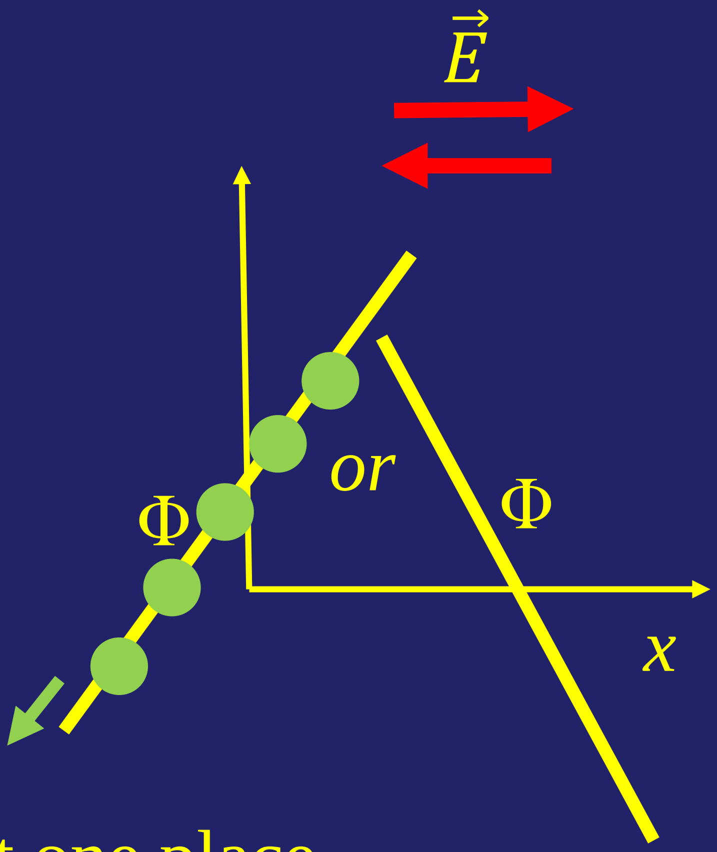
It is the unique solution

# Electrostatic $\Phi$ has no maxima or minima

In 1 dimension
$$\frac{\partial^2}{\partial x^2}\Phi = 0$$

solves to:
$$\Phi = ax + b$$

$$\vec{E} = -\vec{\nabla}\Phi = -\mathrm{a}$$



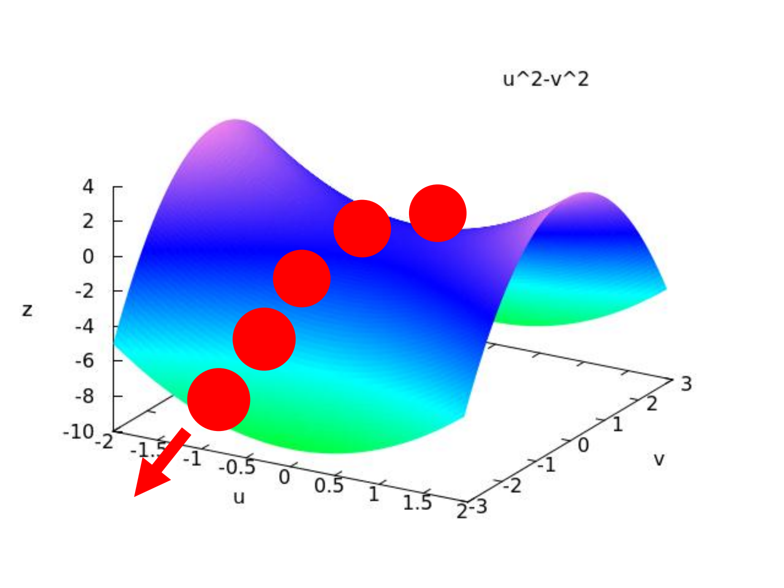

Electrostatic potential *cannot*
hold a charge in static equilibrium at one place

# Electrostatic Φ has no extrema in higher dimensions

In 2 dimensions $$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = 0$$

$$\frac{\partial^2}{\partial x^2}\Phi = -\frac{\partial^2}{\partial y^2}\Phi = a$$

solves to: $\Phi(x, y) = \frac{a}{4}(x^2 - y^2)$




<u>Earnshaw's theorem:</u>
Electrostatic potential *cannot* hold a charge in static equilibrium at one place

# How to solve Laplace equation

"Formal" solution in 2 Cartesian dimensions

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = 0$$

$Let\ \Phi(x,y) = X(x)Y(y)$ Separation of variables

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0 \longrightarrow *\frac{1}{XY} \longrightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = k^2$$

# We get 2nd order ODE for *X(x) and Y(y)*

$$\frac{d^2X}{dx^2} - k^2X = 0$$

$$\frac{d^2Y}{dx^2} + k^2Y = 0$$

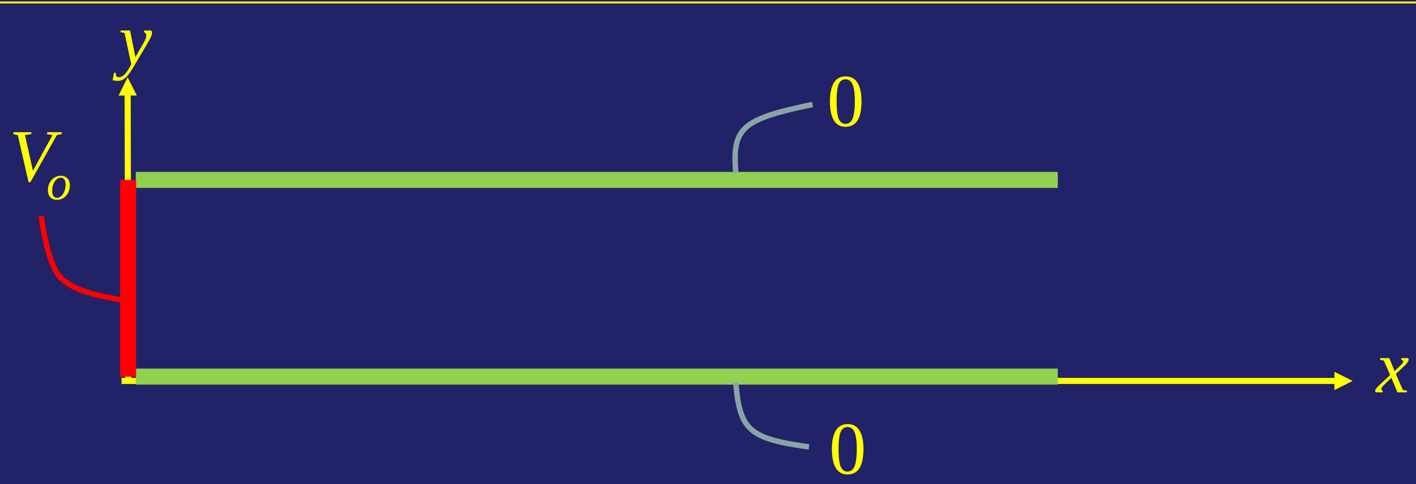
$$X(x) = Ae^{kx} + Be^{-kx}$$

$$Y(y) = Csin(ky) + Dcos(ky)$$

$$\Phi(x, y) = \{Ae^{kx} + Be^{-kx}\}\{Csin(ky) + Dcos(ky)\}$$

Constants *A,B,C,D (and k) determined by boundary conditions*

# Sample calculation in 2-D *(x,y)*




*Boundary conditions:*

1) $\Phi(x,0) = 0$ 2) $\Phi(x,\pi) = 0$

3) $\Phi(0,y) = \mathrm{V}_0$ 4) $\Phi(x,y) = 0 \; as \; x \to \infty$

$$\Phi(x,y) = \{Ae^{kx} + Be^{-kx}\}\{Csin(ky) + Dcos(ky)\}$$

# Boundary conditions give *A*,*B*,*C*,*D*,*k*

$1)\ \Phi(x,0)=0$ $\qquad$ $2)\ \Phi(x,\pi)=0$

$3)\ \Phi(0,y)=\mathrm{V}_0$ $\qquad$ $4)\ \Phi(x,y)=0\ as\ x\to\infty$

$$\Phi(x,y)=\{Ae^{kx}+Be^{-kx}\}\{Csin(ky)+Dcos(ky)\}$$

*(4)→A=0* $\qquad$ *(1)→D=0*

$$\Phi(x,y)=BCe^{-kx}sin(ky)$$

$(2)\to\sin(k\pi)=0\to \mathrm{k}=\text{integer}$

# Boundary conditions give *A,B,C,D,k*

$$\Phi(x,y) = \sum_n E_n e^{-nx} sin(ny)$$

$$3)\ \Phi(0,y) = \mathrm{V}_0 \rightarrow \quad V_0 = \sum_n E_n sin(ny)$$

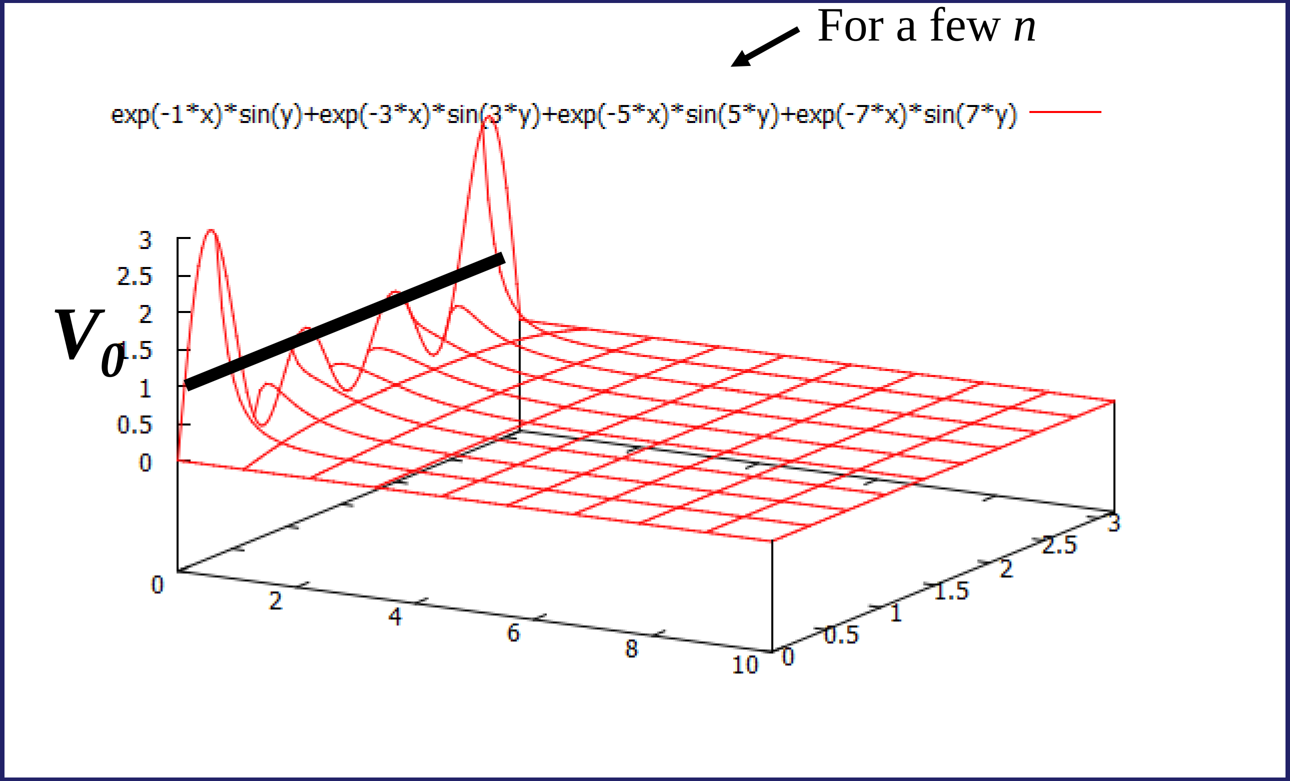
Multiply both sides by *sin(my)* and integrate from 0 to *π*

$$\int_0^\pi V_0 \sin(my)\, dy = \sum_n E_n \underbrace{\int_0^\pi \sin(my)\, sin(ny)\ dy}_{\frac{\pi}{2}\delta_{mn}} \quad \rightarrow E_n = \frac{4V_0}{n\pi}$$

***odd n***

$$\boxed{\Phi(x,y) = \sum_{n=1,3,5\ldots}^{\infty} \frac{4V_0}{n\pi} e^{-nx} sin(ny)}$$

# Φ(x,y) *contours for our toy setup*



# Separation of variables in spherical coordinates is similar

$$\nabla^2\Phi = 0$$

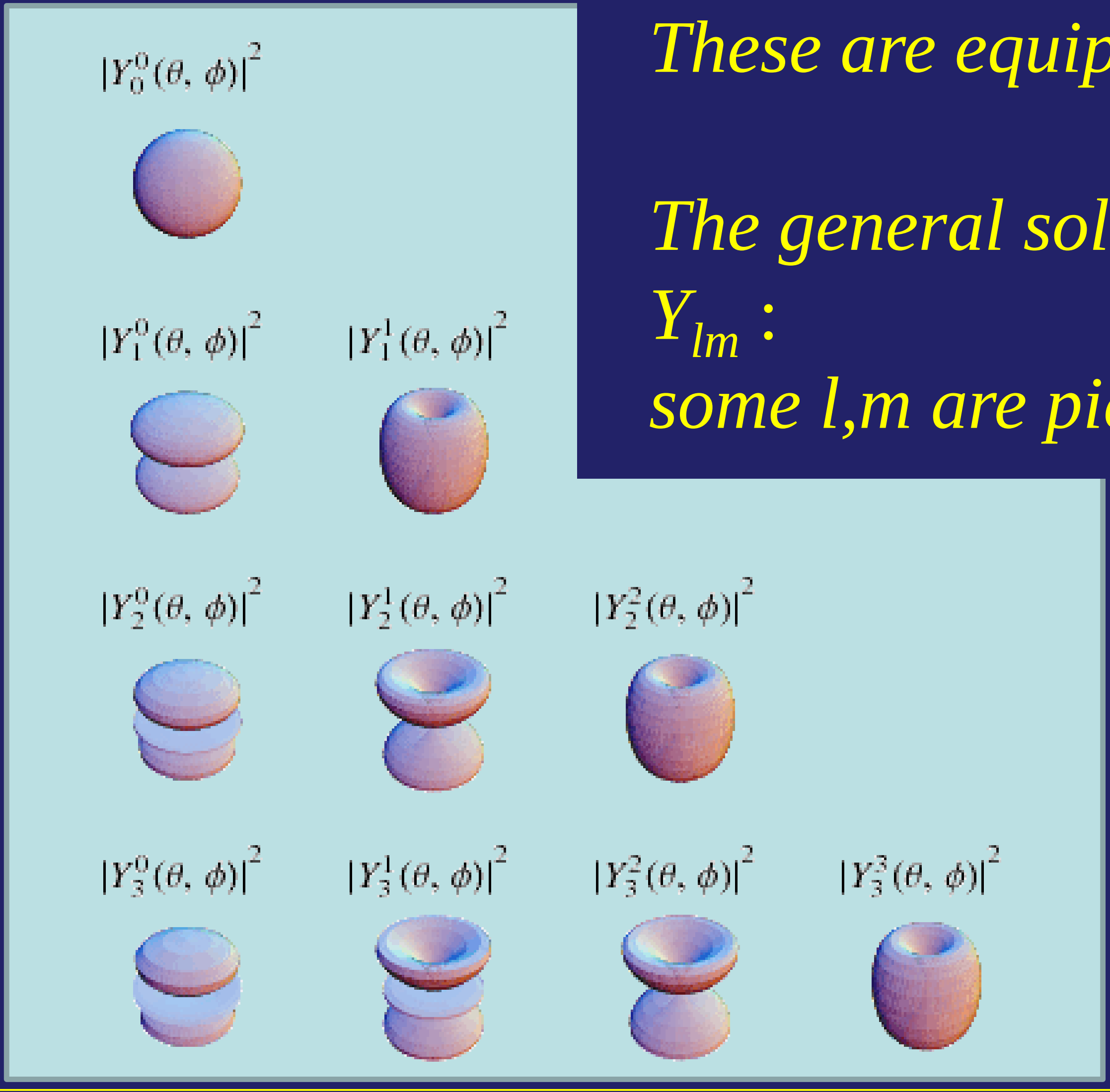
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2 sin\theta}\left(sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2\Phi}{\partial\phi^2}\right)$$

$$\Phi(r,\theta,\phi) = R(r)P(\theta)F(\phi) \quad \rightarrow \textit{separate variables}$$

*Griffiths 3.3.2:*

$$\Phi(r,\theta,\phi) = \sum_{l,m}\left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\phi)$$

# What do the potential contours look like?




*These are equipotential surfaces.*

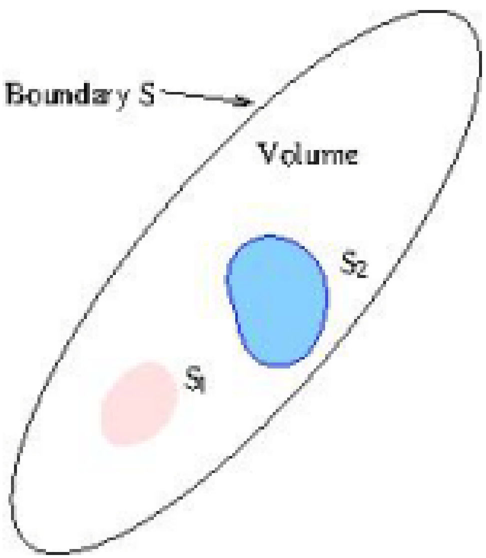
*The general solution is a sum over $Y_{lm}$ :*
*some l,m are picked by boundary counditions*

If a charge is placed in this electrostatic $\Phi$, can it be held stable?

# More mathematics in tutorial problems

# Appendix: Proof that solution to Laplace's equation is unique

To prove the uniqueness theorem, let us assume that contrary to the assertion made in the theorem, there exist two solutions $\varphi_1$ and $\varphi_2$ of either Poisson's or Laplace's equation which satisfy the same set of boundary conditions on surfaces $S_1, S_2, \ldots$ and the boundary S. The conditions, as stated above, may be either of Dirichlet type or Neumann type :




$$\varphi_1|_{S_1} = \varphi_2|_{S_1}, \varphi_1|_{S_2} = \varphi_2|_2, \ldots \varphi_1|_S = \varphi_2|_S$$

OR

$$\left.\frac{\partial\varphi_1}{\partial n}\right|_{S_1} = \left.\frac{\partial\varphi_2}{\partial n}\right|_{S_1}, \left.\frac{\partial\varphi_1}{\partial n}\right|_{S_2} = \left.\frac{\partial\varphi_2}{\partial n}\right|_{S_2}, \ldots, \left.\frac{\partial\varphi_1}{\partial n}\right|_S = \left.\frac{\partial\varphi_2}{\partial n}\right|_S$$

Let us define a new function $\varphi = \varphi_1 - \varphi_2$. In view of the fact that $\varphi_1$ and $\varphi_2$ satisfy the same boundary conditions, the boundary condition satisfied by $\varphi$ are $\varphi|_{S_1} = \varphi|_{S_2} = \cdots = \varphi|_S = 0$ (Dirichlet) OR $\left.\frac{\partial\varphi}{\partial n}\right|_{S_1} = \left.\frac{\partial\varphi}{\partial n}\right|_{S_2} = \ldots \left.\frac{\partial\varphi}{\partial n}\right|_S = 0$ (Neumann).

Further, whether $\varphi_1$ and $\varphi_2$ satisfy Poisson's or Laplace's equation, their difference $\varphi = \varphi_1 - \varphi_2$ satisfies Laplace's equation.

In order to prove the Uniqueness theorem we will use Green's First Identity, derived in Module 1, which states that for two arbitrary scalar fields $\phi$ and $\psi$, the following identity holds,

$$\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)d^3r = \oint_S \phi\frac{\partial\psi}{\partial n}dS$$

where S is the boundary defining the volume V. We choose $\phi = \psi = \varphi$, to get,

$$\int_V (\varphi\nabla^2\varphi + |\nabla\varphi|^2)d^3r = \oint_S \varphi\frac{\partial\varphi}{\partial n}dS$$

# Appendix: Proof that solution to Laplace's equation is unique

Since on the surface , either the Dirichlet or the Neumann boundary condition is valid, the right hand side of the above is zero everywhere. (Note that the surface consists of $S + S_1 + S_2 + \cdots$, with the direction of normal being outward on S and inward on the conductor surfaces enclosed).

Since $\varphi$ satisfies Laplace's equation, we are then left with $\int_V |\nabla\varphi|^2 \, d^3r = 0$. The integrand, being a square of a field is positive everywhere in the volume and its integral can be zero only if the integral itself is identically zero. Thus we have, $\nabla\varphi = 0$ , which leads to $\varphi =$ constant everywhere on the volume. If Dirichlet boundary condition is satisfied, then $\varphi = 0$ on the surface and therefore, it is zero everywhere in the volume giving, $\varphi_1 = \varphi_2$. If, on the other hand, Neumann boundary condition is satisfied, we must have $\frac{\partial\varphi}{\partial n} = 0$, i.e. $\varphi = \varphi_1 - \varphi_2 =$ constant everywhere. Since the constant can be chosen arbitrarily, we take the constant to be zero and get $\varphi_1 = \varphi_2$.