MA-108 Ordinary Differential Equations

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9th March, 2015
D1 - Lecture 4

Solution of 1st order linear ODE

Theorem (Existence and Uniqueness for 1st order linear ODE)

Assume $p(x)$ and $f(x)$ are continuous on $I = (a, b)$. If $x_0 \in I$, then IVP

$$y' + p(x)y = f(x), \quad y(x_0) = y_0 \in \mathbb{R}$$

has a unique solution $y = \phi(x)$ on the interval I.

The solution of the homogeneous part $y' + p(x)y = 0$ is given by

$$y_1(x) = e^{-\int p(x)\,dx}$$

By variation of parameters method, the solution of ODE is $y = uy_1$, where $u'(x) = f(x)/y_1(x)$. Therefore

$$y(x) = e^{-\int p(x)\,dx}\left(\int f(x)\,e^{\int p(x)\,dx}\,dx + C\right)$$

Solution of 1st order linear ODE

- The interval of existence and uniqueness of the solution of IVP

$$y' + p(x)y = f(x), \quad y(x_0) = y_0 \in \mathbb{R}$$

is independent of y_0.

- The uniqueness condition implies that the one parameter family of solutions,

$$y(x) = e^{-\int p(x)\,dx} \left(\int f(x)\, e^{\int p(x)\,dx}\, dx + C \right)$$

is, in fact, a general solution on the interval I.

- Any solution of the IVP is obtained from the general solution for some scalar C.

Interval of validity of an ODE

A set $S \subset \mathbb{R}$ is **open** if for any $x \in S$, there exist $\epsilon > 0$ such that $(-\epsilon + x, x + \epsilon) \subset S$.

An open set $S \subset \mathbb{R}$ is **connected** if $\alpha, \beta \in S$ and $x \in \mathbb{R}$ such that $\alpha < x < \beta$, then $x \in S$.

A connected open sets in $\mathbb{R}$ is called an **(open) interval** and they are of the form (a, b) for $-\infty \leq a < b \leq \infty$.

Let $\Omega = (0,1) \cup (1,2)$. Then Ω is not connected. It is union of two disjoint open intervals. Let

$$f : (0,1) \to \mathbb{R} \;\; \text{and} \;\; g : (1,2) \to \mathbb{R}$$

be n-times differential functions. Define

$$h : \Omega \to \mathbb{R} \;\; \text{as} \;\; h|_{(0,1)} = f \;\; \text{and} \;\; h|_{(1,2)} = g$$

Then h is n-times differential function.

Interval of validity is a connected open interval

Assume the general solution of

$$y' = f(x, y)$$

is defined on the interval $\Omega = (0, 1) \cup (1, 2)$. Let $x_0 \in (0, 1)$ and $x_1, x_2 \in (1, 2)$. Let

$$y_0(x) : (0, 1) \to \mathbb{R}$$

be a solution of $y' = f(x, y)$ with initial condition $y(x_0) = a_0$. Let

$$y_1(x), y_2(x) : (1, 2) \to \mathbb{R}$$

be solutions of $y' = f(x, y)$ with initial conditions

$$y(x_1) = a_1 \quad \text{and} \quad y(x_2) = a_2$$

respectively.

Interval of validity is a connected open interval

Let us define

$$h_1(x), h_2(x) : \Omega \to \mathbb{R} \ \text{ as}$$

$$h_1(x) = h_2(x) = y_0(x) \ \text{ for } \ x \in (0,1) \ \text{ and}$$

$$h_1(x) = y_1(x), \quad h_2(x) = y_2(x) \ \text{ for } \ x \in (1,2)$$

Then h_1 and h_2 are solutions of $y' = f(x,y)$ on $\Omega = (0,1) \cup (1,2)$ with same initial condition $y(0) = a_0$.

Remark. Therefore, uniqueness of solution of an IVP:

$$y' = f(x,y), \quad y(x_0) = a_0$$

means there exist a unique solution on a (connected) open interval containing x_0.
The interval of validity for a solution of an IVP will always be an open interval (connected).

Example

Solve

$$y' + (\cot x)y = x\ \csc x$$

The functions $p(x) = \cot x$ and $f(x) = x \csc x$ are both continuous except at points $x = n\pi$ for integers n.

Let's find solutions of ODE on the intervals $(n\pi, (n+1)\pi)$.

A solution of homogeneous part is

$$y_1(x) = e^{-\int p(x)\,dx} = e^{-\int \cot x\,dx} = e^{-\ln|\sin x|} = \frac{1}{\sin x}$$

Therefore the solution of ODE is $y(x) =$

$$y_1(x)\left(\int \frac{f(x)}{y_1(x)}\,dx + C\right) = \frac{1}{\sin x}\left(\int x \csc x\ \sin x\,dx + C\right)$$

$$= \frac{1}{\sin x}\left(\int x\,dx + C\right) = \frac{1}{\sin x}\left(\frac{x^2}{2} + C\right)$$

Example continued..

If we put the initial condition $y(\pi/2) = 1$ in the general solution

$$y(x) = \frac{1}{\sin x}\left(\frac{x^2}{2} + C\right)$$

then

$$1 = \frac{\pi^2}{8} + C \implies C = 1 - \frac{\pi^2}{8}$$

Thus the solution of IVP is

$$y(x) = \frac{x^2}{2\sin x} + \frac{(1 - \frac{\pi^2}{8})}{\sin x}$$

The interval of validity of this solution is $(0, \pi)$. $\square$

Solution in terms of integral

Example. Solve the IVP $\quad y' - 2xy = 1, \quad y(0) = y_0$

The solution of the homogeneous part is

$$y_1(x) = e^{\int -p(x)\,dx} = e^{\int 2x\,dx} = e^{x^2}$$

The general solution is

$$y(x) = y_1(x)\left(\int \frac{f(x)}{y_1(x)}\,dx + C\right) = e^{x^2}\left(\int e^{-x^2}dx + C\right)$$

Since the initial is at $x_0 = 0$, rewrite the general solution as

$$y(x) = e^{x^2}\left(\int_0^x e^{-t^2}\,dt + C\right)$$

$y(0) = y_0$ gives $C = y_0$. The IVP has solution (defined on $\mathbb{R}$)

$$y(x) = e^{x^2}\left(\int_0^x e^{-t^2}\,dt + y_0\right)$$

$\square$

Solution of 1st order Non-Linear ODE

Existence and Uniqueness for Non-Linear ODE.

(a) (Existence) Assume $f(x,y)$ is continuous on an **open rectangle** $R := \{(x,y) \in \mathbb{R}^2 | \, a < x < b, \; c < y < d\}$ that contains (x_0, y_0). Then IVP: $\boxed{y' = f(x,y), \quad y(x_0) = y_0 \quad (*)}$ has at least one solution on some interval $(a_1, b_1) \subset (a,b)$ containing x_0.

(b) (Uniqueness) If both f and $\partial f / \partial y$ are continuous on R, then IVP (*) has a unique solution on some interval $(a', b') \subset (a,b)$ containing x_0.

Remark. (a) is an existence theorem. It guarantees a solution on some interval containing x_0, but does not give any information on how to find the solution or how to find the interval of validity. In this case, IVP can have more than one solution.

(b) is a uniqueness theorem. It guarantees that IVP has a unique solution on some interval $(a', b') \subset (a, b)$ containing x_0. However, if $(a', b') \neq (-\infty, \infty)$, then IVP may have more than one solution on a larger interval containg (a', b').

For example. it may happen that $b' < \infty$ and two solutions y_1, y_2 are defined on some interval (a', b_1) with $b_1 > b'$, and have different values for $b' < x < b_1$.

Thus the graph of y_1 and y_2 branch off in different directions at $x = b'$.

In this case, since $y_1 = y_2$ on (a', b'), by continuity, $y_1(b') = y_2(b') := \overline{y}$.

Solution of 1st order Non-Linear ODE

Now y_1 and y_2 are both solutions of the IVP:

$$y' = f(x, y), \quad y(b') = \overline{y} \qquad (**)$$

they differ on every open interval containing b'.

Therefore, f or $\partial f / \partial y$ must have a discontinuity at some point in each open rectangle that contains $(b', \overline{y}) \in \mathbb{R}^2$.

Why?

If not, then by uniqueness theorem, (**) will have a unique solution on some open interval containing b', a contradiction.

$\square$

Example

Ex. Consider the IVP

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y(x_0) = y_0 \qquad (*)$$

If

$$f(x, y) = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad \text{then}$$

$$\frac{\partial f}{\partial y} = \frac{-2y}{1 + x^2 + y^2} + \frac{-2y(x^2 - y^2)}{(1 + x^2 + y^2)^2} = \frac{-2y(1 + 2x^2)}{(1 + x^2 + y^2)^2}$$

Since $f(x, y)$ and $\partial f/\partial y$ are continuous for all $(x, y) \in \mathbb{R}^2$, by existence and uniqueness theorem, if (x_0, y_0) is arbitrary, then $(*)$ has a unique solution on some open interval containing x_0. $\square$

Ex. Consider the IVP $\quad y' = \dfrac{x^2 - y^2}{x^2 + y^2}, \quad y(x_0) = y_0 \qquad (*)$

If $$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad \text{then}$$

$$\frac{\partial f}{\partial y} = \frac{-2y}{x^2 + y^2} + \frac{-2y(x^2 - y^2)}{(x^2 + y^2)^2} = \frac{-4x^2 y}{(x^2 + y^2)^2}$$

Here $f(x,y)$ and $\partial f / \partial y$ are continuous for all $(x,y) \in \mathbb{R}^2$, except at $(0,0)$.

If $(x_0, y_0) \neq (0,0)$, then there is an open rectangle R containing (x_0, y_0) but not containing $(0,0)$.

Since $f(x,y)$ and $\partial f / \partial y$ are continuous on R, by existence and uniqueness theorem, if $(x_0, y_0) \neq (0,0)$, then $(*)$ has a unique solution on some open interval containing x_0. $\square$