Tutorial-6, MA 106 (Linear Algebra)

Linear Algebra and its Applications by Gilbert Strang

This tutorial sheet consists of the problems based on 2.6, 3.4 and 4.2. This includes:
Problem Set 2.6: $[2, 4, 5, 11, 12, 17, 18, 27, 30, 31, 36, 44, 50]$
Problem Set 3.4: $[1, 10, 23, 16, 29, 32]$
Review Exercises Chapter 3: $[3.2, 3.7, 3.15, 3.19, 3.29, 3.31]$
Problem Set 4.2: [4, 9, 12, 13, 14, 17, 19, 29]
Problem Set 4.3 [5, 6, 7, 9, 10, 15, 20, 21, 26, 35]

Section 2.6

1. On the space $\mathcal{P}_3$ of cubic polynomials, what matrix represents d^2/dt^2? Construct the 4 by 4 matrix from the standard basis 1, t, t^2, t^3. Find its nullspace and column space. What do they mean in terms of polynomials?

2. Suppose A is a linear transformation from the x-y plane to itself. Why does $A^{-1}(x + y) = A^{-1}x + A^{-1}y$? If A is represented by the matrix M, explain why A^{-1} is represented by M^{-1}.

3. What 3 by 3 matrices represent the transformations that
   (a) project every vector onto the $x - y$ plane?
   (b) reflect every vector through the $x - y$ plane?
   (c) rotate the $x - y$ plane through 90°, leaving the z-axis alone?
   (d) rotate the $x - y$ plane, then $x - z$, then $y - z$, through 90°.?
   (e) carry out the same three rotations, but each one through 180°.?

4. The matrix $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ yields a shearing transformation, which leaves the y-axis unchanged. Sketch its effect on the x-axis, by indicating what happens to $(1, 0)$ and $(2, 0)$ and $(1, 0)$ and how the whole axis is transformed.

5. Which of these transformations on $\mathbb{R}^3$, satisfy $T(v + w) = T(v) + T(w)$, and which satisfy $T(cv) = cT(v)$?
   (a) $T(v) = v/\|v\|$.
   (b) $T(v) = v_1 + v_2 + v_3$.
   (c) $T(v) = (v_1, 2v_2, 3v_3)$.
   (d) $T(v)$ =largest component of v.

6. Find the range and kernel (those are new words for the column space and nullspace) of T.
   (a) $T(v_1, v_2) = (v_2, v_1)$.
   (b) $T(v_1, v_2, v_3) = (v_1, v_2)$.
   (c) $T(v_1, v_2) = (0, 0)$.
   (d) $T(v_1, v_2) = (v_1, v_1)$.

7. Suppose T transposes every matrix M. Try to find a matrix A that gives $AM = M^T$ for every M. Show that no matrix A will do it. Is this a linear transformation that doesnt come from a matrix?

8. (a) What matrix transforms $(1, 0)$ into $(2, 5)$ and transforms $(0, 1)$ to $(1, 3)$?

(b) What matrix transforms $(2, 5)$ to $(1, 0)$ and $(1, 3)$ to $(0, 1)$?

(c) Why does no matrix transform $(2, 6)$ to $(1, 0)$ and $(1, 3)$ to $(0, 1)$?

9. (a) What matrix M transforms $(1, 0)$ and $(0, 1)$ to (r, t) and (s, u)?

   (b) What matrix N transforms (a, c) to (b, d) to $(1, 0)$ and $(0, 1)$?

   (c) What conditions on a, b, c and$d$ will make the previous part impossible?

10. True or false: If we know $T(v)$ for n different nonzero vectors in $\mathbb{R}^2$, then we know $T(v)$ for every vector in $\mathbb{R}^n$.

Section 3.4

11. Project $b = (0, 3, 0)$ onto each of the orthonormal vectors $a_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ and $a_2 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ and then find its projection p onto the plane of a_1 and a_2.

12. Project the vector $b = (1, 2)$ onto two vectors that are not orthogonal, $a_1 = (1, 0)$ and $a_2 = (1, 1)$. Show that, unlike the orthogonal case, the sum of the two one- dimensional projections does not equal b.

13. Find a third column so that the matrix

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & \\ 1/\sqrt{3} & 2/\sqrt{14} & \\ 1/\sqrt{3} & -3/\sqrt{14} & \end{bmatrix}$$

is orthogonal. It must be a unit vector that is orthogonal to the other columns; how much freedom does this leave? Verify that the rows automatically become orthonormal at the same time.

14. If the vectors q_1, q_2, q_3 are orthonormal, what combination of q_1 and q_2 is closest to q_3?

15. Find an orthonormal set q_1, q_2, q_3 for which q_1, q_2, span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

Which fundamental subspace contains q_3 ? What is the least-squares solution of $Ax = b$ if $b = [1\ 2\ 7]^T$?

16. Apply Gram-Schmidt to $(1, -1, 0)$, $(0, 1, -1)$, and $(1, 0, -1)$, to find an orthonormal basis on the plane $x_1 + x_2 + x_3 = 0$. What is the dimension of this subspace, and how many nonzero vectors come out of Gram-Schmidt?

17. Find orthogonal vectors A, B, C by Gram-Schmidt from $a = (1, -1, 0, 0)$, $b = (0, 1, -1, 0)$, $c = (0, 0, 1, -1)$. A, B, C and a, b, c are bases for the vectors perpendicular to $d = (1, 1, 1, 1)$.

18. Construct the projection matrix P onto the space spanned by $(1, 1, 1)$ and $(0, 1, 3)$.

19. If Q is orthogonal, is the same true of Q^3?

20. The system $Ax = b$ has a solution if and only if b is orthogonal to which of the four fundamental spaces?

21. Find an orthonormal basis for the plane $x - y + z = 0$, and find the matrix P that projects onto the plane. What is the nullspace of P.

22. CT scanners examine the patient from different directions and produce a matrix giving the densities of bone and tissue at each point. Mathematically, the problem is to recover a matrix from its projections. in the 2 by 2 case, can you recover the matrix A if you know the sum along each row and down each column?

23. Find an orthonormal basis for $\mathbb{R}^3$ starting with the vector $(1,1,1)$.

Section 4.2

24. For each n, how many exchanges will put $(R_n, \ldots, R_1)$ into the normal order $(R_1, \ldots, R_n)$? Find $det(P)$ for $n \times n$ permutation matrix with 1's on the reverse diagonal.

25. True or False, with reason if true and counterexample if false.

    (a) If A and B are identical except that $b_{11} = 2a_{11}$, then $det(B) = 2\,det(A)$.

    (b) The determinant is the product of pivots.

    (c) If A is invertible and B is singular, then $A + B$ is invertible.

    (d) If A is invertible and B is singular, then AB is singular.

    (e) The determinant of $AB - BA$ is zero.

26. For which values of λ, $A - \lambda I = \begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$ is singular?

27. If every row of A adds to zero, prove that $det(A) = 0$. If every row adds to 1, prove that $det(A - I) = 0$. Show by an example that this does not imply $det(A) = 1$.

28. Suppose that $CD = -DC$ and find the flaw in the following argument: Taking determinants gives $det(C)\,det(D) = -det(D)\,det(C)$, so either $det(C) = 0$ or $det(D) = 0$. Thus $CD = -DC$ is only possible if C or D is singular.

29. Find these determinants by Gaussian elimination:

$$det \begin{pmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{pmatrix}, \qquad det \begin{pmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{pmatrix}$$

30. What is wrong with this proof that projection matrices have $det(P) = 1$?

$$P = A(A^TA)^{-1}A^T, \qquad \text{so} \qquad |P| = |A|\frac{1}{|A^T||A|}|A^T| = 1$$

Section 4.3

31. Let A_n be $n \times n$ matrix which is $(1,1,1)$ tri-diagonal.

$$A_1 = (1),\ A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ A_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

    (a) Let $D_n = det(A_n)$. Expand along first row to show that $D_n = D_{n-1} - D_{n-2}$.

    (b) Starting from $D_1 = 1, D_2 = 0$, find $D_3, \ldots, D_8$. Find D_{1000}.

32. (a) Find the LU factorization, the pivots, and determinant of 4×4 matrix whose entries are $a_{ij} =$ smaller of i and j.

    (b) Find the determinant if $a_{ij} =$ smaller of n_i and n_j, where $n_1 = 2, n_2 = 6, n_3 = 8, n_4 = 10$. Can you give a general rule for any $n_1 \leq n_2 \leq n_3 \leq n_4$?

33. In a 5×5 matrix, does a $+$ sign or $-$ sign go with $a_{15}\, a_{24}\, a_{33}\, a_{42}\, a_{51}$ down the reverse diagonal?

34. Suppose the matrix A is fixed, except that a_{11} varies from $-\infty$ to $+\infty$. Give examples in which $det(A)$ is always zero or never zero. Then show from the cofactor expansion that otherwise $det(A) = 0$ for exactly one value of a_{11}.

35. Compute the determinant of A_2, A_3, A_4. Can you predict A_n?

$$A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

36. (a) Place the smallest number of zeros in a 4×4 matrix that will guarantee that $det(A) = 0$.

    (b) Place as many zeros as possible while still showing $det(A) = 0$.

37. Show that $det(A) = 0$, regardless of the five nonzeros marked with x. Find rank of $A = \begin{pmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{pmatrix}$.

38. (a) If $a_{11} = a_{22} = a_{33} = 0$, how many of the six terms in $det(A_{3\times3})$ will be zero?

    (b) If $a_{11} = a_{22} = a_{33} = a_{44} = 0$, how many of the 24 terms in $det(A_{4\times4})$ will be zero?

39. Let A_n be $n \times n$ matrix which is $(-1, 2, -1)$ tri-diagonal. Let B_n be same as A_n except that $b_{11} = 1$. Find $det(B_n)$.

40. With 2 by 2 blocks, you cannot always use block determinants.

$$det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = det(A)\, det(D), \quad \text{but} \quad det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq det(A)det(D) - det(C)det(B)$$

    (a) Why is first statement true. Somehow B does not enter.

    (b) Show by example that equality fails when C enters.

    (c) Show by example that the answer $det(AD - CB)$ is also wrong.

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