PH 103: Electricity and Magnetism
Tutorial sheet 1S : Vector Calculus

1. Using arrows of proper magnitude and direction, sketch each of the following two dimensional vector functions:

   (a) $(\hat{\imath} + \hat{\jmath})/\sqrt{2}$

   (b) $y\,\hat{\imath}$

   (c) $x\,\hat{\imath} - y\,\hat{\jmath}$

   (d) $y\,\hat{\imath} + xy\,\hat{\jmath}$

   (e) $(y\,\hat{\imath} + x\,\hat{\jmath})/\sqrt{x^2 + y^2}, \quad (x, y) \neq (0, 0)$

2. Write a formula for a vector function in three dimensions which is in the positive radial direction and whose magnitude is 1.

3. Write a formula for a vector function in two dimensions which is tangential to a circle centered at the origin and whose magnitude at ay point (x, y) is equal to its distance from the origin.

4. The height of a hill (in feet) is given by:

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

   where y is the distance (in miles) north and x is the distance east of a chosen origin.

   (a) Where is the top of the hill located ?

   (b) How high is the hill?

   (c) How steep is the slope of the hill 1mile north and 1 mile east of the origin? In what direction is it steepest at that point?

5. A force defined by $F = A(y^2\,\hat{\imath} + 2x^2\,\hat{\jmath})$ is exerted on a particle which is initially at the origin of the co-ordinate system. A is a positive constant. We transport the particle on a triangular path defined by the points $(000), (100), (110)$ in the counterclockwise direction.

   (a) How much work does the force do when the particle travels around the path? Is this is conservative force?

   (b) The particle is placed at rest right at the origin. Is this a stable situation? Give any argument (mathematical, physical, intuitive) to justify the stability (or instability) of this situation.

6. Give arguments to show that the components of the gradient $\vec{\nabla}T$ can be considered to be components of a three-dimensional vector. First give a physical argument using the equation for ΔT. Also show that the components transform the *correct way* (similar to the position vector) when the coordinate system is rotated by an angle θ.

7. Evaluate the following integrals which involve the Dirac Delta-function.

   (a) $\int_0^5 \cos x\ \delta(x-\pi)dx$

   (b) $\int_{-\infty}^{\infty} \ln(x+3)\ \delta(x+2)dx$

   (c) $\int_V (r^4 + r^2(\vec{r}\cdot\vec{c}) + c^4)\ \delta^3(\vec{r}-\vec{c})d\tau$. This is a 3-dimesional volume integral, $d\tau$ is the volume element, $\delta^3()$ is a 3-dimensional Dirac Delta-function. V is a sphere of radius 6 centered about the origin, $\vec{c} = 5\,\hat{\imath} + 3\,\hat{\jmath} + 2\,\hat{k}$.

8. To calculate the magnitude of the electric potential due to a surface charge distribution σ, one can use the formula:

$$\phi = \frac{\sigma}{r} dS$$

   Your task is to set up the integrals to calculate the potential due to a annular ring of inner radius a and outer radius b on the axis of the ring at a distance z from the centre of the ring. In Cartesian co-ordinates:

   (a) What is the area element dS?

   (b) What is the distance r?

   (c) How many integrals do you need?

   (d) What are their limits of integration? Write out the integral. (Do not evaluate it though)

   Now set up the integral in cylindrical coordinates.

   (a) What is the area element dS?

   (b) What is the distance r?

   (c) How many integrals do you need? What are their limits of integration?

   (d) Evaluate the integral.

9. Calculate directly the divergence of the function $\vec{v} = \hat{r}/r$ where $r = \sqrt{x^2+y^2+z^2}$. Check your result using the divergence theorem with a sphere of radius R centered at the origin.

10. Repeat the above problem for the function $\vec{v} = \hat{r}/r^2$. How will you explain the inconsistency in the results?