MA-106 Linear Algebra

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D1 - Lecture 4

Application of $A = LU$

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -8 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -12 & -5 \\ 1 & -6 & 3 \end{pmatrix}$.

To solve $Ax = b$, we can solve two triangular systems $\boxed{Lc = b \text{ and } Ux = c}$. Then $Ax = LUx = Lc = b$.

Take $b = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$. First solve $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$.

We get $c_1 = 1$, $c_2 = 4$, $c_3 = 0$.

Now solve $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -8 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$.

We get $x_3 = 0$, $x_2 = -1/2$, $x_1 = 2$. □

Computing inverse of a matrix

Compute the inverse of the following invertible matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$$

If we write

$$A^{-1} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

where x_i is the i-th column of A^{-1}, then

$$AA^{-1} = \begin{pmatrix} Ax_1 & Ax_2 & Ax_3 \end{pmatrix} = I$$

gives three systems of linear equations

$$Ax_1 = e_1, \quad Ax_2 = e_2, \quad Ax_3 = e_3$$

where e_i is the i-th column of I. Since the coefficient matrix A is same, we can solve them simultaneously as follows:

Calculation of A^{-1} : Gauss-Jordan Method

$$(A \mid e_1 \quad e_2 \quad e_3) = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} \mathbf{2} & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} \mathbf{2} & 1 & 1 & 1 & 0 & 0 \\ 0 & -\mathbf{8} & -2 & -2 & 1 & 0 \\ 0 & 0 & \mathbf{1} & -1 & 1 & 1 \end{array}\right)$$

$= \ (U \mid L^{-1})$, since $A = LU$.

Note that $A^{-1} = U^{-1}\, L^{-1}$, hence conver $(U|L^{-1})$ to $(I|U^{-1}L^{-1})$.

Calculation of A^{-1} continues . . .

$$(U \mid L^{-1}) = \left(\begin{array}{ccc|ccc} \mathbf{2} & 1 & 1 & 1 & 0 & 0 \\ 0 & -\mathbf{8} & -2 & -2 & 1 & 0 \\ 0 & 0 & \mathbf{1} & -1 & 1 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} \mathbf{2} & 1 & 0 & 2 & -1 & -1 \\ 0 & -\mathbf{8} & 0 & -4 & 3 & 2 \\ 0 & 0 & \mathbf{1} & -1 & 1 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} \mathbf{2} & 0 & 0 & 12/8 & -5/8 & -6/8 \\ 0 & -\mathbf{8} & 0 & -4 & 3 & 2 \\ 0 & 0 & \mathbf{1} & -1 & 1 & 1 \end{array}\right)$$

Divide by pivots $\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 12/16 & -5/16 & -6/16 \\ 0 & 1 & 0 & 4/8 & -3/8 & -2/8 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$

$$= (I \mid U^{-1}L^{-1}) \quad = \quad (I \mid A^{-1}) \qquad \square$$

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Transpose A^T of a Matrix A

1. Definition: The i-th row of A is the i-th column of A^T and vice-versa. Hence if $A_{ij} = a$, then $(A^T)_{ji} = a$.
2. Example: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & 1 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & 0 \\ 2 & -2 \\ 3 & 1 \end{pmatrix}$.
3. If A is $m \times n$, then A^T is $n \times m$.
4. If A is upper triangular, then A^T is lower triangular.
5. $(A^T)^T = A, \quad (A+B)^T = A^T + B^T$.
6. $\boxed{(AB)^T = B^T A^T}$.

   Proof. ${(AB)^T}_{ij} = (AB)_{ji} = A^j.B_i = B_i.A^j = (B^T)^i.{A^T}_j$
7. $\boxed{(A^{-1})^T = (A^T)^{-1}}$. Use $AA^{-1} = I$ and take transpose.

Symmetric Matrix

- If $A^T = A$, then A is called a symmetric matrix.
- **Examples:** $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are symmetric.
- If A and B are symmetric, then AB may NOT be symmetric. In above example, $AB = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ is not symmetric.
- If A is symmetric and invertible, then A^{-1} is symmetric. Since $(A^{-1})^T = (A^T)^{-1} = A^{-1}$.
- For any matrix R, $R\,R^T$ is symmetric.

LDU decomposition for square matrix A

Assume no row exchange is needed in Gaussian elimination of A. Then $\boxed{A = LU'}$, where L is lower triangular with diagonal entries 1 and U' is upper triangular with non-zero entries on diagonal as pivots of A.

Example. (1) $U' = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} = DU$

(2) $U' = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} = DU$

If no row exchange is needed in A, then $\boxed{A = LU' = LDU}$, where D is a diagonal matrix with non-zero entries as pivots of A, L and U are lower and upper triangular matrix with 1 on diagonals.

• If A is invertible and $A = LDU$, then this decomposition of A is unique.

Hint. $A = LDU = L'D'U' \implies L'^{-1}LD = D'U'U^{-1}$.

From this conclude that $D = D'$, $L = L'$ and $U = U'$. $\square$

• If $A = LDU$, A is invertible and symmetric, then $U = L^T$.

Proof. Since $A = A^T$, we get $LDU = U^T D^T L^T$.

By uniqueness of decomposition, $L = U^T$. $\square$

• Assume $A = LDU$ and A is not invertible. Is this decomposition unique? No.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} I$$

for any scalar a.

Example

Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ be a symmetric matrix. Then

$$E_{21}(1/2)\,A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$E_{32}(2/3)\,E_{21}(1/2)\,A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{pmatrix}$$

If $L = E_{21}(-1/2)\,E_{32}(-2/3) = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{pmatrix}$,

then $\boxed{A = LDL^T}$, where D is the diagonal matrix with diagonal entries as pivots of A, namely $2,\ 3/2,\ 4/3$.