MA-106 Linear Algebra

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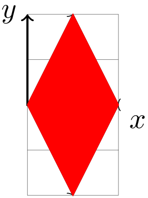
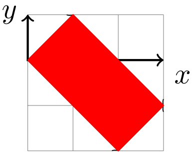
D1 - Lecture 15

1. *Test for invertibility:* A is invertible $\iff det(A) \neq 0$.
2. If A is $n \times n$, then $|det(A)| =$ volume of the box (in n-dimensional space $\mathbb{R}^n$) with edges as rows of A.
3. **Example. 1** The volume (area) of a line in $\mathbb{R}^2 = 0$.

   2. The determinant of $A = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} = \boxed{-4}$.

   (3) Let's compute the volume of the box (parallelogram) with edges as rows / columns of A.





$\boxed{= 4}$

Defining Properties of Determinants

The determinant is a function

$$det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$$

which is defined (uniquely) by its three basic properties.

1. $det(I) = 1$.
2. the sign of determinant is reversed by a row exchange, i.e. if B is obtained from A by exchanging two rows, then $det(B) = -det(A)$.
3. det is linear in each row separately, i.e. if we fix $n-1$ rows, say $A^2, \dots, A^n$, then

$$det\begin{pmatrix} - \\ A^2 \\ \vdots \\ A^n \end{pmatrix} : \mathbb{R}^n \to \mathbb{R}$$

is linear. There are n such functions (one for each row).

Checking Defining Properties: 2×2 case

Known to us: $det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$

1. $\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$
2. $\det \begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - da = -(ad - bc) = -det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$
3. $det \begin{pmatrix} a + a' & b + b' \\ c & d \end{pmatrix} = (a + a')d - (b + b')c =$
   $(ad - bc) + (a'd - b'c) \quad = \quad det \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \det \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$, and
   $det \begin{pmatrix} ta & tb \\ c & d \end{pmatrix} = t(ad - bc) = t\, det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
4. Similarly, check linearity property in second row.

Induced Properties of Determinants: 2×2 case

1. If B and C are two matrices, then

   $det(B+C) = det(B) + det(C)$. (False)

   Infact
   $$det(B+C) = det\begin{pmatrix} B^1 + C^1 \\ B^2 + C^2 \end{pmatrix}$$
   $$= det\begin{pmatrix} B^1 \\ B^2 + C^2 \end{pmatrix} + det\begin{pmatrix} C^1 \\ B^2 + C^2 \end{pmatrix} \quad \text{(by linearity)}$$
   $$= det(B) + det\begin{pmatrix} B^1 \\ C^2 \end{pmatrix} + det\begin{pmatrix} C^1 \\ B^2 \end{pmatrix} + det(C) \quad \text{(by linearity)}$$

2. $det(tB) = t\, det(B)$ (False)

   Infact $det(tB) = det\begin{pmatrix} tB^1 \\ tB^2 \end{pmatrix} = t\, det\begin{pmatrix} B^1 \\ tB^2 \end{pmatrix} = t^2\, det(B)$.

3. If B is $n \times n$, then $\boxed{det(tB) = t^n\, det B}$.

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10. 140050005 RUPANSHU GANVIR
11. 140050014 SHREY KUMAR
12. 140050025 SHREYANSH BARODIYA
13. 140050034 BHOOKYA NAVEEN ABSENT
14. 140050043 GUJJULA CHANUKYA VARDHAN
15. 140050054 RONDI ABHINAV
16. 140050081 SUMITH
17. 140050082 RAVI TEJA

Properties of Determinants continued ...

1. Assume two rows of A are equal (say i-th and j-th rows). Let B be obtained from A by $\boxed{R_i \longleftrightarrow R_j}$, then $B = A$. Hence
$$det(A) = det(B) = -det(A) \implies \boxed{\det(A) = 0}.$$

2. Assume B is obtained from A by $\boxed{R_i \mapsto R_i + aR_j}$. Then
$$det(B) = det\begin{pmatrix} A^i + aA^j \\ A^j \end{pmatrix} = det(A) + det\begin{pmatrix} aA^j \\ A^j \end{pmatrix} = det(A).$$

3. Assume one row of A is zero (say A^i). Let B be got from A by $\boxed{R_i \mapsto R_i + R_j}$, then $B^i = B^j$. Hence $det(A) = \det(B) = 0$.

4. If $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is diagonal, then $det(A) = ab$. (Use linearity).

1. If $A = (a_{ij})$ is triangular, then $det(A) = a_{11} \dots a_{nn}$.

   Proof. If all a_{ii} are non-zero, then by elementary row operations, A reduces to a diagonal matrix $\begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix}$ whose determinant is $a_{11} \dots a_{nn}$.

   If atleast one diagonal entry is zero, then elimination will produce a zero row. Hence $det(A) = 0$.

2. If A is singular, then $det(A) = 0$.

   Elimination produces a zero row in U, where $PA = LU$. Hence $det(A) = \pm\, det(U) = 0$.

3. If A is invertible, then $det(A) \neq 0$.

   Elimination produces n-pivots, say $d_1, \dots, d_n$. Then

$$det(A) = \pm\ det(U) = \pm\ d_1 \dots d_n \neq 0.$$

$$\boxed{det(AB) = det(A)\,det(B)}$$

We may assume that A, B are invertible.
Hint: For fixed B, show that the function d defined by

$$d(A) = det(AB)/det(B)$$

satisfies the following properties

1. $d(I) = 1$.
2. If we interchange two rows of A, then d changes its sign.
3. d is a linear function in each row of A.

Then d is the unique determinant function det. Therefore

$$det(AB) = det(A)\,det(B)$$

$\square$

Properties of Determinants continued . . .

$$\boxed{det(A) = det(A^T)}$$

- If A is singular, then A^T is singular and we get $0 = 0$.
- Hence we assume A is non-singular.
- We get a permutation matrix P such that
$$PA = LDU \implies A^T P^T = U^T D^T L^T$$
- Since U and L are triangular with diagonal entries 1, we get
$$det(U) = 1 = det(U^T) \text{ and } det(L) = 1 = det(L^T)$$
- D is diagonal, hence $D^T = D$.
- Since $PP^T = I$ and $detP = \pm 1$, we get $det(P) = det(P^T)$.
- From above, we get $det(A) = det(A^T)$. $\square$

First Formula for Determinant

If A is invertible, then $\boxed{PA = LDU}$. Hence

$$detA = det(P).det(D) = \pm\, detD = \pm \text{ (product of pivots)}.$$

Example: Let $A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & . & . & -1 \\ & & & -1 & 2 \end{pmatrix}$ be $n \times n$ matrix which is $(-1, 2, -1)$ tri-diagonal.

Pivots of A are $2/1,\ 3/2,\ \ldots,\ (n+1)/n$ (without row exchange). Hence

$$A = LDU = L \begin{pmatrix} 2 & & & \\ & 3/2 & & \\ & & \ddots & \\ & & & (n+1)/n \end{pmatrix} U$$

$$\text{Therefore} \quad det(A) = 2\,(\frac{3}{2})\ \ldots\ (\frac{n+1}{n}) = n+1$$

Formula for Determinant - 2×2 case

Write $(a,b) = (a,0) + (0,b)$, the sum of vectors in coordinate directions. Similarly write (c,d). By linearity property,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} =$$

$$\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

For $n \times n$ matrix, each row splits into n coordinate directions, so the expansion of $det(A)$ has n^n terms. When two rows are in same coordinate direction, that term will be zero. Examples.

$$\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = - \begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix} = -bc$$

The non-zero terms have to come in different columns. So, there will be $n!$ such terms.

Formula for Determinant: 3×3 case:

If $A = (a_{ij})$ is 3×3 matrix, then

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{11}a_{23}a_{32} \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix}$$

$$+ a_{12}a_{23}a_{31} \begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32} \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix} + a_{13}a_{22}a_{31} \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}$$

Formula for Determinant: 3×3 case continued ...

$$= a_{11}\, a_{22}\, a_{33}\,(1) + a_{11}\, a_{23}\, a_{32}\,(-1) + a_{12}\, a_{21}\, a_{33}\,(-1)$$

$$+ a_{12}\, a_{23}\, a_{31}\,(1) + a_{13}\, a_{21}\, a_{32}\,(1) + a_{13}\, a_{22}\, a_{31}\,(-1)$$

$$= \sum_{\text{all permutations } P} (a_{1\alpha}\, a_{2\beta}\, a_{3\gamma})\; det(P)$$

where P runs over all permutation matrices.

If $(\alpha, \beta, \gamma) = (2, 3, 1)$, then $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = P_{13} P_{12}$.

Here $det(P) = (-1)^2 = 1$.