MA-108 Ordinary Differential Equations

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D1 - Lecture 7

Numerical Methods

We have seen some methods to solve $y' = f(x, y)$ with $y(x_0) = y_0$. Today we will see some methods to find an approximate solution.

Ex. Solve $y' = 1 + 2xy, \quad y(0) = 3.$

The solution is given by

$$y(x) = y_1(x)\,[\int_0^x \frac{f(x)}{y_1}\,dx + C\,] = e^{x^2}\,[\int_0^x e^{-x^2}\,dx + C\,].$$

The solution to IVP is $y = e^{x^2}\,[\int_0^x e^{-x^2}\,dx + 3\,]$.

Q. How do we actually find a solution? Using the direction field, we can understand the solution.

Finding the direction field involved estimating what the tangent to the solution curve looks like at every point.

Euler's Method uses the same idea to give a numerical approximation to the solution.

Euler's method

Let $y' = f(x, y), \quad y(x_0) = y_0$. Choose equally distanced points $x_i = x_0 + h\ i$ for $i = 1, \dots, k$ and let y_i is approximation to $y(x_i)$.

Then, Euler's method gives an approximate solution to $y(x)$ at these points. The idea is to approximate the integral curve at these point by the line, tangent to it at that point.

At (x_0, y_0), the tangent line is given by

$$y(x) = y_0 + f(x_0, y_0)(x - x_0).$$

Then an approximation for $y(x_i)$ is given by

$$y_i = y(x_{i-1}) + hf(x_{i-1}, y(x_{i-1}))$$

Then $y_1 = y_0 + hf(x_0, y_0)$ is an approximation for $y(x_1)$. In general, we do not know $y(x_{i-1})$ and hence define

$$y_i = y_{i-1} + hf(x_{i-1}, y_{i-1}).$$

Examples

Let $y' = \dfrac{1+x}{1-y^2} = f(x,y), \quad y(2) = 3; \quad h = 0.1.$
Using $y_i = y_{i-1} + hf(x_{i-1}, y_{i-1})$, Euler's method gives

$$
\begin{aligned}
y_1 &= 3 + f(2,3)\,0.1 = 2.9625 \\
y_2 &= y_1 + f(2.1,\ 2.9625)\,0.1 = 2.92263583 \\
y_3 &= y_2 + f(2.2,\ 2.92263583)\,0.1 = 2.88020564 \\
y_4 &= y_3 + f(2.3,\ 2.90661298)\,0.1 = 2.8349728 \\
y_5 &= y_4 + f(2.4,\ 2.8349728)\,0.1 = 2.78665724 \\
y_6 &= y_5 + f(2.5,\ 2.78665724)\,0.1 = 2.73492387 \\
y_7 &= y_6 + f(2.6,\ 2.73492387)\,0.1 = 2.67936667 \\
y_8 &= y_7 + f(2.7,\ 2.67936667)\,0.1 = 2.61948649
\end{aligned}
$$

Errors

Let us solve $y' = \dfrac{1+x}{1-y^2} = f(x,y), \quad y(2) = 3$; by separable method.

$$y - \frac{y^3}{3} = x + \frac{x^2}{2} + C; \quad C = -10.$$

Let us compare our solution by Euler method's approximate solution up to four decimals.

X	Y(X) Approx	Y(X) Exact	— Error —
2.1	2.9625	2.9613162498	0.0011837502
2.2	2.92263583	2.9201289618	0.0025068682
2.3	2.88020564	2.876207308467	0.003998331533
2.4	2.8349728	2.833552901083	0.001419898917
2.5	2.78665724	2.7790111491	0.0076460909
2.6	2.73492387	2.7250089161	0.0099149539

Order of Error

Euler' method has two kinds of errors. One because of approximating $y(x_i)$ by y_i, called a truncation error.
The other, because of evaluating, say up to 8 decimals, called the roundoff error.

The truncation error at the ith step is given by

$$T_i = y(x_{i+1}) - y(x_i) - hf(x_i, y(x_i))$$

Assume that f, f_x and f_y are continuous and bounded for all (x, y) in some open rectangle. Then y'' exists, since
$y'(x) = f(x, y(x))$;

$$y''(x) = f_x(x, y(x)) + f_y(x, y(x))y' = [f_x + f_y f](x, y(x))$$

Then $y''(x)$ is also bounded in the region, say $[x_0, b]$. Taylor's approximation theorem says that

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(\overline{x_i}), \text{ for some } \overline{x_i} \in (x_i, x_{i+1})$$

Since $y'(x_i) = f(x_i, y(x_i))$, Taylor's expansion gives

$$y(x_{i+1}) - y(x_i) - hf(x_i, y(x_i) = \frac{h^2}{2} y''(\overline{x_i})$$

But y'' is bounded, say by M, therefore truncation error

$$T_i = y(x_{i+1}) - y(x_i) - hf(x_i, y(x_i)) \leq M\frac{h^2}{2}.$$

This says that the local truncation error is of the order of h^2 or $O(h^2)$.

Thus if the step size h is halved, then the error should decrease by a factor of 4. However, this is not all the error. This error is attained at every x_i. Since it will now take twice as many steps to get to the same point, the accumulated error will increase.

Taking this into account, it will show that the total truncation error, in the Euler method is to the order of h i.e. $O(h)$.

1. 140050020 GOVIND LAHOTI
2. 140050032 SUMAN SWAROOP
3. 140050038 GUDIPATI KRISHNA CHAITANYA
4. 140050054 RONDI ABHINAV
5. 140050072 MUTYALA SOWMYA
6. 140050082 RAVI TEJA A
7. 130050071 AMADALA SAROJ RAJABSENT
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9. 140020015 SARDA ABHISHEK RAJESHKUMAR
10. 140020027 SAUMIL AGARWAL
11. 140020038 DHAMALE ASHUTOSH MADHUKAR
12. 140020045 MOHIT SINGHAL
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14. 140020068 SUPREET MEENAABSENT
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16. 140020087 AMRITESH ARYAN
17. 140050010 VISHAL MEENA

Improved Euler Method

Even though the idea in Euler's method is simple, this is not usually the recommended method of approximation.

We note, the main task in using the numerical method, is to compute y_{i+1} using previous y_i's, in particular $f(x_i, y_i)$. We want a method which gives us a good approximation with fewer computations of $f(x_i, y_i)$.

The Improved Euler method, approximates the value of y for $x \in [x_i, x_{i+1}]$, by the line $y(x_i) + m_i(x - x_i)$ where

$$m_i = \frac{f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))}{2}$$

is the average of slopes of intgegral curve at x_i and x_{i+1}. Thus we define

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right].$$

Improved Euler Method: Example

But we do not know the value of y_{i+1}. We replace this by the y_{i+1}, we would have computed in the Euler method. That is,

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1},\, y_i + hf(x_i, y_i))\right].$$

Ex. Let $y' = \dfrac{1+x}{1-y^2}$, $\quad y(2) = 3; \quad h = 0.1.$ Then

$$\begin{aligned} y_1 &= 3 + \frac{0.1}{2}(f(2,3) + f(2.1,\, 3 + 0.1\, f(2,3)) = 2.961317914 \\ y_2 &= 2.961317914 + \frac{0.1}{2}(f(2.1,\, 2.967747619) + \\ &\quad f(2.2,\, 2.961317914 + 0.1 f(2.1, --)) = 2.920132727 \end{aligned}$$

X	Y(X) EM	Y(X) IEM	Y(X) Exact
2.1	2.9625	2.961317914	2.9613162498
2.2	2.92263583	2.920132727	2.9201289618

Improved Euler Method: Example

Clearly, this is already an improvement over Euler Method.

It can be shown that the local truncation error in the Improved Euler Method is $O(h^3)$ (this is $O(h^2)$ in Euler's mathod).

The total truncation error is $O(h^2)$ (this is $O(h)$ in Euler's method).

Exercise. Find approximate values of e by applying Euler method and Improved Euler method to the problem,

$$y' = y, \quad y(0) = 1.$$

Use a step size of $h = 0.5$.

Runge Kutta Method

A more accurate approximation is given by the Runge-Kutta method. This is the most widely used approximation method. In this method, the local truncation error is $O(h^5)$ and the total truncation error is $O(h^4)$.

It uses **super slope**. Here y_{i+1} is given by the formula

$$y_{i+1} = y_i + \frac{h}{6}(A_i + 2B_i + 2C_i + D_i), \quad \text{where}$$

$$\begin{aligned}
A_i &= f(x_i, y_i) \quad \text{Euler's slope} \\
B_i &= f\left(x_i + \frac{h}{2},\ y_i + \frac{h}{2}A_i\right) \\
C_i &= f\left(x_i + \frac{h}{2},\ y_i + \frac{h}{2}B_i\right) \\
D_i &= f(x_i + h,\ y_i + hC_i)
\end{aligned}$$

Second Order Linear ODE's

Now that we're sort of comfortable with first order ODE's (both linear and non-linear), let's step up the difficulty level. We'll look at second order linear ODE's. Recall that a general second order linear ODE is of the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

An ODE of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

is called a second order linear ODE in standard form.
Though there is no formula to find all the solutions of such an ODE, we study the existence, uniqueness and number of solutions of such ODE's.

Second Order Linear ODE's

If $r(x) \equiv 0$ in the equation above, i.e.,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

then the ODE is said to be homogeneous. Otherwise it is called nonhomogeneous. Remember this usage is in the spirit of MA 106.

Initial Value Problem

An initial value problem of a second order linear ODE is of the form:

$$y'' + p(x)y' + q(x)y = r(x);\; y(x_0) = a, y'(x_0) = b,$$

where $p(x)$, $q(x)$ and $r(x)$ are assumed to be continuous on an interval I with $x_0 \in I$.

Solving IVP's

Let

$$C(I) = \{f : I \to \mathbb{R} \mid f \text{ is continuous}$$

$$C^1(I) = \{f : I \to \mathbb{R} \mid f, f' \text{ are continuous}$$

$$C^2(I) = \{f : I \to \mathbb{R} \mid f, f', f'' \text{ are continuous}$$

. Check: $C(I), C^1(I), C^2(I)$ are vector spaces with addition and scalar multiplication defined as:

$$(f + g)(x) = f(x) + g(x),\ x \in I,$$

$$(k \cdot f)(x) = kf(x),\ k \in \mathbb{R}, x \in I.$$