MA-108 Ordinary Differential Equations

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D1 - Lecture 12

Recall: We stated existence and uniqueness theorem for n-th order linear homogeneous IVP.
Proved dimension theorem in n-th order.
We are considering constant coefficient equations.

Constant Differential Operators

Ex. $D^2 - 5D + 6 = (D-3)(D-2)$ as a linear transformation from $\mathcal{C}^2(I) \to \mathcal{C}(I)$, i.e. for any $y \in \mathcal{C}^2(I)$,
$(D-3)(D-2)y = (D-3)(y' - 2y) = (y'' - 5y' + 6y)$ and
$(D-2)(D-3)y = (D-2)(y' - 3y) = y'' - 5y' + 6y$.
Hence $(D-2)(D-3) = (D-3)(D-2)$.

Ex. Check that $(D+1)(D^2+D+1) = (D^2+D+1)(D+1)$ as a function from $\mathcal{C}^3(I) \to \mathbb{R}$. i.e. for each $y \in \mathcal{C}^3(I)$,
$(D+1)(D^2+D+1)y = (D^2+D+1)(D+1)y$.

• The main point is that $D^r \circ D^s = D^s \circ D^r = D^{r+s}$.
Hence if $L = \sum_{i=0}^n a_{n-i} D^i$ and $M = \sum_{i=0}^m b_{m-i} D^i$ are differential operators with constant coefficient, then

$$L(M(f)) = M(L(f)),\ \forall f \in \mathcal{C}^{m+n} \implies LM = ML$$

Note that in $LM = ML$, it is important that L and M are constant coefficient equations.

Ex. If $L = D + xI$ and $M = D + 1$, then
$LM(f) = L(f' + f) = (f'' + f') + x(f' + f)$.
$ML(f) = M(f' + xf) = (f'' + f + xf') + (f' + xf) = LM(f) + f$.

Ex. Solve $y^{(3)} - 7y' + 6y = 0$. Here,

$$L = D^3 - 7D + 6 = (D - 1)(D - 2)(D + 3).$$

Note that

$$e^x \in N(D - 1),\ e^{2x} \in N(D - 2),\ e^{-3x} \in N(D + 3).$$

Thus, $e^x, e^{2x}, e^{-3x} \in \mathrm{Ker}(L)$. Hence, $\{e^x, e^{2x}, e^{-3x}\}$ is a basis of Ker L. Thus, the general solution is of the form

$$c_1e^x + c_2e^{2x} + c_3e^{-3x}$$

Remark: The above example illustrates the general case of P_L having distinct real roots. If

$$P_L(x) = a_0(x - r_1)\dots(x - r_n),$$

with $r_i \in \mathbb{R}$ distinct, then a basis for Ker L is

$$\{e^{r_1 x},\ e^{r_2 x},\ \dots,\ e^{r_n x}\}.$$

Ex. Solve $y^{(4)} - 16y = 0.$

Here, $L = D^4 - 16I.$. Hence
$P_L(x) = x^4 - 16 = (x^2 - 4)(x^2 + 4)$ and
$L = (D - 2I)(D + 2I)(D^2 + 4I).$

We know that $N(D - 2I) = \text{Span}\{e^{2x}\}$,
$N(D + 2I) = Span\{e^{-2x}\}$ and
$N(D^2 + 4I) = Span\{\cos 2x, \sin 2x\}$.

Therefore a basis of $N(L)$ is given by
$\{e^{2x}, e^{-2x}, \cos 2x, \sin 2x\}$ and the general solution is of the form

$$c_1e^{2x} + c_2e^{-2x} + c_3 \cos 2x + c_4 \sin 2x.$$

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Extra slide

Theorem

Suppose $L = A_1A_2\dots A_k$, where A_i are linear differential operators with constant coefficients and $N(A_i)\cap N(A_j) = 0$ for all $i \neq j$. Let B_i be a basis for $N(A_i)$. Then $\cup B_i$ is a basis for $N(A)$.

Proof: From Dimension theorem, we know that $\dim N(L) = \text{deg} P_L$. Now $P_L = P_{A_1}\dots P_{A_k}$.

Then $\deg P_L = \deg P_{A_1} + \dots + \deg P_{A_k}$. Therefore, $\dim N(L) = \dim N(A_1) + \dots + N(A_k)$.

Further since $N(A_i)\cap N(A_j) = 0$ for all $i \neq j$, $\cup B_i$ is linearly independent and has $\dim N(A_1) + \dots + \dim N(A_k) = \dim N(L)$ elements.

Therefore, $\cup B_i$ is a basis for $N(L)$

Constant Differential Operators

Q. What if $P_L(x)$ has some repeated real roots?

In the $n = 2$ case, $m_1 = m_2$ gave us just one solution $e^{m_1 x}$. The other one we found by the method of looking for a solution of the form vf. This process gave us $xe^{m_1 x}$.

Let us begin with the following result.

Proposition

For any real number r, the functions

$$u_1(x) = e^{rx}, u_2(x) = xe^{rx}, \ldots, u_m(x) = x^{m-1}e^{rx}$$

are linearly independent and

$$u_1(x), \ldots, u_m(x) \in \mathrm{Ker}((D - r)^m).$$

Constant Differential Operators

Proof. That these functions are linearly independent is obvious, since $\{1, x, x^2, \ldots, x^m\}$ is linearly independent (e^{rx} is non-zero). We need to show that these functions are in Ker $(D-r)^m$.

When $m = 1$, we need to show

$$u_1(x) = e^{rx} \in \mathrm{Ker}((D-r)),$$

which is true, since

$$(D-r)(e^{rx}) = re^{rx} - re^{rx} = 0.$$

Suppose $m = 2$. Since u_1 is in Ker of $(D-r)$, it's in Ker of $(D-r)^2$. What about $u_2 = xe^{rx}$?

$$\begin{aligned} (D-r)^2(xe^{rx}) &= (D-r)(D-r)(xe^{rx}) \\ &= (D-r)(xre^{rx} + e^{rx} - rxe^{rx}) \\ &= (D-r)(e^{rx}) = 0. \end{aligned}$$

Constant Differential Operators

Use induction to prove general case. Assume

$$u_1, u_2, \ldots, u_{m-1} \in \mathrm{Ker}((D-r)^{m-1}),$$

and we need to show that

$$u_1, u_2, \ldots, u_m \in \mathrm{Ker}((D-r)^m).$$

Clearly

$$u_1, u_2, \ldots, u_{m-1} \in \mathrm{Ker}((D-r)^{m-1}) \subseteq \mathrm{Ker}((D-r)^m).$$

To show that u_m is also in $\mathrm{Ker}((D-r)^m)$, compute $(D-r)^m(x^{m-1}e^{rx})$

$$\begin{aligned}
&= (D-r)^{m-1}(D-r)(x^{m-1}e^{rx}) \\
&= (D-r)^{m-1}(x^{m-1}re^{rx} + (m-1)x^{m-2}e^{rx} - rx^{m-1}e^{rx}) \\
&= (D-r)^{m-1}((m-1)x^{m-2}e^{rx}) = 0.
\end{aligned}$$

Constant Differential Operators

Therefore, a basis for solution space of $(D-r)^m$ is

$$e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}$$

Thus, if

$$P_L(x) = (x-r_1)^{e_1}(x-r_2)^{e_2}\ldots(x-r_\ell)^{e_\ell},$$

where $\sum_{i=1}^{\ell} e_i = n$, then a basis of Ker L is given by

$$e^{r_1x}, \ldots, x^{e_1-1}e^{r_1x}, e^{r_2x}, \ldots, x^{e_2-1}e^{r_2x}, \ \ldots, e^{r_\ell x}, \ldots, x^{e_\ell-1}e^{r_\ell x}.$$

The point is that the above functions are linearly independent and since dim Ker $L = n$, these form a basis.
Ex: Check that the above functions are linearly independent.

Ex: Find the general solution of the ODE:

$$L(y) = (D^3 - D^2 - 8D - 12)(y) = 0.$$

We have

$$P_L(x) = x^3 - x^2 - 8x - 12 = (x-2)^2(x+3),$$

and therefore,

$$L = (D-2)^2(D+3).$$

Thus the general solution is

$$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-3x},$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Constant Differential Operators

Ex: Find the general solution of the ODE:

$$L(y) = (D^6 + 2D^5 - 2D^3 - D^2)(y) = 0.$$

Now,

$$L = D^2(D-1)(D+1)^3.$$

Ker D^2 is $\{1, x\}$ Ker $D-1$ is $\{e^x\}$.
Ker $(D+1)^3$ is $\{e^{-x}, xe^{-x}, x^2e^{-x}\}$.
Thus, the general solution is

$$c_1 + c_2x + c_3e^x + c_4e^{-x} + c_5xe^{-x} + c_6x^2e^{-x},$$

with $c_i \in \mathbb{R}$.

Constant Differential Operators: complex roots

Assume $P_L(x)$ has some complex roots. In the 2nd order case, if $m_1 = a + \imath b, m_2 = a - \imath b$, then $y_1 = e^{ax}\cos bx$ and $y_2 = e^{ax}\sin bx$ were the basis for $N(L)$.

If $P_L(x)$ has a complex root $a + \imath b$, then it also has $a - \imath b$ as a root. Thus,

$$(x - (a + \imath b))(x - (a - \imath b)) = (x - a)^2 + b^2$$

is a factor of $P_L(x)$.
Null space of $(D - a)^2 + b^2$ has a basis

$$\{e^{ax}\cos bx, e^{ax}\sin bx\} \subset N(L)$$

Constant Differential Operators

If $a \pm \imath b$ is a root of $P_L(x)$ of multiplicity m, then $((D-a)^2+b^2)^m$ is a factor of $P_L(x)$.

Can we find the null space of $((D-a)^2+b^2)^m$?

Ex. Check that

$$e^{ax}\cos bx, xe^{ax}\cos bx, \ldots, x^{m-1}e^{ax}\cos bx,$$
$$e^{ax}\sin bx, xe^{ax}\sin bx, \ldots, x^{m-1}e^{ax}\sin bx.$$

are a basis for null space of $((D-a)^2+b^2)^m$.

Constant Differential Operators

Ex: Find the general solution of

$$y^{(5)} - 9y^{(4)} + 34y^{(3)} - 66y^{(2)} + 65y' - 25y = 0.$$

The characteristic polynomial is

$$(x-1)(x^2-4x+5)^2.$$

The roots are

$$1, 2 \pm \imath, 2 \pm \imath.$$

Hence, the general solution is

$$y = c_1 e^x + e^{2x}[c_2 \cos x + c_3 \sin x + c_4 x \cos x + c_5 x \sin x],$$

where $c_i \in \mathbb{R}$.

Examples

Ex: Find the fundamental set of solutions to

$$D^3(D-2I)^2(D^2+4I)^2y=0$$

The fundamental set will be given by

$$\{1, x, x^2, e^{2x}, xe^{2x},\ \cos 2x, \sin 2x, x\cos 2x, x\sin 2x\}$$

Solving the Non-homogeneous Equation

Consider $L(y) = y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x)$, where $p_1, p_2, \ldots, p_n, r$ are continuous on an interval I.
We would like to solve this.
To get a general solution of the non-homogeneous equation, it is enough to solve the homogeneous equation completely and to somehow get one solution of the non-homogeneous equation.

The proof of $n = 2$ case goes through here. We discussed the variation of parameters method to find the particular solution, which will generalise to the n-order case.

But before that note in the example, $y'' + 6y' + 5y = e^x$, y_p was ce^x and in $y'' + 6y' + 5y = e^{-x}$, y_p was of the form cxe^{-x}. Why?

Annihilator Method

The Annihilator method or method of undetermined coefficients helps us in finding a particular solution of a non-homogeneous equation.

Example: Find a particular solution of

$$y^{(4)} - 16y = x^4 + x + 1 = r(x).$$

Here, $\quad L = D^4 - 16,$

and let us take $A = D^5$. Then $Ar(x) = 0$.

We say A annihilates or kills $r(x)$.

Hence a solution y of $L(y) = r(x)$ is also a solution of

$$D^5(D^4 - 16) = 0.$$

$AL = D^5(D^4 - 16)$ has characteristic equation

$$x^5(x^4 - 16) = x^5(x-2)(x+2)(x^2+4).$$

Thus, a general solution of $(AL)(y) = 0$ is of the form

$$c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4 + c_6e^{2x} + c_7e^{-2x} + c_8\cos 2x + c_9\sin 2x.$$

Here $c_6e^{2x} + c_7e^{-2x} + c_8\cos 2x + c_9\sin 2x$ is a solution of the homogeneous part $(D^4 - 16)y = 0$.

We want a particular solution y_p for $y^{(4)} - 16y = x^4 + x + 1$ and its clear that it will satisfy $ALy = 0$.

This implies that we can take $y_p = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4$, since all the other terms are solutions to the corresponding homogenous ODE.

To find c_i's in $y_p = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4$, solve
$y_p^{(4)} - 16y_p = x^4 + x + 1$.
Then $24c_5 - 16(c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4) = x^4 + x + 1$.

Equating the coefficients, we get

$$\begin{aligned} 24c_5 - 16c_1 &= 1 \\ -16c_2 &= 1 \\ -16c_3 &= 0 \\ -16_c4 &= 0 \\ -16c_5 &= 1 \end{aligned}$$

This gives $c_3 = c_4 = 0$, $c_5 = c_2 = -1/16$,
$-16c_1 = 1 - 24c_5 = 1 + 3/2 = 5/2$, hence $c_1 = -5/32$.
Therefore $y_p = -\frac{5}{32} - \frac{1}{16}x - \frac{1}{16}x^4$.

Examples

Ex. Solve $y^{(4)} - 4y'' = e^x + x^2$.
Let $L = D^4 - 4D^2 = D^2(D-2)(D+2)$. Let $z(x)$ and $w(x)$ be such that $Lz = e^x$ and $Lw = x^2$. Then
$L(z+w) = e^x + x^2$.

Let us first solve $Lz = e^x$. We know that e^x is a solution of $My = (D-I)y = 0$.

Now, $MLz = (D-I)D^2(D-2I)(D+2I)z = 0$.
Clearly z satisfies this equation. Hence z will be of the form $z = c_1 + c_2x + c_3e^{2x} + c_4e^{-2x} + c_5e^x$.

But $\{1, x, e^{2x}, e^{-2x}\}$ are all solution to $Ly = 0$ and therefore, $z = c_5e^x$ for some $c_5 \in \mathbb{R}$.

Plugging $z = c_5e^x$ into the equation $y^{(4)} - 4y'' = e^x$, we have $c_5 - 4c_5 = 1 \implies c_5 = -1/3$. Thus $\boxed{z = (-1/3)e^x}$.

Let's solve $Lw = x^2$, where $L = (D^4 - 4)$.
Note that x^2 is a solution to $Ny = D^3y = 0$.

Then $NLw = D^3D^2(D - 2I)(D + 2I)w = 0$. Clearly $w = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4 + c_6e^{2x} + c_7e^{-2x}$.

But $c_1 + c_2x + +c_6e^{2x} + c_7e^{-2x}$ are solutions to $Ly = 0$.
Therefore, $w = c_3x^2 + c_4x^3 + c_5x^4$.

Substituting in $Lw = (D^4 - D^2)w = x^2$, we get

$$24c_5 - 2c_3 + 6c_4x + 12c_5x^2 = x^2$$

This implies, $24c_5 - 2c_3 = 0$, $c_4 = 0$, $c_5 = 1/12$ and $c_3 = 1$.

Therefore, $w = x^2 + \frac{1}{12}x^4$.

Hence a particular solution to $Ly = e^x + x^2$ is given by

$$y_p = z + w = -\frac{1}{3}e^x + x^2 + \frac{1}{12}x^4$$

Summary: Anhilator Method

- Given a linear differential operator L with constant coefficients, we want to solve $Ly = r(x)$.
- We find a particular solution as follows.
- We first find linear a differential operators M which have the property that $M(r(x)) = 0$.
- Find a basis for the solution space of $MLy = 0$.
- Pick those elements in the basis which are not solutions to $Ly = 0$.
- Set y_p to be a linear combination of these particular basis elements and solve $Ly_p = r(x)$ for the constants.
- A general solution to $Ly = r$ is given by $y_p + z$, where z is a general solution to $Ly = 0$.

Examples

Find the form of particular solution to the following ODEs.

- $y'' + 9y' = 6$.
- $y'' + 2y' + y = 4e^x \sin 2x$.
- $y'' + y = x \sin x$
- $y'' + 9y = x^2 e^{3x}$
- $y'' + 9y = x^2 e^{3x} \cos 3x$
- $y^{(4)} - y^{(3)} - y'' + y' = x^2 + 4 + x \sin x$.
- $y^{(4)} - 2y'' + y = x^2 e^x + e^{2x}$
- $y^{(4)} + 2y'' + y = 3 \sin x - 5 \cos x$.
- $y^{(4)} - y^{(3)} - y'' + y' = x^2 + 4 + x \sin x$.
- $y^{(4)} - 2y'' + y = x^2 e^x + e^{2x}$.