MA-106 Linear Algebra

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D1 - Lecture 18

Recall:

- The problem of solving linear system of 1-st order ODE with constant coefficients $\dfrac{du}{dt} = Au$ reduces to solving the eigenvalue problem $Ax = \lambda x$.
- The eigenvalues of A are roots of characteristic polynomial $\det(A - \lambda I)$ and the eigenspace associated to eigenvalue λ is $N(A - \lambda I)$.
- $\lambda = 0$ is an eigenvalue of $A \Leftrightarrow A$ is singular.
  So A is non-singular $\Leftrightarrow 0$ is not an eigenvalue of A.
- If A is a diagonal matrix, then eigenvalues of A are $a_{11}, a_{22}, \ldots, a_{nn}$ and the associated eigenvectors are $e_1, \cdots, e_n$.
- Eigenvectors need not form a basis of $\mathbb{R}^n$. **Ex:** $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- **Def.** Let A and B be square matrices such that $S^{-1}AS = B$ for an invertible matrix S. Then A and B are called **similar**.

- Recall: Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be linear map $L(x, y) = (x + y, x + 2y)$. Let $\mathcal{B} = \{(1, 0)^T, (0, 1)^T\}$ and $\mathcal{B}' = \{(1, 1)^T, (1, -1)^T\}$ be two bases of $\mathbb{R}^2$. Let $A = [L]_{\mathcal{B}}^{\mathcal{B}}$ and $B = [L]_{\mathcal{B}'}^{\mathcal{B}'}$. Then A and B are similar. Consider

$$\mathbb{R}^2_{\mathcal{B}'} \xrightarrow{id} \mathbb{R}^2_{\mathcal{B}} \xrightarrow{L} \mathbb{R}^2_{\mathcal{B}} \xrightarrow{id} \mathbb{R}^2_{\mathcal{B}'}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \; S = [id]_{\mathcal{B}'}^{\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; B = S^{-1}AS = \begin{pmatrix} \frac{5}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}.$$

  We can find B directly also

$$L(1, 1) = (2, 3) = \frac{5}{2}(1, 1) + \frac{-1}{2}(1, -1)$$
$$L(1, -1) = (0, -1) = \tfrac{-1}{2}(1, 1) + \tfrac{1}{2}(1, -1)$$

- If A and B are similar, then they have same characteristic polynomial, $det(A - \lambda I) = det(B - \lambda I)$, hence same eigenvalues.

- **Def.** A square matrix A is called diagonalizable if A is similar to a diagonal matrix Λ, i.e., $S^{-1}AS = \Lambda$ for some S.
  In this case, the eigenvalues of A are the diagonal entries of Λ.

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Diagonalization of a Matrix

Q: What is the advantage of a basis of $\mathbb{R}^n$ consisting of eigenvectors?

Theorem: $\boxed{\text{Eigenvectors diagonalize a matrix}}$

Assume an $n \times n$ matrix A has a basis consisting of eigenvectors $\{x_1, \ldots, x_n\}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$.
Consider the invertible matrix $S = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}$ with columns x_i.

Then $S^{-1}AS =$ diagonal matrix $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$.

Proof. (2×2 case). $Ax_i = \lambda_i x_i$. Hence

$$AS = A \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} Ax_1 & Ax_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 \end{bmatrix} = S\Lambda.$$

Therefore $S^{-1}AS = \Lambda$, i.e., A is similar to a diagonal matrix.

Caution: $\Lambda S = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda_1 a & \lambda_1 b \\ \lambda_2 c & \lambda_2 d \end{bmatrix} \neq S\Lambda.$

Diagonalization: Example

Ex: $A = \begin{bmatrix} 1 & 5 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix}$ is triangular.

$\det(A - \lambda I) = (1-\lambda)(2-\lambda)(3-\lambda)$.
Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

Note: If A is triangular, its eigenvalues are sitting on the diagonal

$$\text{Eigenvectors: } x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} -7 \\ -4 \\ 1 \end{bmatrix}.$$

Further, $\{x_1, x_2, x_3\}$ is a basis of $\mathbb{R}^3$.
Hence $S = [x_1 \;\; x_2 \;\; x_3]$ is invertible , and

$$AS = \begin{bmatrix} Ax_1 & Ax_2 & Ax_3 \end{bmatrix} = [x_1 \;\; 2x_2 \;\; 3x_3] = S\Lambda, \text{ where } \Lambda = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}.$$

Thus $S^{-1}AS = \Lambda$, i.e., A is diagonalizable.

Diagonalization and Change of Basis

With A as before, $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $Tx = Ax$ is linear. If $\mathcal{S} = \{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$, then
$[T]_{\mathcal{S}}^{\mathcal{S}} = \begin{bmatrix}[Te_1]_{\mathcal{S}} & [Te_2]_{\mathcal{S}} & [Te_3]_{\mathcal{S}}\end{bmatrix} = A.$

Recall that for $x_1 = (1,0,0)^T$, $x_2 = (5,1,0)^T$ and $x_3 = (-7,-4,1)$,

$$Tx_1 = x_1, \quad Tx_2 = 2x_2, \quad Tx_3 = 3x_3.$$

Furthermore, $\mathcal{B} = \{x_1, x_2, x_3\}$ is a basis of $\mathbb{R}^3$. **Q:** What is $[T]_{\mathcal{B}}^{\mathcal{B}}$?

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix}[Tx_1]_{\mathcal{B}} & [Tx_2]_{\mathcal{B}} & [Tx_3]_{\mathcal{B}}\end{bmatrix} = \Lambda = \begin{bmatrix}1 & & \\ & 2 & \\ & & 3\end{bmatrix}.$$

Consider $$\mathbb{R}^3_{\mathcal{B}} \xrightarrow{id} \mathbb{R}^3_{\mathcal{S}} \xrightarrow{T} \mathbb{R}^3_{\mathcal{S}} \xrightarrow{id} \mathbb{R}^3_{\mathcal{B}}$$

Change of basis formula:

$$[id]_{\mathcal{S}}^{\mathcal{B}}\, [T]_{\mathcal{S}}^{\mathcal{S}}\, [id]_{\mathcal{B}}^{\mathcal{S}} = [T]_{\mathcal{B}}^{\mathcal{B}} \text{ , i.e.,} \qquad [id]_{\mathcal{S}}^{\mathcal{B}}\, A\, [id]_{\mathcal{B}}^{\mathcal{S}} = \Lambda.$$

Observe: $[id]_{\mathcal{B}}^{\mathcal{S}} = \begin{bmatrix}[id(x_1)]_{\mathcal{S}} & [id(x_2)]_{\mathcal{S}} & [id(x_3)]_{\mathcal{S}}\end{bmatrix} := S.$

i.e., the change of basis formula gives: $S^{-1}AS = \Lambda$.

Thus diagonalization of a matrix is the same as finding a basis w.r.t. which the matrix is diagonal.

When is A Diagonalizabe

- If $x_1, \dots, x_r$ are eigenvectors of A associated to distinct eigenvalues $\lambda_1, \dots, \lambda_r$, then $x_1, \dots, x_r$ are linearly independent.

  Proof. Suppose $x_1, \dots, x_r$ are linearly dependent. Choose a linear relation involving minimum number of x_i's, say
  (1) $\quad a_1x_1 + \dots + a_tx_t = 0. \quad (1 < t \le r,\ t$ is minimal, $a_i \neq 0)$

  Apply A to get $\quad a_1\lambda_1x_1 + \dots + a_t\lambda_tx_t = 0 \qquad (2)$

  $\lambda_1 \cdot (1) - (2)$ gives $\quad a_2(\lambda_1 - \lambda_2)x_2 + \dots + a_t(\lambda_1 - \lambda_t)x_t = 0,$

  which contradicts the minimality of t. $\square$

- If A has n distinct eigenvalues, then A is diagonalizable.

  Proof. If $x_1, \dots, x_n$ are eigenvectors associated to distinct eigenvalues $\lambda_1, \dots, \lambda_n$, then $\{x_1, \dots, x_n\}$ is a linearly independent set.
  Then $S = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}$ is invertible, and $S^{-1}AS = \Lambda$ as earlier.
  Hence A is diagonalizable. $\square$

When is A Diagonalizabe

- A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

  Proof. We have seen ($\Leftarrow$). Let's prove ($\Rightarrow$).

  Assume $S = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}$ is an invertible matrix such that

  $$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

  Then $AS = S\Lambda$, i.e. $\begin{bmatrix} Ax_1 & \dots & Ax_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \dots & \lambda_n x_n \end{bmatrix}$.

  Therefore $x_1, \dots, x_n$ are eigenvectors of A. They are linearly independent since S is invertible. $\square$

$\boxed{\text{The columns of the diagonalizing matrix } S \text{ are eigenvectors of } A}$

S need not be unique, e.g., replace x_1 by $2x_1$ etc.

Diagonalizability: Non-examples

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has repeated eigenvalues $1, 1$.

The eigenvectors of A are $\begin{bmatrix} y \\ 0 \end{bmatrix}$. Theorefore A does not has a basis consisting of eigenvectors, so A is not diagonalizable.

Ex: Similarly, for any $a \in \mathbb{R}$, the matrix $\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$ has repeated eigenvalues a, a and is not diagonalizable.

Eigenvalues of AB and $A+B$

- If λ is an eigenvalue of A, μ is an eigenvalue of B, is $\lambda\mu$ an eigenvalue of AB?

  Proof. False Proof. $\qquad ABx = A(\mu x) = \mu(Ax) = \lambda\mu x.$

  This is false since A and B may not have same eigenvector x.

- **Ex:** $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

  The eigenvalues of A and B are $0, 0$ and that of AB are $1, 0$.

- Eigenvalues of $A+B$ are NOT $\lambda + \mu$.

  In above example, $A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $1, -1$.

- If A and B have same eigenvectors associated to λ and μ, then $\lambda\mu$ and $\lambda+\mu$ are eigenvalues of AB and $A+B$ respectively.

Q: When do A and B have the same eigenvectors?

Simultaneous Diagonalizability

- Assume $\boxed{A \text{ and } B \text{ are diagonalizable}}$. Then A and B have same eigenvector matrix S if and only if $AB = BA$.

- **Proof.** ($\Rightarrow$) Assume $S^{-1}AS = \Lambda_1$ and $S^{-1}BS = \Lambda_2$, where Λ_1 and Λ_2 are diagonal matrices.

  Then $AB = (S\Lambda_1 S^{-1})(S\Lambda_2 S^{-1}) = S(\Lambda_1\Lambda_2)S^{-1}$
  and $BA = S(\Lambda_2\Lambda_1)S^{-1}$.

  Since $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$, we get $AB = BA$.

- (Part of $\Leftarrow$) Assume $AB = BA$.
  If $Ax = \lambda x$, then $ABx = B(Ax) = B(\lambda x) = \lambda Bx$.
  If $Bx = 0$, then x is an eigenvector of B, associated to $\mu = 0$.
  If $Bx \neq 0$, then x and Bx both are eigenvectors of A, associated to λ.

  Special case: Assume all the eigenspaces of A are one dimensional.
  Then $Bx = \mu x$ for some scalar $\mu \Rightarrow x$ is an eigenvector of B.

  We will not prove the general case. $\square$