PH108

Lecture 17: Magnetostatics – 2

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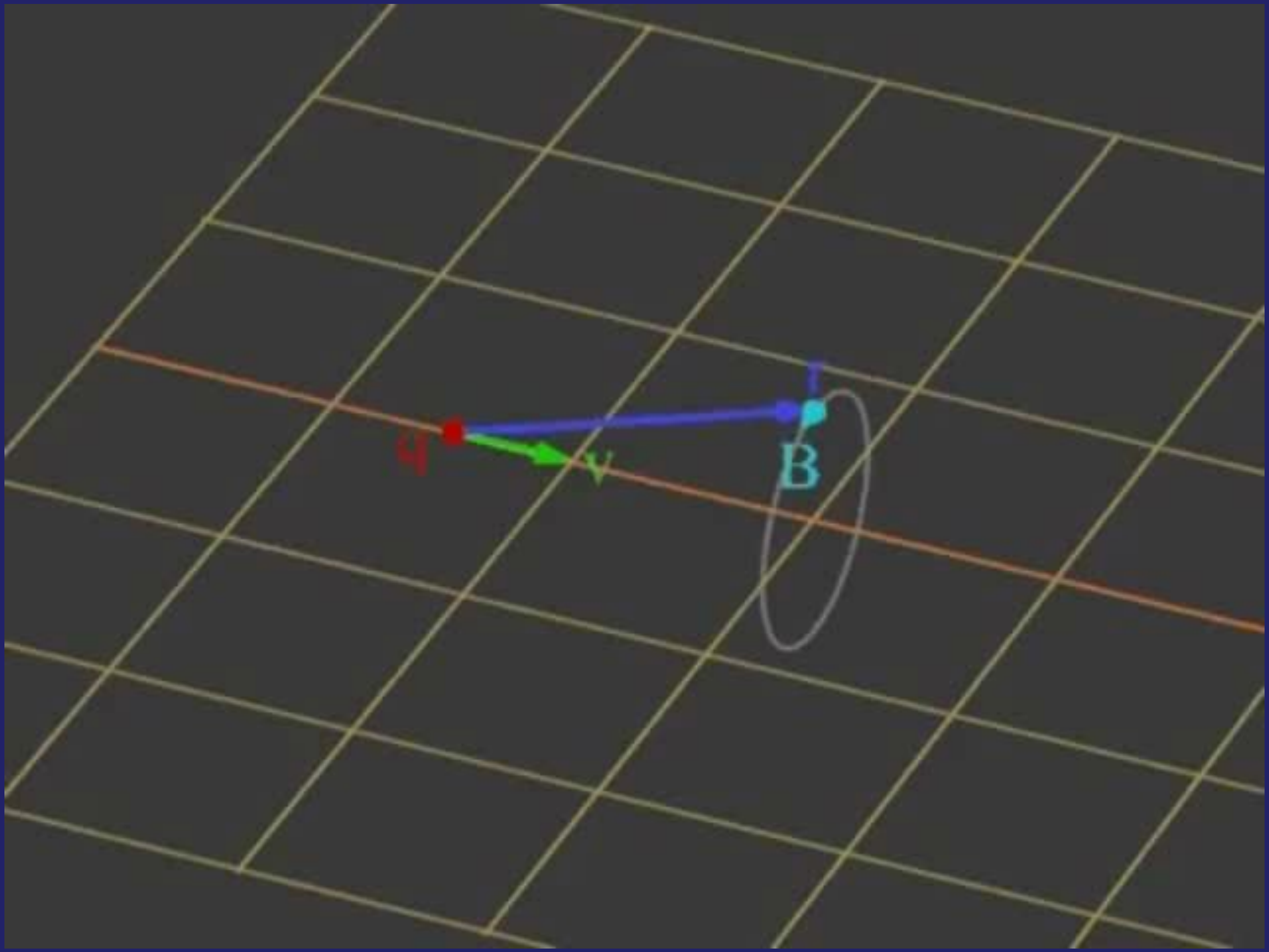
$\vec{B}(r,t)$ is a vector field



www.youtube.com/watch?v=eWaA9RiLsno

$\vec{B}(r,t)$ is a vector field

Any vector field is defined by its divergence and curl

Given $\vec{B} = \frac{\mu_0}{4\pi} \frac{q\,\vec{v} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$

What is $(\vec{\nabla} \cdot \vec{B})$ and what is $(\vec{\nabla} \times \vec{B})$?

Recall:

starting from $F_{electric} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}\,\hat{r}$ we got $\vec{\nabla} \cdot \vec{E}$, $\vec{\nabla} \times \vec{E}$

Magnetostatic ‘Gauss’ Law = Biot-Savart

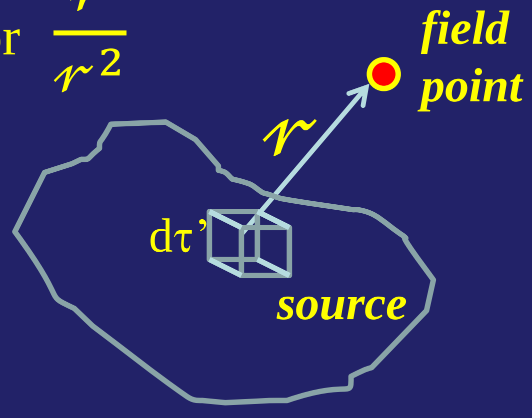
$$\overrightarrow{dB} = \frac{\mu_0}{4\pi}\frac{\vec{J}\times\vec{\mathscr{r}}}{\mathscr{r}^3}\,d\tau'$$

From unit vector $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

field coordinate

source coordinate



$$\vec{B} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d\tau$$

Use $\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$

$$\vec{B} = \frac{\mu_0}{4\pi}\int\vec{\nabla}_r\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\times\vec{J}(\vec{r}')d\tau$$

Gauss law (should) indicate magnetic charge

We have an integrand : $\vec{\nabla} f \times \vec{A}$

$$\vec{\nabla}_r \left(\frac{1}{|\vec{r}-\vec{r}'|}\right) \times \vec{J}(\vec{r}')d\tau$$

Use the identity $\vec{\nabla} \times f\,\vec{A} = \underbrace{f\vec{\nabla} \times \vec{A}}_{0} - \underline{\vec{A} \times \vec{\nabla} f}$

$$0 \quad \frac{1}{|\vec{r}-\vec{r}'|}\,\vec{\nabla}_r \times \vec{J}(r')$$

$\vec{\nabla}_r$ w.r.t $\vec{r}$, so no effect on $\vec{J}(r')$

$$\vec{B} = \frac{\mu_0}{4\pi}\,\vec{\nabla} \times \int \frac{\vec{J}}{|\vec{r}-\vec{r}'|}\,d\tau$$

Divergence of Curl $\equiv 0$

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

No magnetic charge!

$\vec{\nabla} \times \vec{B}$ gives Ampere's Law

Instinctively, from right hand rule, we know $\vec{\nabla} \times \vec{B} \neq 0$

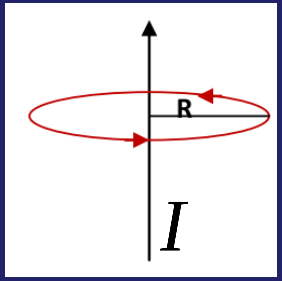
$\vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla} \times \int \frac{\vec{J}}{|\vec{r}-\vec{r}'|} d\tau$ from previous slide

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla} \times \vec{\nabla} \times \int \frac{\vec{J}}{|\vec{r}-\vec{r}'|} d\tau$$

$Use\ \vec{\nabla} \times (\vec{\nabla} \times A) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$ and some vector math….

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{B} = \frac{\mu_0 I}{2\pi R} \hat{\phi}$$



In integral form: $\oint \vec{B} \cdot \vec{dl} = \mu_0 I$

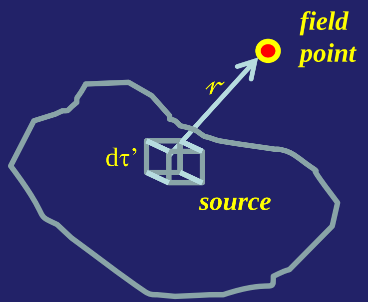
Ampere's Law

We have solved the FIRST problem

1) Calculate the magnetic field due to a set of moving charges, without worrying about *other* charges nearby

2) Calculate the effect of a magnetic field on a set of charges

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J} \times \vec{\mathscr{r}}}{\mathscr{r}^3}\, d\tau'$$



$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \qquad \oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 I$$

Force of a magnetic field on a particle

A particle of charge q moves with velocity $\vec{v}$
in a magnetic field $\vec{B}$

$$\vec{F}_m = q(\vec{v} \times \vec{B})$$

in electric field $\vec{E}$

$$\vec{F}_e = q\vec{E}$$

Total Lorentz Force $\vec{F}_m = q(\vec{E} + \vec{v} \times \vec{B})$

Question

A magnetic field $\vec{B}$ is created by a source charge q moving with velocity $\vec{v}_1$

A *second* charge q nearby is also moving with velocity $\vec{v}_2$

The magnetic force experienced by the *second* charge is

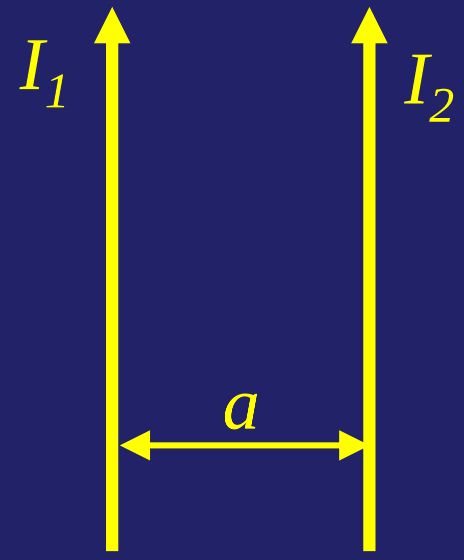
$$\vec{F}_m = q(\vec{v} \times \vec{B})$$

The velocity $\vec{v}$ in the magnetic force refers to:

A) $\vec{v} = \vec{v}_1$ B) $\vec{v} = \vec{v}_2$ C) Both/either

Example: Force between two wires

Two wires with current $I_1\hat{y}$ and $I_2\hat{y}$

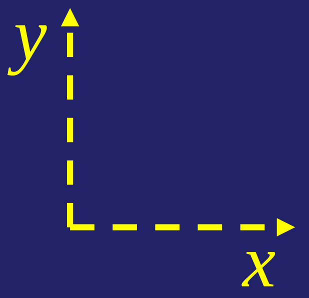


$\vec{B}_1$ due to $\mathrm{I}_1\hat{k}$:

$$\vec{B}_1 = \frac{\mu_0 I_1}{2\pi a}(-\hat{z})$$

In second wire $q\vec{v} = I_2\,\hat{y}$

$$So \quad \vec{F}_{12} = \frac{\mu_0 I_1 I_2}{2\pi a}(-\hat{x})$$

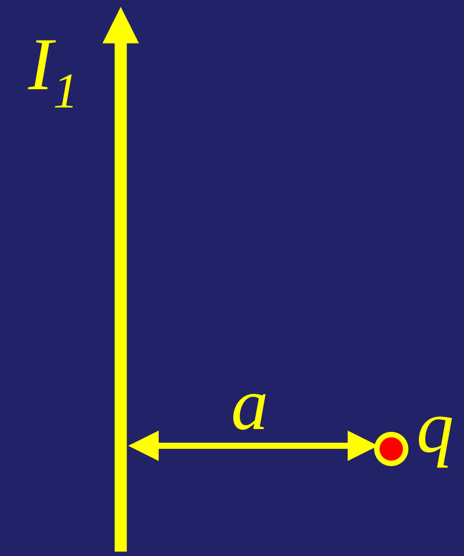


Two wires attract each other

Question 1

A wire carries current $I_1\hat{y}$

A stationary point charge q is placed at distance a from the wire

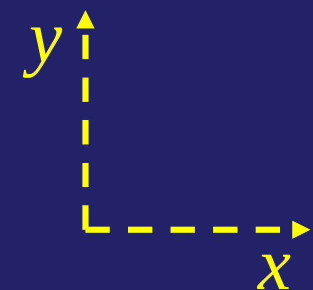


Does q feel an ELECTRIC force?

A) YES

B) NO

C) MAYBE



WHY?

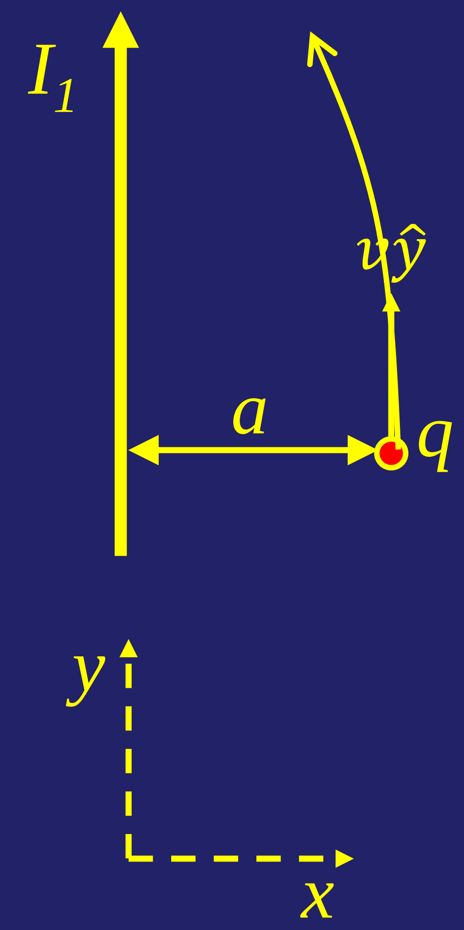
$\because \vec{\nabla} \times \vec{E} = 0$

For any element $d\lambda$ of I_1 the quantity of charge flowing in is *equal* to the charge flowing out: so $\int = 0$

Question 2

A wire carries current $I_1\hat{y}$

A point charge q moves with velocity $v\hat{y}$ at distance a parallel to the wire



Does q feel a MAGNETIC force?

A) YES

B) NO

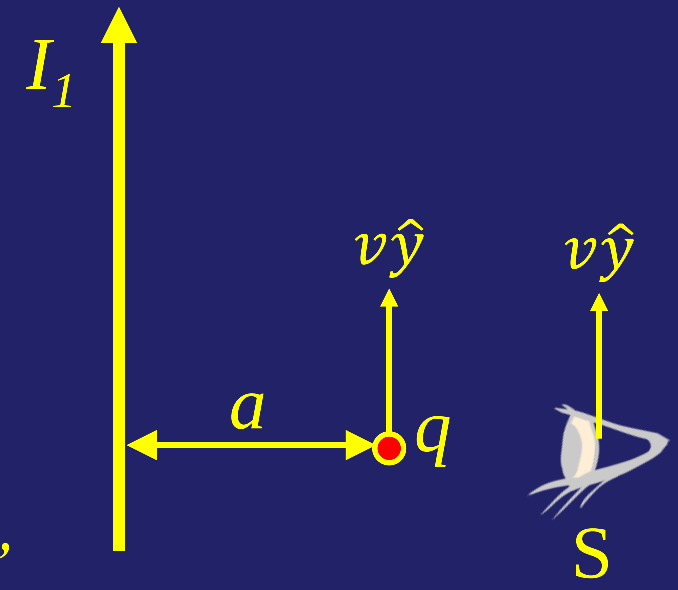
C) MAYBE

$\vec{F}_m = \vec{v} \times \vec{B}$ WILL DEFLECT the particle's x position

Question 3

Same setup as Question 2

Observer S moves with velocity
$v\hat{y}$ parallel to the wire at distance b



In the observer's frame of reference,
Does q feel any force?

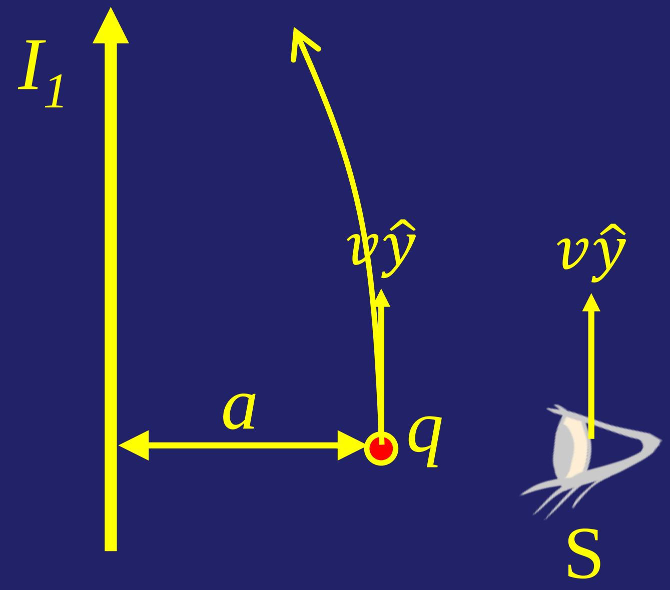
A) YES

B) NO $\because\ \vec{v}_S = 0$ (see Questions 1 and 2)

C) MAYBE

Question 4

(combining questions 2 & 3)



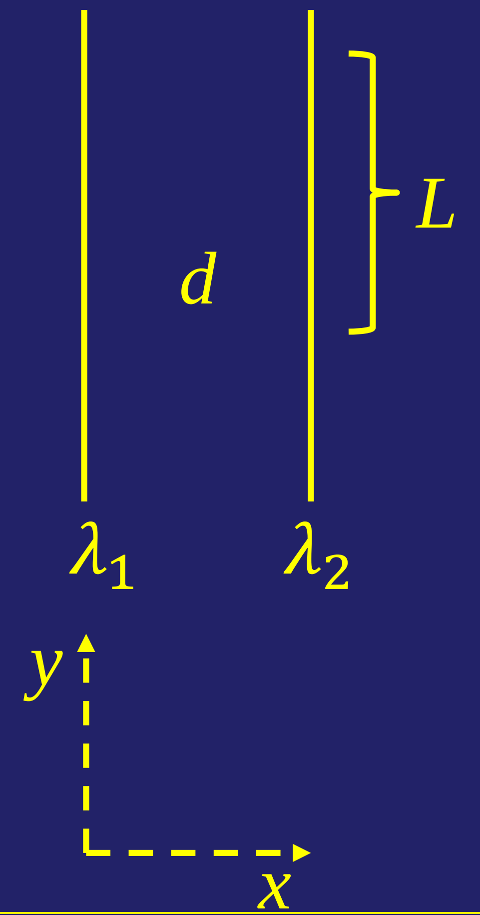
If q does not feel any force in the observer's frame of reference,

WHY DOES TRAJECTORY OF q SHIFT in x ??

How do we resolve the paradox of Question 4?

The resolution requires us to carefully look at the frames of reference for observers. In particular, we will use the concept of 'length contraction' as seen by an observer moving close to the speed of light

Consider two STATIC line charges λ_1 & λ_2



$\vec{E}$ due to λ_1 at location of λ_2 :

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda_1}{d}\ \hat{x}$$

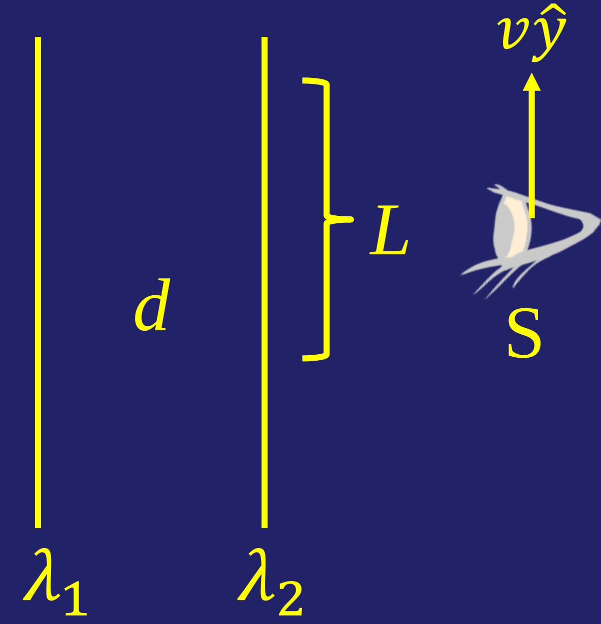
$Force$ due to λ_1 on length 'L' of λ_2 :

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{\lambda_1\lambda_2 L}{d}\ \hat{x}$$

An external observer S moves at velocity $v\,\hat{y}$

In S frame L is seen to be $\frac{\mathrm{L}}{\gamma}$

So the linear charge is distributed over a shorter length. i.e. $\lambda_1 \to \gamma\lambda_1, \lambda_2 \to \gamma\lambda_2$



$In\ S\ frame, the\ Electric\ Force$ is:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\vec{F}_s = \frac{1}{4\pi\epsilon_0}\frac{\gamma\lambda_1\gamma\lambda_2 L/\gamma}{d}\ \hat{x} = \gamma\ \vec{F}$$

In special relativity Lorentz transforms, a transverse force should transform as: $\vec{F}' = \dfrac{\vec{F}}{\gamma}$

The anomaly can be solved if observer S invokes the existence of another force – a magnetic force $\vec{F}_m$ such that:

$$\vec{F}_s + \vec{F}_m = \vec{F}/\gamma \qquad \vec{F}_m = \frac{\vec{F}}{\gamma} - \gamma\,\vec{F} = -\gamma\,\vec{F}\,\frac{v^2}{c^2}$$

Force $\vec{F}_m$ is attractive : this makes sense, because in observer frame S, the two static charges λ_1, λ_2 are two currents $I_1\hat{y}$, $I_2\hat{y}$

So the two observers (you and S) observe the *same* force, but disagree on the nature of the force.

You think it is $\vec{F}_{electric}$;

Observer S thinks it is $\vec{F}_{electric} + \vec{F}_{magnetic}$