PH108

Lecture 11:

Method of Images – plane conductor

Pradeep Sarin
Department of Physics

Supplementary material:
method of images for spherical conductors – Lecture 12 + problems in tutorials

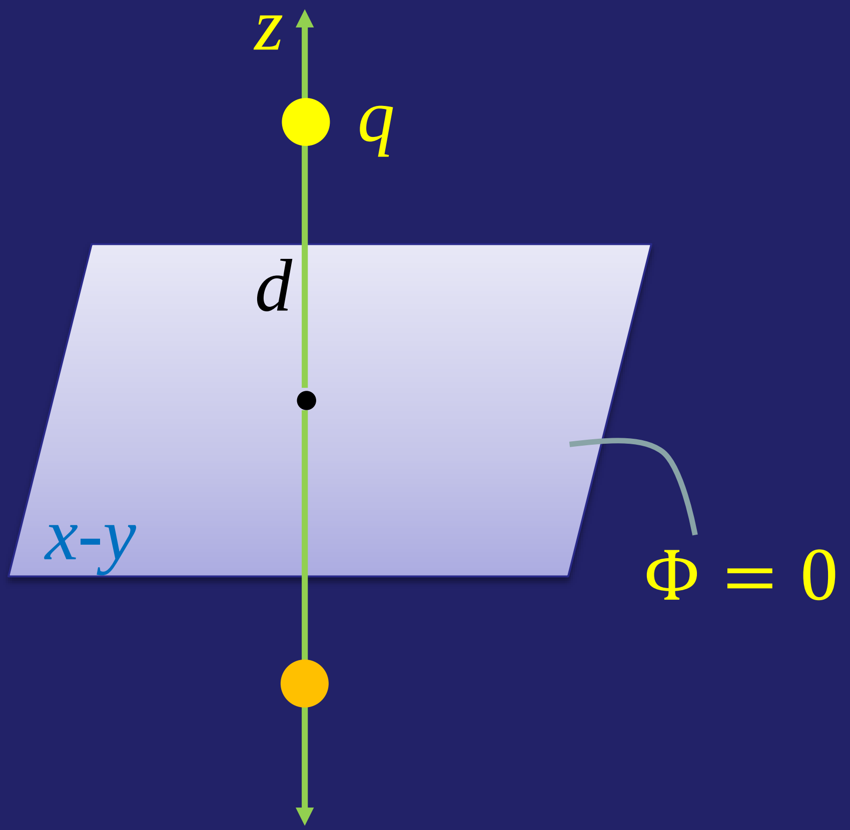
Method of images is an intuitive method to solve Laplace's equation

Recall: solution to $\nabla^2\Phi = 0$ is *unique*

If we develop a systematic intuitive method
to solve for Φ with given boundary conditions,
its good enough – we will get the right Φ

The method of images works for <u>conductors</u> with arbitrary charge distribution around them

Point charge above a grounded conductor

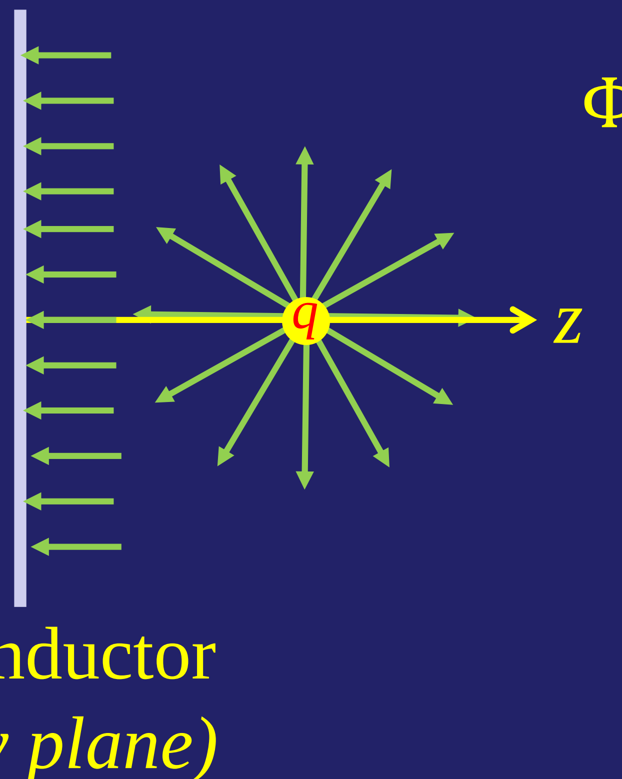


infinite conductor in *x*-*y* plane

Point charge q is placed at $z=d$ above the plane

<u>Problem:</u> Determine $\Phi(x,y,z)$ for $z>0$ *except* at $z=d$

Intuition: visualize the field lines

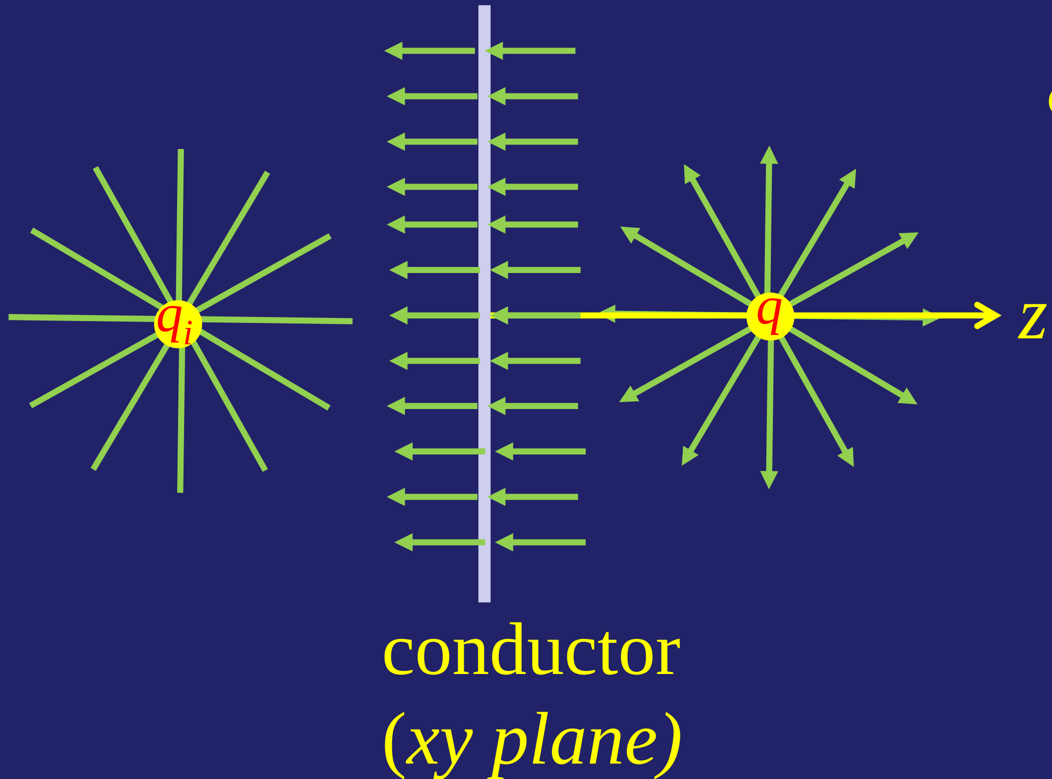


$\Phi(z > 0) = ?$

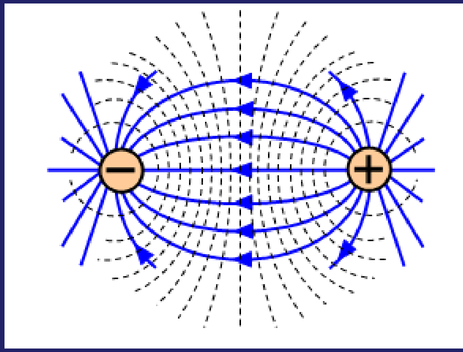
Φ and $\vec{E}$ are continuous functions

To make field lines match, induced σ on conductor is $\sigma(x,y)$

Intuition *suggests* an image charge

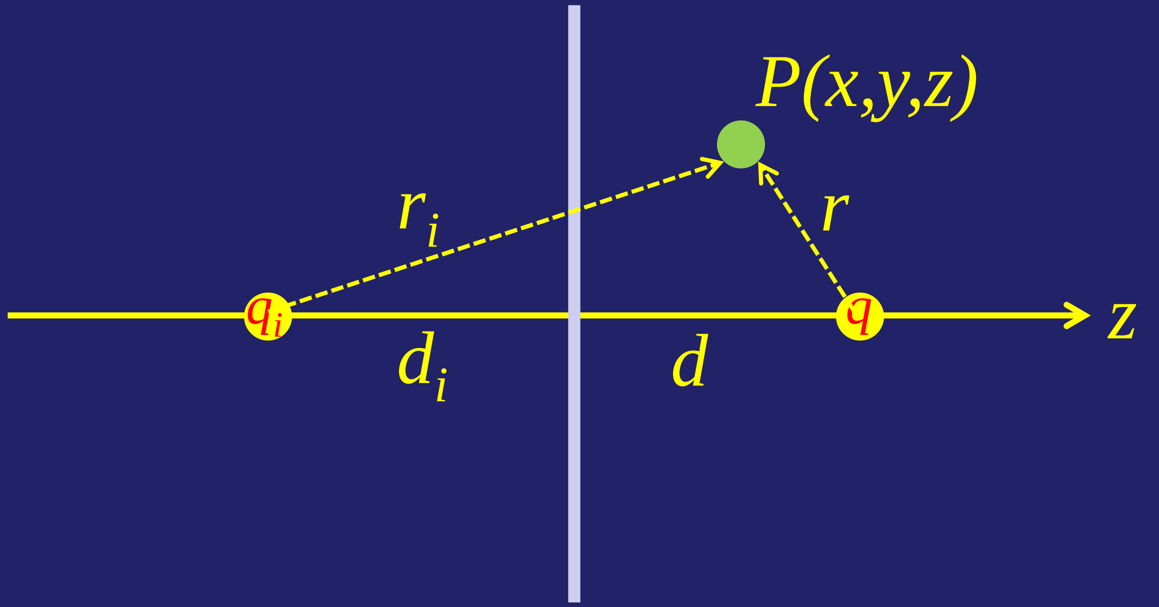


$\Phi(z > 0) = ?$



Note: the image charge q_i is completely fictitious!
We only care about $\Phi(z > 0)$, ignore $z<0$

Calculate Φ by superposition of q and q_i



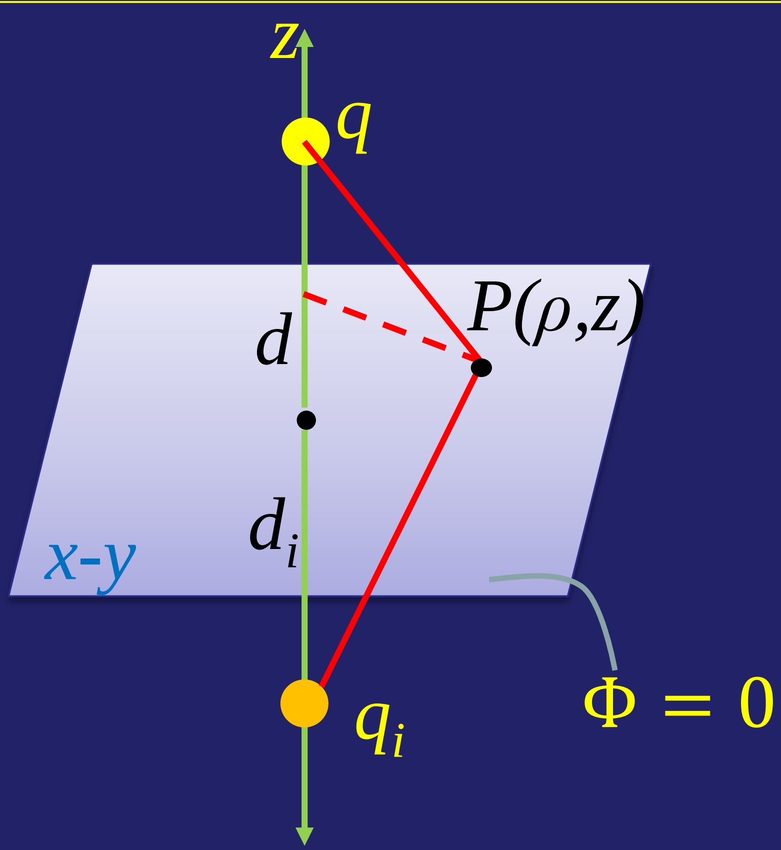
$$\Phi_q = \frac{q}{4\pi\epsilon_0 r}$$

$$\Phi_{qi} = \frac{q_i}{4\pi\epsilon_0 r_i}$$

Satisfies $\nabla^2\Phi = 0$ everywhere in $z>0$ except at location of q $(r=0)$ where we don't care

$$\nabla^2\left(\frac{1}{r}\right) \equiv 4\pi\delta^3(\vec{r})$$

Determine q_i , d_i by boundary conditions



$$\Phi_q = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{\rho^2 + (z-d)^2}}$$

$$\Phi_{qi} = \frac{q_i}{4\pi\epsilon_0 r_i}\frac{1}{\sqrt{\rho^2 + (z+d_i)^2}}$$

$$\Phi(\rho, z=0) = 0$$

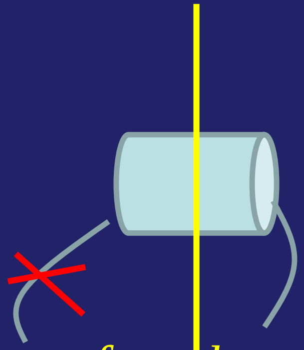
$$\rightarrow q_i = -q \; and \; d_i = d$$

Use Φ to determine $\vec{E}$, induced σ

$$\Phi(z>0)=\frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{\rho^2+(z-d)^2}}-\frac{1}{\sqrt{\rho^2+(z+d)^2}}\right]$$

$$\vec{E}=-\vec{\nabla}\Phi=-\left(\hat{\rho}\frac{\partial}{\partial\rho}+\hat{k}\frac{\partial}{\partial z}\right)\Phi$$

$$\sigma_{induced}=\epsilon_0\,E_z=-\frac{q}{2\pi}\frac{d}{(\rho^2+d^2)^{\frac{3}{2}}}$$



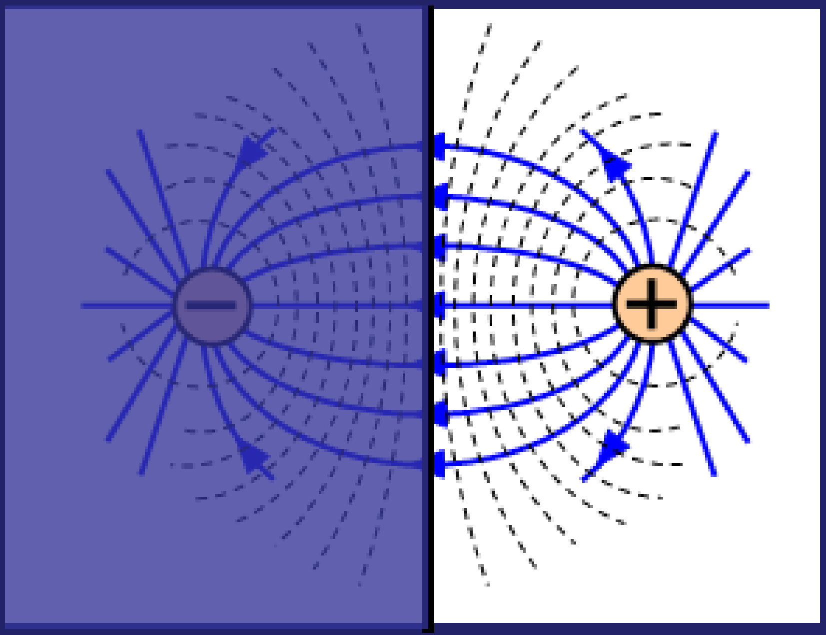
Note: usually $E_z=\frac{\sigma}{2\epsilon_0}$ at conductor surface, but we ignore the E_z on the 'image' side, hence factor of $E_z=\frac{\sigma}{\epsilon_0}$

Electric field lines obey intuition

$$\vec{E} = \frac{q}{4\pi\epsilon_0} *$$

$$\hat{\rho}\left\{\frac{\rho}{(\rho^2+(z-d)^2)^{\frac{3}{2}}} - \frac{\rho}{(\rho^2+(z+d)^2)^{\frac{3}{2}}}\right\}$$

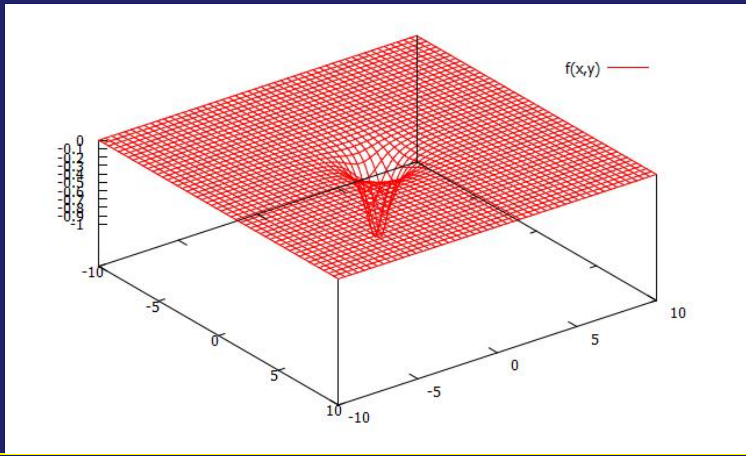
$$+\hat{k}\left\{\frac{(z-d)}{(\rho^2+(z-d)^2)^{\frac{3}{2}}} - \frac{(z+d)}{(\rho^2+(z+d)^2)^{\frac{3}{2}}}\right\}$$



$\boxed{\vec{E}\textit{ valid only for z>0}}$

$$\sigma_{induced} = -\frac{q}{2\pi}\frac{d}{(\rho^2+d^2)^{\frac{3}{2}}}$$

is maximum under the *real* charge

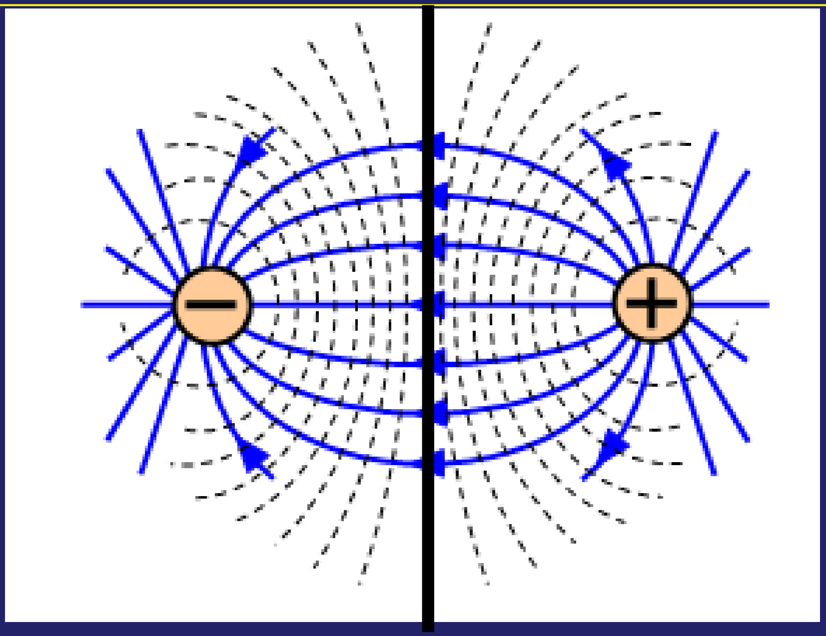


What is the force between q and conductor plate?

$$\vec{E} = \frac{q}{4\pi\epsilon_0} *$$

$$\boxed{\hat{\rho}\left\{\frac{\rho}{(\rho^2+(z-d)^2)^{\frac{3}{2}}} - \frac{\rho}{(\rho^2+(z+d)^2)^{\frac{3}{2}}}\right\}}$$

$$+\hat{k}\left\{\boxed{\frac{(z-d)}{(\rho^2+(z-d)^2)^{\frac{3}{2}}}} - \frac{(z+d)}{(\rho^2+(z+d)^2)^{\frac{3}{2}}}\right\}$$



→ Force due to field of image charge –q at $\rho=0$, $z=d$

$$\vec{F} = -\frac{q^2}{16\pi\epsilon_0 d^2}\hat{k}$$

Force by an *infinite* conductor on a point charge!

Note: the image charge is *not real!*

Total electrostatic energy: $W = \int_{\infty}^{d} \frac{q^2}{16\pi\epsilon_0 z^2} dz$

$$W = \frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$$

This is ½ the electrostatic energy of charges $+q$, $-q$ at distance $2d$

a) The energy is stored in the field: the field exists only in right half $z>0$

b) It doesn't cost any energy to assemble the induced charge – induced charge is moved ⊥ to electric field

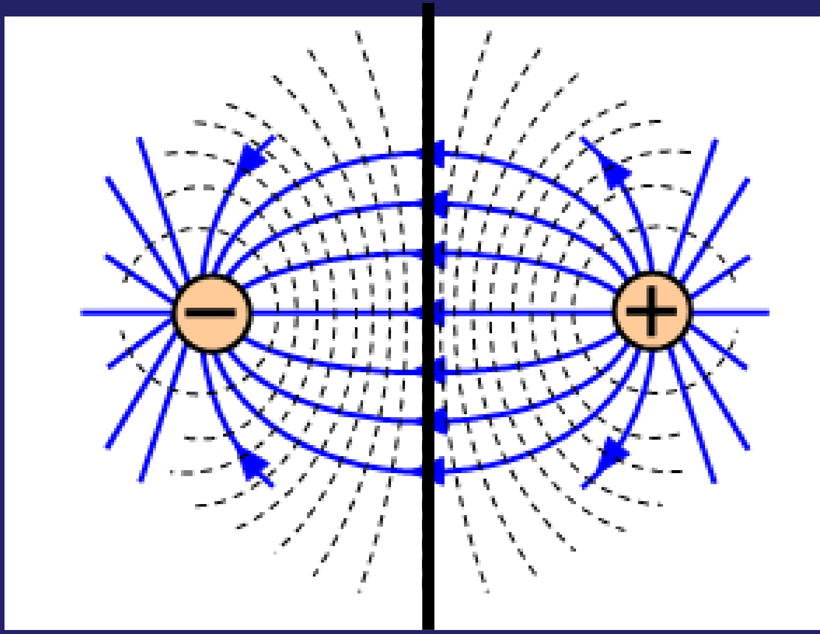


Image charge does not exert any force on the conductor

The image charge is a calculation tool used to compute $\Phi(z>0)$

NOT REAL: it represents the total effect of the induced charge

Logic thread of today's lecture

To solve Laplace's equation $\nabla^2\Phi = 0$ we used:

a) Uniqueness of solution

b) Intuition

For point charge above a grounded conductor, we visualized the field lines and existence of image charge

Checked that $\Phi_{\text{real}} + \Phi_{\text{induced}}$ satisfies Laplace eqn

Used boundary condition $\Phi(z=0)=0$ to find location of image

With Φ, found $\vec{E}$ *and* $\sigma_{induced}$