MA-108 Ordinary Differential Equations

M.K. Keshari

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

7th April, 2015
D1 - Lecture 16

Recall: We computed inverse Laplace transform of $P(s)/Q(s)$, where P, Q are polynomials in s and degree of P is $<$ degree of Q, using partial fractions of P/Q, linearity of L^{-1} and Laplace transforms of known functions.

We computed $L(f') = sL(f) - f(0)$.

$L(f'') = s^2L(f) - sf(0) - f'(0)$ etc.

We can solve IVP of constant coefficient ODE using Laplace transform.

Let's begin with an example.

Ex: Solve $\quad y'' + 2y' + 2y = 1,\ y(0) = -3, y'(0) = 1.$

The equation has a unique solution ϕ defined on all of $\mathbb{R}$. Assume ϕ is of exponential of order s_0. Then for all $s \geq s_0$,

$$\begin{aligned}
L(\phi'') + 2L(\phi') + 2L(\phi) &= L(1) \\
(s^2 L(\phi) - s\phi(0) - \phi'(0)) + 2(sL(\phi) - \phi(0)) + 2L(\phi) &= \frac{1}{s} \\
(s^2 + 2s + 2)L(\phi) - (s+2)\phi(0) - \phi'(0) &= \frac{1}{s} \\
((s+1)^2 + 1)L(\phi) + 3(s+2) - 1 &= \frac{1}{s} \\
L(\phi) = \frac{1 - (3s+5)s}{((s+1)^2+1)s} &= F(s)
\end{aligned}$$

We want to compute $L^{-1}(F(s))$. We use partial fractions.

$$F(s) = \frac{-3s^2 - 5s + 1}{((s+1)^2 + 1)s} = \frac{A}{s} + \frac{B(s+1) + C}{(s+1)^2 + 1}$$

$$\implies -3s^2 - 5s + 1 = A((s+1)^2 + 1) + (B(s+1) + C)s$$

Let $s = 0, -1, 1$, to get the following equations.

$$1 = 2A, \quad 3 = A - C, \quad -7 = 5A + 2B + C$$

This implies $A = 1/2, \quad C = -5/2 \quad$ and $\quad B = -7/2$.

$$\implies L(\phi) = \frac{1}{2s} - \frac{7(s+1)}{2((s+1)^2 + 1)} - \frac{5}{2((s+1)^2 + 1)}$$

$$\implies \phi(t) = \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t$$

$\square$

Ex. More generally, to solve a constant coefficient IVP

$$y'' + py' + qy = r(t), \quad y(0) = a,\ y'(0) = b, \ \ p, q \in \mathbb{R}$$

let ϕ be the unique solution, which has a Laplace transform for all $s \geq s_0$. Applying Laplace transform, we get

$$(s^2 L(\phi) - s\phi(0) - \phi'(0)) + p(sL(\phi) - \phi(0)) + qL(\phi) = L(r)$$

$$\implies (s^2 + ps + q)L(\phi) = L(r) + sa + b + pa$$

We can simply this to an equation $L(\phi) = F(s)$ and compute the inverse Laplace transform of F, to get $\phi(t)$.

Remark. Although the unique solution exist on $\mathbb{R}$, Laplace transform gives solution only on $[0, \infty)$.

Unit Step Function

Let us consider IVP with constant coefficients, where the forcing function $r(t)$ is piecewise continuous. To solve it using Laplace transform, we need to find Laplace transform of piecewise continuous functions. To do this in a systematic way, let us begin with a definition.

Definition

The **unit (or Heaviside) step function** is defind as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Replacing t by $t - a$, we get

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

Ex. Express them in terms of unit step functions.

- Ramp Function $= \begin{cases} 0, & 0 < t < a \\ t-a, & t > a \end{cases} = (t-a)u(t-a)$.

- $f(t) = \begin{cases} \sin t, & 0 < t < t_0 \\ t, & t \geq t_0 \end{cases} = \sin t + u(t-t_0)(t - \sin t)$.

- $f(t) = \begin{cases} \sin t, & 0 < t < t_0 \\ \cos t, & t_0 \leq t \leq t_1 \\ t, & t > t_1 \end{cases}$

  $= \sin t + u(t-t_0)(\cos t - \sin t) + u(t-t_1)(t - \cos t)$.

- $f(t) = \begin{cases} f_1, & 0 \leq t < t_1 \\ f_2, & t_1 \leq t < t_2 \\ . & . \\ f_n, & t_{n-1} \leq t \end{cases}$

  $= f_1 + u(t-t_1)(f_2 - f_1) + \ldots + u(t-t_{n-1})(f_n - f_{n-1})$.

1. 140020046 SHRIKANT MUNDRA ABSENT
2. 140020054 PRANAY LADIWALA
3. 140020063 NAVEEN
4. 140020074 GURMEET SINGH BEDI
5. 140020079 SURBHI SAHU
6. 140020094 ANAY TRIPATHI
7. 140020104 GAURAV JAIN
8. 140020115 NIKHIL PURSHOTTAM MISKIN
9. 140050031 C VISHWESH
10. 140050036 ANGAJALA HEMANTH KUMAR
11. 140050047 Y PUSHYARAG
12. 140050061 RISHABH VIJAY CHAVHAN
13. 140050076 VAKACHARLA PRAMOD
14. 140020122 RATANJOT SINGH
15. 140020024 PRANSHU MAHENDRA JAIN
16. 140020047 ADITYA S PUNEKAR
17. 140020076 PRANAY NAHAR
18. 140020091 KANCHAN KHETAN

Q. Why write a piecewise continuous function in terms of unit step functions? Because it simplifies computing its Laplace transform.

Theorem (Second Shifting Theorem)

Let $g(t)$ be defined for $t \geq 0$. Assume $L(g(t+a))$ exists for $s > s_0$, where $a \geq 0$. Then $L(u(t-a)g(t))$ exists for $s > s_0$, and

$$L(u(t-a)g(t)) = e^{-sa}L(g(t+a)).$$

Proof.

$$\begin{aligned} L(u(t-a)g(t)) &= \int_0^\infty e^{-st}u(t-a)g(t)\,dt \\ = \int_a^\infty e^{-st}g(t)\,dt &= \int_0^\infty e^{-s(x+a)}g(x+a)\,dx \\ &= e^{-sa}\,L(g(t+a)) \end{aligned}$$

Theorem (Second Shifting Theorem)

If $a \geq 0$ and $L(f)$ exists for $s > s_0$, then $L(u(t-a)f(t-a))$ exists for $s > s_0$ and

$$L(u(t-a)f(t-a)) = e^{-as}L(f(t)) = e^{-as}F(s).$$

Ex. $L(u(t-a)) = e^{-as}L(1) = \dfrac{e^{-as}}{s}$.

Ex. If $f(t) = t^2 + 1$, find $L(u(t-1)f(t))$.

$$\begin{aligned} L(u(t-1)f(t)) &= e^{-s}L(f(t+1)) \\ &= e^{-s}L((t+1)^2 + 1) \\ &= e^{-s}L(t^2 + 2t + 2) \\ &= e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s}\right) \end{aligned}$$

Ex. Find Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < 2 \\ -2t+1, & 2 \le t < 3 \\ 3t, & 3 \le t < 5 \\ t-1, & t \ge 5 \end{cases}.$$

Write $f(t)$ in terms of unit step functions as $f(t) =$

$$1+u(t-2)(-2t+1-1)+u(t-3)(3t-(-2t+1))+u(t-5)(t-1-3t)$$

$$= 1 - 2u(t-2)t + u(t-3)(5t-1) - u(t-5)(2t+1).$$

Laplace transform $L(f) =$

$$L(1) - e^{-2s}L(t+2) + e^{-3s}L(5(t+3)-1) - e^{-5s}L(2(t+5)+1)$$

$$= L(1) - e^{-2s}L(t+2) + e^{-3s}L(5t+14) - e^{-5s}L(2t+11)$$

$$= \frac{1}{s} - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + e^{-3s}\left(\frac{5}{s^2} + \frac{14}{s}\right) - e^{-5s}\left(\frac{2}{s^2} + \frac{11}{s}\right).$$

Ex. Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 \le t < \frac{\pi}{2} \\ \cos t - 3\sin t, & \frac{\pi}{2} \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$$

Use

$$L(u(t-a)f(t)) = e^{-sa}L(f(t+a)).$$

Write

$$f(t) = \sin t + u(t - \frac{\pi}{2})(\cos t - 4\sin t) + u(t-\pi)(2\cos t + 3\sin t).$$

$$L(f) = \frac{1}{s^2+1} - e^{-\pi s/2}\left(\frac{1+4s}{s^2+1}\right) - e^{-\pi s}\left(\frac{3+2s}{s^2+1}\right)$$

Ex.: Find inverse Laplace transform of $H(s) = \frac{e^{-2s}}{s}$.

Use the fact that $L(u(t-a)f(t-a)) = e^{-as}L(f(t))$.

If $f(t) = 1$, then $L(f) = \frac{1}{s}$ and $f(t-2) = 1$.

Hence $L^{-1}(H) = u(t-2)f(t) = u(t-2)$.

Ex. Find inverse Laplace transform of $H(s) = \frac{e^{-2s}}{s^2}$.

Here $a = 2$, $F(s) = \frac{1}{s^2} = L(t)$. So $f(t) = t$.

Now $L^{-1}\left(\frac{e^{-2s}}{s^2}\right) = u(t-2)(t-2)$.

Ex. Find inverse Laplace transform of $H(s) = \dfrac{e^{-2s}}{s-3}$.

Here $F(s) = \dfrac{1}{s-3} = L(e^{3t})$.

Hence $L^{-1}\left(\dfrac{e^{-2s}}{s-3}\right) = u(t-2)e^{3(t-2)}$.

Ex. Find inverse Laplace transform of $H(s) = \dfrac{e^{-2s}}{(s-3)^2}$.

Here $F(s) = \dfrac{1}{(s-3)^2} = L(te^{3t})$. So $f(t) = te^{3t}$.

Now $L^{-1}\left(\dfrac{e^{-2s}}{(s-3)^2}\right) = u(t-2)(t-2)e^{3(t-2)}$.

Ex. Find inverse Laplace transform of

$$F(s) = e^{-s}\frac{1}{2s} - e^{-2s}\frac{(s+1)}{((s+1)^2+1)}.$$

$$L^{-1}\left(\frac{1}{2s}\right) = \frac{1}{2}, \quad L^{-1}\left(\frac{7(s+1)}{2((s+1)^2+1)}\right) = e^{-t}\sin t$$

Hence

$$L^{-1}(F(s)) = \frac{1}{2}u(t-1) - u(t-2)e^{-(t-2)}\sin(t-2)$$

$$= \begin{cases} 0, & 0 \le t < 1 \\ \frac{1}{2}, & 1 \le t < 2 \\ -e^{-(t-2)}\sin(t-2) - \frac{1}{2}, & t \ge 2 \end{cases}$$

IVP with peicewise continuous forcing functions

Consider the differential equation of the form

$$y'' + 3y' + 2y = \begin{cases} e^{x} & 0 < x \leq 2 \\ e^{-x} & 2 < x \end{cases}, y(0) = 1, y'(0) = -1.$$

From what we know, this IVP has a unique solution in the interval $(0, 2)$ and if the IVP was defined on $x_0 \in (2, \infty)$ then we would have a unique solution on $(2, \infty)$.

But its still possible to get a solution which is continuous on $[0, \infty)$. Let y_1 be the unique solution to the given IVP on $[0, 2)$. Then evaluate $y_1(2)$ and $y_1'(2)$.

Define a new IVP as
$y'' + 3y' + 2y = e^{-x}, \quad y(2) = y_1(2), \quad y'(2) = y_1'(2)$. This has a unique solution y_2 on $[2, \infty)$.