MA-108 Ordinary Differential Equations

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D1 - Lecture 9

Recall:

Theorem (Uniqueness Theorem to homogeneous IVP)

Consider the homogeneous IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = a, y'(x_0) = b,$$

where $p(x)$ and $q(x)$ are continuous on an interval I containing x_0. Then there is a unique solution to the IVP on I.

Theorem (Dimension Theorem)

If $p(x), q(x)$ are continuous on an open interval I, then the set of solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0 \qquad (1)$$

on I is a vector space of dimension 2. Any basis $\{y_1, y_2\}$ of solutions of (1) is called a **fundamental solutions** *of (1)* $\square$

Proof of Dimension Theorem

If y_1 and y_2 are solutions of (1), then $c_1y_1 + c_2y_2$ is also a solution of (1). To see this,

$$(c_1y_1 + c_2y_2)'' + p(x)(c_1y_1 + c_2y_2)' + q(x)(c_1y_1 + c_2y_2) =$$

$$c_1[y_1'' + p(x)y_1' + q(x)y_1] + c_2[y_2'' + p(x)y_2' + q(x)y_2] = 0$$

Thus the solution space is a vector space. Now

(i) we need to produce two linearly independent solutions, say f and g, and

(ii) show that any other solution is a linear combination of f and g.

Proof of Dimension Theorem Continued ...

(i) Proof of existence of f and g

Fix $x_0 \in I$. Let $y_1 = f(x)$ be the unique solution of the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 1,\ y'(x_0) = 0$$

y_1 exists on I by uniqueness theorem. Similarly, let $y_2 = g(x)$ be the unique solution of the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0,\ y'(x_0) = 1$$

We need to show that f, g are linearly independent. Assume

$$af(x) + bg(x) \equiv 0 \implies af'(x) + bg'(x) \equiv 0$$

for some scalars a and b. Evaluate at $x = x_0$, we get

$$a = 0, \quad b = 0$$

This proves f and g are linearly independent. Now we show **(ii)** that any solution is a linear combination of f and g.

Proof of Dimension Theorem Continued ...

Let $h(x)$ be an arbitrary solution of the given ODE. We want to find c and d in $\mathbb{R}$ such that

$$h(x) = cf(x) + dg(x) \implies h'(x) = cf'(x) + dg'(x) \quad \text{on} \quad I$$

Therefore, evaluating at $x = x_0$ gives

$$h(x_0) = cf(x_0) + dg(x_0) = c \ \text{ and } \ h'(x_0) = cf'(x_0) + dg'(x_0) = d$$

Let $\widetilde{h}(x) = h(x_0)f(x) + h'(x_0)g(x)$. Then $\widetilde{h}(x)$ is a solution of

$$y'' + p(x)y' + q(x)y = 0, \quad \widetilde{h}(x_0) = h(x_0), \ \widetilde{h}'(x_0) = h'(x_0) \quad (2)$$

Since $h(x)$ is also a solution of IVP (2), by uniqueness theorem, $\widetilde{h} = h$. Thus any solution is a linear combination of f and g. Therefore the solution space is 2-dimensional. $\square$

Nonhomogeneous 2nd order linear ODE

Consider 2nd order linear ODE $y'' + p(x)y' + q(x)y = r(x)(1)$ with $p(x), q(x), r(x)$ continuous on open interval I.

The homogeneous part is $\quad y'' + p(x)y' + q(x)y = 0 \quad (2)$

We have seen that solution space of (2) is a 2-dimensional vector space.

(i) Suppose y_1 is a solution of (1) and y_2 is a solution of (2). Then $y_1 + y_2$ is a solution of (2). To see this

$(y_1 + y_2)'' + p(x)(y_1 + y_2)' + q(x)(y_1 + y_2) =$

$(y_1'' + p(x)y_1' + q(x)y_1) + (y_2'' + p(x)y_2' + q(x)y_2)$

$= r(x) + 0 = r(x)$.

(ii) Fix a solution y_1 of (1). If y is a solution of (1), then $y = y_1 + y_2$, for some solution y_2 of (2).

To see this, note that $y - y_1$ is a solution of (2). Call $y - y_1 = y_2$. Then $y = y_1 + y_2$.

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Wronskian and Linear Independence

Given two solutions f and g of $y'' + p(x)y' + q(x)y = 0$. How to check whether f and g are linearly independent?
We have seen in numerical method that evaluating a solution at some point may not be possible. We start with a definition for this purpose.

Definition

Let f and g be two differentiable functions on I. The **Wronskian** of $f(x)$ and $g(x)$ is a function defined by

$$W(f, g; x) := \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x).$$

Proposition

Suppose $I = (a, b)$ and $f(x), g(x) \in \mathcal{C}^1(I)$ are linearly dependent. Then, $W(f, g; x) = 0$ for all $x \in I$.

Theorem (Abel's Formula)

Assume $p(x)$ and $q(x)$ are continuous on $I = (a, b)$. Let $f(x)$ and $g(x)$ be solutions of $y'' + p(x)y' + q(x)y = 0$.
Then Wronskian of $f(x)$ and $g(x)$ is given by

$$W(f, g; x) = W(f, g; a)\, e^{-\int_a^x p(t)dt},$$

for any $a \in I$.

- Thus $W(x_0) = 0$ for some $x_0 \in I \implies W(x) \equiv 0$ on I.
- Similarly, $W(x_0) \neq 0$ for some $x_0 \in I \implies W(x)$ does not take zero value on I.

Ex. Consider ODE $x^2y'' + xy' - 4y = 0$.

Here $y_1 = x^2$ and $y_2 = \dfrac{1}{x^2}$ are solutions. Compute the Wronskian $W(y_1, y_2; x)$.

Direct method: $W = y_1y_2' - y_1'y_2 = x^2(\dfrac{-2}{x^3}) - (2x)\dfrac{1}{x^2} = \dfrac{-4}{x}$.

Let's verify Abel's Formula: If x_0 and x both are in $(-\infty, 0)$ or in $(0, \infty)$, then

$$W(x) = W(x_0)\, exp\left[-\int_{x_0}^{x} p(t)\, dt\right] = W(x_0)\, exp\left[\int_{x_0}^{x} \frac{-1}{t}\, dt\right]$$

$$= W(x_0)\, exp\left[-(\ln|x| - \ln|x_0|)\right] = W(x_0)\, exp(\ln \frac{x_0}{x})$$

$$= \frac{-4}{x_0}\,\frac{x_0}{x} = -4/x$$

Theorem

Consider

$$y'' + p(x)y' + q(x)y = 0,$$

where $p(x)$ and $q(x)$ are continuous on $I = (a, b)$. Suppose f and g are solutions on I. Then f and g are linearly independent on I if and only if $W(f, g; x)$ has no zeros in I.

Proof. (i) $(\Rightarrow)$. It is enough to show that if $W(x_0) = 0$ for some $x_0 \in I$, then f and g are linearly dependent.

Since f, g are linearly independent, $f(x_0) \neq 0$ for some $x_0 \in I$.

Choose an open interval J containing x_0 such that f does not take zero value on J. On J, we have:

$$\left(\frac{g}{f}\right)'(x) = \left(\frac{fg' - f'g}{f^2}\right)(x) = \frac{W(f, g; x)}{f^2(x)} = 0$$

since $W(x_0) = 0 \implies W(x) = W(x_0)e^{\int_{x_0}^{x} -p(t)dt} \equiv 0.$

Proof continued ...

$$\left(\frac{g}{f}\right)' \equiv 0 \quad \text{on} \quad J \implies \frac{g}{f} = k$$

a constant on J. Hence $g(x) = kf(x)$ on J.

But we want $g(x) = kf(x)$ on I. For this, consider the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, y'(x_0) = 0 \quad (*)$$

$y_1 \equiv 0$ and $y_2 = g - kf$ both are solutions of (*). By uniqueness theorem, $y_1 = y_2$ on I. Hence $g(x) = kf(x)$ on I.

Now we have to prove $(\Leftarrow)$. It is enough to show that if f and g are linearly dependent, then $W(f, g, ; x) \equiv 0$. This is proved earlier. $\square$

Wronskian and Linear Independence

Remarks:

1. The continuity of $p(x)$ and $q(x)$ is required in the above theorem. Consider the DE

$$x^2y'' - 4xy' + 6y = 0.$$

   Then, x^2 and x^3 are linearly independent solutions, but $W(x^2, x^3; 0) = 0$.

2. $W(f, g; a) = 0$ for some a with $\{f, g\}$ linearly independent implies that f and g together are not solutions of an ODE on any interval containing a.

3. Similar is the case if Wronskian is zero at a point and not identically zero.

Second Order Linear ODE's

Consider second order linear homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0.$$

As we remarked earlier, there is no general method to find a basis of solutions. However, if we know one non-zero solution $f(x)$, then we have a method to find another solution $g(x)$ such that $f(x)$ and $g(x)$ are linearly independent.
To find such a $g(x)$, set

$$g(x) = v(x)f(x).$$

We'll choose v such that f and g will be linearly independent. Can v be a constant? No. Now for g to be a solution of the given ODE, we need $g'' + p(x)g' + q(x)g = 0.$

$$\implies (vf)'' + p(x)(vf)' + q(x)(vf) = 0.$$

$$\begin{aligned}
&\implies (v'f + vf')' + p(v'f + vf') + qvf = 0\\
&\implies (v'' + 2v'f' + vf'') + p(v'f + vf') + qvf = 0\\
&\implies v(f'' + pf' + qf) + v'(2f' + pf) + v''f = 0\\
&\implies v'(2f' + pf) + v''f = 0
\end{aligned}$$

Thus,

$$\frac{v''}{v'} = -\frac{2f' + pf}{f} = -\frac{2f'}{f} - p.$$

Therefore,

$$\ln |v'| = \ln\left(\frac{1}{f^2}\right) - \int p dx;$$

$$\implies v = \int \frac{e^{-\int p dx}}{f^2} dx.$$

Second Order Linear ODE's

Claim: f and vf are linearly independent.

Enough to check Wronskian!

$$W(f, vf) = f(v'f + f'v) - f'vf$$

$$= f^2 v' = f^2 \frac{e^{-\int p dx}}{f^2} = e^{-\int p dx} \neq 0$$

Theorem

If y_1 is one solution of $y'' + p(x)y' + q(x)y = 0$, then another solution is given by

$$y_2(x) = v y_1(x) = \left(\int \frac{e^{-\int p dx}}{y_1^2} dx\right) y_1(x)$$

Second Order Linear ODE's

Example: Find all solutions of

$$x^2y'' + xy' - y = 0.$$

Given that $f(x) = x$ is one solution.

Write this in standard form:

$$y'' + \frac{y'}{x} - \frac{y}{x^2} = 0.$$

Let $g = vf = vx$ be another solution. Then,

$$v(x) = \int \frac{e^{-\int p dx}}{f^2} dx = \int \frac{e^{-\int \frac{dx}{x}}}{x^2} dx = \int \frac{dx}{x^3} = -\frac{1}{2x^2}.$$

Hence, any solution is of the form $cx + \frac{d}{x}$ for $c, d \in R$.

If $a, b, c \in \mathbb{R}$ with $a \neq 0$, then

$$ay'' + by' + cy = F(x),$$

is called a **constant coefficient equation**. We will begin with homogeneous constant coefficient equation

$$ay'' + by' + cy = 0. \quad (*)$$

In this case, all solutions are defined $(-\infty, \infty)$. Why? Therefore, we will omit the reference to the interval on which the solution is defined. It is always $(-\infty, \infty)$.
Suppose e^{mx} is a solution of (*), where m is a constant. Then,

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0,$$

The quadratic polynomial

$$p(m) = am^2 + bm + c$$

is the **characteristic polynomial** of (*), and

2nd Order Linear ODE's with constant coeff.

$p(m) = 0$ is the **characteristic equation**. Therefore, e^{mx} is a solution of (*) if and only if $p(m) = 0$.
The roots of the characteristic equation are given by

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We consider three cases:

Case 1: When $b^2 - 4ac > 0$. Then characteristic equation has two distinct real roots.

Case 2: When $b^2 - 4ac = 0$. Then characteristic equation has two repeated real roots.

Case 1: When $b^2 - 4ac < 0$. Then characteristic equation has two distinct complex roots which are conjugates.

Distinct real roots case

Example. Find general solution of $y'' + 6y' + 5y = 0 \quad (1)$.

The characteristic polynomial is
$p(m) = m^2 + 6m + 5 = (m+1)(m+5)$.
Thus roots of characteric equation are -1 and -5. Thus
$y_1 = e^{-x}$ and $y_2 = e^{-5x}$ are solutions of (1).
Since $\dfrac{y_1}{y_2}$ is non-constant, $\{y_1, y_2\}$ is a fundamental solutions of (1). Therefore, the general solution of (1) is

$$y(x) = c_1 e^{-x} + c_2 e^{-5x}$$

Ex. Solve IVP $y'' + 6y' + 5y = 0, \quad y(0) = 3, y'(0) = 1$.

From the general solution, $\begin{aligned} c_1 + c_2 &= 3 \\ -c_1 - 5c_2 &= 1 \end{aligned}$

This gives $c_2 = -1$ and $c_1 = 4$. Thus the solution to IVP is

$$y(x) = 4e^{-x} - e^{-5x}$$

A repeated real root case

Example. Find general solution of $y'' + 6y' + 9y = 0 \quad (1)$.

Characteristic polynomial $p(m) = m^2 + 6m + 9 = (m+3)^2$.

The characteristic equation has repeated roots $-3, -3$. Hence $y_1 = e^{-3x}$ is one solution. For other solution, let $y = ue^{-3x}$.

$$y' = u'e^{-3x} - 3ue^{-3x} \quad \text{and} \quad y'' = u''e^{-3x} - 6u'e^{-3x} + 9ue^{-3x}$$

$$(1) \Rightarrow \; e^{-3x}[(u'' - 6u' + 9) + 6(u' - 3u) + 9u] = u''e^{-3x} = 0.$$

Therefore $y = uy_1$ is a solution if and only if $u'' = 0$. Hence $u = c_1 + c_2 x$. Therefore the general solution is

$$y(x) = e^{-3x}(c_1 + c_2 x)$$

Ex. Solve IVP $y'' + 6y' + 9y = 0, \quad y(0) = 3, \; y'(0) = 1.$

We get $c_1 = 3$ and $1 = -3(3) + c_2$ gives $c_2 = 10$.