PH108

Lecture 19:

Magnetic Vector Potential $\vec{A}$

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Recall main steps of electrostatics

Recall that we introduced the electric potential Φ

because $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

is one equation in three unknowns (E_x , E_y , E_z)

Using $\vec{\nabla} \times \vec{E} = 0 = \oint \vec{E} \cdot \overrightarrow{dl}$ (Lecture 5)

We defined the electric potential Φ ; found $\vec{E} = -\vec{\nabla}\Phi$

(Lecture 6)

$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$ $\rightarrow$ Solves to: $\rightarrow$ $\Phi(\mathrm{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')}{\mathscr{r}} d\tau'$

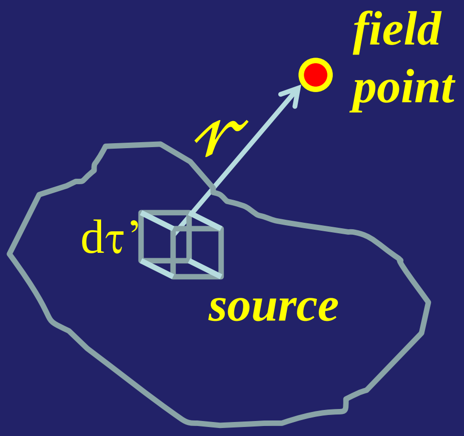
With boundary conditions

Why do we need a magnetic potential ?

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \longrightarrow \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = 0 \quad \longrightarrow \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau'$$



Integrand is a vector cross product

This is difficult to calculate unless there is some symmetry

Magnetic vector potential $\vec{A}$ is magnetic counterpart of Electric potential Φ

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{X}) \equiv 0$ for any $\vec{X}$ and $\vec{\nabla} \cdot \vec{B} = 0$

So we DEFINE: $\vec{B} = \vec{\nabla} \times \vec{A}$ Like $\vec{E} = -\vec{\nabla}\Phi$

What is $\vec{A}$? How do I determine $\vec{A}$?

Any vector field is completely defined by its Curl & Div

So we need to evaluate $(\vec{\nabla} \times \vec{A})$ and $(\vec{\nabla} \cdot \vec{A})$

$(\vec{\nabla} \times \vec{A}) = \vec{B}$ is a physical quantity

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J}$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \qquad B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

x component

$$\frac{\partial}{\partial \mathrm{y}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - \frac{\partial}{\partial \mathrm{z}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) = \mu_0 J_x$$

$$- \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} \right) + \frac{\partial}{\partial \mathrm{x}} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) = \mu_0 J_x$$

similar with y, z components…

We have to choose $\vec{\nabla} \cdot \vec{A}$

$\vec{\nabla} \times (\underbrace{\vec{\nabla} \times \vec{A}}) = \mu_0 \vec{J}$ simplifies to:

$$-\vec{\nabla}^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \mu_0 \vec{J}$$

$= \vec{B}$ is a physical quantity

is a math function

We <u>choose</u> $\vec{\nabla} \cdot \vec{A} = 0$

"Coulomb gauge"

Why is it OK to choose $\vec{\nabla} \cdot \vec{A} = 0$?

Suppose for some configuration we get $\vec{\nabla} \cdot \vec{A} \neq 0$

We can transform $\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\,\psi$

…for any scalar field ψ, $\because \vec{\nabla} \times \vec{\nabla}\,\psi \equiv 0$

So ψ will not affect the value of $\vec{B}$

So $\vec{\nabla} \cdot \vec{A}' = \vec{\nabla} \cdot \vec{A} + \nabla^2\psi = 0$

If we can find ψ such that $\nabla^2\psi = -\vec{\nabla} \cdot \vec{A}$

This is OK, because solution to Poisson eqn must exist

See appendix to Lecture 19 on Moodle for a more 'physics-y' justification

Solution for $\vec{A}$ looks like Poisson eqn

In the Coulomb gauge, $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J}$
Simplifies to:

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J}$$

$\frac{\partial^2 A_x}{\partial x^2} = -\mu_0 J_x$ etc... Three equations in three unknowns

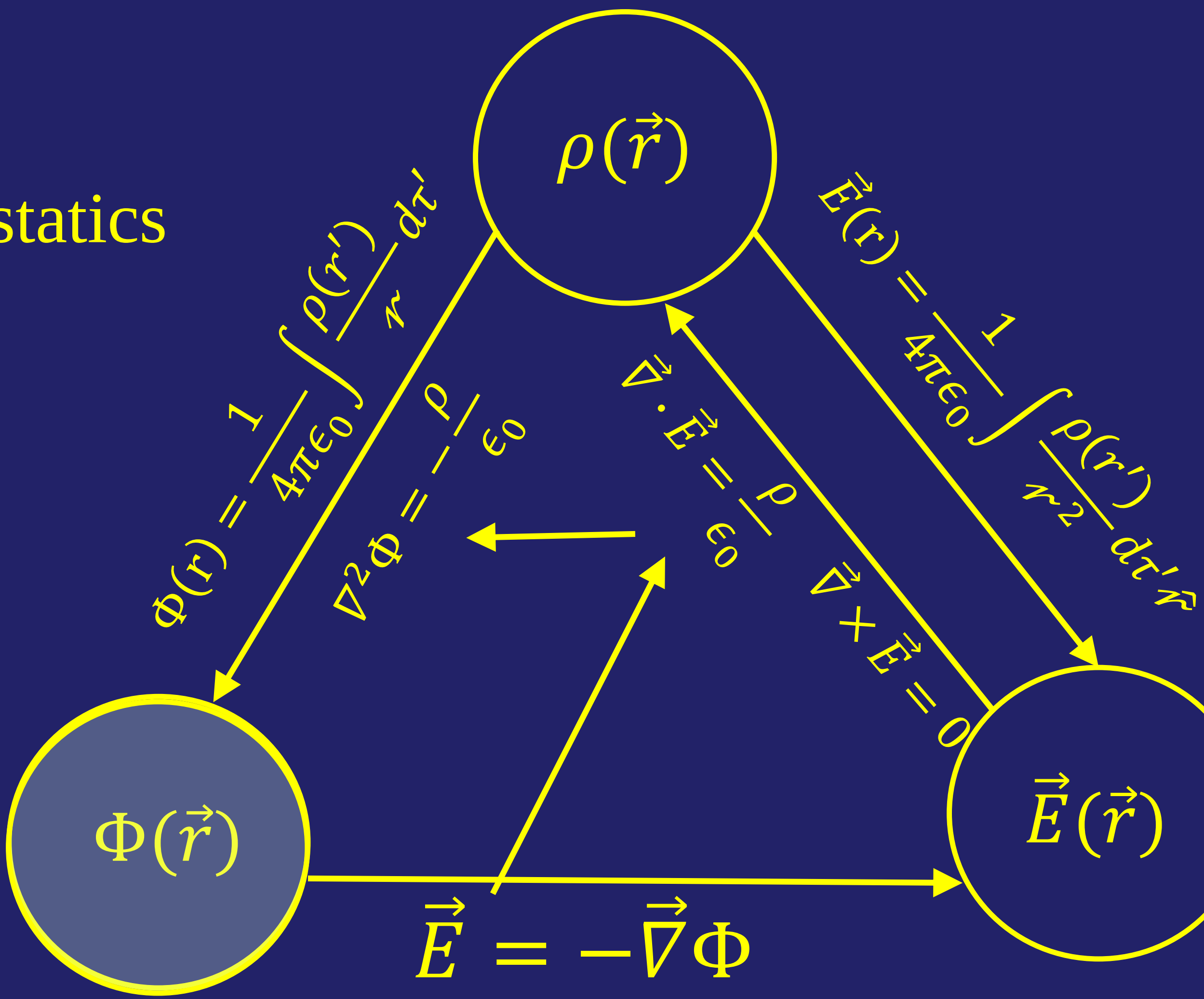
From the similar Poisson equation for electric potential $\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$

$$\vec{A}(r) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau'$$

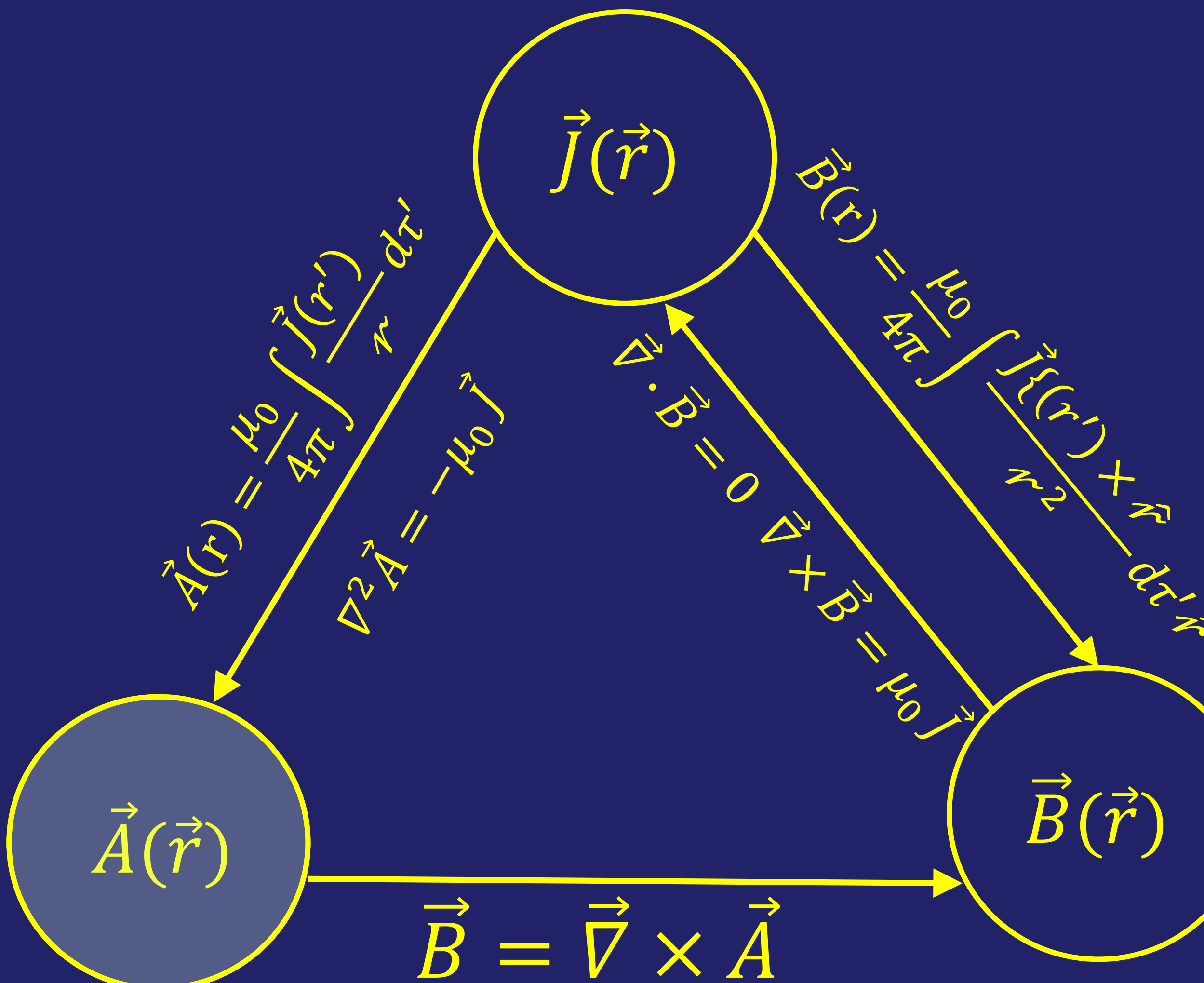
Where does $\vec{A}$ fit in magnetostatics?

Recall

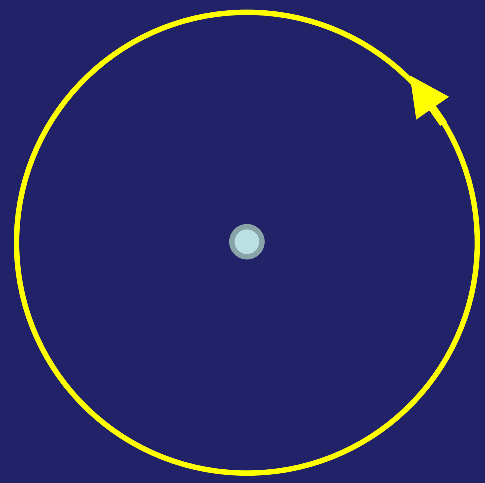
Electrostatics



What is the use of $\vec{A}$?



Question: Example



Wire carrying current I is perpendicular to screen

Magnetic field $\vec{B}$ is shown. <u>Can you calculate</u> $\vec{A}$ with the formula? $\vec{A}(\mathrm{r}) = \frac{\mu_0}{4\pi}\int \frac{I dz\,\hat{z}}{\mathscr{r}}\, d\tau'$

A) YES

B) NO

I extends to infinity!

Summary

$$\rho(r') \xleftrightarrow{\text{sources}} \vec{J}(r')$$

$$\vec{E}(r) \quad \longleftrightarrow \quad \vec{B}(r)$$

Physical Quantities

i.e. can measure their effect:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$\Phi(r) \quad \longleftrightarrow \quad \vec{A}(r)$$

Mathematical tools

to calculate $\vec{E}$ and $\vec{B}$

Exception: $\vec{A}$ has a physical significance in, for example, the 'Aharanov Bohm' effect