MA - 106, Lecture - 20.

Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

1. Find the row reduced form of A.

$$A \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. rank $A = 2$.
3. Null space of $A = \text{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \right\}$

4. $\begin{pmatrix} 0 \\ 1 \\ 1/2 \end{pmatrix}$ in a particular solution of $Ax = b$.

All solutions are $\begin{pmatrix} 0 \\ 1 \\ 1/2 \end{pmatrix} + a \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

5. $b = A \begin{pmatrix} 0 \\ 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 2 \\ 1/2 \end{pmatrix}$.

6. Find left null space of A

$$x^T \begin{pmatrix} \text{col 1.} & \text{col.2} \\ & \end{pmatrix} = 0$$

7 Char poly of $A = \det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix}$

$= (1-\lambda)(2-\lambda)(1-\lambda) + (-1)(2-\lambda)$

$= (2-\lambda)\left((1-\lambda)^2 - 1\right)$

$= (2-\lambda)(-\lambda)(2-\lambda)$

Eigenvalues: 2, 2, 0.

<u>Qu1</u> Is A diagonalizable.

Yes, since A is symmetric.

Write an orthogonal matrix Q s.t. $Q^T A Q = \Lambda$ is diagonal.

Find eigenvectors.

<u>$\lambda = 2$.</u> $(A - 2I)x = 0$ $\quad v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 0$, $\quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\|v_1\| = 1, \quad \|v_2\| = \sqrt{2}, \quad \|v_3\| = \sqrt{2}$.

$$Q = \begin{bmatrix} v_1 & \frac{v_2}{\sqrt{2}} & \frac{v_3}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$Q^T A Q = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 0 \end{bmatrix}$$

Remark: Q is not unique.

<u>Qn:</u> Is $Ax = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is consistent.

<u>Ans:</u> No.

Find the least square solution. $\hat{x}$.

For $\hat{x}$, solve

$$A^T A \hat{x} = A^T \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Theorem: (Jordan-Canonical Form)

Assume A is 2×2.

Let $1, 1$ are eigenvalues of A

Then $\exists$ S : invertible st.

$$S^{-1}AS = \text{one of the following}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

<u>3×3 Case.</u> Eigen-values : 1, 1, 2

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}, \quad \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & \\ \hline 0 & & 2 \end{array}\right)$$

Eigen-Values are 1, 1, 1.

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \left(\begin{array}{cc|c} 1 & 1 & \\ 0 & 1 & \\ \hline & & 1 \end{array}\right), \quad \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$$

$=$

Q4: $A : 3 \times 3$.

$$x^2 + x + 1 = p(x).$$

$$p(A) = A^2 + A + I \quad ; \quad 3\times 3.$$

- Let λ be an eigenvalue of A.

Check $p(\lambda)$ is an eigenvalue of $p(A)$

- Converse. Let μ be an eigenvalue of $p(A)$.

<u>Claim</u> $\mu = p(\lambda)$, λ: eigen-value of A.