PH108

Lecture 22:

Maxwell’s equations – Displacement current

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Recall: Modifications of electromagnetostatics if $\vec{B} = \vec{B}(r,t)$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = 0$$

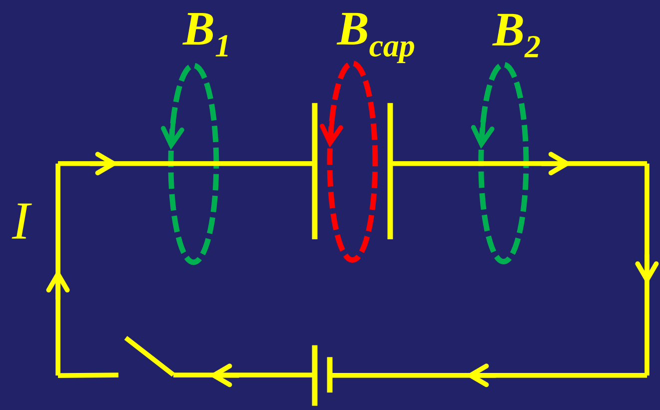
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

If magnetic flux varies in time (Lecture 21)

$\rho(r)$ and $\vec{J}(r)$ create electric field. How is $\vec{B}$ affected if $\vec{J}(r) = \vec{J}(r,t)$?

When switch is closed, current *I* charges a capacitor



Take Amperian loops of dia. *d:*

$$B_1 = \frac{\mu_0 I}{2\pi d} \qquad B_2 = \frac{\mu_0 I}{2\pi d}$$

But *in* the capacitor:

$$B_{cap} = \frac{\mu_0 I_{enclosed}}{2\pi d} = 0$$

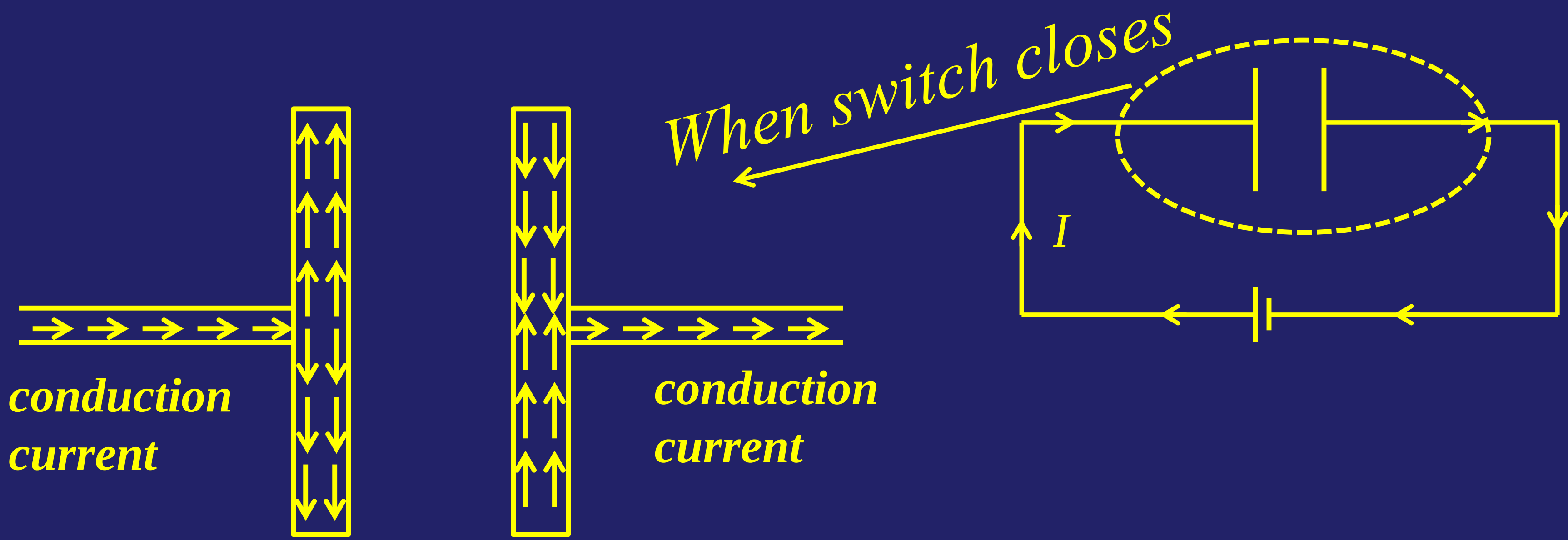
(a) Violates ΔB boundary condition

(b) *I* seems to vanish at the capacitor plates

$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$ goes for a toss!

(c) Gets worse for an observer co-moving with *I*

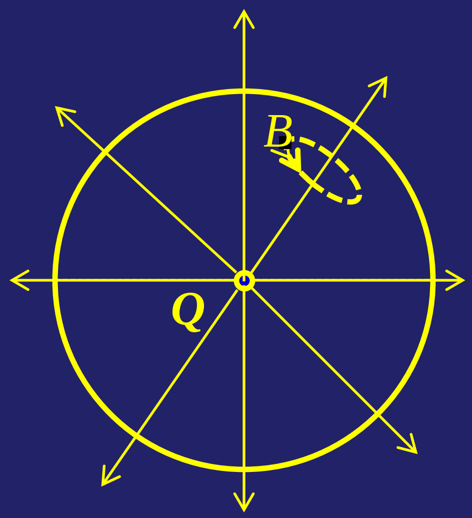
We have to look carefully at the current



$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$ needs to be examined carefully

Consider a current 'source'

Q is a source that *emits* charged particles q isotropically



$\rightarrow$ Q is a source of radially symmetric current: $\frac{\partial Q}{\partial t} = J$

Take an Amperian loop on the spherical surface enclosing Q

I_{encl} for loop $\neq 0$,So there is a magnetic field B around the loop

But the loop can be of any size! $\rightarrow I_{encl}$ can have any value

We have no way to uniquely determine B !

Maxwell's fix for Ampere's law

The problem is: $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla} \cdot \vec{J}$

We need to make the RHS vanish

$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \rightarrow$ $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 (-\frac{\partial \rho}{\partial t})$

Where there is ρ, there must be $\vec{E}$ $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = -\vec{\nabla} \cdot (\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t})$$

We need to put $\vec{E}$ into $\vec{\nabla} \times \vec{B}$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ is very small compared to $\mu_0 \vec{J}$

It's effect is hard to measure (first done by Hertz in 1888)

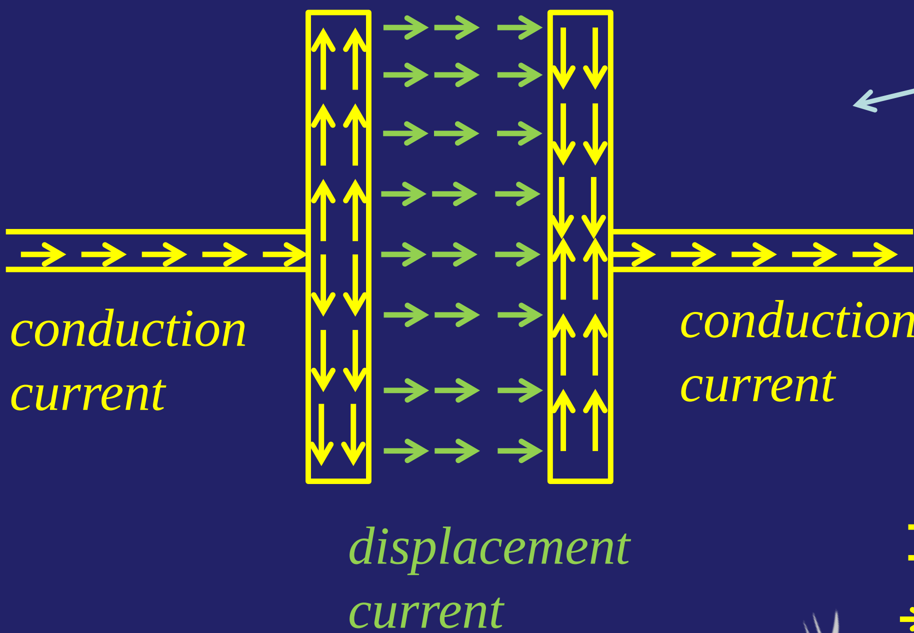
The effect becomes noticeable if the observer is moving

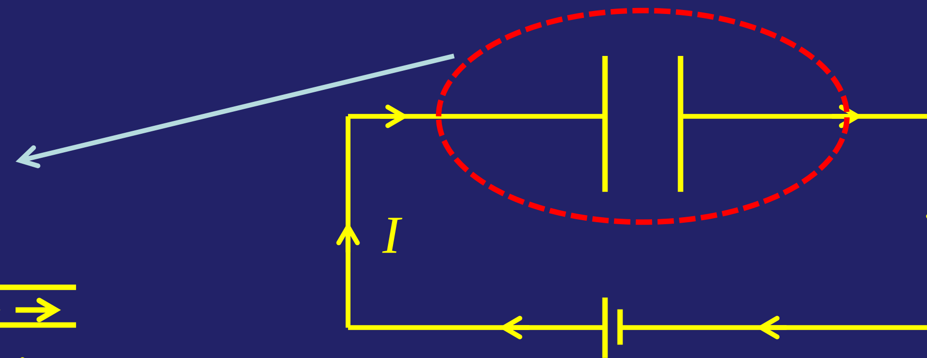
Or, equivalently, the fields $\vec{E}, \vec{B}$ are moving → wave propagation

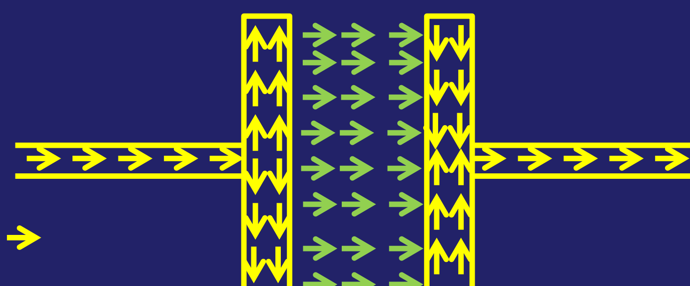
What is the physical effect of $\vec{E}$ in $\vec{\nabla} \times \vec{B}$?

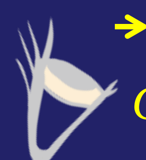
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

(1) Resolve the problem of vanishing current in capacitor





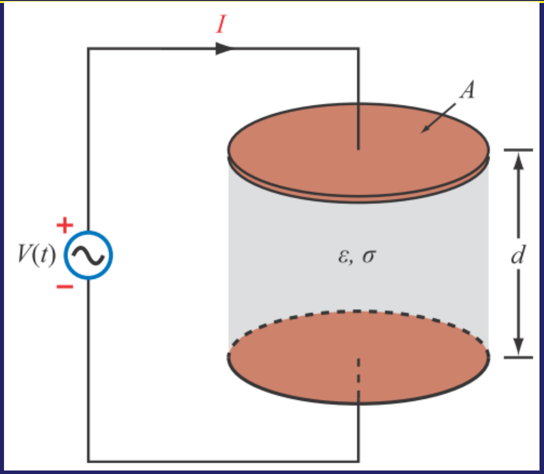




Observer moving with velocity of current sees static charge – so no magnetic field

Sees Electric field in capacitor → displacement current

How to calculate I_d in capacitors with dielectric



Circular parallel plate capacitor

Filled with dielectric ϵ , conductance σ

Charged by time varying voltage *V(t)*

The dielectiric is "imperfect" : conдunctance $\sigma \rightarrow R = \dfrac{d}{\sigma A}$

Conduction current: $I_C = \dfrac{V(t)}{R} = \dfrac{V(t)\sigma A}{d}$

$\mathrm{D} = \epsilon \mathrm{E}$

$E = \dfrac{V}{d}$ Displacement current: $I_d = A\dfrac{\partial D}{\partial t}$ $I_d = \dfrac{A\epsilon}{d}\dfrac{\partial V}{\partial t}$

A real capacitor is = R in parallel with C

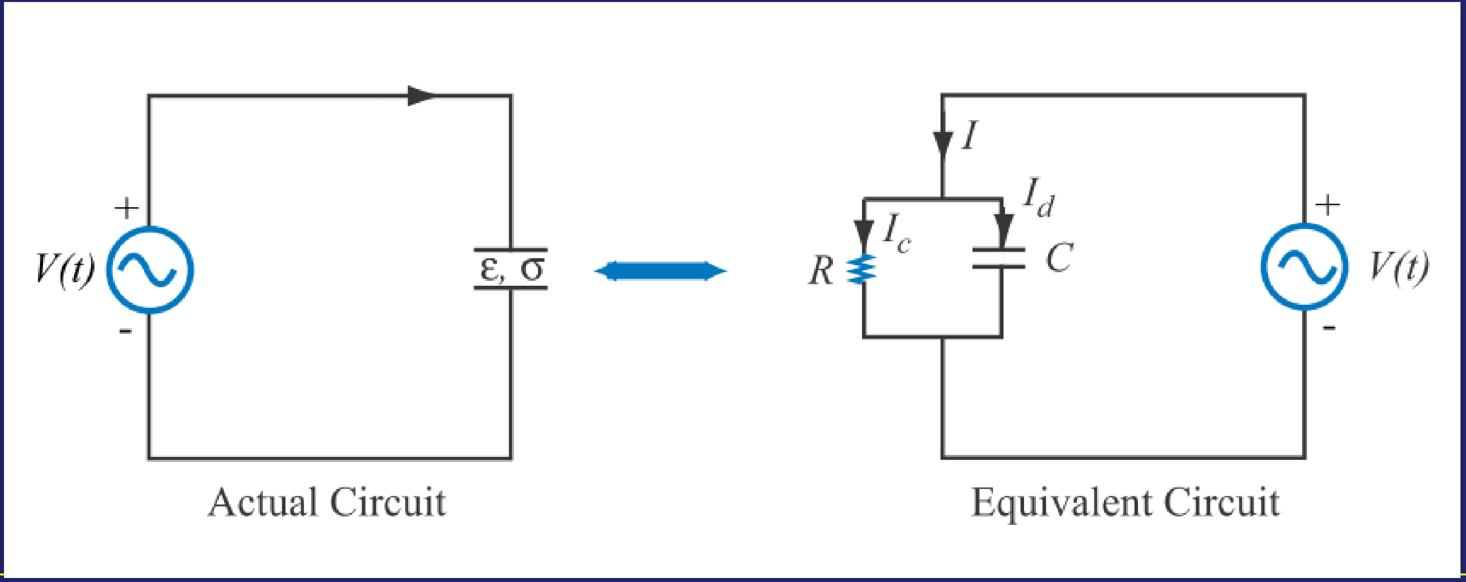
$$I_C = \frac{V(t)}{R} = \frac{V(t)\sigma A}{d}$$

$$I_d = \frac{A\epsilon}{d}\frac{\partial V}{\partial t}$$

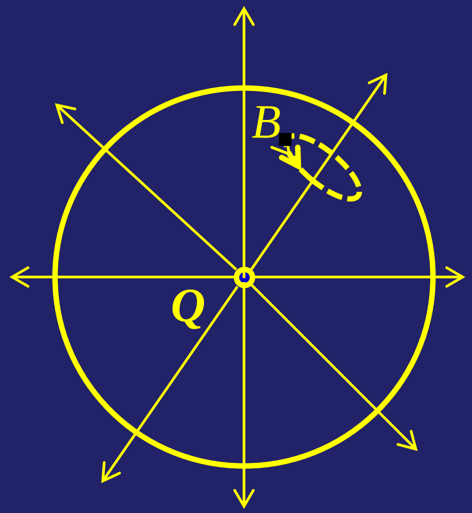
$$Q = CV \rightarrow \frac{dQ}{dt} = I = C\frac{dV}{dt}$$

$$R = \frac{d}{\sigma A}$$

$$C = \frac{A\epsilon}{d}$$



Displacement current resolves loop on sphere problem



$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2}$$

$$\frac{\partial E}{\partial t} = \epsilon_0 \frac{1}{4\pi\epsilon_0 r^2}\frac{\partial Q}{\partial t}$$

$$= -\frac{1}{4\pi\epsilon_0 r^2}\, 4\pi r^2 J = -\frac{1}{\epsilon_0} J$$

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$4\pi r^2 J = -\frac{\partial Q}{\partial t}$$

Hence:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0\vec{J} - \mu_0\epsilon_0 * \frac{1}{\epsilon_0}\vec{J} = 0$$

With $\vec{\nabla} \cdot \vec{B} = 0$

$$\boxed{\vec{B} = 0}$$

Complete Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Physics motivation: measurable quantities (force) should not depend on observer frame of reference

Resolution: *E* and *B* fields are a function of reference frame, but the final measurable force is independent of frame