MA-108 Ordinary Differential Equations

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D1 - Lecture 13

Recall: If $L = D^n + a_1(x)D^{n-1} + \ldots + a_n(x)I$, then null space of L is of dimension n. This is Dimension theorem.

We saw Abel's theorem;

If $L = \sum a_i D^i$ and $M = \sum b_j D^j$, then $LM = ML$.

As a consequence, if $L = L_1 L_2$ such that L_1, L_2 have **no common factor**, then null space of L is spanned by null spaces of L_1 and L_2.

If $L = (D-c)^r\left((D-a)^2 + b^2\right)^s$ with $r, s > 0$, $b > 0$ then we can write a basis for null space of L. This is

$\{e^{cx}, xe^{cx}, \ldots, x^{r-1}e^{cx}\}$
$\cup\{e^{ax}\cos bx, e^{ax}\sin bx,\ \ldots, x^{s-1}e^{ax}\cos bx, x^{s-1}e^{ax}\sin bx\}$

Theorem

Suppose $L = A_1 A_2 \ldots A_k$, where A_i are linear differential operators with constant coefficients and $N(A_i) \cap N(A_j) = 0$ for all $i \neq j$. Let B_i be a basis for $N(A_i)$. Then $\cup B_i$ is a basis for $N(A)$.

Proof: From Dimension theorem, we know that $\dim N(L) = \text{deg} P_L$. Now $P_L = P_{A_1} \ldots P_{A_k}$.

Then $\deg P_L = \deg P_{A_1} + \ldots + \deg P_{A_k}$. Therefore, $\dim N(L) = \dim N(A_1) + \ldots + \dim N(A_k)$.

Further since $N(A_i) \cap N(A_j) = 0$ for all $i \neq j$, $\cup B_i$ is linearly independent and has $\dim N(A_1) + \ldots + \dim N(A_k) = \dim N(L)$ elements.

Therefore, $\cup B_i$ is a basis for $N(L)$

Annihilator or Undetermined Coefficient Method

The Annihilator method or **method of undetermined coefficients** helps us in finding a particular solution of a non-homogeneous equation with constant coefficients when the annihilator of $r(x)$ is a differential operator with constant coefficients.

• If $r_1(x) = a_0x^n + a_1x^{n-1} + a_n$ is a polynomial of degree n with $a_i \in \mathbb{R}$, $a_0 \neq 0$, then its annihilator is D^{n+1}.

• The annihilator of $r_2(x) = a_1 \cos bx + a_2 \sin bx$ is $(D^2 + b^2)$.

• The annihilator of $r_3(x) = e^{ax}(a_1 \cos bx + a_2 \sin bx)$ is $(D - a)^2 + b^2$.

• The annihilator of $r_4(x) = x^r e^{ax}(a_1 \cos bx + a_2 \sin bx)$ is $[(D - a)^2 + b^2]^{r+1}$.

• The annihilator of $r_1(x) + r_2(x)$ is $D^{n+1}(D^2 + b^2)$.

Ex. Find the form of particular solution to the following ODEs.

- $y'' + 9y' = 6$.
  Annihilator A of r is D, $L = D^2 + 9D$. Hence $AL = D^2(D+9)$. Take $y_p = cx$. $\square$

- $y'' + 2y' + y = 4e^x \sin 2x$.
  $L = (D+1)^2$; $A = (D-1)^2 + 4$. Take $y_p = e^x[c_1 \cos 2x + c_2 \sin 2x]$. $\square$

- $y'' + y = x \sin x$
  $L = D^2 + 1$; $A = (D^2+1)^2$. Take $y_p = c_1 x \cos x + c_2 x \sin x + c_3 x^2 \cos x + c_4 x^2 \sin x$. $\square$

- $y'' + 9y = x^2 e^{3x}$
  $L = D^2 + 9$; $A = (D-3)^3$. Take $y_p = e^{3x}[c_1 + c_2 x + c_3 x^2]$ $\square$

Ex. Find the form of particular solution y_p.

- $y'' + 9y = xe^{3x}\cos 3x$
  $L = D^2 + 9$, $A = ((D-3)^2 + 9)^2$. y_p will come from A.
  $\square$

- $y^{(4)} - y^{(3)} - y'' + y' = x^2 + 4 + x\sin x$.
  $L = D^4 - D^3 - D^2 + D = D(D-1)^2(D+1)$,
  $A = D^3(D^2+1)^2$. y_p is lin comb of
  $\{x, x^2, x^3, \cos x, \sin x, x\cos x, x\sin x\}$. $\square$

- $y^{(4)} - 2y'' + y = x^2e^x + e^{2x}$
  $L = D^4 - 2D^2 + 1 = (D^2-1)^2 = (D+1)^2(D-1)^2$,
  $A = (D-1)^3(D-2)$, $AL = (D+1)^2(D-1)^5(D-2)$.
  So y_p is linear combinations of

$$x^2e^x, x^3e^x, x^4e^x, e^{2x}$$

$\square$

Variation of Parameters

The variation of parameters method generalizes to nth order linear ODE $y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x)$ where p_i's and r are continuous on I.
Let $y_1, \ldots, y_n$ be a basis of solutions of homogeneous part.
Assume the particular solution y_p is given by

$$y_p = v_1(x)y_1 + v_2(x)y_2 + \ldots + v_n(x)y_n$$

Assume

$$\begin{array}{rcl} v_1'y_1 + \ldots + v_n'y_n & = 0 & (1) \\ v_1'y_1' + \ldots + v_1'y_n' & = 0 & (2) \\ \vdots & & \vdots \\ v_1'y_1^{(n-2)} + \ldots + v_n'y_n^{(n-2)} & = 0 & (n-1) \end{array}$$

Compute $y_p', \ldots, y_p^{(n)}$ and put in the ODE, we get

$$v_1'y_1^{(n-1)} + \ldots + v_n'y_n^{(n-1)} = r(x) \qquad (n)$$

Variation of Parameters

Thus,

$$\begin{bmatrix} y_1 & y_2 & \cdot & y_n \\ y_1' & y_2' & \cdot & y_n' \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdot & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ \cdot \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ r(x) \end{bmatrix}.$$

Use Cramer's rule to solve for $v_1', v_2', \ldots, v_n'$, and thus get $v_1, v_2, \ldots, v_n$, and form $\boxed{y = v_1y_1 + v_2y_2 + \ldots + v_ny_n}$, here v_i' is given by

$$v_i' = \frac{\begin{vmatrix} y_1 & \cdot & y_{i-1} & 0 & y_{i+1} & \cdot & y_n \\ y_1' & \cdot & y_{i-1}' & 0 & y_{i+1}' & \cdot & y_n' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & \cdot & y_{i-1}^{(n-1)} & r(x) & y_{i+1}^{(n-1)} & \cdot & y_n^{(n-1)} \end{vmatrix}}{W(y_1, \ldots, y_n; x)}$$

Ex: Solve $y^{(3)} - y^{(2)} - y^{(1)} + y = r(x)$.
Here $L = D^3 - D^2 - D + 1 = (D-1)^2(D+1)$.
Hence, a basis of solutions is $\{e^x, xe^x, e^{-x}\}$.
We need to calculate $W(x)$. Use Abel's formula:

$$W(x) = W(0)\, e^{-\int_0^x p_1(t)dt} = W(0) \cdot e^x.$$

$$W(x) = \begin{vmatrix} e^x & xe^x & e^{-x} \\ e^x & e^x + xe^x & -e^{-x} \\ e^x & 2e^x + xe^x & e^{-x} \end{vmatrix}.$$

$$\implies W(0) = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 4.$$

Hence,

$$W(x) = 4e^x.$$

$$W_1(x) = \begin{vmatrix} 0 & xe^x & e^{-x} \\ 0 & e^x + xe^x & -e^{-x} \\ r(x) & 2e^x + xe^x & e^{-x} \end{vmatrix} = -r(x)(2x+1).$$

Similarly,

$$W_2(x) = 2r(x),\ W_3(x) = r(x)e^{2x}.$$

Therefore, a particular solution y_p is given by $y_p =$

$$e^x \int_0^x \frac{-r(t)(2t+1)}{4e^t} dt + xe^x \int_0^x \frac{2r(t)}{4e^t} dt + e^{-x} \int_0^x \frac{r(t)e^{2t}}{4e^t} dt.$$

Ex. $r(x) = 1$, $r(x) = x$, $r(x) = e^x$.

1. 140020020 VIDWANS NIRAJ ASHUTOSH
2. 140050053 BODDEDA JAGADEESH
3. 140020033 TANUL RAJHANS CHIWANDE
4. 140020086 PINTU RAJ
5. 140020107 KUMAR SPANDAN SARDAR
6. 140020055 HIMANSHI ARORA
7. 140020068 SUPREET MEENAABSENT
8. 140020059 RAJAT YADAV ABSENT
9. 140020060 SHASHI KANT KUMAR
10. 140050062 POOSA NIHAL
11. 140020050 CHANDRAPREET SINGH
12. 140020077 HARSH GUPTA ABSENT
13. 13I190003 ASHWANI KUMAR ABSENT
14. 13I190005 AKASH PRADEEP BHATTACHARJEEABSENT
15. 140020083 SIDDHARTH BIRLA
16. 140050029 B AVINASH
17. 140050051 K SINDHURA
18. 140050080 AJMEERA SRINATH NAIK

Laplace Transforms

This method illustrates a general problem solving technique in mathematics; transform a difficult problem into an easier one, solve the later problem and use its solution to get a solution of the original problem.

Laplace tranform converts an IVP for a constant coefficient equation, into an algebraic equation whose solution is used to solve the IVP.

We have already seen some methods to solve IVP. Laplace transform is especially useful when we are dealing with discontinuous **forcing functions** $r(x)$.

For example, when $r(x)$ is piece-wise continuous function, by earlier method, we need to solve IVP on each piece where $r(x)$ is continuous. Laplace transform gives solution in one step.

Laplace Transforms

To define Laplace transform; let's define an **improper integral.** If g is integrable over the interval $[a, T]$ for every $T > a$, then an improper integral of g over $[a, \infty)$ is defined as

$$\int_a^\infty g(t)\,dt = \lim_{T\to\infty} \int_a^T g(t)\,dt$$

We say that the improper integral **converges** to the limit value, if the limit exists and is finite; otherwise we say that the improper integral **diverges** or **does not exist**.

Ex. Let $f(t) = e^{ct}$, $t \geq 0$ and $c \neq 0$ constant. Then

$$\int_0^\infty e^{ct}\,dt = \lim_{T\to\infty}\int_0^T e^{ct}\,dt = \lim_{T\to\infty}\frac{1}{c}(e^{cT}-1)$$

Therefore the integral converges to $-1/c$ if $c < 0$; and diverges if $c > 0$. If $c = 0$, then $f(t) = 1$ and the integral again diverges.

Ex. Let $f(t) = 1/t$ for $t \geq 1$. Then

$$\int_1^\infty \frac{1}{t}\,dt = \lim_{T\to\infty}\int_1^T \frac{dt}{t} = \lim_{T\to\infty}\ln T$$

the improper integral diverges.

Laplace Transforms

Definition

Let $f(t)$ be defined for $t \geq 0$ and let s be a real number. The **Laplace transform** of f, denoted by $F(s)$, is defined as

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

for those values of s for which the improper integral converges.

Note that s is a parameter and t is a variable of integration.

The Laplace transform can be thought of as an operator L that transforms a function $f(t)$ into the function $F(s)$. We will write it as

$$F = L(f) \quad \text{or} \quad F(s) = L(f(t)) \quad \text{or} \quad f(t) \leftrightarrow F(s).$$

Exercise. If $F(s_0)$ is defined, then $F(s)$ is defined $\forall s > s_0$.