MA-106 Linear Algebra

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D1 - Lecture 7

Recall: We have seen how to find all solutions of $Ax = b$.

$$x_{\text{Complete}} = x_{\text{Particular}} + x_{\text{NullSpace}}$$

We defined Column space $C(A)$ of A.

$b \in C(A)$ if and only if $Ax = b$ is consistent.

If $b \in C(A)$, then to write b as a linear combination of columns of A, we need to solve $Ax = b$.

We defined Vector space $(V, +, .)$

It is closed under linear combinations with following properties

$+$ is commutative, associative, additive identity 0, additive inverse

scalar multiplication . is associative, $1.v = v$, distributive.

Examples of Vector spaces

1. $M = \{2 \times 2$ matrices with real entries$\}$ is a vector space.
2. $V = C([0,1], \mathbb{R})$ is a vector space.
   In V, vectors are continuous functions $f : [0,1] \to \mathbb{R}$.
   The zero vector is the zero function.
3. $P = \{$polynomial functions $: \mathbb{R} \to \mathbb{R}\}$ is a vector space.
   A typical element of P: $a_0 + a_1 x + \ldots + a_n x^n : \mathbb{R} \to \mathbb{R}$ with $a_i \in \mathbb{R}$. The zero vector is the zero polynomial.
4. Let M, V and P be vector spaces considered above.
   Let $W = \{(m, v, p) : m \in M, v \in V, p \in P\}$.
   Define addition and scalar multiplication componentwise.
   Then W is a vector space. What is the zero vector in W?
5. $T = \{2 \times 2$ matrices with entries in $W\}$ is a vector space with componentwise addition and scalar multiplication.

Subspaces

A subspace of a vector space is a non-empty subset which itself satisfies the requirements of a vector space:

$$\boxed{\text{Linear combinations stay in the subspace.}}$$

Thus, if V is a vector space, and W is a non-empty subset, then W is a subspace of V if:

$$\boxed{\overline{x},\ \overline{y} \text{ in } W, \quad u,\ v \text{ in } \mathbb{R}, \Rightarrow u \cdot \overline{x} + v \cdot \overline{y} \text{ are in } W.}$$

1. $\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$ is not a subspace of $\mathbb{R}^2$. Why?
2. The set of 2×2 symmetric matrices is a subspace of M. So is the set of 2×2 lower triangular matrices.
3. The set of invertible 2×2 matrices is not a subspace of M. Why?

Examples: Subspaces of $\mathbb{R}^n$

- $\{0\}$: The zero subspace.
- $\mathbb{R}^n$ itself.
- A line not passing through the origin is not a subspace of $\mathbb{R}^2$. Why? e.g., $x - y = 1$.
- A line L passing through the origin is a subspace of $\mathbb{R}^2$. Why? e.g., $x - y = 0$.
- Let A be an $m \times n$ matrix.
  The null space of A, $N(A)$, is a subspace of $\mathbb{R}^n$.
  The column space of A, $C(A)$, is a subspace of $\mathbb{R}^m$.
  Recall: They are both closed under linear combinations.

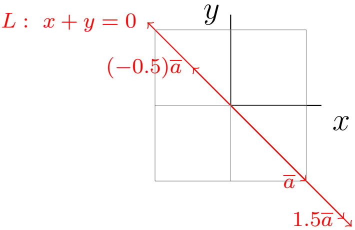
Note: A subspace must contain the 0 vector!

Examples: Subspaces of $\mathbb{R}^2$

What are the subspaces of $\mathbb{R}^2$?

- $V = \{\begin{pmatrix}0 & 0\end{pmatrix}^T\}$.
- $V = \mathbb{R}^2$.

  Suppose V conatins a non-zero vector, say $a = \begin{pmatrix}-1 & 1\end{pmatrix}^T$.

  Hence V must contain the entire line $L: \ x+y=0$, i.e., all multiples of $\overline{a}$.
- Thus if V is non-zero and not $\mathbb{R}^2$, then it is a line passing through the origin. Why?

Examples: Subspaces of $\mathbb{R}^2$

Let V be a subspace of $\mathbb{R}^2$ containing $v_1 = \begin{pmatrix}-1 & 1\end{pmatrix}^T$. Then V must contain the entire line $L: \; x + y = 0$.
If $V \neq L$, it contains a vector v_2, which is not a multiple of v_1, say $v_2 = \begin{pmatrix}0 & 1\end{pmatrix}^T$.

Observe: $A = \begin{pmatrix}v_1 & v_2\end{pmatrix} = \begin{pmatrix}-1 & 0 \\ 1 & 1\end{pmatrix}$ has two pivots,

$\Rightarrow$ A is invertible.
$\Rightarrow$ for any v in $\mathbb{R}^2$, $Ax = v$ is solvable,
$\Rightarrow$ v is in $C(A)$,
$\Rightarrow$ v can be written as a linear combination of v_1 and v_2.
$\Rightarrow$ v is in V, i.e., $V = \mathbb{R}^2$

To summarise: A subspace of $\mathbb{R}^2$, which is non-zero, and not a line passing through the origin, is $\mathbb{R}^2$.

Linear Span: Definition

Given a collection S of vectors v_1, $v_2, \ldots, v_n$ in a vector space V, the *linear span* of S, denoted Span(S) or Span$\{v_1, \ldots, v_n\}$, is the set of all linear combinations of v_1, $v_2, \ldots, v_n$, i.e.,

$$\boxed{\text{Span}(S) = \{v = a_1 \cdot v_1 + \cdots + a_n \cdot v_n, \text{ for scalars } a_1, \ldots, a_n\}.}$$

Remark: Span(S) is a subspace of V. Why?

• Span$\{v_1, \ldots, v_n\} = C(A)$ for $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$.
Thus $v \in$ Span$\{v_1, \ldots, v_n\} \Leftrightarrow Ax = v$ is consistent.

• Let $v_1, \ldots, v_n \in \mathbb{R}^n$ and $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$. Then
A is invertible $\Leftrightarrow$ A has n pivots
$\Leftrightarrow Ax = v$ is consistent for every v in $\mathbb{R}^n$
$\Leftrightarrow$ Span$\{v_1, \ldots, v_n\} = \mathbb{R}^n$.

Linear Span: Examples

Examples:

1. Span$\{0\} = \{0\}$.
2. If $v \neq 0$ is a vector, Span$\{v\} = \{a \cdot v$, for scalars $a\}$. Goemetrically: Span$\{v\}$ = the line in the direction of v passing through the origin.
3. Span$\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \mathbb{R}^2$.
4. The columns $e_1, \ldots, e_n$ of the $n \times n$ identity matrix I span $\mathbb{R}^n$.
5. If A is $m \times n$, then Span$\{A_1, \ldots, A_n\} = C(A)$.
6. If $v_1, \ldots, v_k$ are all the special solutions of A, then Span $\{v_1, \ldots, v_k\} = N(A)$.

Remark: All of the above are subspaces.

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Linear Span: Examples

Example: Is v in Span$\{v_1, v_2, v_3, v_4\}$, where:

$$v = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix},\ v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},\ v_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix},\ v_3 = \begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix},\ v_4 = \begin{pmatrix} 5 \\ 12 \\ 13 \end{pmatrix}?$$

Let $A = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$. Then

$$(A \,|\, b) \to \left(\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array}\right).$$

Linear Span: Examples

Recall $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$

$Ax = b$ is solvable $\Leftrightarrow 5b_1 - b_2 - b_3 = 0$.

$\Rightarrow v = (1, 0, 4)^T$ is not in Span$\{v_1, v_2, v_3, v_4\}$, and

$w = \begin{pmatrix} 1 & 0 & 5 \end{pmatrix}^T = 4v_1 + (-1)v_3$ is in it.

Observe: $v_2 = 2v_1$ and $v_4 = 2v_1 + v_3$.

Hence v_2, v_4 are in Span$\{v_1, v_3\}$.

Therefore, Span$\{v_1, v_3\}$ = Span$\{v_1, v_2, v_3, v_4\} = C(A) =$ the plane $P : (5x - y - z = 0)$.

Q: Is the span of two vectors always a plane?

Linear Independence: Definition

Q: Is the span of two vectors always a plane? **A:** Not always.

If v is a multiple of $w \neq 0$, then Span$\{v, w\}$ = Span$\{w\}$, which is a line through the origin.

Q: If v and w are not on the same line through the origin?
A: Yes. v, w are examples of the following:

The vectors v_1, $v_2, \ldots, v_n$ in a vector space V, are *linearly independent* if

$$\boxed{a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \Rightarrow a_1 = 0,\ a_2 = 0,\ \ldots,\ a_n = 0}.$$

Observe: If $A = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$, then v_1, $v_2, \ldots, v_n$ are linearly independent $\Leftrightarrow$ $Ax = x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0$ has only the trivial solution $\Leftrightarrow$ $N(A) = 0$.

added for clarification

The line in $\mathbb{R}^2$ not passing through origin is not a vector space. e.g. If L is the line $x - y = 1$. Then $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ are on L, but their sum $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is not on L. Hence L is not closed under linear combination. So L is not a vector space.