Tutorial-5, MA 106 (Linear Algebra)
Linear Algebra and its Applications by Gilbert Strang

This tutorial sheet consists of the problems based on 3.1,3.2 and 3.3. This includes:
Problem Set 3.1: $[1, 2, 4, 5, 7, 12, 15, 16, 18, 19, 21, 25(a), (c), (d), 26, 38, 41, 42, 49]$
Problem Set 3.2: $[1(a), 4, 10, 14, 17, 22, 24]$
Problem Set 3.3: $[2, 3, 8, 12, 13, 26]$

- Section 3.1

1. Find the length of $a = (2, 2, 1)$,and write two independent vectors that are perpendicular to a.

2. Give an example in $\mathbb{R}^2$ of linearly independent vectors that are not orthogonal. Also, give an example of orthogonal vectors that are not independent.

3. How do we know that the i^{th} row of an invertible matrix B is orthogonal to the j^{th} column of B^{-1} , if $i \neq j$?

4. Which pairs are orthogonal among the vectors v_1, v_2, v_3, v_4?

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

5. Find a vector x orthogonal to the row space of A, and a vector y orthogonal to the column space, and a vector z orthogonal to the nullspace:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}.$$

6. Find a basis for the orthogonal complement of the row space of A:

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 1 & 4 \end{bmatrix}.$$

   Split $x = (3, 3, 3)$ into a row space component x_r and a nullspace component x_n.

7. Find a matrix whose row space contains $(1, 2, 1)$ and whose nullspace contains $(1, -2, 1)$, or prove that there is no such matrix.

8. Find all vectors that are perpendicular to $(1, 4, 4, 1)$ and $(2, 9, 8, 2)$.

9. If $S = \{0\}$ is the subspace of $\mathbb{R}^4$ containing only the zero vector, what is S? If S is spanned by $(0, 0, 0, 1)$, what is $S^\perp$? What is $(S^\perp)^\perp$?

10. Why are these statements false?

    (a) If V is orthogonal to W, then $V^\perp$ is orthogonal to $W^\perp$.

    (b) V orthogonal to W and W orthogonal to Z makes V orthogonal to Z.

11. Let P be the plane in $\mathbb{R}^2$ with equation $x + 2y - z = 0$. Find a vector perpendicular to P. What matrix has the plane P as its nullspace, and what matrix has P as its row space?

12. Construct a matrix with the required property or say why that is impossible.

    (a) Column space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, null space contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

    (b) Row space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, null space contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

    (c) $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution and $A^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

13. If $AB = 0$ then the columns of B are in ______ of A. The rows of A are in the ______ of B. Why cannot A and B be 3×3 matrices of rank 2?

14. If S is the subspace of $\mathbb{R}^3$ containing only the zero vector, what is $S^\perp$? If S is spanned by $(1, 1, 1)$, what is $(S^\perp)$? If S is spanned by $(2, 0, 0)$ and $(0, 0, 3)$, what is $S^\perp$?

15. Suppose V is the whole space $\mathbb{R}^4$. Then $V \perp$ contains only the vector ______ . Then $(V^\perp)^\perp$ is . So $(V^\perp)^\perp$ is the same as ______ .

16. Suppose S is spanned by the vectors $(1, 2, 2, 3)$ and $(1, 3, 3, 2)$. Find two vectors that span $S^\perp$. This is the same as solving $Ax = 0$ for which A?

17. Why is each of these statements false?

    (a) $(1, 1, 1)$ is perpendicular to $(1, 1, -2)$, so the planes $x + y + z = 0$ and $x + y - 2z = 0$ are orthogonal subspaces.

    (b) The subspace spanned by $(1, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 1)$ is the orthogonal complement of the subspace spanned by $(1, -1, 0, 0, 0)$ and $(2, -2, 3, 4, -4)$.

    (c) Two subspaces that meet only in the zero vector are orthogonal.

- Section 3.2

18. Is the projection matrix P invertible? Why or why not?

19. What matrix P projects every point in $\mathbb{R}^3$ onto the line of intersection of the planes $x + y + t = 0$ and $x - t = 0$?

20. Project the vector b onto the line through a. Check that $e = b - \text{proj}_a(b)$ is perpendicular to a:

    (i) $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (ii) $b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $a = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$.

21. Project $b = (1, 0, 0)^T$ onto the lines through $a_1 = (-1, 2, 2)^T$, $a_2 = (2, 2, -1)^T$, and $a_3 = (2, -1, 2)^T$. Add the three projections $p_1 + p_2 + p_3$.

22. Project the vector $b = (1, 1)^T$ onto the lines through $a_1 = (1, 0)^T$ and $a_2 = (1, 2)^T$. Draw the projections p_1 and p_2 and add $p_1 + p_2$. The projections do not add to b because ______.

- Section 3.3

23. Suppose the values $b_1 = 1$ and $b_2 = 7$ at times $t_1 = 1$ and $t_2 = 2$ are fitted by a line $b = Dt$ *through the origin.* Solve $D = 1$ and $2D = 7$ by least squares, and sketch the best line.

24. Solve $Ax = b$ by least squares, and find $p = A\hat{x}$ if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

25. If P is the projection matrix onto a k-dimensional subspace S of the whole space R^n, what is the column space of P and what is its rank?

26. If V is the subspace spanned by $(1, 1, 0, 1)^T$ and $(0, 0, 1, 0)^T$, find
    (a) a basis for the orthogonal complement $V^{\perp}$.
    (b) the projection matrix P onto V.
    (c) the vector in V closest to the vector $b = (0, 1, 0, -1)^T$ in $V^{\perp}$.

27. Find the best straight-line fit (least squares) to the measurements:
    $b = 4$ at $t = -2$, $b = 3$ at $t = -1$, $b = 1$ at $t = 0$ and $b = 0$ at $t = 2$.

    Then find the projection of $b = (4, 3, 1, 0)$ onto the column space of $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$.

28. A middle-aged man was stretched on a rack to lengths $L = 5$, 6, and 7 feet under applied forces of $F = 1$, 2, and 4 tons. Assuming Hookes law $L = a + bF$, find his normal length a by least squares.

- Extra Problems

29. A certain experiment produces the data $(1, 7.9)$, $(2, 5.4)$and $(3, -0.9)$. Describe the model that produces a least squares fit of these points by a function of the form

$$y = A\cos x + B\sin x.$$

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