MA-108 Ordinary Differential Equations

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D1 - Lecture 11

Recall: we discussed $y'' + ay' + by = 0$, $a, b \in \mathbb{R}$.

Cauchy Euler equation $x^2y'' + axy' + by = 0$.

All solutions of $y'' + p(x)y' + q(x)y = r(x)$ is $y_p + c_1y_1 + c_2y_2$, where y_1, y_2 are solution of homogeneous part.

By variation of parameter method, y_p is given by

$$y_p = y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)}\, dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)}\, dx.$$

n^{th} order differential equations

An n-order linear ODE is a differential equation of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = g(x),$$

where a_i's and g are continuous on some open interval I.

An n-order linear ODE in standard form is given by

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x).$$

As in 2nd order case, an n-order linear ODE is said to be **homogeneous** if $r \equiv 0$ and **non-homogeneous** otherwise.

Recall: solution space of a 2nd order linear homogenous ODE

$$y'' + p(x)y' + q(x)y = 0$$

is equivalent to the null space of the linear transformation $L : \mathcal{C}^2(I) \to \mathcal{C}(I)$ given by

$$L = D^2 + p(x)D + q(x)Id, \quad \text{where} \quad D = \frac{d}{dx}$$

Define the vector space

$$C^n(I) = \{f : I \to \mathbb{R} \mid f, f^1, \dots, f^{(n)} \text{ are continuous}\}.$$

Define linear transformation $L : C^n(I) \to C(I)$ as

$$L = D^n + p_1 D^{n-1} + \ldots + p_{n-1} D + p_n I,$$

Then the solution space to the ODE

$$y^{(n)} + p_1(x) y^{(n-1)} + \ldots + p_n(x) y = 0$$

is given by the null space of L.
We would like to prove that Dimension of $N(L) = n$.

In order to do this, we need the existence and uniqueness theorem in the n-th order case. The proof of the dimension theorem will then follow as before.

Existence and Uniqueness theorem

Def. An initial value problem for an n-order ODE is given by

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0 \qquad (1)$$

$$y(x_0) = k_0,\ y^1(x_0) = k_1,\ \ldots,\ y^{n-1}(x_0) = k_{n-1} \quad (2)$$

where p_i's are continuous on an open interval I and $x_0 \in I$.

Theorem (Existence and Uniqueness)

If $p_i(x)$ are continuous throughout an open interval I containing x_0, then the IVP defined above has a unique solution on I.

Just as in the linear case, the solution is unique over all of the interval on which the coefficient functions are defined.

Constant Coefficients

The method of constant coefficients generalises to the n-case.

Ex. Solve $y''' - 7y' - 6y = 0$.
We notice that this is same as the solution space of

$$Ly = (D^3 - 7D - 6I)y = 0$$

But

$$L = (D^3 + 6D^2 + 11D - 6I)$$

$$= (D - 3)(D^2 + 3D + 2I)$$

$$= (D - 3)(D + 1)(D + 2)$$

Now, note that if y is such that $(D + 2)y = 0$, then $Ly = 0$.

$$L = (D-3)(D+1)(D+2)$$

$$= (D+2)(D+1)(D-3) = (D-3)(D+2)(D+1)$$

If $(D+1)y = 0$ or $(D-3)y = 0$, then $Ly = 0$.

This gives us that $f(x) = e^{-x}$, $g(x) = e^{-2x}$ and $h(x) = e^{3x}$ are all solutions to the given ODE.

By dimension theorem, if this is linearly independent, then they give basis for all solutions. How do we check that?

Let if possible,

$$af + bg + ch \equiv 0$$

$$\implies af' + bg' + ch' \equiv 0 \quad \text{and} \quad af'' + bg'' + ch'' \equiv 0$$

Wronskian

This is equivalent to

$$\begin{bmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let $W(f,g,h;x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$.

Then

$$W(f,g,h;0) = \begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = -30 + 12 - 2 = -20 \neq 0.$$

$\implies$ f, g and h are linearly independent solutions of the ODE $y''' - 7y' - 6y = 0$ on any interval I containing 0.

Let $f_1, f_2, \ldots, f_n \in C^n(I)$. Define their **Wronskian,**

$$W(f_1, \ldots, f_n; x) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ \vdots & \vdots & \ldots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix}$$

As in the second order case, we have the following theorem.

Theorem

Let

$$L = D^n + p_1(x)D^{n-1} + \ldots + p_n(x)I$$

where $p_1, \ldots, p_n$ are continuous on an open interval I. Let $y_1, \ldots, y_n$ be solutions to the linear ODE $Ly = 0$. Let $x_0 \in I$. Then the Wronskian of $\{y_1, \cdots, y_n\}$ is given by

$$W(x) = W(x_0)\mathit{exp}\left(-\int_{x_0}^{x} p_1(t)\ dt\right)$$

where $x \in I$. Thus W either has no zeros on I or $W \equiv 0$ on I.

Then we have the following theorem, analogous to the second order case.

Theorem

Let

$$L = D^n + p_1(x)D^{n-1} + \ldots + p_n(x)I$$

where $p_1, \ldots, p_n$ *are continuous on an open interval* I. *Let* $y_1, \ldots, y_n$ *be solutions to the linear ODE* $Ly = 0$. *Then the following statements are equivalent.*

1. *The set* $\{y_1, \ldots y_n\}$ *is a fundamental set of solutions of* $Ly = 0$ *on* I, *i.e.* $\{y_1, \ldots, y_n\}$ *is linearly independent on* I.
2. *The Wronskian of* $\{y_1, \ldots, y_n\}$ *is nonzero at some point on* I.
3. *The Wronskian of* $\{y_1, \ldots, y_n\}$ *is nonzero for all* $x \in I$.

Dimension theorem

We can prove the dimension theorem for, the nth order case, as in the second order case.

Dimension Theorem. Let

$$L = D^n + p_1(x)D^{n-1} + \ldots + p_{n-1}(x)D + p_n(x)I$$

where p_i's are continuous on an open interval I. Then dimension of $N(L) = n$.

Proof. Let $x_0 \in I$ and let $\{e_1, \ldots, e_n\}$ be the standard basis vectors of $\mathbb{R}^n$.
By existence and uniqueness theorem, the IVP

$$Ly = 0; \quad (y(x_0), y'(x_0), \ldots, y^{(n)}(x_0)) = e_i$$

has a unique solution y_i on I for all $i = 1, \ldots n$.

Clearly $W(y_1, \ldots, y_n; x_0) = 1$ is non-zero. Therefore $\{y_1, \ldots, y_n\}$ is linearly independent solutions.

Further, assume y is any solution to $Ly = 0$. Then

$$z(x) = y(x_0)y_1(x) + y'(x_0)y_2(x) + \ldots + y^{(n-1)}(x_0)y_n(x)$$

is a solution of $L(y) = 0$. Since

$$z(x_0) = y(x_0),\ z'(x_0) = y'(x_0),\ \ldots,\ z^{(n-1)}(x_0) = y^{(n-1)}(x_0)$$

using the existence and uniqueness theorem to IVP, we get

$$y(x) \equiv z(x) \quad \text{on} \quad I$$

This proves the dimension theorem. $\square$

Def. Consider

$$L = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Thus for any function $f \in C^n(I)$,

$$L(f) = a_0 f^{(n)} + a_1 f^{(n-1)} + \ldots + a_{n-1} f' + a_n f.$$

Such an L is called a **constant coefficient differential operator**.

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Constant Differential Operators

Let

$$L = \sum_{i=0}^{n} a_{n-i} D^i, \quad \text{and} \quad M = \sum_{i=0}^{m} b_{m-i} D^i$$

be differential operators. Since,

$$D^r \cdot D^s = D^s \cdot D^r = D^{r+s},$$

for $r, s \geq 0$, we have:

$$L(M(f)) = M(L(f)),$$

for any f for which these make sense; in particular for $f \in C^\infty(I)$. Here $C^\infty(I) = \{f : I \to \mathbb{R}$ such that $f^{(n)}$ exists for all $n \geq 0$ $\}$.

Ex. Solve $y''' - 3y'' + 7y' - 5y = 0 \quad (1)$. Here

$$L = D^3 - 3D^2 + 7D - 5 = (D-1)(D^2 - 2D + 5)$$

$$= (D-1)((D-1)^2 + 4))$$

As remarked, we note that these linear transformations can be composed in a different order and solutions to $(D-1)y = 0$ and $[(D-1)^2 + 4]y = 0$ give solutions to $Ly = 0$.

Thus e^x, $e^x \cos 2x$ and $e^x \sin 2x$ are all solutions to the ODE (1).

Verify that these three solutions are linearly independent, to get that general solution to $Ly = 0$ are of the form

$$y(x) = e^x[c_1 + c_2 \cos 2x + c_3 \sin 2x].$$

Let us see how to do this in general.

Constant Differential Operators

Consider

$$L = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n, \quad a_i \in \mathbb{R}$$

We define the characteristic polynomial to be

$$P_L(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n.$$

Theorem

Let L and M be two constant coefficient linear differential operators. Then,

1. $L = M$ *if and only if* $P_L(x) = P_M(x)$.
2. $P_{L+M}(x) = P_L(x) + P_M(x)$.
3. $P_{LM}(x) = P_L(x) \cdot P_M(x)$.
4. $P_{\lambda L}(x) = \lambda \cdot P_L(x)$, *for every* $\lambda \in \mathbb{R}$.

Proof: (2), (3) and (4) are obvious. They follow from the definition of the characteristic polynomial.
Proof of (1): Suppose

$$L = \sum_{i=0}^{n} a_{n-i} D^i, \quad M = \sum_{i=0}^{m} b_{m-i} D^i.$$

Then,

$$P_L(x) = \sum_{i=0}^{n} a_{n-i} x^i, \quad P_M(x) = \sum_{i=0}^{m} b_{m-i} x^i.$$

Thus, $P_L = P_M$ clearly implies $L = M$.
Conversely, suppose $L = M$. Then $L(f) = M(f)$ for any $f \in C^\infty(I)$, in particular for $f(x) = e^{rx}$; i.e.,

$$\sum_{i=0}^{n} a_{n-i} r^i e^{rx} = \sum_{i=0}^{m} b_{m-i} r^i e^{rx}.$$

Cancel e^{rx} to conclude that $P_L = P_M$.

Constant Differential Operators

Corollary: Let L, M, N be constant coefficient linear differential operators such that

$$P_L(x) = P_M(x) \cdot P_N(x) \implies L = MN.$$

Proof.

$$P_L = P_M \cdot P_N = P_{MN} \implies L = MN.$$

Corollary: If $P_L(x) = a_0(x - r_1) \dots (x - r_n)$, then,

$$L = a_0(D - r_1) \dots (D - r_n).$$

Ex: $D^2 - 5D + 6 = (D - 3)(D - 2)$ as differential operators.