MA-108 Ordinary Differential Equations

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Recall: Second Shifting Theorem:

$$L(u(t-a)g(t)) = e^{-sa} L(g(t+a))$$

Using this theorem, we can solve IVP with piecewise continuous forcing functions.

We introduced convolution of two functions f and g as

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

Convolution Theorem:

$$L(f * g) = F(s)G(s)$$

The convolution theorem provides a formula for solution of an IVP with unspecified forcing function.

Examples

EXAMPLE: Give a formula for the solution of the IVP.

$$y'' + 2y' + 2y = f(t), \quad y(0) = a,\ y'(0) = b$$

Taking Laplace transform gives,

$(s^2 + 2s + 2)Y(s) = F(s) + b + as + 2a$. Therefore,

$$Y(s) = \frac{1}{s^2 + 2s + 2}F(s) + \frac{b + a + a(s+1)}{s^2 + 2s + 2}$$

$$L^{-1}\left(\frac{1}{s^2 + 2s + 2}\right) = e^{-t}\sin t,$$

Hence

$$y(t) = \int_0^t f(t-\tau)e^{-\tau}\sin\tau\, d\tau + e^{-t}\left[(b+a)\sin t + a\cos t\right]$$

Evaluating Convolution Integrals

Def. An integral of the form $\int_0^t f(\tau)g(t-\tau)\,d\tau$ is called a **convolution integral**.

Ex. Evaluate the integral

$$h(t) = \int_0^t (t-\tau)^5 \tau^7 \, d\tau$$

We could do it by expanding the integrand. Let's do it using convolution theorem.

$$h(t) = t^5 * t^7, \; H(s) = L(t^5)L(t^7) = \frac{5!\,7!}{s^6 s^8} = \frac{5!\,7!}{s^{14}}$$

Therefore,

$$h(t) = L^{-1}\left(\frac{5!\,7!}{s^{14}}\right) = \frac{5!\,7!}{13!} t^{13}$$

Ex. Evaluate the following integral

$$h(t) = \int_0^t \sin a(t-\tau)\cos b\tau \, d\tau, \quad |a| \neq |b|$$

Note that $h(t) = (\sin at) * (\cos bt)$. Hence

$$\begin{aligned} H(s) &= L(\sin at)L(\cos bt) \\ &= \frac{a}{s^2+a^2}\frac{s}{s^2+b^2} \\ &= \frac{a}{b^2-a^2}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) \end{aligned}$$

Therefore,

$$h(t) = \frac{a}{b^2-a^2}(\cos at - \cos bt)$$

Volterra Integral Equations

An integral equation of the form

$$y(t) = f(t) + \int_0^t k(t-\tau)y(\tau)\,d\tau$$

is called a **Volterra integral equation**. Here $f(t)$ and $k(t)$ are known functions and y is unknown.

We can solve them using convolution theorem.
Taking Laplace transform, we get

$$Y(s) = F(s) + K(s)Y(s) \implies Y(s) = \frac{F(s)}{1-K(s)}$$

Ex. Solve the integral equation

$$y(t) = 1 + 2\int_0^t e^{-2(t-\tau)} y(\tau)\, d\tau$$

Taking Laplace transform, we get

$$Y(s) = \frac{1}{s} + \frac{2}{s+2} Y(s)$$

This gives $Y(s)\left(1 - \frac{2}{s+2}\right) = Y(s)\frac{s}{s+2} = \frac{1}{s}$

$$Y(s) = \frac{1}{s} + \frac{2}{s^2} \implies y(t) = 1 + 2t$$

Additional Properties of Laplace Transform

Assume $L(f(t))$ is defined for $s > s_0$, then

1. $L\left(\int_0^t f(\tau)\,d\tau\right) = \dfrac{F(s)}{s}, \quad s > \max\{0, s_0\}$.
2. $L(tf(t)) = -F^{(1)}(s), \quad s > s_0$.
3. $L\left(\dfrac{f(t)}{t}\right) = \int_s^\infty F(s')ds', \quad s > s_0$.
4. Assume f is piecewise continuous and of exponential order. Then
   $(i) \lim_{s\to\infty} F(s) = 0, \qquad (ii) \lim_{s\to\infty} sF(s)$ is bounded.
5. Assume f and f' both are piecewise continuous and of exponential order. Then $\lim_{s\to\infty} sF(s) = f(0)$.
6. If f is piecewise continuous and periodic of period T, then $L(f(t)) = \dfrac{1}{1-e^{-sT}} \int_0^T f(T)e^{-st}dt, \; s > 0$

Theorem

If $F(s)$ exists for $s > s_0$, then

$$L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s}, \quad s > \mathit{max}\{0, s_0\}$$

Proof.

$$L\left(\int_0^t f(\tau)d\tau\right) = L(f * 1) = L(f)L(1) = \frac{F(s)}{s}$$

for $s > \max\{0, s_0\}$.

Ex. Compute $L^{-1}\left(\dfrac{1}{s^{n+1}}\right)$.

Since $L(t) = \dfrac{1}{s^2}$, $L\left(\int_0^t t\,dt\right) = \dfrac{1}{s^3}$, i.e. $L(t^2) = \dfrac{2}{s^3}$.

$L\left(\int_0^t t^2\,dt\right) = \dfrac{2}{s^4} \implies L(t^3) = \dfrac{3!}{s^4}$.

Proceeding by induction, we get $L(t^n) = \dfrac{n!}{s^{n+1}}$.

Ex. Find $L^{-1}\left(\dfrac{1}{s^2(s^2+1)}\right)$.

Since $L(\sin t) = \dfrac{1}{s^2+1}$,

$$L^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = \int_0^t \int_0^t \sin t\,dt$$

$$= \int_0^t (1-\cos t)\,dt = t - \sin t$$

Theorem

If $F(s)$ exists for $s > s_0$, then

$$L(tf(t)) = -\frac{dF(s)}{ds}, \quad s > s_0.$$

In general, $\quad L(t^k f(t)) = (-1)^k F^{(k)}(s), \quad s > s_0, \ k > 0.$

Proof.

$$\begin{aligned} \frac{dF(s)}{ds} &= \frac{d}{ds}\left(\int_0^\infty f(t)e^{-st}\,dt\right) \\ = \int_0^\infty \frac{\partial}{\partial s}(e^{-st})f(t)\,dt &= \int_0^\infty -te^{-st}f(t)\,dt \\ &= -L(tf(t)). \end{aligned}$$

How to justify this interchanging of differentiation and integration?

Differentiation under the Integral sign

Suppose we need to differentiate the function

$$F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$$

with respect to x. Assume $a(x)$ and $b(x)$ and their derivatives are continuous for $x_0 \leq x \leq x_1$. Further $f(x,t)$ and $\dfrac{\partial}{\partial x} f(x,t)$ are continuous (in both t and x) in some open rectangle containing $x_0 \leq x \leq x_1$ and $a(x) \leq t \leq b(x)$.

Then for $x_0 \leq x \leq x_1$:

$$\frac{d}{dx} F(x) = f(x, b(x))\, b'(x) - f(x, a(x))\, a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\, dt\,.$$

Search for "Leibniz Integral Rule".

Ex. Find $L^{-1}\left(\dfrac{s}{(s^2+4)^2}\right)$.

If $F(s) = \dfrac{1}{s^2+4}$, then $f(t) = \dfrac{1}{2}\sin 2t$. Hence

$L(tf(t)) = -\dfrac{dF(s)}{ds} = \dfrac{2s}{(s^2+4)^2}$.

Therefore, $L^{-1}\left(\dfrac{s}{(s^2+4)^2}\right) = \dfrac{1}{4}t\sin 2t$.

Exercise. Find $L^{-1}\left(\dfrac{s}{(s^2+4)^3}\right)$.

Theorem

If $F(s)$ *exists for* $s > s_0$, $\lim_{t\to 0} \dfrac{f(t)}{t}$ *exists, then* $L\left(\dfrac{f}{t}\right)$ *exists, and*

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s')\,ds', \quad s > s_0$$

Proof. $\int_s^\infty F(s')\,ds' =$

$$\int_s^\infty \left(\int_0^\infty f(t)e^{-s't}\,dt\right) ds' \;=\; \int_0^\infty f(t)\left(\int_s^\infty e^{-s't}\,ds'\right) dt$$

$$= \int_0^\infty \frac{f(t)}{t}\,e^{-st}\,dt \;=\; L\left(\frac{f(t)}{t}\right)$$

By Fubini's Theorem, if $\int\int |f(x,y)|\,dx\,dy$ converges, then

$$\int\int f(x,y)\,dx\,dy = \int\int f(x,y)\,dy\,dx$$

Ex. Find $L^{-1}(F(s))$, where $F(s) = \ln\left(\frac{s-a}{s-b}\right)$, where $a \neq b$ are real numbers.

$\frac{dF(s)}{ds} = \frac{1}{s-a} - \frac{1}{s-b} = G(s)$, say. If $s_0 = \max\{a, b\}$, then

$g(t) = L^{-1}\left(\frac{1}{s-a} - \frac{1}{s-b}\right) = e^{at} - e^{bt}$ exists.

Since $\lim_{t\to 0} \frac{g(t)}{t} = \lim_{t\to 0} \frac{e^{at} - e^{bt}}{t} = a - b$ exists, we get

$$\begin{aligned} L\left(\frac{g(t)}{t}\right) &= \int_s^\infty G(s')\, ds' = \int_s^\infty \left(\frac{1}{s'-a} - \frac{1}{s'-b}\right) ds' \\ &= \ln\left(\frac{s'-a}{s'-b}\right)\Big|_s^\infty = -\ln\left(\frac{s-a}{s-b}\right) \end{aligned}$$

Therefore, $L^{-1}\left(\ln\left(\frac{s-a}{s-b}\right)\right) = -\frac{g(t)}{t} = \frac{e^{bt} - e^{at}}{t}$.

Theorem

If f is piecewise continuous and of exponential order, then

$$(i)\ \lim_{s\to\infty} F(s) = 0, \quad (ii)\ \lim_{s\to\infty} sF(s) < \infty.$$

Proof. $|f(t)| \leq Me^{s_0 t}$ for $t \geq t_0$. Further we may assume $|f(t)| \leq K$ for $t \in [0, t_0]$. Hence

$$\begin{aligned}
|F(s)| &= \left| \int_0^\infty f(t)e^{-st}\,dt \right| \leq \int_0^\infty |f(t)|e^{-st}\,dt \\
&= \int_0^{t_0} |f(t)|e^{-st}\,dt + \int_{t_0}^\infty |f(t)|e^{-st}\,dt \\
&\leq \int_0^{t_0} Ke^{-st}\,dt + \int_{t_0}^\infty Me^{-(s-s_0)t}\,dt \\
&= K\frac{1-e^{-st_0}}{s} + \frac{M}{s-s_0}, \quad \text{for all } s > s_0
\end{aligned}$$

$$\implies \lim_{s\to\infty} F(s) = 0, \text{ and } \lim_{s\to\infty} sF(s) = K + M < \infty$$

Ex. Does there exist a function $f(t)$ which is piecewise continuous and of exponential order, such that $L(f(t)) = 1$? No. Since then $\lim_{s\to\infty} F(s) = 0$.

May be there exist some function $f(t)$ which is either not piecewise continuous or not of exponential order, and $L(f(t)) = 1$. Answer is Yes. Dirac delta function or impluse function has this property.

Exercise Find L^{-1} of (i) $\left(\frac{1}{s}\tanh s\right)$, (ii) $\ln\left(\frac{s^2+1}{s^2+s}\right)$, (iii) $\ln\left(1 \pm \frac{1}{s^2}\right)$.

Find if $\lim_{s\to\infty} sF(s) \to f(0)$. If not, then state why.