Tutorial-6, MA 108 (ODE)
Spring 2015, IIT Bombay

1. Find the Laplace transform of the following functions using the definition of Laplace transform.

   (a) $f(t) = \begin{cases} 1, & 0 \leq t < 4 \\ t, & t \geq 4 \end{cases}$

   (b) $f(t) = \begin{cases} t^2, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$

2. Find the Laplace transform of the following functions using the Laplace transform of step functions.

   (a) $f(t) = \begin{cases} te^t, & 0 \leq t < 1 \\ e^t, & t \geq 1 \end{cases}$

   (b) $f(t) = \begin{cases} t, & 0 \leq t < 1 \\ t^2, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases}$

3. Find the inverse Laplace transform of the following functions.

   (a) $H(s) = \dfrac{e^{-\pi s}(1-2s)}{s^2+4s+5}$

   (b) $H(s) = \dfrac{1}{s} - \dfrac{2}{s^3} + e^{-2s}\left(\dfrac{3}{s} - \dfrac{1}{s^2}\right) + e^{-3s}\left(\dfrac{4}{s} + \dfrac{3}{s^2}\right)$

4. Solve the following IVPs using Laplace transform.

   (a) $y'' - y = \begin{cases} e^{2t}, & 0 \leq t < 2 \\ 1, & t \geq 2 \end{cases} \quad y(0) = 3, \;\; y'(0) = -1.$

   (b) $y'' - 5y' + 4y = \begin{cases} 1, & 0 \leq t < 1 \\ -1, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases} \quad y(0) = 3, \;\; y'(0) = -5.$

   (c) $y'' + 9y = \begin{cases} \cos t, & 0 \leq t < \dfrac{3\pi}{2} \\ \sin t, & t \geq \dfrac{3\pi}{2} \end{cases} \quad y(0) = 0, \;\; y'(0) = 0.$

   (d) $y'' + y = \begin{cases} t, & 0 \leq t < \pi \\ -t, & t \geq \pi \end{cases} \quad y(0) = 0, \;\; y'(0) = 0.$

(e) $y'' - 3y' + 2y = \begin{cases} 0, & 0 \le t < 2 \\ 2t - 4, & t \ge 2 \end{cases}$ $\quad y(0) = 0, \quad y'(0) = 0.$

(f) $y'' + 2y' + y = \begin{cases} e^t, & 0 \le t < 1 \\ e^t - 1, & t \ge 1 \end{cases}$ $\quad y(0) = 3, \quad y'(0) = -1.$

(g) $y'' + 2y' + 2y = \begin{cases} t^2, & 0 \le t < 1 \\ -t, & 1 \le t < 2 \\ -1, & t \ge 3\pi \end{cases}$ $\quad y(0) = 2, \quad y'(0) = -1.$

5. Solve the IVP and find a formula in terms of f for the solution that does not involve any step functions and represents y on each interval of continuity of f

   (a) $y'' + y = f(t) \quad y(0) = 0, \quad y'(0) = 0;$
   $f(t) = m + 1, \quad m\pi \le t < (m+1)\pi, \quad m = 0, 1, \ldots.$

   (b) $y'' + y = f(t) \quad y(0) = 0, \quad y'(0) = 0;$
   $f(t) = (-1)^m, \quad m\pi \le t < (m+1)\pi, \quad m = 0, 1, \ldots.$

   (c) $y'' - y = f(t) \quad y(0) = 0, \quad y'(0) = 0;$
   $f(t) = m + 1, \quad m\pi \le t < (m+1)\pi, \quad m = 0, 1, \ldots.$
   Hint: You will need the formula for $1 + r + \ldots + r^m = \dfrac{1 - r^{m+1}}{1 - r} \quad (r \ne 1).$

   (d) $y'' + 2y' + 2y = f(t) \quad y(0) = 0, \quad y'(0) = 0;$
   $f(t) = (m+1)(\sin t + 2\cos t), \quad 2m\pi \le t < 2(m+1)\pi, \quad m = 0, 1, \ldots.$

6. Let $0 = t_0 < t_1 < \ldots < t_n$. Suppose f_m is continuous on $[t_m, \infty)$ for $m = 1, \ldots n$. Let

$$f(t) = \begin{cases} f_m(t), & t_m \le t < t_{m+1} \quad m = 1, \ldots, n-1 \\ f_n(t), & t \ge t_n \end{cases}$$

Show that the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \ y'(0) = k_1.$$

as defined for peicewise continuous forcing functions is given by

$$f(t) = \begin{cases} z_0(t), & 0 \le t < t_1 \\ z_0 + \ldots + z_m(t), & t_m \le t < t_{m+1} \quad m = 1, \ldots, n-1 \\ z_0 + \ldots + z_n(t) & t \ge t_n \end{cases}$$

where z_0 is a solution of

$$az'' + bz' + cz = f_0(t), \quad z(0) = k_0, \ z'(0) = k_1$$

and z_m is a solution of

$$az'' + bz' + cz = f_m(t) - f_{m-1}(t), \quad z(t_m) = 0, \ z'(t_m) = 0$$

for $m = 1, \ldots, n$.

7. Express the following inverse transform as an integral.

(a) $\dfrac{1}{s^2(s^2+4)}$

(b) $\dfrac{s}{s^2(s^2+4)}$

(c) $\dfrac{s}{(s+2)(s^2+9)}$

(d) $\dfrac{1}{(s+1)^2(s^2+4s+5)}$

(e) $\dfrac{1}{s^2(s-2)^3}$

8. Find the Laplace transform

(a) $\int_0^t \sin a\tau \cos b(t-\tau)\ d\tau$.

(b) $\int_0^t \sinh a\tau \cosh b(t-\tau)\ d\tau$.

(c) $e^t \int_0^t \sin \omega\tau \cos \omega(t-\tau)\ d\tau$.

(d) $e^t \int_0^t e^{2\tau} \sinh(t-\tau)\ d\tau$.

(e) $\int_0^t (t-\tau)^4 \sin 2\tau\ d\tau$.

(f) $\int_0^t (t-\tau)^7 e^{-\tau} \sin 2\tau\ d\tau$.

9. Find a formula for the solutions of the IVP.

(a) $y'' + 3y' + y = f(t), \quad y(0) = 0,\ y'(0) = 0.$

(b) $y'' + 4y = f(t), \quad y(0) = 0,\ y'(0) = 0.$

(c) $y'' + 6y' + 9y = f(t), \quad y(0) = 0,\ y'(0) = -2.$

(d) $y'' + \omega^2 y = f(t), \quad y(0) = a,\ y'(0) = b.$

(e) $y'' - 5y' + 6y = f(t), \quad y(0) = 1,\ y'(0) = 3.$

10. Solve the integral equation

    (a) $y(t) = t - \int_0^t (t-\tau)y(\tau)\ d\tau.$

    (b) $y(t) = 1 + 2\int_0^t \cos(t-\tau)y(\tau)\ d\tau.$

    (c) $y(t) = t + \int_0^t y(\tau)e^{-(t-\tau)}\ d\tau.$

11. Use the convolution theorem to solve the integral

    (a) $\int_0^t (t-\tau)^7\tau^8\ d\tau$

    (b) $\int_0^t (t-\tau)^6\tau^7\ d\tau$

    (c) $\int_0^t e^{-\tau}\sin(t-\tau)\ d\tau$

12. Show that $f * g = g * f$.

13. Show that if $p(s) = as^2 + bs + c$ has distinct real zeros r_1 and r_2 then the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \ y'(0) = k_1$$

    is

$$\begin{aligned} y(t) &= k_0 \frac{r_2 e^{r_1 t} - r_2 e^{r_2 t}}{r_2 - r_1} + k_1 \frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1} \\ &\quad + \frac{1}{a(r_2 - r_1)} \int_0^t (e^{r_2 \tau} - e^{r_1 \tau}) f(t-\tau)\ d\tau \end{aligned}$$

14. For the above problem find a formula for the solution if the roots of $p(s)$ are repeated and is given by r, and when the roots are complex $\lambda \pm i\omega$.