Tutorial-2, MA 106 (Linear Algebra)
Linear Algebra and its Applications by Gilbert Strang

Problem Set 1.4 [9, 10, 15, 16, 21, 22, 27, 35, 43, 46]

1. The product of two lower triangular matrices is again lower triangular (all its entries above the main diagonal are zero). Confirm this with a 3×3 example, and then explain how it follows from the laws of matrix multiplication.

2. Suppose A commutes with every 2×2 matrix ($AB = BA$) and in particular

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{commutes with} \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Show that $a = d$ and $b = c = 0$. If $AB = BA$ for all matrices B, then A is a multiple of the identity matrix.

3. By trial and error find examples of 2 by 2 matrices such that
   (a) $A^2 = -I$, A having only real entries.
   (b) $B^2 = 0$, although $B \neq 0$.
   (c) $CD = -DC$, not allowing the case $CD = 0$.
   (d) $EF = 0$, although no entries of E or F are zero.

4. The first row of AB is a linear combination of all the rows of B. What are the coefficients in this combination, and what is the first row of AB, if

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}?$$

5. The matrix that rotates the $x - y$ plane by an angle θ is $A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Verify that $A(\theta_1)\,A(\theta_2) = A(\theta_1 + \theta_2)$ from the identities for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$. What is $A(\theta)$ times $A(-\theta)$?

6. Suppose $a_{33} = 7$ and the third pivot is 5. If you change a_{33} to 11, what is the third pivot? If you change a_{33} to $------$, there is zero in the third pivot position.

7. What three matrices $E_{21},\ E_{31},\ E_{32}$ put $A = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{pmatrix}$

into triangular form U? Multiply those E's to get one matrix M that does the elimination $MA = U$.

8. The parabola $y = a + bx + cx^2$ goes through the points $(x, y) = (1, 4), (2, 8), (3, 14)$. Find and solve a matrix equation for the unknowns (a, b, c).

9. What rows or columns or matrices do you multiply to find

   (a) the third column of AB?

   (b) the first row of AB?

   (c)the entry in row 3, column 4 of AB?

   (d)the entry in row 1, column 1 of CDE?

10. Multiply AB using columns times rows

$$AB = \begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 3 & 0 \end{pmatrix} + - - - - \quad = \quad - - - - - -$$

Problem Set 1.5 [3, 7, 9, 15, 18, 21, 27, 30, 34, 40, 41]

11. What multiple l_{32} of row 2 of A will elimination subtract from row 3 of A? Use the factored form of A. What will be the pivots? Will a row exchange be required?

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{pmatrix}$$

12. Factor A into LU and write down the upper triangular system $Ux = c$ which appears after elimination, for

$$Ax = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$$

13. Factor A into LU for $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$.

14. How could you factor A into a product UL, upper triangular times lower triangular? Would they be the same factors as in $A = LU$?

15. Solve as two triangular system, without multiplying LU to find A:

$$LUx = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

16. What three elimination matrices E_{21}, E_{31}, E_{32} put A into upper triangular form $E_{32}E_{31}E_{21}A = U$? Factor A into LU, where $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix}$$

17. Compute L and U for the symmetric matrix $A = \begin{pmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{pmatrix}$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

18. Find the triple factorization $A = LDU$ for the symmetric matrix $A = \begin{pmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{pmatrix}$.

19. For which numbers c is $A = LU$ impossible with three pivots?

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

20. Which permutation P makes PA upper triangular? Which permutations make $P_1 A P_2$ lower triangular? Multiplying A on the right side by P_2 exchanges the $-------- --$ of A.

$$A = \begin{pmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{pmatrix}$$

21. If P_1 and P_2 are permutation matrices, so is $P_1 P_2$. This still has the rows of I in some order. Give examples with $P_1 P_2 \neq P_2 P_1$ and $P_3 P_4 = P_4 P_3$.

Problem Set 1.6 [4, 7, 10, 12, 19, 22, 24, 44]

22. Find the inverses (in any legal way) of

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 4 & 5 \end{pmatrix}$$

23. (a) Find the inverses of the permutation matrices

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(b) Explain for permutations why P^{-1} is always the same as P^T. Show that the 1's are in the right place to give $PP^T = I$.

24. Use Gauss-Jordan method to invert $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

25. Compute the symmetric LDL^T factorization of $A = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{pmatrix}$.

26. If A is invertible, which properties of A remain true for A^{-1}.

    (a) A is triangular, (b) A is symmetric, (c) All entries of A are whole numbers (integers), (d) all entries are rationals (fractions).

27. Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible? How would you find B^{-1} from A^{-1}.

28. If A and B are square matrices, show that $I - AB$ is invertible if $I - BA$ is invertible. Start from $B(I - AB) = (I - BA)B$.

29. This matrix has a remarkable inverse. Find A^{-1} by elimination on $[A \mid I]$. Extend it to 5×5 "alternating matrix in 1, -1" and guess its inverse.

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Review Exercises:

30. (a) There are sixteen 2 by 2 matrices whose entries are 1's and 0's. How many are invertible.

    (b) If you put 1's and 0's at random into the entries of a 10 by 10 matrix, is it more likely to be invertible or singular?

31. If possible, find 3 by 3 matrices B such that

    (a) $BA = 2A$ for every A.

    (b) $BA = 2B$ for every A.

    (c) BA has the first and last rows of A reversed.

    (d) BA has the first and last columns of A reversed.

32. Starting with a first plane $u + 2v - w = 6$, find the equation for

    (a) the parallel plane through the origin.

    (b) a second plane that also contains the points $(6, 0, 0)$ and $(2, 2, 0)$.

    (c) a third plane that meets the first and second in the point $(4, 1, 0)$.

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