MA-106 Linear Algebra

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D1 - Lecture 16

Recall: determinant is a function $det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ which is uniquely defined by its three basic properties.

1. $det(I) = 1$.
2. the sign of determinant is reversed by a row exchange,
3. det is linear in each row separately,

Some Properties of determinant.

If two rows of A are equal, then $det(A) = 0$.

If B is obtained from A by $R_i \mapsto R_i + aR_j$, then $det(B) = det(A)$.

If $A = (a_{ij})$ is triangular, then $det(A) = a_{11} \dots a_{nn}$.

If A is singular, then $det(A) = 0$.

$det(AB) = det(A)\, det(B)$

$det(A) = det(A^T)$

If A is invertible, then $detA = \pm$ product of pivots.

Recall:

Formula for determinant of a 3×3 matrix $A = (a_{ij})$

$det(A) =$

$$a_{11}a_{22}a_{33}\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{11}a_{23}a_{32}\begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix}$$

$$+a_{12}a_{23}a_{31}\begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32}\begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}$$

$$= \sum_{\text{all permutations } P} (a_{1\alpha}\, a_{2\beta}\, a_{3\gamma})\ det(P)$$

where (α, β, γ) runs over all permutations of $\{1, 2, 3\}$, $\{e_1, e_2, e_3\}$ are standard basis of $\mathbb{R}^3$ and P runs over all permutation matrices $\begin{pmatrix} (e_\alpha)^T \\ (e_\beta)^T \\ (e_\gamma)^T \end{pmatrix}$

Formula for Determinant: $n \times n$ case

For $n \times n$ matrix $A = (a_{ij})$,

$$det(A) = \sum_{\text{all permutations } P} (a_{1i_1} \, \ldots \, a_{ni_n}) \, det(P)$$

The sum is over $n!$ permutations of numbers $(1, \ldots, n)$.

Here $(i_1, i_2, \ldots, i_n)$ runs over all permutations of $\{1, 2, \ldots, n\}$, $\{e_1, \ldots, e_n\}$ are standard basis of $\mathbb{R}^n$ and P runs over all the permutation matrix $\begin{pmatrix} (e_{i1})^T \\ \vdots \\ (e_{i_n})^T \end{pmatrix}$.

If P is a permutation matrix, then $det(P) = +1$, if the number of row exchanges in P needed to get I is even and $det(P) = -1$ if it is odd.

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3. 140020046 SHRIKANT MUNDRA ABSENT
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8. 140020101 RAHUL CHOUDHARY
9. 140020110 SLOKA AMBATI
10. 140050014 SHREY KUMAR
11. 140020121 GURJOT SINGH WALIA
12. 140050007 NEELADRISHEKHAR KANJILAL
13. 140050019 RISHABH AGARWAL
14. 140050021 YATHANSH KATHURIA
15. 140050037 LAKUMARAPU NIKHIL KUMAR
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Cofactor, expansion of det along first row

Let C_{1j} be the coefficient of a_{1j} in the expansion

$$det(A) = \sum_{\text{all permutations } P} (a_{1i_1} \dots a_{ni_n})\, det(P)$$

Then $\boxed{det(A) = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}}$.

Note that $C_{1j} = det \begin{pmatrix} 0 & . & 1 & . & 0 \\ a_{21} & . & a_{2j} & . & a_{2n} \\ \vdots & . & \vdots & . & \vdots \\ a_{n1} & . & a_{nj} & . & a_{nn} \end{pmatrix}$

the matrix obtained from A by replacing its first row by $e_j^T = (0, \dots, 1, \dots, 0)$ (1 at j-th place).

Hence $\boxed{C_{1j} = (-1)^{j-1}\, det(M_{1j})}$, where M_{ij} is the matrix obtained from A by deleting its 1-st row and j-th column.

Example - 3×3 case.

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} 1 & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + a_{12} \begin{vmatrix} & 1 & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} & & 1 \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Expansion of $det(A)$ along its i-th row

If C_{ij} is the coefficient of a_{ij} in the formula of $det(A)$, then

$$\boxed{det(A) \;=\; a_{i1}\, C_{i1} + \,\ldots + a_{in}\, C_{in}.}$$

By $i-1$ row exchange on A, get the matrix $B = \begin{pmatrix} A^i \\ A^1 \\ \vdots \\ A^{i-1} \\ A^{i+1} \\ \vdots \\ A^n \end{pmatrix}$

Since $det(A) = (-1)^{i-1}\, det(B)$, we get

$$C_{ij}(A) = (-1)^{i-1} C_{1j}(B) = (-1)^{i-1}(-1)^{j-1}\, det(M)$$

where M is obtained from B by deleting 1-st row and j-th column.
Hence M is obtained from A by deleting i-th row and j-th column.
Write M as M_{ij}. Then $\boxed{C_{ij} = (-1)^{i+j}\, det(M_{ij})}$

Expansion of $det(A)$ along j-th column of A

Note that $$C_{ij}(A^T) = C_{ji}(A).$$

Hence, if we write $A^T = (b_{ij})$, then

$$\begin{aligned} det(A) &= det(A^T) \\ &= b_{j1}\, C_{j1}(A^T) + \ldots + b_{jn}\, C_{jn}(A^T) \\ &= a_{1j}\, C_{1j}(A) + \ldots + a_{nj}\, C_{nj}(A) \end{aligned}$$

This is the expansion of $det(A)$ along j-th column of A.

Example:

$$F_n = \begin{vmatrix} 1 & -1 & & & \\ 1 & 1 & -1 & & \\ & 1 & 1 & -1 & \\ & & . & . & . \\ & & & 1 & 1 \end{vmatrix}$$

$(1, 1, -1)$ tri-diagonal $n \times n$ matrix. Expanding along row 1, we get

$$F_n = F_{n-1} + (-1)^{1+2}(-1) \begin{vmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & 1 & 1 & -1 & \\ & . & . & . & \\ & & & 1 & 1 \end{vmatrix}$$

$$= F_{n-1} + F_{n-2}, \qquad \text{by expanding along first column.}$$

Since $F_1 = 1$, $F_2 = 2$, we get F_n's as the sequence of integers $1, 2, 3, 5, 8, 13, \ldots$, known as Fibonacci sequence.

Applications of det: (1) Computing A^{-1}

If $C = (C_{ij})$: cofactor matrix of A, then $\boxed{A\,C^T = det(A)\,I}$ i.e.

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & \dots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \dots & C_{nn} \end{pmatrix} = \begin{pmatrix} det(A) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & det(A) \end{pmatrix}$$

Proof. We have seen that $a_{i1}C_{i1} + \dots + a_{in}C_{in} = det(A)$. Now

$$a_{11}C_{21} + a_{12}C_{22} + \dots + a_{1n}C_{2n} = det \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{11} & \dots & a_{1n} \\ a_{31} & \dots & a_{3n} \\ \cdot & & \cdot \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = 0.$$

Similarly, if $i \neq j$, then $a_{i1}C_{j1} + a_{i2}C_{j2} + \dots + a_{in}C_{jn} = 0$. $\square$

Remark. If A is invertible, then $\boxed{A^{-1} = \frac{1}{det(A)} C^T}$.

For $n \geq 4$, this is Not a good formula to find A^{-1}. Use elimination to find A^{-1}. This formula is of theoretical importance.

Example

Find the inverse of $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

$$det(A) = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + (-1)^{1+2}(-1)\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}$$

$$= 2(3) + (-2) = 4$$

$C_{11} = 3,\ C_{12} = (-1)(-2) = 2,\ C_{13} = 1,$

$C_{21} = (-1)(-2) = 2,\ C_{22} = 4, C_{23} = (-1)(-2) = 2,$

$C_{31} = 1,\ C_{32} = (-1)(-2) = 2,\ C_{33} = 3.$

Hence

$A^{-1} = |A|^{-1}\, C^T = (1/4)\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix}$

Check. $AA^{-1} = I$.

Application of det: (2) Solving $Ax = b$

Cramer's rule: If A is invertible, the $Ax = b$ has a unique solution

$$x = A^{-1}b = |A|^{-1} C^T b = |A|^{-1} \begin{pmatrix} C_{11} & \dots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \dots & C_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Hence $\quad x_j = |A|^{-1} (b_1 C_{1j} + b_2 C_{2j} + \dots + b_n C_{nj}) = |A|^{-1} det(B_j),$

where $det(B_j)$ is expanded along j-th column of

$$B_j := \begin{pmatrix} a_{11} & . & a_{1,j-1} & b_1 & a_{1,j+1} & . & a_{1n} \\ \vdots & . & \vdots & \vdots & \vdots & . & \vdots \\ a_{n1} & \cdot & a_{n,j-1} & b_n & a_{n,j+1} & \cdot & a_{nn} \end{pmatrix}$$

Remark: Use elimination to solve $Ax = b$. Cramer's rule is of theoretical importance.

$\boxed{\text{Assume rows of } A \text{ are orthogonal}}$. Then

$$AA^T = \begin{pmatrix} A^1 \\ \vdots \\ A^n \end{pmatrix} \begin{pmatrix} A^1 & \dots & A^n \end{pmatrix} = \begin{pmatrix} l_1^2 & & 0 \\ & \ddots & \\ 0 & & l_n^2 \end{pmatrix}$$

where $l_i = \sqrt{|A^i.A^i|}$ is the length of A^i. Since $det(A) = det(A^T)$, we get $\boxed{|det(A)| = l_1 \dots l_n}$.

Since the edges of the box spanned by rows of A are at right angles, the volume of the box

$$\begin{aligned} &= \text{ the product of lengths of edges} \\ &= |det(A)|. \end{aligned}$$

Assume $\boxed{\text{edges are Not at right angles}}$. We will consider 2×2 case. The volume of the parallelogram spanned by rows A^1, A^2

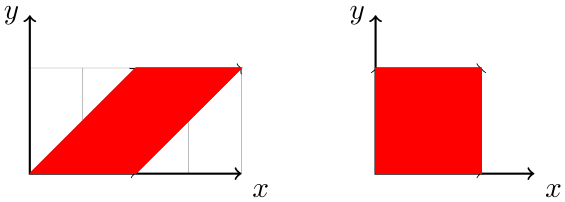
$= \text{base} \;.\; \text{height}$

$= \text{length of } A^1 \;.\; \text{length of projection vector } A^2 - proj_{A^1} A^2$

$$= det \begin{pmatrix} A^1 \\ A^2 - proj_{A^1} A^2 \end{pmatrix}, \quad \text{(since two vectors are orthogonal)}$$

$$= det(A)$$

Example. Let $A = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.



$\boxed{= 4}$

Extra reading: Application (4): Formula for Pivots

We can find when elimination is possible without row exchange.

Observation: If row exchange is not required in A, then first k pivots are determined by top-left $k \times k$ submatrix $\widetilde{A}_k$ of A.

Example. If $A = (a_{ij})_{3\times 3}$, then $\widetilde{A}_1 = (a_{11})$, $\widetilde{A}_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\widetilde{A}_3 = A$.

- Assume pivots are $d_1, \ldots, d_n$. Then
- $det(\widetilde{A}_1) = a_{11} = d_1$
- $det(\widetilde{A}_2) = d_1\, d_2 = \det(A_1)\, d_2$
- $det(\widetilde{A}_3) = d_1\, d_2\, d_3 = det(A_2)\, d_3 \quad \ldots$
- If $det(\widetilde{A}_k) = 0$, then we need a row exchange in elimination.
- Otherwise the k-th pivot is $\boxed{d_k = det(\widetilde{A}_k)/det(\widetilde{A}_{k-1})}$

Example. Find if row exchange is required in Gauss Elimination of A and find pivots.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 9 \end{pmatrix}$$

Solution. $\widetilde{A}_1 = (1), \quad \widetilde{A}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}, \quad \widetilde{A}_3 = A$

$$det(\widetilde{A}_1) = 1, \quad det(\widetilde{A}_2) = 3$$

$$det(\widetilde{A}_3) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 3 & 8 \end{vmatrix} = 15$$

Therefore row exchange is not required in elimination and

the pivots are $d_1 = 1, \ d_2 = 3/1 = 3, \ d_3 = 15/3 = 5.$ $\square$