PH 103 : Electricity and Magnetism

Tutorial Sheet 6 : **Mutlipole Expansion and Dielectrics**

Multipole Expansion

1. An electric dipole $\vec{p}$ is fixed at the origin, making an angle β with the z-axis. Calculate the electric flux through the hemispherical surface of radius R resting on the x-O-y plane and centred at the origin. [Ans.: $p\cos\beta/2\epsilon_0 R$]

2. A dipole $\vec{p} = p\,\hat{k}$ is fixed at the origin. Calculate the work done in moving a point charge +Q from the point A$(r, \theta = \pi/2)$ to the point B$(r, \theta = 0)$ [r and θ are polar coordinates] in two different ways: (a) from the potential at the two points, and (b) by calculating the line integral of E from A to B.

   [Ans.: $Qp/4\pi\epsilon_0 r^2$]

3. A circular area of radius a lies in the x-y plane, and is centered at the origin. It has a charge density $\sigma(r, \theta) = \sigma_0 r\cos\theta$. Obtain the multipole expansion up to the quadrupole term for a point P located at $(0, 0, z)$, with $z \gg a$. (Ans: monopole term=zero, dipole moment $\vec{p} = \hat{\imath}\,\sigma_0\pi a^4/4$).

4. Two pure dipoles $\vec{p}_1$ and $\vec{p}_2$ are oriented in space such that $\vec{p}_1$ has position vector $\vec{r}_1$ with respect to $\vec{p}_2$. The angle between $\vec{p}_1$ and $\vec{r}_1$ and $\vec{p}_2$ and $\vec{r}_2$ are θ' and θ, respectively, while the angle between the two dipoles is ϕ. Show that the mutual interaction energy between the two dipoles is $U = p_1 p_2\left(\cos\phi - 3\cos\theta\cos\theta'\right)/4\pi\epsilon_0 r^3$.

5. A sphere of radius a centered at the origin carries a volume charge density $\rho(r, \theta) = \rho_0 a\,(a - 2r)\sin\theta/r^2$ where ρ_0 is a constant, and r, θ are the usual spherical polar coordinates. Find the approximate potential (up to the quadrupole term) at large distances for points on the z-axis. [Ans. zero up to dipole]

6. Four point charges $-2q$, $3q$, $-2q$ and q are placed at the points (0,a,0), (0,0,a), (0,$-a$,0) and (0,0,$-a$), respectively. Using spherical polar coordinates, calculate the approximate potential (up to the quadrupole term), for large distances from the origin. [Ans. : zero up to the dipole term]

7. A thin ring of radius a carrying a constant linear charge density λ is kept in the x-O-y plane with centre as origin. Using spherical coordinates, calculate the first three terms in the multipole expansion for the potential ($r \gg a$, $\theta = 0$) of the ring. Why is the term proportional to $1/r^2$ missing in the expansion. [Ans. : $V \simeq \frac{a\lambda}{2\epsilon_0 r}\left(1 - \frac{a^2}{2r^2}\right)$]

Dielectrics

1. A uniformly polarized dielectric sphere of radius R has a frozen-in polarization $\vec{P} = P_0\hat{k}$. Calculate the surface (σ_b) and volume (ρ_b) charge densities. Calculate the electric field inside the sphere. Find the potential at any point outside the sphere ($r \gg R$) where r is

the distance from the centre of the sphere which coincides with the origin of the coordinate system. [Ans. $\sigma_b = P_0 \cos\theta$, $E_{in} = (P_0/3\epsilon_0)(-\hat{k})$, $V_{out} = (P_0 R^3 \cos\theta)/(3\epsilon_0 r^2)$].

2. An unpolarized dielectric sphere of radius R is placed in a uniform electric field $\vec{E} = E_0\hat{k}$. Find the electric field both inside as well as outside the sphere by solving the Laplace's equation.

3. A thick spherical shell of inner radius a and outer radius b has a "frozen in" polarization $P(r) = \frac{k}{r}\hat{r}$ where k is a constant and r is the distance from the centre. Find $\vec{D}$ and $\vec{E}$ everywhere. Also find σ_b and ρ_b. [Ans. $\sigma_b = -k/a$ at $r = a$, $\sigma_b = +k/b$ at $r = b$ $\rho_b = -3a/r^2$, $\vec{D} = 0$ everywhere, $E(a < r < b) = (k/\epsilon_0 r)(-\hat{r})$, $E = 0$ everywhere else].

4. A spherical capacitor consists of an inner conducting sphere of radius R_1 and an outer concentric conducting shell of radius R_3. The region $R_1 \leq r \leq R_2$ is filled with a material of dielectric constant κ. The inner conductor has a charge Q while the outer conducting shell which is grounded has a charge –Q. Calculate (i) the displacement vector $\vec{D}$, the electric field $\vec{E}$ and the potential in the two regions $R_1 \leq r \leq R_2$ and $R_2 \leq r \leq R_3$, (ii) the polarization vector $\vec{P}$ and the bound surface (σ_b) and volume (ρ_b) charge densities in the dielectric. [Ans. $\overrightarrow{D}(r \leq R_1) = \frac{Q}{4\pi r^2}$]

5. A conducting sphere of radius R having charge $+Q$ is surrounded by a dielectric medium with a dielectric constant that varies as $\kappa = 3R/r$ and extends from $r = R$ to $r = 2R$ (where r is the radial distance). Calculate

   (a) the polarization vector $\vec{P}$

   (b) the bound volume charge densities in the dielectric and

   (c) the total polarization charge on the surface $r = 2R$.

   [Ans. $\vec{P} = \frac{Q}{4\pi r^2}(1 - \frac{r}{3R})\hat{r}$, $\rho_b = Q/12\pi a R r^2$, $Q/3$]

6. A dielectric rod in the shape of a right circular cylinder of length $2L$ and radius R is polarized in the direction of its length. The polarization P is uniform and has a magnitude P_0. Taking the origin at the centre of the cylinder, calculate the electric field at a point z, where $|z| < L/2$.

7. Consider a dielectric with displacement vector given by $\vec{D}_0 = \epsilon_0\vec{E}_0 + \vec{P}$, where all the three quantities are uniform.

   (a) A small spherical cavity is hollowed out of the material. Find the electric field $\vec{E}$ and $\vec{D}$ at the centre of the cavity. [Ans. $\vec{E} = \vec{E}_0 + \frac{\vec{P}}{3\epsilon_0}$, $\vec{D} = \vec{D}_0 - \frac{2}{3}\vec{P}$]

   (b) Find the electric field $\vec{E}$ and $\vec{D}$ inside a needle-shaped cavity running parallel to the direction of polarization. [Ans. $\vec{E} = \vec{E}_0$, $\vec{D} = \vec{D}_0 - \vec{P}$]

   (c) Find the electric field $\vec{E}$ and $\vec{D}$ for a thin wafer shaped cavity with its plane perpendicular to the direction of polarization. [Ans. $\vec{E} = \vec{E}_0 + \frac{\vec{P}}{3\epsilon_0}$, $\vec{D} = \vec{D}_0$]

8. A conducting sphere of radius R is half submerged in a linear homogeneous semi infinite liquid dielectric medium of dielectric constant κ. The sphere is at a potential V_0. Assuming that there is no bound charge at the liquid-air interface, calculate

   (a) the potential at a point outside the sphere

   (b) the electric field, the displacement vector and the bound charge density in the dielectric and

   (c) total free charge on the conductor.

   [Ans. $V = V_0R/r$, $\vec{E} = \frac{V_0R}{r^2}\hat{r}$, $\vec{D} = \epsilon_0\kappa\vec{E}$, $\sigma_b = \epsilon_0(\kappa - 1)V_0/R$, $\rho_b = 0$, $Q_{free,tot} = 2\pi\epsilon_0(\kappa + 1)V_0R$].

9. Obtain the boundary conditions that the field vectors must satisfy at the interface between two dielectrics. If the electric field vector $\vec{E}_1$ in a semi-infinite medium of permittivity ϵ_1 makes an angle θ with the interface with a second infinite dielectric medium of permittivity ϵ_2, find the angle ϕ that the field vector $\vec{E}_2$ in the second medium makes with the interface. [Ans. $\tan\phi = (\epsilon_2/\epsilon_1)\tan\theta$]

10. The plane $z = 0$ marks the boundary between the free space ($z > 0$) and an infinite dielectric medium ($z < 0$) with dielectric constant $\kappa = 8$. The electric field in the free space, immediately next to the interface is $\vec{E} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$. Determine the electric field on the dielectric side of the interface. [Ans. $\vec{E} = 2\hat{\imath} + 3\hat{\jmath} + 0.5\hat{k}$].