MA-106 Linear Algebra

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D1 - Lecture 5

Echelon Form

Recall: If A is an $n \times n$ matrix, then $PA = LU$, where P is a permutation matrix, L is lower triangular, U is upper triangular, and all of size $n \times n$.

Q: What happens when A is not a square matrix?

Let $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$. By elimination, we see:

$$A \to \begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U.$$

Thus $E_{32}(1)E_{31}(-3)E_{21}(-2)A = U$. Therefore $A = LU$,

where $L = E_{21}(2)E_{31}(3)E_{32}(-1) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}$.

Echelon Form

If A is $m \times n$, we can find P, L and U such that $\boxed{PA = LU}$. In this case, L and P will be $m \times m$ and U will be $m \times n$.

U has the following properties:

1. Pivots are the 1st nonzero entries in their rows.
2. Entries below pivots are zero, by elimination.
3. Each pivot lies to the right of the pivot in the row above.
4. Zero rows are at the bottom of the matrix.

U is called the *echelon form* of A.

Possible 2×2 echelon forms: Let $\bullet =$ pivot entry.

$$\begin{pmatrix} \bullet & * \\ 0 & \bullet \end{pmatrix}, \begin{pmatrix} \bullet & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \bullet \\ 0 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Row Reduced (Echelon) Form

To obtain the row reduced form R of a matrix A:
1) Get the echelon form U.
2) Make the pivots 1.
3) Make the entries above the pivots 0.

Excercise: Find all possible 2×2 row reduced forms.

Example. Let $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$. Then

$U = \begin{pmatrix} \mathbf{1} & 2 & 3 & 5 \\ 0 & 0 & \mathbf{2} & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Divide by pivots: $\begin{pmatrix} \mathbf{1} & 2 & 3 & 5 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Row reduced form of A: $R = \begin{pmatrix} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

U and R are used to solve $Ax = 0$ and $Ax = b$.

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Null Space: Solution of $Ax = 0$

Let A be $m \times n$. The Null Space of A, denoted $N(A)$, is the set of all vectors x in $\mathbb{R}^n$ such that $Ax = 0$.

Key Point: $Ax = 0$ has the same solutions as $U\,x = 0$, which has the same solutions as $R\,x = 0$, i.e.,

$$\boxed{N(A) = N(U) = N(R)}.$$

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}. \quad R\,x = \begin{pmatrix} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix}.$$

$R\,x = 0$ gives $t + 2u + 2w = 0$ and $v + w = 0$.
i.e., $t = -2u - 2w$ and $v = -w$.

Null Space: Solution of $A\overline{x} = 0$

$Rx = 0$ gives $t = -2u - 2w$ and $v = -w$,
t and v are *dependent* on the values of u and w.
u and w are *free* and *independent*, i.e., we can choose any value for these two variables.

Special solutions:

$u = 1$ and $w = 0$, gives $x = \begin{pmatrix} -2 & 1 & 0 & 0 \end{pmatrix}^T$.
$u = 0$ and $w = 1$, gives $x = \begin{pmatrix} -2 & 0 & -1 & 1 \end{pmatrix}^T$.
The null space contains all possible linear combinations of the special solutions:

$$x = \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2u - 2w \\ u \\ -w \\ w \end{pmatrix} = u \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

Rank of A

$Ax = 0$ always has a solution: the trivial one, i.e., $x = 0$.

Main Q1: When does $Ax = 0$ have a non-zero solution?

A: When there is at least one free variable,
i.e., not every column of R contains a pivot.

To keep track of this, if R is row reduced form of A, then
$\boxed{\text{rank}(A) = \text{number of columns containing pivots in } R}$.

If A is $m \times n$ and rank$(A) = r$, then

- rank$(A) \leq \min\{m, n\}$.
- no. of dependent variables $= r$.
- no. of free variables $= n - r$.
- $Ax = 0$ has only the 0 solution $\Leftrightarrow r = n$.
- $m < n \Rightarrow Ax = 0$ has non-zero solutions.

Rank of A

True/False: If $m \geq n$, then $Ax = 0$ has only the 0 solution.False

Example: $R = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ when $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$.

The number of columns containing pivots in R is 2. So, rank$(A) = 2$.

R contains a 2×2 identity matrix, namely the rows and columns corresponding to the pivots.

This is the row reduced form of the corresponding submatrix $\begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix}$ of A, which is invertible, since it has 2 pivots.

Thus, rank$(A) = r \Rightarrow A$ has an $r \times r$ invertible submatrix, but not of larger size.

Finding $N(A)$: Solving $Ax = 0$

Let A be $m \times n$. To solve $Ax = 0$, find R and solve $Rx = 0$.

1. Find free or independent variables i.e. columns in R without pivots (u and w in example).
2. Find pivot or dependent variables i.e. columns in R with pivots (t and v in example).
3. No free variables i.e., rank$(A) = n \Rightarrow N(A) = 0$.
4. If rank$(A) < n$, obtain a special solution:
   Set one free variable $= 1$, and other free variables $= 0$.
   Solve $Rx = 0$ to obtain values of pivot variables.
5. Find special solutions for each free variable.
6. $N(A) =$ space of linear combinations of special solutions.
7. This information is stored in a compact form in:

   Null Space Matrix: Its columns are Special solutions.

Linear Combinations in $N(A)$

Example: $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$, $x = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $y = \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

are in $N(A)$. Check that the following vectors are in $N(A)$:

$x + y = \begin{pmatrix} -4 & 1 & -1 & 1 \end{pmatrix}^T$,

$-3 \cdot x = \begin{pmatrix} 6 & -3 & 0 & 0 \end{pmatrix}^T$.

Remark: Let A be an $m \times n$ matrix and $u, v \in \mathbb{R}$.

- $\boxed{N(A) \text{ contains vectors from } \mathbb{R}^n}$.
- If $x, y \in N(A)$ i.e. $Ax = 0$ and $Ay = 0$, then

$A(ux + vy) = u(Ax) + v(Ay) = 0$ i.e. $ux + vy \in N(A)$.

i.e. a linear combination of vectors in $N(A)$ is also in $N(A)$.

$\boxed{\text{Thus } N(A) \text{ is \textit{closed under} linear combinations}}$.