PH108

Lecture 13: Electric potential at large distance: Multipole expansion

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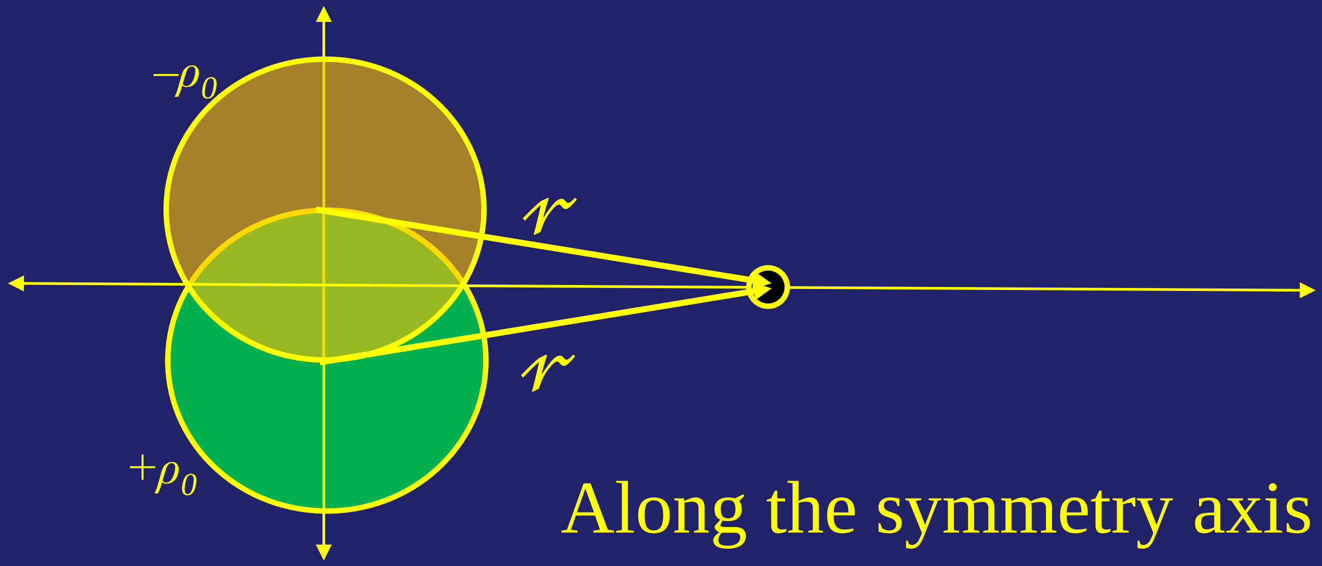
Supplementary reading: Griffiths, Section 3.4

Problem: determine the potential outside due to two intersecting spheres

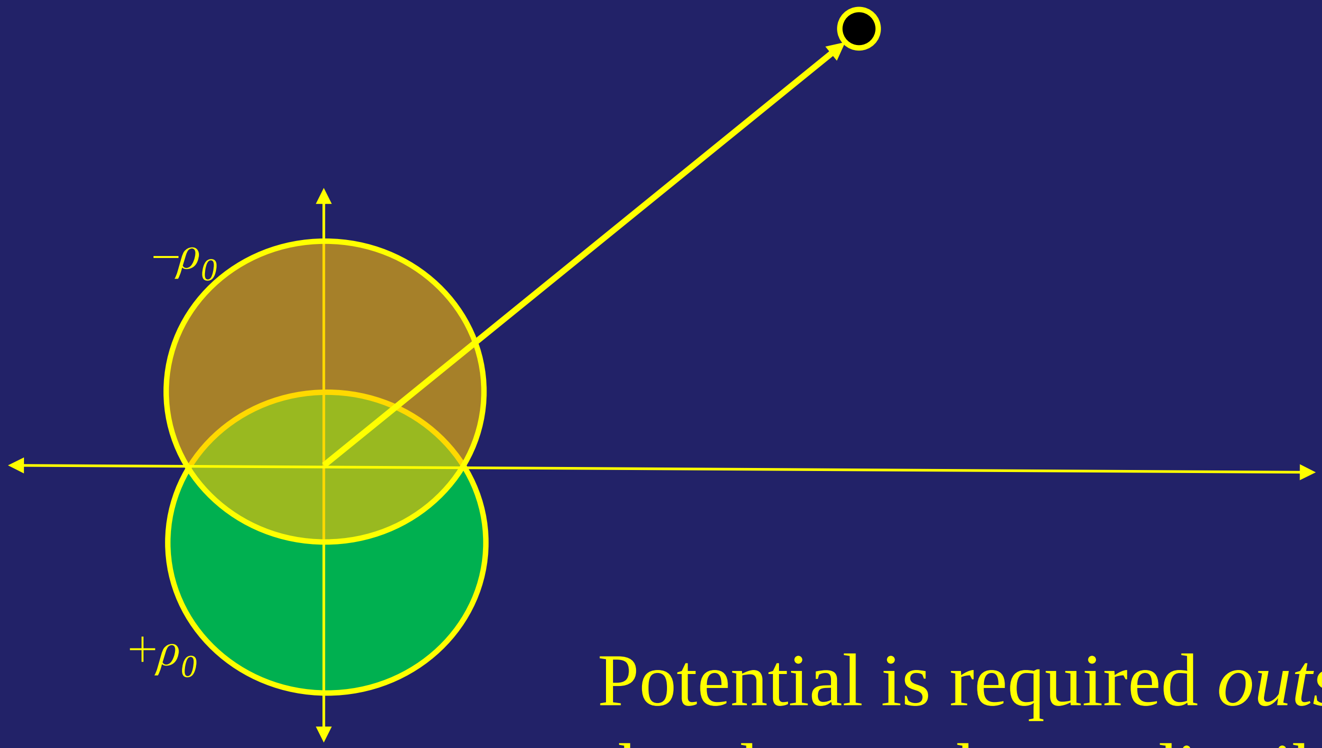
$$\int \rho_o = q$$



Along the symmetry axis is easy:

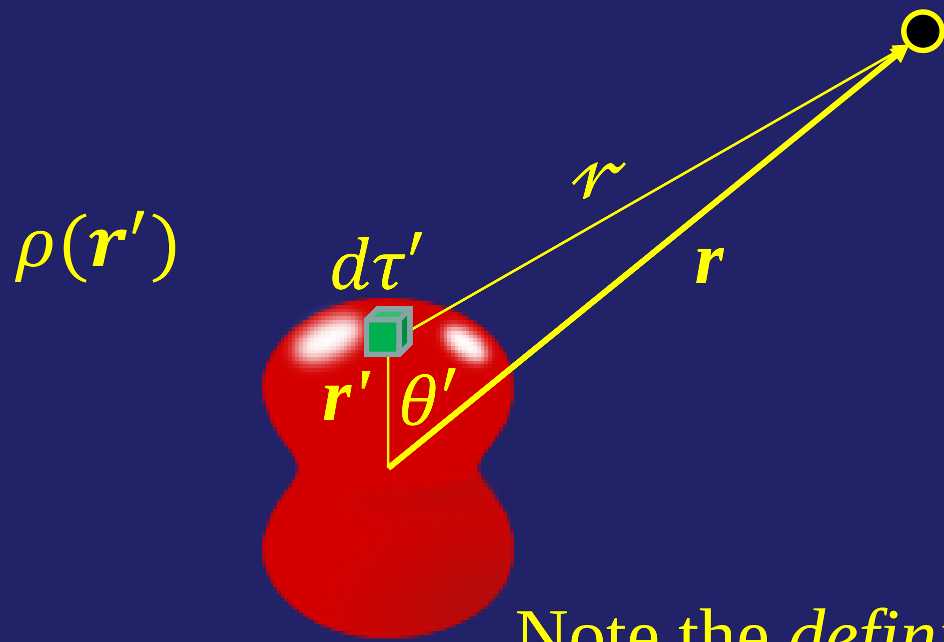
$$\frac{1}{4\pi\epsilon_0}\left(-\frac{q}{\mathcal{r}}+\frac{q}{\mathcal{r}}\right)=0$$

Problem: determine the potential due to two intersecting spheres *everywhere*



Potential is required *outside* the charge charge distribution

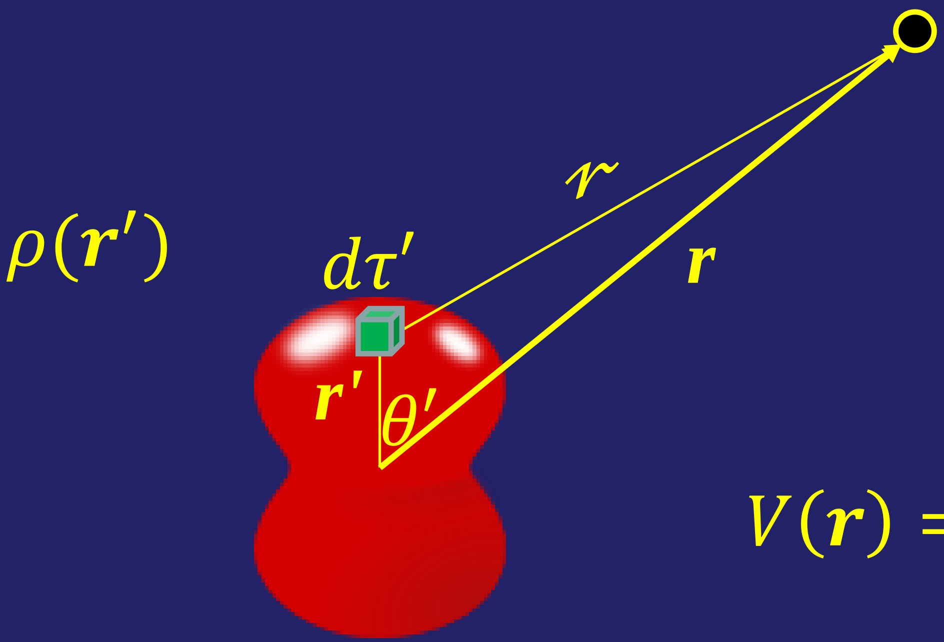
Problem: determine the potential due to *arbitrary* charge *everywhere*



<u>Note the *definition* of coordinate system:</u>

- Origin is taken *inside* ρ
- Polar axis is taken at $d\tau'$
- We will integrate over $d\tau'$
- No φ symmetry is assumed

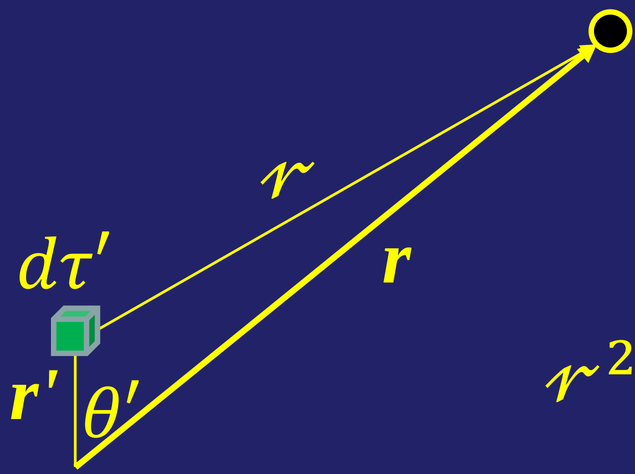
Apply superposition principle



$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\boldsymbol{r}')d\tau'$$

$\mathscr{r}$ changes as you roam over $d\tau'$

Express $\mathscr{r}$ integrable in the integrand



$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'cos\theta'$$

We want $r \gg r'$ so: $$\mathscr{r}^2 = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}cos\theta'\right]$$

$$\mathscr{r} = r\sqrt{1+\delta}$$

$$\delta = \frac{r'}{r}\left(\frac{r'}{r} - 2cos\theta'\right)$$

$\mathscr{r}$ evaluates to a sum of Legendre polynomials

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\delta)^{-\frac{1}{2}} = \frac{1}{r}\left(1 - \frac{1}{2}\delta + \frac{3}{8}\delta^2 - \frac{5}{16}\delta^3 + \ \ldots\right)$$

$$\delta = \frac{r'}{r}\left(\frac{r'}{r} - 2cos\theta'\right)$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1 + \left(\frac{r'}{r}\right)cos\theta' + \left(\frac{r'}{r}\right)^2 \frac{1}{2}(3\cos^2\theta' - 1) + \left(\frac{r'}{r}\right)^3 \ldots\right.$$

$$\boxed{\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(cos\theta')}$$

Potential at (r,θ) is a sum of $\frac{1}{r^{n+1}}$ terms

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\theta')\rho(\boldsymbol{r}')d\tau'$$

Monopole $$\frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\boldsymbol{r}')d\tau'$$

Dipole $$+\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\theta'\rho(\boldsymbol{r}')d\tau'$$

Quadrupole $$+\frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\boldsymbol{r}')d\tau'$$

Potential at (r,θ) is a sum of INFINITE $\frac{1}{r^n}$ terms

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\theta')\rho(\boldsymbol{r}')d\tau'$$

We started with origin inside ρ

We did not assume any φ symmetry

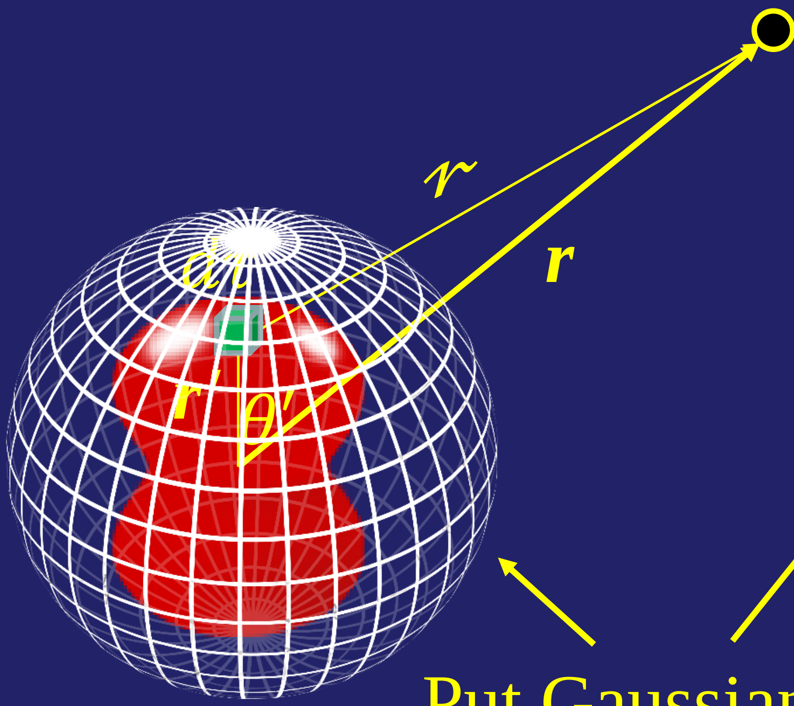
The final expression is *exact* for $r \gg r'$

it depends on sum of powers of $\frac{1}{r}$

r is distance from origin

Consider the first term for *n=0*

Monopole $$\frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\boldsymbol{r}')d\tau'$$



$$\int \rho(\boldsymbol{r}')d\tau' = Q$$

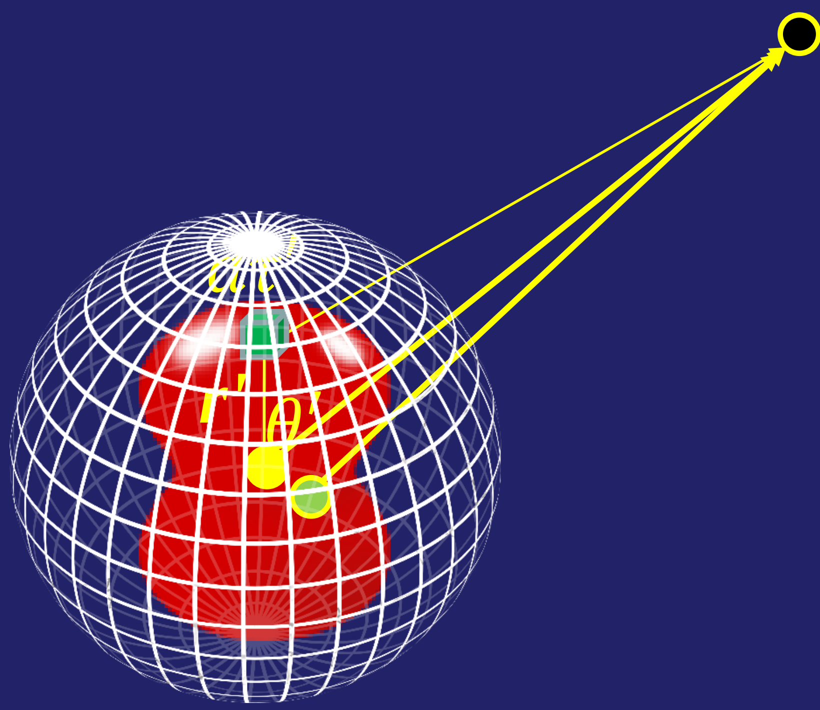
$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

Put Gaussian sphere to enclose $\rho(r)$,

Then *V(r)* at large *r* is what we expect from Gauss Law

Question:

Monopole $\dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r}\int \rho(\mathbf{r}')d\tau'$ $\qquad \int \rho(\mathbf{r}')d\tau' = Q$



If the origin is <u>different</u>

The potential is:

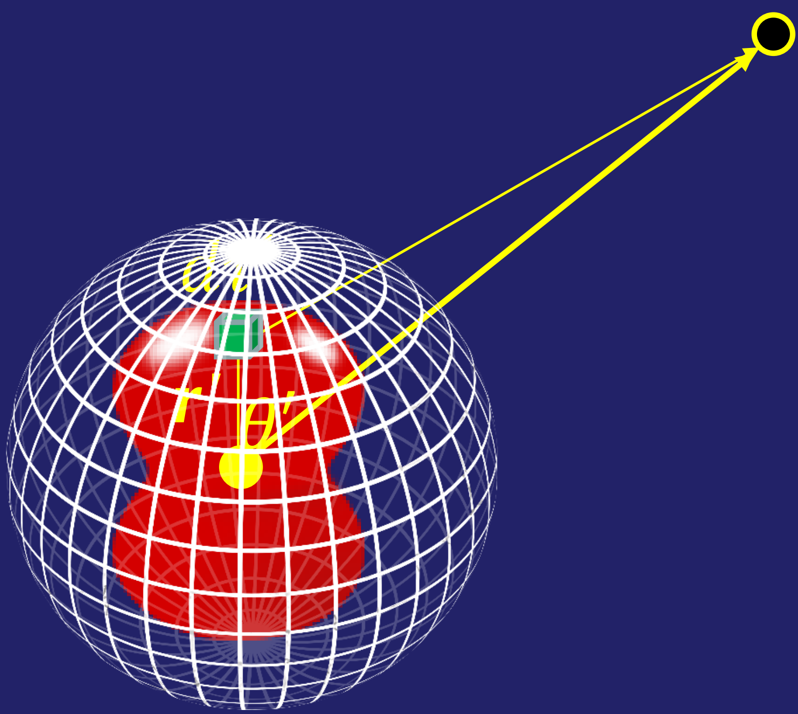
A) $V(r) = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r}$
B) $V(r) = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}$
C) Something else

Depends on $\rho(r)$: eg higher multipole moments will appear for point charge

Question: Net charge is 0

IF $\int \rho(\boldsymbol{r}')d\tau' = 0$

The potential is:

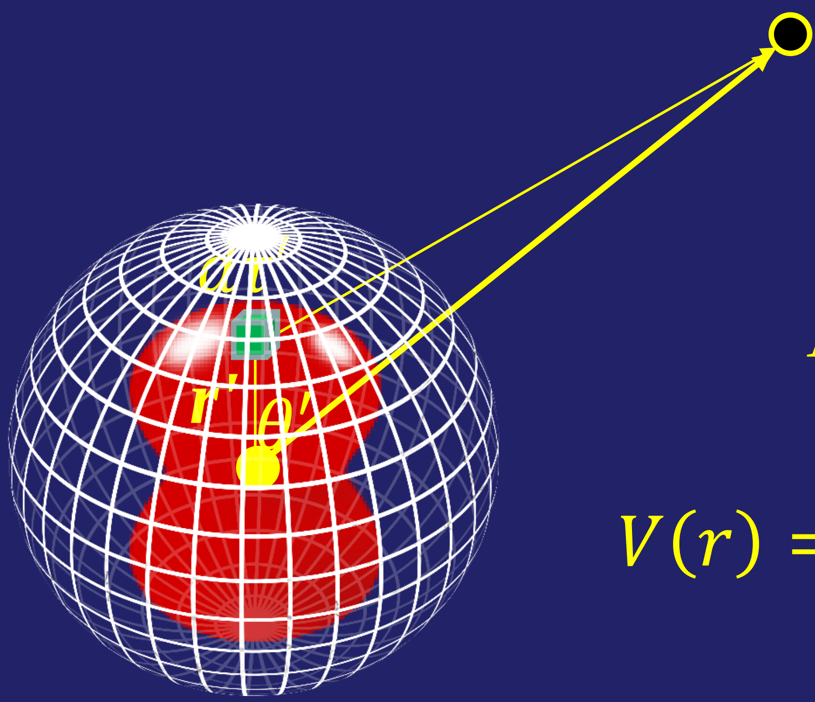


A) $V(r) \sim \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^2}$
B) $V(r) \sim \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}$
C) Something else

Question: Net charge is 0

IF $\int \rho(\boldsymbol{r}')d\tau' = 0$

The potential is:



dominated by next order

term $\frac{1}{r^2}$

A)

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\underbrace{\int r'cos\theta\rho(r')d\tau'}_{\text{Dipole moment}}$$

Logic thread of today's lecture

Potential due to arbitrary charge distribution at large r is to be evaluated

The exact solution is a sum over $n = 0$ *to* ∞
of charge distribution integrated with n^{th} Legendre Poly
each term is weighted with $\frac{1}{r^{n+1}}$

Monopole $\sim \frac{1}{r}$ Dipole $\sim \frac{1}{r^2}$ Quadrupole $\sim \frac{1}{r^3}$ ...

Choice of origin makes a difference!