MA-108 Ordinary Differential Equations

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D1 - Lecture 15

1. $L(1) = \frac{1}{s}, \quad L(t) = \frac{1}{s^2}, \quad L(t^n) = \frac{n!}{s^{n+1}}, \; s > 0.$
2. $L(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}, \quad L(\cos \omega t) = \frac{s}{s^2 + \omega^2}, \; s > 0.$
3. $L(\sinh \omega t) = \frac{\omega}{s^2 - \omega^2}, \quad L(\cosh \omega t) = \frac{s}{s^2 - \omega^2}, \; s > |b|.$
4. $L(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}, \; s > a.$
5. $L(e^{at} \sin \omega t) = \frac{\omega}{(s-a)^2 + \omega^2}, \; s > a.$
6. $L(e^{at} \cos \omega t) = \frac{s-a}{(s-a)^2 + \omega^2}, \; s > a.$
7. $L(e^{at} \sinh bt) = \frac{b}{(s-a)^2 - b^2}, \; s > a + |b|.$
8. $L(e^{at} \cosh bt) = \frac{s-a}{(s-a)^2 - b^2}, \; s > a + |b|.$

Inverse Laplace Transform

If $L(f(t)) = F(s)$ is the Laplace transform of f, then we say f is an **inverse Laplace transform** of F, write $f = L^{-1}(F)$.

In order to solve an IVP using Laplace transform, we need to find inverse Laplace transforms. The formula for inverse Laplace transform uses complex function theory.

$$L^{-1}(F(s)) = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} e^{T(\sigma + i\zeta)} F(\sigma + i\zeta)\, d\zeta,$$

where σ is suitably defined. In this course, we will use the table of Laplace transform to find inverse transform.

Linearity Property: If $F_1, \ldots, F_r$ are Laplace transforms and $c_i \in \mathbb{R}$, then

$$L^{-1}(c_1 F_1 + \ldots + c_r F_r) = c_1 L^{-1}(F_1) + \ldots + c_r L^{-1}(F_r).$$

Inverse Laplace Transform

Ex. $L^{-1}\left(\dfrac{1}{s^2-1}\right) = \sinh t,\ L^{-1}\left(\dfrac{s}{s^2+9}\right) = \cos 3t.$

Ex.

$$L^{-1}\left(\frac{8}{s+5}+\frac{7}{s^2+3}\right) = L^{-1}\left(\frac{8}{s+5}\right) + L^{-1}\left(\frac{7}{s^2+3}\right)$$

Note: $L(f) = F \implies L\left(e^{at}f(t)\right) = F(s-a), \quad L(1) = 1/s.$

$$= 8e^{-5t} + \frac{7}{\sqrt{3}} \sin\left(\sqrt{3}t\right).$$

Ex. $L^{-1}\left(\dfrac{3s+8}{s^2+2s+5}\right) = L^{-1}\left(\dfrac{3(s+1)+5}{(s+1)^2+4}\right)$

$$= e^{-t}L^{-1}\left(\frac{3s+5}{s^2+4}\right) = e^{-t}L^{-1}\left(\frac{3s}{s^2+4}\right) + e^{-t}L^{-1}\left(\frac{5}{s^2+4}\right)$$

$$= e^{-t}\left[3\cos 2t + \frac{5}{2}\sin 2t\right].$$

If P, Q are polynomials with deg $P \leq$ deg Q, then L^{-1} of $P(s)/Q(s)$ is found, by finding partial fractions, using **Heaviside Method**.

Ex. (Q is product of distinct linear factors)

Let $F(s) = \dfrac{6+(s+1)(s^2-5s+11)}{s(s-1)(s-2)(s+1)}$. Find $L^{-1}(F(s))$.

The partial fraction of $F(s)$ is of the form

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} + \frac{D}{s+1}.$$

$$A = F(s)s|_{s=0} = \frac{6+(s+1)(s^2-5s+11)}{(s-1)(s-2)(s+1)}|_{s=0} = \frac{17}{2},$$

$$B = F(s)(s-1)|_{s=1} = \frac{6+(s+1)(s^2-5s+11)}{s(s-2)(s+1)}|_{s=1}$$

$$= \frac{6+2.7}{-2} = -10,$$

$$C = F(s)(s-2)|_{s=2} = \frac{6+(s+1)(s^2-5s+11)}{s(s-1)(s+1)}|_{s=2}$$

$$= \frac{6+3.5}{6} = \frac{7}{2},$$

$$D = F(s)(s+1)|_{s=-1} = \frac{6+(s+1)(s^2-5s+11)}{s(s-1)(s-2)}|_{s=-1}$$

$$\frac{6}{-6} = -1.$$

$$L^{-1}\left(F(s)\right) = L^{-1}\left(\frac{17}{2s} - \frac{10}{s-1} + \frac{7}{2(s-2)} - \frac{1}{s+1}\right)$$

$$= \frac{17}{2} + -10e^{t} + \frac{7}{2}e^{2t} - e^{-t}$$

Ex. (Q is power of a linear factor)

Let $F(s) = \dfrac{s^2 - 5s + 7}{(s+2)^3}$. Find $L^{-1}(F(s))$.

The partial fraction of $F(s)$ is of the form

$$F(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}.$$

To find A, B, C, expand the numerator of $F(s)$ in powers of $(s+2)$.

$$\begin{aligned} s^2 - 5s + 7 &= ((s+2)-2)^2 - 5((s+2)-2) + 7 \\ &= (s+2)^2 - 9(s+2) + 21. \end{aligned}$$

Thus $A = 1$, $B = -9$, $C = 21$.

Therefore, $L^{-1}(F(s)) = L^{-1}\left(\dfrac{1}{s+2} - \dfrac{9}{(s+2)^2} + \dfrac{21}{(s+2)^3}\right)$

$$= e^{-2t} L^{-1}\left(\frac{1}{s} - \frac{9}{s^2} + \frac{21}{s^3}\right) = e^{-2t}\left(1 - 9t + \frac{21}{2}t^2\right).$$

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Ex. Let $F(s) = \dfrac{8+3s}{(s^2+1)(s^2+4)}$. Find $L^{-1}(F(s))$.

The partial fraction of $F(s)$ is of the form

$$\frac{8+3s}{(s^2+1)(s^2+4)} = \frac{A+Bs}{s^2+1} + \frac{Cs+D}{s^2+4}.$$

Equate the powers of s in

$$8+3s = (A+Bs)(s^2+4) + (C+Ds)(s^2+1)$$

and solve to get A, B, C, D.

We have a simpler method in this particular case, here Q is a polynomial in s^2, put $x = s^2$ in

$$\frac{1}{(x+1)(x+4)} = \frac{1}{3}\left(\frac{1}{x+1} - \frac{1}{x+4}\right)$$

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

Hence

$$F(s) = \frac{8+3s}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{8+3s}{s^2+1} - \frac{8+3s}{s^2+4}\right)$$

Therefore,

$$L^{-1}(F(s)) = L^{-1}\left(\frac{8}{3(s^2+1)} + \frac{s}{s^2+1} - \frac{8}{3(s^2+4)} - \frac{s}{s^2+4}\right)$$

$$= \left(\frac{8}{3}\sin t + \cos t - \frac{4}{3}\sin 2t - \cos 2t\right).$$

Laplace transform of Derivatives

Our goal is to apply Laplace transforms to differential equations. So we want to know the Laplace transform of derivative of a function. Consider

$$\int_0^T e^{-st} f'(t)\, dt = f(t)e^{-st}\Big|_0^T - \int_0^T (-s)e^{-st} f(t)\, dt$$

$$= f(T)e^{-sT} - f(0) + s\int_0^T e^{-st} f(t)\, dt$$

If f is of exponential order s_0, then as $T \to \infty$, $f(T)e^{-sT} \to 0$ and $\int_0^\infty e^{-st} f'(t)\, dt = L(f)$.

Thus we have the following theorem.

Theorem

Let f be continuous on $[0, \infty)$ and of exponential order s_0. Let f' be piecewise continuous on $[0, \infty)$. Then the Laplace transform for f' exists for $s > s_0$ and is given by

$$L(f') = sL(f) - f(0)$$

We do not need f' to be of exponential order.

Proof : If f' was continuous on $[0, \infty)$, the proof is done on last slide. If f' is only piecewise continuous with $t_1 < t_2 < \ldots < t_n$ being the discontinuities in $[0, T]$, then

$$\int_0^T e^{-st} f(t)\, dt = \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} e^{-st} f'(t)\ dt$$

$$= \sum_{i=1}^{n} \left[f(t)e^{-st}\Big|_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} (-s)e^{-st} f(t)\ dt \right]$$

$$= f(t_n)e^{-st_n} - e^{-st_0}f(t_0) + s\int_{t_0}^{t_n} e^{-st}f(t)$$

Noting that $t_0 = 0$ and $t_n = T$ and allowing $T \to \infty$, we get

$$L(f') = sL(f) - f(0)$$

Ex. Let us compute $L(\cos\omega t)$ using that

$$L(\sin\omega t) = \frac{\omega}{s^2+\omega^2}.$$

For $f(t) = \sin\omega t$, use $L(f') = sL(f) - f(0)$. Then

$$\begin{aligned} L(\omega\cos\omega t) &= s\frac{\omega}{s^2+\omega^2} - 0 \\ \omega L(\cos\omega t) &= s\frac{\omega}{s^2+\omega^2} \\ L(\cos\omega t) &= \frac{s}{s^2+\omega^2} \end{aligned}$$

Q. How does this help us solve initial value problems? Consider the ODE

$$y' + y = 0, \quad y(0) = 5.$$

We already know that the solution is given by $y = 5e^{-x}$. Let us verify this using Laplace transform.

Let us assume that the given equation has a solution ϕ and it is of exponential order s_0 for some s_0.

Then $L(\phi' + \phi) = L(0) \implies sL(\phi) - \phi(0) + L(\phi) = 0.$

This says that $L(\phi) = \dfrac{5}{s+1}$.

Applying inverse Laplace transform, we get that $\phi(x) = 5e^{-x}$.

Remark. Solving IVP with Laplace transform requires initial conditions at $t = 0$.

We have have the following result about $L(f^{(n)})$.

Theorem

Let $f,\ f',\ \ldots, f^{(n-1)}$ be continuous and $f^{(n)}$ be piecewise continuous on $[0,\infty)$. Let $f,\ f',\ \ldots, f^{(n-1)}$ be of exponential order s_0 for some s_0. Then the Laplace transforms of $f,\ f',\ \ldots, f^{(n-1)},\ f^{(n)}$ exists and

$$L(f^{(n)}) = s^n L(f) - f^{(n-1)}(0) - s f^{(n-2)} - \ldots - s^{n-1} f(0).$$

We do not need that $f^{(n)}$ be of exponential order.

Ex: Consider $y'' + 4y = 3\sin t,\ \ y(0) = 1,\ \ y'(0) = -1.$

We know this equation has a unique solution ϕ on $\mathbb{R}$. Assume it is of exponential order $s_0 \geq 0$. Applying Laplace transform on $[0,\infty)$, we get that for all $s > s_0$

$$L(\phi'') + 4L(\phi) = 3\frac{1}{s^2+1}$$

$$
\begin{aligned}
L(\phi'') + 4L(\phi) &= \frac{3}{s^2+1} \\
(s^2L(\phi) - s\phi(0) - \phi'(0)) + 4L(\phi) &= \frac{3}{s^2+1} \\
(s^2+4)L(\phi) - s + 1 &= \frac{3}{s^2+1}
\end{aligned}
$$

$$
\begin{aligned}
L(\phi) = \frac{3}{(s^2+1)(s^2+4)} &+ \frac{s-1}{s^2+4} \\
L(\phi) = \frac{1}{s^2+1} - \frac{2}{s^2+4} &+ \frac{s}{s^2+4}
\end{aligned}
$$

Therefore,

$$\phi(t) = \sin t - \sin 2t + \frac{1}{2}\cos 2t.$$

From uniqueness theorem, this is the solution on all of $\mathbb{R}$.

Ex: Solve $\quad y'' + 2y' + 2y = 1,\ y(0) = -3, y'(0) = 1.$

The equation has a unique solution ϕ defined on all of $\mathbb{R}$. Assume ϕ is of exponential of order s_0. Then for all $s \geq s_0$,

$$\begin{aligned}
L(\phi'') + 2L(\phi') + 2L(\phi) &= L(1) \\
(s^2 L(\phi) - s\phi(0) - \phi'(0)) + 2(sL(\phi) - \phi(0)) + 2L(\phi) &= \frac{1}{s} \\
(s^2 + 2s + 2)L(\phi) - (s+2)\phi(0) - \phi'(0) &= \frac{1}{s} \\
((s+1)^2 + 1)L(\phi) + 3(s+2) - 1 &= \frac{1}{s} \\
L(\phi) = \frac{1 - (3s+5)s}{((s+1)^2+1)s} &= F(s)
\end{aligned}$$

We want to compute $L^{-1}(F(s))$. We use partial fractions.