MA-106 Linear Algebra

M.K. Keshari

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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D1 - Lecture 9

Recall

• A subset $B = \{v_1, \ldots, v_n\}$ of V is basis of V if
(1) it is linearly independent and
(2) Span$(B) = V$.

• Equivalently, B is a basis of V
$\Leftrightarrow$ B is a maximal linearly independent set in V
$\Leftrightarrow$ B is a minimal spanning set of V.

• Every v in V can be uniquely written as a linear combination of $\{v_1, \ldots, v_n\}$.

• A vector space can have many basis. But the number of vector in a basis is independent of the basis chosen.

Dimension of a Vector Space

If $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are basis of V, then $m = n$.

This is called the *dimension* of V.

$$\boxed{\text{dim}(V) = \text{number of elements in a basis of } V.}$$

- $\text{dim}(\{0\}) = 0$.
- $\text{dim}(\mathbb{R}^n) = n$.
- If $\mathbf{L}$ is a line through origin, then $\text{dim}(\mathbf{L}) = 1$.
- If $\mathbf{P}$ is a plane through origin, then $\text{dim}(\mathbf{P}) = 2$.
- A basis for M, the vector space of 2×2 matrices is $\{e_{11}, e_{12}, e_{21}, e_{22}\}$, where

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Ex. Show that they are linearly independent and Span M.

Hence $\text{dim}(M) = 4$.

- Find a basis of the space of $m \times n$ matrices. Their dimension is mn.
- The dimension of the real vector space $\mathbb{C}$ is 2.
- The dimension of the complex vector space $\mathbb{C}$ is 1.
- Let $V = C([0, 2\pi], \mathbb{R})$ and

$W =$ subspace Span$(1, \sin^2 x, \cos^2 x, \sin^4 x)$.
Find a basis of W.

$\sin^2 x + \cos^2 x = 1 \Rightarrow W =$ Span$(\sin^2 x, \cos^2 x, \sin^4 x)$.
L.I. : Assume $c_1 \sin^2 x + c_2 \cos^2 x + c_3 \sin^4 x = 0$.
$x = 0$ gives $c_2 = 0$. Hence $c_1 \sin^2 x + c_3 \sin^4 x = 0$.
Ex. Show that $c_1 = c_3 = 0$. Hence dimension of W is 3.

- Find the dimension of $C(A)$, where A is 3×4 matrix with (i, j)-th entry as $i^2 + j^2$.

1. 140020007 NIKHIL MOTIBHAI SUMERA
2. 140020008 SIDDHANT GANDHI
3. 140020051 AMAN VIJAY
4. 140020052 VAISHNAVI KHANDELWAL
5. 140020062 SATYAM AGNIHOTRI
6. 140020070 DIMPAL
7. 140020085 RISHU KUMAR
8. 140020094 ANAY TRIPATHI
9. 140020117 AKHIL NASSER
10. 140050018 SHREY RAJESH
11. 140050001 MRIDUL SAYANA
12. 140050010 VISHAL MEENA
13. 140050022 KUSHAL BABEL
14. 140050032 SUMAN SWAROOP
15. 140050050 UPPARA RAGHUVEER
16. 140050080 AJMEERA SRINATH NAIK
17. 140050087 ASHNA GAUR

The Four Fundamental Subspaces

Let A be an $m \times n$ matrix. Associated to A, we have four subspaces:

- The column space of A: $C(A) = \{v : Ax = v \text{ is consistent}\}$.
- The null space of A: $N(A) = \{x : Ax = 0\}$.
- The row space of $A = \text{Span}\{A^1, \ldots, A^m\} = C(A^T)$.
- The left null space of $A = \{x : xA = 0\} = N(A^T)$.

Note:

$N(A)$ and $C(A^T)$ are subspaces of $\mathbb{R}^n$, and
$C(A)$ and $N(A^T)$ are subspaces of $\mathbb{R}^m$.

Example: Let $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$.

Find the four fundamental subspaces of A, their bases and dimensions.

For $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$, reduced form $R = \begin{pmatrix} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

$$N(A) = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -2b - 2d \\ b \\ -d \\ d \end{pmatrix} = b \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

$$= \text{Span}\left\{ w_1 = \begin{pmatrix} -2 & 1 & 0 & 0 \end{pmatrix}^T, w_2 = \begin{pmatrix} -2 & 0 & -1 & 1 \end{pmatrix}^T \right\}.$$

w_1, w_2 are linearly independent

$\Rightarrow$ they form a basis for $N(A)$

$\Rightarrow$ $\dim(N(A)) = 2$.

$\boxed{\text{A basis for } N(A) \text{ is the set of special solutions.}}$

$\boxed{\dim(N(A)) = \text{ number of free variables} = n \text{ - rank}(A)}$

The Big Four: $C(A)$

For $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$, reduced form $R = \begin{pmatrix} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Let $A = \begin{pmatrix} v_1 & \cdots & v_4 \end{pmatrix}$. Recall $C(A) =$ Span$\{v_1, v_3\}$
$\{v_1, v_3\}$ are linearly independent, since $(v_1 \ v_3)$ has 2 pivots.
$\Rightarrow \{v_1, v_3\}$ is a basis for $C(A)$.

Observe: 1. v_1 and v_3 correspond to the pivot columns of A.
2. Pivot columns of R contains an identity submatrix.

• In general, since $N(A) = N(R)$, if R_i is a non-pivot column of R, then x_i is a free variable. Take the special solution x with $x_i = 1$ and other free variables $= 0$.

• Then $Rx = 0 = Ax$ gives that i-th column A_i of A is a linear combination of pivot columns of A.

• Therefore, pivot columns of A span $C(A)$. Claim: they are basis of $C(A)$.

• Assume $\text{rank}(A) = r$. If A' is the matrix with pivot columns, then its reduced form R' contains an identity matrix $I_{r\times r}$.

• Therefore, $N(A') = N(R') = 0 \implies$ columns of A' (pivot columns of A) are linearly independent.

$\implies$ pivot columns of A is a basis for $C(A)$.

$\implies \dim(C(A)) =$ no. of pivots of A.

$$\boxed{\text{A basis for } C(A) \text{ is given by the pivot columns of } A.}$$

$$\boxed{\dim(C(A)) = \text{no. of pivots of } A = \text{rank}(A).}$$

Rank-Nullity Theorem: Let A be an $m \times n$ matrix. Then

$$\text{rank}(A) + \dim(N(A)) = n.$$

The Big Four: $C(A^T)$

$C(A^T) = \{$ linear combinations of rows of $A\}$
$= \{$ linear combinations of rows of R: row reduced form of $A\}$
$= \{$ linear combinations of pivot rows of $R\}$,
(since other rows of R are zero).

Pivot rows of R are linearly independent.

e.g. $R = \begin{pmatrix} \mathbf{1} & 0 & 1 & 0 & 1 \\ 0 & \mathbf{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ has 3 pivots.

Clearly first three rows of R are linearly independent, since they contain identity submatrix.

Pivot rows of R: linearly independent $+$ their span is $C(A^T)$.

Hence $\boxed{\text{pivot rows of } R \text{ are a basis of } C(A^T)}$.

Hence $\text{rank}(A^T) = \dim(C(A^T)) =$ no. of pivot rows $R =$ no. of pivots of $A = \text{rank}(A)$.

Therefore $\boxed{\text{rank}(A) = \text{rank}(A^T)}$

e.g. For $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$, $R = \begin{pmatrix} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Therefore a basis for $C(A^T)$ is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\dim(C(A^T)) = 2$.

The Big Four: $N(A^T)$

If A is $m \times n$, then A^T is $n \times m$.

By Rank-Nullity Theorem,

rank$(A^T)+$ dim$(N(A^T)) = m$. Hence:

$$\boxed{\text{dim}(N(A^T)) = m - \text{rank}(A).}$$

E.g. For $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$, rank$(A) = 2$, hence

dim$(N(A^T)) = 3 - 2 = 1$.

Find a basis for $N(A^T)$.

Added after lecture

Some examples of complex vector spaces are
1. $\mathbb{C}^n = \{(z_1, \ldots, z_n) | z_i \in \mathbb{C}\}$ with component wise addition and scalar multiplication.
2. $C([0,1], \mathbb{C})$ complex valued continuous functions on $[0,1]$. Vector addition and scalar multiplication is an in real case.
3. $M(\mathbb{C}) = \{2 \times 2$ matrices with complex entries $\}$. The vector addition and scalar multiplication are component wise.

Ex. 1. A basis for $\mathbb{C}^2$ as a complex vector space is $\{(1,0),(0,1)\}$.
2. A basis for $\mathbb{C}^2$ as a real vector space is
$\{(1,0),(i,0),(0,1),(0,i)\}$.
3. Dimension of $M(\mathbb{C})$ as a complex vector space is 4. Write down a basis.
4. Dimension of $M(\mathbb{C})$ as a real vector space is 8. Write down a basis.

Fact. If V is a complex vector space of dimension n then V is also a real vector space of dimension $2n$.