PH108

Lecture 23:

Maxwell Equations: Electromagnnetic Waves

Pradeep Sarin
Department of Physics

Supplementary reading: Griffiths Sec 9.1, 9.2 (*not* 9.3)

Maxwell Equations : time dependent $\vec{E}, \vec{B}$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Physics motivation: measurable quantities (force) should not depend on observer frame of reference

time dependent $\vec{E}, \vec{B}$ **with NO $\boldsymbol{\rho}, \vec{\boldsymbol{J}}$**

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

These are FOUR coupled first order differential equations

Use $\vec{\nabla} \times (\vec{\nabla} \times \vec{X}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{X}) - \nabla^2 \vec{X}$ to uncouple them

2nd order differential equation for $\vec{E}$

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial(\vec{\nabla} \times \vec{B})}{\partial t} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2 \vec{E};$$

$$\boxed{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$$

2nd order differential equation for $\vec{E}$ and $\vec{B}$

These are called wave equations

LHS has space $[L]^{-2}$

RHS has time $[T]^{-2}$

$$[\mu_0 \epsilon_0] = [L^{-2} T^2] \rightarrow \frac{1}{velocity^2}$$

$\vec{E}, \vec{B}$ "waves" "travel" in free space with velocity $v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

$$\boxed{\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}}$$

$$\boxed{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$$

What does it mean: $\vec{E}, \vec{B}$ "travel" ?

$\vec{E}(\vec{r})$ is <u>vector field</u>

For every point $\vec{r}$ in space, there is a vector $\vec{E}$

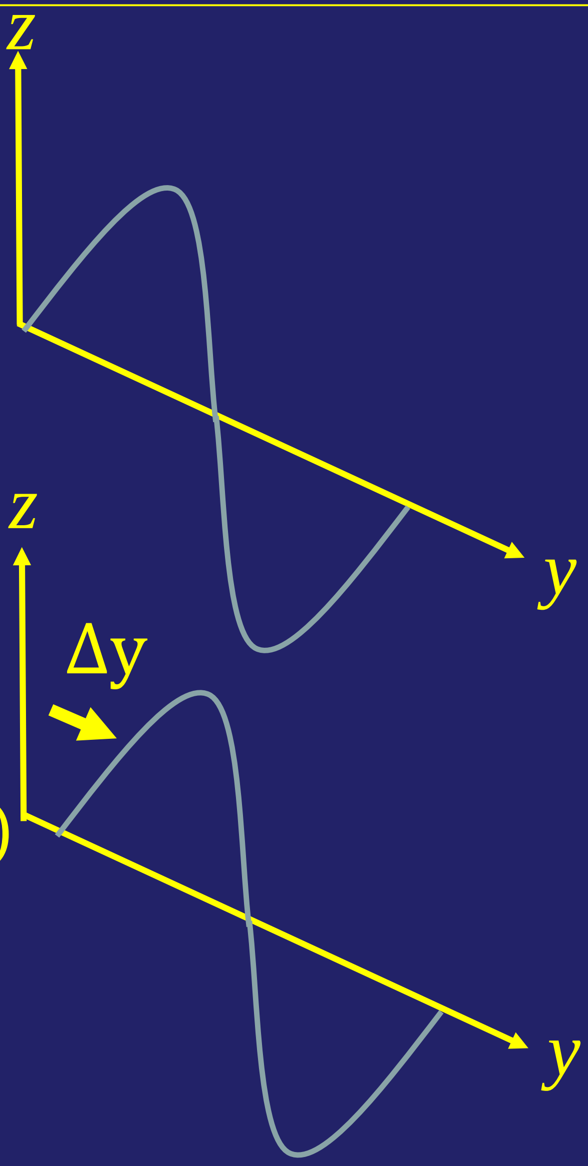
at time t

at time $t + \Delta t$ the vectors $\vec{E}$ are translated by $\Delta\vec{r}$

Velocity of travel is $v = \frac{\Delta\vec{r}}{\Delta t}$

… Similarly $\vec{B}(\vec{r})$ is <u>vector field</u> …

Example: $\vec{E} = \hat{z}\, E_0 \sin(y - vt)$



Electric field is in $\hat{z}$ direction

At $t + \Delta t$:

$$\sin[(y + \Delta y) - v(t + \Delta t)] = \sin(y - vt)$$

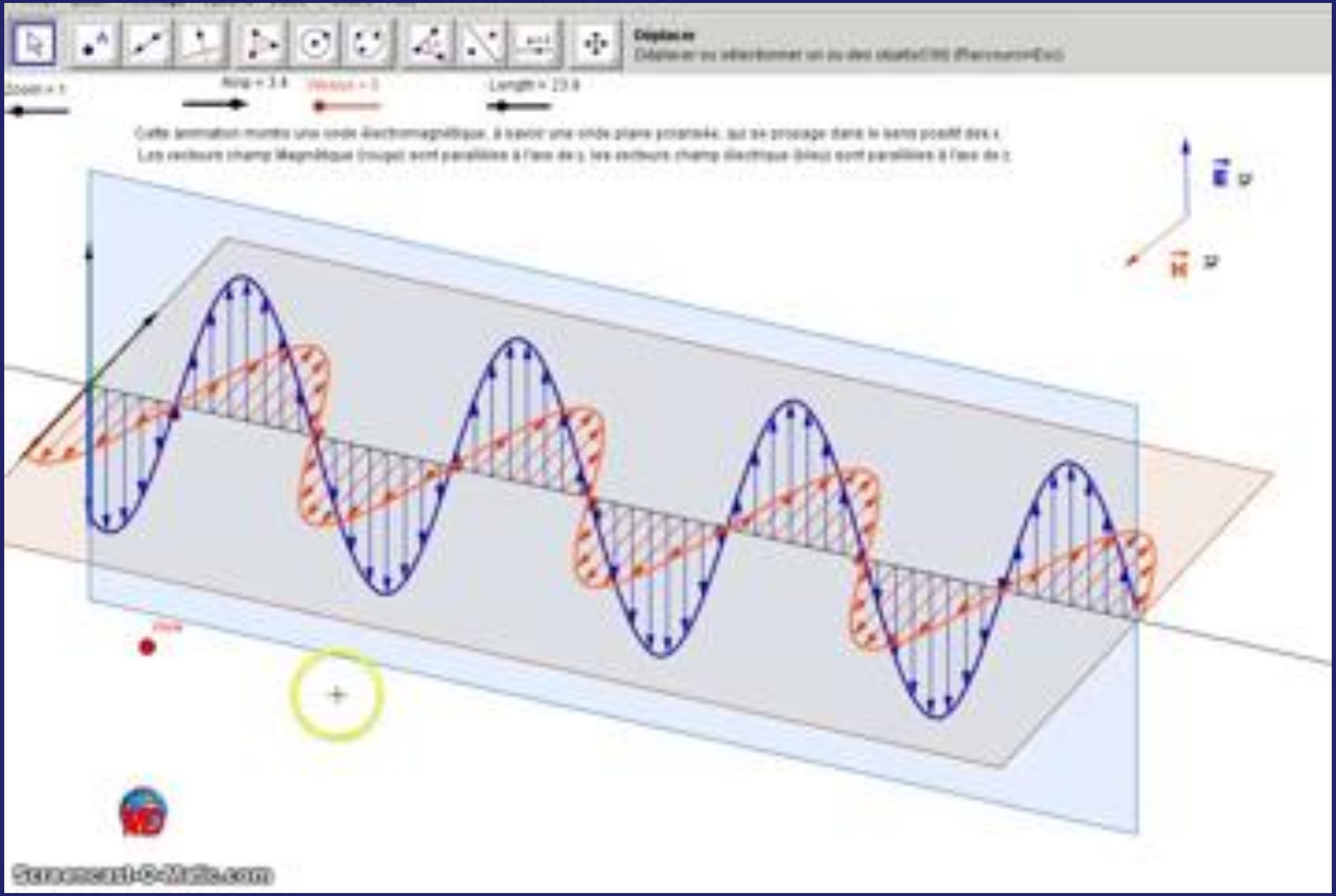
$$IF\ \ \Delta y = v\Delta t$$

It is easy to show that $\vec{E} = \hat{z}E_0 \sin(y - vt)$ satisfies Maxwell's equations

IF $\vec{B} = \hat{x}B_0 \sin(y - vt)$ and $v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

$\vec{E}, \vec{B}$ wave propagation in free space

3D vector visulations: Geogebra : http://tube.geogebra.org



General Solutions to the wave equation

$$\vec{E}(r,t) = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$\vec{k}$ points in the direction of travel

$$\vec{B}(r,t) = \vec{B}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$v = \frac{\omega}{|\vec{k}|}$ is the velocity

$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \equiv c$ in free space : velocity of light in free space

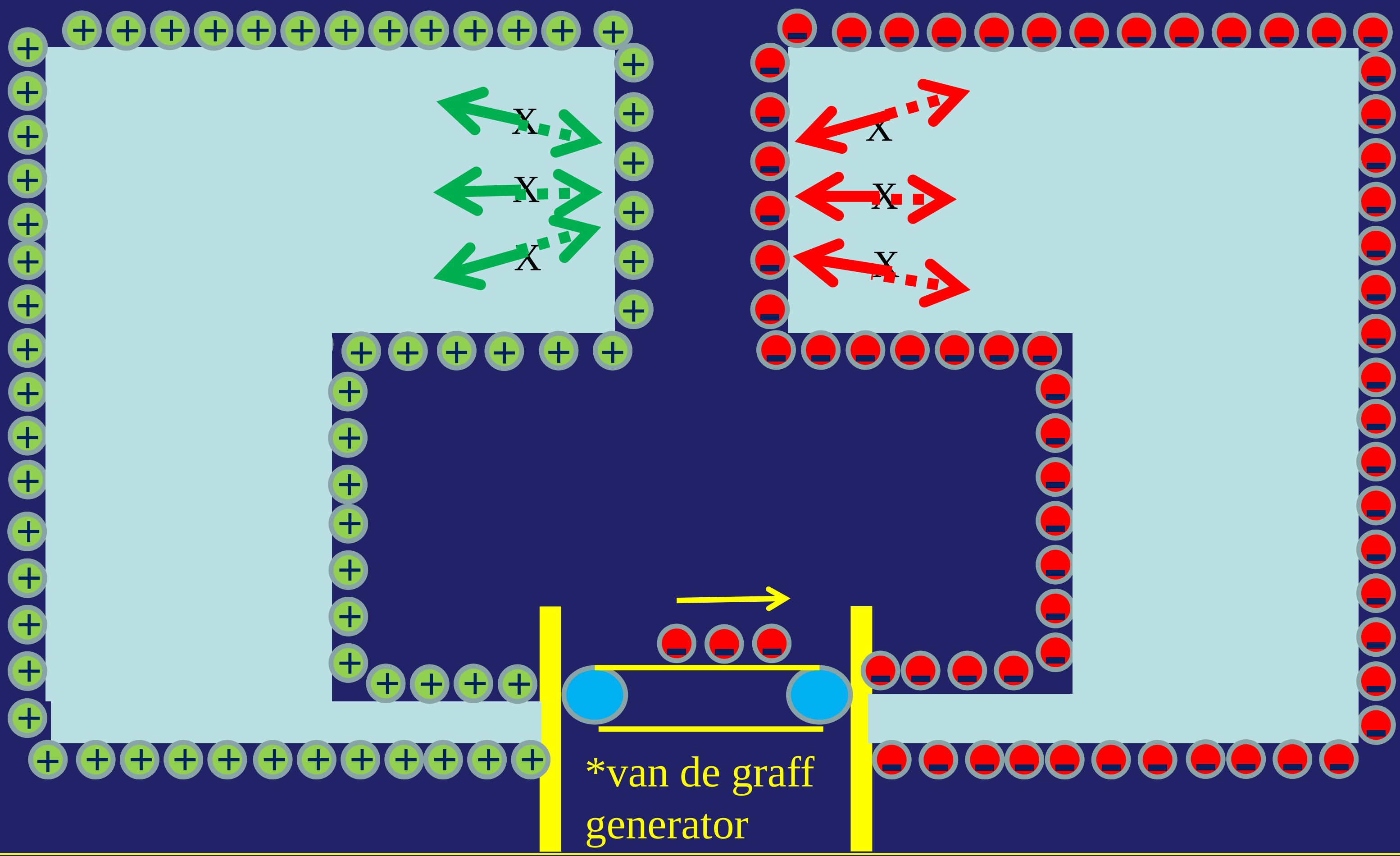
$v = \frac{1}{\sqrt{\mu \epsilon}}$ in a real material $\equiv$ velocity of light in material

... more in Griffiths Sec 9.1, 9.2

Why are we doing wave in the last lecture?

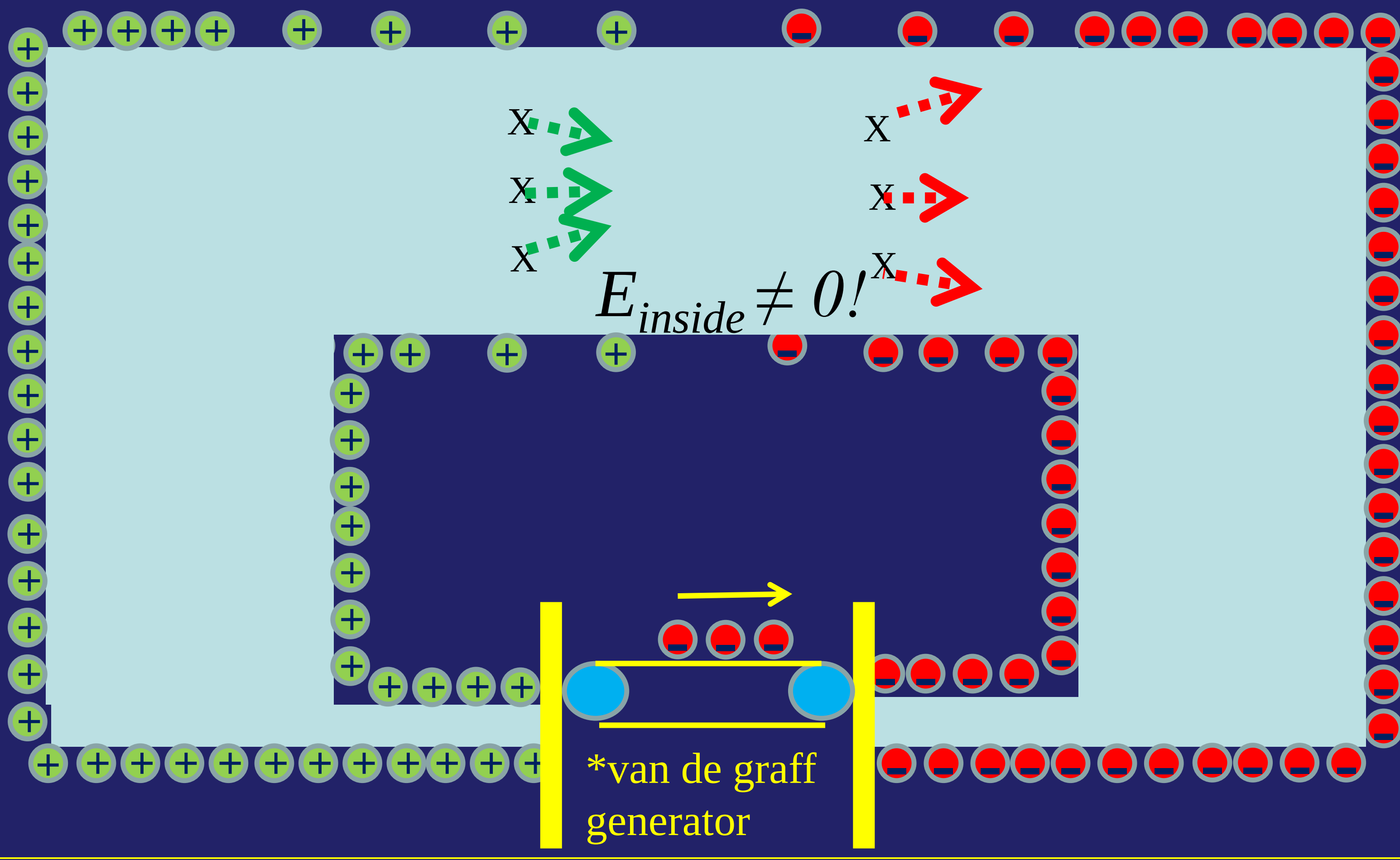
Walk into a room: flick the switch→ Light bulb goes ON

No free charge : EM wave propagation in a conductor



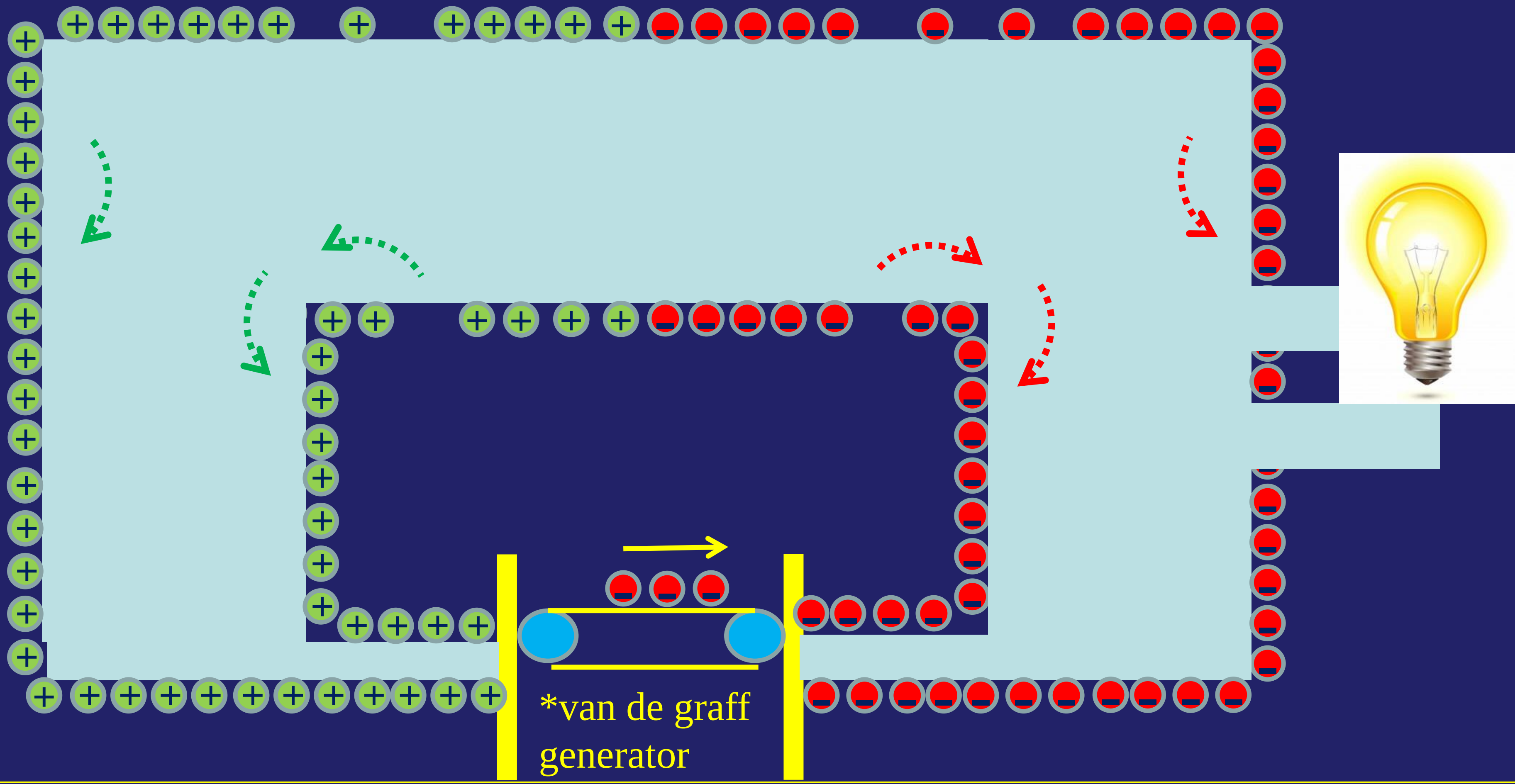
Switch closed

For a short time, $E_{inside} \neq 0$ $\quad E = E(r,t) \sim \delta(r,t)$



Electric field travels in the conductor

$E(r,t) \sim \delta(r,t)$ travels at velocity $v = \frac{1}{\sqrt{\mu\epsilon}}$ $\rightarrow$ bulb turns ON



THANK YOU