MA-108 Ordinary Differential Equations

M.K. Keshari

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

3rd March, 2015
D1 - Lecture 2

Basic Concepts

Recall: an ODE is a functional equation
$F(x, y, y^1, \ldots, y^{(n)}) = 0.$

We defined order, linear and non-linear for an ODE.

A linear ODE of order n is of the form

$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = b(x),$

where $a_0, a_1, \ldots, a_n, b$ are functions of x and $a_0(x) \neq 0$.

An explicit solution to an ODE $F(x, y(x), \ldots, y^{(n)}(x)) = 0$ is a (real valued) function which satisfies the equation in an open interval.

- $L\dfrac{d^2Q(t)}{dt^2} + R\dfrac{dQ(t)}{dt^2} + \dfrac{1}{C}Q(t) = E(t)$. Second order linear ODE
- $\alpha^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial u(x,t)}{\partial t}$ PDE
- $y' + 2y = x^3 e^{-2x}$. first order linear ODE
- $3x^2y^2 + 2x^3y\dfrac{dy}{dx} = 0$ first order non-linear ODE .
- $x^2y + 2x^3\dfrac{dy}{dx} = 0$ first order homogeneous linear ODE .

Solutions to IVP

- An ODE of order n with n no. of initial conditions $y(x_0) = y_0, \ldots, y^{(n-1)}(x_0) = y_{n-1}$, is called an **initial value problem** (IVP).
- For example, our rat-owl problem with initial condition $R(0) = 100$ becomes an IVP.
  The solution $R(t) = Ce^{kt} + 300/k$ satisfying initial condition $R(0) = 100 = C + 300/k$ is
  $R(t) = (100 - 300/k)e^{kt} + 300/k$.
- The graph of a particular solution of an ODE is called a **solution curve**.
- The graph $x^2 + y^2 = 9$ (*) satisfies $yy' + x = 0$.
  Both $y_1(x) = \sqrt{9 - x^2}$ and $y_2(x) = -\sqrt{9 - x^2}$ are solutions of the ODE $yy' + x = 0$ on the intervals $(-3, 3)$.
  (*) is an example of an *implicit solution* of an ODE.

Implicit Solutions

Definition: An **implicit solution** to an ODE $F(x, y(x), \ldots, y^{(n)}(x)) = 0$ is an equation $g(x, y) = 0$ which satisfies the differential equation and gives an explicit solution $y(x)$ of ODE on some interval.

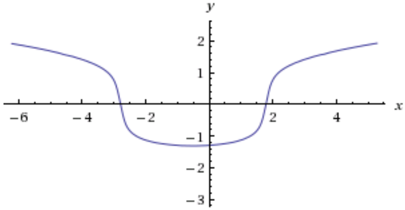
• The graph of an implicit solution is called an **integral curve**.

• Note $x^2 + y^2 + 9 = 0$ formally satisfies the ODE $y'y + x = 0$. But this is not an implicit solution, as it does not give an explicit solution on any interval.

- $y^5 + y - x^2 - x - C = 0$ –(*) is an implicit solution to the ODE $(5y^4 + 1)y' = 2x + 1$. To check this, plot the implicit solution, this will be the integral curve for ODE for each value of C.

Any portion of graph (integral curve) which defines a function is an explicit solution of ODE.

Any differential function satisfying (*) is a solution.



This is the plot of
$y^5 + y - x^2 - x + 5 = 0$

Differential Equation: Direction fields

We will begin our study by discussing first order ODE of the form $y' = f(x, y)$.

We will soon see that f being continuous in a rectangle is sufficient to guarantee the existence of solution. However, there is no general algorithm to solve this ODE.

Suppose that $f(x, y)$ is defined in a *region* (open) $D \subset \mathbb{R}^2$. If $y = \phi(x)$ is a solution curve and (x_0, y_0) is a point on it, then the slope at (x_0, y_0) is $f(x_0, y_0)$.

Solution by software: If we want to plot the solution in a rectangle using computer, then choose many points (x, y) in that rectangle and find the slope $f(x, y)$ at those points. Draw a small line segment at each point with given slope. Finally, plot an approximate solution curve passing through those points.

Solution by human:

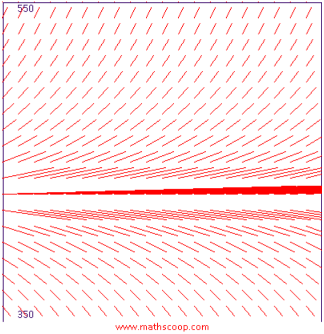
Along the curves $f(x,y) = c$, where c is a constant, the slopes of solution curves are constant. These curves are called the **direction field** or the *slope field* for ODE $y' = f(x,y)$.

Draw direction fields $f(x,y) = c$ for different values of c.

Draw line segments on direction field $f(x,y) = c$ with slope c.

Draw approximate solution curves, passing through those points with given slope.

Direction Field: Examples

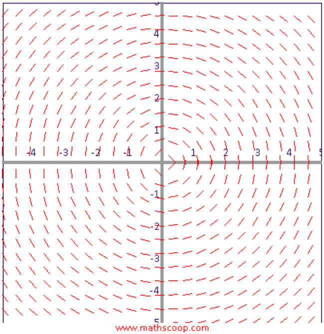


Equation : 0.7y -300

Consider $y' = 0.7y - 300$ from the Rat-Owl problem Then we can see that all the solution curves diverge from the critical point which will give us the equilibrium state for the population.

- $y = \frac{300}{.7}$ is a stable solution.
- If $y(0) > \frac{300}{.7}$, there is a population explosion
- If $y(0) < \frac{300}{.7}$, then population vanishes as t is large.
- We can find solution curves with given initial conditions by sketching a curve along the slopes.

Direction Field: Examples



Equation : -x/y

Consider $y' = -\dfrac{x}{y}$ which we discussed before.

Then the slope field gives us various integral curves to the equation

Note to draw integral curves, draw the direction fields, which is got by plotting $f(x, y) = c$ for various constants c.
Then draw integral curves as curves which intersects each direction field at given slopes

Examples

Example: Find the curve through the point $(1, 1)$ in the xy-plane having at each of its points, the slope $-\frac{y}{x}$.
The relevant ODE is

$$y^1 = -\frac{y}{x}.$$

By inspection,

$$y = \frac{c}{x}$$

is its general solution for an arbitrary constant c; i.e., a family of hyperbolas. The initial condition given is

$$y(1) = 1,$$

which implies $c = 1$. Hence the particular solution for the above problem is

$$y = \frac{1}{x}.$$

Homogeneous/NonHomogeneous

The equation $y' = 0.7y - 300$ is an example of a linear non-homogeneous ODE.

A first order linear ODE can be written in the form
$y' + p(x)y = f(x)$.
If $f = 0$, the ODE is called **homogeneous**.

How can we solve the equation $y' = 0.7y - 300$.

The solution to its complementary homogeneous equation $y' = 0.7y$ is given by $y = Ce^{0.7x}$.

Variation of Parameters

We can solve non-homogeneous linear ODE using the solution to the complimentary homogeneous ODE.
Assume we want to solve $y' + p(x)y = g(x)$.

- Let y_1 be a solution of $y' + p(x)y = 0$.
- Let $y = u\ y_1$. Substitute it in the given equation.

$$\begin{aligned} u'y_1 + uy_1' + p(x)uy_1 &= g(x) \\ \implies u'y_1 &= g(x) \\ \implies u' &= \frac{g(x)}{y_1} \end{aligned}$$

  Note we need to consider y_1 in an interval where it does not take zero values.

- Solve for u by integrating $\dfrac{g(x)}{y_1}$ with respect to x.

Linear Non-homogeneous equation: Example

Variation of Parameter method: The solution of $y' + p(x)y = g(x)$ is $\boxed{uy_1}$, where y_1 is a solution of $y' + p(x)y = 0$, and u is obtained by solving $u' = \dfrac{g(x)}{y_1}$.

Example. Solve $y' = 0.7y - 300$.

$$\begin{aligned} \text{Let} \quad y &= ue^{0.7x} \\ \Longrightarrow u' &= -300\, e^{-0.7x} \\ \Longrightarrow u &= \frac{300}{0.7}\, e^{-0.7x} + C \\ \Longrightarrow u &= 428.6\, e^{-0.7x} + C \end{aligned}$$

Therefore, $y = 428.6 + Ce^{0.7x}$ is a general solution.

Variation of Parameters

Solve $y' + 2xy = xe^{-x^2}$ given that e^{-x^2} is a solution to the equation $y + 2xy = 0$.

Using the variation of parameters method, set $y = ue^{-x^2}$ and solve for $u' = \dfrac{xe^{-x^2}}{e^{-x^2}}$.

Then, $u = \dfrac{x^2}{2} + C$. Hence $y = \dfrac{x^2}{2} + Ce^{-x^2}$.

Remark: Note this is similar to the situation, when we discussed solutions to the matrix equation $Ax = b$ and $Ax = 0$. General solution to the non-homogeneous equation are given by particular solution to the equation $+$ general solution to the homogeneous equation.

Caution: So far we have not said whether the solutions we obtain give us all possible solutions to the given ODE.

Separation of Variables: Example

How did we solve $y' + 2xy = 0$?

Let us rewrite this equation as $\frac{1}{y}\frac{dy}{dx} = -2x$.

We can solve these equations by separating the variables.

Note: we are already assuming that y cannot take zero value on the interval where the solution will be defined.

$$\begin{aligned} \frac{1}{y}\,dy &= -2x\,dx \\ \int \frac{1}{y}\,dy &= \int -2x\,dx \\ \ln|y| &= -x^2 + C \\ y &= Ce^{-x^2} \end{aligned}$$

Separation of Variables: General Method

Let $y' = f(x,y)$ be a differential equation. Rewrite it as $M(x,y) + N(x,y)\,\dfrac{dy}{dx} = 0$. Such an M and N always exist by choosing $M = f$ and $N = 1$.

The equation is said to be **separable** if it is possible to choose M and N such that M is a function in only x and N is a function only in y. Assume ODE is separable.

Let H_1 and H_2 be antiderivatives of M and N respectively. Then $H_1'(x) = M(x)$ and $H_2'(y) = N(y)$. Then our ODE is

$$\begin{aligned}
H_1'(x) + H_2'(y)\,\frac{dy}{dx} &= 0. \\
\frac{dH_1(x)}{dx} + \frac{dH_2(y)}{dy}\,\frac{dy}{dx} &= 0. \\
\frac{d}{dx}[H_1(x) + H_2(y(x))] &= 0.
\end{aligned}$$

Separation of Variables: Example

Thus we get that the solution to the ODE will satisfy the equation

$$H_1(x) + H_2(y(x)) = C.$$

In general, the separable variables method only gives us the implicit solution to the given ODE.

Example. Solve $y' = \dfrac{3x^2 - 1}{3 + 2y}$.