MA-106 Linear Algebra

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D1 - Lecture 8

Instructions for the Quiz

- Short Quiz for 3 marks will be held on Thursday, 22/01/15 from 8:30 - 8:40 PM.
- You should take the quiz in your assigned tutorial section. Not doing so will result in a 0 on the quiz.
- If you are still unaware of your assigned section, please email me.
- No EXTRA time will be given to students who show up late in the Quiz.
- Any such requests, failure to comply with tutors or instructions will result in a 0 on the Quiz.
- Any instances of copying will result in a 0 on the quiz for all the students involved and will be reported to the Academic Disciplinary committee.
- There will be no make ups for the short quizzes.

Linear Independence: Definition

Recall: 1. We defined subspace of a vector space.
2. Span$(v_1, \dots, v_n)$ = smallest subspace containing $v_1, \dots, v_n$.

• Let v and w be vectors in $\mathbb{R}^n$ and $w \neq 0$.
1. If $v = cw$, then Span$\{v, w\}$ = Span$\{w\}$: line through 0.
2. If $v \neq cw$ for any c, then Span(v, w): plane through 0.

Definition. The vectors v_1, $v_2, \dots, v_n$ in a vector space V, are *linearly independent* if

$$\boxed{a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \Rightarrow a_1 = 0,\ a_2 = 0,\ \dots,\ a_n = 0}.$$

Observe: If $V = \mathbb{R}^m$ and $A = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$ is $m \times n$ matrix, then $Ax = x_1v_1 + \ldots + x_nv_n$. Hence

v_1, $v_2, \dots, v_n$ are linearly independent
$\Leftrightarrow Ax = 0$ has only the trivial solution
$\Leftrightarrow N(A) = 0$.

Linear Independence: Examples from $\mathbb{R}^m$

1. The zero vector 0 is not linearly independent. $(1.0 = 0)$
2. If $v \neq 0$, then it is linearly independent.
3. v, w are not linearly independent $\Leftrightarrow$ one is a multiple of the other $\Leftrightarrow$ they lie on the same line through 0.
4. More generally, $v_1, \ldots, v_n$ are not linearly independent
$\Leftrightarrow c_1 v_1 + \ldots + c_n v_n = 0$ with atleast one $c_i \neq 0$
$\Leftrightarrow v_i$ is a linear combination of the other v_j's
$\Leftrightarrow v_i$ is in Span$\{v_j : j = 1, \ldots n, j \neq i\}$.
5. Let $A = \begin{pmatrix} A_1 & \ldots & A_n \end{pmatrix}$ be $m \times n$. Then
rank$(A) = n \Leftrightarrow A$ has n pivots $\Leftrightarrow N(A) = 0$
$\Leftrightarrow A_1, \cdots, A_n$ are linearly independent.
In particular, if A is $n \times n$, then
$A_1, \cdots, A_n$ are linearly independent $\Leftrightarrow A$ has n pivots
$\Leftrightarrow A$ is invertible.

Linear Independence: Examples from $\mathbb{R}^m$

Definition. The vectors $v_1, \ldots, v_n$ are *linearly dependent* if they are not linearly independent, i.e., if

$$\boxed{Ax = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} x = 0 \text{ has non-trivial solutions.}}$$

Example: The vectors v_1, v_2, v_3, v_4 are linearly dependent, where $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$, $v_3 = \begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix}$, $v_4 = \begin{pmatrix} 5 \\ 12 \\ 13 \end{pmatrix}$.

Recall: If $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$,

then $N(A) \neq 0$, so v_1, v_2, v_3, v_4 are not independent.

A non-trivial linear combination is
$2v_1 + (-1)v_2 + 0v_3 + 0v_4 = 0$.

More generally, if $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^m$, then $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is $m \times n$.

If $n > m$, then $\text{rank}(A) \leq m < n \Rightarrow N(A) \neq 0$.

If $(x_1, \ldots, x_n)^T \neq 0 \in N(A)$, then $x_1 v_1 + \ldots + x_n v_n = 0$.
$\Rightarrow v_1, \ldots, v_n$ is linearly dependent.

$\boxed{\text{In } \mathbb{R}^m\text{, any set with more than } m \text{ vectors is linearly dependent.}}$

Examples from other vector spaces

1. Let $V = C([0,1], \mathbb{R})$. Are vectors $\sin x, \cos x \in V$ are linearly independent?
Assume $c_1 \sin x + c_2 \cos x = 0$, where 0 is the zero vector in V.
LHS is zero vector means it takes zero value for all $t \in [0,1]$.
Take $t = 0$, we get $c_2 = 0$.
Take $t = 1$ in $c_1 \sin x = 0$, we get $c_1 = 0$.
Hence $\sin x$ and $\cos x$ are linearly independent vectors in V.

2. $V_1 = C([-1,1], \mathbb{R})$. Show that vectors x and $|x|$ are linearly independent in V_1.

3. $P =$ vector space of polynomial functions : $\mathbb{R} \to \mathbb{R}$. Show that vectors $1, x, x^2$ are linearly independent in P.

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2. 140020017 RAHUL AGARWAL
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9. 140050004 YOGESH KUMAR MEENA
10. 140050013 NAVEEN KUMAR
11. 140050018 SHREY RAJESH
12. 140050030 KONKYANA SAHIL
13. 140050042 MALLELA SAI ARAVIND
14. 140050055 NAVULURI SRI SURYA
15. 140050063 THALLAPALLY SAHITH
16. 140050075 SURENDER SINGH LAMBA
17. 140050078 KARNATI VENKATA NAGA

Basis: Example

Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$, $v_3 = \begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix}$, $v_4 = \begin{pmatrix} 5 \\ 12 \\ 13 \end{pmatrix}$.

Then $v_2 = 2v_1$, and $v_4 = 2v_1 + v_3 \Rightarrow$ Span$\{v_1, v_2, v_3, v_4\}$ = Span$\{v_1, v_3\} \Rightarrow$ we only need v_1 and v_3 to span $C(A)$, v_2 and v_4 are unnecessary.

Observe:

1. Span(v_1) or Span(v_3) is a line, and $C(A)$ is a plane $\Rightarrow$ both v_1 and v_3 are necessary to span $C(A)$, i.e., $\{v_1, v_3\}$ is a *minimal spanning set* for $C(A)$.
2. v_1 and v_3 are linearly independent and span $C(A)$.
3. If $v \in$ Span$\{v_1, v_3\}$, then $v = c_1 v_1 + c_3 v_3$ for $c_1, c_3 \in \mathbb{R}$. Hence v_1, v_3, v are linearly dependent, i.e., $\{v_1, v_3\}$ is a *maximal linearly independent set* in $C(A)$.

This is an example of a *basis* of $C(A)$.

Basis: Definition

A subset $B = \{v_1, \ldots, v_n\}$, of a vector space V, is basis of V, if (1) it is linearly independent and (2) Span$(B) = V$.

Equivalent conditions for a basis:

A subset B of V is a basis of V

$\Leftrightarrow$ B is a maximal linearly independent set in V

$\Leftrightarrow$ B is a minimal spanning set of V.

Remarks/Examples:

- Every vector space V has a basis.
- By convention, the empty set is a basis for $V = \{0\}$.
- $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$.
- $\{e_1, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$, called the *standard basis*.

Basis: Properties

- Basis depends on scalars chosen!

A basis for real vector space $\mathbb{C}$ is $\{1, i\}$ and

a basis for a complex vector space $\mathbb{C}$ is $\{1\}$.

- Let $B = \{v_1, \ldots, v_n\}$: basis for V and $v \in V$.

Span$(B) = V \Rightarrow v = a_1v_1 + \cdots + a_nv_n$ for scalars $a_1, \ldots, a_n$.

This expression for v is $\boxed{\text{unique}}$.

Assume $a_1v_1 + \ldots + a_nv_n = c_1v_1 + \ldots + c_nv_n$

$\implies (a_1 - c_1)v_1 + \ldots + (a_n - c_n)v_n = 0.$

Linear independence of $v_1, \ldots, v_n$

$\implies a_i - c_i = 0$ for all i

$\implies a_i = c_i$ for all i.

Let $B = \{v_1, \ldots, v_n\}$: basis for V

$$\boxed{\text{Every } v \text{ in } V \text{ can be uniquely written}}$$
$$\boxed{\text{as a linear combination of } \{v_1, \ldots, v_n\}.}$$

Q: Is a basis itself unique? **A:** No.

e.g., The columns of any $n \times n$ invertible matrix form a basis for $\mathbb{R}^n$.

e.g. $\{e_1, e_2\}$ is a basis for $\mathbb{R}^2$, so is $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

Find three other examples.

The number of vectors in each basis of $\mathbb{R}^2$ is 2. Not a coincidence!

It will be done in next class without proof

If $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are both basis of V, then $m = n$. This is called the *dimension* of V.

Proof. Assume $n > m$. Since Span$(w_1, \ldots, w_m) = V$, $v_1 = c_1 w_1 + \ldots + c_n w_n$ with atleast one $c_i \neq 0$. After renaming w_i's, we may assume $c_1 \neq 0$.

Span$(v_1, w_2, \ldots, w_n) = V$, since it contains w_1. Claim: $v_1, w_2, \ldots, w_n$ is a basis. To see this, assume $a_1 v_1 + a_2 w_2 + \ldots + a_n w_n = 0$ with some $a_i \neq 0$. If $a_1 = 0 \implies w_2, \ldots, w_n$ is not linearly independent. If $a_1 \neq 0$ implies Span$(w_2, \ldots, w_n) = V$, a contradiction.

Since $v_1, w_2, \ldots, w_n$: basis of V, $v_2 = c_1 v_1 + c_2 w_2 + \ldots + c_n w_n$. Since $v_2 \neq c_1 v_1$, $c_i \neq 0$ for some $i > 1$. We may assume $c_2 \neq 0$ (after renaming w_i's). Then $v_1, v_2, w_3, \ldots, w_n$ is a basis of V.

Proceed as above, we get $v_1, \ldots, v_m, w_{m+1}, \ldots, w_n$ is a basis of V. This is a contradiction.