MA-106 Linear Algebra

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D1 - Lecture 1

Some Class Policies

EVALUATION: 50 marks are waiting to be earned:

Two In-Tutorial Quizzes	2×3 marks
Main Quiz	12 marks
Final	32 marks
Total	**50** marks

ATTENDANCE:

- Attendance in the first week of classes is mandatory.
- Attendance $< 80\% \implies$ you *may* be awarded a DX grade.
- Attendance bonus for everyone: 5% (2.5 marks).
- Absent during random in-class attendance: -1 mark.

ACADEMIC HONESTY: Be honest. Do not violate the academic integrity of the Institute. Any form of academic dishonesty will invite severe penalties.

What is Linear Algebra?

Twitter version: It is the theory of solving simultaneous linear equations in a finite number of unknowns.

Example:

Q: Suppose you have 10, 3, 6 and 1 currency notes respectively of denominations Rs. 10, Rs. 20, Rs. 50 and Rs. 100. You buy food worth Rs. 260 at the Gulmohar cafeteria. How many notes of each denomination will you need to pay?

A: Give names to unknowns: Let t, u, v and w be number of notes of denominations Rs. 10, Rs. 20, Rs. 50 and Rs. 100 respectively that you will pay.
Want to solve: $\boxed{10t + 20u + 50v + 100w = 260.}$
Restrictions: $t \leq 10$, $u \leq 3$, $v \leq 6$ and $w \leq 1$.

$[t, u, v, w] \leq [10, 3, 6, 1]$

Want to solve: $\boxed{10t + 20u + 50v + 100w = 260.}$
A possible solution: $(t, u, v, w) = (1, 0, 3, 1)$.
Is this solution unique? No. e.g., $(10, 3, 2, 0)$.

Are the following solutions permissible:
$(1, 0, 1, 2)$? No. We need $w \leq 1$.
or $(-1, 1, 5, 0)$? Only if the shopkeeper gives back change.
or $(0, 0, 5.2, 0)$? No. We need integer values.

Key note:
In general, we are looking for all possible solutions to the given system, i.e., <u>without any constraints</u>, unlike the introductory example.

An Example

Solve the system: $(1)\;\; x + 2y = 3, \qquad\qquad (2)\;\; 3x + y = 4.$

Elimination of variables:

Eliminate x by $(2) - 3 \times (1)$ to get $y = 1$.

Cramer's Rule (determinant): $y = \dfrac{\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = \frac{4-9}{1-6} = 1$

In either case, back substitution gives $\quad x = 1$

We could also solve for x first and use back substitution for y.

Comparison: For a large system, say 100 equations in 100 variables, elimination method is preferred, since computing the determinants of a 101 matrices of size 100×100 is time-consuming.

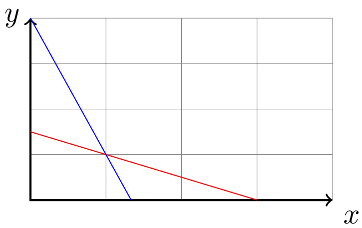
Row method:

$$\boxed{x+2y=3}$$

and

$$\boxed{3x+y=4}$$

represent lines in $\mathbb{R}^2$ passing through $\boxed{(0,3/2) \quad \text{and} \quad (3,0)}$ and through $\boxed{(0,4) \quad \text{and} \quad (4/3,0)}$ respectively.



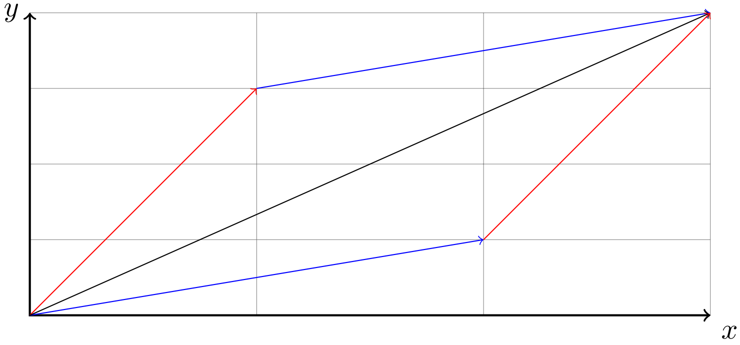
The intersection of the two lines is the unique point $(1,1)$. Hence $x=1$ and $y=1$ is the solution of above system of linear equations.

Geometry of linear equations

Column method: The system is $x\begin{pmatrix}1\\3\end{pmatrix} + y\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}.$

We need to find a *linear combination* of the column vectors on LHS to produce the column vector on RHS.

Geometrically this is same as completing the parallelogram with given directions and diagonal.



Here $x = 1$, $y = 1$ will work.

3 equations in 3 variables

Row method: A linear equation in 3 variables represents a plane in a 3 dimensional space $\mathbb{R}^3$.

Example: (1)

$$\boxed{\text{x+2y+3z=6}}$$

represents the plane passing through
$(0,0,2), \quad (0,3,0), \quad (6,0,0).$

Example: (2)

$$\boxed{\text{x+2y+3z=12}}$$

represents a plane passing through $(0,0,4),(0,6,0),(12,0,0).$

The two planes are parallel to each other.

3 equations in 3 variables

- Solving 3 by 3 system by the **row method** means finding intersection of three planes, say P_1, P_2, P_3.

  This is same as the intersection of a line L (given by P_1, P_2) with plane P_3.

- If the line L does not intersects with the plane P_3, then the linear system has no solution.

- If the line L is contained in the plane P_3, then the system has infinitely many solutions.

  In this case, every point of L is a solution.

- Workout some examples.

Linear Combinations

Column method: Consider the 3×3 system:
x+2y+3z=6, -2x+3y=1, -x+5y+2z=6. Equivalently,

$$x \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + y \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + z \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$$

We want a *linear combination* of the column vectors on LHS which is equal to RHS.
Observe: 1. $x = 1, y = 1, z = 1$ is a solution. **Q:** Is it unique?
2. Since each column represents a vector in $\mathbb{R}^3$ from origin, we can find the solution geometrically, as in the 2×2 case.
Q: Can we do the same when number of variables are > 3?

$\boxed{\text{Solve the system by other techniques.}}$

Gaussian Elimination

Example: $2x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + 2z = 9.$

Algorithm: Eliminate x from last 2 equations by $(2) - 2(1)$, and $(3) + (1)$ to get the *equivalent system:*

$$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 8y + 3z = 14$$

The first *pivot* is 2, second pivot is -8. Eliminate y from the last equation to get an equivalent *triangular system*:

$$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad z = 2$$

Solve this triangular system by *back substitution*, we get

$$z = 2, \quad y = 1, \quad x = 1$$

Observe: This is the only possible solution!

Singular case: No solution

Example: $\mathbf{2}x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + z = 9.$

$\boxed{\text{Step 1}}$ Eliminate x (using the 1st pivot 2) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 8y + 2z = 14$

$\boxed{\text{Step 2}}$: Eliminate y (using the 2nd pivot -8) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 0 = 2.$

The last equation shows that there is no solution,
i.e., the system is *inconsistent*.

Geometric reasoning: In Step 1, notice we get two distinct parallel planes $8y + 2z = 12$ and $8y + 2z = 14$.
They have no point in common.

Note: The planes in the original system were not parallel, but in an equivalent system, we get two distinct parallel planes!

Singular Case: Infinitely many solutions

Example: $\mathbf{2}x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + z = 7.$

$\boxed{\text{Step 1}}$ Eliminate x (using the 1st pivot 2) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 8y + 2z = 12$

$\boxed{\text{Step 2}}$: Eliminate y (using the 2nd pivot -8) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 0 = 0.$

There are only two equations. For every value of z, values for x and y are obtained by back-substitution, e.g, $(1, 1, 2)$ or $\left(\frac{7}{4}, \frac{3}{2}, 0\right)$. Hence the system has infinitely many solutions.

Geometric reasoning: In Step 1, notice we get two parallel planes $-8y - 2z = 12$ and $8y + 2z = 12$.

They give the same plane. Hence we are looking at the intersection of the two planes, $2x + y + z = 5$ and $8x + 2z = 12$, which is a line.

Matrix form for solving $Ax = b$

One good way to write forward elimination is in matrix form.

Let us write the above algorithm as follows.

Note that the last column is the RHS column vector b.

$$\begin{pmatrix} 2 & 1 & 1 & | & 5 \\ 4 & -6 & 0 & | & -2 \\ -2 & 7 & 2 & | & 9 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 & | & 5 \\ 0 & -8 & -2 & | & -12 \\ 0 & 8 & 3 & | & 14 \end{pmatrix} \to$$

$$\begin{pmatrix} 2 & 1 & 1 & | & 5 \\ 0 & -8 & -2 & | & -12 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

Solution is $z = 2$, $y = 1$, $x = 1$.

Remarks

- Pivots can not be zero. (since we need to divide by it)
- If the first pivot (coefficient of x in 1st equation) is zero, then interchange it with next equation so that you get a non-zero first pivot.

  Similar remark is applicable for other pivots.
- Consider system of n equations in n variables.

  If the system has n pivots (non-zero), then system has a unique solution.

  If the system has atmost $n-1$ pivots, then the system will have infinitely many solutions, provided it is consistent.