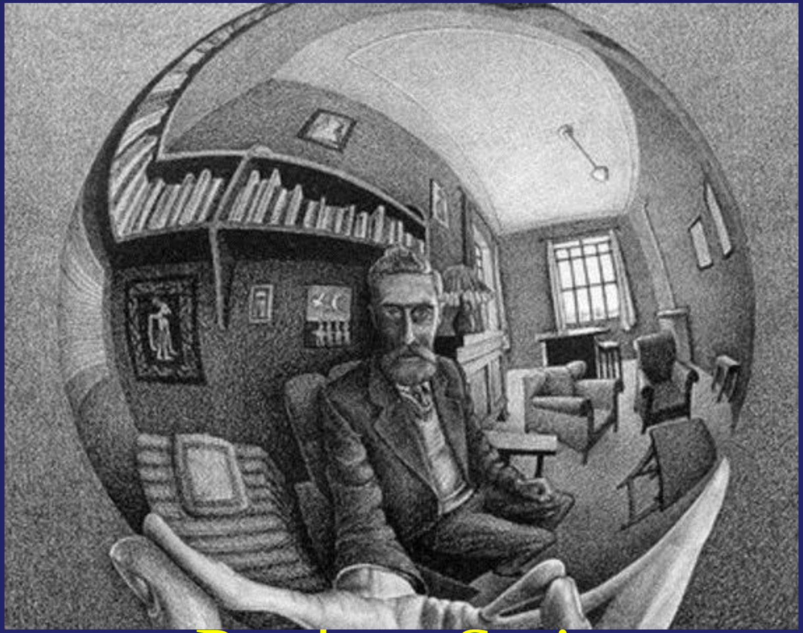
PH108

Lecture 12:

Method of images – spherical conductor



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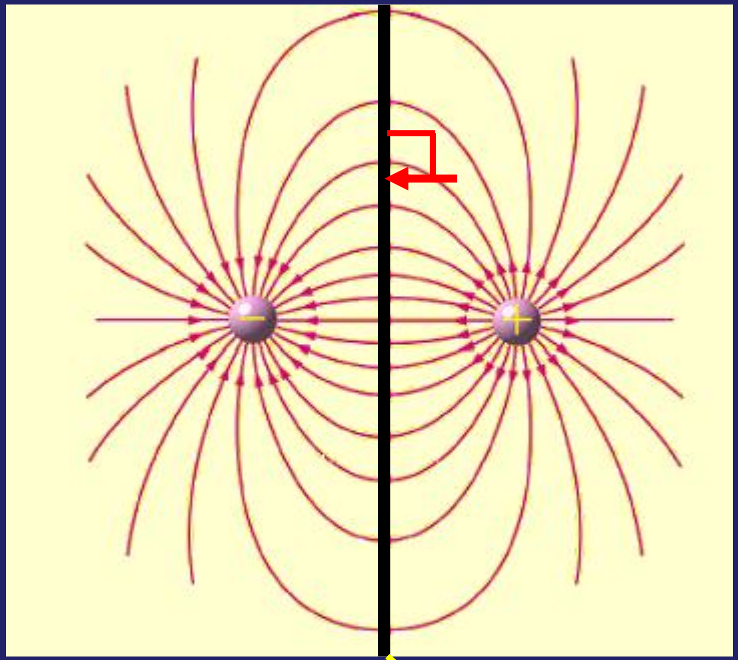
Recall: point charge and a plane conductor

Image charge q_i

$q_i = - q$

$d_i = d$

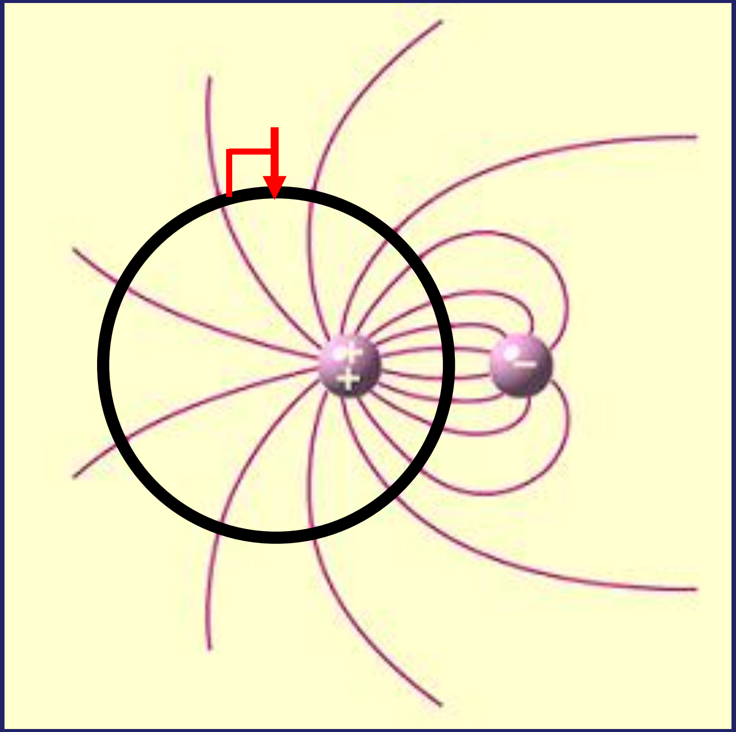


Real charge q

Boundary condition: V=0

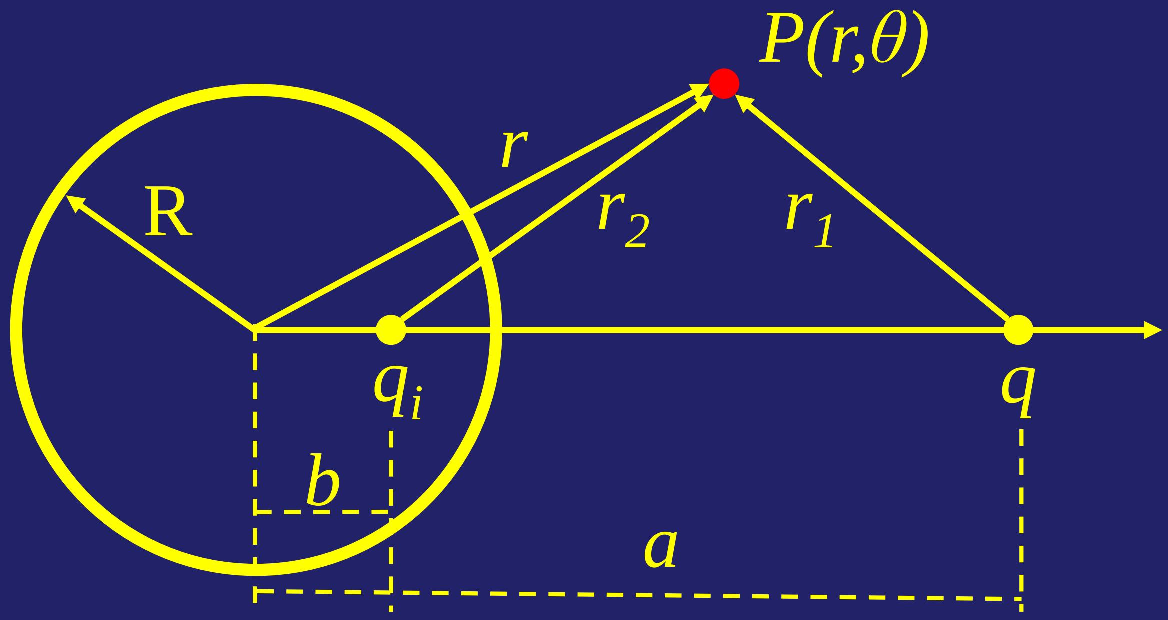
$\vec{E} \perp$ surface at conductor

Image charge changes if the conductor (‘mirror’) is deformed into a sphere



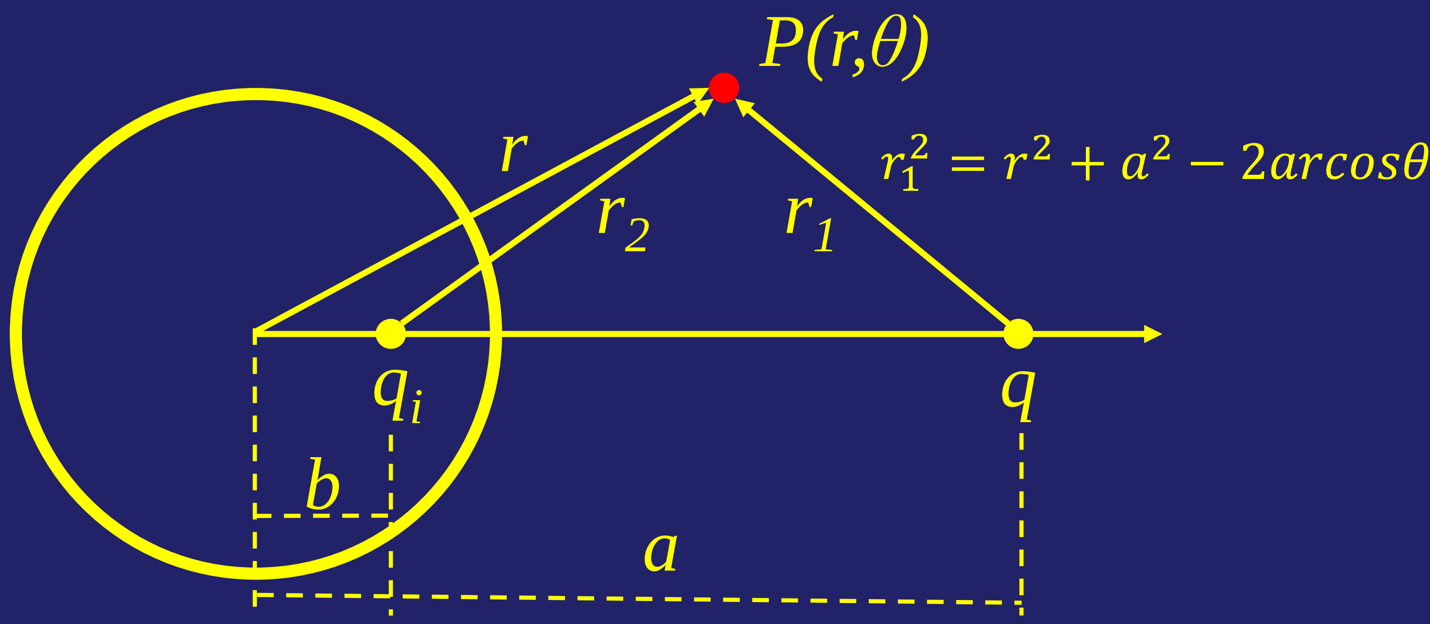
$\vec{E} \perp$ surface at conductor: $q_i \neq -q$ $d_i \neq d$

Calculate $\Phi(r, \theta)$ outside a conducting sphere



$$\Phi_P = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} + \frac{q_i}{r_2}\right)$$

Use triangle trigonometry



$$\Phi_P = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{r^2 + a^2 - 2arcos\theta}} + \frac{q_i}{\sqrt{r^2 + b^2 - 2brcos\theta}}\right)$$

Use Boundary condition $\Phi(r = R, \theta) = 0$

$$\left(\frac{q}{\sqrt{R^2 + a^2 - 2aR\cos\theta}} + \frac{q_i}{\sqrt{R^2 + b^2 - 2bR\cos\theta}}\right) = 0$$

$$q^2(R^2 + b^2 - 2bR\cos\theta) = q_i^2(R^2 + a^2 - 2aR\cos\theta)$$

valid for all θ

$$\rightarrow q^2 bR = q_i^2 aR$$

$$\boxed{b = \frac{R^2}{a}}$$

$$\boxed{q_i = q\sqrt{\frac{b}{a}}}$$

With b and q_i can write $\Phi(r,\theta), \vec{E}(r,\theta)$

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2+a^2-2arcos\theta}} - \frac{R}{\sqrt{r^2a^2+R^4-2aRcos\theta}}\right)$$

$$\vec{E} = -\vec{\nabla}\Phi = -\hat{r}\frac{q}{4\pi\epsilon_0}\left(\frac{-r+acos\theta}{(r^2+a^2-2arcos\theta)^{\frac{3}{2}}} + \frac{R(a^2R \;\; - R^2acos\theta)}{(r^2a^2+R^4-2R^2arcos\theta)^{\frac{3}{2}}}\right)$$

$$\vec{E}_\theta = 0$$

Induced charge on conductor surface:

$$\sigma_{induced}(R,\theta) = \epsilon_0\,\vec{E}(R,\theta)$$

Question

$$\sigma_{induced} = \frac{R^2 - a^2}{(R^2 + a^2 - 2aR\cos\theta)^{\frac{3}{2}}}$$

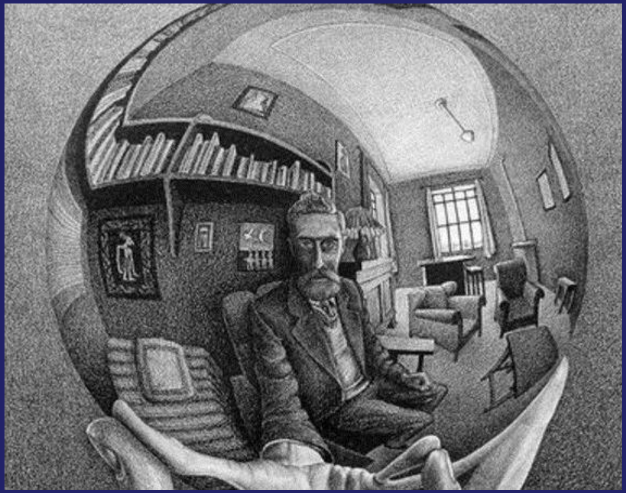
The TOTAL charge induced on the sphere surface is:

A) Less than

B) More than

C) Equal to

$\Bigg\}$ … real charge q?



Induced charge is LESS than real charge

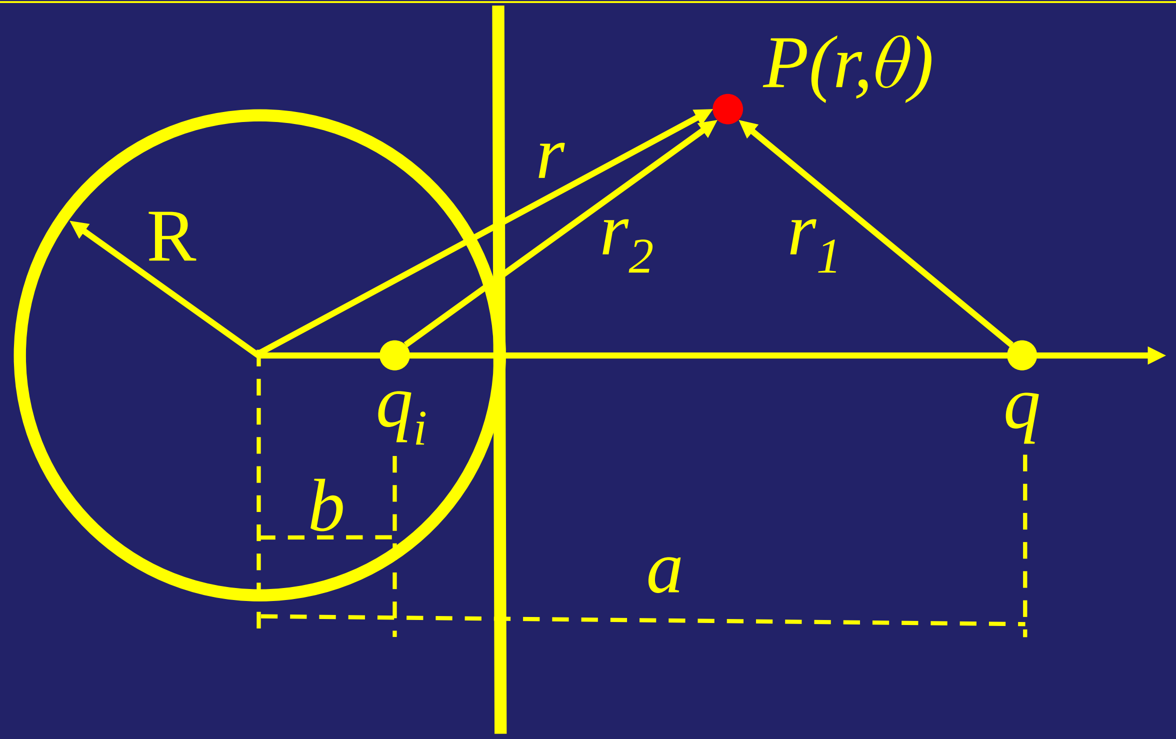
$$\sigma_{induced} = \int_{\theta=0}^{\pi} \frac{R^2 - a^2}{(R^2 + a^2 - 2aR\cos\theta)^{\frac{3}{2}}}$$

A) Less than … real charge q

Image charge: $q_i = q\sqrt{\frac{b}{a}}$

Induced charge: $Q_{induced} = \int \sigma_{induced} = -q\frac{R}{a}$ $a > R,$ $< q$

What happens when we revert to a plane conductor?

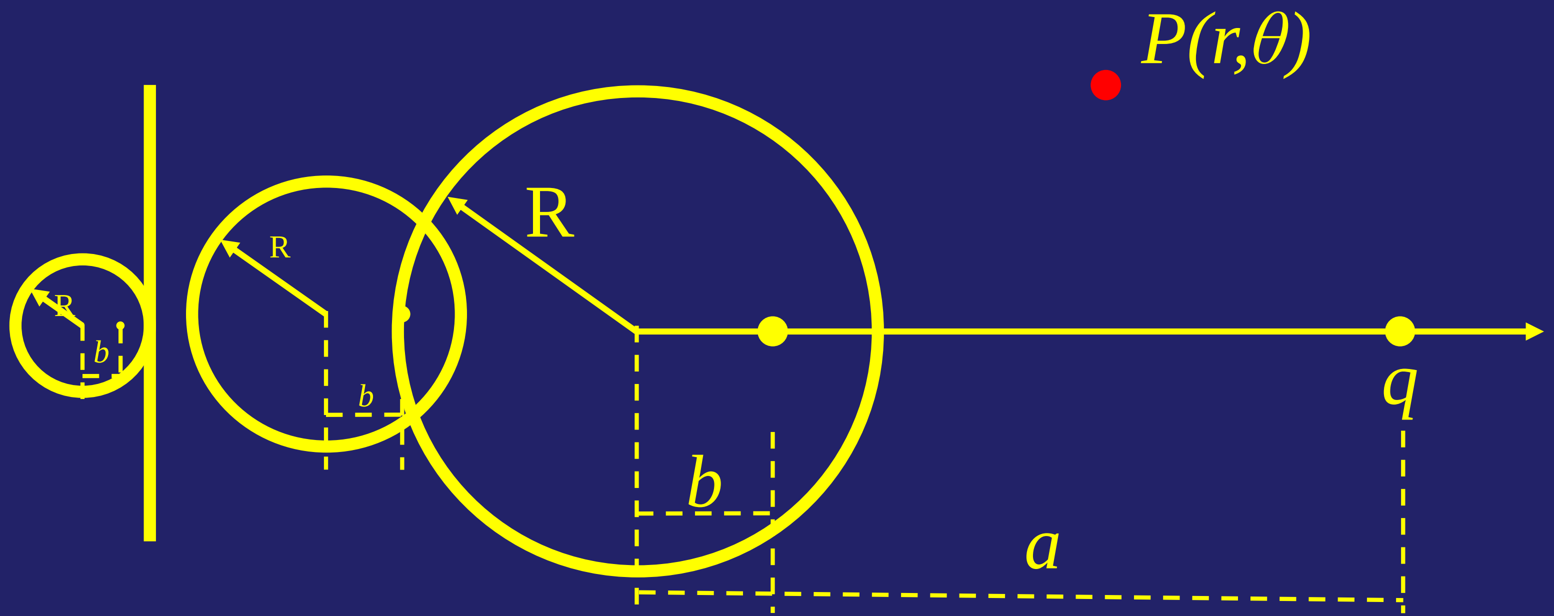


$$b = \frac{R^2}{a}$$

$$Q_{induced} = -q\frac{R}{a}$$

As $R \to \infty$ we *should* get a plane conductor

Must evaluate the sphere → plane carefully



Origin shifts to extreme left as the sphere ‘unfolds’

In the limiting case, the surface *becomes* the origin
→ measure a, b with respect to the surface

Sphere → plane limit works OK

Let $d = (a - R)$ be position of real charge q

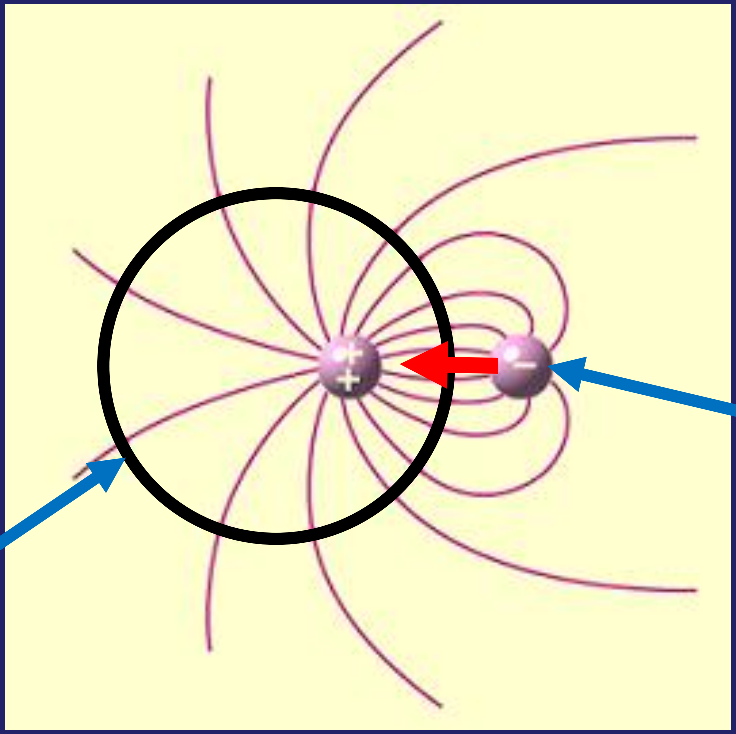
$$Q_{induced} = -q\frac{R}{a} = -q\frac{R}{d+R} \rightarrow -q \;\; as\ R \rightarrow \infty$$

Let $b' = (R - b); b = \frac{R^2}{a}$

$$\rightarrow b' = R - \frac{R^2}{d+R} = \frac{Rd}{d+R} \rightarrow d\ as\ R \rightarrow \infty$$

What is the practical use of method of images?

This is a healthy cell with charge



This is a pathogen with charge

$\nabla^2 \Phi = 0 \rightarrow$ no maxima, minima of Φ

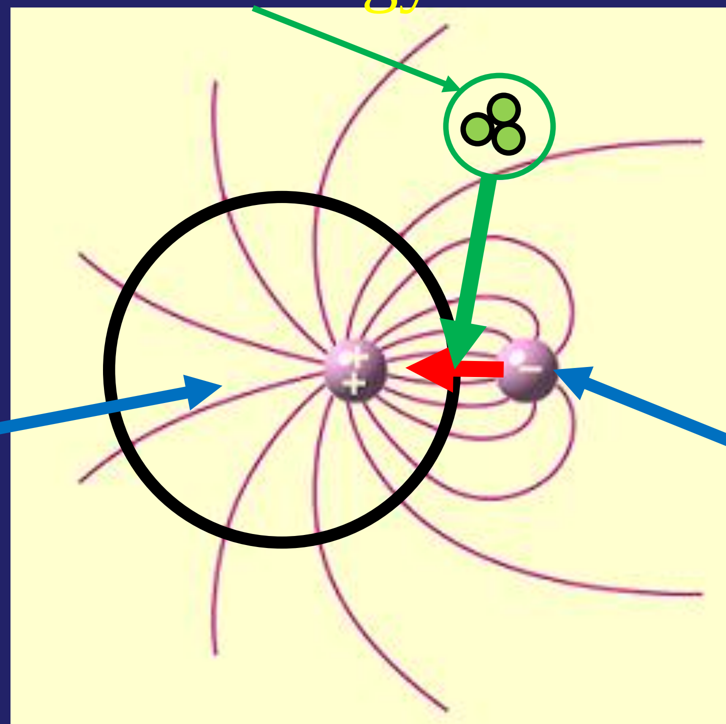
pathogen attaches to the cell $\longrightarrow$ disease

Design of medicines

You design a molecule charge with *lower* energy

Also use medicine to tune the electrolyte around cell → tune ρ_0 background charge in $\nabla^2\Phi = \rho_0$ in your favor.

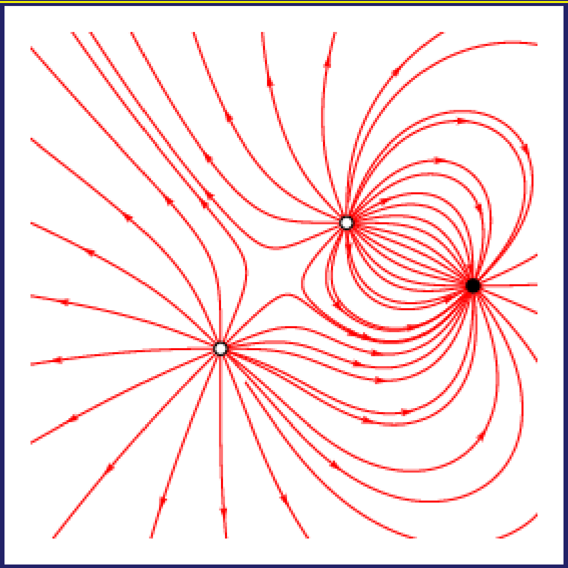
You know the charge distribution in healthy cell



This is a pathogen with charge

Medicine attaches to cell preferentially and blocks site for pathogen → blocks disease

This is a difficult problem in general!



Real world molecules may not have spherical, ϕ symmetry
Charges will be distributed in a complex manner
Distribution may fluctuate as function of time & temperature

Complex numerical calculations are typically needed, with no guarantee that a stable optimum medicine molecule exists. Pathogens can mutate to different charge configurations