MA-108 Ordinary Differential Equations

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D1 - Lecture 17

Recall:

We solved IVP constant coefficient equations of order 2, using Laplace transform.

We introduced unit step or Heaviside functions.

Second shifting theorem $L(u(t-a)f(t-a)) = e^{-sa}F(s)$.

Computed Laplace transforms of piecewise continuous functions using unit step functions and 2nd shifting theorem.

We can compute inverse Laplace transforms of functions involving e^{-sa} terms using second shifting theorem.

We will start with constant coefficient equations of order 2 with piecewise continuous forcing functions.

IVP with peicewise continuous forcing functions

Ex. Consider the differential equation of the form

$$y'' + 3y' + 2y = \begin{cases} e^t, & 0 < t \leq 2 \\ e^{-t}, & 2 < t \end{cases}, y(0) = 1, y'(0) = -1.$$

From what we know, this IVP has a unique solution in the interval $(0, 2)$ and if the IVP was defined on $t_0 \in (2, \infty)$ then we would have a unique solution on $(2, \infty)$.

But it's still possible to get a solution which is continuous on $[0, \infty)$. Let y_1 be the unique solution to the given IVP on $[0, 2)$. Then evaluate $y_1(2)$ and $y_1'(2)$.

Define a new IVP as

$$y'' + 3y' + 2y = e^{-t}, \quad y(2) = y_1(2), \quad y'(2) = y_1'(2).$$

This has a unique solution y_2 on $[2, \infty)$.

This gives us a solution $y(t)$ of original IVP on $(0, \infty)$ as

$$y(t) = \begin{cases} y_1(t), & 0 \leq t < 2 \\ y_2(t), & 2 \leq t < \infty \end{cases}$$

such that y, y' are continuous on $[0, \infty)$.

Note that, since $r(t)$ is discontinuous at $t = 2$, $r(2+) = e^{-2}$ and $r(2-) = e^2$; though, y and y' are continuous on $(0, \infty)$, y'' is not defined at $t = 2$.

Fact. We can not find any solution of IVP on an interval I, if I contains $t = 2$, which is a jump discontinuity of $r(t)$.

So we need to define what we mean by a solution of IVP

$$ay'' + by' + cy = r(t),\ a, b, c \in \mathbb{R},\ y(0) = k_0,\ y'(0) = k_1$$

on $(0, \infty)$, when r has jump discontinuity.

We state the following theorem, without proof, which motivates our definition. Proof should be clear.

Theorem

Let f be a peciewise continuous function with jump discontinuities at $t_1, t_2, \ldots, t_n$. Let k_0 and k_1 are arbitrary real numbers. Consider the ODE

$$ay'' + by' + cy = f(t). \qquad (*)$$

Then there is a unique function y defined on $[0, \infty)$ such that

1. $y(0) = k_0$ *and* $y'(0) = k_1$.
2. y *and* y' *are continuous on* $[0, \infty)$.
3. y'' *is defined on every open subinterval* I *of* $[0, \infty)$ *that does not contain any of the points* $t_1, \ldots, t_n$.
4. y *satisfies* $(*)$ *on every such subinterval* I *of* $(0, \infty)$.
5. y'' *has limits from the right and left at* $t_1, \ldots, t_n$.

Ex. Solve the IVP

$$y'' + y = \begin{cases} 1, & 0 \le t < \pi/2 \\ -1, & \pi/2 \le t < \infty \end{cases}, \quad y(0) = 2, y'(0) = -1$$

Let $y_1(t)$ be the solution of $y'' + y = 1,\ y(0) = 2, y'(0) = -1.$

$$y_1(t) = 1 + \cos t - \sin t$$

Compute $y_1(\pi/2) = 0,\ y_1'(\pi/2) = \ -1.$

Let $y_2(t)$ be solution of $y'' + y = -1,\ y(\frac{\pi}{2}) = 0, y'(\frac{\pi}{2}) = -1.$

$$y_2(t) = \ -1 + \cos t + \sin t$$

The solution of IVP is

$$y(t) = \begin{cases} 1 + \cos t - \sin t, & 0 \le t < \frac{\pi}{2} \\ -1 + \cos t + \sin t, & t \ge \frac{\pi}{2} \end{cases}$$

EXAMPLE Let us solve previous problem using Laplace transform.

$$y''+y = f(t) = \begin{cases} 1, & 0 \leq t < \pi/2 \\ -1, & \pi/2 \leq t < \infty \end{cases}, \; y(0) = 2, \; y'(0) = -1$$

$$f(t) = 1 + (-1-1)u\left(t - \frac{\pi}{2}\right) = 1 - 2u\left(t - \frac{\pi}{2}\right)$$

Let the ODE has a solution ϕ such that ϕ and ϕ' are continuous.

$$\begin{aligned} L(\phi'') + L(\phi) &= L(f(t)) \\ s^2L(\phi) - \phi'(0) - s\phi(0) + L(\phi) &= L\left(1 - 2u(t - \frac{\pi}{2})\right) \\ (s^2+1)L(\phi) + 1 - 2s &= \frac{1}{s} - 2e^{-\pi s/2}\frac{1}{s} \\ L(\phi) = (1 - 2e^{-\pi s/2})\frac{1}{s(s^2+1)} &+ \frac{2s-1}{s^2+1} \end{aligned}$$

Use partial fractions to get $\dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$.

Taking inverse Laplace transform of

$$L(\phi) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right) - 2e^{-\pi s/2}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{2s-1}{s^2+1}$$

we get $\phi(t) =$

$$1-\cos t-2u\left(t-\frac{\pi}{2}\right)+2u\left(t-\frac{\pi}{2}\right)\cos\left(t-\frac{\pi}{2}\right)+2\cos t-\sin t$$

Simplifying we have,

$$\begin{aligned}\phi(t) &= 1+\cos t-\sin t-2u\left(t-\frac{\pi}{2}\right)(1-\sin t)\\ &= \begin{cases} 1+\cos t-\sin t, & 0 \le t < \frac{\pi}{2}\\ -1+\cos t+\sin t, & t \ge \frac{\pi}{2}\end{cases}\end{aligned}$$

Check that ϕ and ϕ' are continuous and ϕ'' has left and right limit at $\pi/2$.

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Ex. Solve the IVP $y'' + y = f(t),\ y(0) = 0,\ y''(0) = 0,$ where

$$f(t) = \begin{cases} 0, & 0 \le t < \dfrac{\pi}{4} \\ \cos 2t, & \dfrac{\pi}{4} \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

$$f(t) = u\left(t - \frac{\pi}{4}\right)\cos 2t - u(t-\pi)\cos 2t$$

Let us compute $L(f(t))$ first.

$$\begin{aligned} L(f) &= L\left(u\left(t - \frac{\pi}{4}\right)\cos 2t\right) - L(u(t-\pi)\cos 2t) \\ &= e^{-\pi s/4} L\left(\cos 2\left(t + \frac{\pi}{4}\right)\right) - e^{-\pi s} L\left(\cos 2(t+\pi)\right) \\ &= e^{-\pi s/4} L(-\sin 2t) - e^{-\pi s} L(\cos 2t) \\ &= -\frac{2e^{-\pi s/4}}{s^2+4} - \frac{s e^{-\pi s}}{s^2+4} \end{aligned}$$

Example continued...

Taking Laplace transform, IVP

$y'' + y = f(t),\ y(0) = 0,\ y'(0) = 0$ gives

$$Y(s)(s^2+1) = L(f), \quad \text{where } L(y(t)) = Y(s)$$

We get $Y(s)$

$$\begin{aligned}
&= \frac{1}{s^2+1}\left[-\frac{2e^{-\pi s/4}}{s^2+4} - \frac{se^{-\pi s}}{s^2+4}\right] \\
&= -e^{-\pi s/4}\frac{2}{(s^2+1)(s^2+4)} - e^{-\pi s}\frac{s}{(s^2+1)(s^2+4)} \\
&= -\frac{2e^{-\pi s/4}}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right] - \frac{e^{-\pi s}}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right] \\
&= e^{-\pi s/4}H_1(s) + e^{-\pi s}H_2(s)
\end{aligned}$$

$$y(t) = u\left(t - \frac{\pi}{4}\right)h_1\left(t - \frac{\pi}{4}\right) + u(t-\pi)h_2(t-\pi)$$

Example continued...

Let us find out $h_1(t)$ and $h_2(t)$.

$$h_1(t) = L^{-1}\left(\frac{-2}{3(s^2+1)} + \frac{2}{3(s^2+4)}\right) = \frac{-2}{3}\sin t + \frac{1}{3}\sin 2t$$

$$h_2(t) = L^{-1}\left(\frac{-s}{3(s^2+1)} + \frac{s}{3(s^2+4)}\right) = \frac{-1}{3}\cos t + \frac{1}{3}\cos 2t$$

Therefore,

$$\begin{aligned}
y(t) &= u\left(t-\frac{\pi}{4}\right)\left[\frac{-2}{3}\sin(t-\pi/4) + \frac{1}{3}\sin 2(t-\pi/4)\right] \\
&+ u(t-\pi)\left[\frac{-1}{3}\cos(t-\pi) + \frac{1}{3}\cos 2(t-\pi)\right] \\
&= u(t-\pi/4)\left[\frac{-\sqrt{2}}{3}(\sin t - \cos t) - \frac{1}{3}\cos 2t\right] \\
&+ \frac{1}{3}u(t-\pi)(\cos t + \cos 2t)
\end{aligned}$$

Example continued...

$$y(t) = \begin{cases} 0, & 0 \le t < \dfrac{\pi}{4} \\ \dfrac{-\sqrt{2}}{3}(\sin t - \cos t) - \dfrac{1}{3}\cos 2t, & \dfrac{\pi}{4} \le t < \pi \\ \dfrac{-\sqrt{2}}{3}\sin t + \dfrac{1+\sqrt{2}}{3}\cos t, & t \ge \pi \end{cases}$$

Check that y, y' are continuous and y'' has left and right limits at $\pi/4$ and π.

Convolution

Consider IVP

$$ay'' + by' + cy = f(t),\ y(0) = 0,\ y'(0) = 0$$

Taking Laplace transform gives

$$(as^2 + bs + c)Y(s) = F(s)$$

$$\implies Y(s) = F(s)G(s),\ \text{where } G(s) := \frac{1}{as^2 + bs + c}$$

Till now, we were finding $y(t) = L^{-1}(Y(s))$, for known forcing function, by partial fraction method.

Q. What if $f(t)$ is unknown function? Can we get a formula for $y(t) = L^{-1}(F(s)G(s))$ in terms of $f(t)$?

Convolution : $L^{-1}(FG)$

Ex. Consider IVP $\quad y' - ay = f(t),\ y(0) = 0.$

Solution is $y = ue^{at}$, where $u' = e^{-at} f(t)$.

Using $u(0) = 0$, we get $u(t) = \int_0^t e^{-a\tau} f(\tau)\, d\tau$.

Therefore, the solution of IVP is

$$y(t) = e^{at} \int_0^t e^{-a\tau} f(\tau)\, d\tau = \int_0^t e^{a(t-\tau)} f(\tau)\, d\tau$$

Let us use Laplace transform to solve same IVP. We get

$$(s-a)Y(s) = F(s) \implies Y(s) = F(s)\frac{1}{s-a}$$

If we write $G(s) = \dfrac{1}{s-a}$, then $g(t) = e^{at}$ and the solution $y(t)$ can be written as

$$L^{-1}(F(s)G(s)) = y(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau$$

Definition (Convolution)

The convolution $f * g$ of two functions f and g is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

Previous example showed that when $g(t) = e^{at}$, then $L^{-1}(F(s)G(s)) = f * g$ or $F(s)G(s) = L(f * g)$. This is true in general.

Exercise.

(1) $f * g = g * f$.

(2) $f * (g_1 + g_2) = f * g_1 + f * g_2$

(3) $(f * g) * h = f * (g * h)$

(4) $f * 0 = 0 * f = 0$

(5) $f * 1 \neq f$, e.g. $\sin t * 1 = 1 - \cos t$.

Theorem (Convolution Theorem)

If $L(f) = F(s)$ and $L(g) = G(s)$, then $L(f * g)$ exists, and

$$L(f * g) = L\left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right) = F(s)G(s)$$

Proof. Let us assume that the Laplace transform of $f * g$ exists. We will prove the formula.

$$L(f * g) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)\ d\tau\right)\ dt$$

Reversing the order of integration gives us the following

$$= \int_0^\infty f(\tau)\left(\int_\tau^\infty e^{-st} g(t-\tau)\ dt\right) d\tau$$

$$= \int_0^\infty f(\tau)\left(\int_0^\infty e^{-s(x+\tau)} g(x)\ dx\right) d\tau$$

$$\begin{aligned} L(f*g) &= \int_0^\infty f(\tau)\left(\int_0^\infty e^{-s(x+\tau)}g(x)\ dx\right) d\tau \\ &= \int_0^\infty f(\tau)e^{-s\tau}(G(s))d\tau \\ &= G(s)F(s) \end{aligned}$$

EXAMPLE: Let us verify the result for $f(t) = e^{at}$ and $g(t) = e^{bt}$.

In this case,

$$F(s)G(s) = \left(\frac{1}{s-a}\right)\left(\frac{1}{s-b}\right) = \frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right)$$

Example continued ...

$$
\begin{aligned}
(f * g)(t) &= \int_0^t f(\tau)g(t-\tau)\, d\tau \\
&= \int_0^t e^{a\tau}e^{b(t-\tau)}\, d\tau \\
&= \int_0^t e^{(a-b)\tau}e^{bt}\, d\tau \\
&= e^{bt}\left(\frac{e^{(a-b)t}}{a-b} - \frac{1}{a-b}\right) \\
&= \frac{e^{at}}{a-b} - \frac{e^{bt}}{a-b} \\
L(f * g) &= \frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right) \\
&= F(s)G(s) \\
&= L(f)L(g)
\end{aligned}
$$

The convolution theorem provides a formula for solution of an IVP with constant coefficient second order equation with unspecified forcing function.

Ex. Solve IVP $y'' + 3y' + 2y = f(t), \quad y(0) = 0,\ y'(0) = 0.$

Applying Laplace transform, we get

$$(s^2 + 3s + 2)L(y) = L(f(t)) = F(s)$$

This gives $Y(s) = F(s)G(s) = L((f * g)(t))$, where

$$G(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+2)(s+1)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Therefore $g(t) = e^{-t} - e^{-2t}$. Therefore

$$y(t) = (f*g)(t) = f*(e^{-t} - e^{-2t}) = \int_0^t f(t-\tau)(e^{-\tau} - e^{-2\tau})\, d\tau$$