PH108

Lecture 04

Vector calculus – an applications review

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Today’s objective

a. <u>Your</u> objective: how to solve problems

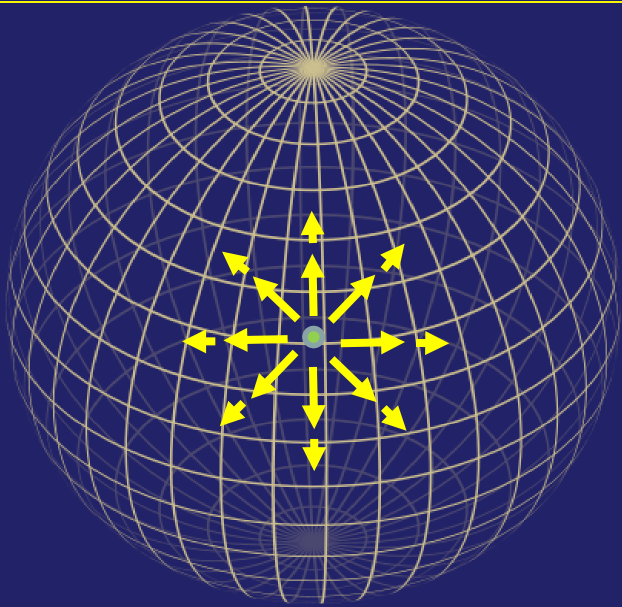
1) Given a charge distribution, determine $\vec{E}(x, y, z)$ everywhere

2) Given $\vec{E}(x, y, z)$, determine its effect on a charge distribution

b. <u>My</u> objective: Prepare you to *find* problems

Consider the first formula: Gauss's law

$$\oiint_S \vec{E} \cdot d\vec{\sigma} = \frac{q}{\epsilon_0}$$



Makes 'intuitive sense'

For a point charge q at the origin,
All $\vec{E} \sim \hat{r}$ field lines point radially outward & d$\vec{\sigma} \sim \hat{r}$

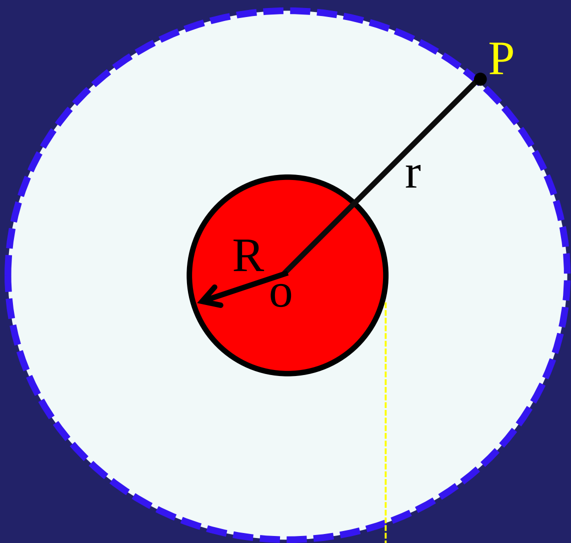
The integration is trivial: $\oiint E \ r^2 sin\theta d\theta d\phi \, \hat{r} \cdot \hat{r} = \frac{q}{\epsilon_0}$

$$\rightarrow E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

But we knew this already from Coulomb's law!

Gauss's Law is great for symmetrical charge

Sphere of radius R containing uniform charge density ρ

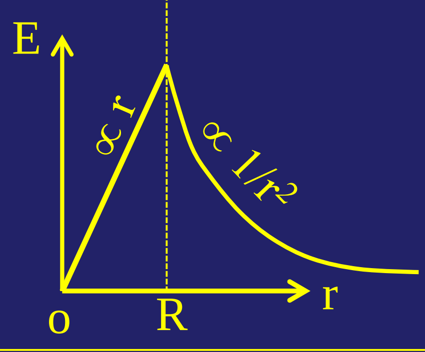


Draw a spherical surface of radius *r* centred at O :
This is also called a 'Gaussian surface' - a fancy name for a *closed* surface around a charge distribution to which we are about to apply Gauss's law

Flux through the spherical Gaussian surface is:

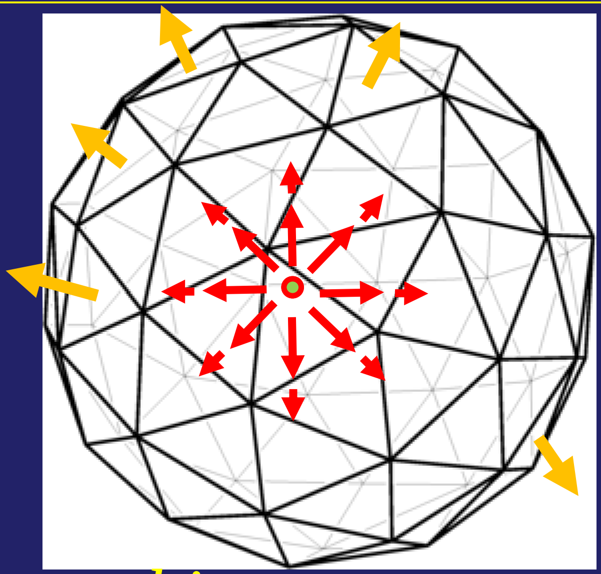
$$\oiint_S \vec{E} \cdot d\vec{\sigma} = \frac{q_{encl}}{\epsilon_0}$$

$\vec{E}$ and $d\vec{\sigma}$ along $\hat{r} \rightarrow$ LHS $= 4\pi r^2 E$ RHS $= \rho \frac{4}{3} \pi R^3$



$$\vec{E}_P = \frac{1}{4\pi\epsilon_0} \frac{q_{encl}}{r^2}$$

Consider an *arbitrary* surface surrounding a point charge

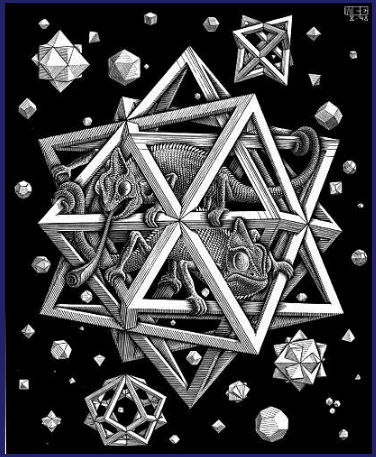


Problem: Find the field $\vec{E}$ on *each* piece of the (closed) surface

Each surface piece $\Delta\vec{\sigma}_i$ has direction

$$\sum_{i=1}^{n} \overrightarrow{E_i} \cdot \Delta\overrightarrow{\sigma_i} = \frac{q}{\epsilon_0}$$

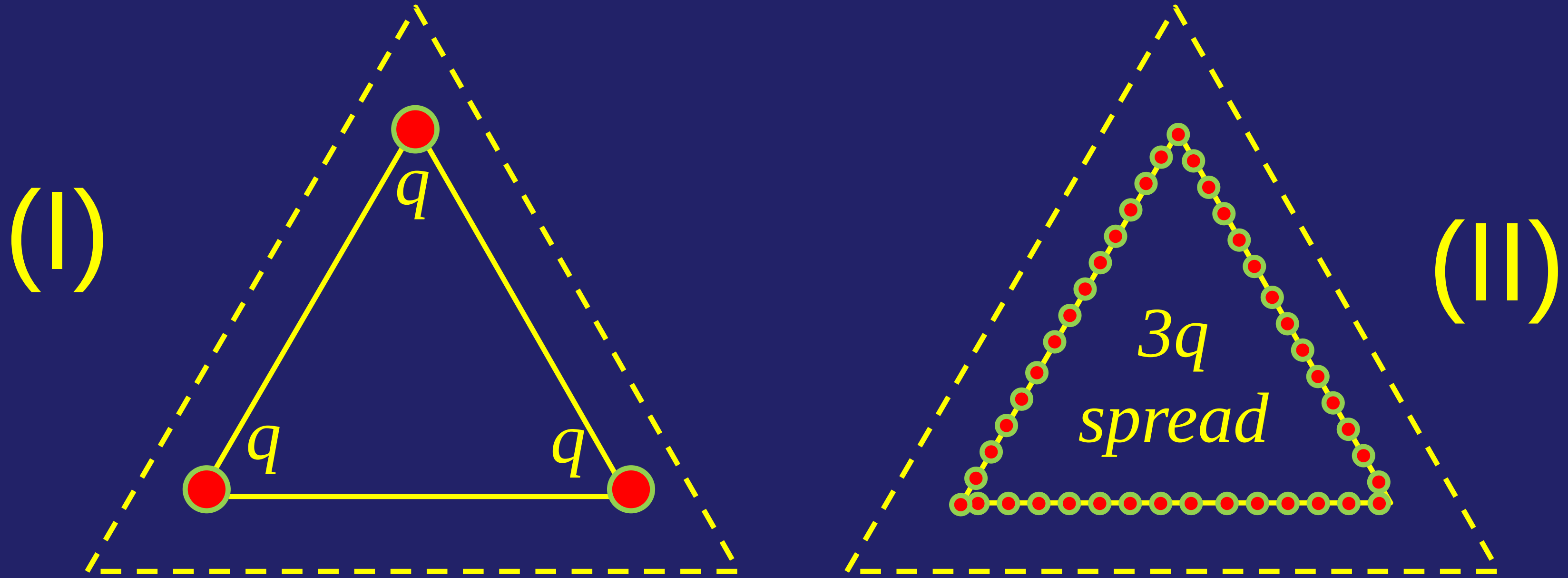
or this..



1 equation in *n* unknowns … not very useful!

QUESTION

In TWO dimensions, consider equilateral triangle with the following two charge distributions



The MAGNITUDE of E on the dashed line is:

1. uniform in (I) *only*
2. uniform in (II) *only*
3. uniform in (I) & (II) *and same*
4. uniform in (I) & (II) *but different values for* (I) & (II)
5. varies everywhere

Gauss law is a 'descriptive' formula
We want a a 'prescriptive' formula

For this, we define a mathematical quantity:

Divergence of a vector field $\overrightarrow{W}$

is the net outward flux

through a closed surface

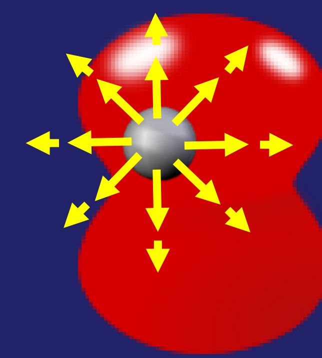
enclosing a volume

as the volume $\rightarrow 0$

$$\vec{\nabla} \cdot \overrightarrow{W} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \oint \overrightarrow{W} \cdot \overrightarrow{dS}$$

Divergence resolves the effect of a *continuous* charge distribution $\rho(r)$

Pick a small piece Δv centered at *(x,y,z)*



Mean charge over Δv is $\bar{\rho}(x,y,z)$

Total charge in Δv is $\bar{\rho}\Delta v$

Gauss's Law: $$\oiint_{s\ over\ \Delta v} \vec{E}\cdot d\vec{\sigma} = \frac{\bar{\rho}\Delta v}{\epsilon_0}$$

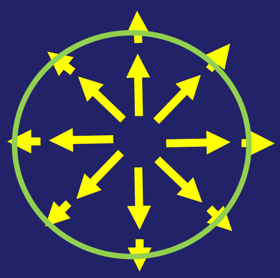
$\rho(x,y,z)$

$$\underbrace{\lim_{\Delta v \to 0} \frac{1}{\Delta v} \oiint \vec{E}\cdot d\vec{\sigma}}_{\equiv\, \vec{\nabla}\cdot\vec{E}} = \frac{\bar{\rho}}{\epsilon_0}$$

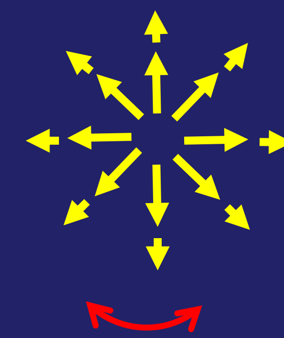
Differential form of Gauss' Law

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}}$$

Divergence theorem relates flux at a surface to divergence in the volume



$$\oiint \vec{E}\cdot d\vec{\sigma} = \iiint_V \vec{\nabla}\cdot\vec{E}\,dv$$



"intuitively obvious"

Number of field lines crossing an enclosed surface

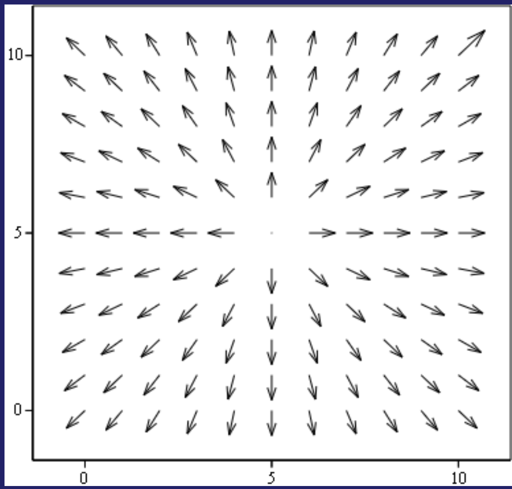
How much the field lines diverge inside the volume

Detailed proof not necessary for our applications

Look up any text on vector calculus, eg *Apostol*

QUESTION

Consider a vector field $\vec{E} = \frac{q}{\epsilon_0}\vec{r}$



The charge density $\rho(r)$ that produces this field is:

1. q
2. qr
3. qr^2
4. $3q$

Some formulae for divergence in different co-ordinate systems

Cartesian:

$$\vec{\nabla}\cdot\vec{E} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$

Polar:

$$\vec{\nabla}\cdot\vec{E} = \left(\frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} + \frac{1}{rsin\theta}\frac{\partial(sin\theta E_\theta)}{\partial\theta} + \frac{1}{rsin\theta}\frac{\partial E_\phi}{\partial\phi}\right)$$

Cylindrical:

$$\vec{\nabla}\cdot\vec{E} = \left(\frac{1}{\rho}\frac{\partial(\rho E_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}\right)$$

Logic thread of today's lecture

There is Gauss's law: flux over a closed surface ~ $q_{enclosed}$

Gauss's law works great for symmetrical charge.

We worked out spherical ρ – look for other examples in tutorials

Divergence of a vector field *seems* to be a more useful quantity than the field itself for general case
We have not seen why yet

Divergence theorem relates divergence in volume to flux on surface: this seems useful – we will see how