Tutorial-7, MA 106 (Linear Algebra)

Linear Algebra and its Applications by Gilbert Strang

This tutorial sheet consists of the problems based on 4.4, 5.1 and 5.2. This includes:
Problem Set 4.3:[3, 5, 7, 9, 15, 18, 23, 26, 27, 32, 34]

Problem Set 5.1: $[1-4, 7, 9, 11, 13, 16, 20, 26, 30, 34]$

Problem Set 5.2:$[2, 4, 6, 7, 8, 9, 12, 14, 17, 19, 26, 29, 30, 31]$

Section 4.3

1. Predict in advance and confirm by elimination, the pivot entries of $A = \begin{pmatrix} 2 & 1 & 2 \\ 4 & 5 & 0 \\ 2 & 7 & 0 \end{pmatrix}$.

2. Use the cofactor matrix C to invert matrices A and B.
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

3. Find x, y, z by Cramer's rule:
$$x + 4y - z = 1,\ x + y + z = 0,\ 2x + 3z = 0$$

4. Find the determinant when a vector x replaces j-th column of I.

5. If the right side b is the last column of A, solve the 3×3 system $Ax = b$. Explain how each determinant in Cramer's rule leads to your solution x.

6. If all the cofactors are zero, how do you know that A has no inverse? If non of the cofactors are zero, is A sure to be invertible? Find the rank of such an A.

7. Suppose $det(A) = 1$ and you know all the cofactors. How will you find A.

8. L is lower triangular and S is symmetric. Assume they are invertible.
$$L = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}, \quad S = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}$$
   (a) Which three cofactors of L are zero. (L^{-1} is lower triangular)
   (b) Which three pairs of cofactors of S are equal. (S^{-1} is symmetric)

9. The parallelogram with sides $(2, 1)$ and $(2, 3)$ has the same area as the parallelogram with sides $(2, 2)$ and $(1, 3)$. Find those area by determinants and say why they must be equal.

10. (a) The corners of a triangle are $(2, 1), (3, 4), (0, 5)$. What is the area.
    (b) A new corner at $(-1, 0)$ makes it lopsided (four sides). Find the area.

11. The Hadamard matrix H has orthogonal rows. The box is a hypercube in $\mathbb{R}^4$. Find $det(H)$.
$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Section 5.1

12. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Verify that the trace equals the sum of the eigenvalues, and the determinant equals their product.

13. With the same matrix A, solve the differential equation $du/dt = Au$, $u(0) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$. What are the two pure exponential solutions?

14. . If we shift to $A - 7I$, what are the eigenvalues and eigenvectors and how are they related to those of A ?

$$B = A - 7I = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}$$

15. Solve $du/dt = Pu$, when P is a projection: $P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ and $u(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

16. Give an example to show that the eigenvalues can be changed when a multiple of one row is subtracted from another. Why is a zero eigenvalue not changed by the steps of elimination?

17. Find the eigenvalues and eigenvectors of

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

Check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace and $\lambda_1\lambda_2\lambda_3$ equals the determinant.

18. The eigenvalues of A equal the eigenvalues of A^T. This is because $\det(A-\lambda I)$ equals $\det(A^T-\lambda I)$. That is true because ______. Show by an example that the eigenvectors of A and A^T are not the same.

19. (a) Construct 2 by 2 matrices such that the eigenvalues of AB are not the products of the eigenvalues of A and B, and the eigenvalues of $A + B$ are not the sums of the individual eigenvalues.

    (b) Verify, however, that the sum of the eigenvalues of $A+B$ equals the sum of all the individual eigenvalues of A and B, and similarly for products. Why is this true?

20. Choose the third row of the companion matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \_ & \_ & \_ \end{bmatrix}$$

so that its characteristic polynomial is $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$.

21. Find the rank and all four eigenvalues for both the both the matrix of ones and the checker board matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

22. Suppose A has eigenvalues $0, 3, 5$ with independent eigenvectors u, v, w.

(a) Give a basis for the nullspace and a basis for the column space.

(b) Find a particular solution to $Ax = v + w$. Find all solutions.

(c) Show that $Ax = u$ has no solution. (If it had a solution, then ______ would be in the column space.)

23. If A has $\lambda_1 = 4$ and $\lambda_2 = 5$, then $\det(A - \lambda I) = (\lambda - 5)(\lambda - 4) = \lambda^2 - 9\lambda + 20$. Find three matrices that have trace $= 9$, determinant 20, and $\lambda = 4, 5$.

24. The matrix A is singular with rank 1 . Find all the eigenvalues and eigenvectors.

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}$$

25. When P exchanges rows 1 and 2 and columns 1 and 2, the eigenvalues dont change. Find eigenvectors of A and PAP for $\lambda = 11$:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \text{ and } PAP = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}$$

Section 5.2

26. (a) If $A^2 = I$, what are the possible eigenvalues of A?

(b) If this A is 2 by 2, and not I or $-I$, find its trace and determinant.

(c) If the first row is $(3, -1)$, what is the second row?

27. If $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. Find A^{100} by diagonalising A.

28. Find all the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and write two different diagonalising matrices S.

29. Suppose $A = uv^T$ is a column times a row (a rank-1 matrix).

(a) By multiplying A times u, show that u is an eigenvector. What is λ ?

(b) What are the other eigenvalues of A (and why)?

(c) Compute trace(A) from the sum on the diagonal and the sum of λ s.

30. Which of the following cannot be diagonalised?

$$A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

31. Suppose A has eigenvalues $1, 2, 4$. What is the trace of A^2? What is the determinant of $(A^{-1})^T$?

32. If the eigenvalues of A are 1, 1, 2, which of the following are certain to be true? Give a reason if true or a counterexample if false:

(a) A is invertible.

(b) A is diagonalizable.

(c) A is not diagonalizable.

33. Show by direct calculation that the trace of AB and BA are same where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} q & r \\ s & t \end{bmatrix}$$

Deduce that $AB - BA = I$ is impossible (except in infinite dimensions).

34. True or false: If the n columns of S (eigenvectors of A) are independent, then

(a) A is invertible.

(b) A is diagonalizable.

(c) S is invertible.

(d) S is diagonalizable.

35. Suppose $A = S\Lambda S^{-1}$. What is the eigenvalue matrix for $A+2I$? What is the eigenvector matrix? Check that $A + 2I = (\ )(\ )(\ )^{-1}$.

36. The matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalisable because the rank of $A - 3I$ is ? ____. Change one entry to make A diagonalizable. Which entries could you change?

37. Find Λ and S to diagonalize A in the next problem. What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of this limiting matrix you see the ? ______.

38. $A^k = S\Lambda^k S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ?______. Does $A^k \to 0$ or $B^k \to 0$?

$$A = \begin{bmatrix} .6 & .4 \\ .4 & .6 \end{bmatrix} \text{ and } B = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}$$

39. . Diagonalize A and compute $S\Lambda^k S^{-1}$ to prove this formula for A^k :

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ has } A^k = \frac{1}{2}\begin{bmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{bmatrix}$$

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