MA-108 Ordinary Differential Equations

M.K. Keshari

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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D1 - Lecture 19

1. $L\left(e^{-at}f(t)\right) = F(s+a), \quad s > s_0 + a.$
2. $L(u(t-a)f(t-a)) = e^{-as}F(s), \quad s > s_0, a > 0.$
3. $L\left(\int_0^t f(\tau)\,d\tau\right) = \dfrac{F(s)}{s}, \quad s > \max\{0, s_0\}.$
4. $L(tf(t)) = -F^{(1)}(s), \quad s > s_0.$
5. $L\left(\dfrac{f(t)}{t}\right) = \int_s^\infty F(s')ds', \quad s > s_0.$
6. Assume f is piecewise continuous and of exponential order. Then
   $(i)\lim_{s\to\infty} F(s) = 0, \qquad (ii)\lim_{s\to\infty} sF(s)$ is bounded.
7. Assume f and f' both are piecewise continuous and of exponential order. Then $\lim_{s\to\infty} sF(s) = f(0)$.
8. If f is piecewise continuous and periodic of period T, then $L(f(t)) = \dfrac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$, $s > 0$

Theorem

Assume f and f' both are piecewise continuous and of exponential order. Then

$$\lim_{s\to\infty} sF(s) = f(0).$$

Proof. Since

$$L(f'(t)) = sL(f(t)) - f(0)$$

Since f and f' both are piecewise continuous and of exponential order, we get

$$\lim_{s\to\infty} L(f'(t)) = 0, \text{ and } \lim_{s\to\infty} sF(s) < \infty$$

Therefore,

$$\lim_{s\to\infty} sF(s) = f(0)$$

Ex. Let $f(t) = L^{-1}\left(\dfrac{1 - s(5+3s)}{s((s+1)^2+1)}\right)$. Find $f(0)$.

We can find $f(t)$ by partial fraction. Hence we know that f and f' are continuous and of exponential order. Therefore,

$$\begin{aligned} f(0) &= \lim_{s\to\infty} sF(s) \\ &= \lim_{s\to\infty} \frac{1 - s(5+3s)}{((s+1)^2+1)} \\ &= \lim_{s\to\infty} \frac{1 - 5s - 3s^2}{s^2+2s+2} \\ &= -3 \end{aligned}$$

Theorem

If f is piecewise continuous and periodic of period T, then

$$L(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T f(T) e^{-st}\, dt, \;\; s > 0$$

Proof.

$$\begin{aligned} L(f(t)) &= \int_0^T f(t)e^{-st}\, dt + \int_T^{2T} f(t)e^{-st}\, dt + \ldots \\ &= \int_0^T f(t)e^{-st}\, dt + \int_0^T f(t+T)e^{-s(t+T)}\, dt + \ldots \\ &= \int_0^T f(t)e^{-st}\, dt \left(1 + e^{-sT} + e^{-2sT} + \ldots\right) \\ &= \frac{1}{(1-e^{-sT})} \int_0^T f(t)e^{-st}\, dt\,, \;\; s > 0 \end{aligned}$$

Ex. Find the Laplace transform of periodic function

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}, \quad f(t+2) = f(t)$$

$$\begin{aligned}
L(f(t)) &= \frac{1}{(1-e^{-2s})} \int_0^2 f(t)e^{-st}\,dt \\
&= \frac{1}{(1-e^{-2s})} \int_0^1 te^{-st}\,dt \\
&= \frac{1}{(1-e^{-2s})} \left[t\frac{e^{-st}}{-s}\Big|_0^1 - \int_0^1 \frac{e^{-st}}{-s}\,dt \right] \\
&= \frac{1}{(1-e^{-2s})} \left[\frac{e^{-s}}{-s} - \frac{1}{s^2}(e^{-s}-1) \right]
\end{aligned}$$

Exercise. Find Inverse Laplace transform and varify, whether $\lim_{s\to\infty} sF(s) = f(0)$. If not, then state why it is not.

$$(i)\quad \frac{s}{(s^2+a^2)^2}, \qquad (ii)\quad \frac{s}{(s^2-a^2)^2}$$

$$(iii)\quad \frac{s^2}{(s+1)^3}, \qquad (iv)\quad \frac{1}{\sqrt{s+1}}$$

$$(v)\quad \frac{s}{(s-a)^{3/2}}, \qquad (vi)\quad \frac{s}{(s+4)^6}$$

$$(vii)\quad \frac{e^{-s}}{s^5}, \qquad (viii)\quad \frac{e^{-2s}}{(s+1)^2}$$

$$(ix)\quad \ln\left(1+\frac{a^2}{s^2}\right), \qquad (x)\quad \ln\left(1-\frac{a^2}{s^2}\right),$$

$$(xi)\quad \frac{1}{s(1-e^{-s})}, \qquad (xii)\quad \frac{1}{s(1+e^{-s})}$$

$$(xiii)\quad \frac{s}{(s^2+1)^{3/2}}, \qquad (xiv)\quad \frac{1}{(s+1)(1-e^{-2s})}$$

Constant coefficient equations with Impulses

In this section, we will consider ODE $ay'' + by' + cy = f(t)$, $a, b, c \in \mathbb{R}$, and $f(t)$ represents a force that is very large for a short time and zero otherwise. Such forces are **impulsive**.

Impulsive forces occur when two objects collide, e.g. the force due to hammer blow. Since it will be difficult to know $f(t)$, how do we solve our ODE.

If f is an integrable function and $f(t) = 0$ outside $[t_0, t_0 + h]$, then $I = \int_{t_0}^{t_0+h} f(t)\, dt$ is called the **total impulse** of f. This is total area under the graph of f.

The idea is that if h is very small, then $y(t)$ will be sensitive to I, but rather insensitive to shape of f. Therefore, if we vary the shape of f, but keep the area same, then $y(t)$ will change very little.

Constant coefficient equations with Impulses

Thus we replace f by a simple rectangular pulse $f(t) = I/h$ on $[t_0, t_0 + h]$ and zero outside $[t_0, t_0 + h]$.

Now the ODE gets simplified, but the solution still depends on h, which may be difficult to find for f. So we take the ideal situation, and make $h \to 0$ to eliminate h.

Define a rectangular pulse having unit impulse as

$$D(t - t_0; h) = \begin{cases} 1/h, & t \in [t_0, t_0 + h] \\ 0, & \text{otherwise} \end{cases}$$

Here D is for physicist P.A. Dirac, who developed the idea of impulsive forces in 1929.

$$\delta(t - t_0) := \lim_{h \to 0} D(t - t_0; h) = \begin{cases} \infty, & t = t_0 \\ 0, & \text{otherwise} \end{cases}$$

$\delta(t - t_0)$ is not an ordinary function, but we can think of it as idealised *point unit impulse* focused at $t = t_0$.

Constant coefficient equations with Impulses

Let us prove that if g is continuous at $t = t_0$, then

$$\lim_{h\to 0}\int_0^\infty g(t)D(t-t_0;h)\,dt = g(t_0)$$

Proof. we may assume that $g(t)$ is continuous on $[t_0, t_0+h]$, by taking h small. Then

$$\begin{aligned}\lim_{h\to 0}\int_0^\infty g(t)D(t-t_0;h)\,dt &= \lim_{h\to 0}\int_{t_0}^{t_0+h} g(t)\frac{1}{h}\,dt \\ = \lim_{h\to 0}\frac{1}{h}g(t_0+h_1)h, \text{ for some } & \quad h_1\in[0,h], \text{ by MVT} \\ &= g(t_0)\end{aligned}$$

In particular, $\int_0^\infty \delta(t-t_0)\,dt = 1$, take $g = 1$.

The "generalised function" $\delta(t-t_0)$ is called **Dirac delta function** or **unit impulse function**.

Let's try to define the meaning of solution of IVP

$$ay'' + by' + cy = \delta(t - t_0),\ y(0) = 0,\ y'(0) = 0, t_0 > 0 \qquad (*)$$

Theorem

Fix $t_0 \geq 0$. For each $h > 0$, let y_h be the solution of IVP

$$ay_h'' + by_h' + cy_h = D(t - t_0; h),\ y_h(0) = 0,\ y_h'(0) = 0, \textit{ Then}$$

$$\lim_{h \to 0+} y_h(t) = u(t - t_0)w(t - t_0),$$

where

$$w(t) = L^{-1}\left(\frac{1}{as^2 + bs + c}\right)$$

Therefore the solution of $(*)$ is defined as

$$y(t) = u(t - t_0)w(t - t_0), \quad \text{where } w(t) = L^{-1}\left(\frac{1}{as^2 + bs + c}\right)$$

Note $w(t) = L^{-1}\left(\dfrac{1}{as^2+bs+c}\right)$ is the solution of the IVP

$$aw'' + bw' + cw = 0,\ w(0) = 0,\ w'(0) = \frac{1}{a}$$

Infact $w(t)$ is defined on $(-\infty, \infty)$, and is given by

$$w = \frac{e^{m_2 t} - e^{m_1 t}}{a(m_2 - m_1)}, \quad w = \frac{1}{a} t e^{m_1 t}, \quad \frac{1}{a\omega} e^{\lambda t} \sin \omega t$$

depending on whether the characteristic polynomial $p(m) = am^2 + bm + c$ has distinct real roots m_1, m_2, or repeated real roots m_1, m_1, or complex conjugate roots $\lambda \pm i\omega$.

This means, if $t_0 > 0$, then the solution $\boxed{y(t) = u(t - t_0)w(t - t_0)}$ of the IVP

$$ay'' + by' + cy = \delta(t - t_0),\ y(0) = 0,\ y'(0) = 0, t_0 > 0 \qquad (*)$$

is defined on $(-\infty, \infty)$ and has the following properties.

$$\begin{aligned} y(t) &= 0 \text{ for all } t < t_0 \\ ay'' + by' + cy = 0 &\text{ on } (-\infty, t_0) \text{ and } (t_0, \infty) \\ y'(t_0-) = 0, &\qquad y'(t_0+) = \frac{1}{a} \end{aligned}$$

When $t_0 = 0$, $y'(0-)$ is not defined, so in this case

$$y(t) = u(t - t_0)w(t - t_0) = u(t)w(t)$$

is a solution of

$$ay'' + by' + cy = \delta(t),\ y(0) = 0,\ y'(0+) = 0$$

Ex. Solve $y'' + 2y' + y = \delta(t - t_0),\ y(0) = 0,\ y'(0) = 0$. Here

$$\begin{aligned} w(t) &= L^{-1}\left(\frac{1}{s^2 + 2s + 1}\right) \\ &= L^{-1}\left(\frac{1}{(s+1)^2}\right) \\ &= e^{-t}t \end{aligned}$$

Therefore, the solution is given by

$$\begin{aligned} y(t) &= u(t - t_0)w(t - t_0) \\ &= u(t - t_0)e^{-(t-t_0)}(t - t_0) \end{aligned}$$

Ex. Solve

$y'' + 6y' + 5y = 3e^{-2t} + 2\delta(t-1),\ y(0) = -3,\ y'(0) = 2.$

If $y_1(t)$ is a solution of

$y'' + 6y' + 5y = 3e^{-2t},\ y(0) = -3,\ y'(0) = 2$, then

$y_1(t) = c_1e^{-t} + c_2e^{-5t} - e^{-2t}$, where $c_1 = -\dfrac{5}{2}$ and $c_2 = \dfrac{1}{2}$.

The solution of IVP is $y(t) = y_1 + y_2$, where y_2 is a solution of

$y'' + 6y' + 5y = 2\delta(t-1),\ y(0) = 0,\ y'(0) = 0.$ Hence

$$\begin{aligned} w(t) &= 2L^{-1}\left(\frac{1}{s^2+6s+5}\right) \\ &= \frac{1}{2}\left(\frac{1}{s+1} - \frac{1}{s+5}\right) \\ &= \frac{1}{2}(e^{-t} - e^{-5t}) \end{aligned}$$

$$y(t) = -\frac{5}{2}e^{-t} + \frac{1}{2}e^{-5t} - e^{-2t} + \frac{1}{2}u(t-1)\left(e^{-(t-1)} - e^{-5(t-1)}\right)$$

Exercise. Solve the following problems.

1. $y'' + 3y' + 2y = 6e^{2t} + 2\delta(t-1), \quad y(0) = 2, \ y'(0) = -6.$
2. $y'' + y = \sin 3t + 2\delta(t - \pi/2), \qquad y(0) = 1, \ y'(0) = -1.$
3. $y''+2y'+2y = \delta(t-\pi)-3\delta(t-2\pi), \ y(0) = -1, \ y'(0) = 2.$
4. $y'' + 4y = f(t) + \delta(t - 2\pi), \qquad y(0) = 0, \ y'(0) = 1$ and
$$f(t) = \begin{cases} 1, & 0 \le t < \pi/2 \\ 2, & t \ge \pi/2 \end{cases}$$
5. $y'' + 4y' + 4y = -\delta(t), \qquad y(0) = 1, \ y'(0+) = 5.$
6. Find a solution not involving unit step function which represents y on each suninterval of $[0, \infty)$ on which the forcing function is zero.

   (a) $y'' - y = \sum_{k=1}^{\infty} \delta(t-k), \qquad y(0) = 0, \ y'(0) = 1$

   (b) $y'' - 3y' + 2y = \sum_{k=1}^{\infty} \delta(t-k), \ y(0) = 0, \ y'(0) = 1$