MA-108 Ordinary Differential Equations

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D1 - Lecture 8

Recall: We saw some numerial methods to solve $y' = f(x, y)$ with $y(x_0) = y_0$.

Let h be the step size, i.e. $x_i = x_0 + hi$.

Euler's method uses $y_i = y_{i-1} + hf(x_{i-1}, y_{i-1})$.

Improved Euler Method uses

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1},\, y_i + hf(x_i, y_i))\right].$$

Runge Kutta Method is the most widely used approximation method.

For MA 108 exam purpose, we will only consider Euler and Inproved Euler.

Now we will consider 2nd order linear ODE.

Example

Ex. Solve $y'' + y = 0$. This is an example of a second order linear ODE. . How to solve this?

Observe that $\sin x$ and $\cos x$ satisfy this equation. Any scalar multiple of sine or cosine function is also a solution. . In fact, every linear combination $c_1 \sin x + c_2 \cos x$ is a solution.

Ex. Solve $y'' - y = 0$. It is easy to see that e^x is a solution. But, so is e^{-x}! Again every linear combination $c_1 e^x + c_2 e^{-x}$ is a solution.

Are these all the solutions? If not, what are the other solutions, and how to find them? If yes, why are these the only solutions?

Solving IVP's

The function $\quad L : C^2(I) \to C(I) \quad$ defined by

$$L(f) = f'' + p(x)f' + q(x)f$$

is a linear transformation. The null space of L, $N(L) = \{f \in C^2(I) \mid L(f) = 0\}$ consists of solutions of ODE

$$y'' + p(x)y' + q(x)y = 0.$$

Theorem (Uniqueness Theorem to homogeneous IVP)

Consider the homogeneous IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = a, y'(x_0) = b,$$

where $p(x)$ and $q(x)$ are continuous on an interval $I = (c, d)$ containing x_0. Then there is a unique solution to the IVP on I.

Ex. Find the largest interval where the ODE

$$x^2y'' + xy' - 4y = 0$$

with initial condition $y(x_0) = y_0$ has a unique solution.
Write the ODE in standard form

$$y'' + p(x)y' + q(x)y = 0$$

where $p(x) = \dfrac{1}{x}$ and $q(x) = \dfrac{-4}{x^2}$. Since $p(x)$ and $q(x)$ are continuous on $(-\infty, 0) \cup (0, \infty)$, the IVP has has a unique solution on $(-\infty, 0)$ if $x_0 < 0$ and on $(0, \infty)$ if $x_0 > 0$.

- Verify that $y_1 = x^2$ is a solution of ODE on $(-\infty, \infty)$ and $y_2 = \dfrac{1}{x^2}$ is a solution on $(-\infty, 0) \cup (0, \infty)$

Ex. Solve IVP $x^2y'' + xy' - 4y = 0, \quad y(1) = 2, y'(1) = 0.$

Since $y(x) = c_1 x^2 + c_2 \dfrac{1}{x^2}$ is a solution of ODE. Find c_1 and c_2 using initial conditions.

We get $c_1 + c_2 = 2, \quad 2c_1 - 2c_2 = 0.$

This gives $c_1 = 1$ and $c_2 = 1.$

Thus solution of IVP is

$$y(x) = x^2 + \frac{1}{x^2}$$

which is unique on the interval $(0, \infty)$. $\square$

Ex. Solve $x^2y'' + xy' - 4y = 0, \quad y(-1) = 2, y'(-1) = 0$

Dimension Theorem

Given a vector space, what's the most important thing about it? Dimension. We would like to know, what's the dimension of $N(L)$?

DIMENSION THEOREM:

Dimension of $N(L) = 2 =$ order of the ODE.
In other words, if $p(x), q(x)$ are continuous on an open interval I, then the set of solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0$$

on interval I is a vector space of dimension 2. $\square$

• The theorem says that once you know that e^x and e^{-x} are solutions of $y'' - y = 0$, any other solution will be of the form $y(x) = c_1 e^x + c_2 e^{-x}$.

• Similarly, any solution of $y'' + y = 0$ are of the form $y(x) = c_1 \sin x + c_2 \cos x$.

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Basis of Solutions

- Consider $y'' + p(x)y' + q(x)y = 0 \quad (1)$,
  where p, q are continuous on I. Let $y_1(x)$ and $y_2(x)$ be solutions of (1) on I. We say $\{y_1, y_2\}$ are **fundamental solutions** of (1), if any solution of (1) can be written as $c_1y_1(x) + c_2y_2(x)$ on I for some scalars c_1, c_2.
- We will see that if y_1, y_2 are linearly independent solutions of (1), then they are fundamental solutions.

- Consider a second order linear DE
  $y'' + p(x)y' + q(x)y = r(x) \quad (2)$
  The DE $\quad y'' + p(x)y' + q(x)y = 0 \quad (1)$
  is the homogeneous second order linear DE corresponding to (2).
  Suppose y_1 is a solution of (1) and y_2 is a solution of (2), then $y_1 + y_2$ is a solution of (2). (Why?).

Basis of Solutions

Dimension theorem told us that the solution space of ODE

$$y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional. Hence a set of fundamental solutions exist. How to find them?

Theorem

Let f, g be two solutions of $y'' + p(x)y' + q(x)y = 0$, where p and q are continuous on I. Let $(f(x_0), f'(x_0))$ and $(g(x_0), g'(x_0))$ be linearly independent vectors in $\mathbb{R}^2$, for some $x_0 \in I$. Then the solution space is the linear span of f and g.

Proof. Let $h(x)$ be a solution of the given ODE. We want to find c and d in $\mathbb{R}$ such that

$$h(x) = cf(x) + dg(x).$$

Basis of Solutions

This implies

$$h(x_0) = cf(x_0) + dg(x_0)$$
$$h'(x_0) = cf'(x_0) + dg'(x_0).$$

Thus,

$$\begin{pmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{pmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} h(x_0) \\ h'(x_0) \end{bmatrix}.$$

As the column vectors

$$\begin{bmatrix} f(x_0) \\ f'(x_0) \end{bmatrix} \& \begin{bmatrix} g(x_0) \\ g'(x_0) \end{bmatrix}$$

are linearly independent, the matrix

$$W(x_0) = \begin{pmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{pmatrix}$$

is invertible.

Basis of Solutions

Therefore,

$$c = \frac{\begin{vmatrix} h(x_0) & g(x_0) \\ h'(x_0) & g'(x_0) \end{vmatrix}}{\det W(x_0)}, \quad d = \frac{\begin{vmatrix} f(x_0) & h(x_0) \\ f'(x_0) & h'(x_0) \end{vmatrix}}{\det W(x_0)}.$$

(What's this method called? Cramer's rule) Now,

$$u(x) = h(x) - cf(x) - dg(x)$$

satisfies the given DE, and

$$u(x_0) = 0 = u'(x_0).$$

But the constant function $u(x) \equiv 0$ also satisfies the IVP. Thus,

$$h(x) = cf(x) + dg(x)$$

by the uniqueness theorem to 2nd order homogeneous IVP.

Wronskian and Linear Independence

The dimension theorem says that if $p(x)$ and $q(x)$ are continuous in an interval I then the solutions form a vector space spanned by two linearly independent solutions. How to check for linear independence? We start with a definition.

The Wronskian of any two differentiable functions $f(x)$ and $g(x)$ is defined by

$$W(f,g;x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}.$$

Ex 1. Find Wronskian of e^x and e^{-x} at $x = 0$.
$W(e^x, e^{-x}, 0) = e^x(-e^{-x}) - e^{-x}e^x|_{x=0} = -2.$

2.
$W(\sin x, \cos x, 0) = \sin x(-\sin x) - \cos x(\cos x)|_{x=0} = -1.$

Wronskian and Linear Independence

Theorem

The Wronskian of any two solutions $f(x), g(x)$ of

$$y'' + p(x)y' + q(x)y = 0$$

is given by

$$W(f, g; x) = W(f, g; a)e^{-\int_a^x p(t)dt},$$

for any $a \in I$.

Proof. Set $W(f, g; x) = W(x)$. Then,

$$\begin{aligned} W(x) &= (fg' - f'g)(x) \\ W'(x) &= (fg'' - f''g)(x). \end{aligned}$$

Wronskian and Linear Independence

Now,

$$f'' = -p(x)f' - q(x)f$$
$$g'' = -p(x)g' - q(x)g.$$

Thus,

$$\begin{aligned} W'(x) &= -fpg' - fqg + gpf' + gqf \\ &= -p(fg' - f'g) \\ &= -pW(x). \end{aligned}$$

Hence,

$$W(x) = ce^{-\int_a^x p(t)dt},$$

for a constant c. For $x = a$, we get $W(a) = c$. Hence,

$$W(f, g; x) = W(f, g; a)e^{-\int_a^x p(t)dt}.$$

Wronskian and Linear Independence

Proposition

Suppose $f(x)$ and $g(x)$ are linearly dependent and differentiable. Then, $W(f, g; x) = 0$.

Proof. As $f(x)$ and $g(x)$ are linearly dependent, there exist $c, d \in \mathbb{R}$, not both 0, such that

$$cf(x) + dg(x) = 0.$$

Thus,

$$cf'(x) + dg'(x) = 0.$$

Hence,

$$\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, $W(f, g; x) = 0$ (Why?).

Wronskian and Linear Independence

Note: The converse is not true. For instance, if $f(x) = x^2$ and

$$g(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0, \end{cases}$$

then, check that $W(f, g; x) = 0$ for all $x \in \mathbb{R}$, but f and g are linearly independent. In the next slide, we'll see a correct formulation of the converse.

Wronskian and Linear Independence

Theorem

Suppose $f(x)$ and $g(x)$ are solutions of

$$y'' + p(x)y' + q(x)y = 0$$

where $p(x)$ and $q(x)$ are continuous on an interval I. Then,

1. *$f(x)$ and $g(x)$ are linearly dependent on I if and only if $W(f, g; a) = 0$ for some $a \in I$.*
2. *If $W(f, g; a) = 0$ for some $a \in I$, then $W \equiv 0$ on I.*

Thus, if there exists $b \in I$ such that $W(f, g; b) \neq 0$, then f and g are linearly independent on I.