MA-108 Ordinary Differential Equations

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D1 - Lecture 5

Recall: Uniqueness of a solution $y(x)$ of IVP:
$y' = f(x, y), \; y(x_0) = a_0$ means $y(x)$ is unique on a (connected) open interval containing x_0.

The interval of validity for a solution of an IVP will always be an open interval (connected).

(Existence) IVP $y' = f(x, y), \; y(x_0) = y_0$ has a solution on some interval containing x_0, if $f(x, y)$ is continuous on an open rectangle around (x_0, y_0).

(Uniqueness) IVP $y' = f(x, y), \; y(x_0) = y_0$ has a unique solution on some interval containing x_0, if f and $\partial f/\partial y$ are continuous on an open rectangle around (x_0, y_0).

Example

Ex. Consider the IVP

$$y' = \frac{x+y}{x-y}, \quad y(x_0) = y_0 \qquad (*)$$

If

$$f(x,y) = \frac{x+y}{x-y}, \quad \text{then} \quad \frac{\partial f}{\partial y} = \frac{2x}{(x-y)^2}$$

Here $f(x,y)$ and $\partial f/\partial y$ are continuous everywhere except on the line $y = x$.

If $x_0 \neq y_0$, there is an open rectangle R containing (x_0, y_0) that does not intersect with the line $y = x$.
Since $f(x,y)$ and $\partial f/\partial y$ are continuous on R, by existence and uniqueness theorem, if $x_0 \neq y_0$, then $(*)$ has a unique solution on some open interval containing x_0. $\square$

Ex. Consider the IVP

$$y' = \frac{10}{3}\,xy^{2/5}, \quad y(x_0) = y_0 \quad (*)$$

Q1. For what (x_0, y_0), $(*)$ has a solution?
Q2. For what (x_0, y_0), $(*)$ has a unique solution on some open interval that contains x_0?

Here $$f(x,y) = \frac{10}{3}\,xy^{2/5} \text{ and } \frac{\partial f}{\partial y} = \frac{4}{3}\,xy^{-3/5}$$

• Since $f(x,y)$ is continuous for all $(x,y) \in \mathbb{R}^2$, IVP $(*)$ has a solution for all (x_0, y_0).

• Since $f(x,y)$ and $\partial f/\partial y$ both are continuous for all (x,y) with $y \neq 0$. If $y_0 \neq 0$, there is an open rectangle R containing (x_0, y_0) s.t. f and $\partial f/\partial y$ are continuous on R. Hence IVP $(*)$ has a unique solution on some open interval containing x_0. $\square$

Ex. Consider $f(x, y) = \frac{10}{3} xy^{2/5}$ and $\partial f / \partial y = \frac{4}{3} xy^{-3/5}$.

Since $\partial f / \partial y$ is not defined for $y = 0$, it is discontinuous if $y = 0$. Hence the uniqueness theorem does not apply when $y_0 = 0$.
Hence the IVP

$$y' = \frac{10}{3} xy^{2/5}, \quad y(0) = 0 \qquad (*)$$

may have more than one solution on every open interval containing $x_0 = 0$. We will show that this is true!

By inspection $y \equiv 0$ is a solution of $(*)$.

We will show that there are non-zero solutions to the IVP (*).

Let y be a non-zero solution of $y' = \frac{10}{3} xy^{2/5}$.

Example continued ...

Then separating variables, we get

$$\frac{y'}{y^{2/5}} = \frac{10}{3}\,x$$

Integrating it, we get

$$\frac{5}{3}\,y^{3/5} = \frac{5}{3}\,(x^2 + C)$$

$$\Longrightarrow \; y(x) = (x^2 + C)^{5/3} \qquad (**)$$

Since we divided by $y^{2/5}$ to separate variables this solution is legitimate only on the open intervals where $y(x)$ does not take zero values.

However, (**) is defined for all (x, y). Differentiating it, gives

$$y' = \frac{5}{3}\,(x^2 + C)^{2/3}(2x) = \frac{10}{3}\,xy^{2/5}, \qquad \forall\, x \in (-\infty, \infty)$$

Example continued ...

Thus

$$y(x) = (x^2 + C)^{5/3} \text{ satisfies } y' = (10/3)y^{2/5}$$

on $(-\infty, \infty)$ for all C.

Now $y(0) = 0$ in $y(x) = (x^2 + C)^{5/3}$ gives $C = 0$.

Thus $\boxed{y(x) = x^{10/3}}$ is another solution of IVP (*).

Thus we have two solutions of IVP

$$y' = (10/3)y^{2/5}, \quad y(0) = 0 \qquad (*)$$

namely $y \equiv 0$ and $y(x) = x^{10/3}$.

We can construct two more solutions of IVP (*). How? $\square$

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Ex. The IVP $\quad y' = \dfrac{10}{3}\, xy^{2/5}, \quad y(0) = -1 \quad (*)$ has a unique solution on some open interval containing $x_0 = 0$.

Find a solution and the largest open interval (a, b) on which this solution is unique.

Let $y(x)$ be any solution of IVP. Since $y(0) = -1$, there is an open interval I containing $x_0 = 0$ such that y takes non-zero values on I. In this case, the general solution of ODE $y' = (10/3)\, xy^{2/5}$, is given by (solved earlier)

$$y(x) = (x^2 + C)^{5/3}$$

From $y(0) = -1$, we get $C = -1$. Hence

$$y(x) = (x^2 - 1)^{5/3} \quad \text{for} \quad x \in I \qquad (**)$$

Note $y(x) = (x^2 - 1)^{5/3}$ is a solution of $y' = (10/3)\, xy^{2/5}$, which is defined on $(-\infty, \infty)$.

Hence every solution of (*), which does not take zero-value, is given by $y(x) = (x^2 - 1)^{5/3}$ on open interval $(-1, 1)$.

This is the **unique** solution of IVP (*) on $(-1, 1)$. Why? Use Existence and Uniqueness theorem for all $x_0 \in (-1, 1)$ and initial condition $y(x_0) = (x_0^2 - 1)^{5/3}$.

$(-1, 1)$ is the **largest** interval on which IVP (*) has a **unique** solution. To see this, note that we can define another solution

$$y_1(x) = \begin{cases} (x^2 - 1)^{5/3} & , \quad -1 \leq x \leq 1 \\ 0 & , \quad |x| > 1 \end{cases}$$

This also shows that the largest interval on which the solution of (*) is unique is $(-1, 1)$. This solution can be extended on larger interval $(-\infty, \infty)$ by y_1 and y both. $\square$

Exercise. Find largest interval where $y' = \dfrac{10}{3} x y^{2/5}, y(0) = 1$ has a unique solution.

Transforming Non-Linear into Separable ODE

A non-linear differential equation $y' + p(x)y = f(x)y^r \quad (*)$, where $r \in \mathbb{R} - \{0, 1\}$ is said to be a **Bernoulli Equation**. For $r = 0, 1$, it is linear.

If y_1 is a non-zero solution of $y' + p(x)y = 0$, then putting $y = uy_1$ in (*), we get

$$\begin{aligned} u'y_1 + uy_1' + puy_1 &= fu^r y_1^r \\ \implies u'y_1 &= fu^r y_1^r \\ \implies \frac{u'}{u^r} &= f(x)(y_1(x))^{r-1} \\ \implies \frac{u^{-r+1}}{-r+1} &= \int f(x)(y_1(x))^{r-1}\,dx + C. \end{aligned}$$

Ex. Solve $y' - y = xy^2$.

Converting Non-Linear into Separable ODE

Consider $y' = f(x, y)$.
Substitute $y = uy_1$, where $y_1(x)$ is known function and $u(x)$ unknown.

$$u'y_1(x) + uy_1'(x) = f(x, uy_1(x)),$$
$$\implies u'y_1(x) = f(x, uy_1(x)) - uy_1'(x).$$

If $f(x, uy_1(x)) = q(u)y_1'(x)$ for some function u, then

$$u'y_1(x) = (q(u) - u)y_1'(x) \text{ is separable.}$$

After checking for constant solutions $u = u_0$ s.t. $q(u_0) = u_0$, we can separate variables to obtain

$$\frac{u'}{q(u) - u} = \frac{y_1'(x)}{y_1(x)}$$

Homogeneous Non-Linear Equations

Def. A differential equation $y' = f(x, y)$ is said to be **homogeneous** if it can be written as $y' = q(y/x)$.

Substitute $y = vx$, where v is an unknown function, we get

$$v'x + v = q(v) \text{ a separable ODE.}$$

Example. Solve $xy' = y + x \qquad\qquad (*)$.

Rewrite it as $y' = \dfrac{y}{x} + 1$. This is homogeneous ODE.

Substitute $y = vx$. We get $v'x + v = v + 1 \implies v'x = 1$.

By integration, $v(x) = \ln|x| + C$.

Thus the solution to (*) is $y = x(\ln|x| + C)$. $\square$

Example: Solve

$$x^2y' = y^2 + xy - x^2$$

Write the ODE as

$$y' = \frac{y^2 + xy - x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1$$

This is homogeneous. Substitute $y = vx$ to get

$$v'x+v = v^2+v-1 \implies \frac{v'}{v^2-1} = \frac{1}{2}\left(\frac{1}{v-1} - \frac{1}{v+1}\right)v' = \frac{1}{x}$$

Integration gives

$$\frac{1}{2}\left(\ln|v-1| - \ln|v+1|\right) = \ln|x| + C_1$$

$$\implies \frac{v-1}{v+1} = Cx^2 \implies v = \frac{1+Cx^2}{1-Cx^2}$$

ODE $y' = (y/x)^2 + (y/x) - 1$

Therefore

$$y = x\,\frac{1 + Cx^2}{1 - Cx^2}$$

Q. Are these all the solutions? No!

Both $y = x$ and $y = -x$ are also solutions, but only $y = x$ can be obtained from the general solution.

The solutions $\quad y = x\dfrac{1 + Cx^2}{1 - Cx^2} \quad$ were obtained in the intervals not containing 0.

Does this mean that the only solutions to the ODE, in an interval containing zero, are $y = x$ or $y = -x$?

Interval of Validity

Note that $y = x\dfrac{1+Cx^2}{1-Cx^2}$ is continuous at $x = 0$ and satisfies the ODE $x^2y' = y^2 + xy - x^2$ trivially at that point, since $y(0) = 0$.

In fact, for arbitrary $C_1, C_2 \in \mathbb{R}$, the function

$$y(x) = \begin{cases} x\,\dfrac{1+C_1x^2}{1-C_1x^2} & \text{if } x < 0 \\[2ex] x\,\dfrac{1+C_2x^2}{1-C_2x^2} & \text{if } x \geq 0 \end{cases}$$

is differentiable and satisfies the ODE $x^2y' = y^2 + xy - x^2$ with $y(0) = 0$. Thus this IVP has infinitely many solutions one for each choice of C_1, C_2.

We have noted before that the interval of validity will depend on C and hence on the initial condition.

An IVP

Let $C < 0$. Then $1 - Cx^2$ is positive and non-zero for all x
Let $C \geq 0$. Then $1 - Cx^2$ is non-zero if $x \neq 1/\sqrt{C}$.

Example. Solve IVP $\quad x^2y' = y^2 + xy - x^2, \; y(1) = 2 \qquad (*)$
Find the interval of validity of the solution.

If $y(1) = 2$, then $\dfrac{1+C}{1-C} = 2 \implies C = 1/3$.
Existence and Uniqueness theorem says, (*) has a unique solution

$$y(x) = x\,\frac{3+x^2}{3-x^2}$$

on an open interval $(a, b) \subset (-\sqrt{3}, \sqrt{3})$ containing $x_0 = 1$.
The solution is valid over the open set $\mathbb{R} - \{\pm\sqrt{3}\}$.

If possible, find the largest interval, in which this solution is unique!

We may presume this ought to be $(-\sqrt{3}, \sqrt{3})$.

However, as noted before, for any $C \in \mathbb{R}$

$$y(x) = \begin{cases} x\,\dfrac{1+Cx^2}{1-Cx^2} & \text{if } a < x < 0 \\[2ex] x\,\dfrac{3+x^2}{3-x^2} & \text{if } 0 \le x < \sqrt{3} \end{cases}$$

where, $a = \dfrac{-1}{\sqrt{C}}$ if $C > 0$ and $a = -\infty$ if $C < 0$,

is clearly a solution.

Thus the largest open interval in which IVP (*) has a *unique solution* is $(0, \sqrt{3})$. $\square$

Examples

Describe the method to solve the following differential equation and find solution.

- $y' = \dfrac{x^2 + 3x + 2}{y - 2}, y(1) = 4$ non-linear, Separable
- $(x-2)(x-1)y' - (4x-3)y = (x-2)^3$ Linear non-homogeneous
- $(1 + x^2)y' + 2xy = \dfrac{1}{(1 + x^2)y}$ Bernoulli Equation
- $y' = \dfrac{2x + y + 1}{x + 2y - 4}$ Can be converted to a separable equation, use substitution $X = x - 2$, $Y = y - 3$.
- $3x^2y^2 + 6x^3y\dfrac{dy}{dx} = 0$. Exact equation