MA-106 Linear Algebra

M.K. Keshari

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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D1 - Lecture 14

A map $T : V \to W$ is a linear transformation if $T(c_1v + c_2w) = c_1T(v) + c_2T(w)$.

A linear transformation maps subspace to a subspace.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Infact the columns of the matrix A is $T(e_1), \ldots, T(e_n)$.

Coordinate Vectors: General Case

If $\mathcal{B} = \{v_1, \ldots, v_n\}$ is an ordered basis of a vector space V, then any $v \in V$ can be uniquely written as $v = a_1 v_1 + \cdots + a_n v_n$, for scalars $a_1, \ldots, a_n$. We say that $[v]_{\mathcal{B}} = (a_1, \ldots, a_n)^T$ is the coordinate vector of v w.r.t. the basis $\mathcal{B}$. Thus

$\boxed{\text{A vector in } V \text{ can be identified with its coordinate vector in } \mathbb{R}^n}$.

Example. $\mathcal{P}_2$ is a 3 dimensional vector space with an ordered basis $B_2 = \{1, x, x^2\}$.

The coordinate vector of $v = a + bx + cx^2 \in \mathcal{P}_2$ is $[v]_{B_2} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Using the correspondence $\mathcal{P}_2 \longleftrightarrow \mathbb{R}^3$ defined as

$$v = a + bx + cx^2 \in \mathcal{P}_2 \longleftrightarrow [v]_{B_2} \in \mathbb{R}^3$$

we can identify $\mathcal{P}_2$ with $\mathbb{R}^3$.

Matrix of a Linear Transformation

Identify $\mathcal{P}_2$ with $\mathbb{R}^3$ by $v \in \mathcal{P}_2 \mapsto [v]_{B_2} \in \mathbb{R}^3$; $B_2 = \{1, x, x^2\}$. Similarly, we identify $\mathcal{P}_1$ with $\mathbb{R}^2$ using the ordered basis $B_1 = \{1, x\}$, i.e., $v \in \mathcal{P}_1 \leftrightarrow [v]_{B_1} \in \mathbb{R}^2$.

Consider $S : \mathcal{P}_2 \to \mathcal{P}_1$ defined as $S(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x$.

Want to represent S with a matrix A, using these identifications.

Therefore $S(1) = 0, S(x) = 1, S(x^2) = 2x \implies$

$Ae_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = [0]_{B_1},\ Ae_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [1]_{B_1},\ Ae_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = [2x]_{B_1} \Rightarrow$

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. **Notation:** $\boxed{A = [S]_{B_2}^{B_1}.}$

Observe: If $v = a + bx + cx^2$, then

$A([v]_{B_2}) = A\left(\begin{bmatrix} a & b & c \end{bmatrix}^T\right) = \begin{bmatrix} b \\ 2c \end{bmatrix} = [b + 2cx]_{B_1}$. Thus,

$\boxed{\text{fixing bases, we can identify a linear transformation with a matrix.}}$

Matrix of a Linear Transformation

Recall $S : V_B \to W_{B'}$ linear $\Rightarrow$ associated matrix := $\boxed{[S]_B^{B'}}$

Consider the following bases $B_1 = \{e_1, e_2\}$, and $B_2 = \{v_1 = (1,1)^T, v_2 = (-1,1)^T\}$ of $\mathbb{R}^2$. Then $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y)^T = (x-y, x+y)^T$ is linear.

Q: What is $[T]_{B_1}^{B_1}$? $\quad Te_1 = (1,1)^T = 1 \cdot e_1 + 1 \cdot e_2$ and

$$Te_2 = (-1,1)^T = (-1) \cdot e_1 + 1 \cdot e_2 \Rightarrow [T]_{B_1}^{B_1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Q: What is $[T]_{B_1}^{B_2}$? $\ Te_1 = (1,1)^T = 1 \cdot v_1 + 0 \cdot v_2$ and

$$Te_2 = (1,-1)^T = 0 \cdot v_1 + 1 \cdot v_2 \Rightarrow [T]_{B_1}^{B_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Q: Check: $[T]_{B_2}^{B_1} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ and find $[T]_{B_2}^{B_2}$.

$\boxed{\text{Matrix form of a linear transformation depends on the bases chosen}}$

1. 140020012 RUDRAJIT DAS
2. 140020015 SARDA ABHISHEK RAJESHKUMAR
3. 140020049 DEEPAK KUMAR MEENA
4. 140020050 CHANDRAPREET SINGH
5. 140020080 MANISH KUMAR
6. 140020081 RAM PAL DAHIYA
7. 140020112 GALI BABU VATAN ABSENT
8. 140020113 ROHITH KUMAR AJMEERA
9. 140020115 NIKHIL PURSHOTTAM MISKIN
10. 140050015 MOHIT VYAS
11. 140050016 ANKUR POONIYA ABSENT
12. 140050045 NANDIGAM PAVAN KUMAR
13. 140050061 RISHABH VIJAY CHAVHAN
14. 140050076 VAKACHARLA PRAMOD
15. 140050085 GOWTHAM B
16. 140050084 ANIKET MURHEKAR

Composition of L.T. $\rightarrow$ product of matrices

Let U, V, W be finite dimensional real vector spaces.

Let $S : U \rightarrow V$ and $T : V \rightarrow W$ be linear transformations.

Then the composition $T \circ S : U \rightarrow W$ is linear.

Question. What does a composition of linear transformations give in terms of matrices?

Answer. Matrix multiplication.

If $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $B : \mathbb{R}^m \rightarrow \mathbb{R}^r$ are linear,
then their composition

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m \xrightarrow{B} \mathbb{R}^r$$

is BA.

Recall $S : U \to V$ and $T : V \to W$ are linear.
Let B_1, B_2, B_3 as an ordered basis of U, V, W respectively.
Let $A = [S]_{B_1}^{B_2}$ and $B = [T]_{B_2}^{B_3}$. Then $\boxed{[T \circ S]_{B_1}^{B_3} = BA}$.

Definition. We say that a linear transformation $T : U \to V$ is an isomorphism if T is one-one and onto.

Ex. If $T : U \to V$ is an isomorphism, then there exist a linear transformation $T_1 : V \to U$ such that $T_1 \circ T = id_U$ and $T \circ T_1 = id_V$. T_1 is called the inverse of T and is unique.

Ex. If $T : U \to V$ is an isomorphism, then $[T]_{B_1}^{B_2}$ is an invertible matrix. Further, if $T_1 : V \to U$ is the inverse of T, then $[T_1]_{B_2}^{B_1}$ is the inverse of $[T]_{B_1}^{B_2}$. See $U_{B_1} \xrightarrow{T} V_{B_2} \xrightarrow{T_1} U_{B_1}$.

Change of Basis transformation

Let $T : V \to W$ be linear. Let B_1, B_1' be two basis of V and B_2, B_2' be two basis of W.

Let $A = [T]_{B_1}^{B_2}$ and $B = [T]_{B_1'}^{B_2'}$. **Q.** How is A and B related?

Since the composition $V_{B_1'} \xrightarrow{id} V_{B_1} \xrightarrow{T} W_{B_2} \xrightarrow{id} W_{B_2'}$ is T.

Note that $S_1 = [id]_{B_1'}^{B_1}$ and $S_2 = [id]_{B_2}^{B_2'}$ are invertible matrices.
Further, $[id]_{B_2'}^{B_2} = S_2^{-1}$. Hence by composition formula,

$[T]_{B_1'}^{B_2'} = S_2 A S_1$.

Most Important Case. Let $T : V \to V$ be a linear transformation. Let B and B' be basis of V. We want to relate $A = [T]_B^B$ and $A' = [T]_{B'}^{B'}$.

If $S = [id]_{B'}^{B}$, then S is invertible matrix with $S^{-1} = [id]_B^{B'}$.
Using $V_{B'} \xrightarrow{id} V_B \xrightarrow{T} V_B \xrightarrow{id} V_{B'}$, we get $\boxed{A' = S^{-1}AS}$.

Finding matrices of Linear transformations: Example 2

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by
$T(\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T) = \begin{bmatrix} x_1 + x_2 & x_2 - x_1 & x_2 \end{bmatrix}^T$.
Let $\mathcal{B}_1 = \{(u_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, u_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T\}$ and
$\mathcal{B}_2 = \{w_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T, w_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, w_3 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T\}$ be the basis for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Find the matrix for the transformation T with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$.
We know that every vector in $v \in \mathbb{R}^2$ can be written in terms of $\mathcal{B}_1$, that is, $v = s_1 u_1 + s_2 u_2$ or $[v]_{\mathcal{B}} = \begin{bmatrix} s_1 & s_2 \end{bmatrix}^T$.
Then, $T(v) = s_1 T(u_1) + s_2 T(u_2)$. As before, we would like to write this as a vector in terms of the basis $\mathcal{B}_2$.

If $T(u_1) = a_{11} w_1 + a_{21} w_2 + a_{31} w_3$ or
$[T(u_1)]_{\mathcal{B}_2} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \end{bmatrix}^T$ then
$[s_1 T(u_1)]_{\mathcal{B}_2} = \begin{bmatrix} s_1 a_{11} & s_1 a_{21} & s_1 a_{31} \end{bmatrix}^T$.

Then $([T(v)]_{\mathcal{B}_2})= \begin{bmatrix} [T(u_1)]_{\mathcal{B}_2} & [T(u_2)]_{\mathcal{B}_2} \end{bmatrix} \begin{bmatrix} s_1 & s_2 \end{bmatrix}^T$
$= [T]_{\mathcal{B}_1}^{\mathcal{B}_2} v_{\mathcal{B}_2} = A[v]_{\mathcal{B}_2}$.

Recall:$T(\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T) = \begin{bmatrix} x_1 + x_2 & x_2 - x_1 & x_2 \end{bmatrix}^T$.
$\mathcal{B}_1 = \{\begin{bmatrix} 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 \end{bmatrix}^T\}$ and
$\mathcal{B}_2 = \{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T\}$

$$\begin{aligned} [T(u_1)]_{\mathcal{B}_2} &= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T = -w_1 - 2w_2 + 2w_3 \\ [T(u_2)]_{\mathcal{B}_2} &= \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}^T = -w_2 + 2w_3 \end{aligned}$$

Then $A' = [T]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{bmatrix} -1 & 0 \\ -2 & 1 \\ 2 & 2 \end{bmatrix}$.

Let us verify how the two matrices representing T with respect to different bases, that is,

How are $A = [T]_{\mathcal{S}_1}^{\mathcal{S}_2} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$ and $A' = [T]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{bmatrix} -1 & 0 \\ -2 & 1 \\ 2 & 2 \end{bmatrix}$

related.

$$S_1 = [id]_{\mathcal{S}_1}^{\mathcal{B}_1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad S_2 = [id]_{\mathcal{B}_2}^{\mathcal{S}_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Verify that $A = [T]_{\mathcal{S}_1}^{\mathcal{S}_2} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= S_2 A' S_1.$$

Remarks

Let V and W denote finite dimensional real vector spaces, with fixed ordered basis $\mathcal{B}$ of V.

- Every linear transformation $T : V \to W$ can be represented by a matrix A for fixed ordered bases of V and W.
- If $T : V \to V$ is a linear transformation and $\mathcal{B}'$ is an ordered basis of V, then $[T]_{\mathcal{B}}^{\mathcal{B}} = S^{-1}[T]_{\mathcal{B}'}^{\mathcal{B}'}S$ where $S = [id]_{\mathcal{B}'}^{\mathcal{B}}$. We say $[T]_{\mathcal{B}}^{\mathcal{B}}$ and $[T]_{\mathcal{B}'}^{\mathcal{B}'}$ are said to be **similar.**
- Let dim $V = n$. Every vector $v \in V$ can be represented by a column vector $[v]_{\mathcal{B}} \in \mathbb{R}^n$. This representation is unique.
- We get a linear transformation $T : V \to \mathbb{R}^n$, $T(v) = [v]_{\mathcal{B}}$.
- The linear transformation T is one-one and onto. Hence it defines a **linear isomorphism**.
- Every real vector space of dimension n is isomorphic to $\mathbb{R}^n$.