PH108

Lecture 03

Co-ordinate systems

Quick review

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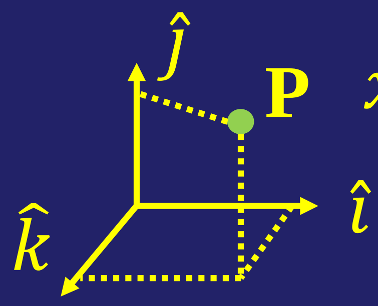
Supplementary reading: 'Div, Grad, Curl and all that' - Chapter 2

Coordinate systems

Cartesian: (x, y, z)

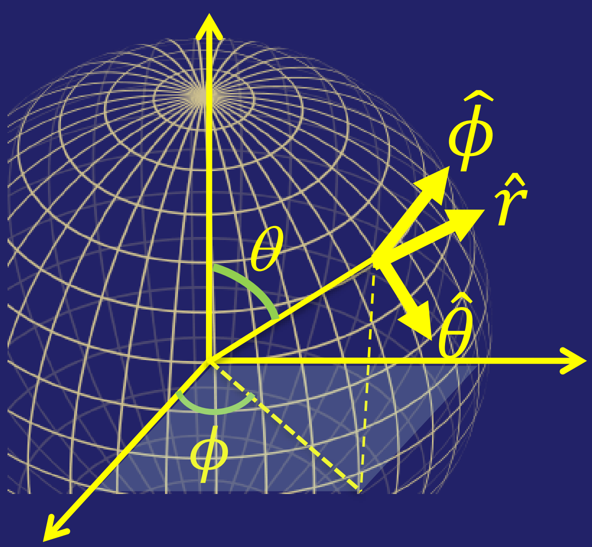
$(\hat{\imath}, \hat{\jmath}, \hat{k})$



$x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Polar: (r, θ, ϕ)

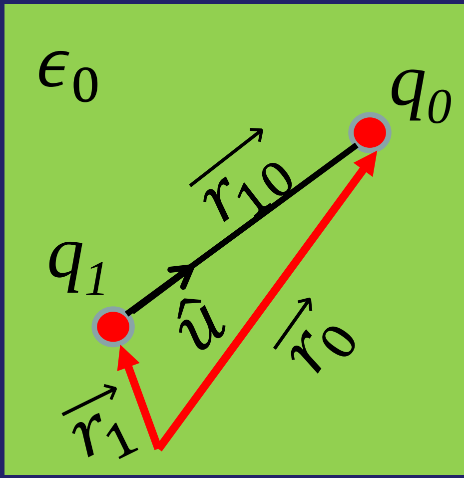
$(\hat{r}, \hat{\theta}, \hat{\phi})$



Cylindrical: (r, ϕ, z)

$(\hat{r}, \hat{\phi}, \hat{k})$

How are unit vectors defined?



Recall: force **BY** q_1 **ON** q_0 is given by:

$$\overrightarrow{F_{10}} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_0}{r_{10}^2}\hat{u}$$

Unit vector *in direction* of $\vec{r}$: $\hat{u} \equiv \frac{\overrightarrow{r_{10}}}{|\overrightarrow{r_{10}}|}$

$$\overrightarrow{r_{10}} = \overrightarrow{r_0} - \overrightarrow{r_1} = |\overrightarrow{r_0} - \overrightarrow{r_1}|\,\hat{u}$$

Why are unit vectors useful?

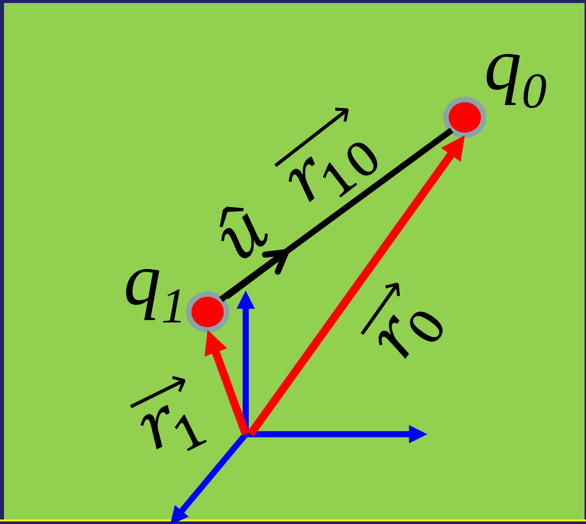
Vector math becomes *independent* of coordinate systems

You can use a coordinate system suitable for simplifying the vector math

HOW? In any particular coordinate system, project the vector onto the coordinate axes

Example: POSITION VECTOR $\vec{r}$

$$\vec{r} = (\vec{r} \cdot \widehat{u_1})\widehat{u_1} + (\vec{r} \cdot \widehat{u_2})\widehat{u_2} + (\vec{r} \cdot \widehat{u_3})\widehat{u_3}$$



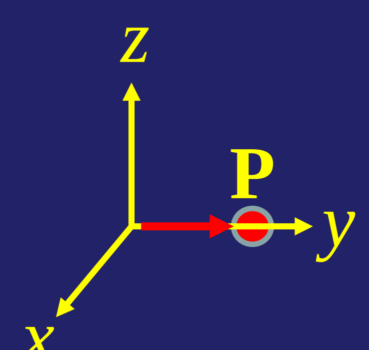
Questions

A) What are the *dimensions* of the unit vector?

1) $M^1 L^1 T^1$
2) $M^0 L^1 T^1$
3) $M^0 L^{-1} T^0$
4) $M^0 L^0 T^0$

B) In <u>polar</u> coordinates what is the <u>position</u> vector $\vec{r}$ of the point P shown at (x,y,z) = (0, 2*m*, 0)

1) $\vec{r} = 2m\,\hat{r} + \pi\,\hat{\theta}$
2) $\vec{r} = 2m\,\hat{r}$
3) $\vec{r} = 2m\,\hat{r} + \pi\,\hat{\theta} + \pi\,\hat{\phi}$
3) something else

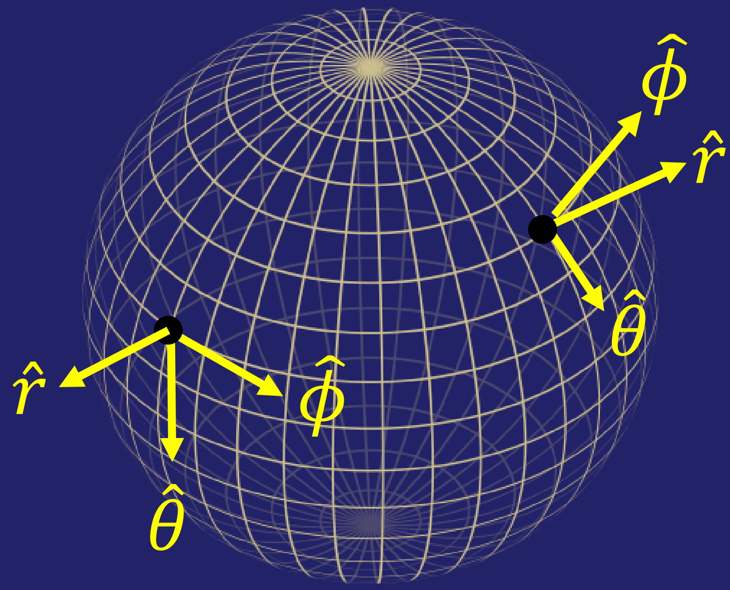


Unit vectors of polar coordinates change direction as you move to different points

Cartesian unit vectors are *fixed* $\hat{\imath}, \hat{\jmath}, \hat{k}$

constant magnitude and constant direction

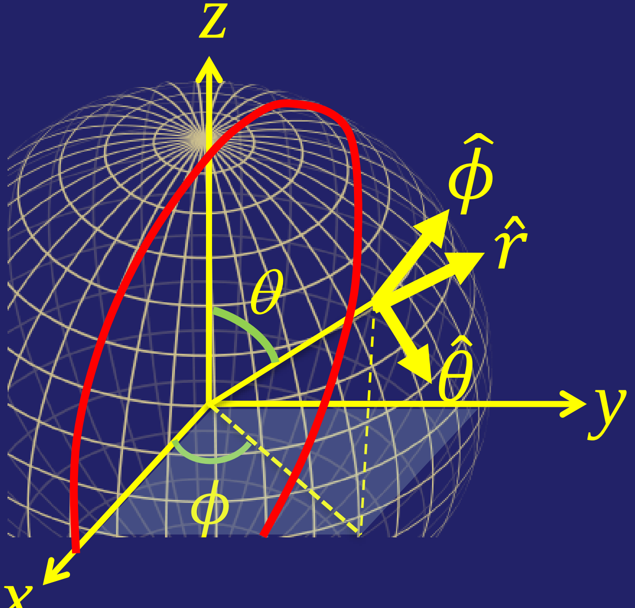
Polar unit vectors *change direction* with position



$(\hat{r}, \hat{\theta}, \hat{\phi})$ change direction → have to evaluate differential changes carefully!

Consider the position vector $\vec{r} = r\hat{r}$ Velocity: $v = \frac{d\vec{r}}{dt}$

If the particle is moving on a great circle, it makes sense to use polar coordinates

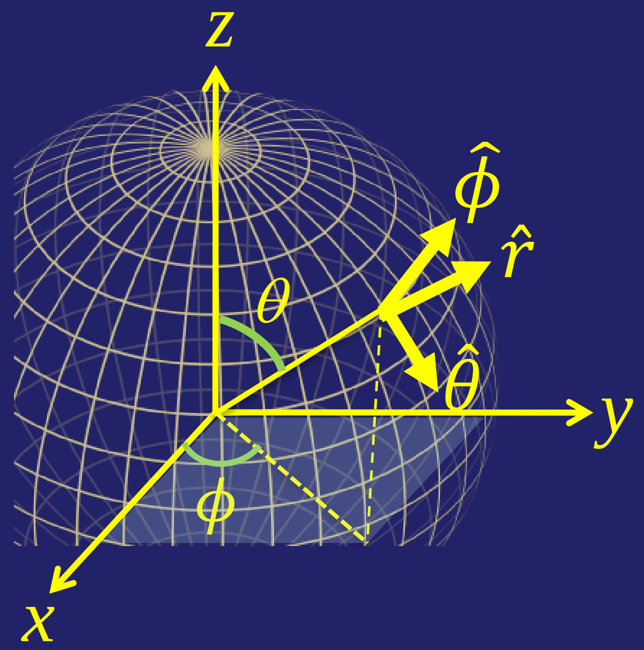


$$v = \frac{d}{dt}(\vec{r}) = \frac{\mathrm{d}}{\mathrm{dt}}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{\mathrm{d}\hat{r}}{\mathrm{dt}}$$

! $\hat{r}$ changes as a function of time

Project $(\hat{r}, \hat{\theta}, \hat{\phi})$ onto unit vectors that *do not* change as a function of time

$$\hat{r} = sin\theta cos\phi\ \hat{\imath} + sin\theta sin\phi\ \hat{\jmath} + cos\theta\ \hat{k}$$
$$\hat{\theta} = cos\theta cos\phi\ \hat{\imath} + cos\theta sin\phi\ \hat{\jmath} + sin\theta\ (-\hat{k})$$
$$\hat{\phi} = sin\phi\ (-\hat{\imath}) + cos\phi\ \hat{\jmath}$$



For the great circle in *x-z plane*, $\phi = 0$

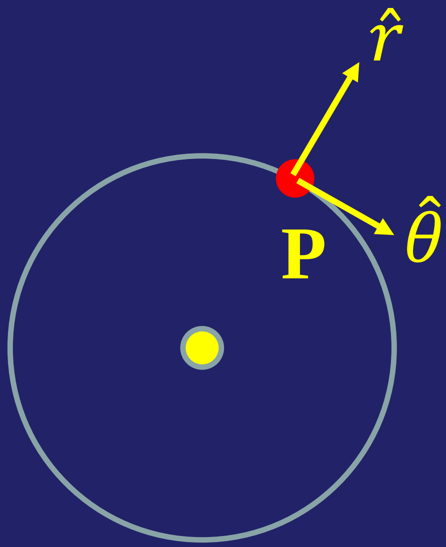
$$\frac{\mathrm{d}\hat{r}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}\left(sin\theta\ \hat{\imath} + cos\theta\ \hat{k}\right) = \frac{\mathrm{d}\theta}{dt}\hat{\theta}$$

$$v = \frac{dr}{dt}\hat{r} + r\frac{\mathrm{d}\theta}{\mathrm{dt}}\hat{\theta}$$

So $v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

Question

Consider a particle **P** moving along a circle of *constant* radius *r*



Its <u>velocity in polar coordinates</u> is:

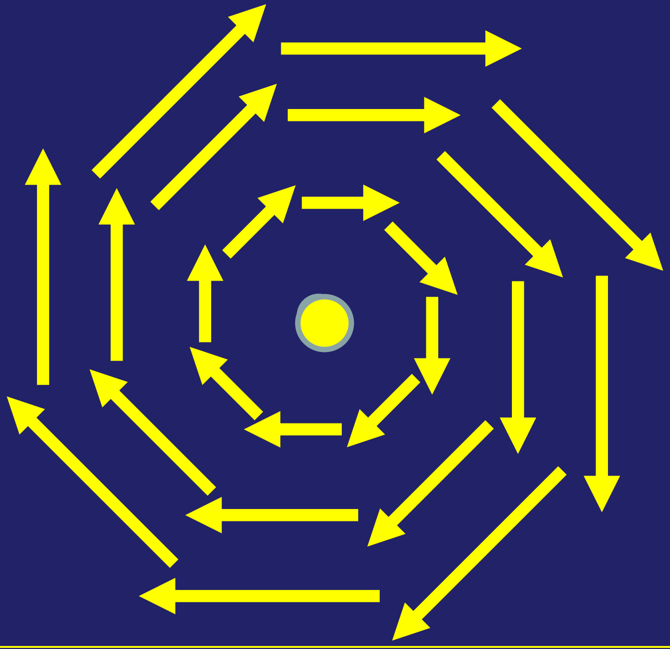
1) $v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

2) $v = r\hat{r} + r\dot{\theta}\hat{\theta}$

3) $v = r\dot{\theta}\hat{\theta}$

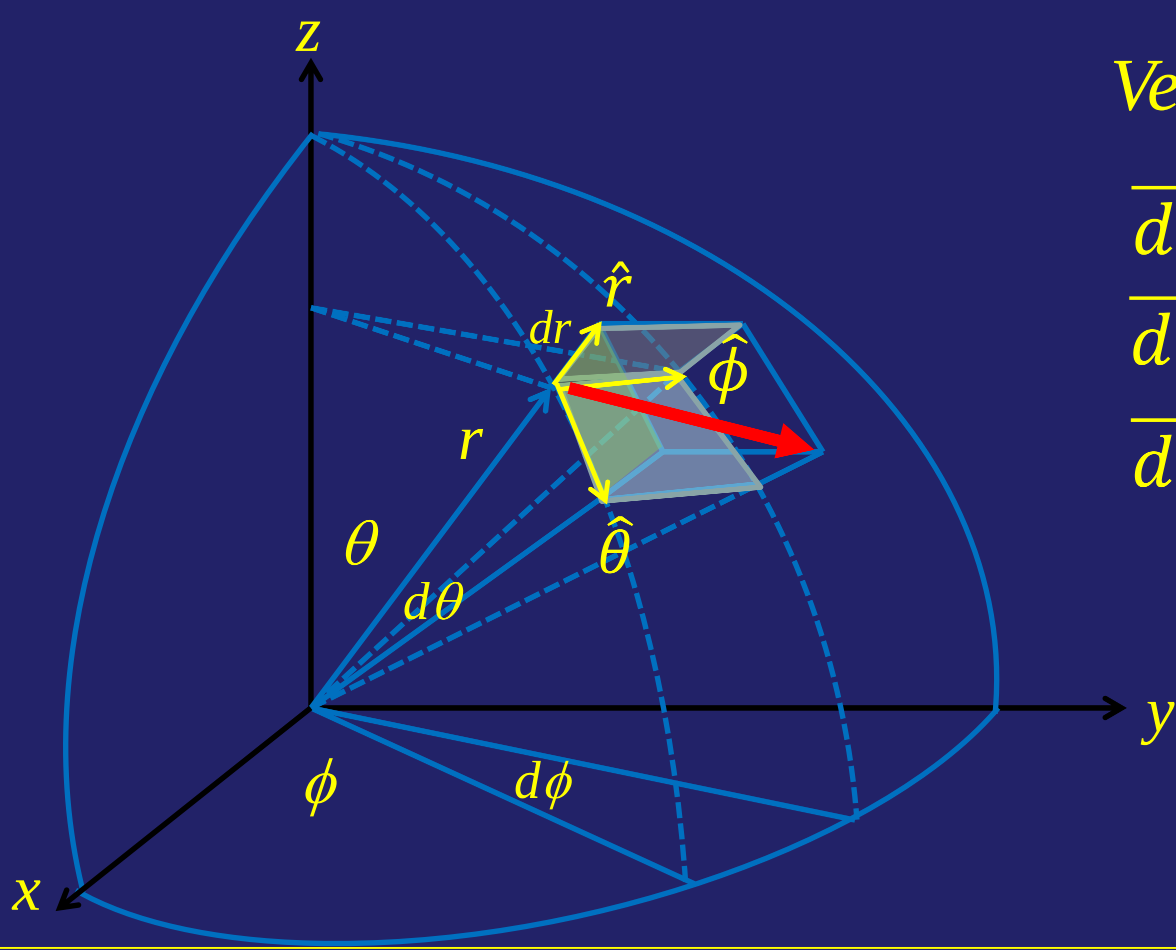
4) $v = \dot{r}\hat{r}$

Velocity vector field



Line, Area and Volume in polar coordinates

$$\overrightarrow{dl} = dr\,\hat{r} + rd\theta\,\hat{\theta} + rsin\theta d\phi\hat{\phi}$$



Vector Area elements:

$$\overrightarrow{d\sigma_r} = r^2 sin\theta d\theta d\phi\,\hat{r}$$
$$\overrightarrow{d\sigma_\theta} = rsin\theta drd\phi\,\hat{\theta}$$
$$\overrightarrow{d\sigma_\phi} = rdrd\theta\,\hat{\phi}$$

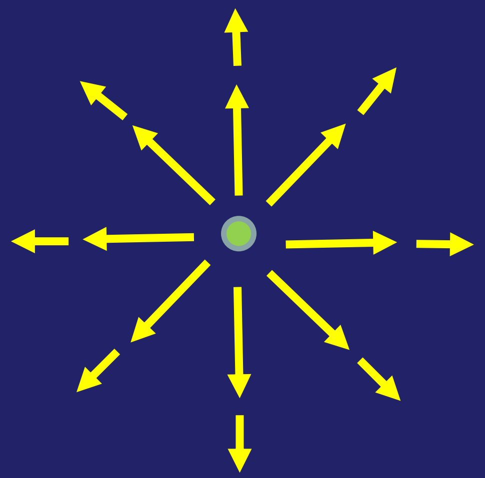
Volume element dV

$dV = r^2 sin\theta\, dr\, d\theta\, d\phi$

What is the use of all this math?

Consider a point charge q

Determine the electric field at a distance r



Think of q as a 'source' of a vector field $\boldsymbol{E}$

A trivial calculation...

All field lines leaving the source must cross the sphere

↑

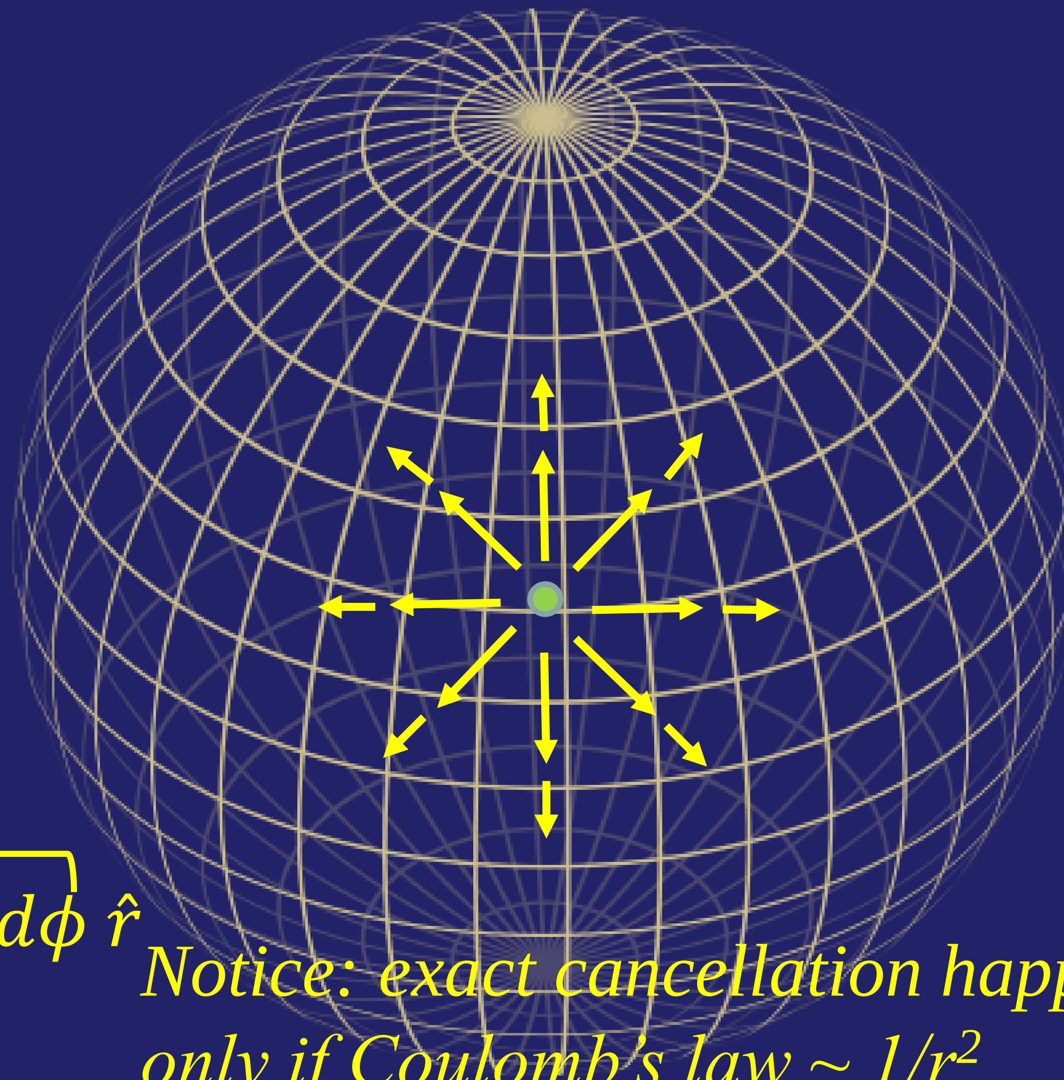
intuition

↓

Gauss's Law

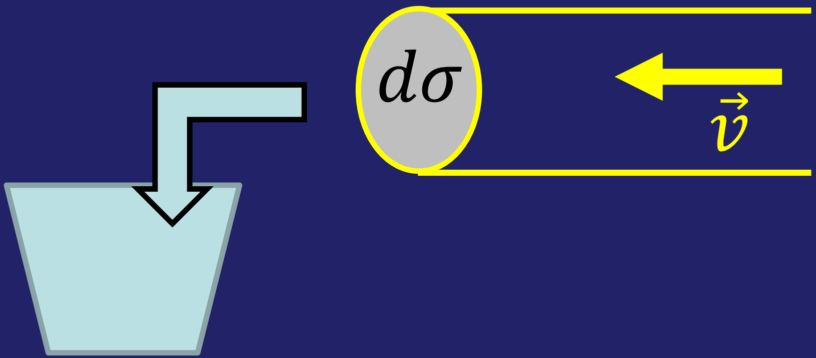
$$\oint\!\!\!\!\oint_S \vec{E}\cdot\overrightarrow{d\sigma_r} = \frac{q}{\epsilon_0}$$

$$\frac{1}{\cancel{4\pi}\epsilon_0}\frac{q}{\cancel{r^2}}\hat{r} \qquad \cancel{r^2}\overbrace{sin\theta d\theta d\phi}^{\cancel{4\pi}}\,\hat{r}$$



Notice: exact cancellation happens only if Coulomb's law ~ $1/r^2$

What is the flux of a vector field?



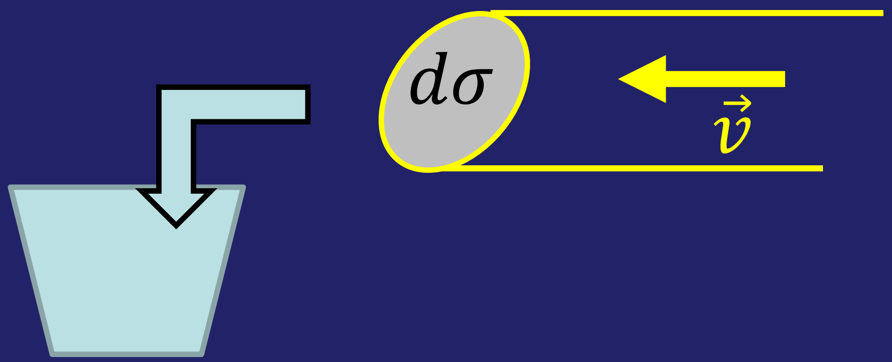
What is the rate at which water accumulates in the bucket?

i.e volume of water crossing the surface of area ΔS in time Δt?

ie Flux Φ?

$$\Phi = \frac{(v\Delta t)d\sigma}{\Delta t} = v\, d\sigma$$

Flux (latin root = flow)

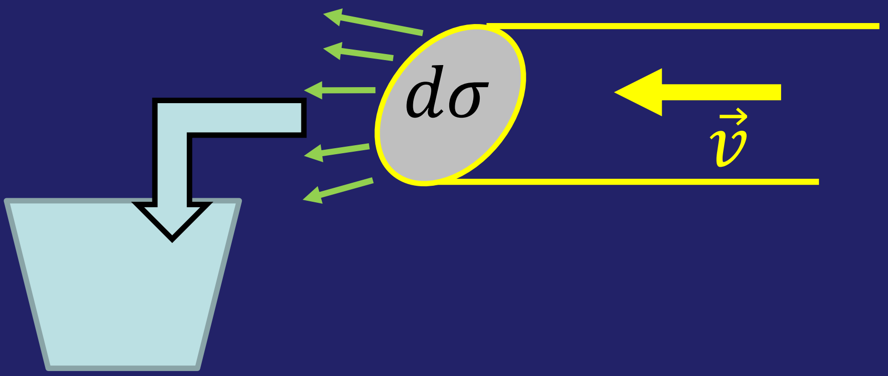


What if the pipe is sliced at an angle?

What is the rate at which water accumulates in the bucket?

ie Flux Φ? $\Phi = \vec{v} \cdot \overrightarrow{d\sigma}$

Flux: non trivial case



What if the velocity varies from point to point?

What is the rate at which water accumulates in the bucket?

ie Flux Φ?

$$\Phi = \int_S \vec{v} \cdot \overrightarrow{d\sigma}$$

Define: Divergence

Divergence of a vector field

is the net outward flux

through a closed surface

enclosing a volume

as the volume → 0

$$\nabla \cdot \overrightarrow{W} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \oint \overrightarrow{W} \cdot \overrightarrow{dS}$$

Worked examples in tutorials!