MA-106 Linear Algebra

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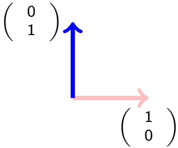
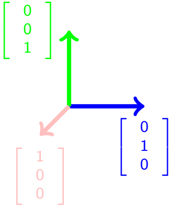
D1 - Lecture 10

Orthogonal or perpendicular vectors

The standard basis for $\mathbb{R}^2$



are vectors of length 1 and perpendicular to each other.
Similarly the standard basis of $\mathbb{R}^3$



are vectors of length 1 and perpendicular to each other.

Orthogonal vectors

Recall : the length of a vector $v = (x_1, x_2)^T \in \mathbb{R}^2$ is given by

$$\|v\| = \sqrt{x_1^2 + x_2^2} = \sqrt{v^T v}.$$

By Pythagores theorem, $v, w \in \mathbb{R}^2$ are perpendicular to each other if and only if

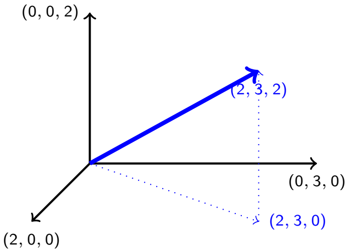
$$\begin{aligned}
\|v\|^2 + \|w\|^2 &= \|v - w\|^2 \\
v^T v + w^T w &= (v - w)^T (v - w) \\
&= (v^T - w^T)(v - w) \\
&= v^T v - w^T v - v^T w + w^T w \\
v^T v + w^T w &= v^T v - 2v^T w + w^T w \quad (\text{since } w^T v = v^T w \ )
\end{aligned}$$

Therefore, v and w are perpendicular to each other if and only if

$$v^T w = 0.$$

We will generalise this to $\mathbb{R}^3$ and then to $\mathbb{R}^n$.

length of a vector in $\mathbb{R}^3$ and $\mathbb{R}^n$



Apply Pythagores theorem to the blue triangle to get

$$||(2,3,2)|| = \sqrt{||(2,3,0)||^2 + ||(0,0,2)||^2} = \sqrt{2^2+3^2+2^2}.$$

Generalize using induction to define the length of a vector $v = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ as

$$\begin{aligned} \|v\| &= \sqrt{\|(x_1, \cdots, x_{n-1}, 0)\|^2 + \|(0, 0, \cdots, x_n)\|^2} \\ &= \sqrt{x_1^2 + \cdots + x_{n-1}^2 + x_n^2} = \sqrt{v^T v}. \end{aligned}$$

Inner product and Orthogonal set

Define the **inner product** (dot product) of two vectors $x, y \in \mathbb{R}^n$ as $x.y := x^T y$. Note that

- $x^T y = y^T x = x_1 y_1 + \cdots + x_n y_n$.
- (Bilinearity) $(x+y)^T z = x^T z + y^T z$.
  $(c\,x)^T y = c\,(x^T y) = x^T (c\,y)$.
- $x^T x = \|x\|^2 \geq 0$ and $x^T x = 0$ if and only if $x = 0$.

Definition

As done in $\mathbb{R}^2$, two vectors $v, w \in \mathbb{R}^n$ are orthogonal or perpendicular if and only if $\boxed{v^T w = 0}$ (consider Span of v, w).

A set of vectors $\{v_1, v_2, \cdots\} \subseteq \mathbb{R}^n$ is an **orthogonal set** if $v_i \neq 0$ for all i and $v_i^T v_j = 0$ for all $i \neq j$.

An orthogonal set $\{v_1, v_2, \cdots\} \subseteq \mathbb{R}^n$ with $\|v_i\| = 1$ for all i is called an **orthonormal set**.

Matrices with columns as orthogonal set

1. $$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

2. $$A = \begin{pmatrix} 2 & 1 & -2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{pmatrix}.$$

3. Use 1 and -1 only.
$$B = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

4. We will show that an orthogonal set is always linearly independent. Hence above matrices are invertible.

5. From A and B, we can get matrices with orthonormal columns by replacing each column vector v by $v/\|v\|$.

Theorem

Every orthogonal set in $\mathbb{R}^n$ is a linearly independent set.

Proof.

Let $\{v_1, \cdots, v_k\}$ be an orthogonal set in $\mathbb{R}^n$, i.e. $v_i \neq 0$ and $v_i^T v_j = 0$ for $i \neq j$.
Assume for some $a_1, \cdots, a_k \in \mathbb{R}$,

$$\begin{aligned}
a_1 v_1 + a_2 v_2 + \cdots + a_k v_k &= 0 \\
\implies (a_1 v_1 + a_2 v_2 + \cdots + a_k v_k)^T v_1 &= 0 v_1 = 0 \\
\implies (a_1 v_1^T + a_2 v_2^T + \cdots + a_k v_k^T) v_1 &= 0 \\
\implies a_1 v_1^T v_1 + a_2 v_2^T v_1 + \cdots + a_k v_k^T v_1 &= 0 \\
\implies a_1 v_1^T v_1 &= 0 \\
\implies a_1 &= 0
\end{aligned}$$

Similarly, we can show that $a_2 = \cdots = a_n = 0$.
Hence $\{v_1, \cdots, v_k\}$ is linearly independent. $\square$

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Orthogonal Subspaces

Definition

Let V and W be subspaces of $\mathbb{R}^n$. We say V and W are orthogonal to each other (notation: $V \perp W$) if every vector in V is orthogonal to every vector in W.

Example

$V = \text{Span of } \{v_1, \ldots, v_r\}$, $W = \text{Span } \{w_1, \ldots, w_t\}$. If $v_i^T w_j = 0$ for all i, j, then $V \perp W$.

Proof. If $v = a_1 v_1 + a_2 v_2 \in V$, $w = b_1 w_1 + b_2 w_2 \in W$, then

$$\begin{aligned} v^T w &= (a_1 v_1^T + a_2 v_2^T)(b_1 w_1 + b_2 w_2) \\ &= a_1 b_1\, v_1^T w_1 + a_1 b_2\, v_1^T w_2 + a_2 b_1\, v_2^T w_1 + a_2 b_2\, v_2^T w_2 \\ &= 0 \end{aligned}$$

using bilinearity.

Example

Let V be yz-plane (screen wall) and W be xz-plane (side wall). Are V and W orthogonal to each other (as subspaces of $\mathbb{R}^3$)?

No. The vector $e_3 = (0, 0, 1) \neq 0$ lies in V and W both .

Remark. If V and W are orthogonal subspaces of $\mathbb{R}^n$, then $V \cap W = (0)$. It is a necessary condition.

Is it sufficient also? No.

Example

$V =$ Span of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $W =$ Span of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Then $V \cap W = (0)$, but V and W are not orthogonal, since $\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$.

Example. $V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$, $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

Then V is orthogonal to W.

Proof. It is enough to see that both generators of V are orthogonal to the given generator of W.

Question. Can we enlarge W to W' such that $V \perp W'$ and $\dim V + \dim W' = \dim \mathbb{R}^4 = 4$.

If $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$, then $V =$ row space of A and $W \subset N(A)$.
By Rank-Nullity Theorem: $\text{rank}(A) + \text{nullity}(A) = 4$.
Since $\text{rank}(A) = 2$, we get $\text{Nullity}(A) = 2$.

Therefore if $W' = N(A)$, then $V \perp W'$ and $\dim V + \dim W' = \dim \mathbb{R}^4$.

By inspection, $W' = \text{Span}\{(1, -1, 1, 0)^T, (0, 0, 0, 1)^T\}$.

Theorem (Fundamental Theorem of Orthogonality - 1)

Let A be a $m \times n$ matrix.

1. The row space of A, $C(A^T)$ is orthogonal to $N(A)$.

2. The column space is orthogonal to the left nullspace, $N(A^T)$.

Example. Assume $\text{rank}(A) = 1$. Then A has 1 pivot. Therefore the column space has dimension 1 and row space has dimension 1.

For $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$, $C(A^T) = \text{Span}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $N(A) = \text{Span}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$,

$C(A) = \text{Span}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $N(A^T) = \text{plane } y_1 + 2y_2 + 3y_3 = 0$.

$\dim(C(A^T)) + dim(N(A)) = 1 + 1 = 2$,
$\dim(C(A)) + \dim(N(A^T)) = 1 + 2 = 3$.

Observe: $N(A) =$ set of all vectors othogonal to row space.
$N(A^T) =$ set of all vectors othogonal to column space.