MA-106 Linear Algebra

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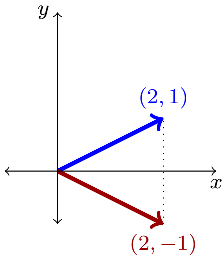
Summary

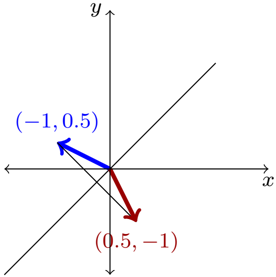
Let us collate our knowledge of special matrices so far:

- A square matrix A is invertible $\iff$ $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.
- An orthogonal matrix Q has the property that $Q^T Q = I$.
- A projection matrix is a symmetric matrix P such that $P^2 = P$.
- If A is $m \times n$ matrix A, then the map $A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $A\mathbf{x} = \mathbf{b}$ gives a correspondence between vectors in $\mathbb{R}^n$ and vectors in $\mathbb{R}^m$.
- If A was orthogonal, then $m = n$ and $\|\mathbf{b}x\| = \|\mathbf{b}\|$. Since $\|\mathbf{b}\|^2 = \|A\mathbf{x}\|^2 = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = \|\mathbf{x}\|^2$.
- If A is a projection matrix, then $A^T A = A$. Hence
  $\|A\mathbf{x}\|^2 = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^T A\mathbf{x} \leq \|\mathbf{x}\|\|A\mathbf{x}\|$
  (Cauchy-Schwarz inequality: $|v^T w| \leq \|v\|\|w\|$)
  $\implies \|A\mathbf{x}\| \leq \|\mathbf{x}\|$.

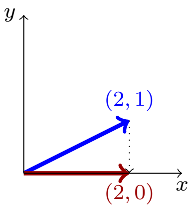
Linear Transformation



Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$. Thus A reflects vectors along the X-axis.



Let $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $B \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$. Thus, B reflects vectors along the line $x = y$.



Let $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $P \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$. Thus P projects vectors onto the X-axis.

Note that under these transformations, lines get mapped to lines. More generally linear combinations get mapped to linear combinations, that is,

$$A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1 A\mathbf{x}_1 + c_2 A\mathbf{x}_2.$$

Let V and W be vector spaces. Define a **linear transformation** $T : V \to W$ to be a function which maps linear combinations of vector to the linear combinations of their images, that is,

$$T(c_1 v + c_2 w) = c_1 T(v) + c_2 T(w)$$

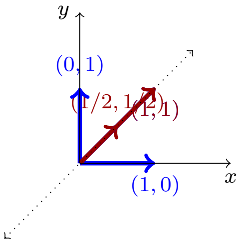
Examples

Which of the following are linear transformations?

- Let A be an $m \times n$ matrix. Define $f : \mathbb{R}^n \to \mathbb{R}^m$ by $f(\mathbf{x}) = A\mathbf{x}$.
- Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $g(x_1, x_2, x_3) = (x_1, x_2, 0)$.
- Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $h(x_1, x_2, x_3) = (x_1, x_2, 5)$. Note a linear transformation must map the zero vector to the zero vector.
- Let $R : \mathbb{R}^2 \to \mathbb{R}^4$ be defined as $R(x_1, x_2) = (x_1, 0, x_2, x_4^2)$. A linear transformation should map a subspace to a subspace.
- Let $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be defined as $T(x_1, x_2, \ldots) = (x_1 + x_2, x_2 + x_3, \ldots,)$.
- Let $S : \mathcal{P}_2 \to \mathcal{P}_1$ be defined as $S(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x$.
  Taking derivatives of polynomials is linear.

Consider the following linear transformations. Describe their behaviour on vectors.

- $P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. This transforms a vector $\begin{pmatrix} x & y \end{pmatrix}^T$ onto the vector $\begin{pmatrix} (x+y)/2 & (x+y)/2 \end{pmatrix}^T$. What does that mean geometrically?



This transforms the vector $\begin{pmatrix} 1 & 0 \end{pmatrix}^T$ to $\begin{pmatrix} 1/2 & 1/2 \end{pmatrix}^T$. This transforms the vector $\begin{pmatrix} 0 & 1 \end{pmatrix}^T$ to $\begin{pmatrix} 1/2 & 1/2 \end{pmatrix}^T$. This transforms the vector $\begin{pmatrix} 1 & 1 \end{pmatrix}^T$ to $\begin{pmatrix} 1 & 1 \end{pmatrix}^T$.

This projects a vector onto the line along $\begin{pmatrix} 1 & 1 \end{pmatrix}^T$.

- $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

  We can verify what happens on the standard basis vectors $\begin{pmatrix}1 & 0\end{pmatrix}^T$ and $\begin{pmatrix}0 & 1\end{pmatrix}^T$.

  This rotates a vector by 45 degrees.

- $G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

  Observing the transformation on the standard basis vectors, we see G projects a vector on the XY-plane.

- Recall $g : \mathbb{R}^3 \to \mathbb{R}^3$ given by $g(x_1, x_2, x_3)^T = (x_1, x_2, 0)^T$.

  and $G \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = g(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$.

Matrices of Linear transformations

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

More generally, let V and W be finite dimensional vector spaces with bases B and B'. Then any linear transformation $T : V \to W$ can be described using an appropriate matrix.

We will explain this using the following examples.

- Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by
  $T(\begin{pmatrix} x_1 & x_2 \end{pmatrix}^T) = \begin{pmatrix} x_1 + x_2 & x_2 - x_1 & x_2 \end{pmatrix}^T$.
- Let $S : \mathcal{P}_2 \to \mathcal{P}_1$ be defined as
  $S(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x$.

Finding matrices of Linear transformations

Recall $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} x_1 + x_2 \\ x_2 - x_1 \\ x_2 \end{pmatrix}$.

T is a linear transformation, so it preserves linear combinations,
$T(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = T(x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = x_1 T(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) + x_2 T(\begin{pmatrix} 0 \\ 1 \end{pmatrix})$.

It is therefore sufficient to know values of T on a basis.

If $A = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$, then $A(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = v_1$ and $A(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = v_2$.

Set $T(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} := v_1$ and $T(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} := v_2$.

Then $A\mathbf{x} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ -x_1 + x_2 \\ x_2 \end{pmatrix} = T(\mathbf{x})$

Finding matrices of Linear transformations

Recall that if $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, then $v = ae_1 + be_2 + ce_3$,

i.e. in the representation of v as a linear combination of (ordered) basis $\mathcal{S} = \{e_1, e_2, e_3\}$, a is the coefficient of e_1, b is the coefficient of e_2, c is the coefficient of e_3.

We say that $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the coordinate of vector $v \in \mathbb{R}^3$ with respect to standard (ordered) basis $\mathcal{S}$ and denote it by $[v]_{\mathcal{S}}$.

If we take another basis $\mathcal{S}' = \{e_2, e_3, e_1\}$ of $\mathbb{R}^3$, then same vector $v = be_2 + ce_3 + ae_1$. Hence the coordinate of v w.r.t. basis $\mathcal{S}'$ is $[v]_{\mathcal{S}'} = \begin{pmatrix} b \\ c \\ a \end{pmatrix}$.

If B is a (ordered) basis of a vector space V of dimension n, then any vector in V can be uniquely expressed as a linear combination of basis vector.

$\boxed{\text{Vectors in } V \text{ can be identified with its coordinate vector in } \mathbb{R}^n}$.

Example. $\mathcal{P}_2$ is a 3 dimensional vector space with an ordered basis $B_2 = \{1, x, x^2\}$.

Basis is not unique, $B_2' = \{1, x, x + x^2\}$ is another basis.

Fix basis B_2. If $v = a + bx + cx^2 \in \mathcal{P}_2$, then its coordinate vector $[v]_{B_2} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. The coordinate vector $[v]_{B_2'} = \begin{pmatrix} a \\ b - c \\ c \end{pmatrix}$, since $a1 + (b - c)x + c(x + x^2) = v$.

Using the correspondence $\mathcal{P}_2 \longleftrightarrow \mathbb{R}^3$ defined as

$v \in \mathcal{P}_2 \mapsto [v]_{B_2} \in \mathbb{R}^3$ and $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \mapsto a + bx + cx^2 \in \mathcal{P}_2$,

we can identify $\mathcal{P}_2$ with $\mathbb{R}^3$.