Tutorial-4, MA 108 (ODE)
Spring 2015, IIT Bombay

1. Find the general solution using the annihilator method (method of undetermined coefficients).

   (a) $y'' - 2y' - 3y = e^x(-8 + 3x)$.

   (b) $y'' + y = e^{-x}(2 - 4x + 2x^2) + e^{3x}(8 - 12x - 10x^2)$.

   (c) $y'' + 3y' - 2y = e^{-2x}[(4 + 20x)\cos 3x + (26 - 32x)\sin 3x]$.

   (d) $y'' + 2y' + y = 8x^2 \cos x - 4x \sin x$.

2. Let $P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$. Let y_1 be a solution to the corresponding homogeneous equation. Then making the substitution uy_1 in the differential gives a second order equation of the form $Q_0(x)u'' + Q_(x)u' = F$. This is really a first order equation in variable $z = u'$ and can be solved using the variation of parameters method. This is called the method of **reduction of order.**

   In the following exercise, given one fundamental solution of the non-linear ODE, solve the differential equation using the method of reduction of order. Also, find the other fundamental solution to the corresponding homogeneous equation.

   (a) $x^2y'' + xy' - y = 4/x^2; \quad y_1 = x$

   (b) $y'' + 4xy' + (4x^2 + 2)y = 8e^{x(x+2)}; \quad y_1 = e^{-x^2}$.

   (c) $x^2y'' - 3xy' + 4y = 4x^4; \quad y_1(x) = x^2$

   (d) $4x^2y'' - 4x(x + 1)y' + (2x + 3)y = 4x^{5/2}e^{2x}, \quad y_1 = x^{1/2}$.

   (e) $xy'' - y' + 4x^3y = 0, \ \ x > 0; \quad y_1(x) = \sin x^2$.

   (f) $x^2y'' + xy' + (x^2 - 0.25)y = 0 \ \ x > 0; \quad y(x) = x^{-1/2}\sin x$.

3. Solve the IVP given that y_1 is a solution to corresponding homogeneous equation

   (a) $x^2y'' - 3xy' + 4y = 4x^4, \quad y(-1) = 7, \ y'(-1) = 8; \ y_1 = x^2$.

   (b) $(3x - 1)y'' - (3x + 2)y' - (6x - 8)y = 0, \quad y(0) = 2, y'(0) = 3; \ y_1 = e^{2x}$.

4. This exercise presents a method for evaluating the integral

$$y = \int e^{\lambda x}(P(x)\cos \omega x + Q(x)\sin \omega x)\ dx$$

   where $\omega \neq 0$ and

$$P(x) = p_0 + p_1x + \ldots + p_kx^k, \qquad Q(x) = q_0 + q_1x + \ldots + q_kx^k.$$

   (a) Show that $y = e^{\lambda x}u$, where

$$u' + \lambda u = P(x)\cos \omega w + Q(x)\sin \omega x.$$

(b) Show that the previous equation has a particular solution of the form

$$u_p = A(x)\cos\omega x + B(x)\sin\omega x$$

where,

$$A(x) = A_0 + A_1 x + \ldots + A_k x^k \qquad B(x) = B_0 + B_1 x + \ldots + B_k x^k$$

and the pairs of coefficients $(A_k, B_k),\ (A_{k-1}, B_{k-1}), \ldots (A_0, B_0)$ can be computed successively as the solutions of pairs of equations obtained by equating the coefficients of $x^r \cos\omega x$ and $x^r \sin\omega x$ for $r = k, k-1, ..., 0$.

(c) Conclude that

$$\int e^{\lambda x}(P(x)\cos\omega x + Q(x)\sin\omega x) = e^{\lambda x}(A(x)\cos\omega x + B(x)\sin\omega x)$$

where c is a constant of integration.

(d) Evaluate $\int x^2 \cos x\ dx$ and $\int x^3 e^x \sin x\ dx$ using the above method.

5. The non-linear first order equation

$$y' + y^2 + p(x)y + q(x) = 0$$

is called the Riccati equation. Assume that p and q are continuous.

(a) Show that y is a solution of this equation if and only if $y = z'/z$, where

$$z'' + p(x)z' + q(x)z = 0.$$

(b) Show that the general solution of the Ricatti equation is

$$y = \frac{c_1 z_1' + c_2 z_2'}{c_1 z_1 + c_2 z_2}$$

where $\{z_1, z_2\}$ is a fundamental set of solutions for the equation for second order homogeneous equation obtained by change of variables.

(c) Does the formula imply that the first order non-linear equation has a two-parameter family of solutions? Explain your answer.

(d) Use the above method to solve the equation $x^2(y' + y^2) - x(x+2)y + (x+2) = 0$; given $y_1 = 1/x$ is a solution to the corresponding homogeneous equation.

6. In the absence of damping the motion of a spring-mass system satisfies the initial value problem

$$mu'' + ku = 0, \quad u(0) - a \quad u'(0) = b.$$

Solve this IVP.

7. The motion of a certain spring-mass system is governed by the ODE $u'' + 0.125u' + u = 0$, where u is measured in feet and t in seconds. If $u(0) = 2$ and $u'(0) = 0$, determine the position of the mass at any time.

8. Find a particular solution using variation of parameters method.

   (a) $y'' - 2y' + y = 14x^{3/2}e^x$.
   (b) $4y'' + y = 2\sec(t/2)$.
   (c) $y'' - 5y' + 6y = g(x)$.
   (d) $y'' - y = \dfrac{4e^{-x}}{1 - e^{-2x}}$
   (e) $x^2y'' - x(x+2)y' + (x+2)y = 2x^3 \quad x > 0$ where the fundamental set of solutions to the corresponding homogeneous equation is $\{x, xe^x\}$
   (f) $x^2y'' + xy' + (x^2 - 0.25)y = 3x^{3/2}\sin x \quad x > 0$. Note in an earlier exercise you have computed the solutions to the corresponding homogeneous equation.
   (g) $(1 - x)y'' + xy' - y = 2(x - 1)^2e^{-x}, \quad 0 < x < 1; \quad y_1(x) = e^x, \; y_2(x) = x$.
   (h) $y'' + y = \sec x \tan x$.
   (i) $y'' - 3y' + 2y = \sin e^{-x}$.

9. Find a general solution to the following differential equations, IVP where mentioned.

   (a) $y''' - y = 0$.
   (b) $y^{(4)} + 64y = 0$.
   (c) $y^{(5)} + y^{(4)} + y''' + y'' + y' + y = 0$.
   (d) $y''' - 2y'' + 4y' - 8y = 0, \quad y(0) = 0, \; y'(0) = -2, \; y''(0) = 0$
   (e) $y''' - 6y'' + 12y' - 8y = 0, \quad y(0) = 1, \; y'(0) = -1, \; y''(0) = -4$
   (f) $y^{(4)} + 2y''' - 2y'' - 8y' - 8y = 0, \quad y(0) = 5, \; y'(0) = -2, \; y''(0) = 6, \; y'''(0) = 8$.
   (g) $y^{(4)} + 2y'' + y = 0$.

10. Find the fundamental set of solutions for the following equations.

    (a) $(D^2 + 9)^3D^2y = 0$.
    (b) $D^3(D - 2)^2(D^2 + 4)^2y = 0$.
    (c) $[(D - 1)^4 - 16]y = 0$

11. Use the method of reduction of order to solve $(2-t)y''' + (2t-3)y'' - ty' + y = 0$. $t < 2$; given that $y_1(t) = e^t$ is a solution.

12. Find the general solution.

    (a) $y''' - y'' - y' + y = 2e^{-t} + 3$
    (b) $y^{(4)} - 4y'' = 3t + \cos t$.
    (c) $y''' - y'' - y' + y = e^x(7 + 6x)$.
    (d) $4y^{(4)} - 11y'' - 9y' - 2y = -e^x(1 - 6x)$.
    (e) $y''' + 3y'' + 4y' + 12y = 8\cos 2x - 16\sin 2x$.
    (f) $y^{(4)} + 3y''' + 2y'' - 2y' - 4y = -e^{-x}(\cos x - \sin x)$

13. Find a particular solution using Anhilator method. Write down the Anhilator explicitly. Do not evaluate the coefficients.

    (a) $y''' - 2y'' + y' = t^3 + 2e^t$
    (b) $y^{(4)} - y''' + y'' + y' = t^2 + 4 + t \sin t$.
    (c) $y^{(4)} + 4y'' = \sin 2t + te^t + 4$.
    (d) $y''' - 2y'' + y' - 2y = -e^x[(9 - 5x + 4x^2) \cos 2x - (6 - 5x - 3x^2) \sin 2x]$
    (e) $y^{(4)} - 7y''' + 18y'' - 20y' + 8y = e^{2x}(3 - 8x - 5x^2)$.
    (f) $y^{(4)} + 5y''' + 9y'' + 7y' + 2y = e^{-x}(30 + 24x) - e^{-2x}$.