MA-106 Linear Algebra

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D1 - Lecture 6

Recall: We have seen row reduced form R of a matrix A.

We saw how to find the null space $N(A) = N(R)$ which is the solution space of $Ax = 0$.

We defined rank of A which is the number of pivots of A.

Now we will see how to find the solution space of $Ax = b$.

Caution: If $b \neq 0$, then solving $Ax = b$ is Not the same as solving $U\,x = b$ or $R\,x = b$.

Example: $Ax = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix} \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = b.$

Convert $Ax = b$ to $U\,x = c$ and then to $R\,x = d$.

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array}\right) \to \left(\begin{array}{cccc|c} \mathbf{1} & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & -2 & -2 & b_3 - 3b_1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} \mathbf{1} & 2 & 3 & 5 & b_1 \\ 0 & 0 & \mathbf{2} & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array}\right)$$

System is consistent $\Leftrightarrow b_3 + b_2 - 5b_1 = 0$ i.e. $b_3 = 5b_1 - b_2$

Solving $Ax = b$ or $U\,x = c$ or $R\,x = d$

$Ax = b$ has a solution $\Leftrightarrow b_3 = 5b_1 - b_2$.

Example. There is no solution when $b = \begin{pmatrix}1 & 0 & 4\end{pmatrix}^T$.

Suppose $b = \begin{pmatrix}1 & 0 & 5\end{pmatrix}^T$. Then $(A\,|\,b) \to (U\,|\,c) =$

$$\left(\begin{array}{cccc|c} \mathbf{1} & 2 & 3 & 5 & b_1 \\ 0 & 0 & \mathbf{2} & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array}\right) = \left(\begin{array}{cccc|c} \mathbf{1} & 2 & 3 & 5 & 1 \\ 0 & 0 & \mathbf{2} & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} \mathbf{1} & 2 & 3 & 5 & 1 \\ 0 & 0 & \mathbf{1} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} \mathbf{1} & 2 & 0 & 2 & 4 \\ 0 & 0 & \mathbf{1} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = (R\,|\,d)$$

$Ax = b$ is reduced to solving $U\,x = c = \begin{pmatrix}1 & -2 & 0\end{pmatrix}^T$,
which is further reduced to solving $R\,x = d = \begin{pmatrix}4 & -1 & 0\end{pmatrix}^T$.

Solving $Ax = b$ ... continued

Solving $\boxed{Ax = b}$ is reduced to solving $\boxed{Rx = d}$, i.e., we want to solve

$$\begin{pmatrix} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$$

i.e., $t = 4 - 2u - 2w$ and $v = -1 - w$

Set the free variables $u = w = 0$ to get $t = 4$ and $v = -1$.

A particular solution: $x_P = \begin{pmatrix} 4 & 0 & -1 & 0 \end{pmatrix}^T$.

Ex: Check it is a solution! i.e. check $Ax = b$.

Observe: In $Rx = d$, the vector d gives values for the pivot variables, when the free variables are 0.

Solving $Ax = b$ . . . continued

From $Rx = d$, we get $t = 4 - 2u - 2w$ and $v = -1 - w$, where u and w are free.

Complete set of solutions to $Ax = b$ is

$$\begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 4 - 2u - 2w \\ u \\ -1 - w \\ w \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 0 \\ -1 \\ 0 \end{pmatrix} + u \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$= x_{\text{particular}} + x_{\text{NullSpace}}$$

General Solution of $Ax = b$

To solve $Ax = b$ completely, reduce to $Rx = d$. Then:

1. Find $x_{\text{NullSpace}}$, i.e., $N(A)$, by solving $Rx = 0$.

2. Set free variables $= 0$ and solve $Rx = d$ for pivot variables.

This is a particular solution: $x_{\text{particular}}$.

3. Complete solutions: $\boxed{x_{\text{complete}} = x_{\text{particular}} + x_{\text{NullSpace}}}$

Ex: Verify geometrically for a 1×2 matrix, say $A = \begin{pmatrix} 1 & 2 \end{pmatrix}$.

The Column Space of A

Main Q2: When does $Ax = b$ have a solution?

If $Ax = b$ has a solution, then there exist $x_1, \dots, x_n \in \mathbb{R}$ such that $\begin{pmatrix} A_1 & \cdots & A_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 A_1 + \cdots + x_n A_n = b$, i.e.,

b can be written as a linear combination of the columns of A.

The *column space* of A, denoted by $C(A)$

is the set of all linear combinations of the columns of A

$= \{b$ in $\mathbb{R}^m$ such that $Ax = b$ is consistent$\}$.

Finding $C(A)$: Consistency of $Ax = b$

Example: Let $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$. Then

$Ax = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}^T$ has a solution if and only if $\boxed{-5b_1 + b_2 + b_3 = 0}$. Therefore $C(A)$ is a plane in $\mathbb{R}^3$ passing through the origin with normal vector $\begin{pmatrix} -5 & 1 & 1 \end{pmatrix}^T$.

- $a = \begin{pmatrix} 1 & 0 & 4 \end{pmatrix}^T$ is not in $C(A)$ as $Ax = a$ is inconsistent.
- $b = \begin{pmatrix} 1 & 0 & 5 \end{pmatrix}^T$ is in $C(A)$ as $Ax = b$ is consistent.

To write b as a linear combination of columns of A, solve $Ax = b$.

$x = \begin{pmatrix} 4 & 0 & -1 & 0 \end{pmatrix}^T$ is a solution of $Ax = \begin{pmatrix} 1 & 0 & 5 \end{pmatrix}^T$. Hence

$$\boxed{\begin{pmatrix} 1 & 0 & 5 \end{pmatrix}^T = 4A_1 + (-1)A_3.}$$

Linear Combinations in $C(A)$

We have $\begin{pmatrix}1 & 0 & 5\end{pmatrix}^T = 4A_1 + (-1)A_3$.

Q: Can we write b as a different combination of $A_1, \ldots, A_4$?
Yes. Since $\begin{pmatrix}-2 & 1 & 0 & 0\end{pmatrix}^T$ is in $N(A)$, $-2A_1 + A_2 = 0$.
Hence for any scalar c

$$\begin{pmatrix}1 & 0 & 5\end{pmatrix}^T = 4A_1 + (-1)A_3 + c(-2A_1 + A_2)$$

- If b is in $C(A)$, b can be written as a linear combination of the columns of A in as many ways as the solutions of $Ax = b$.
- If $b_1, b_2 \in C(A)$, then for any scalars c_1, c_2,

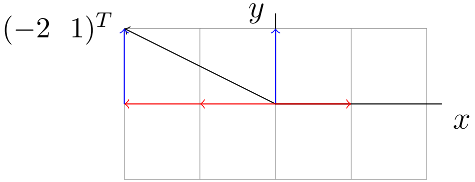
$$c_1 b_1 + c_2 b_2 \in C(A)$$

$\boxed{\text{Thus, } C(A) \text{ is } \textit{closed under} \text{ linear combinations.}}$

Vector Spaces: $\mathbb{R}^{\mathbf{n}}$

We begin with $\mathbb{R}^1$, $\mathbb{R}^2, \ldots, \mathbb{R}^n$, etc., where $\mathbb{R}^n$ consists of all column vectors of length n, i.e.,
$\mathbb{R}^n = \{\overline{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^T$, where $x_1, \ldots, x_n$ are in $\mathbb{R}\}$.
We can add two vectors, and we can multiply vectors by real numbers, i.e., we can take linear combinations in $\mathbb{R}^n$.

Examples:
$\mathbb{R}^1$ is the real line, $\mathbb{R}^3$ is the usual 3-dimensional space, and $\mathbb{R}^2$ is represented by the x-y plane; the x and y co-ordinates are given by the two components of the vector.



Vector Spaces: Definition

Defn. A non-empty set V is a vector space if it is *closed under* vector addition (i.e., if $\overline{x}$, $\overline{y}$ are in V, then $\overline{x}+\overline{y}$ must be in V) and scalar multiplication, (i.e., if $\overline{x}$ is in V, u is in $\mathbb{R}$, then $u \cdot \overline{x}$ must be in V).

Equivalently,

$$\boxed{\overline{x}, \overline{y} \text{ in } V,\ u, v \text{ in } \mathbb{R},\ \implies u \cdot \overline{x} + v \cdot \overline{y} \text{ must be in } V.}$$

• A vector space is a triple $(V, +, \cdot)$ with vector addition $+$ and scalar multiplication $\cdot$

• The elements of V are called vectors and the scalars are chosen to be real numbers (for now).

• If the scalars are allowed to be complex numbers, then V is a *complex* vector space.

Vector Spaces: Properties

Let $\overline{x}$, $\overline{y}$ and $\overline{z}$ be vectors, u and v be scalars. The vector addition and scalar multiplication are also required to satisfy:

- $\overline{x} + \overline{y} = \overline{y} + \overline{x}$ Commutativity of addition
- $(\overline{x} + \overline{y}) + \overline{z} = \overline{x} + (\overline{y} + \overline{z})$ Associativity of addition
- There is a unique vector 0, such that $\overline{x} + 0 = \overline{x}$
  Existence of additive identity
- For each $\overline{x}$, there is a unique $-\overline{x}$ such that $\overline{x} + (-\overline{x}) = 0$
  Existence of additive inverse
- $1 \cdot \overline{x} = \overline{x}$ Existence of unity
- $(uv) \cdot \overline{x} = u \cdot (v \cdot \overline{x})$ Associativity of scalar multiplication
- $(u + v) \cdot \overline{x} = u \cdot \overline{x} + v \cdot \overline{x}, \quad u \cdot (\overline{x} + \overline{y}) = u \cdot \overline{x} + u \cdot \overline{y}$
  Distributivity

Vector Spaces: Examples

1. $V = 0$, the space consisting of only the zero vector.
2. $V = \mathbb{R}^n$, the n-dimensional space.
3. $V = \mathbb{R}^\infty$, vectors with infinite number of components, e.g., $\overline{x} = (1, 1, 2, 3, 5, 8, \ldots)$
4. $V = M$, the set of 2×2 matrices.
   Q: Is this the same as $\mathbb{R}^4$?
5. $V = C([0, 1], \mathbb{R})$, the set of continuous real-valued functions from closed interval $[0, 1]$.
   e.g., x^2, e^x. Is $\frac{1}{x}$ a vector in V? N0.
   How about $\frac{1}{x-2}$? Yes.
   vector addition and scalar multiplication are pointwise:
   $(f + g)(x) = f(x) + g(x)$ and $(u \cdot f)(x) = uf(x)$.

New frame added for clarification

1. Let $V = C([0,1],\mathbb{R})$. Then V is a vector space. Any element in a vector space is called a vector. In V, vectors are continuous functions $f : [0,1] \to \mathbb{R}$. The zero element of V is the zero function.
2. The set P of all polynomial functions from $\mathbb{R} \to \mathbb{R}$ is a vector space. A typical element of P is a polynomial function $a_0 + a_1 x + \ldots + a_n x^n : \mathbb{R} \to \mathbb{R}$ with $a_i \in \mathbb{R}$. The zero element of P is zero polynomial.
3. Let M, V and P be vector spaces considered above. Let W be the set consisting of ordered tuple (m,v,p), where $m \in M, v \in V, p \in P$. Define addition and scalar multiplication componentwise. Then W is a vector space.