PH108

Lecture 14: Multipole expansion – 2

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Supplementary reading: Griffiths Section 3.4
Any good reference on Dirac delta functions, eg: `http://goo.gl/NAOofd`

Multipole expansion of *V(r)* due to *ρ*

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\theta')\rho(\boldsymbol{r}')d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\boldsymbol{r}')d\tau' \quad \text{Monopole}$$

$$+\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\theta'\rho(\boldsymbol{r}')d\tau' \qquad \text{Dipole}$$

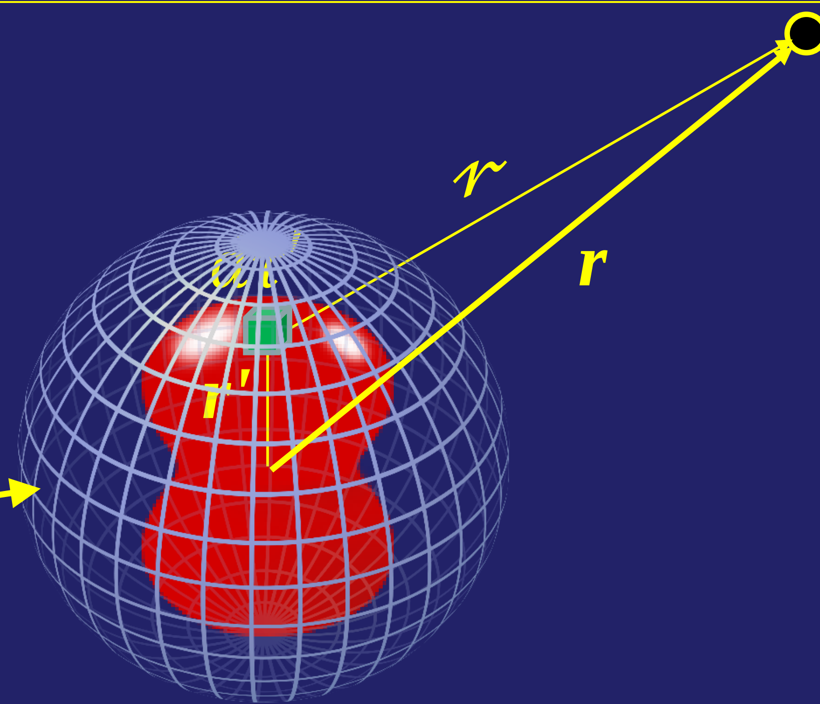
$$+\frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\boldsymbol{r}')d\tau' \quad \text{Quadrupole}$$

+ … *higher order 'pole' terms* $\frac{1}{r^4}, \frac{1}{r^5}$ …

We will look at the first few terms in more detail

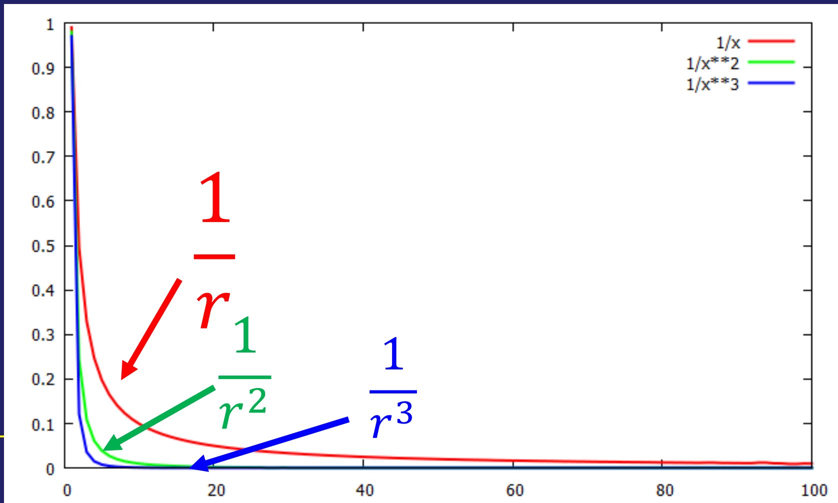
Monopole : Net total charge

$$\frac{1}{4\pi\epsilon_0}\frac{1}{r}\underbrace{\int \rho(\boldsymbol{r}')d\tau'}_{=\,Q}$$



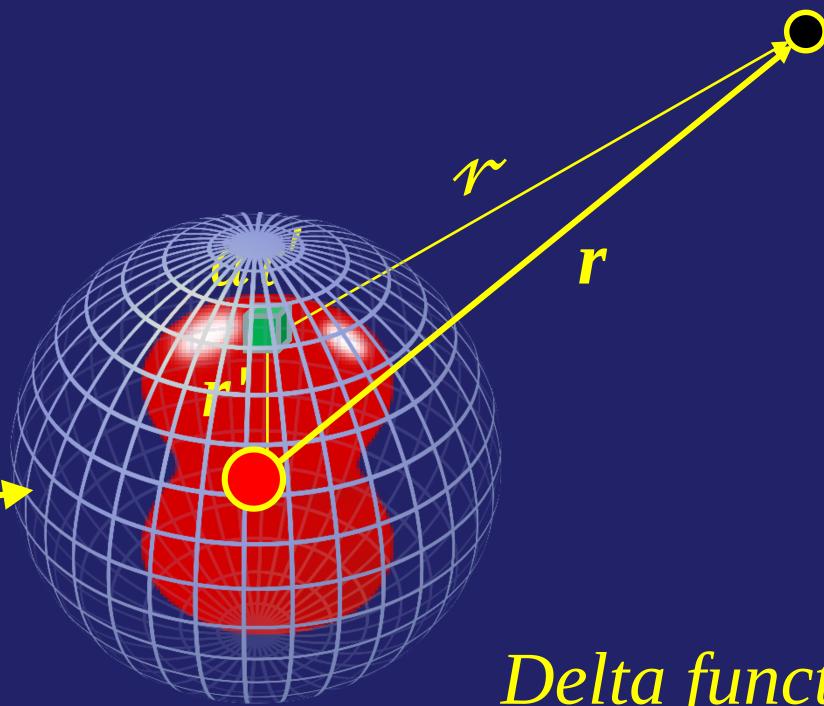
At large *r*: $V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$

is the 'dominant' term



Monopole : Net total charge

$$\frac{1}{4\pi\epsilon_0}\frac{1}{r}\underbrace{\int \rho(\boldsymbol{r}')d\tau'}_{= Q}$$



Delta function

$\delta(x) \to \infty \; at \; x = 0, elsewhere = 0$

$\int \delta(x)dx = 1 \qquad \int f(x)\delta(x)dx = f(0)$

If Q is point charge at origin,
Monopole V*(r)* is EXACT

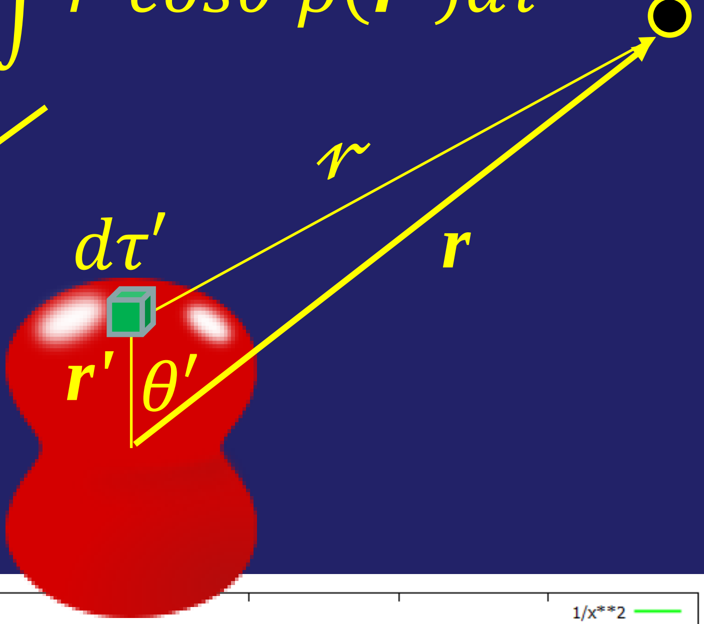
Point charge at origin: $\rho(r) = Q\delta(r) \to \int \to Q$; All higher multipoles have $\int r^n\delta(r)dr = 0$

Note: for 3D, replace $\delta(r)$ by $\delta^3(r)$ & volume integrals

Dipole term dominates if *Q*= *0*

$$\frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \cancel{\rho(\mathbf{r}')}d\tau' + \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'cos\theta'\rho(\mathbf{r}')d\tau'$$

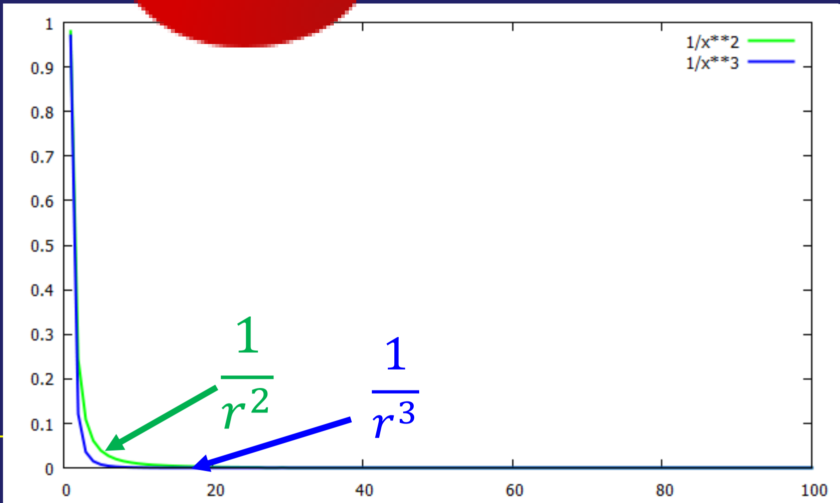
0



$$r'cos\theta' = \hat{r}\cdot\vec{r}'$$

$$\hat{r}\cdot\underbrace{\int \vec{r}'\rho(\vec{r}')d\tau'}_{\equiv\ \vec{p}\ \text{Dipole moment}}$$

$$\boxed{V_{dipole}(r) = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2}}$$



How to calculate dipole moment? Examples …

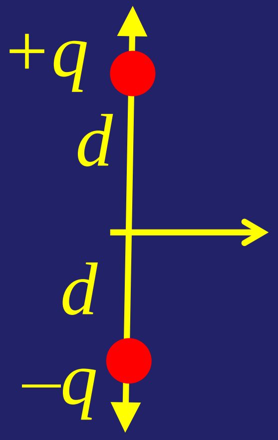
$$\int \vec{r}'\rho(\vec{r}')d\tau'$$

This is a *vector* quantity because of $\vec{r}'$

This is a *volume integral* over $\rho(\vec{r}')$

Example 1: Two equal and opposite point charges

Located at $\vec{r} = +d\hat{k}$ and $\vec{r} = -d\hat{k}$



$$\rho(\vec{r}') = q\delta^3\left(\vec{r} - d\hat{k}\right) - q\delta^3\left(\vec{r} + d\hat{k}\right)$$

$$\vec{p} = \int \vec{r}'\rho(\vec{r}')d\tau' = qd\hat{k} - q\left(-d\hat{k}\right) = 2qd\hat{k}$$

Dipole moment of collection of point charges located at $\vec{r}_1', \vec{r}_2'$ ...

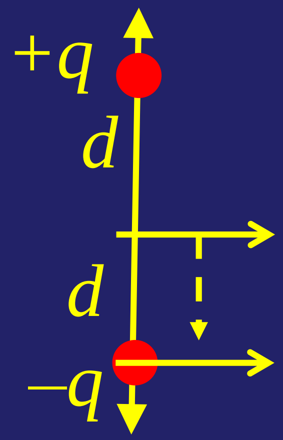
$$\rho(\vec{r}') = q_1\delta^3(\vec{r}' - \vec{r}_1') + q_2\delta^3(\vec{r}' - \vec{r}_2') + \cdots$$

$\int \vec{r}'\rho(\vec{r}')d\tau'$ will convert each $\delta^3(\vec{r}' - \vec{r}_i')$ into $\vec{r}_i'$

$$\vec{p} = \int \vec{r}'\rho(\vec{r}')d\tau' = \sum q_i\vec{r}_i'$$

Choice of origin affects dipole moments

$$\vec{p} = qd\hat{k} - q\left(-d\hat{k}\right) = 2qd\hat{k}$$



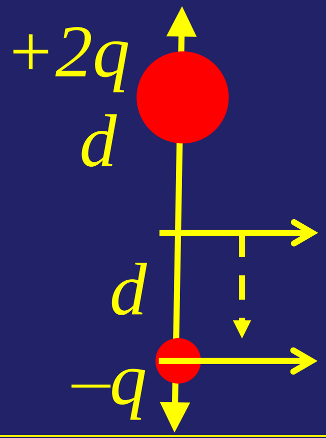
Shift origin to $-d\hat{k}$

New co-ordinates are $+2d\hat{k}$ and 0

$$\vec{p}_{new} = +q2d\hat{k} - q(0) = 2qd\hat{k} = \vec{p}_{old}$$

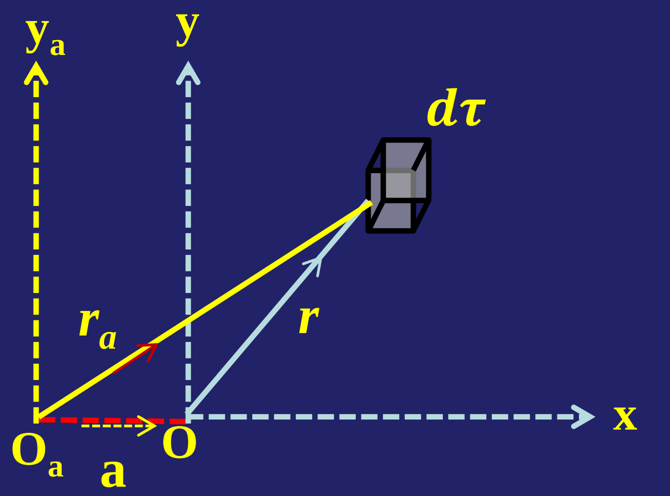
If charges are $+2q, \ -q$

$$\vec{p}_{new} = +2q2d\hat{k} - q(0) = 4qd\hat{k} \boxed{\neq \vec{p}_{old}}$$



In general, dipole moment is independent of coordinates if total $Q=0$

$$\vec{p} = \int \vec{r}\rho d\tau$$



Change origin so $\vec{r} \to \vec{r} - \vec{a}$

$$\vec{p}_{new} = \int (\vec{r} - \vec{a})\rho d\tau$$

Then $\vec{p}_{new} = \vec{p}_{old}$

$$\vec{p}_{new} = \int \vec{r}\rho d\tau - \vec{a}\int \rho d\tau$$

If $Q_{tot}=0$

Why no 'tripoles' ?

It does not matter *how many* point charges are present

As long as $\sum q_i \vec{r}_i' \neq 0$ the dominant term in multipole expansion is $\sim \frac{1}{r^2}$

$$\vec{p} = \int \vec{r}'\rho(\vec{r}')d\tau' = \sum q_i \vec{r}_i'$$

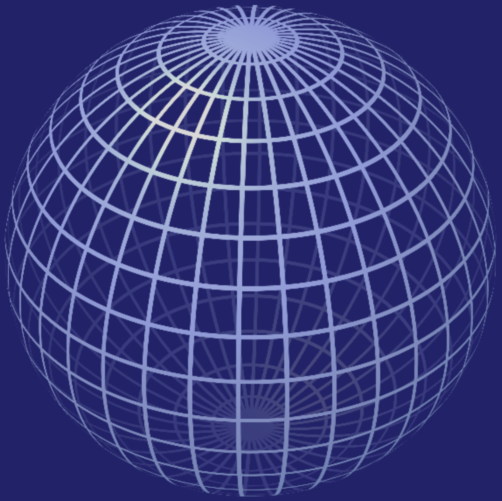
$$\boxed{V_{dipole}(r) = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2}}$$

Other examples of dipole moment

Spherical SHELL with surface charge $\sigma = \sigma_0 cos\theta$

All volume $\int \ldots d\tau$ become surface $\int \ldots dS$

Will it have monopole moment?



Will it have dipole moment?

Visualize the charge density

Spherical shell charge: dipole moment

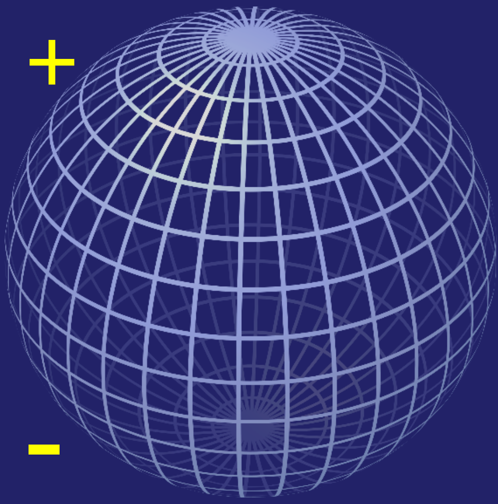
Spherical SHELL with surface charge $\sigma = \sigma_0 cos\theta$

All volume $\int \ldots d\tau$ become surface $\int \ldots dS$

Will it have monopole moment?

NO

$$\int \sigma dS = \int \sigma_0 cos\theta \; R^2 sin\theta d\theta d\phi = 0$$



Spherical shell charge: dipole moment

Spherical SHELL with surface charge $\sigma = \sigma_0 cos\theta$

All volume $\int \ldots d\tau$ become surface $\int \ldots dS$

Will it have dipole moment?

YES

$$\vec{p} = \int \vec{r}\sigma dS = \int (Rsin\theta \cancelto{0}{cos\phi}\hat{\imath} + Rsin\theta \cancelto{0}{sin\phi}\hat{\jmath} + Rcos\theta\hat{k}) * \sigma_0 cos\theta\, R^2 sin\theta d\theta d\phi$$

$$\boxed{\vec{p} = \sigma_0 \frac{4}{3}\pi R^3\, \hat{k}}$$

If there is ϕ symmetry, $\vec{p} \sim \hat{k}$

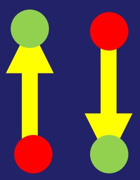
When do higher multipoles matter?

When $\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')d\tau' = 0$, next term $\sim \frac{1}{r^3}$ dominates

Example: for point charges, if $\sum q_i\vec{r}_i' = 0$

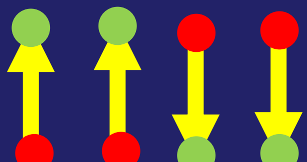
This happens if:

Two dipoles with equal and opposite $\vec{p}$ cancel



Quadrupole
"4 charges"

Two quadrupoles cancel



Octupole
"8 charges"

More poles in tutorials !

Details of the Dirac delta function sketched on slide 5

(These are the useful working formulae. For a review: `http://goo.gl/NAOofd`

The role of a $\delta(x - x_0)$ function is to 'pick' out the value of $f(x)$ at $x = x_0$

This is how it works in 1,2,3 dimensions:

1D: $\int_{-\infty}^{+\infty} f(x)\delta(x - x_0) = f(x_0)$ with normalization $\int_{-\infty}^{+\infty} \delta(x) \equiv 1$

2D: on a circle of radius *R*:

$$\iint_{r.\theta=0}^{r=\infty,\theta=2\pi} f(r,\theta)\delta^2(r-R)rdrd\theta = \int_0^{2\pi} Rf(R,\theta)\ d\theta$$

Represent a ring with uniform charge $\lambda : f(r,\theta)\delta^2(r-R) = \lambda\delta^2(r-R) \rightarrow 2\pi\lambda R$

3D: on a spherical shell of radius *R*: surface charge σ_0

$$\iiint_{r=0,\theta=0,\phi=0}^{r=\infty,\theta=\pi,\phi=2\pi} f(r,\theta,\phi)\delta^3(r-R)r^2 sin\theta d\theta d\phi = 4\pi R^2\sigma_0$$