MA-108 Ordinary Differential Equations

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D1 - Lecture 14

Recall: we defined an improper integral

$$\int_a^{\infty} g(t)\,dt = \lim_{T\to\infty} \int_a^{T} g(t)\,dt$$

its convergence and divergence.

The Laplace transform of $f(t),\ t \geq 0$ is

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

defined for those values of s for which the improper integral converges.

Ex. Find the Lapalace transform $F(s)$ of $f(t) = 1$.

$$F(s) = \int_0^\infty e^{-st}\, dt$$

$$= \lim_{T\to\infty} \frac{1}{s}(1 - e^{-sT})$$

$F(s) \to \dfrac{1}{s}$ for $s > 0$ and diverges for $s < 0$. For $s = 0$ also $F(s)$ diverges. We write this as

$$L(1) = \frac{1}{s}, \quad s > 0 \quad \text{or} \quad 1 \leftrightarrow \frac{1}{s}, \quad s > 0$$

Convention. Instead of writing $\lim_{T\to\infty}$ everytime, we will write directly as

$$\int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st}|_0^\infty = \begin{cases} \dfrac{1}{s} & , \quad s > 0 \\ \infty & , \quad s < 0 \end{cases}$$

Laplace Transforms

Ex. Find Laplace transform of $f(t) = t$.

For $s \leq 0$, $F(s)$ diverges. For $s > 0$,

$$F(s) = \int_0^\infty e^{-st} t \, dt$$

$$= -\frac{1}{s} t e^{-st} |_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt$$

$$= \frac{1}{s^2}$$

Thus

$$L(t) = \frac{1}{s^2}, \quad s > 0$$

Exercise.

- For $a \in \mathbb{R}$, $L(e^{at}) = \dfrac{1}{s-a}$, $s > a$.
- $L(te^{at}) = \dfrac{1}{(s-a)^2}$, $s > a$.
- For $n \geq 1$, $L(t^n) = \dfrac{n!}{s^{n+1}}$, $s > 0$
- For $\omega \in \mathbb{R}$, $L(\sin \omega t) = \dfrac{\omega}{s^2+\omega^2}$, $s > 0, \omega \in \mathbb{R}$

$$L(\cos \omega t) = \frac{s}{s^2+\omega^2}, \; s > 0$$

Theorem (Linearity property)

Suppose $L(f_i)$ is defined for $s > s_i$ for $1 \leq i \leq n$. Let s_0 be maximum of s_i's and $c_i \in \mathbb{R}$. Then

$$L(c_1 f_1 + \ldots + c_n f_n) = c_1 L(f_1) + \ldots + c_n L(f_n), \quad s > s_0$$

Ex. We know $L(e^{at}) = \dfrac{1}{s-a},\ s > a$. Then for $b \neq 0$.

$$L(\cosh bt) = L\left(\frac{e^{bt} + e^{-bt}}{2}\right) = \frac{1}{2}\left(\frac{1}{s-b} + \frac{1}{s+b}\right)$$

First one is defined for $s > b$ and second for $s > -b$. Hence

$$L(\cosh bt) = \frac{s}{s^2 - b^2},\ s > |b|$$

Ex. $L(\sinh bt) = L\left(\dfrac{e^{bt} - e^{-bt}}{2}\right) = \dfrac{b}{s^2 - b^2},\ s > |b|$

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Theorem (First Shifting Theorem)

If $L(f(t)) = F(s)$ *for* $s > s_0$, *then* $L(e^{at}f(t)) = F(s-a)$ *for* $s > s_0 + a$.

Proof.

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad s > s_0$$

$$\implies F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt, \quad s - a > s_0$$

$$\implies F(s-a) = L(e^{at} f(t)), \quad s > a + s_0$$

Ex. $L(1) = \dfrac{1}{s},\ s > 0 \implies L(e^{at}) = \dfrac{1}{s-a},\ s > a$

Ex. $L(t^n) = \dfrac{n!}{s^{n+1}},\ s > 0 \implies L(t^n e^{at}) = \dfrac{n!}{(s-a)^{n+1}},\ s > a$

Ex. $L(\sin \omega t) = \dfrac{\omega}{s^2+\omega^2},\quad L(\cos \omega t) = \dfrac{s}{s^2+\omega^2},\quad s > 0$

$$\implies L(e^{at} \sin \omega t) = \frac{\omega}{(s-a)^2+\omega^2},\ s > a$$

$$L(e^{at} \cos \omega t) = \frac{s-a}{(s-a)^2+\omega^2},\ s > a$$

Ex. $L(e^{at} \sinh bt) = \dfrac{b}{(s-a)^2-b^2},\ s > a + |b|$

Ex. $L(e^{at} \cosh bt) = \dfrac{s-a}{(s-a)^2-b^2},\ s > a + |b|$

Existence of Laplace transform

Not every function has a Laplace transform. Since

$$\int_0^\infty e^{-st} e^{t^2} \, dt = \infty$$

for every real s, the function $f(t) = e^{t^2}$ does not have a Laplace transform. We would like to know which functions have Laplace transform. Recall

$$f(t_0+) = \lim_{t \to t_0+} f(t); \quad f(t_0-) = \lim_{t \to t_0-} f(t)$$

Then $\lim_{t \to t_0} f(t)$ exists $\Leftrightarrow$ $f(t_0+)$ and $f(t_0-)$ both exists and are equal.
Let $f : (a,b) \to \mathbb{R}$. Then f is continuous at $t_0 \in (a,b) \Leftrightarrow$ $f(t_0+) = f(t_0-) = f(t_0)$.

Existence of Laplace transform

Def. If $f(t_0+) \neq f(t_0-)$, but both limit exists, then f has a **jump discontinuity** at t_0, and $f(t_0+) - f(t_0-)$ is called the jump in f at t_0.

Ex. Define f as $f(t) = 1$ for $t < 0$ and $f(t) = -1$ for $t \geq 0$. Then f has jump discontinuity at 0, and the jump in f at $t_0 = 0$ is -2.

Def. Assume $f(t_0+) = f(t_0-)$ and both exists. Assume either $f(t_0)$ is not defined, or $f(t_0)$ is defined but $f(t_0) \neq f(t_0+) = f(t_0-)$. Then f has a **removable discontinuity** at t_0.

Ex. If $f(t) = 1$ for $t \neq 0$ and $f(0) = 0$, then f has removable discontinuity at 0.

Ex. If $f(t) = \dfrac{t^2 - 1}{t - 1}$, then f has removable discontinuity at 1.

Existence of Laplace transform

Remark. If f and g are integrable on $[a, b]$ and they differ only at finitely many points, then

$$\int_a^b f(t)dt = \int_a^b g(t)dt$$

In particular, $L(f) = L(g)$, if exists.

Definition

(a) A function $f : [0, T] \to \mathbb{R}$ is called **piecewise continuous**, if $f(0+)$ and $f(T-)$ are finite and f is continuous on $(0, T)$ except possibly at finite many points, where f may have jump discontinuity or removable discontinuity.

(b) A function $f : [0, \infty) \to \mathbb{R}$ is called **piecewise continuous**, if it is so on $[0, T]$ for every $T > 0$.

Existence of Laplace transform

If f is piecewise continuous on closed interval $[a, b]$, then

$$\int_a^b f(t)dt$$

exists.
If f is piecewise continuous on $[0, \infty)$, then so is $e^{-st}f(t)$.
Hence

$$\int_0^T e^{-st}f(t)dt$$

exists for every $T > 0$. But piecewise continuity of f does not guarantee that the Laplace transform

$$\int_0^\infty e^{-st}f(t)dt = \lim_{T\to\infty}\int_0^T e^{-st}f(t)dt$$

exists for some $s \in (s_0, \infty)$. For example, take $f(t) = e^{t^2}$.

Existence of Laplace transform

The reason is that e^{t^2} grows too rapidly in comparison to e^{st} for any fixed s.

Definition

A function f is of **exponential order** s_0, if there exist constants M and t_0 such that

$$|f(t)| \leq Me^{s_0 t}, \quad t \geq t_0.$$

We say f is of **exponential order**, if f is so of order s_0 $\forall s_0$.

Ex. $f(t) = e^{t^2}$ is not of exponential order.

$$\lim_{t\to\infty} \frac{e^{t^2}}{Me^{s_0 t}} = \lim_{t\to\infty} \frac{1}{M} e^{t^2 - s_0 t} = \infty$$

So $e^{t^2} > Me^{s_0 t}$ for large t, for any fixed s_0, M.

Existence of Laplace transform

Theorem

If f is piecewise continuous on $[0, \infty)$ and of exponential order s_0, then Laplace transform $L(f)$ is defined for $s > s_0$.

Proof. Assume $|f(t)| \leq Me^{s_0 t}, \quad t \geq t_0$. We need to show that the integral

$$\int_0^\infty e^{-st} f(t)dt = \int_0^{t_0} e^{-st} f(t)dt + \int_{t_0}^\infty e^{-st} f(t)dt$$

converges. The first integral exists and is finite, since f is piecewise continuous. For $s > s_0$,

$$|e^{-st} f(t)| \leq e^{-st} M e^{s_o t} = M e^{-(s-s_o)t}$$

Thus the second integral converges, since it is dominated by a convergent integral. Therefore $L(f)$ exists.

Existence of Laplace transform

Ex. If f is bounded on $[t_0, \infty)$, say

$$|f(t)| \leq M, \quad t \geq t_0$$

then f is of exponential order $s_0 = 0$. For example, $\sin \omega t$ and $\cos \omega t$ are of exponential order 0. Thus $L(\sin \omega t)$ and $L(\cos \omega t)$ exists for $s > 0$ (already seen).

Exercise. If $\lim_{t\to\infty} e^{-s_0 t} f(t)$ exists and is finite, then show that f is of exponential order s_0.

Ex. If $\alpha \in \mathbb{R}$ and $s_0 > 0$, then $\lim_{t\to\infty} e^{-s_0 t} t^{\alpha} = 0$

Hence t^{α} is of exponential order s_0 for any $s_0 > 0$.
Q. Does this mean $L(t^{\alpha})$ exists for any $\alpha \in \mathbb{R}$. No. We need piecewise continuity for $t \geq 0$.

If $\alpha \geq 0$, then t^{α} is continuous on $(0, \infty)$, hence $L(t^{\alpha})$ exists for $\alpha \geq 0$.

Ex. Find Laplace transform of piecewise linear function

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ e^{-t}, & t \geq 1 \end{cases}$$

Solution.

$$L(f) = F(s) = \int_0^\infty e^{-st} f(t) dt$$

$$= \int_0^1 e^{-st} dt + \int_1^\infty e^{-st} e^{-t} dt$$

$$= -\frac{1}{s} e^{-st}\Big|_0^1 + \frac{-1}{s+1} e^{-(s+1)t}\Big|_1^\infty$$

$$= \begin{cases} \dfrac{1 - e^{-s}}{s} + \dfrac{e^{-(s+1)}}{s+1} & , \quad s > -1, s \neq 0 \\ 1 + \dfrac{1}{e} & , \quad s = 0 \end{cases}$$