# Tutorial-1, MA 108 (Linear Algebra)

1. Verify that the function is a solution of the differential equation on some interval, for any choice of the arbitrary constants appearing in the function.

   (a) $y = ce^{2x}; \quad y' = 2y.$

   (b) $y = e^{t^2} \int_0^t e^{-s^2}\, ds + e^{t^2}, \quad y' - 2ty = 1.$

   (c) $y = \dfrac{1}{2} + ce^{-x^2}; \quad y' + 2xy = x.$

   (d) $y = \tan\left(\frac{x^3}{3} + c\right); \quad y' = x^2(1 + y^2).$

2. Let $a$ be a nonzero real number

   (a) Verify that if $c$ is an arbitrary constant then
   $$y = (x - c)^a$$
   is a solution of $y' = ay^{(a-1)/a}$ on $(c, \infty)$.

   (b) Suppose $a < 0$ or $a > 1$. Can you think of a solution of the given differential equation which is not of the given form?

3. Verify that
   $$y = \begin{cases} e^x - 1, & x \geq 0 \\ 1 - e^{-x}, & x < 0 \end{cases}$$
   is a solution of
   $$y' = |y| + 1$$
   on $(-\infty, \infty)$.

4. Solve the initial value problem

   (a) $xy' + (1 + x\cot x)y = 0, \ y(0) = 2.$

   (b) $y' - \dfrac{2x}{1 + x^2} y = 0, \ y(0) = 2.$

   (c) $xy' + 2y = 8x^2, \quad y(1) = 3.$

5. Find the general solution for the following equations.

(a) $(x-2)(x-1)y' - (4x-3)y = (x-2)^3$.

(b) $x^2y' + 3xy = e^x$

6. Let $xy' - 2y = -1$.

   (a) Find a general solution to the above problem on $\mathbb{R} - \{0\}$.

   (b) Show that $y$ is a general solution for the above ODE if and only if

$$y = \begin{cases} \frac{1}{2} + c_1x^2, & x \geq 0 \\ \frac{1}{2} + c_2x^2, & x < 0 \end{cases}$$

   where $c_1, c_2$ are arbitrary constants.

   (c) Conclude that all solutions of the ODE on $\mathbb{R}$ are solutions of the initial value problem $xy' - 2y = -1 \quad , y(0) = \frac{1}{2}$

   (d) Show that if $x_0 \neq 0$ and $y_0$ is arbitrary, then the initial value problem $xy' - 2y = -1 \quad , y(x_0) = y_0$ has infinitely many solutions on $\mathbb{R}$. Why does this not contradict existence and uniqueness theorem for linear ODEs?

7. Assume that all the following DE's are defined on $(a, b)$.

   (a) Prove: If $y_1$ and $y_2$ are solutions of

$$y' + p(x)y = f_1(x)$$

   and

$$y' + p(x)y = f_2(x)$$

   respectively, and $c_1$ and $c_2$ are constants, then $y = c_1y_1 + c_2y_2$ is a solution of

$$y' + p(x)y = c_1f_1(x) + c_2f_2(x).$$

   (b) Show that if $y_1$ and $y_2$ are solutions of the non homogeneous equation

$$y' + p(x)y = f(x)$$

   then $y_1 - y_2$ is a solution of the homogeneous equation

$$y' + p(x)y = 0.$$

(c) Show that $y_1$ is a solution of the non-homogeneous linear equation and $y_2$ a solution of the complimentary homogeneous linear equation then $y_1 + y_2$ is solution to non-homogeneous linear equation .

8. Solve the following problem by noting that $g'(y)y' + p(x)g(y) = f(x)$ can be solved by changing variable to $z = g(y)$.

   (a) $\dfrac{xy'}{y} + 2\ln y = 4x^2$.

   (b) $\dfrac{y'}{(1+y)^2} - \dfrac{1}{x(1+y)} = -\dfrac{3}{x^2}$.

9. A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If $Q(t)$ is the amount present at time $t$, then $dQ/dt = -rQ$, where $r > 0$ is the decay rate.

   (a) If 100mg of thorium-234 decays to 82.04mg in 1week, determine the decay rate.

   (b) Find an expression for the amount of thorium-234 present at any time $t$.

   (c) Find the time required for the thorium-234 to decay to one-half its original amount.

10. Following steps give a method to derive the equation of the motion of a pendulum as given in class. Assume that the rod is rigid and weightless, that the mass is a point mass, and that there is no friction or drag anywhere in the system.

    (a) Assume that the mass is in an arbitrary displaced position, indicated by the angle $\theta$ . Draw a free-body diagram showing the forces acting on the mass.

    (b) Apply Newtons law of motion in the direction tangential to the circular arc on which the mass moves. Then the tensile force in the rod does not enter the equation. Observe that you need to find the component of the gravitational force in the tangential direction. Observe also that the linear acceleration, as opposed to the angular acceleration, is $Ld^2\theta/dt^2$, where $L$ is the length of the rod.

    (c) Simplify the result from part (b) to obtain the equation in $\theta$.

11. Solve the following

    (a) $y(1 + x^3)y' = x^2$.

    (b) $y' = (\cos^2 x)(\cos^2 2y)$.

(c) $(1+y^2)y' = x^2$.

12. Some of the results requested in the following problems can be obtained either by solving the given equations analytically or by plotting numerically generated approximations to the solutions. Try to form an opinion as to the advantages and disadvantages of each approach.

 (a) Find the explicit solution of the following IVP, plot the graph and determine the interval in which the solution is defined.

$$(1+2y)y' = 2x \quad y(0) = -2$$

 (b) Solve the initial value problem

$$y' = \frac{(1+3x^2)}{3y^2-6y}, \quad y(0) = 1.$$

 and determine the interval in which the solution is valid. Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

 (c) Solve the initial value problem

$$y' = 2\cos 2x/(3+2y), \quad y(0) = -1$$

 and determine where the solution attains maximum value.

13. Show that the following equations are homogeneous, that is, of the from $y' = q(y/x)$. Solve the differential equation.

 (a) $\dfrac{dy}{dx} = \dfrac{x+3y}{x-y}$

 (b) $(x^2+3xy+y^2)dx - x^2dy = 0$.

 (c) $y' = \dfrac{x^3+y^3}{xy^2}, \quad y(1) = 3$

14. Solve using substitution.

 (a) $t^2y' + 2ty - y^3 = 0 \quad t > 0$.

 (b) $y' = \epsilon y - \sigma y^3, \quad \epsilon > 0, \sigma > 0$.

(c) $x^2y' + 2y = 2e^{1/x}y^{1/2}$.

(d) $xy' + y = x^4y^4, \quad y(1) = 1/2$.

15. A body falling in a relatively dense fluid, oil for example, is acted on by three forces (see Figure 2.3.5) (page 65, Boyce and DiPrima ): a resistive force $R$, a buoyant force $B$, and its weight $w$ due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a, the resistive force is given by Stokess law, $R = 6\pi\mu a|v|$, where $v$ is the velocity of the body, and $\mu$ is the coefficient of viscosity of the surrounding fluid.

    (a) Find the limiting velocity of a solid sphere of radius $a$ and density $\rho$ falling freely in a medium of density $\rho'$ and coefficient of viscosity $\mu$.

    (b) In 1910 R. A. Millikan studied the motion of tiny droplets of oil falling in an electric field. A field of strength $E$ exerts a force $Ee$ on a droplet with charge $e$. Assume that $E$ has been adjusted so the droplet is held stationary ($v = 0$) and that $w$ and $B$ are as given above. Find an expression for $e$. Millikan repeated this experiment many times, and from the data that he gathered he was able to deduce the charge on an electron.

16. A generalized Riccati equation is of the form $y' = P(x) + Q(x)y + R(x)y^2$. (If $R \equiv 1$, then it is called a Riccati equation.) Let $y_1$ be a known solution and $y$ an arbitrary solution of the ODE. Let $z = y - y_1$. Show that $x$ is a solution of a Bernoulli equation with $n = 2$.

    Further solve
    $$y' = e^{2x} + (1 - 2e^x)y + y^2; \quad y_1 = e^x.$$

17. In each of following problems determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

    (a) $y' + (\tan\ t)y = \sin\ t \quad y(\pi) = 0$.

    (b) $(4 - t^2)y' + 2ty = 3t^2, \quad y(1) = -3$.

18. In each of the following problems state on which rectangles the hypotheses of existence and uniqueness theorem for ODEs are satisfied.

    (a) $y'(1 - t^2 + y^2) = \ln|ty|$.

(b) $\dfrac{dy}{dt} = \dfrac{1+t^2}{3y - y^2}$.

19. In each of following problems solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value $y_0$.

    (a) $y + y^3 = 0, \quad y(0) = y_0$.

    (b) $y' = \dfrac{t^2}{y(1+t^3)} \quad y(0) = y_0$

20. (a) Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are solutions of the initial value problem
$$y' = \frac{-t + (t^2 + 4y)^{1/2}}{2}, \quad y(2) = -1$$
    Where are these solutions valid?

    (b) Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of the existence uniqueness theorem for ODE.

    (c) Show that $y = ct + c^2$, where $c$ is an arbitrary constant, satisfies the differential equation in part (a) for $t \geq -2c$. If $c = -1$, the initial condition is also satisfied, and the solution $y = y_1(t)$ is obtained. Show that there is no choice of $c$ that gives the second solution $y = y_2(t)$.