MA-108 Ordinary Differential Equations

M.K. Keshari

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

2nd March, 2015
D1 - Lecture 1

Welcome!

Welcome back! We'll study the topic of ordinary differential equations in the next six or seven weeks. It's a very important area of mathematics with extremely useful applications in the real world.

Ordinary Differential Equations - ODE - MA 108.

differential - so be comfortable with MA 105.

important tools from MA 106; so be comfortable with 106 as well.

non-ordinary? - MA 207 - PDE.

Class Information

- Instructor : Manoj K. Keshari
- Office : 204 D, Dept of Math.
- Office Hours : Wednesday 9:30 - 10:30 AM
- Email : keshari@math.iitb.ac.in
- Reference Text : Elementary Differential Equations by William Trench available freely at ramanujan.math.trinity.edu/wtrench/texts/index.shtml
  Elementary Differential Equations and Boundary Value Problems by Boyce and DiPrima
- There will be a **Quiz** of 30 mks.
- **Final** exam will be 65 marks.
- Total 100 mks to be earned!
- Attendance will account for 5 marks. You will loose 2.5 marks if you are not present when your names is called out in Random attendance.

Example: A falling object

Suppose an object is falling under the force of gravity.
The drag force which acts in the opposite direction to gravity can be assumed to be proportional to the velocity of the object.
The drag force is given by νv where v the velocity at time t and ν is the drag coefficient given in terms of mass per unit time.

The constant ν depends on the particular object.

By Newton's second law of motion

$$m\frac{dv}{dt} = mg - \nu v$$

How to solve this ODE?

Rat-Owl Problem

Consider a simple prey-predator problem.
Let R denote the population of mice in a given rural area at a given time.
In the absence of predators, we assume that the population of rats increases at a rate proportional to its current population.
Then

$$\frac{dR}{dt} = kR$$

where k is the growth rate, say per months.
If we know that the owls prey on the rats and kill 10 of them everyday. Then

$$\frac{dR}{dt} = kR - 300.$$

How would we solve this?

Rat-Owl Problem

Consider the simple equation $\frac{dR}{dt} = kR$.

We know that the exponential function $R = e^{kt}$ satisfies this equation. This we can solve by taking anti-derivatives.
In fact, any ce^{kt} is a solution.

Questions:

- Are these all the solutions?
- How do I solve a variant of this problem?
$$\frac{dR}{dt} = kR - 300.$$
- Will such an equation always have a solution?
If not, then when will it have a solution?
- We need to be able to classify the kind of differential equations that we can solve!

Basic Concepts

Let $y(x)$ denote a function in the variable x. An ODE is an equation containing one or more derivatives of an unknown function y.

Note that the equation may contain y itself (the 0^{th} derivative), and known functions of x (including constants).

In other words, an ODE is a relation between the derivatives $y, y^1, \ldots, y^{(n)}$ and functions of x:

$$F(x, y, y^1, \ldots, y^{(n)}) = 0.$$

ODE's occur naturally in physics, Biology, engineering, finance and so on.
Can you give some obvious examples?
Velocity and acceleration being derivatives, often give rise to DE's.

Basic Concepts

Given an ODE, you would like to solve it. At least try to solve it. We just finished a course trying to solve a system of linear equations. Now we want to attempt solving differential equations.

Questions:

(i) What's a solution?

(ii) Does an equation always have a solution? If so, how many?

(iii) Can the solutions be expressed in a nice form? If not, how to get a feel for it?

(iv) How much can we proceed in a systematic manner? How do we rate the difficulty of a DE?

order - first, second, ..., n^{th}, ...

linear or non-linear?

Order - the degree of the highest derivative in the ODE.

Linear - When do we say that $F(x, y, y^1, \ldots, y^{(n)}) = 0$ linear? Write

$$F(x, y, y^1, \ldots, y^{(n)}) = 0$$

as

$$L(y)(x) = L(y, y^1, \ldots, y^{(n)}) = f(x),$$

and now L is a linear transformation from

$$\mathcal{C}^n = \text{the space of functions which are at least } n\text{-times differentiable}$$

to

$$\mathcal{F} = \text{the space of functions.}$$

If these functions are defined on an open set $\Omega \subseteq \mathbb{R}$, we say that the ODE is defined on Ω. (Why do we take an open set?)

Basic Concepts

Question: How does a linear ODE of order n look?
Let $x_0 \in \Omega$. Associated to this point, there is a natural linear map from $\mathcal{C}^n(\Omega)$ to $\mathbb{R}^{n+1}$. Evaluation? This is

$$y(x) \mapsto (y(x_0), y^1(x_0), \ldots, y^{(n)}(x_0)).$$

Now:

$$\begin{tikzcd}
\mathcal{C}^n(\Omega) \arrow[r, "\text{eval}(x_0)"] \arrow[d, "L"'] & \mathbb{R}^{n+1} \arrow[d] \\
\mathcal{F}(\Omega) \arrow[r, "\text{eval}(x_0)"'] & \mathbb{R}
\end{tikzcd}$$

Basic Concepts

Thus, corresponding to

$$y \mapsto L(y)(x_0),$$

we have a $1 \times (n+1)$ matrix, say,

$$\begin{bmatrix} a_n(x_0) & a_{n-1}(x_0) & \dots & a_0(x_0) \end{bmatrix}$$

such that

$$L(y)(x_0) = \begin{bmatrix} a_n(x_0) & a_{n-1}(x_0) & \dots & a_0(x_0) \end{bmatrix} \cdot \begin{bmatrix} y(x_0) \\ y^1(x_0) \\ \vdots \\ y^{(n)}(x_0) \end{bmatrix}.$$

Thus, a linear ODE of order n is of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = b(x),$$

where $a_0, a_1, \ldots, a_n, b$ are functions of x and $a_0(x) \neq 0$.

Basic Concepts

Thus, if the linear transformation

$$L : \mathcal{C}^n(\Omega) \to \mathcal{F}(\Omega)$$

has the property that

$$L(y)(x_0) = T_{x_0}((y(x_0), y^1(x_0), \ldots, y^{(n)}(x_0))$$

for a linear transformation $T_{x_0} : \mathbb{R}^{n+1} \to \mathbb{R}$ for every $x_0 \in \Omega$, then L is of the form

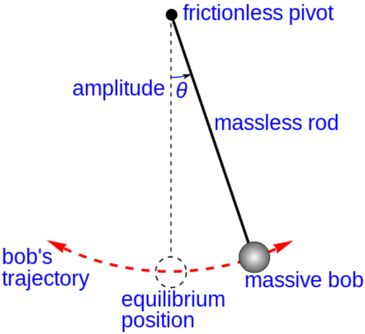
$$L(y) = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y$$

for some functions $a_0(x), \ldots, a_n(x)$.

Remark. An arbitrary linear transformation $L : \mathcal{C}^n(\Omega) \to \mathcal{F}(\Omega)$ need not be of this form!

Examples

The motion of an oscillating pendulum.
Consider an oscillating pendulum of length L. Let θ be the angle it makes with the vertical direction.



The physical system satisfies the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

This is an example of a *non-linear second order* differential equation.

Examples

- $L\dfrac{d^2Q(t)}{dt^2} + R\dfrac{dQ(t)}{dt^2} + \dfrac{1}{C}Q(t) = E(t)$. Second order ODE
- $\alpha^2\dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial u(x,t)}{\partial t}$ PDE
- $y' + 2y = x^3 e^{-2x}$. first order ODE
- $3x^2y^2 + 2x^3y\dfrac{dy}{dx} = 0$ first order non-linear ODE .
- $x^2y + 2x^3\dfrac{dy}{dx} = 0$ first order homogeneous linear ODE .

Hide and Seek

One day, Einstein, Newton, and Pascal meet up and decide to play a game of hide and seek.

Einstein volunteered to count. As Einstein counted, eyes closed, to 100, Pascal ran away and hid, but Newton stood right in front of Einstein and drew a one meter by one meter square on the floor around himself.

When Einstein opened his eyes, he immediately saw Newton and said "I found you Newton," but Newton replied, "No, you found one Newton per square meter. You found Pascal!".

Solutions of ODE : Examples

- How to solve $y^{(n)} = f(x)$ for a continuous f?
  We can integrate the equation n-times to get a solution.
- Consider $\frac{dR}{dt} = kR - 300$. Verify that $Ce^{kt} + \frac{300}{k}$ is a solution. The solution exists and is defined for all $t \in \mathbb{R}$.
- Consider $y' + \dfrac{1}{x}y = 0$. The equation is defined only when $x \neq 0$. Consider solutions y which take non-zero values on the intervals $(-\infty, 0)$ and $(0, \infty)$ then, we can rewrite the equation as $\dfrac{y'}{y} = -\dfrac{1}{x}$. By integrating, we get $\ln|y| = -\ln|x| + k$, i.e., $|y| = \dfrac{e^k}{|x|}$: general solution. The solution is $y = \pm\dfrac{c}{x}$ on $(-\infty, 0) \cup (0, \infty)$.

Solutions to ODE

Definition: An **explicit solution** to an ODE $F(x, y(x), \ldots, y^{(n)}(x)) = 0$ is a (real valued) function which satisfies the equation in an open interval, that is, an n times differentiable function y which satisfies the above equation on some open interval (a, b) is an explicit solution.

- Both $\sin t$ and $\cos t$ satisfy the equation $y'' + y = 0$. In fact, any linear combination of the two functions will also give a solution which is defined on $(-\infty, \infty)$.
- The function $y(x) = x - 1$ will satisfy this equation at $x = 1$ but this is *not a solution* to ODE.

Solutions to IVP

- An ODE of order n with n no. of initial conditions $y(x_0) = y_0, \ldots, y^{(n-1)}(x_0) = y_{n-1}$, is called an **initial value problem** (IVP).
- For example, our rat-owl problem with initial condition $R(0) = 100$ becomes an IVP.
  The solution $R(t) = Ce^{kt} + 300/k$ satisfying initial condition $100 = C + 300/k$ is
  $R(t) = (100 - 300/k)e^{kt} + 300/k$.
- The graph of a particular solution of an ODE is called a **solution curve**.
- The graph $x^2 + y^2 = 9$ (*) satisfies $yy' + x = 0$.
  Both $y_1(x) = \sqrt{9 - x^2}$ and $y_2(x) = -\sqrt{9 - x^2}$ are solutions of the ODE $yy' + x = 0$ on the intervals $(-3, 3)$.
  (*) is an example of an *implicit solution* of an ODE.

Implicit Solutions

Definition: An **implicit solution** to an ODE $F(x, y(x), \ldots, y^{(n)}(x)) = 0$ is an equation $g(x, y) = 0$ which satisfies the differential equation and gives an explicit solution $y(x)$ of ODE on same interval.

- The graph of an implicit solution is called an **integral curve**.
- Note $x^2 + y^2 + 9 = 0$ formally satisfies the equation $y'y + x = 0$. But this is not an implicit solution, as it does not give an explicit solution on any interval.
- $y^5 + y - x^2 - x - C = 0$ –(*) is an implicit solution to the ODE $(5y^4 + 1)y' = 2x + 1$. To check this, plot the implicit solution, and any portion of graph which defines a function is an explicit solution of ODE.

Any differential function satisfying (*) is a solution.

The geometric representation of (*) gives an integral curve for the ODE for each value of C.