MA-106 Linear Algebra

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D1 - Lecture 2

See www.math.iitb.ac.in/ keshari/MA106.html for lecture slides, Tutorial Sheets and other information.

We will have 1st tutorial on 8-th January, so try the problems before that from Tutorial Sheet 1.

Recall that we have seen Gaussian elimination and notion of pivots.

Question on row and column picture. Consider 4 linear equations in 2 unknowns x and y. Then

(1) the row picture shows 4 lines.

(2) the column picture is in which dimensional space? 4

The equations have no solutions unless the vectors on the RHS are linear combinations of vectors of LHS.

Example. Consider $3x - 2y = b_1$ and $6x - 4y = b_2$.
For which b_1, b_2, the system has a solution.

Only if $2b_1 = b_2$.

Consider the system of linear equations

$$x + by = 0$$
$$x - 2y - z = 0$$
$$y + z = 0.$$

Question 1. Which number b leads to a row exchange in Gaussian elimination.

(2) Which b leads to a missing pivot. In this case find a non-zero solution.

Solution. If $b = -2$, then we need a row exchange in the Gaussian elimination.

If $b = -1$, then system will have only two pivots. After elimination, the system will be

$$x - y = 0 \text{ and } y + z = 0.$$

Hence $(1, 1, -1)$ is a non-zero solution ($b = -1$).

Matrix notation $Ax = b$ for linear systems

- Consider previous system
$$\begin{aligned} 2u + v + w &= 5, \\ 4u - 6v &= -2, \\ -2u + 7v + 2w &= 9. \end{aligned}$$

Write $x = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ for the unknown vector, $A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$ for the coefficient matrix, and $b = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$ for RHS vector.

- If we have n equations in n variables, then size of A is $n \times n$, size of column vector b is n, size of unknown vector x is n.

- If we have m equations in n variables, then size of A is $m \times n$, size of column vector b is m, and size of unknown vector x is n.

- We want to write above system as $Ax = b$. For this we need to define matrix operations.

Matrix addition

We can add two matrices if and only if thay have same size.

Example 1. We know how to add two row or column vector.

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} + \begin{pmatrix} -3 & -2 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 2 \end{pmatrix} \quad \text{(component-wise)}$$

Example 2.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} + \begin{pmatrix} -1 & -4 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 2 & 5 \end{pmatrix}$$

In general, matrix addition is component-wise, i.e. if A^i denotes the i-th row of A and A_i denotes the i-th column of A, then

$$(A+B)^i = A^i + B^i \quad \text{and} \quad (A+B)_i = A_i + B_i.$$

Multiplication of a Matrix and a Vector

(1) a row at a time: $\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = (2u + v + w).$

Therefore we can write the previous system as $(Ax = b)$

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2u + v + w \\ 4u - 6v \\ -2u + 7v + 2w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$$

(2) a column at a time:

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = u \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + v \begin{pmatrix} 1 \\ -6 \\ 7 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

i.e. Ax is a linear combination of columns of $A = \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix}$,

i.e. $\boxed{Ax = uA_1 + vA_2 + wA_3}$.

Matrix Multiplication

Two matrices A and B can be multiplied if and only if no. of columns of A = no. of rows of B.

Column wise When B is a column vector, we already know how to multiply AB. In general, the multiplication is column-wise of B.

Let $B = \begin{pmatrix} B_1 & \dots & B_m \end{pmatrix}$, where B_i is i-th column of B. Then

$$AB = \begin{pmatrix} AB_1 & \dots & AB_m \end{pmatrix}$$

Row wise If A is a row vector, then $AB = \begin{pmatrix} AB_1 & \dots & AB_m \end{pmatrix}$ is a row vector. In general, the multiplication is row-wise of A.

$$AB = \begin{pmatrix} A^1 \\ \vdots \\ A^m \end{pmatrix} B = \begin{pmatrix} A^1 B \\ \vdots \\ A^m B \end{pmatrix}$$

where A^i is the i-th row of A.

Row operations - Elementary matrices

Example: $$E\,x := \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u \\ -2u+v \\ w \end{pmatrix}$$

Matrix multiplication EA is column-wise in A, hence

$$EA = \begin{pmatrix} EA_1 & EA_2 & EA_3 \end{pmatrix}.$$

Therefore EA has same effect on A as the row operation

$$R_2 \mapsto R_2 - 2R_1$$

on the matrix A.

Such a matrix (diagonal entries 1 and atmost one off-diagonal entry non-zero) is called an **elementary matrix**.

Notation $E := E_{21}(-2)$. Similarly define $E_{ij}(\lambda)$.

Permutation matrix

$$P_{12}\, x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ u \\ w \end{pmatrix}$$

Matrix multiplication $P_{12}A$ is column-wise in A, hence

$$P_{12}A = \begin{pmatrix} P_{12}A_1 & P_{12}A_2 & P_{12}A_3 \end{pmatrix}$$

Therefore $P_{12}A$ has the same effect on A as the interchange of its first and second rows.

P_{12} is got by interchanging 1-st and 2-nd columns of identity matrix.

Such a matrix is called a **permutation** (or row exchange) matrix.

Remark: Row operations on A correspond to multiplication by $E_{ij}(\lambda)$ (elementary matrices) or P_{ij} (permutation matrices) on the left of A.

Summary of matrix multiplication AB

- $(AB)_{ij} =$ (ith row of A). (jth column of B)
- jth column of $AB = A.$ (jth column of B)
- ith row of $AB =$ (ith row of A)$.B$

Matrix Properties

- (associative) $\quad (AB)C = A(BC)$
- (distributive) $\quad A(B+C) = AB + AC$

  $$(B+C)D = BD + CD$$

- (non-commutative) $\quad AB \neq BA$, in general. Find examples.