Tutorial-4, MA 106 (Linear Algebra)
Linear Algebra and its Applications by Gilbert Strang

This tutorial sheet consists of the problems based on 2.3 and 2.4. This includes:
Problem Set 2.3: [3, 4, 7, 8 ,10, 12, 16, 18, 19, 22, 25, 33, 36, 40, 44]
Problem Set 2.4: [2, 4, 7, 8, 10, 13, 20, 23, 27, 28, 32, 33]

1. Are the following vectors linearly independent?
   (a) $(1, 3, 2)^T$, $(2, 1, 3)^T$, $(3, 2, 1)^T$.
   (b) $(1, -3, 2)^T$, $(2, 1, -3)^T$, $(-3, 2, 1)^T$.

2. Let $v_1 = (1, 0, 0)^T$, $v_2 = (1, 1, 0)^T$, $v_3 = (1, 1, 1)^T$ and $v_4 = (2, 3, 4)^T$.
   (a) v_1, v_2, v_3, v_4 are linearly dependent because ______.
   (b) Find scalars a_1, a_2, a_3, a_4, not all zero, such that $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$.
   (c) Show that v_1, v_2, v_3 are linearly independent.
   (d) Find all combinations of 3 vectors from v_1, v_2, v_3, v_4, which are linearly independent.
   (e) Compute the rank of $A = (v_1\ \ v_2\ \ v_3\ \ v_4)$, and the dimensions of its four fundamental spaces.

3. Find the largest possible number of independent vectors among: $v_1 = (1, -1, 0, 0)^T$, $v_2 = (1, 0, -1, 0)^T$, $v_3 = (1, 0, 0, -1)^T$, $v_4 = (0, 1, -1, 0)^T$, $v_5 = (0, 1, 0, -1)^T$, $v_6 = (0, 0, 1, -1)^T$. How is this number related to $\mathrm{Span}\{v_1, \ldots, v_6\}$?

4. (a) Two vectors v_1 and v_2 in $\mathbb{R}^4$ will be dependent if and only if ______.
   (b) If v is any vector in $\mathbb{R}^4$, v and 0 are dependent because ______.

5. $x = v + w$ and $y = v - w$ are combinations of v and w. Show that v and w can be written as combinations of x and y. How are $\mathrm{Span}\{v, w\}$ and $\mathrm{Span}\{x, y\}$ related? When is each pair of vectors a basis for its span?

6. Describe the subspace of $\mathbb{R}^3$ spanned by:
   (a) $u_1 = (1, 1, -1)^T$ and $u_2 = (-1, -1, 1)^T$.
   (b) $v_1 = (0, 1, 1)^T$, $v_2 = (1, 1, 0)^T$ and $v_3 = (0, 0, 0)^T$.
   (c) The columns of a 3×5 echelon matrix with 2 pivots.
   (d) All vectors with positive components.

7. Is v in $\mathrm{Span}\{v_1, \ldots, v_n\}$? If yes, write v as a combination of the v_i's.
   (a) $v_1 = (1, 1, 0)^T$, $v_2 = (2, 2, 1)^T$, $v_3 = (0, 0, 2)^T$; $b = (3, 4, 5)^T$.
   (b) $v_1 = (1, 2, 0)^T$, $v_2 = (2, 5, 0)^T$, $v_3 = (0, 0, 2)^T$, $v_4 = (0, 0, 0)^T$; $v = (a, b, c)^T$.

   In each case, find a basis of $\mathrm{Span}\{v_1, \ldots, v_n\}$.

8. Let $\mathbf{P}$ be the plane $x - 2y + 3z = 0$ in $\mathbb{R}^3$.
   (a) Find a basis for $\mathbf{P}$.
   (b) Find a basis for the space of all the vectors perpendicular to $\mathbf{P}$.
   (c) Find a basis for the intersection of $\mathbf{P}$ with the x-y plane.

9. If A is $m \times n$, the columns of A are n vectors in $\mathbb{R}^m$. If they are linearly independent, what is $\text{rank}(A)$? If they span $\mathbb{R}^m$, what is $\text{rank}(A)$? What happens if they are a basis of $\mathbb{R}^m$?

10. Find a basis for each of the following subspaces of $\mathbb{R}^4$.

    (a) All vectors whose components are equal.

    (b) All vectors whose components add to zero.

    (c) All vectors that are perpendicular to $(1,1,0,0)^T$ and $(1,0,1,1)^T$.

11. Find a basis for the null space and column space of $U = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$.

12. Prove that the following are subspaces of M, the set of 3×3 matrices, and find a basis.

    (a) All diagonal matrices.

    (b) All symmetric matrices ($A^T = A$).

    (c) All skew-symmetric matrices ($A^T = -A$).

    (d) All lower triangular matrices.

13. Suppose A is a 5×4 matrix with $\text{rank}(A) = 4$. Show that $Ax = v$ has no solution if and only if the 5×5 matrix $[A|v]$ is invertible. Show $Ax = v$ is solvable when $[A|v]$ is singular.

14. Find the echelon forms U, basis and the dimension of the four fundamental subspaces of:

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{pmatrix}.$$

15. Without computing A, find bases for the 4 fundamental subspaces: $A = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$.

16. Let A be $m \times n$ with rank r. Suppose there are right hand sides b for which $Ax = b$ is not solvable.

    (a) What inequalities must be true between m, n and r?

    (b) Explain why $A^T y = 0$ has non-trivial solutions.

17. Fill in the blanks: Let A be an $m \times n$ matrix, with rank r.

    (a) If A has linearly independent columns, then $r =$ _____, the nullspace is _____, and the row space is _____.

    (b) If $Ax = b$ always has at least one solution, then the solutions to $A^T y = 0$ is/are ____. (Hint: Find r).

    (c) If $m = n = 3$ and A is invertible, then a basis for (i) $N(A)$ is ____, (ii) $C(A)$ is ____, (iii) $N(A^T)$ is ____ and (iv) $C(A^T)$ is ____. Do the same for the 3×6 matrix $B = \begin{pmatrix} A & A \end{pmatrix}$.

    (d) If $m = 7$, $n = 9$ and $r = 5$, then the dimension of (i) $N(A)$ is ____, (ii) $C(A)$ is ____, (iii) $N(A^T)$ is ____ and (iv) $C(A^T)$ is ____.

    (e) If $m = 3$, $n = 4$ and $r = 3$, then $C(A) =$ ____ and $N(A^T) =$ ____.

    (f) If B is obtained by exchanging the first two rows of A, then the fundamental subspaces which remain unchanged are ____.

    (g) With B as above, if $(1,2,3,4)$ is in the left nullspace of A, write down a non-zero vector in the left nullspace of B.

18. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ be vectors such that:

    (i) $\text{Span}\{v_1, v_2, v_3, v_4\} = \mathbb{R}^3$ and (ii) the vectors $\{v_2, v_3, v_4\}$ are linearly independent.

    For each of the following statements, state if it is true or false. Justify.

    (a) The vectors $\{v_1, v_2, v_3, v_4\}$ are linearly independent.
    (b) The vectors $\{v_1, v_2, v_3\}$ form a spanning set for $\mathbb{R}^3$.
    (c) The vectors $\{v_2, v_3\}$ are linearly independent.
    (d) For any other vector v_5 in $\mathbb{R}^3$, the vectors $\{v_1, v_2, v_3, , v_4, v_5\}$ form a spanning set for $\mathbb{R}^3$.
    (e) The vectors $v_2 + v_3$, $v_2 + v_4$, $v_3 + v_4$ are linearly independent.

19. True or False. Give a good reason if it is true, or a counter-example if it is false.

    (a) If the columns of a matrix are dependent, so are the rows.
    (b) The column space of a 2×2 matrix is the same as its row space.
    (c) The column space of a 2×2 matrix has the same dimension as its row space.
    (d) The columns of a matrix are a basis for its column space.
    (e) If $AB = 0$, then $C(B)$ is contained in $N(A)$ (and the row space of A is contained in the left null space of B).
    (f) A and A^T have the same number of pivots.
    (g) A and A^T have the same left null space.
    (h) If the row space equals the column space, then $A = A^T$.
    (i) If $A^T = -A$, then the row space of A equals its column space.
    (j) If the vectors $v_1, \dots, v_n$ span a subspace V, then $\dim(V) = n$.
    (k) If $v_1, \dots, v_n$ are linearly independent in a vector space V, then $\dim(V) \geq n$.
    (l) If W is a subspace of V, then $\dim(W) \leq \dim(V)$.
    (m) $\text{rank}(AB) \leq \text{rank}(A)$. (Hint: How are $C(AB)$ and $C(A)$ related?)
    (n) The intersection of two vector spaces cannot be empty.
    (o) If $Ax = Ay$, then $x = y$.
    (p) If a square matrix A has independent columns, then so does A^2.

20. Let $\mathbb{P} = \{a_0 + a_1X + a_2X^2 + \cdots + a_nX^n : a_i'\text{s are in } \mathbb{R}\}$ be the set of polynomials. Show that $\mathbb{P}$ is a real vector space under usual addition and scalar multiplication. Can you find a linearly independent set of size 2? 3? 50?

21. Let $\mathbb{P}_{\leq 2} = \{a_0 + a_1X + a_2X^2 : a_0, a_1, a_2 \text{ are in } \mathbb{R}\}$ be the set of polynomials of degree two or less.

    (a) Show that $\mathbb{P}_{\leq 2}$ is a subspace of $\mathbb{P}$.
    (b) Show that $\text{Span}\{1, X, X^2\} = \mathbb{P}_{\leq 2}$.
    (c) Find a basis for $\mathbb{P}_{\leq 2}$ and its dimension.

22. Find a basis for $\text{Span}\{cos(x), sin(x)\}$ in $\mathbf{F}$, the vector space of real-valued continuous functions on $[0, 1]$.

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