Tutorial-3, MA 106 (Linear Algebra)
Linear Algebra and its Applications by Gilbert Strang

Problem Set 2.1 [2, 5, 7, 8, 10, 13, 18, 19, 20, 22, 23, 24, 27, 30]

1. Which of the following are subspaces of $\mathbb{R}^3$?

   (a) The plane of vectors (b_1, b_2, b_3) with (i) $b_1 = 0$. (ii) $b_1 = 1$.
   (b) The set of vectors (b_1, b_2, b_3) with $b_2 b_3 = 0$.
   (c) All linear combinations of the vectors $(1, 1, 0)$. and $(2, 0, 1)$.
   (d) The plane of vectors (b_1, b_2, b_3) satisfying $b_3 - b_2 + 3b_1 = 0$.

2. Describe the column space and null space for:

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. Let $\mathbf{P}$ be the plane in $\mathbb{R}^3$ with the equation $x + 2y + z = 6$. What is the equation of the parallel plane $\mathbf{P_0}$ through the origin? Are $\mathbf{P}$ and $\mathbf{P_0}$ subspaces of $\mathbb{R}^3$?

4. Which of the following descriptions are correct? The solutions of $Ax = 0$, where $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ form

   (a) a plane (b) a line (c) a point (d) a subspace of $\mathbb{R}^2$
   (e) a subspace of $\mathbb{R}^3$ (f) the nullspace of A (g) the column space of A.

5. Show that the set of nonsingular 2×2 matrices is not a vector space. Show also that the set of singular 2×2 is not a vector space.

6. The functions $f(x) = x^2$ and $g(x) = 5x$ are vectors in the space $\mathbf{F}$ of real-valued functions on $[0, 1]$. The combination $3f(x) - 4g(x)$ is the vector $h(x) =$ ____.
   Why is $\mathbf{F}$ not a vector space if scalar multiplication is redefined as $c \cdot f(x) = f(cx)$?

7. The matrix $A = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$ is a vector in M, the space of all 2×2 matrices. Write the zero vector in this space, the vectors $\frac{1}{2}A$ and $-A$. What matrices are in the smallest subspace containing A?

8. Let M be vector of 3×3 matrices. Are the following true or false?

   (a) The symmetric matrices in M (i.e., $A = A^T$) form a subspace.
   (b) The skew symmetric matrices in M (i.e., $A = -A^T$) form a subspace.
   (c) The unmatrices in M (i.e., $A \neq A^T$) form a subspace.
   (d) The matrices that have $(1, 1, 1)$ in their nullspace form a subspace.

9. Suppose $\mathbf{P}$ is a plane in $\mathbb{R}^3$ through the origin, and $\mathbf{L}$ is a line in $\mathbb{R}^3$ through the origin. The smallest subspace containing $\mathbf{P}$ and $\mathbf{L}$ is either ____ or ____.

10. If we add an extra column b to a matrix A, then the column space gets larger unless ____. Give an example in which the column space gets larger and an example in which it does not.
    Why is $Ax = b$ solvable exactly when the column space *does not* get larger by including b?

11. Find conditions on the RHS for which the following systems are solvable:

$$(a)\ \begin{pmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

12. Are the following true or false? Briefly explain if it is true, give a counter-example if it is false.

   (a) The vectors b not in the column space of A form a subspace.
   (b) If $C(A)$ contains only the zero vector, then A is the zero matrix.
   (c) The column space of $2A$ is the same as the column space of A.
   (d) The column space of $A - I$ is equal to the column space of A.
   (e) The column space of AB is equal to the column space of A.

13. If A is an invertible 8×8 matrix, then its column space is ____. Why?

14. If the 9×12 system $Ax = b$ is solvable for every b, then $C(A) =$ ____.

15. Construct a 3×3 matrix whose column space contains $(1,1,0)$ and $(1,0,1)$, but not $(1,1,1)$. Construct a 3×3 matrix whose column space is only a line.

Problem Set 2.2

[2, 5, 7, 10, 11, 12, 14, 18, 29, 35, 38, 40, 44, 49, 51, 53, 54, 55, 61, 66, 68]

16. Find the echelon form U, the free variables, and the special solutions:

$$A = \begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

   $Ax = b$ is consistent when b satisfies $b_2 =$ ____. Find the set of complete solutions.

17. Reduce A and B to their echelon forms, find their ranks, the free and the dependent variables.

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

   Find the special solutions to $Ax = 0$ and $Bx = 0$, and their nullspaces.

18. Describe the set of attainable right-hand sides b for $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. What is the rank, and a particular solution?

19. A is $m \times n$ with row reduced form R. Which of the following give a correct definition of rank(A)?

   (a) The number of nonzero rows in R.
   (b) $n - m$.
   (c) n - number of free columns.
   (d) The number of 1's in R.

20. If the r pivot variables come first, the reduced R must look like $R = \begin{pmatrix} I & F \\ 0 & 0 \end{pmatrix}$, where I is $r \times r$, and F is $r \times (n - r)$. What is the null space matrix containing the special solutions?

21. Let $A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Under what conditions on b does $Ax = b$ have a solution? Find two vectors in $N(A)$ and a complete solution to $Ax = b$.

22. Write a 2×2 system $Ax = b$ with many solutions x_n to $Ax = 0$ but no particular solution x_p to $Ax = b$. In your example, which b's allow an x_p?

23. Find rank$(A) = r$ in each of the following. Find an $r \times r$ submatrix of A that is invertible, and find its inverse.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

24. Reduce $Ax = b$ to $Ux = c$ and $Rx = d$. Find the null space, the column space, and the complete set of solutions of $Ax = b$, where $A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$.

25. Under what conditions on b is $Ax = b$ solvable? Find all solutions when that condition holds.

$$x + 2y - 2z = b_1 \qquad 2x + 5y - 4z = b_2 \qquad 4x + 9y - 8z = b_3.$$

26. If $Ax = b$ has infinitely many solutions, why is it impossible for $Ax = c$ (a new constant vector) to have exactly one solution? Is it possible for $Ax = c$ to be inconsistent?

27. If $Ax = b$ has two solutions x_1 and x_2, find:

    (a) two solutions to $Ax = 0$ and (b) another solution to $Ax = b$.

28. Find q (if possible) so that the ranks are (a) 1, (b) 2, (c) 3:

$$A = \begin{pmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 & 3 \\ q & 2 & q \end{pmatrix}.$$

29. Reduce the matrices A and B to their echelon forms U. Find a special solution for each variable and describe all solutions in the nullspace.

$$A = \begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{pmatrix}.$$

    Reduce the echelon forms U to R, find the rank r and draw a box around the $r \times r$ identity matrix in R.

30. Suppose column 4 of a 3×5 matrix is all 0s. Then x_4 is certainly a ____ variable. The special solution corresponding to x_4 is $x =$ ____.

31. Are the following true or false? Briefly explain if it is true, give a counter-example if it is false.

    (a) A square matrix has no free variables.
    (b) An invertible matrix has no free variables.
    (c) An $m \times n$ matrix has no more than n pivot variables.
    (d) An $m \times n$ matrix has no more than m pivot variables.

32. The nullspace of a 3×4 matrix A is the line through $(2, 3, 1, 0)$. What is the rank of A and a complete solution to $Ax = 0$? What is the exact form of the row reduced form R of A?

33. The complete solution to $Ax = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for any real number c. Find A.

34. Explain why A and $-A$ have the same row reduced form R.

35. Construct a matrix whose column space contains $(1, 1, 1)$ and whose nullspace is the line of multiples of $(1, 1, 1, 1)$.

36. Why does no 3×3 matrix have a nullspace that equals its column space?

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