Tutorial-2, MA 108 (ODE)
Spring 2015, IIT Bombay

1. In the following exercises based on the existence and uniqueness theorem find all the (x_0, y_0), for which theorem gives an interval on which the given initial value problem has a solution and (b) and interval around x_0 for which it has a unique solution.

   (a) $y = \dfrac{e^x + y}{x^2 + y^2}$.

   (b) $y' = (x^2 + y^2)y^{1/3}$.

   (c) $y' = \dfrac{\tan y}{x - 1}$

2. Find infinitely many solutions of the initial value problem

$$y' = y^{2/5}, \quad y(0) = 1$$

   on $(-\infty, \infty)$.

3. Let

$$y' = 3x(y-1)^{1/3} \quad y(x_0) = y_0.$$

   (a) For what points (x_0, y_0) does the existence and uniqueness theorem does the above IVP have a solution.

   (b) For what points (x_0, y_0) does the existence and uniqueness theorem does the above IVP have a unique solution in an interval around x_0.

   (c) Let $(x_0, y_0) = (0, 1)$. Find nine solutions for the IVP which differ from each other for values of x in every open interval that contains $x_0 = 0$.

4. (a) From existence and uniqueness theorem, the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(3) = -7.$$

   has a unique solution on some open interval that contains $x_0 = 3$. Determine the largest such open interval, and find the solution on this interval.

   (b) Find infinitely many solutions of the IVP , all defined on $(-\infty, \infty)$.

5. Following may not be separable but can be made separable by substitution.

   (a) $y' = \dfrac{-6x + y - 3}{2x - y - 1}$.

(b) $y' = \dfrac{-x+3y-14}{x+y-2}$.

(c) $xyy' = 3x^6 + 6y^2$.

(d) $x(\ln x)^2 y' = -4(\ln x)^2 + y\ln x + y^2$.

6. Determine if the following equations are exact and solve them.

(a) $(3y\cos x + 4xe^x + 2x^2e^x)\ dx + (3\sin x + 3)\ dy = 0$.

(b) $(\dfrac{1}{x} + 2x)\ dx + (\dfrac{1}{y} + 2y)\ dy = 0$.

(c) $(y\sin(xy) + xy^2\cos(xy))\ dx + (x\sin(xy) + xy^2\cos(xy))\ dy = 0$.

(d) $(y^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)\ dx + (xe^{xy}\cos 2x - 3)\ dy = 0$.

(e) $\dfrac{x}{(x^2+y^2)^{3/2}}\ dx + \dfrac{y}{(x^2+y^2)^{3/2}} = 0$.

7. Solve the IVP and determine in what region the solution is valid.

(a) $(4x^3y^2 - 6x^2y - 2x - 3)dx + (2x^4y - 2x^3)\ dy = 0 \quad y(1) = 3$.

(b) $(y^3-1)e^x\ dx + 3y^2(e^x+1)\ dy = 0,\ \ y(0) = 0$.

(c) $(9x^2 + y - 1)\ dx - (4y - x)\ dy = 0,\ \ y(1) = 0$.

8. Find all the functions M such that the following equation is exact.

$$M(x,y)\ dx + 2xy\sin x\cos y\ dy = 0$$

9. Find all the functions N such that the equation is exact.

$$(\ln(xy) + 2y\sin x\ dx + N(x,y)\ dy = 0.$$

10. Suppose M and N are continuous and have continuous partial derivatives M_y and N_x that satisfy the exactness condition $M_y = N_x$ on an open rectangle R around (x_0, y_0). Show that if (x,y) is in R and

$$F(x,y) = \int_{x_0}^{x} M(s,y_0)\ ds + \int_{y_0}^{y} N(x,t)\ dt.$$

then $F_x = M$ and $F_y = N$.

11. Solve using the previous exercise.

$$(x^2+y^2)\ dx + 2xy\ dy = 0$$

12. Solve the initial value problem

$$y' + \frac{2}{x}y = -\frac{2xy}{x^2 + 2x^2 + 1}, \quad y(1) = -2.$$

13. Solve the following after finding an integrating factor.

    (a) $(27xy^2 + 8y^3)\ dx + (18x^2y + 12xy^2)\ dy0.$

    (b) $-y\ dx + (x^4 - x)\ dy = 0.$

    (c) $y \sin y\ dx + x(\sin y - y \cos y)\ dy = 0.$

    (d) $y(1 + 5 \ln|x|)\ dx + 4x \ln|x|\ dy = 0.$

    (e) $(3x^2y^3 - y^2 + y)\ dx + (-xy + 2x)\ dy = 0.$

    (f) $y\ dx + (2x - ye^y)\ dy = 0.$

    (g) $(a \cos(xy) - y \sin(xy))\ dx + (b \cos(xy) - x \sin(xy))\ dy = 0.$

14. Suppose M , N , M_x , and N_y are continuous for all (x, y) and $\mu = \mu(x, y)$ is an integrating factor for

$$M(x, y)\ dx + N(x, y)\ dy = 0.$$

    Assume that μ_x and μ_y are continuous for all (x, y), and suppose $y = y.(x)$ is a differentiable function such that $\mu(x, y(x)) = 0$ and $\mu_x(x, y(x)) \neq 0$ for all x in some interval I . Show that y is a solution to the above ODE on I.

15. Let $y' + p(x)y = f(x)$. Show that $\mu = \pm e^{\int p(x)\ dx}$ is an integrating factor. Find the explicit solution using this integrating factor.

16. Show that if $(Nx - My)/(xM - yN) = R$, where R depends on the quantity xy only, then the differential equation $M + Ny? = 0$ has an integrating factor of the form $\mu(xy)$. Find a general formula for this integrating factor.

17. Use the previous problem to solve $(3x + \frac{6}{y}) + (\frac{x^2}{y} + 3\frac{y}{x})\frac{dy}{dx} = 0.$

18. Consider the initial value problem $y' = y^{1/3}$, $y(0) = 0$.

    (a) Is there a solution that passes through the point $(1, 1)$? If so, find it.

    (b) Is there a solution that passes through the point $(2, 1)$? If so, find it.

    (c) Consider all possible solutions of the given initial value problem. Determine the set of values that these solutions have at $t = 2$.

19. (a) Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are solutions of the initial value problem

$$y' = \frac{-t + (t^2 + 4y)^{1/2}}{2} \qquad y(2) = -1.$$

Where are these solutions valid?

(b) Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of the Uniqueness theorem.

(c) Show that $y = ct + c^2$ where c is an arbitrary constant, satisfies the differential equation in part (a) for $t \geq -2c$. If $c = -1$, the initial condition is also satisfied, and the solution $y = y_1(t)$ is obtained. Show that there is no choice of c that gives the second solution $y = y_2(t)$.

20. (a) Show that $\phi(t) = e^{2t}$ is a solution of $y' - 2y = 0$ and that $y = c\phi(t)$ is also a solution of this equation for any value of the constant c.

(b) Show that $\phi(t) = 1/t$ is a solution to the equation $y' + y^2 = 0$ for $t > 0$ but that $y = c\phi(t)$ is not a solution of this equation unless $c = 0$ or $c = 1$. Note that the equation of part (b) is nonlinear, while that of part (a) is linear.