Families of Random Variables

Fall 2015

Instructor:

Ajit Rajwade

Topic Overview

- We will study several useful families of random variables that arise in interesting scenarios in statistics.
- Discrete random variables- Bernoulli, Binomial, Poisson, multinomial, hypergeometric
- Continuous random variables- Gaussian, uniform, exponential, chi-square

Bernoulli Distribution

Definition

- Let X be a random variable whose value is 1 when a coin toss produces a heads, and a 0 otherwise. If p is the probability that the coin toss produces a heads, we have:

  $P(X = 1) = p$, $P(X = 0) = 1\text{-}p$

- This is called a **Bernoulli pmf** with parameter p – named after Jacob Bernoulli. X is called a Bernoulli random variable.

- Note here: the coin need not be unbiased any longer!

Jacob Bernoulli: Known for contributions to calculus, probability and geometry

Properties

- $E[X] = p(1) + (1-p)(0) = p$.
- $Var(X) = p(1-p)^2 + (1-p)(0-p)^2 = p(1-p)$
- What's the median?
- What is (are) the mode(s)?

Binomial Distribution

Defintion

- Let X be a random variable denoting the number of heads in a sequence of n independent coin tosses (or *Bernoulli trials*) having *success probability* (i.e. probability of getting a heads) p.

- Then the pmf of X is given as follows:

$$P(X = i) = C(n,i)\,p^{i}(1-p)^{n-i}$$

- This is called the **binomial pmf** with parameters (n,p).

Defintion

- The pmf of X is given as follows:

$$P(X = i) = C(n,i)\,p^{i}(1-p)^{n-i}$$

$$C(n,i) = \frac{n!}{i!(n-i)!}$$

- Explanation: Consider a sequence of trials with *i* successes and *n*-*i* failures. The probability that this sequence occurs is $(1\text{-}p)^{n\text{-}i}$, by the *product rule*. But there are C(*n*,*i*) such sequences – so we add their individual probabilities using the *sum rule*.

Defintion

- Example: In 5 coin tosses, if we had two heads and three tails, the possible sequences are:

  HHTTT, HTHTT, HTTHT, HTTTH, TTTHH, THTTH, TTHHT, TTTHH, THTHT,THHTT

- What's the probability that a sequence of Bernoulli trials produces a success only on the *i*-th trial? Note that this is **not** a binomial distribution.

Defintion

- The pmf of X is given as follows:

$$P(X = i) = C(n,i)\,p^{i}(1-p)^{n-i}$$

$$C(n,i) = \frac{n!}{i!(n-i)!}$$

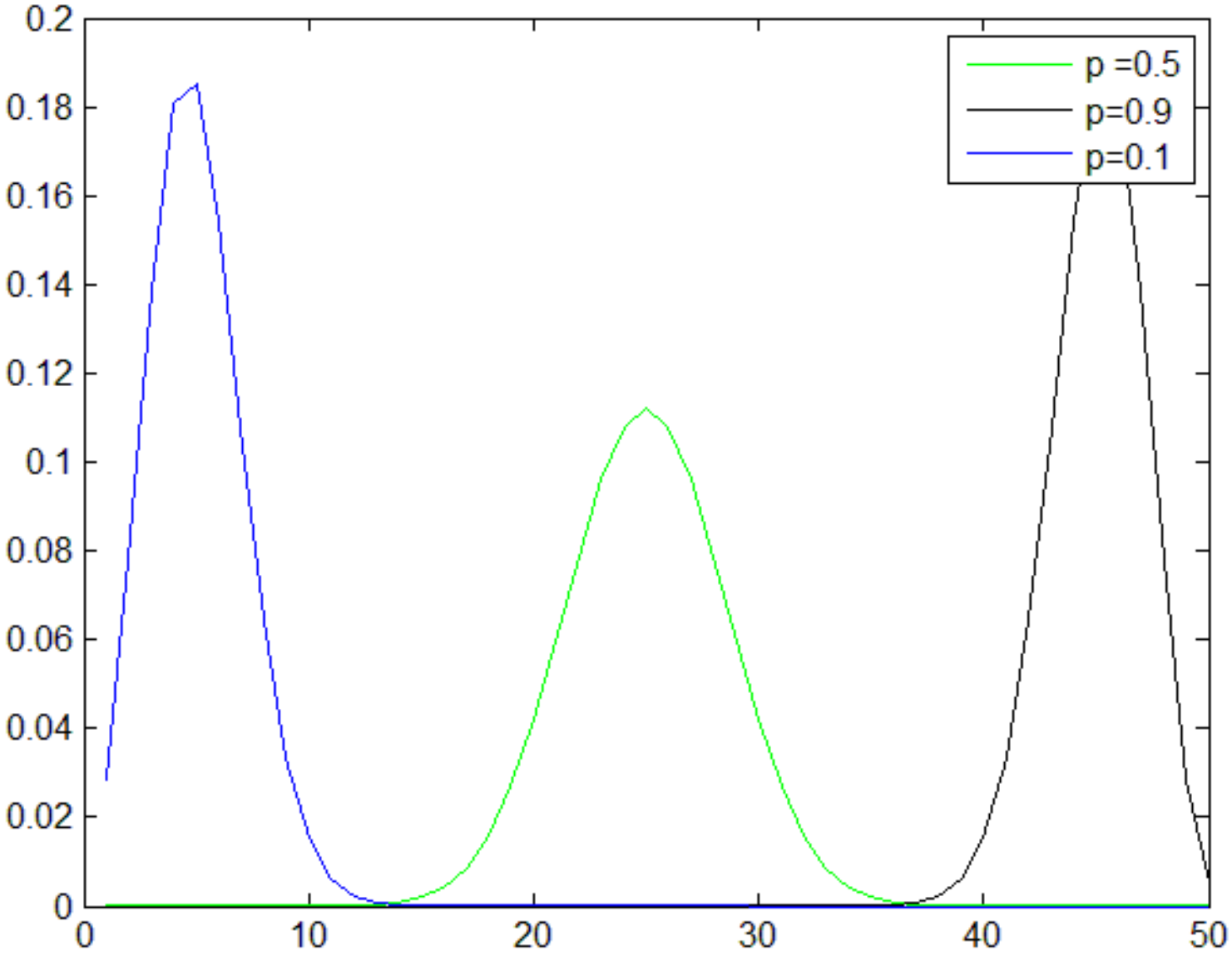
- To verify it is a valid pmf:

$$\sum_{i=1}^{n} P(X = i) = \sum_{i=1}^{n} C(n,i)\,p^{i}(1-p)^{n-i}$$

$$= (p + (1-p))^{n}$$

$$= 1$$

Binomial theorem!


0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
10
20
30
40
50
p =0.5
p=0.9
p=0.1

Example 1

- Let's say you design a smart self-driving car and you have tested it thoroughly. You have determined that the probability that your car collides is p. You go for an international competition and the rules say that you will win a prize if in k difficult tests, your car collides at most once. What's the probability that you will win the prize?

- Answer: X is the number of times your car collides. X is a binomial random variable with parameters (k,p). The probability that you win a prize is

  $P(X <= 1) = (1-p)^k + C(k,1)p(1-p)^{k-1}$.

Example 2

- At least half of an airplane's engines need to function for it to operate. If each engine independently fails with probability p, for what values of p is a 4-engine airplane more likely to operate than a 2-engine airplane?
- Answer: The number of functioning engines (X) is a binomial random variable with parameters $(4,p)$ or $(2, p)$.
- In 4-engine case, the probability of operation is $P(X=2) + P(X=3) + P(X=4) = 6p^2(1-p)^2 + 4p(1-p)^3+(1-p)^4$.
- In 2-engine case, it is $P(X=1)+P(X=2) = 2p(1-p)+(1-p)^2$. Do the math!
- Answer is for $p < 1/3$.

Properties

- Mean: Recall that binomial random variable X is the *sum* of random variables of the following form:

$X_i = 1$ if trial is yields success (this occurs with prob. p)

$= 0$ otherwise

- Hence

$$E(X) = E(\sum_{i=1}^{n} X_i) = np$$

Notice how we are making use of the linearity of the expectation operator. These calculations would have been much harder had you tried to plug in the formulae for binomial distribution.

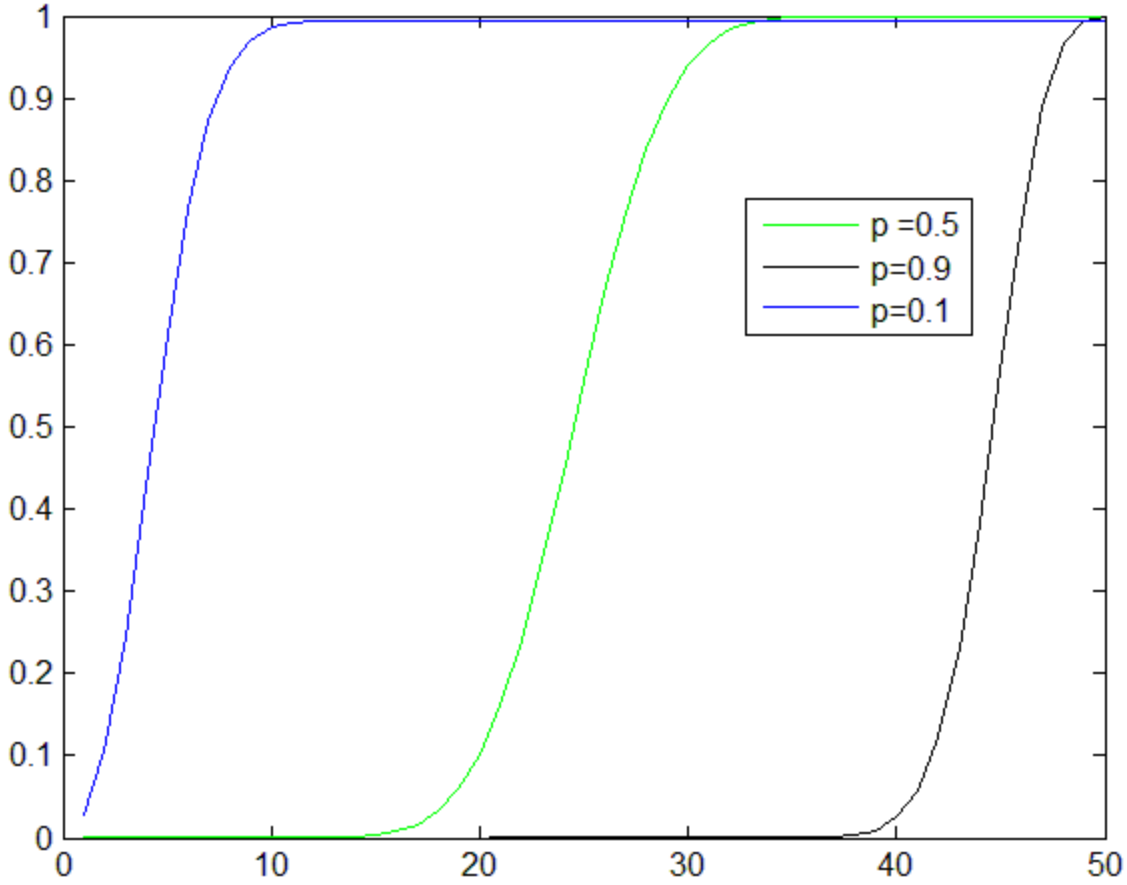
- Variance:

$$Var(X) = Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) = np(1-p)$$

Properties

- The CDF is given as follows:

$$F_X(i) = P(X \leq i) = \sum_{k=0}^{i} C(n,k)\, p^k (1-p)^{n-k}$$


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
p =0.5
p=0.9
p=0.1

Example

Bits are sent over a communications channel in packets of 12. If the probability of a bit being corrupted over this channel is 0.1 and such errors are independent, what is the probability that no more than 2 bits in a packet are corrupted?

If 6 packets are sent over the channel, what is the probability that at least one packet will contain 3 or more corrupted bits?

Let X denote the number of packets containing 3 or more corrupted bits. What is the probability that X will exceed its mean by more than 2 standard deviations?

Example

- We want $P(X = 0) + P(X=1) + P(X=2)$.
- $P(X=0) = (0.1)^0(0.9)^{12} = 0.282$
- $P(X=1) = C(12,1)\ (0.1)^1(0.9)^{11} = 0.377$
- $P(X=2) = C(12,2)\ (0.1)^2(0.9)^{10} = 0.23$
- So the answer is 0.889.

Example

- The probability that 3 or more bits are corrupted is 1-0.889 = 0.111

- Let Y = number of packets with 3 or more corrupted bits. Then we want $P(Y \geq 1) = 1\text{-}P(Y=0)$

  $= 1\text{-}C(6,0)(0.111)^0\ (0.889)^6 = 0.4936.$

- Mean of Y is $\mu = 6(0.111) = 0.666.$
- Standard deviation of Y is $\sigma = [6(0.111)(0.889)]^{0.5} = 0.77.$
- We want $P(Y > \mu + 2\sigma) = P(Y > 2.2) = P(Y \geq 3) = 1\text{-}P(Y=0)\text{-}P(Y=1)\text{-}P(Y=2) = ?$

Example

- We want $P(Y > \mu + 2\sigma) = P(Y > 2.2) = P(Y \geq 3) = 1 - P(Y=0) - P(Y=1) - P(Y=2) = ?$

- $P(Y=0) = C(6,0)\,(0.111)^0(0.889)^6 = 0.4936$
- $P(Y=1) = C(6,1)(0.111)(0.889)^5 = 0.37$
- $P(Y=2) = C(6,2)\,(0.111)^2(0.889)^4 = 0.115$

- $P(Y > \mu + 2\sigma) = 1-(0.4936+0.37+0.115) = 0.0214$

Related distributions

- In a sequence of Bernoulli trials, let X be the random variable for the trial number that gave the first success.

- Then X is called a geometric random variable and its pmf is given as:

$$P(X = i) = p(1-p)^{i-1}$$

- Let X be the number of trials before a total of r failures. Then X is called a **negative binomial random variable**. What is its pmf?

Properties: Mode

- Let $X \sim$ Binomial (n,p).
- Then $P(X = k) \geq P(X = k\text{-}1) \leftrightarrow k \leq (n+1)p$ (prove this).
- Also $P(X = k) \geq P(X = k+1) \leftrightarrow k \geq (n+1)p\text{-}1$ (prove this).
- Any integer-valued k which satisfies both the above conditions will be the mode (why?).

Poisson distribution

Definition – and genesis

- We have seen the binomial distribution before:

$$P(X = i) = \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i}$$

- Here p is the success probability. We can express it in the form

$$p = \frac{\lambda}{n}, \lambda = \text{expected number of successes in } n \text{ trials}$$

- Hence

$$P(X = i) = \frac{n!}{i!(n-i)!} \left(\frac{\lambda}{n}\right)^i (1-\frac{\lambda}{n})^{n-i}$$

Definition – and genesis

- We have

$$P(X=i)=\frac{n(n-1)(n-2)\ldots(n-i+1)}{i!}\left(\frac{\lambda}{n}\right)^{i}(1-\frac{\lambda}{n})^{n-i}$$

- In the limit when $n\to\infty$ and $p\to 0$ such that $\lambda=np$, we have

$$P(X=i)=\frac{\lambda^{i}}{i!}e^{-\lambda}$$

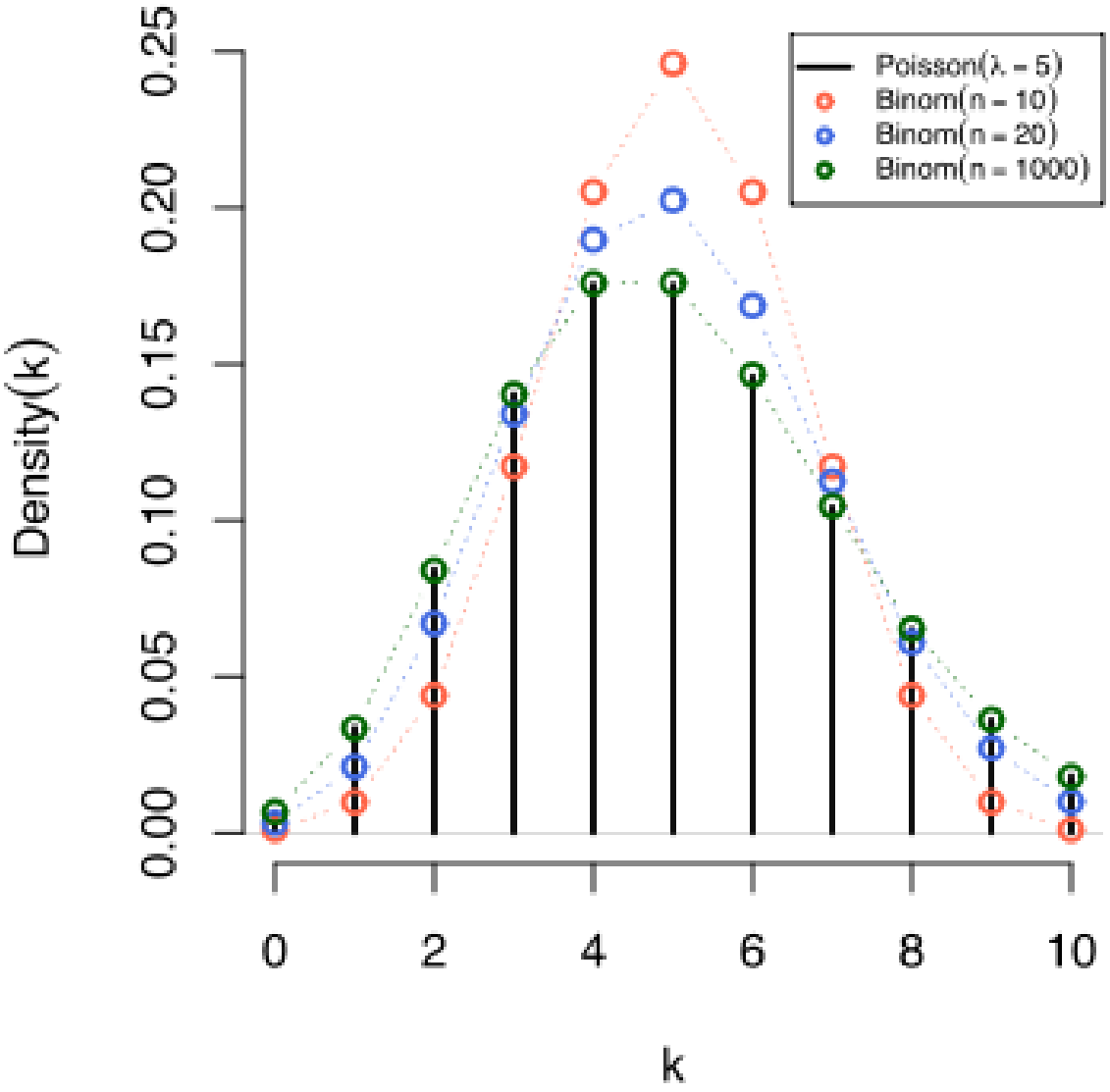
- This is called as the **Poisson pmf** and the above statement is called the **Poisson limit theorem**.

Definition

- The Poisson distribution is used to model the number of successes of a long sequence of independent Bernoulli trials if the *expected number of successes* (i.e. λ) is *known* and *constant*.

- For a Poisson random variable, note that the expected number of successes λ is constant and the **parameter** of the pmf. This is unlike the Binomial pmf for which the success probability *p* of a *single* Bernoulli trial is constant and also a parameter of the pmf.

Simon-Denis Poisson: French mathematician known for his contributions to probability, and also for the Poisson equation, a well known partial differential equation.


Density(k)
0.00
0.05
0.10
0.15
0.20
0.25
0
2
4
6
8
10
k
Poisson(λ = 5)
Binom(n = 10)
Binom(n = 20)
Binom(n = 1000)

Properties

- To double check that it is indeed a valid pmf, we check that:

$$\sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1$$

Using Taylor's series for exponential function about 0

- The afore-mentioned analysis tells us that the expected number of successes is equal to λ. To prove this rigorously – see next slide.

Properties

- The afore-mentioned analysis tells us that the expected number of successes is equal to λ. To prove this rigorously:

$$E(X) = E\left(\sum_{i=0}^{\infty} i e^{-\lambda} \frac{\lambda^i}{i!}\right)$$

$$= E\left(\sum_{i=1}^{\infty} i e^{-\lambda} \frac{\lambda^i}{i!}\right)$$

$$= \lambda e^{-\lambda} E\left(\sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}\right)$$

$$= \lambda e^{-\lambda} E\left(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!}\right)$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Properties

- Variance:

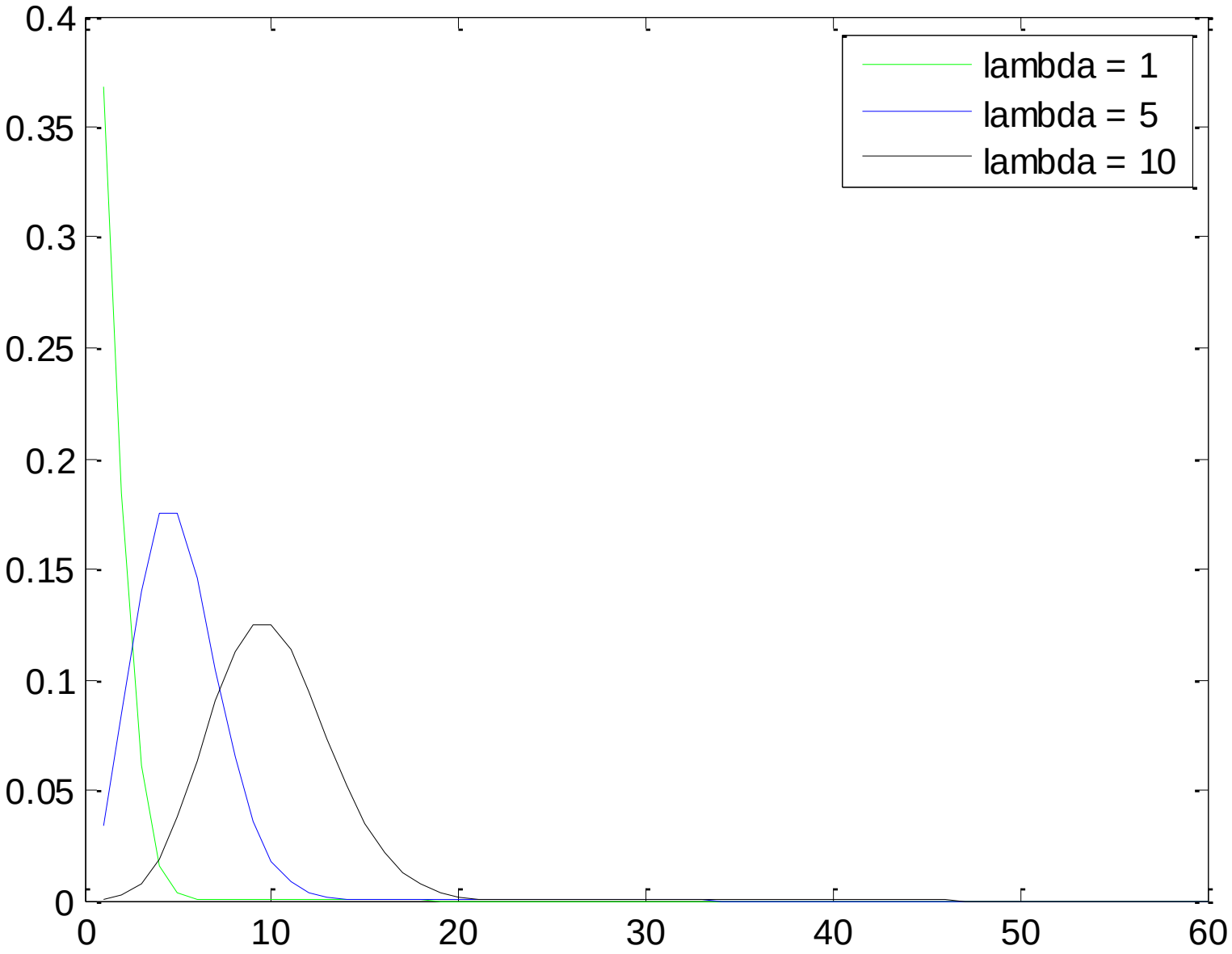
$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = E\left(\sum_{i=0}^{\infty} i^2 e^{-\lambda} \frac{\lambda^i}{i!}\right) = \lambda^2 + \lambda$$

$$\therefore Var(X) = \lambda$$

Detailed proof on the board. Also see here.

Notice: the mean and variance both increase with lambda.



Properties

- Consider independent Poisson random variables X and Y having parameters λ_1 and λ_2 respectively. Then $Z = X+Y$ is also a Poisson random variable with parameter $\lambda_1 + \lambda_2$.

  Detailed proof on the board.

- PMF – recurrence relation:

$$P(X = i) = \frac{\lambda^i}{i!} e^{-\lambda}$$

$$P(X = i+1) = \frac{\lambda^{i+1}}{(i+1)!} e^{-\lambda}$$

$$\frac{P(X = i+1)}{P(X = i)} = \frac{\lambda}{i+1}, P(X = 0) = 0$$

Poisson distribution: examples

- The number of misprints in a book (assuming the probability *p* of a misprint is small, and the number of letters typed is very large)

- Number of traffic rule violations in a typical city in the USA (assuming the probability *p* of a violation is small, and the number of vehicles is very large).

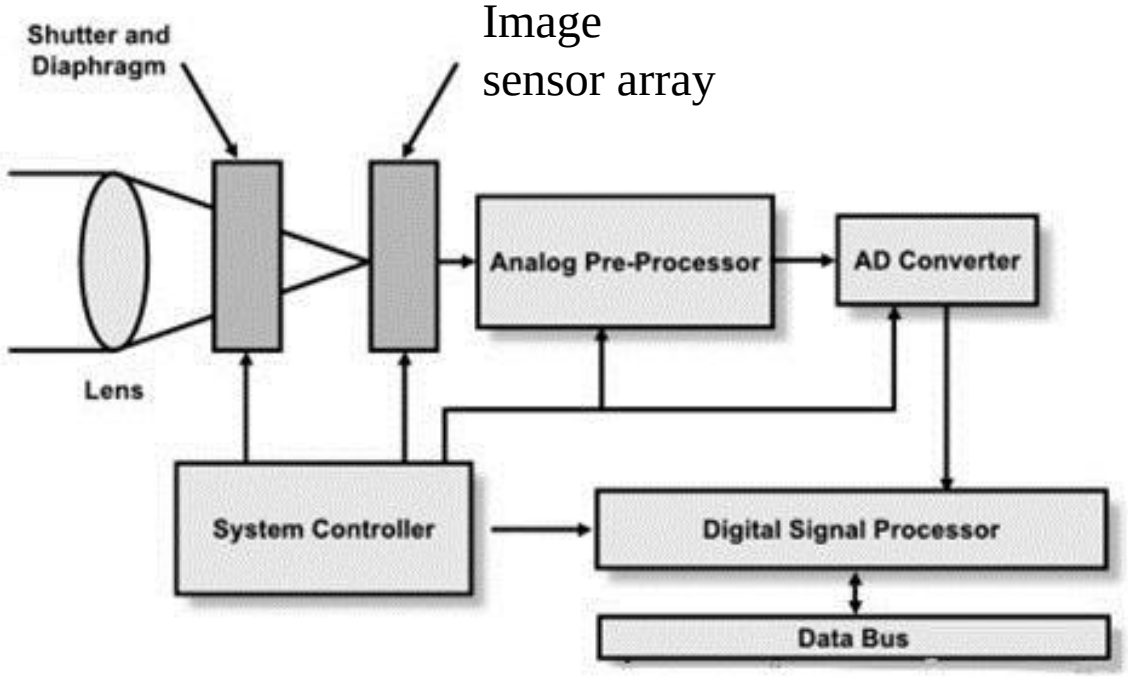
- In general, the Poisson distribution is used to model rare events, even though the event has plenty of "opportunities" to occur. (Sometimes called the **law of rare events** or the **law of small numbers**).

Poisson distribution: examples

- Number of people in a country who live up to 100 years
- Number of wrong numbers dialed in a day
- Number of laptops that fail on the first day of use

Poisson distribution: application

(not for exam)



Poisson distribution: application

(not for exam)

- An image sensor array consists of multiple **elements** – each sensor element measures the **intensity** of light reflected from a point (rather – area element) on a scene.

- A given scene point emits light usually at some average rate (measured as number of **photons** emitted per unit time per unit area).

- The intensity is measured in terms of the number of photons **counted** by the sensor element.

Poisson distribution: application

(not for exam)

- The number of photons *X* incident on a sensor element over time interval *t* (*the exposure time of the camera*) is modelled by the following pmf:

$$P(X = i) = \frac{(\lambda t)^i}{i!} e^{-\lambda t}$$

λ = expected number of photons emitted by the scene point per unit time (proportional to the irradiance/brightness of that scene point)
Parameter of the Poisson distribution = λt = expected number of photons emitted

- Now, the sensor element does not count all the incident photons.
- The number of counted photons is distributed as per Binomial(*n*,*p*) where *n*, the number of incident photons is itself Poisson distributed, and *p* is the success probability of the Bernoulli trials that constitute the Binomial distribution.

Poisson distribution: application

(not for exam)

- It can be proved that the number of photons Y counted by the sensor element over time interval t is modelled by the following pmf:

$$P(Y = i) = \frac{(\lambda tp)^i}{i!} e^{-\lambda tp}$$

Proof on the board!
We will cover this derivation in tutorial and **it is not excluded from the exam!**

- This is called as **thinning of the Poisson distribution by a Binomial distribution**.

- More details here or here.

Poisson distribution: application

(not for exam)

- Note: there is a **separate** Poisson distribution for each image pixel – each with a **different** λ that is directly proportional to the brightness of the scene point.

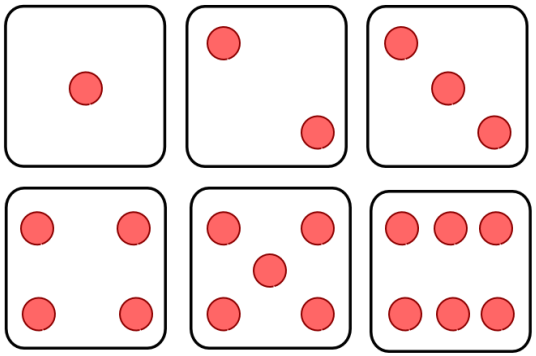
- The variation in the number of counted photons manifests itself as **"image shot noise"** – leading to a grainy appearance, which is even more distinct at lower intensity levels.

Image quality at 9 different average rates of photon emission.

Multinomial distribution

Definition

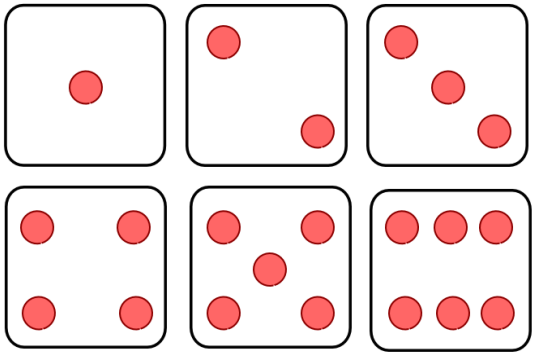
- Consider a sequence of n independent trials each of which will produce one out of k possible outcomes, where the set of possible outcomes is the *same* for each trial.



- Assume that the probability of each of the k outcomes is known and constant and given by $p_1, p_2, \ldots, p_k$.

Definition

- Let **X** be a *k*-dimensional random variable for which the i^{th} element represents the number of trials that produced the i^{th} outcome (also known as the number of *successes* for the i^{th} category)



Eg: in 20 throws of a die, you had 2 ones, 4 twos, 7 threes, 4 fours, 1 five and 2 sixes.

Definition

- Then the pmf of **X** is given as follows:

$$P(\mathbf{X} = (x_1, x_2, ..., x_k))$$
$$= P(X_1 = x_1, X_2 = x_2, ..., X_k = x_k)$$
$$= \frac{n!}{x_1! x_2! ... x_k!} p_1^{x_1} p_2^{x_2} ... p_k^{x_k}$$
$$\forall i, 0 \le p_i \le 1, \sum_{i=1}^{n} p_i = 1, x_1 + x_2 + ... + x_k = n$$

The number of ways to arrange n objects which can be divided into k groups of identical objects. There are x_1 objects of type 1, x_2 objects of type 2, and x_k objects of type k.

- This is called the **multinomial pmf**.

Definition

- The success probabilities for each category, i.e. p_1, p_2,…, p_k are all *parameters* of the multinomial pmf.

- Remember: The multinomial random variable is a vector whose i^{th} component is the number of successes of the i^{th} category (i.e. the number of times that the trials produced a result of the i^{th} category).

Properties

- Mean ***vector***:

$$E(\mathbf{X}) = (np_1, np_2, \ldots, np_k), E(X_i) = np_i$$

- Variance of a component

$$Var(X_i) = Var(\sum_{j=1}^{n} X_{ij}) = \sum_{j=1}^{n} Var(X_{ij}) = np_i(1 - p_i)$$

Assuming independent trials

X_{ij} is a Bernoulli random variable which tells you whether or not there was a success in the i^{th} category on the j^{th} trial

Properties

- For vector-valued random variables, the variance is replaced by the **covariance matrix**. The covariance matrix **C** in this case will have size *k* x *k*, where we have:

$$C(i, j) = E[(X_i - \mu_i)(X_j - \mu_j)] = Cov(X_i, X_j)$$

$$Cov(X_i, X_j) = -np_i p_j$$

Proof : next page

$$Cov(X_i, X_j) = -np_i p_j$$

Proof :

$$X_i = \#\text{successes in category } i = \sum_{k=1}^{n} X_{ik}$$

$$X_j = \#\text{successes in category } j = \sum_{l=1}^{n} X_{jl}$$

These are independent Bernoulli random variables – each representing the outcome of a trial (indexed by k and l)

$$Cov(X_i, X_j) = \sum_{k=1}^{n}\sum_{l=1}^{n} Cov(X_{ik}, X_{jl})$$

By linearity of covariance

$$= \sum_{k=1}^{n}\sum_{l=1,l\neq k}^{n} Cov(X_{ik}, X_{jl}) + \sum_{l=1}^{n} Cov(X_{il}, X_{jl})$$

By independence of trials

$$= 0 + \sum_{l=1}^{n} (E(X_{il}X_{jl}) - E(X_{il})E(X_{jl}))$$

Since in a trial, success can be achieved only in one category

$$= \sum_{l=1}^{n} (0 - p_i p_j) = -np_i p_j$$

$$\text{Now } Var(X_i + X_j) = E[(X_i - \mu_i + X_j - \mu_j)^2]$$
$$= Var(X_i) + Var(X_j) + 2Cov(X_i, X_j)$$

$$\therefore Var(X_i + X_j) = np_i(1 - p_i) + np_j(1 - p_j) - 2np_i p_j$$
$$= n(p_i + p_j)(1 - p_i - p_j)$$

Hypergeometric distribution

Sampling with and without replacement

- Suppose there are k objects each of a different type.
- When you sample 2 objects from these **with replacement**, you pick a particular object with probability $1/k$, and you place it back (***replace*** it).
- The probability of picking an object of another type is again $1/k$.
- When you sample **without replacement**, the probability that your first object was of so and so type is $1/k$. The probability that your second object was of so and so type is now $1/(k\text{-}1)$ because you ***didn't*** put the first object back!

Definition

- Consider a set of objects of which N are of good quality and M are defective.
- Suppose you pick some n objects out of these *without* replacement.
- There are $C(N+M,n)$ ways of doing this.
- Let X be a random variable denoting the number of good quality objects picked (out of a total of n).

Definition

- There are C(N,i)C(M,n-i) ways to pick i good quality objects and n-i bad objects.

- So we have

$$P(X = i) = \frac{C(N,i)C(M,n-i)}{C(N+M,n)}, 0 \le i \le n$$

$$C(a,b) = 0 \text{ if } b > a \text{ or } b < 0$$

Properties

- Consider random variable X_i which has value 1 if the i^{th} trial produces a good quality object and 0 otherwise.

- Now consider the following probabilities:

$$P(X_1 = 1) = \frac{N}{N+M}$$

$$P(X_2 = 1) = P(X_2 = 1 \mid X_1 = 1)P(X_1 = 1) + P(X_2 = 1 \mid X_1 = 0)P(X_1 = 0)$$

$$= \frac{N-1}{N+M-1}\frac{N}{N+M} + \frac{N}{N+M-1}\frac{M}{N+M} = \frac{N}{N+M}$$

In general, $P(X_i = 1) = \frac{N}{N+M}$

Properties

- Note that:

$$X = \sum_{i=1}^{n} X_i$$

Each X_i is a Bernoulli random variable with parameter $p=N/(N+M)$.

$$\therefore E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \frac{nN}{N+M}$$

$$\therefore Var(X) = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} Cov(X_i, X_j)$$

$$Var(X_i) = P(X_i = 1)(1 - P(X_i = 1)) = \frac{NM}{N+M}$$

Properties

- Note that:

$$\therefore Var(X) = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} Cov(X_i, X_j)$$

$$Var(X_i) = P(X_i = 1)(1 - P(X_i = 1)) = \frac{NM}{N+M}$$

$$Cov(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j)$$

$$E(X_i X_j) = P(X_i X_j = 1) = P(X_j = 1, X_i = 1)$$

$$= P(X_j = 1 \mid X_i = 1)P(X_i = 1)$$

$$= \frac{N-1}{N+M-1}\frac{N}{N+M}$$

Properties

- Note that:

$$Cov(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j)$$

$$E(X_i X_j) = P(X_i X_j = 1) = P(X_j = 1, X_i = 1)$$

$$= P(X_j = 1 \mid X_i = 1)P(X_i = 1)$$

$$= \frac{N-1}{N+M-1}\frac{N}{N+M}$$

$$Cov(X_i, X_j) = \frac{N(N-1)}{(N+M)(N+M-1)} - \left(\frac{N}{N+M}\right)^2$$

$$= \frac{-NM}{(N+M)^2(N+M-1)}$$

Properties

- Note that:

$$Cov(X_i, X_j) = \frac{N(N-1)}{(N+M)(N+M-1)} - \left(\frac{N}{N+M}\right)^2$$

$$= \frac{-NM}{(N+M)^2(N+M-1)}$$

$$Var(X) = \frac{nNM}{(N+M)^2} - \frac{n(n-1)NM}{(N+M)^2(N+M-1)}$$

$$= \frac{nNM}{(N+M)^2}\left(1 - \frac{n-1}{N+M-1}\right)$$

$$= np(1-p)\left(1 - \frac{n-1}{N+M-1}\right)$$

Recall: Each X_i is a Bernoulli random variable with parameter $p=N/(N+M)$.

$\approx np(1-p)$ when N and/or M is/are very large

Gaussian distribution

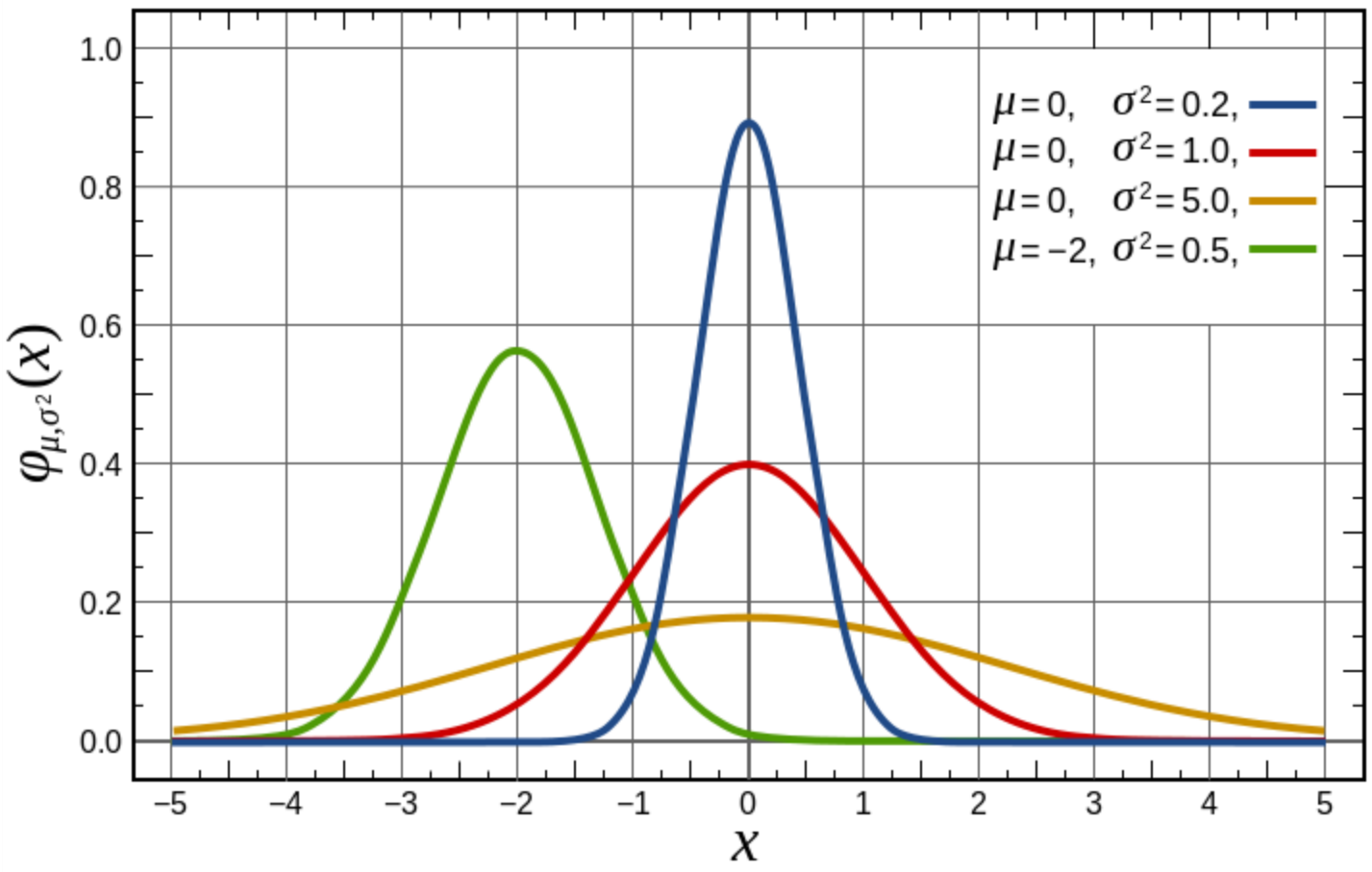
Definition

- A continuous random variable is said to be normally distributed with parameters mean μ and standard deviation σ if it has a probability density function given as:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right) \text{ denoted as}$$

$$\mathcal{N}(\mu, \sigma^2)$$

- This pdf is symmetric about the mean μ and has the shape of the “bell curve”.


$\mu = 0, \quad \sigma^2 = 0.2,$
$\mu = 0, \quad \sigma^2 = 1.0,$
$\mu = 0, \quad \sigma^2 = 5.0,$
$\mu = -2, \quad \sigma^2 = 0.5,$
$\varphi_{\mu,\sigma^2}(x)$
x
1.0
0.8
0.6
0.4
0.2
0.0
−5
−4
−3
−2
−1
0
1
2
3
4
5

Definition

- If μ=0 and σ=1, it is called the **standard normal distribution**

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \text{ denoted as}$$

$$\mathcal{N}(0,1)$$

Properties

- Mean:

$$E(X-\mu)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)e^{-(x-\mu)^2/(2\sigma^2)}dx$$

$$=\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(y)e^{-(y)^2/(2)}dy$$

$$=0---why?$$

$$\therefore E[X]=\mu$$

Properties

- Variance:

$$E((X-\mu)^2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/(2\sigma^2)} dx$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-(y)^2/(2)} dy$$

$$= \sigma^2 \text{ --why?}$$

Properties

If $X \sim N(\mu, \sigma^2)$ and if $Y = aX + b$, then

$Y \sim N(a\mu + b, a^2\sigma^2)$

Proof on board. And in the book.

Properties

- Median = mean (why?)
- Because of symmetry of the pdf about the mean

- Mode = mean – can be checked by setting the first derivative of the pdf to 0 and solving, and checking the sign of the second derivative.

- CDF for a 0 mean Gaussian with variance 1 – is given by:

$$\Phi(x) = F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(x)^2/(2)} dx$$

Properties

- CDF – it is given by:

$$\Phi(x) = F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(x)^2/(2)} dx$$

- It is closely related to the error function erf(x) defined as:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-x^2} dx$$

- It follows that:

$$\Phi(x) = \frac{1}{2}\left[1 + erf\left(\frac{x}{\sqrt{2}}\right)\right]$$

Verify for yourself

Properties

- For a Gaussian with mean μ and standard deviation σ, it follows that:

$$\Phi\left(\frac{x-\mu}{\sigma}\right)=\frac{1}{2}\left[1+erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$$

- The probability that a Gaussian random variable has values from μ-nσ to μ+nσ is given by:

$$\Phi(n)-\Phi(-n)=erf\left(\frac{n}{\sqrt{2}}\right)$$

Properties

- The probability that a Gaussian random variable has values from $\mu - n\sigma$ to $\mu + n\sigma$ is given by:

$$\Phi(n) - \Phi(-n) = erf\left(\frac{n}{\sqrt{2}}\right)$$

n	$\Phi(n)$-$\Phi(-n)$
1	68.2%
2	95.4%
3	99.7%
4	99.99366%
5	99.9999%
6	99.9999998%

Hence a Gaussian random variable lies within $\pm 3\sigma$ from its mean with more than 99% probability

A strange phenomenon

- Let's say you draw $n = 2$ values, called x_1 and x_2, from a [0,1] uniform random distribution and compute:

$$y_j = \sqrt{n}\left(\frac{\sum_{i=1}^{n} x_i}{n} - \mu\right)$$

(where μ is the true mean of the uniform random distribution)

- You repeat this process some 5000 times (say), and then plot the histogram of the computed $\{y_j\}$ values.
- Now suppose you repeat the earlier two steps with larger and larger n.

A strange phenomenon

- Now suppose you repeat the earlier two steps with larger and larger *n*.
- It turns out that as *n* grows larger and larger, the histogram starts resembling a 0 mean Gaussian distribution with variance equal to that of the sampling distribution.
- Now if you repeat the experiment with samples drawn from any other distribution instead of [0,1] uniform random (i.e. you change the sampling distribution).
- The phenomenon still occurs, though the resemblance may start showing up at smaller or larger values of *n*.
- This leads us to a very interesting theorem called the **central limit theorem**.
- Demo code: http://www.cse.iitb.ac.in/~ajitvr/CS215_Fall2015/MATLAB_Code/

Central limit theorem

- Consider $X_1, X_2, \ldots, X_n$ to be a sequence of independent and identically distributed (i.i.d.) random variables each with mean μ and variance $\sigma^2 < \infty$. Then as $n \to \infty$, the *distribution* (i.e. CDF) of the following quantity:

$$Y_n = \sqrt{n}\left(\frac{\sum_{i=1}^{n} x_i}{n} - \mu \right)$$

converges to that of $\mathcal{N}(0, \sigma^2)$. Or, we say Y_n converges in distribution to $\mathcal{N}(0, \sigma^2)$. This is called the **Lindeberg-Levy central limit theorem**.

Central limit theorem

- Note that the random variables $X_1, X_2, \ldots, X_n$ must be independent and identically distributed.

- There is a version of the central limit theorem that requires only independence – and allows the random variables to belong to different distributions. This extension is called the **Lindeberg Central Limit theorem**, and is given on the next slide.

Lindeberg's Central limit theorem

- Consider $X_1, X_2, \ldots, X_n$ to be a sequence of independent random variables each with mean μ_i and variance $(\sigma_i)^2 < \infty$. Then as $n \to \infty$, the distribution of the following quantity:

$$Y_n = \left(\frac{\sum_{i=1}^{n} (x_i - \mu_i)}{\sum_{i=1}^{n} \sigma_i^2} \right)$$

$\mathbf{1}_A : X \to \{0,1\}$

$\mathbf{1}_A(x) = 1 \text{ if } x \in A$

$= 0 \text{ otherwise}$

Indicator function

converges to that of $\mathcal{N}(0, 1)$ provided for every $\varepsilon > 0$

$$\lim_{n \to \infty} \left(\frac{\sum_{i=1}^{n} E[(x_i - \mu_i)^2 . 1_{\{|x_i - \mu_i| > \varepsilon s_n\}}]}{s_n} \right) = 0, s_n = \sum_{i=1}^{n} \sigma_i^2$$

Lindeberg's Central limit theorem

- Informally speaking, the take home message from the previous slide is that the CLT is valid even if the random variables emerge from different distributions.

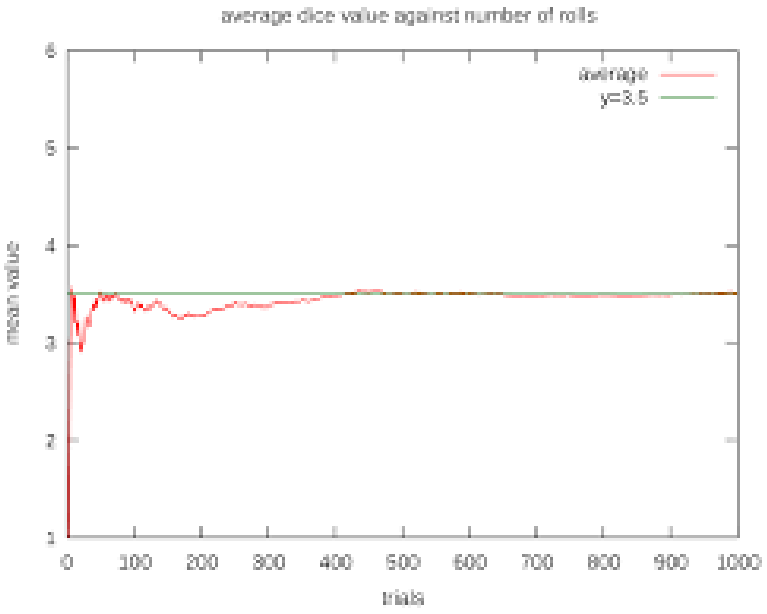
- This provides a major motivation for the widespread usage of the Gaussian distribution.

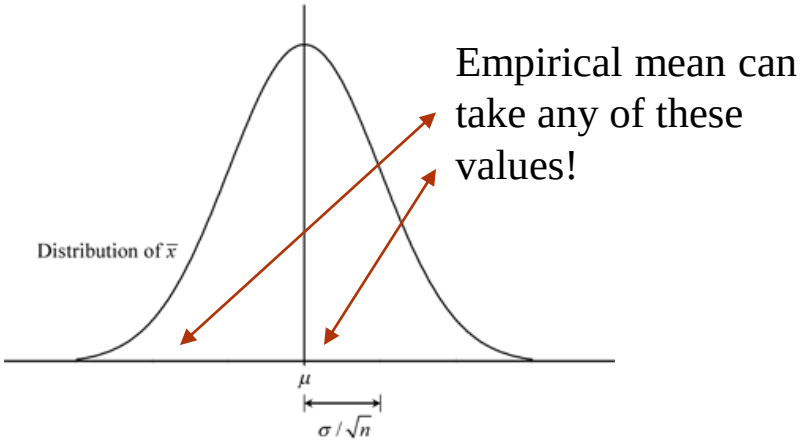
- The errors in experimental observations are often modelled as Gaussian – because these errors often stem from many different independent sources, and are modelled as being weighted combinations of errors from each such source.

Central limit theorem versus law of large numbers

- The law of large numbers says that the empirical mean calculated from a large number of samples is **equal to** (or very close) to the *true mean* μ (of the distribution from which the samples were drawn).

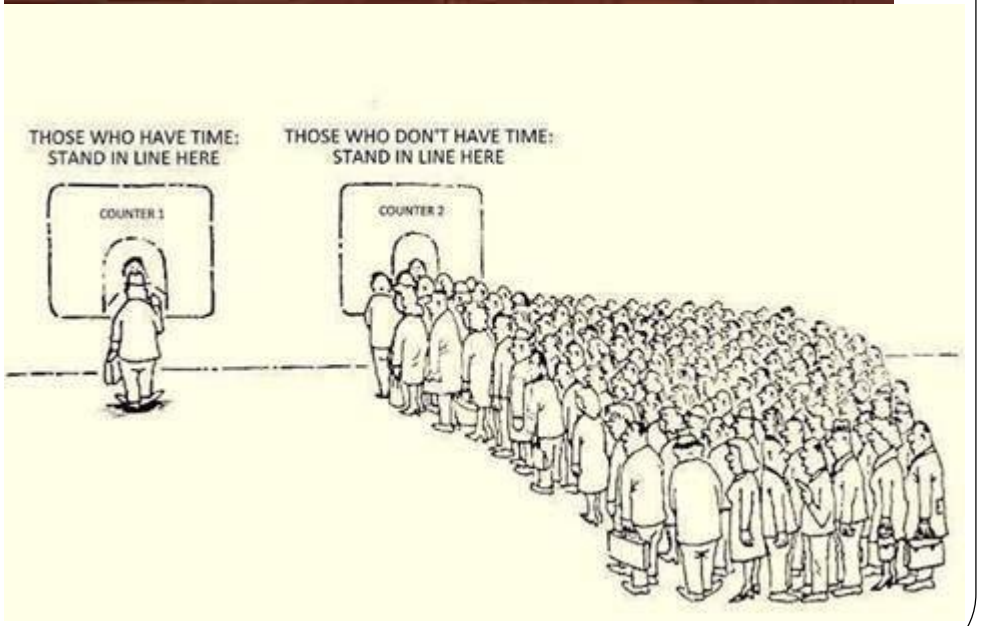
- The central limit theorem says that the empirical mean calculated from a large number of samples is a **random variable** drawn from a Gaussian distribution with mean equal to the true mean μ (of the distribution from which the samples were drawn).





Central limit theorem versus law of large numbers

- Is this a contradiction?



Central limit theorem versus law of large numbers

- The answer is NO!
- Go and look back at the central limit theorem.

$$Y_n = \sqrt{n}\left(\frac{\sum_{i=1}^{n} x_i}{n} - \mu\right) \sim \mathcal{N}(0, \sigma^2)$$

$$\rightarrow \left(\frac{\sum_{i=1}^{n} x_i}{n} - \mu\right) \sim \mathcal{N}(0, \sigma^2 / n) \quad (--\,why\,?)$$

$$\rightarrow \left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \sim \mathcal{N}(\mu, \sigma^2 / n)$$

This variance drops to 0 when n is very large! All the probability is now concentrated at the mean!

Application

- Your friend tells you that in 10000 successive independent unbiased coin tosses, he counted 5200 heads. Is he serious or joking?

- **Answer:** Let X1, X2,…, Xn be the random variables indicating whether or not the coin toss was a success (a heads). These are i.i.d. random variables whose sum is a random variable with mean $n\mu$=10000(0.5) = 5000 and standard deviation $\sigma n^{1/2}$ = sqrt(0.5(1-0.5))sqrt(10000) = 50.

Application

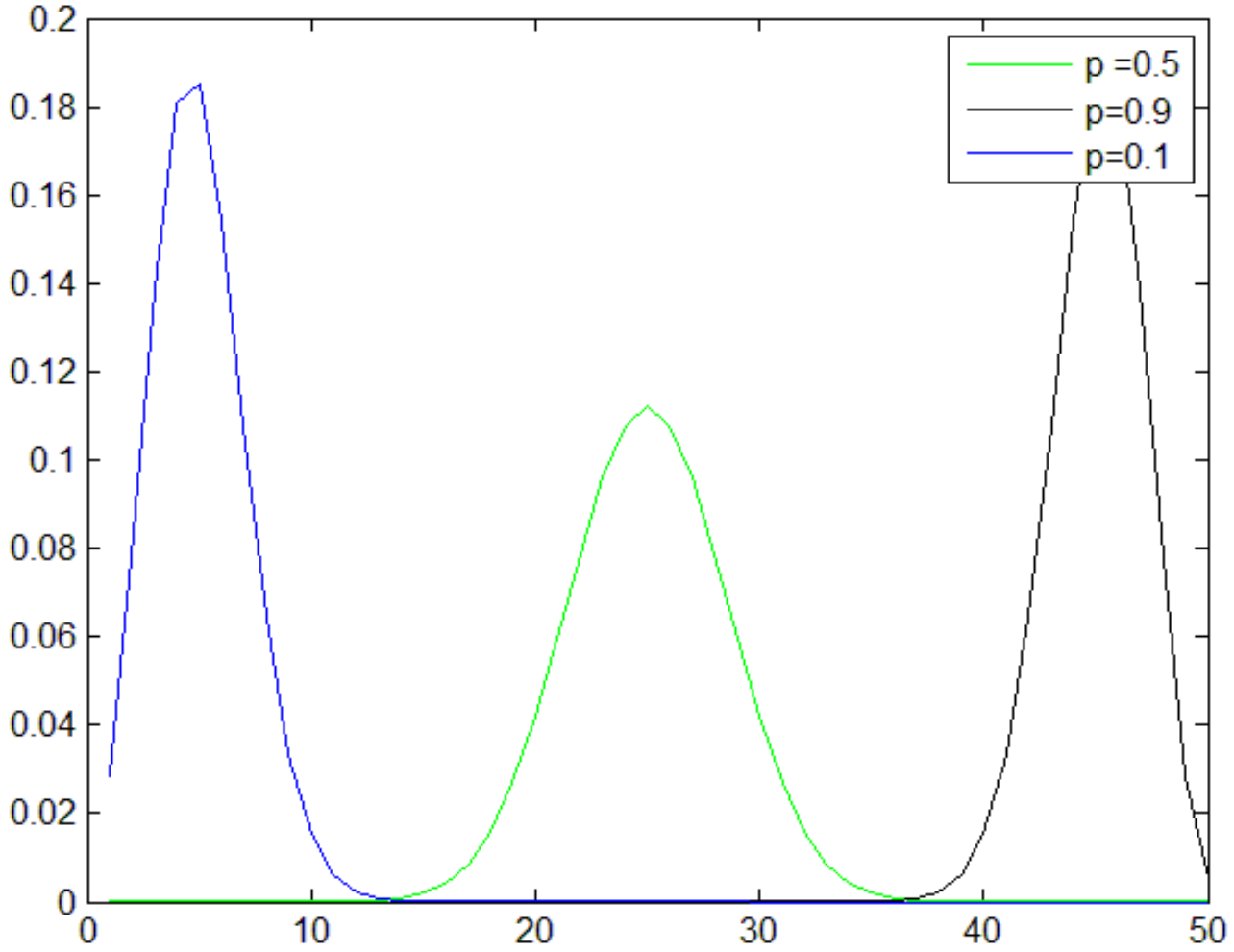
- Your friend tells you that in 10000 successive independent unbiased coin tosses, he counted 5200 heads. Is he serious or joking?

- **Answer:** The given number of heads is 5200 which is 4 standard deviations away from the mean. The chance of that occurring is of the order of 0.00001 (see the slide on error functions). So your friend is (most likely) joking.

- Notice that this answer is much more principled than giving an answer purely based on some arbitrary threshold over $|X\text{-}5000|$.

- You will study much more of this when you do hypothesis testing.

Binomial distribution and Gaussian distribution



Binomial distribution and Gaussian distribution

- The binomial distribution begins to resemble a Gaussian distribution with an appropriate mean for large values of *n*.

- In fact this resemblance begins to show up for surprisingly small values of *n*.

- Recall that a binomial random variable is the number of successes of independent Bernoulli trials

$$X = \sum_{i=1}^{n} X_i, X_i = 1 (\text{heads on } i^{\text{th}} \text{ trial}) \text{ else } 0$$

Binomial distribution and Gaussian distribution

- Each X_i has a mean of p and standard deviation of $p(1-p)$.

- Hence the following random variable is a standard normal random variable by CLT:

$$\frac{X - np}{\sqrt{np(1-p)}}$$

- Watch the animation here.

Binomial distribution and Gaussian distribution

- Another way of stating the afore-mentioned facts is that:

When $n \to \infty$, we have $\forall a,b, a \le b$,

$$P(a \le \frac{X - np}{\sqrt{np(1-p)}} \le b) = \Phi(b) - \Phi(a)$$

where $X \sim Binomial(n, p)$

- This is called the **de Moivre-Laplace theorem** and is a special case of the CLT. But its proof was published almost 80 years before that of the CLT!

Distribution of the sample mean

- Consider independent and identically distributed random variables $X_1, X_2, \ldots, X_n$ with mean μ and standard deviation σ.

- We know that the sample mean (or empirical mean) is a random variable given by:

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

Distribution of the sample mean

- Now we have:

$$E(\overline{X}) = E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \mu$$

$$Var(\overline{X}) = \frac{\sum_{i=1}^{n} Var(X_i)}{n^2} = \frac{\sigma^2}{n}$$

If $X_1, X_2, ..., X_n$ were normal random variables,

then it can be proved that $\overline{X}$ is also a normal random variable.

Otherwise, $\overline{X}$ is approximately normally distributed, as per the central limit theorem.

Distribution of the sample variance

- The sample variance is given by:

$$S^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1} = \frac{\sum_{i=1}^{n} X_i^2 - n\overline{X}^2}{n-1}$$

- The sample standard deviation is S.

Distribution of the sample variance

- The expected value of the sample variance is derived as follows:

$$E((n-1)S^2) = E(\sum_{i=1}^{n} X_i^2 - n\overline{X}^2)$$

$$= nE(X_1^2) - nE(\overline{X}^2)$$

$$E(W^2) = Var(W) + (E(W))^2$$

$$\therefore E((n-1)S^2) = nVar(X_1) + n(E(X_1))^2$$

$$- nVar(\overline{X}) - n(E(\overline{X}))^2$$

Distribution of the sample variance

- The expected value of the sample variance is derived as follows:

$$\therefore E((n-1)S^2) = nVar(X_1) + n(E(X_1))^2$$
$$- nVar(\overline{X}) - n(E(\overline{X}))^2$$

$$\therefore E((n-1)S^2) = n\sigma^2 + n\mu^2$$
$$- n(\sigma^2/n) - n(\mu)^2 = (n-1)\sigma^2$$

$$\therefore E(S^2) = \sigma^2$$

Distribution of the sample variance

- The expected value of the sample variance is derived as follows:

$$\therefore E((n-1)S^2) = (n-1)\sigma^2$$

$$\therefore E(S^2) = \sigma^2$$

If the sample variance were instead defined as

$$S^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n} = \frac{\sum_{i=1}^{n} X_i^2 - n\overline{X}^2}{n},$$

we would have:

$$E(S^2) = \frac{(n-1)\sigma^2}{n}$$

This is **undesirable** – as we would like to have the expected value of the sample variance to equal the true variance! Hence S^2 here above is multiplied by $(n\text{-}1)/n$ to correct for this anomaly giving rise to our strange definition of sample variance. This multiplication by $(n\text{-}1)/n$ is called **Bessel's correction.**

Distribution of the sample variance

- But the mean and the variance alone does not determine the distribution of any random variable.
- So what about the distribution of the sample variance?
- For that we need to study another distribution first – the chi-squared distribution.

Chi-square distribution

- If $Z_1, Z_2, \ldots, Z_n$ are independent standard normal random variables, then the following quantity

$$X = Z_1^2 + Z_2^2 + \ldots + Z_n^2$$

is said to have a chi-square distribution with *n* degrees of freedom and is denoted as follows

$$X \sim \chi_n^2$$

- The formula for this is as follows:

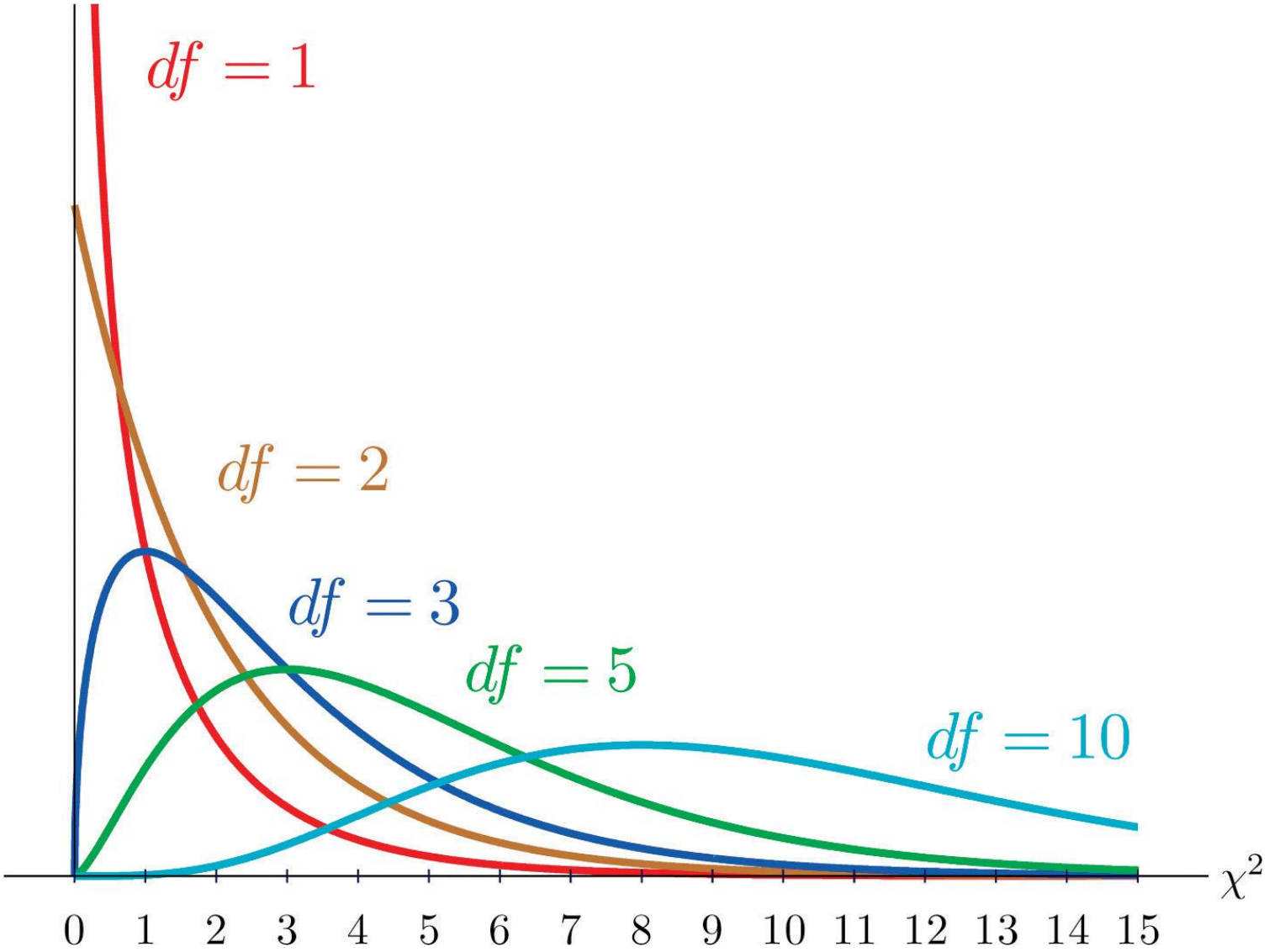
$$f_X(x) = \frac{x^{n/2-1} e^{-x/2}}{2^{n/2}\Gamma(n/2)}$$

$$\Gamma(y) = (y-1)!\,(y \text{ integer})$$

$$= \int_0^\infty x^{y-1} e^{-x} dx$$

Chi-square distribution

- You will be able to obtain the expression for the chi-square distribution when you study transformation of random variables or moment generating functions.


df = 1
df = 2
df = 3
df = 5
df = 10
χ²
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

Additive property

- If X_1 and X_2 are independent chi-square random variables with n_1 and n_2 degrees of freedom respectively, then $X_1 + X_2$ is also a chi-square random variable with $n_1 + n_2$ degrees of freedom. This is called the **additive property**.

- It is easy to prove this property by observing that $X_1 + X_2$ is basically the sum of $n_1 + n_2$ independent normal random variables.

Chi-square distribution

- Tables for the chi-square distribution are available for different number of degrees of freedom, and for different values of the independent variable.

Back to the distribution of the sample variance

$$(n-1)S^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2 = \sum_{i=1}^{n}(X_i - \mu)^2 - n(\overline{X} - \mu)^2$$

$$\therefore \frac{\sum_{i=1}^{n}(X_i - \mu)^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sigma^2} + \frac{n(\overline{X} - \mu)^2}{\sigma^2}$$

$$\therefore \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sigma^2} + \left(\frac{\sqrt{n}(\overline{X} - \mu)}{\sigma}\right)^2$$

The sum of squares of *n* standard normal random variables

The square of a standard normal random variable

Back to the distribution of the sample variance

$$\therefore \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2} + \left(\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}\right)^2$$

The sum of squares of n standard normal random variables

The square of a standard normal random variable

It turns out that these two quantities are **independent** random variables. The proof of this requires multivariate statistics and transformation of random variables, and is deferred to a later point in the course. If you are curious, you can browse this link, but it's not on the exam for now.

Given this fact about independence, it then follows that the middle term is a chi-square distribution with *n*-1 degrees of freedom.

Uniform distribution

Uniform distribution

- A uniform random variable over the interval [a,b] has a pdf given by:

$$f_X(x) = 1/(b-a), if\ a \le x \le b$$
$$= 0 \text{ otherwise}$$

- Clearly, this is a valid pdf – it is non-negative and integrates to 1.

- It is easy to show that its mean and median are equal to $(b+a)/2$.

Applications

- Uniform random variables, especially over the [0,1] interval are very important, in developing to programs to draw samples from other distributions including the Gaussian, Poisson, and others.

- You will study more of this later on in the semester.

- For now, we will study two applications. How do you draw a sample from a distribution of the following form: $P(X = x_i) = p_i, 1 \leq i \leq n,$

$$\sum_{i=1}^{n} p_i = 1$$

Applications

- For now, we will study two applications. How do you draw a sample from a discrete distribution with the following pmf:

$$P(X = x_i) = p_i, 1 \le i \le n, \sum_{i=1}^{n} p_i = 1$$

Draw $u \sim Uniform(0,1)$

If $u \le p_1 \rightarrow$ sampled value is x_1

If $p_1 \le u \le p_1 + p_2 \rightarrow$ sampled value is x_2

.

.

If $p_1 + p_2 + \ldots + p_{n-1} \le u \le p_1 + p_2 + \ldots + p_{n-1} + p_n \rightarrow$ sampled value is x_n

Applications

- Uniform random variables, especially over the [0,1] interval are very important, in developing to programs to draw samples from other distributions including the Gaussian, Poisson, and others.

- You will study more of this later on in the semester.

- For now, we will study how to generate a random permutation of *n* elements. That is what the **“randperm”** function in MATLAB does, and you have used it at least once so far!

Application: generating a random subset

- In fact, we will do something more than randperm – we will develop theory to generate a random subset of size k from a set $A=\{a_1, a_2,\ldots, a_n\}$ of size n, assuming all the C(n,k) subsets are equally likely.

- Let us define the following for each element j ($1 \leq j \leq n$):

Notation for chosen subset

$$I_j = 1 \text{ if } a_j \in B_k, \text{ else } 0$$

- Now we will sequentially pick each element of the subset randomly as follows:

Application: generating a random subset

- Notice that $P(I_1=1) = k/n$. (why?)
- If I_1 is 1, then $P(I_2=1) = (k-1)/(n-1)$. (why?)
- If I_1 is 0, then $P(I_2=1) = k/(n-1)$. (why?)
- Thus $P(I_2=1|I_1) = (k-I_1)/(n-1)$ (why?)
- Side question: what is $P(I_2=1)$?

Application: generating a random subset

- Continuing this way, one can show that:

$$P(I_j \mid I_1, I_2, \ldots, I_{j-1}) = \frac{k - \sum_{i=1}^{j-1} I_i}{n - (j-1)}, 2 \leq j \leq n$$

Application: generating a random subset

- This suggests the following procedure:

$$U_1 \sim Uniform(0,1)$$
$$I_1 = 1, \text{if } U_1 < k/n, \text{else } 0$$
$$U_2 \sim Uniform(0,1)$$
$$I_2 = 1, \text{if } U_2 < (k - I_1)/(n-1), \text{else } 0$$
.

.

$$U_j \sim Uniform(0,1)$$
$$I_j = 1, \text{if } U_j < (k - I_1 - I_2 - \ldots - I_{j-1})/(n - j + 1), \text{else } 0$$

When does this process stop? It stops at step #*j*

If $I_1+I_2+\ldots+I_j = k$ and the random subset B_k contains those indices whose I-values are 1

OR

If the number of unfilled entries in the random subset B_k = number of remaining elements in *A*. In this case B_k = all remaining elements in *A* with index greater than *i*=largest index in B_k. See figure 5.6 of the book.

Exponential Distribution

Motivation

- Consider a Poisson distribution with an average number of successes per unit time given by λ.
- So the number of successes in time t is λt.
- This is actually called a **Poisson process**.
- Now consider the time taken (T) for the first success – this is called as the **waiting time**.

Motivation

- Let X ~ Poisson (λt) for time interval (0,t).

- T is a random variable whose distribution we are going to seek to model here. Then,

$$P(T \le t) = 1 - P(T > t) = 1 - P(X = 0)$$

The probability that the first success occurred after time t = probability that there was no success in the time interval (0,t), i.e. X = 0 in that interval

$$P(T \le t) = 1 - P(T > t) = 1 - P(X = 0)$$

$$= 1 - \frac{e^{-\lambda t}(\lambda t)^0}{0!} = 1 - e^{-\lambda t}$$

Motivation

$$F_T(t) = 1 - e^{-\lambda t}$$

$$f_T(t) = \lambda e^{-\lambda t}, t \geq 0$$

$$(0 \text{ elsewhere})$$

Such a random variable T is called an exponential random variable. It models the waiting time for a Poisson process. It has a parameter λ.

Properties: Mean and Variance

- Mean:

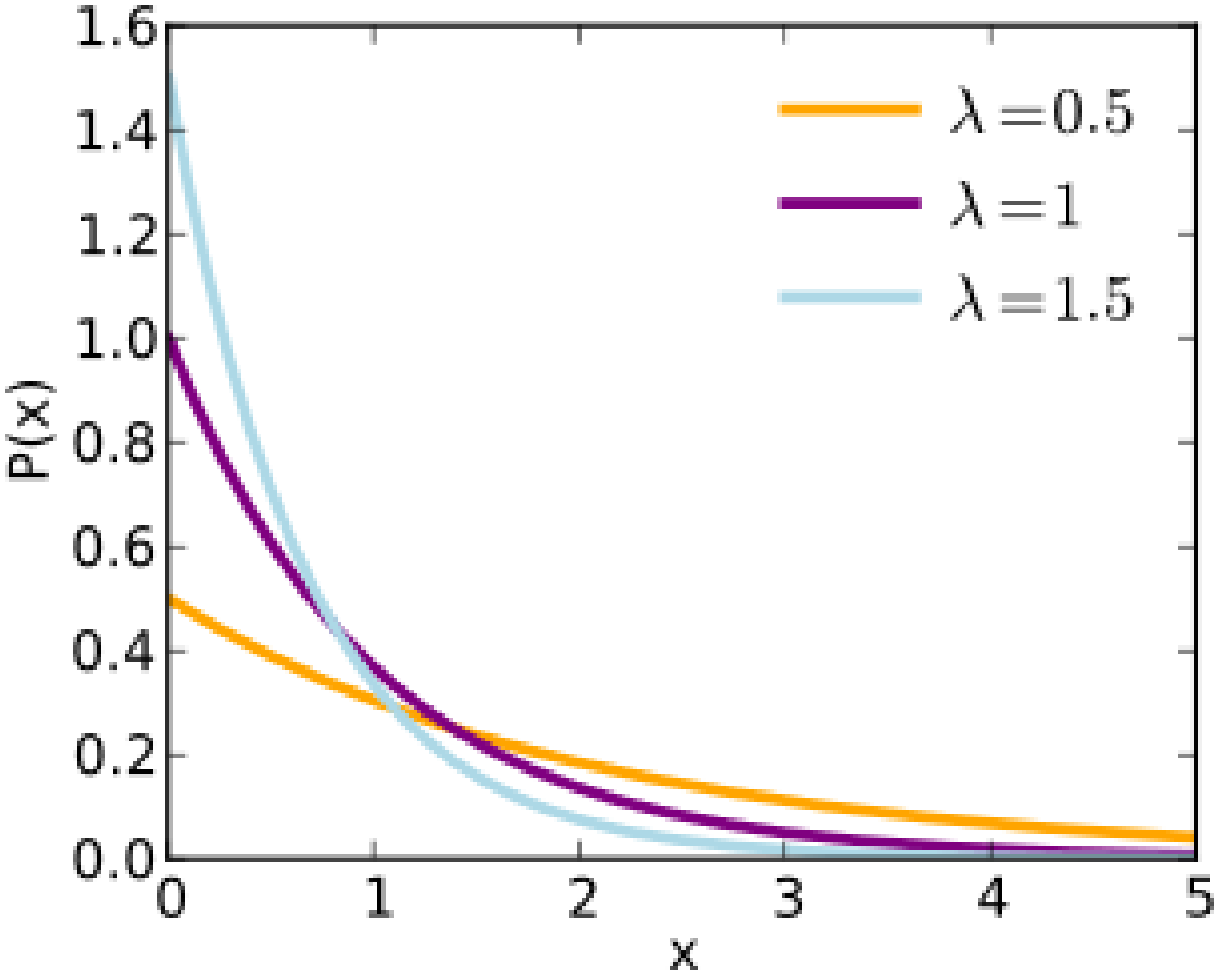
$$E(T) = \int_0^\infty t\lambda e^{-\lambda t} dt = \lambda \int_0^\infty t e^{-\lambda t} dt = \frac{1}{\lambda}$$

This is intuitive – a Poisson process with a large average rate should definitely lead to a lower expected waiting time.

- Variance:

$$Var(T) = E(T^2) - (E(T))^2 = \frac{1}{\lambda^2}$$

$$E(T^2) = \int_0^\infty t^2 \lambda e^{-\lambda t} dt = \frac{2}{\lambda^2}$$


1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
P(x)
0
1
2
3
4
5
x
$\lambda = 0.5$
$\lambda = 1$
$\lambda = 1.5$

Properties: Mode and Median

- Mode: always at 0

- Median

$$\int_0^u \lambda e^{-\lambda t} dt = \frac{1}{2}$$

$$u = \frac{\ln 2}{\lambda}$$

Properties: “Memorylessness”

- A non-negative random variable T is said to be **memoryless** if:

$$\forall s,t \geq 0, P(T > s+t \mid T > t) = P(T > s)$$

- Meaning: This gives the probability that given a waiting time of success of more than t, the waiting time will exceed s+t, i.e. one would have to wait for s more time units for success.

- Another formula (equivalent to the earlier one)

$$\frac{P(T > s+t, T > t)}{P(T > t)} = P(T > s)$$

Properties: “Memorylessness”

- You can easily verify that this holds for the exponential distribution.

$$P(T > t) = e^{-\lambda t}$$

$$P(T > s) = e^{-\lambda s}$$

$$P(T > s + t) = e^{-\lambda (s+t)}$$

Example

- Suppose that the number of miles a car can run before its battery fails is exponentially distributed with an average of α. What is the probability that the car won't fail on a trip of *k* miles given that it has already run for *l* miles?

- Solution: For exponential distribution we know that

$$\frac{P(T > k+l, T > l)}{P(T > l)} = P(T > k) = e^{-k\lambda} = e^{-k/\alpha}$$

Example

- Suppose that the number of miles a car can run before its battery fails is exponentially distributed with an average of α. What is the probability that the car won't fail on a trip of k miles given that it has already run for *l* miles?

- Solution: If the distribution were not exponential, then we have

$$P(T > k+l \mid T > l) = \frac{P(T > k+l, T > l)}{P(T > l)} = \frac{1 - F_T(k+l)}{1 - F_T(l)}$$

Property: Minimum

- Consider independent exponentially distributed random variables $X_1, X_2, \ldots, X_n$ with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $\min(X_1, X_2, \ldots, X_n)$ is exponentially distributed.

$$P(\min(X_1, X_2, ..., X_n) > x) = P(X_1 > x, X_2 > x, ..., X_n > x)$$

$$= \prod_{i=1}^{n} P(X_i > x) \text{ due to independence}$$

$$= \prod_{i=1}^{n} e^{-\lambda_i x}$$

$$= e^{-\sum_{i=1}^{n} \lambda_i x}$$