Q1. Let $X_1, X_2, \ldots X_n$ be independent random variables from $N(\mu, \sigma^2)$. Determine the maximum likelihood estimator for σ^2 when μ is known. What is the expected value of this estimator? Repeat the exercise when μ is known.

μ known

$$JL = \prod_{i=1}^{n} \frac{e^{-(x_i-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \quad \text{(due to independence)}$$

$$JLL = \sum_{i=1}^{n} -\frac{(x_i-\mu)^2}{2\sigma^2} - \log \sigma - \log\sqrt{2\pi}$$

$\gamma = \sigma^2$

$$\cancel{\partial JLL} = \sum_{i=1}^{n} -\frac{(x_i-\mu)^2}{2\gamma} - \frac{1}{2}\log\gamma + \text{constant}$$

$$\frac{\partial JLL}{\partial \gamma} = \sum_{i=1}^{n}\left(\frac{+(x_i-\mu)^2\,(+1)}{2\gamma^2} - \frac{1}{2\gamma}\right) = 0$$

$$\therefore \gamma = \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$$

$E(\hat{\sigma}^2) = \sigma^2 \longrightarrow$ unbiased estimator.

μ unknown

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 \text{ where } \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$E(\hat{\sigma}^2) = \frac{1}{n}\sum_{i=1}^{n} E\left((x_i - \bar{x})^2\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E\left(x_i^2 - 2x_i\bar{x} + \bar{x}^2\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(x_i^2) - \frac{2}{n} E\left(\underbrace{\left(\sum_i \cancel{E}(x_i)\right)}\bar{x}\right)$$

$$+ E(\bar{x}^2)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(x_i^2) - \frac{2}{\cancel{n}} E\left(\cancel{n}\bar{x}\ \bar{x}\right) + E(\bar{x}^2)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(x_i^2) - E(\bar{x}^2)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left\{Var(x_i) + \left(E(x_i)\right)^2\right\} - \left(Var(\bar{x}) + \left(E(\bar{x})\right)^2\right)$$

because $E(x^2) = Var(x) + \left(E(x)\right)^2$ by definition of variance

$$= \frac{1}{n}\left(\sum_{i=1}^{n}\left(\sigma^2 + \mu^2\right)\right) - \left(\frac{n\sigma^2}{n^2} + \mu^2\right)$$

$$= \sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2 = \sigma^2\left(1 - \frac{1}{n}\right)$$

This is not an unbiased estimator.

Q2. Let $X_1, X_2 \ldots X_n$ be independent random variables from a geometric distribution. Determine the ML estimate of the 'p' parameter.

Solution:

$$P(X=k) = (1-p)^{k-1} p$$ (# of failures till the first success)

$$JL = \prod_{i=1}^{n} (1-p)^{k_i - 1} p$$

$$JLL = \sum_{i=1}^{n} (k_i - 1)\log(1-p) + \log p$$

$$\frac{\partial J_{LL}}{\partial p} = 0 = \sum_{i=1}^{n} \left(\frac{-(k_i - 1)}{1-p} + \frac{1}{p} \right)$$

$$\frac{+1}{1-p} \sum_{i=1}^{n} (k_i - 1) = \frac{n}{p}$$

$$\therefore \frac{1}{n} \sum_{i=1}^{n} k_i \cancel{-1} = \frac{1}{p} \cancel{-1}$$

$$\therefore \hat{p} = \frac{n}{\sum_{i=1}^{n} k_i}$$

Q3. What is the CDF for a geometric PMF? Show that a geometric distribution is memory less. (4)

$$P(X=k) = (1-p)^{k-1}\, p$$

$$P(X \geq k) = \sum_{l=k}^{\infty} (1-p)^{l-1}\, p \qquad \Big|\; m = l-k$$

$$= \sum_{m=0}^{\infty} (1-p)^{m+k-1}\, p$$

$$= p(1-p)^{k-1} \sum_{m=0}^{\infty} (1-p)^{m} \cancel{p}$$

$$= p(1-p)^{k-1} \underbrace{\left(\frac{1}{p}\right)} = (1-p)^{k-1}$$

by infinite series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\cdots$$

Note: $P(X > k) = P(X \geq k) - P(X = k)$

$$= (1-p)^{k-1} - p(1-p)^{k-1} = (1-p)^{k}.$$

$$P(X < k) = 1 - (1-p)^{k}.$$

To prove memorylessness, we need to show that

$$P(X > s+t, X > t) = P(X > t)\, P(X > s)$$

ie $P(X > s+t) = P(X > t)\, P(X > s)$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$(1-p)^{s+t} \qquad (1-p)^{s} \quad (1-p)^{t}.$$

$\therefore$ verified.

Q4. let $X_1, X_2 \dots X_n$ be a set of independent random variables from a distribution of unknown mean θ.
Then prove that any estimator of the form

$$\hat{\theta} = \sum_{i=1}^{n} \lambda_i X_i, \quad \sum_{i=1}^{n} \lambda_i = 1$$

is unbiased. For what values of $\{\lambda_i\}$ will such an estimator have least mean squared error?

$$E(\hat{\theta}) = \sum_{i=1}^{n} \lambda_i E(X_i) = \theta \sum_{i=1}^{n} \lambda_i = \theta$$

hence unbiased

For an unbiased estimator,

Mean squared error = variance

$$= \mathrm{Var}(\hat{\theta}) = \sum_{i=1}^{n} \lambda_i^2 \,\mathrm{Var}(X_i)$$

(due to independence of the X_is)

$$= \sum_{i=1}^{n-1} \lambda_i^2 \,\mathrm{Var}(X_i) + \underbrace{\Big(1 - \sum_{i=1}^{n-1} \lambda_i\Big)}_{\lambda_n}{}^2 \,\mathrm{var}(X_n)$$

$$\frac{\partial MSE}{\partial \lambda_i} = \cancel{2}\lambda_i \,\mathrm{var}(X_i) + \cancel{2}\Big(1 - \sum_{i=1}^{n-1}\lambda_i\Big)(-1)\,\mathrm{var}(X_n) = 0$$

$$\lambda_i \,\mathrm{var}(X_i) = \underbrace{\Big(1 - \sum_{i=1}^{n-1}\lambda_i\Big)}_{\lambda_n} \mathrm{var}(X_n)$$

Now $\mathrm{var}(X_i) = \mathrm{var}(X_n)$ as these are variables from the same distribution.

$\therefore \lambda_i = \lambda_n$.

This is true for $\forall i$, $1 \le i \le n-1$

But $\sum_{i=1}^{n} \lambda_i = 1$. Hence $\forall i$, $1 \le i \le n$,

$$\lambda_i = 1/n.$$

$\therefore$ the estimator $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i$ has the least MSE

Addendum to previous problem

If $Var(X_i) = \sigma_i^2$ where $\sigma_i^2 \neq \sigma_j^2$, $i \neq j$, then

$$\lambda_i = \lambda_n \, \sigma_n^2 / \sigma_i^2$$

$$\sum_{i=1}^{n} \lambda_i = \lambda_n \sum_{i=1}^{n} \left(\sigma_n^2 / \sigma_i^2 \right) = 1$$

$$\therefore \lambda_n = \frac{1}{\sigma_n^2 \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right)}$$

$$\therefore \lambda_i = \frac{1}{\sigma_i^2 \left(\sum_{i=1}^{n} 1/\sigma_i^2 \right)}$$

Q5. Let $X_1, X_2 \dots X_n$ be independent geometric random variables. Then show that $Y = \min(X_1, X_2, \dots, X_n)$ is a geometric random variable. What is its parameter?

$$P(Y \geq y) = \prod_{i=1}^{n} P(X_i \geq y)$$

$$= \prod_{i=1}^{n} (1 - p_i)^y \longrightarrow (1)$$

CDF of a geometric r-v. with parameter q is

$$P(X > k) = (1-q)^k \longrightarrow ②$$

Comparing ① and ②, we have

$$q = 1 - \prod_{i=1}^{n} (1-p_i)$$

$\downarrow$

this is the parameter of the PDF.

Q6. Some n students throw their hats in the middle of a room and then collect them randomly. Let Y be a random variable denoting the number of people who managed to collect their own hat. Find $E(Y)$ and $Var(Y)$

Solution:

Let $Y_i = \begin{cases} 1 & \rightarrow i^{th} \text{ person collected his/her hat} \\ 0 & \text{otherwise} \end{cases}$

$$Y = \sum_{i=1}^{n} Y_i. \quad E(Y) = \sum_{i=1}^{n} E(Y_i) = n\left(\frac{1}{n}\right) = 1.$$

$$Var(Y) = \sum_{i=1}^{n} var(Y_i) + \sum_{i}\sum_{j\neq i} Cov(Y_i, Y_j)$$

$$Var(Y_i) = E(Y_i^2) - (E(Y_i))^2$$

$$= \frac{1}{N}\left(1-\frac{1}{N}\right) \longrightarrow$$ Y_i is a Bernoulli r.v. with par. $p = 1/N$ so its var $= p(1-p)$

$$Cov(\cancel{Y_i}, Y_j) = E(Y_iY_j) - E(Y_i)E(Y_j)$$

$$= \frac{1}{N(N-1)} - \frac{1}{N^2} = \frac{1}{N}\left(\frac{1}{N-1} - \frac{1}{N}\right)$$

$$= \frac{1}{N^2}\left(\frac{1}{N-1}\right)$$

$$\therefore Var(Y) = N\left(\frac{1}{N}\right)\left(1-\frac{1}{N}\right) + (N^2-N)\frac{1}{(N-1)N^2}$$

$$= \frac{N-1}{N} + \frac{N\cancel{(N-1)}\,1}{\cancel{(N-1)}\,N^{\cancel{2}}} = 1$$