CS 207: Discrete Structures

Graph theory

Bipartite graphs and subgraphs, cliques and independent sets

Lecture 25
Sept 21 2015

Topic 3: Graph theory

Topics covered in the last two lectures:

- What is a Graph?
- Paths, cycles, walks and trails; connected graphs.
- Eulerian graphs and a characterization in terms of degrees of vertices.
- Bipartite graphs and a characterization in terms of odd length cycles.

Reference: Section 1.1, 1.2 of Chapter 1 from Douglas West.

Topics that will be covered this lecture and next

- Finish proof of characterization of bipartite graphs using odd cycles.
- Subgraphs and degree sum formula.
- Cliques and independent sets.
- Finding a large bipartite subgraph of a given graph

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- Graph representation as a matrix.
- Comparing graphs: isomorphism (eg Peterson's)

Bipartite graphs

Definition

A graph is called bipartite, if the vertices of the graph can be partitioned into $V = X \cup Y$, $X \cap Y = \emptyset$ s.t., $\forall e = (u, v) \in E$,

- either $u \in X$ and $v \in Y$
- or $v \in X$ and $u \in Y$

Example: m jobs and n people, k courses and ℓ students.

- How can we check if a graph is bipartite?
- Can we characterize bipartite graphs?

Characterizing bipartite graphs using cycles.

- Recall: A path or a cycle has length n if the number of edges in it is n.
- A path (or cycle) is call odd (or even) if its length is odd (or even, respectively).

Lemma

Every closed odd walk contains an odd cycle.

Proof: By induction on the length of the given closed odd walk. Exercise!

Characterizing bipartite graphs using cycles.

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Theorem, Konig, 1936

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- ($\implies$) direction is easy.
- Let G be bipartite with $(V = X \cup Y)$. Then, every walk in G alternates between X, Y.

$\implies$ if we start from X, each return to X can only happen after an even number of steps.

$\implies$ G has no odd cycles.

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Proof:

- ($\Longleftarrow$) Suppose G has no odd cycle, then let us construct the bipartition. Wlog assume G is connected.

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- ($\Longleftarrow$) Suppose G has no odd cycle, then let us construct the bipartition. Wlog assume G is connected.
- Let $u \in V$. Break V into

$$\begin{aligned} X &= \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is even}\}, \\ Y &= \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is odd}\}, \end{aligned}$$

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- If there is an edge vv' between two vertices of X or two vertices of Y, this creates a closed odd walk: $uP_{uv}vv'P_{v'u}u$.

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- If there is an edge vv' between two vertices of X or two vertices of Y, this creates a closed odd walk: $uP_{uv}vv'P_{v'u}u$.
- By Lemma, it must contain an odd cycle: contradiction.
- This along with $X \cap Y = \emptyset$ and $X \cup Y = V$, implies X, Y is a bipartition. $\square$

Some basic stuff that we have already seen

Degree-Sum Formula (also called Handshake Lemma!)

For any graph G with vertex set V and edge set E:

$$\sum_{v \in V} d(v) = 2|E|$$

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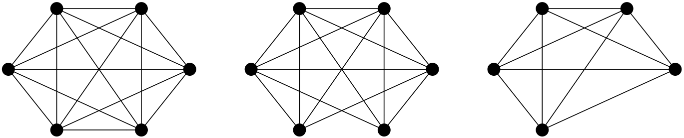
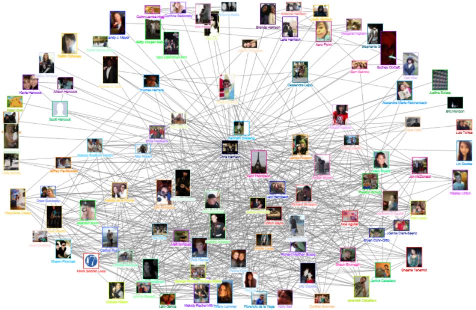
Subgraphs of a graph G

A subgraph H of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ (and the assignment of endpoints to edges in H is same as in G).



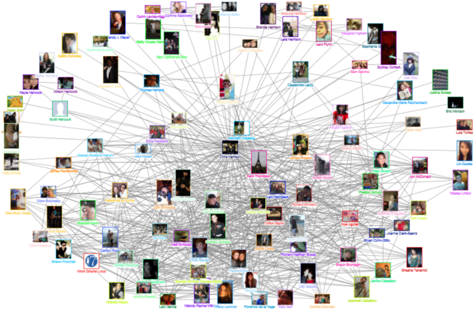
- Why are subgraphs important/interesting?

Cliques and independent sets



- Consider a large social network graph where friends are linked by an edge.
- What is the largest clique of friends?
- If we want to spread a youtube video, how many people should we send it to so that we are guaranteed everyone will see it (assuming friends forward to each other)?

Cliques and independent sets



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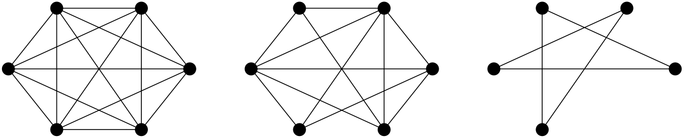

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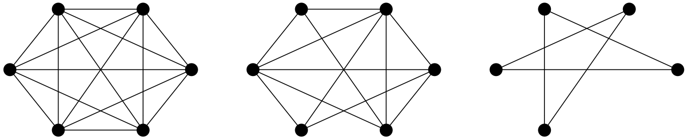

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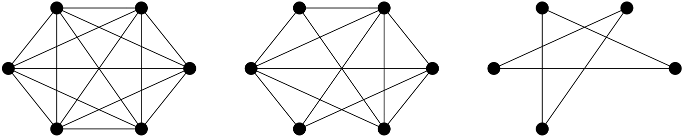

- Thus, a clique in a graph G is a complete subgraph of G.
- An independent set in G is a complete subgraph of $\overline{G}$, where $\overline{G}$ is the complement of G obtained by making all adjacent vertices non-adjacent and vice versa.

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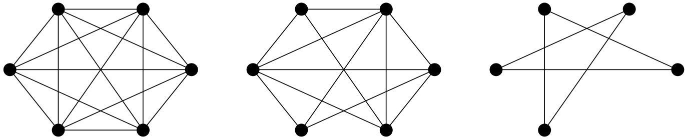

Questions:

- What is the size of the largest clique/independent set in each of the above graphs? In any complete graph?

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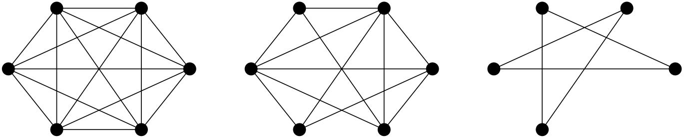

Questions:

- What is the size of the largest clique/independent set in each of the above graphs? In any complete graph?
- Given graph G, integer k, does G have a clique of size k?

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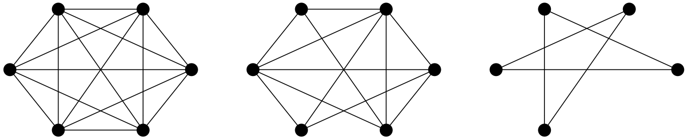

Questions:

- In a graph with 6 vertices, can you always find a clique or an independent set of size 3?

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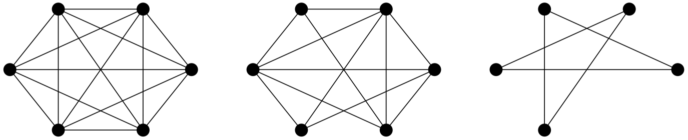

Questions:

- In a graph with 6 vertices, can you always find a clique or an independent set of size 3?
- Yes, because $R(3,3) = 6$!

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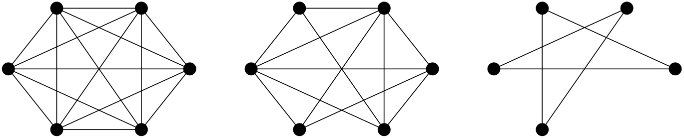

Ramsey's theorem - restated

In any graph with $R(k, \ell)$ vertices, there exists either a clique of size k or an independent set of size ℓ.

Bipartite subgraphs of graphs

- Does a graph G always have a bipartite subgraph?
- How large is it?

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Theorem

Every loopless graph G has a bipartite subgraph with at least $|E|/2$ edges.