CS 207: Discrete Structures

Lecture 20 – Counting and Combinatorics
PHP and its extensions
Ramsey Theory - A search for order in disorder

Aug 31 2015

Recap: Topics in Combinatorics

Counting techniques and applications

1. Basic counting techniques, double counting
2. Binomial theorem, permutations and combinations, Estimating $n!$
3. Recurrence relations and generating functions
4. Principle of Inclusion-Exclusion (PIE) and its applications.
   - Hand-shake Lemma
   - Counting the number of surjections on $[n]$.
   - Number of derangements $-> \frac{1}{e}$.
   - Number of partitions of size k.

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   - Hand-shake Lemma
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   - Number of derangements $- > \frac{1}{e}$.
   - Number of partitions of size k.
5. Pigeon-Hole Principle (PHP) and its applications.
   - Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ which is either increasing or decreasing.
   - The coloring game.

Different variants of PHP

Simplest formulation (Variant 0)

Let $k \in \mathbb{N}$. If $k+1$ (or more) objects are to be placed in k boxes, then at least one box will have 2 (or more) objects.

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PHP (Variant 3)

If $n \geq k_1 + k_2 + \ldots + k_r - r + 1$ objects are colored with r colors, then for some $i \in \{1 \ldots r\}$, there exist k_i objects all of color i.

Back to the coloring game

Lemma

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Proof:

- Let $1, \ldots, 6$ be the points, and red/blue the colors.
- Consider the edges $16, 26, 36, 46, 56$.
- By PHP at least 3 of them must be same color, say $16, 26, 36$ are red.
- Now there are two possibilities:
  - Either one of $12, 23, 31$ is red (then we have a red triangle).
  - Else none of them are red, implies 123 is a blue triangle. $\square$

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- What if there were 5 or lesser nodes?

Another coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

- complete: all pairs of edges are present.
- How do you prove this? Any ideas?
- How is this different from the previous problem?

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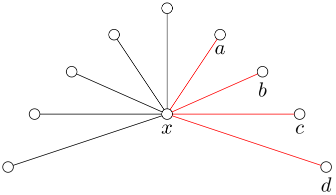

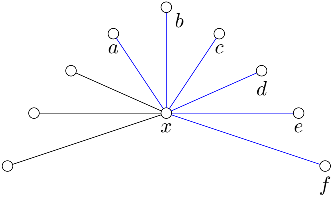

- Consider all edges from some node x.
- By PHP, either 4 edges have red color or 6 have blue.

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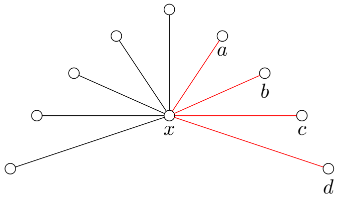

- Case 1: 4 red edges
  - Either one of edges between a, b, c, d is red or all are blue. So, we are done.

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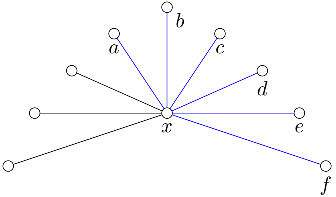

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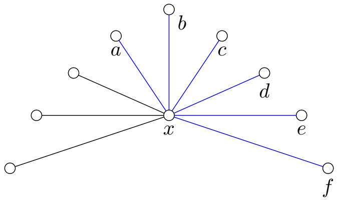

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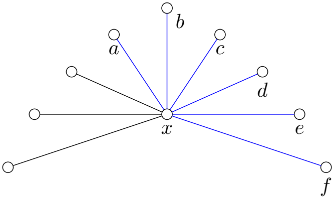

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- And this completes the proof. $\square$

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- That is, does this fail for a graph on 9 nodes?
- Can you find 2-coloring on a graph of 9 nodes such that the statement above does NOT hold?

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Answer: No! In fact, it does hold on 9 nodes!

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- Recall the Handshake lemma!
  - In any graph, the number of nodes having odd degree is even.

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  - In any graph, the number of nodes having odd degree is even.
- Thus, this case is impossible and we are done. □

Can we generalize the above?

In general,

How many nodes should a (complete) graph have so that any 2 coloring of its edges has

- either, a k-sized complete graph with all red edges
- or, a ℓ-sized complete graph with all blue edges

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- Also $R(3, 4) = 9$ (not quite...).

What about $R(k, \ell)$ in general?

Ramsey's theorem

Figure : Frank Plumpton Ramsey (1903-1930)

Ramsey's theorem (simplified version)

For any $k, \ell \in \mathbb{N}$, there exists $R(k, \ell) \in \mathbb{N}$ such that any 2-coloring of a (complete) graph on $R(k, \ell)$ nodes has

- either, a k-sized complete graph with all red edges
- or, a ℓ-sized complete graph with all blue edges

Moreover, we have

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}$$

Edge coloring problems

Summary of results till now

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3. Any 2-coloring of a graph on 9 nodes has either a red triangle or a blue complete graph on 4 nodes.
   - Is 9 the optimal such number? $R(3,4) \leq 9$.
   - (H.W?) Prove that $R(3,4) = 9$!

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   - (H.W?) Prove that $R(3,4) = 9$!

- (H.W) Prove that any 2-coloring of a graph on 18 nodes has a monochromatic complete graph on 4 nodes.
  (hint: you may use any of the above results)