CS 207: Discrete Structures

Lecture 17 – Counting and Combinatorics
Principle of Inclusion Exclusion

Aug 25 2015

Topics in Combinatorics

Basic counting techniques and applications

1. Basic counting techniques, double counting
2. Binomial coefficients and binomial theorem, permutations and combinations.
3. Estimating $n!$
4. Recurrence relations
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6. Today: Principle of Inclusion-Exclusion and its applications.

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Solving recurrence relations

- Number of subsets of a set of size n,
- Fibonacci sequence,
- Catalan numbers: $C(n) = \frac{(2n-2)!}{n!(n-1)!} = \frac{1}{n}\binom{2n-2}{n-1}$

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  - Compare coefficients of x^n in $(1 + x)^{2n} = ((1 + x)^n)^2$.

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- What about the summation? terms $1 \leq i < j \leq m = \binom{m}{2}$

Thus, we have # surjections from $[n]$ to $[m]$ =

$m^n - \binom{m}{1}(m-1)^n + \binom{m}{2}(m-2)^n - \ldots + (-1)^{m-1}\binom{m}{m-1} \cdot 1^n$.

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Proof: (H.W): Prove PIE by induction.

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Applications of PIE

- How many integral solutions does $x_1 + x_2 + x_3 = 11$ have where $0 \leq x_1 \leq 3, 0 \leq x_2 \leq 4, 0 \leq x_3 \leq 6$?
- Number of derangements of a set with n elements
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- Thus, $D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} \ldots + (-1)^n \frac{1}{n!})$

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- In other words, $\forall \delta > 0, \exists N_\delta \in \mathbb{N}$, such that for all $n > N_\delta$, $|\frac{D(n)}{n!} - \frac{1}{e}| \leq \delta$.

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- (H.W.) Prove this by using the Taylor's expansion for e^{-1}.