CS 207: Discrete Structures

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Aug 10, 2015

Lecture 10 – Basic mathematical structures: chains, antichains, lattices

Recap: Partial order relations

Last class we saw

- Partial orders: definition and examples
- Posets, chains and anti-chains
- Graphical representation as Directed Acyclic Graphs
- Topological sorting (application to task scheduling)
- Mirsky's theorem (application to parallel task scheduling)

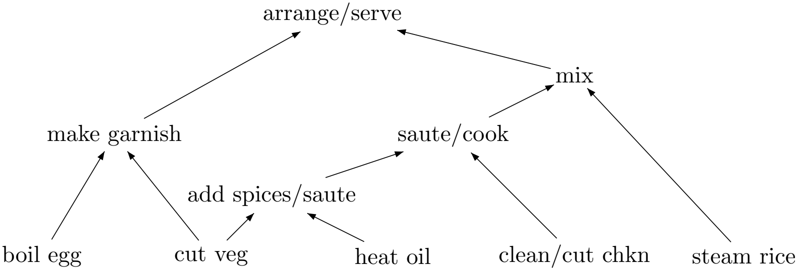
Recall: Partial Orders and Equivalence relations

- A poset is a set S with a partial order $\preceq \ \subseteq S \times S$.
- A totally ordered set is a poset in which every pair of elements is comparable, i.e., $\forall a, b \in S$, either $a \preceq b$ or $b \preceq a$.

Definitions: Let $(S, \preceq)$ be a poset.

- A subset $B \subseteq S$ is called a chain if every pair of elements in B is related by $\preceq$.
- A subset $A \subseteq S$ is called an anti-chain if no two distinct elements of A are related by $\preceq$.

Tasks scheduling as a poset



Theorems

- Every finite poset has a topological sort, i.e., a totally ordered set that is consistent with the poset (H.W).
- Every finite poset has a legal parallel schedule that runs in t steps, where t is the length of the longest chain.

Parallel task scheduling and chains

In fact, we proved:

Theorem

For a finite poset $(S, \preceq)$ with length of longest chain $= t$, we can partition S into t subsets $S_1, \ldots, S_t$ such that $\forall i \in \{1, \ldots, t\}$, $\forall a \in S_i$, if $b \preceq a, b \neq a$ then $b \in S_1 \cup \ldots \cup S_{i-1}$.

Assuming this theorem,

- Observe that we can schedule all of S_i at time i (since we know that all previous tasks were done earlier!).
- Thus, each S_i is an anti-chain.
- This solves the parallel task scheduling problem.

Parallel task scheduling and chains

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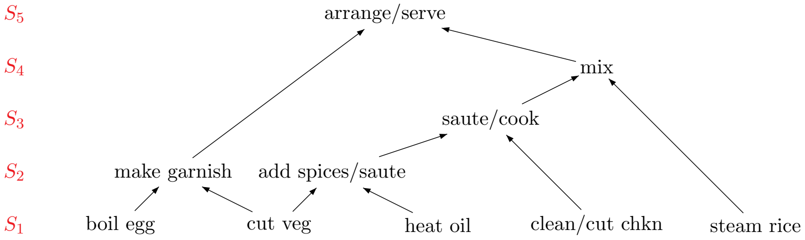
Proof: Put each $a \in S$ in S_i such that i is the length of the longest chain ending at a. Now proof by contradiction:



Parallel task scheduling and chains

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- By defn of S_i, $\exists$ chain of length at least i ending at b.
- But now, $b \preceq a, b \neq a$ implies we can extend the chain to chain of length $\geq i + 1$, ending at a.
- But then a cannot be in S_i. Contradiction. $\square$

Consequences for chains and anti-chains

Since each S_i was an anti-chain, a celebrated result follows...

Corollary (Mirsky's theorem, 1971)

If the longest chain in a poset $(S, \leq)$ is of length t, then S can be partitioned into t anti-chains.

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For all $t > 0$, any poset with n elements must have

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- or an antichain with at least $\frac{n}{t}$ elements.

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Its "dual" is called the Dilworth's theorem...

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Dilworth's Theorem, $\sim$ 1950

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- Of course partition is different from before. But will a similar proof technique work? Try induction!

Minimal and maximal elements

Let $(S, \preceq)$ be a poset.

- An element $a \in S$ is called minimal if, $b \preceq a$ implies $b = a$.
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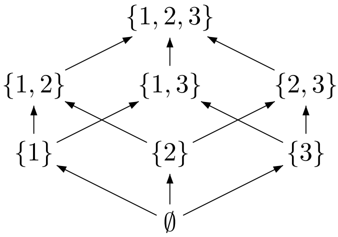

Least upper bounds and greatest lower bounds

Let $(S, \preceq)$ be a poset and $A \subseteq S$.

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- $l \in S$ is called a lower bound of A iff $l \preceq a$ for all $a \in A$.

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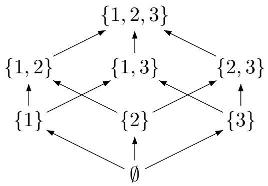

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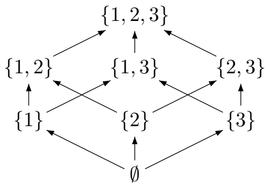

A poset in which every pair of elements has both a lub and a glb is called a lattice.

Lattices and complete lattices

Definitions

- A lattice is a poset in which every pair of elements has both a lub and a glb (in the set), i.e., $\forall x, y \in S$, there exists $l, u \in S$ such that l is the glb and u is the lub of $\{x, y\}$.
- A complete lattice is a poset in which any subset of elements has both a lub and a glb (in the set), i.e., $\forall S' \subseteq S$, there exists $l, u \in S$ such that l is the glb and u is the lub of S'.

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Theorem (Zorn's lemma)

Given a poset $(S, \preceq)$, if every non-empty chain in S has an upper-bound, then S has some maximal element.

A fixed point theorem

- Given two posets $(S, \preceq_s)$ and $(T, \preceq_T)$, $f : S \to T$ is order-preserving or monotonic if for all $a, b \in S$, $a \preceq_S b$ implies $f(a) \preceq_T f(b)$.

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Theorem (Tarski's fixed point theorem)

Let $(S, \preceq)$ be a complete lattice and $f : S \to S$ be a monotonic function. Then the set of fixed points of f is a (non-empty) complete lattice.

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Important result with several applications in many domains of mathematics and CS.