CS 207: Discrete Structures

Graph theory

Stable matchings

Lecture 32-33

Oct 8-9 2015

Topic 3: Graph theory

Topics in Graph theory

1. Basics concepts and definitions.
2. Eulerian graphs: Using degrees of vertices.
3. Bipartite graphs: Using odd length cycles.
4. Connected components: Using cycles.
5. Maximum matchings: Using augmenting paths.
6. Perfect matchings in bipartite graphs: Using neighbour sets. – Hall's theorem
7. Relating matchings to vertex covers.

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6. Perfect matchings in bipartite graphs: Using neighbour sets. – Hall's theorem
7. Relating matchings to vertex covers.
8. Today: Stable matchings...

Matchings and vertex covers: Pop quiz!

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6. The size of the maximum matching equals the size of the minimum vertex cover of G.

A min-max theorem

Theorem (Konig '31, Egervary '31)

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- Show that H has a matching that saturates $Q \cap X$ into $Y \setminus T$, H' has a matching saturating T.
- Together this forms the desired matching (since H, H' are disjoint).

$\square$

Next topic: Stable matchings

Stable matchings

Boys	Girls
• 1	• A
• 2	• B
• 3	• C
• 4	• D
• 5	• E

Stable matchings

	Boys	Girls
$C > B > E > A > D$	• 1	• A : 35214
$ABECD$	• 2	• B : 52143
$DCBAE$	• 3	• C : 43512
$ACDBE$	• 4	• D : 12345
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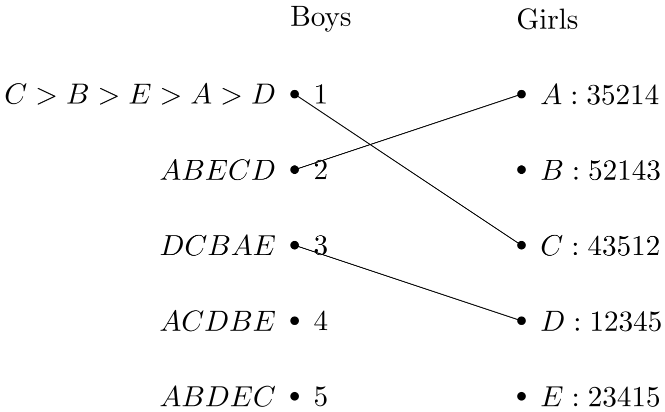
Stable matchings



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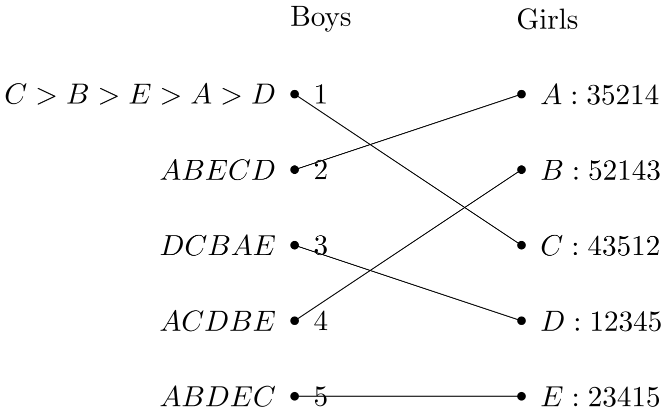

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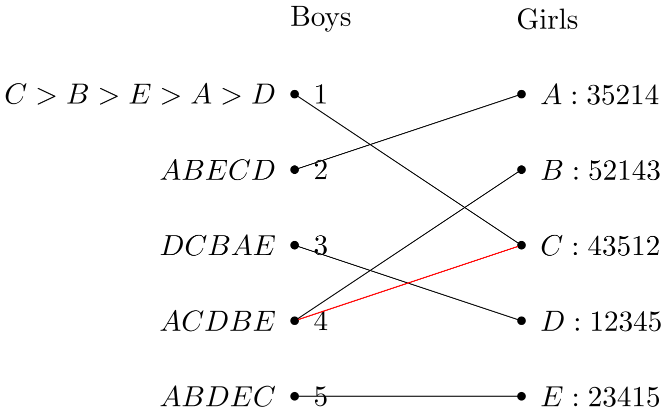

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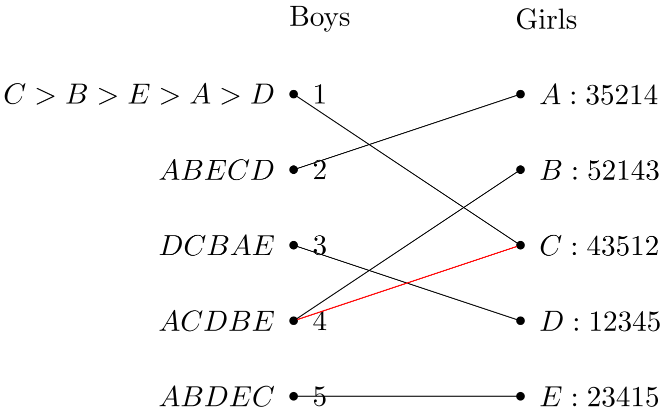

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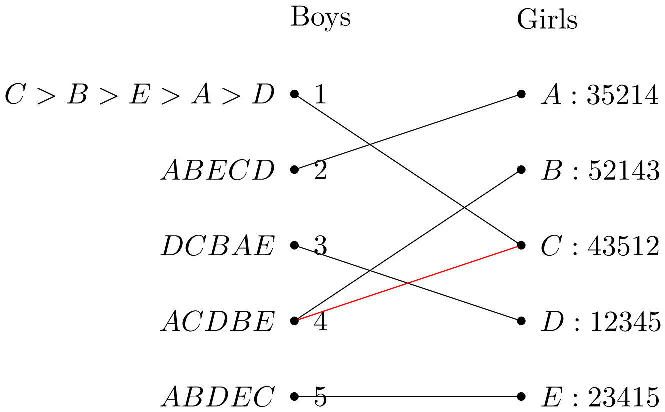

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- Danger! 4 prefers C to B and C prefers 4 to 1. Divorce!
- Qn: Can you match everyone without such Rogue couples?!

More than just a funny puzzle

- College admissions: Original Gale and Shapley paper, 1962.
- Matching hospitals and residents.
- Matching dancing partners.
- Matching students with jobs.

Stable matchings

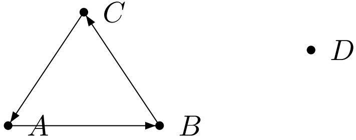
Definition

Given a matching M in a graph with preference lists of nodes.

- Unstable pair: Two vertices x, y such that x prefers y to its assigned vertex and vice versa.
- x, y would be happier by eloping.
- Qn: Find a perfect matching with no unstable pairs. Such a matching is called a Stable Matching.

Roommates Problem

- $A : BCD$
- $B : CAD$
- $C : ABD$
- $D : ABC$



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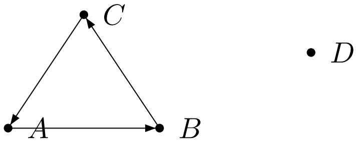

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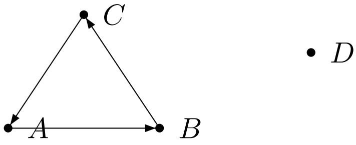

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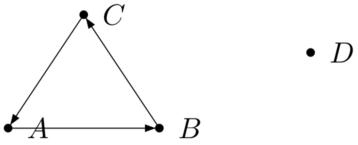

- What can you observe from this?
- Stable matchings don't always exist.
- So, do they exist for bipartite graphs and how can we prove this?

The proposal algorithm

Given: bipartite graph, and preference list for men/women

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- Does this algorithm terminate?
- If yes, does it produce a stable matching when it terminates?

Termination and Correctness of the proposal algo

- Try out the algo on the example.

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    - For each day (except last), at least one woman is crossed off some man's list.
    - As there are n men and each has list of size n, algo must terminate in n^2 days.

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  - W' rejected him only because she preferred some M'' to M.
  - By Lemma 2, she likes her final partner at least as much as M'', so better than M.

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- We will show that W' prefers some other M' and hence no unstable pair.
- Thus no man can be part of an unstable pair, implies stable matching. $\square$

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Conclusion: Propose first!

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- D. Gale and L.S. Shapley, College Admissions and the Stability of Marriage, American Mathematical Monthly 69(1962), pp. 9-14.
- D. Gusfield and R.W. Irving, The Stable Marriage Problem: Structure and Algorithms, MIT Press, 1989.

The 2012 Nobel prize in Economics to Shapley and Roth: "for the theory of stable allocations and the practice of market design".