CS 207: Discrete Structures

Graph theory

Matchings, maximum matchings, augmenting paths

Lecture 29
Oct 1 2015

Recap: Matchings

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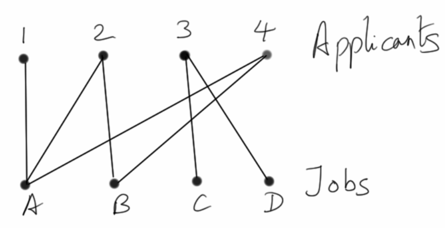

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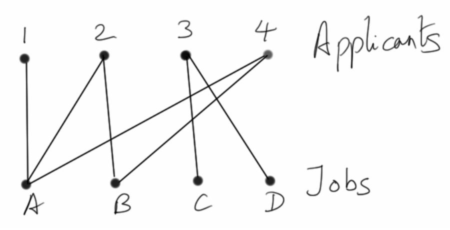

- What are the properties of such an assignment?
- Another practical example: the dating scene!

Recap: Matchings

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Our goal is to characterize perfect matchings

But first...

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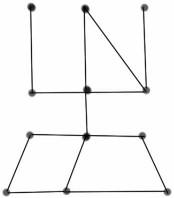

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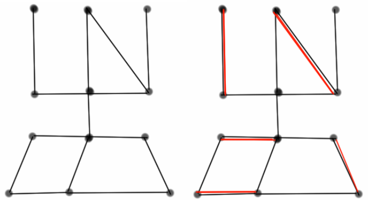

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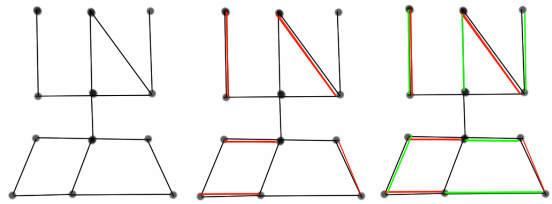

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Theorem

A matching M in G is a maximum matching iff G has no M-augmenting path.

Characterizing maximum matchings

We need a definition and a lemma.

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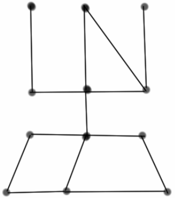
If M, M' are matchings in a graph H, the symmetric difference $M \triangle M'$ is the set of edges which are either in M or in M' but not both, i.e., $M \triangle M' = (M \setminus M') \cup (M' \setminus M)$.



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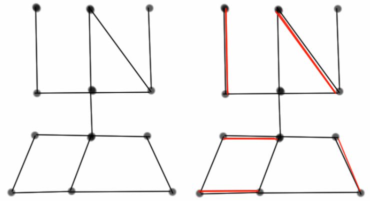

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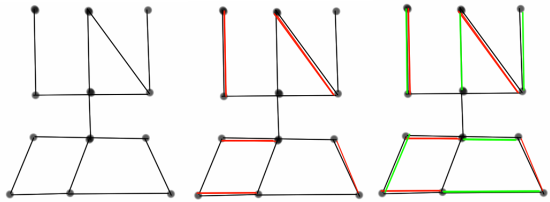

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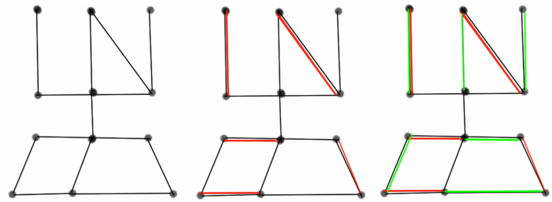

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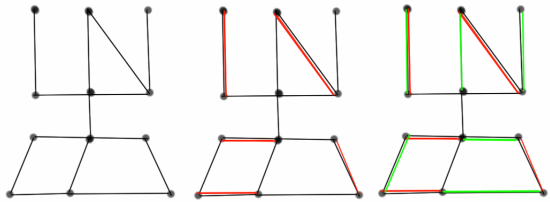

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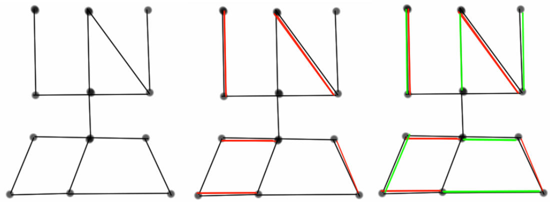

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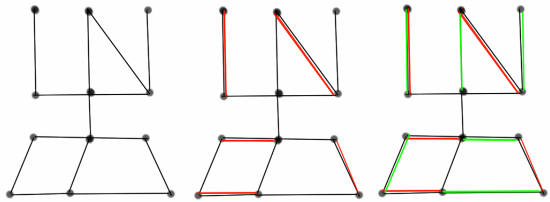

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- Thus, each cycle has even length with equal edges from M and M'. $\square$

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- This component can only be a path that starts and ends with an edge of M''; i.e., it is an M-augmenting path in G.