CS207 (Discrete Structures)
Exercise Problem Set 3

August 8, 2015

Instructions:

- Attempt *all* questions.
- If you have any doubts or you find any typos in the questions, post them on piazza at once!

Relations

1. Give an example for each of the following, if such an example exists. Else prove why it cannot exist.
   (a) A relation that is irreflexive, antisymmetric and not transitive.
   (b) A relation that is neither symmetric nor antisymmetric.
   (c) An antisymmetric relation which has a symmetric relation as its subset.

Equivalence relations

2. Suppose R_1 and R_2 are two equivalence relations on set S.
   (a) Is $R_1 \cap R_2$ an equivalence relation?
   (b) Is $R_1 \cup R_2$ an equivalence relation?
   (c) Let $f : S \to S$ be a function. Then is the relation R_3, defined by aR_3b if $f(a)R_1f(b)$, an equivalence relation?

   For each of the above, if your answer is "yes", you must prove it, and if your answer is "no", you must provide a counterexample.

3. Consider a necklace made of 3 beads, each of which can be either red, white or blue. Let S be the set of all such necklaces. Define the following relation R on S as: $N_1 \; R \; N_2$ iff necklace N_2 can be obtained from necklace N_1 by rotating it (and *not* allowing to flip the necklace).
   (a) Show that R is an equivalence relation.
   (b) What are the equivalence classes of R?
   (c) Is the number of elements in each equivalence class the same? Is there a relationship between the number of elements in an equivalence class of R and the total number of elements in S?
   (d) If in the definition of the relation, we allow flipping of the necklace as well: that is, $N_1 \; R' \; N_2$ iff necklace N_2 can be obtained from necklace N_1 by rotating or flipping it. Is R' an equivalence relation? Why or why not?

Posets, chains and anti-chains

4. Let $(S, \preceq)$ be a (non-empty) poset. We write $a \prec b$ if we have $a \prec b$ and $a \neq b$. An element $a \in S$ is called *maximal* if $\nexists b \in S$ s.t. $a \prec b$. Similarly, an element $a \in S$ is called *minimal* if $\nexists b \in S$ s.t. $b \prec a$.
   (a) Consider the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |\})$. What are its maximal and minimal elements?
   (b) Consider poset $(\mathcal{P}(S), \subseteq)$. What are its maximal and minimal elements?
5. Prove carefully that each finite poset has a topological sort (i.e., a linearization).
6. For all $t > 0$, prove that any poset with n elements must have either a chain of length greater than t or an antichain with at least $\frac{n}{t}$ elements.
7. *Consider a permutation of the numbers from 1 to n arranged as a sequence from left to right on a line. Using Mirsky's theorem done in class, prove that there exists a $\sqrt{n}$-length subsequence of these numbers that is completely increasing or completely decreasing as you move from right to left.

   For example, the sequence 2, 3, 4, 7, 9, 5, 6, 1, 8 has an increasing subsequence of length 3, for example: 2, 3, 4, and a decreasing subsequence of length 3, for example: 9, 6, 1. (Hint: Use the previous question!)