CS 207: Discrete Structures

Instructor : S. Akshay

July 27, 2015

Lecture 04 – Basic Mathematical Structures

Logistics

Tutorial timings

Tuesdays 5.15pm at SIC 301 Kresit building.

Recap of last three lectures

Chapter 1: Mathematical reasoning

- Propositions, predicates.
- Axioms, Theorems and Types of proofs: contradiction, contrapositive, etc.
- Principle of Mathematical Induction
- Well-ordering principle and Strong Induction

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Today:- Chapter 2: Basic Mathematical Structures

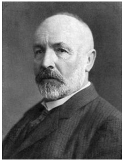
- Finite and infinite sets, Functions
- Relations

Sets

What is a set?

- A set is an unordered collection of objects.
- The objects in a set are called its elements.

Sets



§ I

The Conception of Power or Cardinal Number

By an "aggregate" (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects *m* of our intuition or our thought. These objects are called the "elements" of M.

Figure : Georg Cantor (1845-1918); extract

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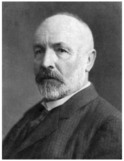

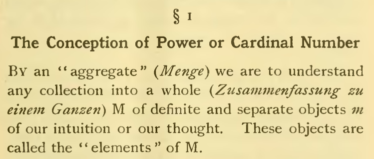

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What is a set?

- A **set** is an unordered collection of objects.
- The objects in a set are called its **elements**.

More formally,

Let P be a property. Any collection of objects that are defined by (or satisfy) P is a set, i.e., $S = \{x \mid P(x)\}$.

Some simple boring stuff about sets

Examples and properties

- We have already seen examples: $\mathbb{Z}, \mathbb{N}, \mathbb{R}$, set of all horses,...
- Let A, B be two sets. Recall the usual definitions:
  - Equality $A = B$, Subset $A \subseteq B$,
  - Cartesian product $A \times B = \{(a, b) \mid a \in A, b \in B\}$
  - Union $A \cup B = \{x \mid a \in A \text{ or } b \in B\}$
  - Intersection $A \cap B = \{x \mid a \in A \text{ and } b \in B\}$
  - Empty set ϕ,
  - Power set of $A = \mathcal{P}(A)=$ set of all subsets of A.
  - If U is the universe, then the complement of A, $\bar{A} = A^c = \{x \in U \mid x \notin A\}$.

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So, what is the difference between $\{\emptyset\}$ and $\emptyset$?

Not so simple...

A barber is a man in town who only shaves those who don't shave themselves.

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Then if $S \in S$, then $S \notin S$ and if $S \notin S$, then $S \in S$!

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How do you resolve this?

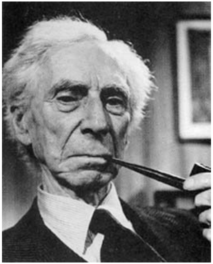

Figure : Bertrand Russell (1872-1970)

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Axiomatic approach to set theory (ZFC!)

Start with a few objects defined. Then for a set A and a property P, $S = \{x \in A \mid P(x)\}$ is a set.

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Why does this definition get rid of Russell's paradox?

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Let $P(x) = x \notin x$. let A be a set and $S = \{x \in A \mid x \notin x\}$.

- if $(S \in S)$: from the definition of S, $S \in A$ and $S \notin S$, which is a contradiction.
- if $(S \notin S)$: from the definition, either $S \notin A$ or $S \in S$. But we have assumed that $S \notin S$. Hence, $S \notin A$. No contradiction!

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- Example: $\{$set of all even natural numbers$\}$ vs $\mathbb{N}$ vs $\mathbb{Q}$ vs $\mathbb{R}$
- Turns out we need functions... but first...

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- And suppose they are all full (like in IIT guest house).

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1. Can you accomodate 1 or finitely many more guests, by shifting around the existing guests?
2. What if infinitely many more guests arrive?
3. What if infinitely many more trains with infinitely many more guests arrive? (H.W)

Functions

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i.e., $f : A \to B$ is a subset R of $A \times B$ such that

(i) $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

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- We write $f(a) = b$ and call b the image of a.
- $Range(f) = \{b \in B \mid \exists a \in A \text{ s.t. } f(a) = b\}$, $Domain(f) = A$

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Composition of functions

- If $g : A \to B$ and $f : B \to C$, then $f \circ g : A \to C$ is defined by $f \circ g(x) = f(g(x))$.

- Defined only if $Range(g) \subseteq Domain(f)$.
- Example: if $f(x) = x^2$, $g(x) = x - x^3$ with $f, g : \mathbb{R} \to \mathbb{R}$, what is $f \circ g(x), g \circ f(x)$?

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Composition of functions is associative

- If $h : A \to B$ and $g : B \to C$ and $f : C \to D$, then $f \circ (g \circ h) = (f \circ g) \circ h$.

Check it! (H.W.)

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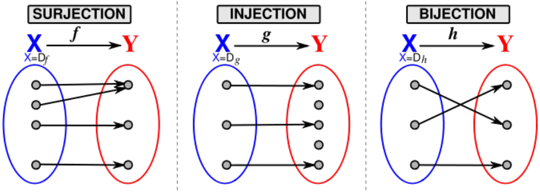
Inverse of a function

- If $f : A \to B$ is a ??? function, then $f^{-1} : B \to A$ defined by $f^{-1}(b) = a$ if $f(a) = b$, is called its inverse.

Comparing (finite and infinite) sets



- Surjective or onto: $f : A \to B$ is surjective if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$.
- Injective or 1-1: $f : A \to B$ is injective if $\forall x, y \in A$, if $f(x) = f(y)$, then $x = y$.
- Bijective or 1-1 correspondence: A function is bijective if it is surjective and injective.

If f is a bijection, then its inverse function exists and $f \circ f^{-1} = f^{-1} \circ f = \mathrm{id}$

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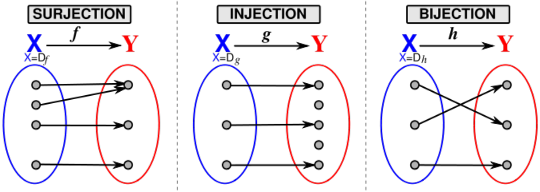

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1. $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = x^2$.
2. $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that $f(x) = x^2$.

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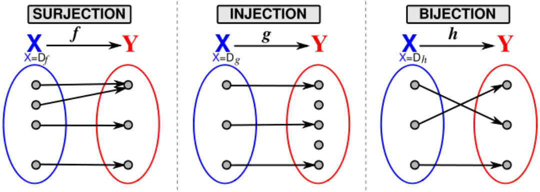

- Surjective or onto: $f : A \to B$ is surjective if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$. – If A, B finite, $|A| \geq |B|$
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Properties of finite and infinite sets

Relative notion of "size"

Thus, two finite/infinite sets have the same "size" iff there is a bijection between them.

Properties of finite and infinite sets

Similarities between finite and infinite sets

- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, then $\exists$ **bij** from B to A.
- $\exists$ **surj** from A to B and $\exists$ **surj** B to A, implies $\exists$ **bij** from A to B.

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- $\exists$ **bij** from A to B, then $\exists$ **bij** from B to A.
- (Schröder-Bernstein Theorem:) $\exists$ **surj** from A to B and $\exists$ **surj** B to A, implies $\exists$ **bij** from A to B. (H.W: Read this!)