CS 207: Discrete Structures

Lecture 12 – Basic counting techniques

Aug 13, 2015

Course Outline

Topics to be covered

1. Proofs and structures
2. Counting and combinatorics
   - Basics of counting
   - Subsets, partitions, Permutations and combinations
   - Recurrence relations and generating functions
   - Principle of Inclusion and Exclusion and its applications
   - Pigeonhole Principle and its extensions
3. Elements of graph theory
4. Introduction to abstract algebra and number theory

Two examples

Letters and envelopes

Given n letters and n addressed envelopes, in how many ways can letters be placed in envelopes so that none of them is in its correctly addressed envelope.

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- Can you ever have a draw?

Basic counting principles illustrated...

- How many reflexive relations are there on a set A of size n?
  - Reflexive relations are ordered pairs of which there are n^2.
  - Of these, all n pairs of (a, a) have be present.
  - Of the remaining, we can choose any of them to be in or out.
  - there are $n^2 - n$ of them, so 2^{n^2-n} of them.
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The product principle

If there are n_1 ways of doing something and n_2 ways of doing another thing, then there are $n_1 \cdot n_2$ ways of performing both actions.

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- How many functions are there from a set of size n to itself?
- If 20 teams play in the IITB-premier league and every game has a winner/loser and loser is always eliminated. How many games are played before a champion is chosen?
- How many subsets does a set A of n elements have?

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- How many subsets does a set A of n elements have?
  - Product principle: two choices for each element, hence $2 \cdot 2 \cdots 2 \cdot 2$ (n-times).
  - Bijection: between $\mathcal{P}(X)$ and n-length sequences over $\{0, 1\}$ (characteristic vector).
  - Induction: Since we already know the answer!
  - Recurrence: $F(n) = 2 \cdot F(n-1), F(0) = 1.$ solve it?
  - Sum principle: Subsets of size $0 +$ subsets of size $1 + \ldots +$ subsets of size $n =$ Total number of subsets.

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Sum Principle

If something can be done in n_1 or n_2 ways such that none of the n_1 ways is the same as any of the n_2 ways, then the total number of ways to do this is $n_1 + n_2$.

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- But, how many subsets of size k does a set of n elements have? This number, denoted $\binom{n}{k}$, is called a binomial coefficient.
- We all know(?) that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Prove it!

Permutations and combinations

Binomial Coefficients. Let n, k be integers s.t., $n \geq k \geq 0$.

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- Equate them! Principle of double counting.
  - if you can't count something, count something else and count it twice over!

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Permutations and combinations

- No. of k-size subsets of set of size n = No. of k-combinations of a set of n (distinct) elements $= \binom{n}{k}$.
- No. of k-size ordered subsets of set of size n = No. of k-permutations of a set of n (distinct) elements.

Simple examples to illustrate "double counting"

Prove the following identities (by only using double counting!)

1. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
2. $\displaystyle\binom{n}{k} = \binom{n}{n-k}$.
3. $\displaystyle k\binom{n}{k} = n\binom{n-1}{k-1}$
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The latter two are in fact recursive definitions for $\binom{n}{k}$. What are the boundary conditions?

A more interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

What will you count here?

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Proof in six steps:

1. Define a relation R: iRj if i and j shook hands.
2. Is this relation symmetric (trans/refl.)? Draw its graph.

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4. Therefore, Total no. of directed edges $= \sum_i m_i$.
5. But now, let X be the total number of handshakes. Clearly this is an integer. Total no. of directed edges $= 2 \cdot X$.
6. This implies, $\sum_i m_i = 2 \cdot X$. Which means that number of i such that m_i is odd is even!

What about partitions?

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What about number of partitions of a set of n distinct elements, denoted B_n?

- Same as number of equivalence relations!
- What are B_3, B_2, B_1? What about B_0?
- What about B_n in general?