CS 207: Discrete Structures

Lecture 14 – Counting and Combinatorics

Aug 18 2015

Last two classes

Basic counting techniques and applications

1. Sum and product, bijection, double counting principles
2. Counting no. of (ordered) subsets, partitions, relations...
3. Handshake lemma

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3. Handshake lemma

4. Binomial coefficients and binomial theorem
5. Pascal's triangle and its applications
6. Permutations and combinations with/without repetitions

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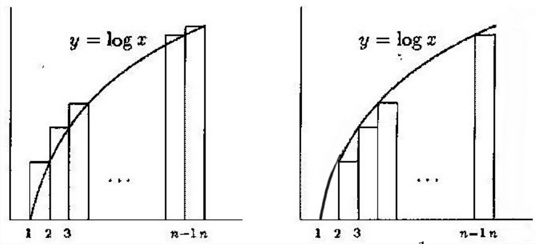
Now, we relate it to natural log function as shown in the figure.



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- r.h.s. $n! \leq e^{(n+1)\log(n)-n+1} = n^{n+1}/e^{n-1} = ne(n/e)^n$. $\square$

Next: Recurrence relations and generating functions

Recall: No. of subsets of a set of n elements

How many subsets does a set A of n elements have?

- Induction
- Product principle: two choices for each element, hence $2 \cdot 2 \cdots 2 \cdot 2$ (n-times).
- Bijection: between $\mathcal{P}(X)$ and n-length $\{0,1\}$-sequences.
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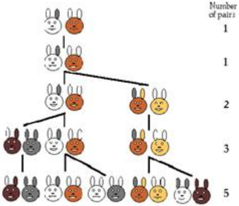
But how do you solve it?

Another example of recurrence: The Fibonacci Sequence



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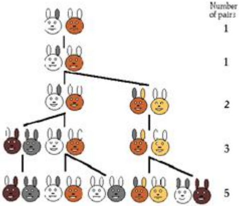

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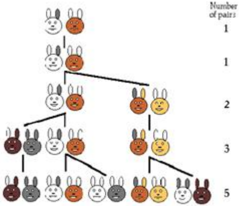

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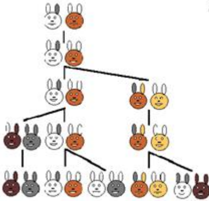

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- But rabbits die!
- Consider $u_n = u_{n-1} + u_{n-2} - u_{n-3}$ where $u_2 = 2, u_1 = u_0 = 1$

Recurrence and linear recurrence relations

Definition

- A recurrence relation for a sequence is an equation that expresses its n^{th} term using one or more of the previous terms of the sequence.
- A linear recurrence relation is of the form

$$u_n = a_{k-1}u_{n-1} + \ldots + a_1 u_{n-k+1} + a_0 u_{n-k}$$

where $a_0, \ldots, a_{k-1} \in \mathbb{R}$ are constants.

- k is called the degree/depth of the sequence.
- The first few (e.g., k elements $u_0, \ldots, u_{k-1}$) are initial conditions and they determine the whole sequence.

Some more examples of recurrences

How many bit strings of length n are there that do not have two consecutive 0's?

- Find a recurrence relation for this
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  In general, let $C(n)$ be the number of ways of doing this.

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- Let $C(n)$ be the number of ways of doing this.
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- Thus, $C(n) = \sum_{i=1}^{n-1} C(i)C(n-i)$ for $n > 1$

- Initial conditions are $C(0) = C(1) = 1$.
- This sequence are called Catalan numbers...

How do we solve such recurrences? We start with the Fibonacci sequence.

An aside: find the Fibonacci sequence!

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 200 252 200 120 45 10 1

- $F(n) = F(n-1) + F(n-2)$.
- $1, 1, 2, 3, 5, 8, 13, \ldots.$
- Can you observe the sum of which terms in the Pascal's triangle gives rise to the terms of the Fibonacci sequence?