15 Simulation

Generating Uniformly-Distributed Random Numbers

– Generate a sequence of (pseudo) random numbers by the following recurrence relation:
$X_{n+1} := (aX_n + c)$ modulo m, $n \geq 0$
generates random integers in the range $[0, m-1]$, *under some conditions*.

– $X_0 =$ seed

– Hull-Dobell Theorem: Conditions for a full period for any arbitrary seed value: (1) c, m are relatively prime. (2) $a-1$ divisible by all prime factors of m. (3) $a-1$ is a multiple of 4, if m is a multiple of 4.

– Choose, $m \gg 1$ to be very large and take X_n/m as an approximation to a uniform real-valued random variable on $(0, 1)$

– Drawback: Resulting sequence will pass formal tests for uniformity of distribution only for a specific choice of parameters (result sensitive to parameters).

– Current state of the art (since 1997): Mersenne twister algorithm.

Generating a Random Integer

– How to generate a random integer in the set $1, 2, \cdots, k$?
1) Generate a random number u uniformly distributed over $(0, 1)$
2) Multiply u by k to give $x = ku \in (0, k)$
3) Take $[x] + 1 = [ku] + 1$, where as the $[\cdot]$ is the integer part of x = largest integer $\leq x$

Note: Do NOT reuse algorithm for generating uniform random number with $m = k + 1$

Generating a Random Permutation

– Consider any arbitrary permutation. Let $X(i)$ denote the element at index i, where $i = 1, \cdots, n$.
– Algorithm:
1) Set $m = n$.
2) Generate a random variable a_m that is equally likely to take any value from $1, \cdots, m$
3) Interchange values of $X(a_m)$ and $X(m)$
4) Set $m \leftarrow m - 1$. Repeat last 2 steps, until $m = 1$.

CDF Transform

– Let X = continuous RV with CDF $F(\cdot)$ that is **strictly increasing**. Thus, $F(\cdot)$ has an inverse function.

– Define $U := F(X)$ as a transformed random variable
– Then, U is a RV with a uniform PDF over $[0, 1]$

– Proof:
Range of values that U can take = range of values CDF $F(\cdot)$ can take = $[0, 1]$.
The CDF of U evaluated at u is, by definition, $P(U \leq u) = P(F(X) \leq u) = P(X \leq F^{-1}(u)) = F(F^{-1}(u)) = u$
Thus, PDF of U is the constant 1 over the interval $[0, 1]$
Q.E.D.

Another way to look at it: Start with a RV U having a uniform PDF over $[0, 1]$
Define $X := G^{-1}(U)$, where $G(Z)$ is a strictly increasing function (CDF) from $[0, 1] \to [0, 1]$. Let $Q(Z)$ (PDF) be the derivative of $G(Z)$
Then, what is the CDF of X ?

Answer: $P(U \leq u) = u = G(G^{-1}(u)) = Q(X \leq G^{-1}(u)) = Q(X \leq x)$
Thus, sampling from $P(U)$ and applying the transformation $G^{-1}(U)$ gives us a RV whose PDF is $Q(X)$

Thus, we can simulate a RV X with a strictly-increasing CDF $F(\cdot)$ by simulating a uniform random variate U on $[0,1]$ and then applying the transformation $F^{-1}(u)$

Example: Let X be uniform on the interval $[a,b]$. Then, $u := \frac{x-a}{b-a}$ is uniform on the interval $[0,1]$.
Proof: The CDF OF X, i.e., $F(x) = \frac{x-a}{b-a}$. Thus, the defined u is the CDF transform of x.

Example: Let X be an exponential random variable. Then its CDF $F(x) = 1 - \exp(-\lambda x)$ for $x \geq 0$.
Thus, the transform $g(\cdot)$ that matches (mass conservation) the CDF T to the CDF of U (uniform random variable over $[0,1]$) is such that $1 - \exp(-\lambda x) = u$, where $x = g(u)$.
Thus, $x = -(1/\lambda)\log(1-u)$ is the relationship, and the transform, that maps U to X. Note that $1-U$ is also a uniform random variable on $[0,1]$.

Example: A Pareto distribution has the PDF $P(x) = \beta\alpha^{\beta}/x^{\beta+1}$, for $x > \alpha$, and $P(x) = 0$ otherwise. This has the CDF $F(x) = 1 - (\alpha/x)^{\beta}$ for $x \geq \alpha$. Thus, the required transformation to simulate a Pareto RV is $x = \alpha/(1-u)^{1/\beta}$.