CS 207: Discrete Structures

Lecture 21 – Counting and Combinatorics
Introduction to Ramsey Theory

Sept 1 2015

Topics in Combinatorics

Last two classes

- Pigeon-Hole Principle (PHP) and its applications
  - Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ which is either increasing or decreasing.
- The coloring game and its variants

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- The coloring game and its variants

Today's class

Generalizing the coloring game

An introduction to Ramsey theory.

Edge coloring problems

Results we saw last class

1. Any 2-coloring of a graph on 6 nodes has either a red triangle or a blue triangle.
   - 6 is the optimal such number. Thus, $R(3,3) = 6$.

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2. Any 2-coloring of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

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1. Any 2-coloring of a graph on 6 nodes has either a red triangle or a blue triangle.
   - 6 is the optimal such number. Thus, $R(3,3) = 6$.
2. Any 2-coloring of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.
3. Any 2-coloring of a graph on 9 nodes has either a red triangle or a blue complete graph on 4 nodes.
   - Is 9 the optimal such number? $R(3,4) \leq 9$.
   - (H.W?) Prove that $R(3,4) = 9$!
4. Any 2-coloring of a graph on 18 nodes has a monochromatic complete graph on 4 nodes.

Ramsey's theorem

Recall:

Definition

For $k, \ell \in \mathbb{N}$, $R(k, \ell)$ denotes the minimum number of nodes such that any 2-coloring of a (complete) graph on $R(k, \ell)$ nodes has

- either, a complete graph on k-nodes with all red edges
- or, a complete graph on ℓ-nodes with all blue edges

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Ramsey's theorem (simplified version)

For all $k, \ell \in \mathbb{N}$, $R(k, \ell)$ exists, i.e., it is finite.

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Ramsey's theorem (simplified version)

For all $k, \ell \in \mathbb{N}$, $R(k, \ell)$ exists, i.e., it is finite. In fact,

$$R(k, \ell) \leq \binom{k+\ell-2}{k-1}$$

Ramsey theory: A search for order in disorder!

Every structure no matter how disordered must contain some regular sub-part!

E.g., any 2-coloring on a complete graph of 10 nodes contains either a complete graph of 3 nodes of one color or a complete graph of 4 nodes of the other color.

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- Suppose in a group of people any two are friends or enemies.

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E.g., any 2-coloring on a complete graph of 10 nodes contains either a complete graph of 3 nodes of one color or a complete graph of 4 nodes of the other color.

- Suppose in a group of people any two are friends or enemies.
- In any set of 10 people there must be either 3 mutual friends or 4 mutual enemies.

Proof of Ramsey's theorem

- What is $R(n, 2) = R(2, n)$?

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Proof:

- By strong induction on $k + \ell$.
- Base case: $R(2, 2) = 2$.

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Proof:

- By strong induction on $k + \ell$.
- Base case: $R(2, 2) = 2$.
- Suppose it is true for all k, ℓ such that $k + \ell < N$. We will show that $R(k, \ell)$ is finite by showing

$$R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1)$$

  where $R(k - 1, \ell)$ and $R(k, \ell - 1)$ exist by induction hypothesis since $k + \ell - 1 < N$.

Proof of Ramsey's theorem contd.

By ind hyp assume that $R(k-1,\ell)$ and $R(k,\ell-1)$ exist. Then,

Claim: $R(k,\ell) \leq R(k-1,\ell) + R(k,\ell-1)$

Proof of Ramsey's theorem contd.

By ind hyp assume that $R(k-1,\ell)$ and $R(k,\ell-1)$ exist. Then,

Claim: $R(k,\ell) \leq R(k-1,\ell) + R(k,\ell-1)$

- i.e., given a 2-colored complete graph with $R(k-1,\ell) + R(k,\ell-1)$ nodes, it has either a complete red graph with k nodes or a complete blue graph with ℓ nodes.

Proof of Ramsey's theorem contd.

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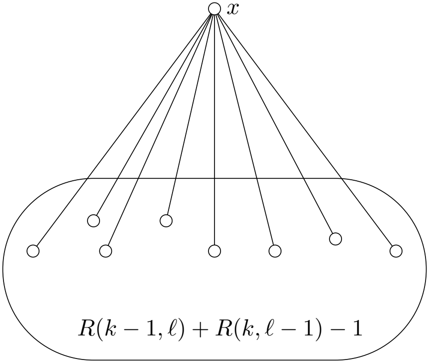
Claim: $R(k,\ell) \leq R(k-1,\ell) + R(k,\ell-1)$

Consider complete graph with $R(k-1,\ell) + R(k,\ell-1)$ nodes.



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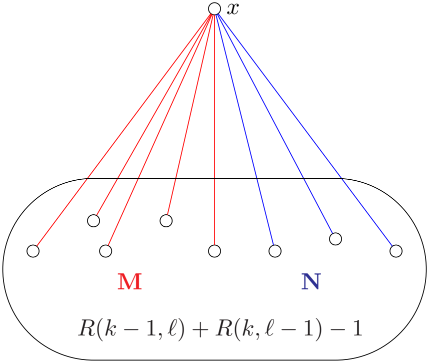

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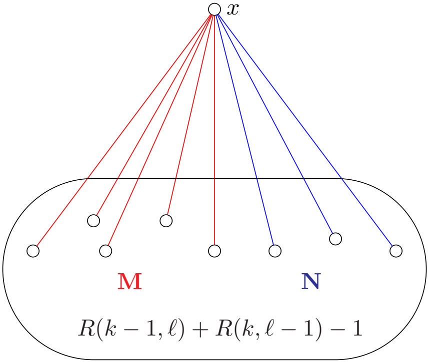

- Clearly $M + N = R(k-1,\ell) + R(k,\ell-1) - 1$.

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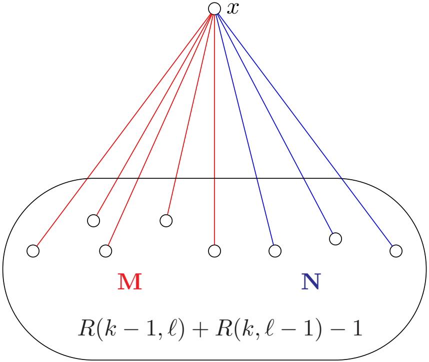

- Clearly $M + N = R(k-1,\ell) + R(k,\ell-1) - 1$.
- By PHP, either $M \geq R(k-1,\ell)$ or $N \geq R(k,\ell-1)$.

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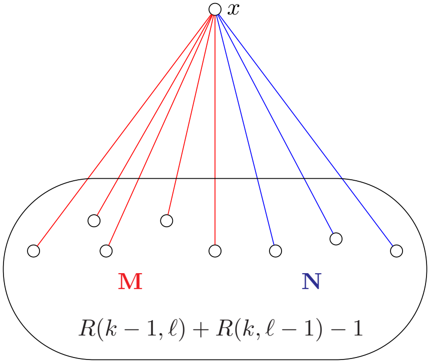

- Case 1: $M \geq R(k-1,\ell)$.

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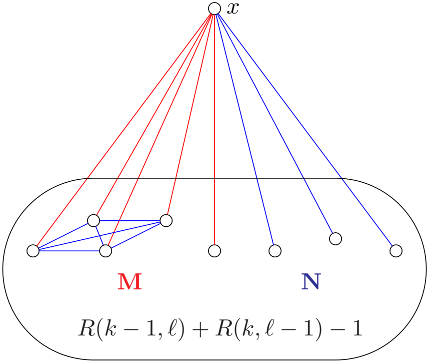

- Case 1: $M \geq R(k-1,\ell)$. Either complete blue graph on ℓ nodes

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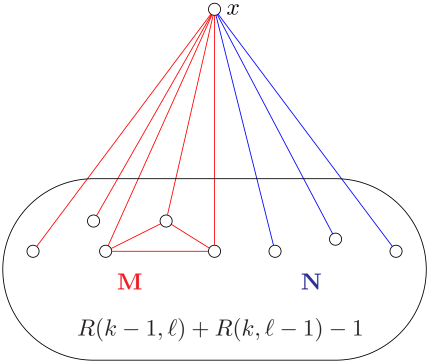

- Case 1: $M \geq R(k-1,\ell)$. Either complete blue graph on ℓ nodes or complete red graph on $k-1$ nodes $+\ x$

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By ind hyp assume that $R(k-1,\ell)$ and $R(k,\ell-1)$ exist. Then,

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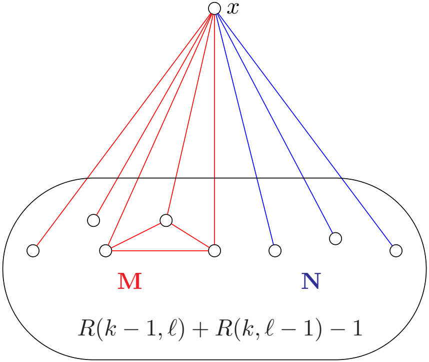

- Case 1: $M \geq R(k-1,\ell)$. ✓
- Case 2: $N \geq R(k,\ell-1)$ leads to similar argument. ✓

Proof of Ramsey's theorem contd.

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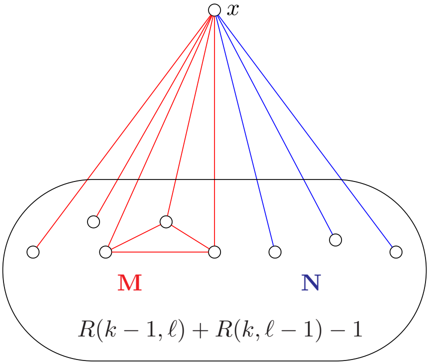

Thus in all cases, we have $R(k,\ell) \le R(k-1,\ell) + R(k,\ell-1)$. $\square$

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Ramsey's theorem (simplified version)

For all $k, \ell \geq 2$, $R(k, \ell)$ exists, i.e., it is finite. Further,

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}$$

Proof:

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Proof: Now, this should be trivial!

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Proof:

- By induction on $k + \ell$ as before.
- Base case for $k = \ell = 2$ is done.
- By what we just showed and induction hypothesis we have:

$$\begin{aligned} R(k,\ell) &\leq R(k-1,\ell) + R(k,\ell-1) \\ &\leq \binom{k+\ell-3}{k-2} + \binom{k+\ell-3}{k-1} = \binom{k+\ell-2}{k-1} \end{aligned}$$

$\square$

Ramsey theory

Some interesting facts

- The general Ramsey theorem extends this to any finite number of colors (not just 2).
- Several applications, vast research area!
- Exact values are known only for 6 or so entries: $R(3,3) = 6$, $R(3,4) = 9$, $R(4,4) = 18$,.... $R(3,8) = 28$ or 29...
- Only bounds are known for rest. (see wiki on this...)

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> *Suppose an evil alien would tell mankind "Either you tell me the value of $R(5,5)$ or I will exterminate the human race." ... It would be best to try to compute it, both by mathematics and with a computer. If he would ask for the value of $R(6,6)$, the best thing would be to destroy him before he destroys us, because we couldn't.*

Two more exercises to try out on Ramsey theory

- We know that a 2-coloring of any graph that contains a complete graph on 6 nodes must contain one monochromatic triangle.

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- (H.W.) Prove that any 2-coloring of a complete graph on 6 nodes in fact must contain 2 monochromatic triangles!

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- (H.W.) Prove that any 2-coloring of a complete graph on 6 nodes in fact must contain 2 monochromatic triangles!
- Are there other graphs that do not contain a complete graph on 6 nodes and yet must have a monochromatic triangle?

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- (H.W.) Prove that any 2-coloring of a complete graph on 6 nodes in fact must contain 2 monochromatic triangles!
- Are there other graphs that do not contain a complete graph on 6 nodes and yet must have a monochromatic triangle?

Further reading

- Books on graph theory: Frank Harary, Douglas West
- Book on Ramsey theory: Ramsey Theory, R.L. Graham , B.L. Rothschild, J. H. Spencer