Tutorial Amal Dani

From the desk of

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① Consider $X \sim \text{Poisson}(\lambda)$ and $P(Y \mid X = \ell) = \text{Binomial}(\ell, p)$ where $0 \le p \le 1$ and $\lambda > 0$. Show that $Y \sim \text{Poisson}(\xi)$ where ξ is a function of λ and p. This process is called as the thinning of a poisson distribution by a binomial distribution. It has realistic applications in image processing.

Solution:

$$P(Y=k) = \sum_{\ell=k}^{\infty} P(Y=k \mid X=\ell)\, P(X=\ell)$$

$$= \sum_{\ell=k}^{\infty} \binom{\ell}{k} p^k (1-p)^{\ell-k} \frac{\lambda^\ell}{\ell!} e^{-\lambda}$$

$$= \frac{e^{-\lambda}}{k!} p^k \lambda^k \sum_{\ell=k}^{\infty} \frac{\ell!}{(\ell-k)!} (1-p)^{\ell-k} \lambda^{\ell-k}$$

$$= \frac{e^{-\lambda}}{k!} (\lambda p)^k \sum_{\ell=k}^{\infty} \frac{(\lambda(1-p))^{\ell-k}}{(\ell-k)!}$$

$$= \frac{e^{-\lambda}}{k!} (\lambda p)^k \sum_{\ell=0}^{\infty} \frac{(\lambda(1-p))^{\ell}}{\ell!}$$

$$= \frac{(\lambda p)^k}{k!} e^{-\lambda} e^{\lambda(1-p)} = \frac{(\lambda p)^k}{k!} e^{-\lambda p}$$

$$= \text{Poisson}(\lambda p)$$

Neelgagan

(2) If X_1 and X_2 are identically distributed random variables, prove that $Cov(X_1 - X_2, X_1 + X_2) = 0$. X_1 and X_2 need not be independent.

Solution:

$$Cov(X_1 - X_2, X_1 + X_2) = E\left((X_1 - X_2 - (\bar{X}_1 - \bar{X}_2))(X_1 + X_2 - (\bar{X}_1 + \bar{X}_2))\right)$$

Where $\bar{X}_1 = E(X_1)$ and $\bar{X}_2 = E(X_2)$.

$$= E\left((X_1 - \bar{X}_1 - (X_2 - \bar{X}_2))(X_1 - \bar{X}_1 + X_2 - \bar{X}_2)\right)$$

$$= E\left((X_1 - \bar{X}_1)^2 - (X_2 - \bar{X}_2)^2\right)$$

$$= E\left((X_1 - \bar{X}_1)^2\right) - E\left((X_2 - \bar{X}_2)^2\right)$$

$= 0$ as the variance of X_1 and X_2 are equal since they are identically distributed.

(3) If U is uniformly distributed on $[0, 1]$, and V is uniformly distributed on $[a, b]$ then express V in terms of U, ie if $V = x + yU$ then express x and y in terms of a and b.

Solution:

$P(U \le u) = u = P(x + yU \le x + yu)$

$\therefore F_V(x + yu) = F_U(u) = u$

for $u = 0$, $F_V(x) = F_U(0) = 0$ and hence $x = a$ (the least possible value of V).

for $u = 1$, we have $F_V(x+y) = F_U(1) = 1$

$\therefore x + y = F_V^{-1}(1) = b$

$\therefore y = b - x = b - a$.

(4) Let X be a random variable denoting the first trial that resulted in a success given a sequence of independent Bernoulli trials. Determine $P(X = k)$ and $E(X)$.

$$P(X = k) = p(1-p)^{k-1}$$

$$E(X) = \sum_{l=0}^{\infty} l\, p(1-p)^{l-1} = \sum_{l=0}^{\infty} l(1-q)q^{l-1}$$

$$= \sum_{l=0}^{\infty} l q^{l-1} - \sum_{l=0}^{\infty} l q^{l}$$

$$= \sum_{l=1}^{\infty} l q^{l-1} - \sum_{l=1}^{\infty} l q^{l}$$

$$= \sum_{l=0}^{\infty} (l+1) q^{l} - \sum_{l=0}^{\infty} l q^{l} = \sum_{l=0}^{\infty} q^{l} = \frac{1}{1-q} = \frac{1}{p}$$

(5) Let U be a uniform random variable $[0,1]$. Let $V = 1/U$. Note that V is another random variable. Find the PDF, CDF and mean of V.

Solution:

$$F_V(v) = P(V \le v) = P(1/U \le v)$$

$$= P(U \geq 1/v) \quad \text{since } v \text{ is always non-negati}$$

$$= 1 - P(U \le 1/v) = 1 - \int_0^{1/v} 1 \cdot du \quad \left(\text{note: } f_U(u) = 1\right)$$

$$= 1 - 1/v.$$

$$f_V(v) = \frac{d}{dv}\left(1 - \frac{1}{v}\right) = \frac{1}{v^2}$$

$$E(V) = \int_{-\infty}^{\infty} v f_V(v)\, dv = \int_1^{\infty} \frac{v}{v^2}\, dv$$

$$= \left(\log v\right)_1^{\infty} = \infty. \quad (\text{Mean does not exist})$$

Is $f_V(v)$ a valid PDF?

$$\int_{-\infty}^{\infty} f_V(v)\, dv = \int_1^{\infty} \frac{dv}{v^2} = \left(\frac{v^{-1}}{-1}\right)_1^{\infty} = 1$$

(6) If $f_{XY}(x,y) = k(x)\, l(y)$, $-\infty < x < \infty$
$-\infty < y - \infty$

then show that X and Y are independent.

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy = k(x) \int_{-\infty}^{\infty} l(y)\, dy$$
$$= k(x)\; c_1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx = l(y) \int_{-\infty}^{\infty} k(x)\, dx$$
$$= l(y)\; c_2$$

where c_1 and c_2 are constants.

Now $\iint_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$

$$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x)\, l(y)\, dx\, dy = \int_{-\infty}^{\infty} k(x)\, dx \int_{-\infty}^{\infty} l(y)\, dy$$
$$= c_1\; c_2 = 1.$$

$\therefore$ $l(y)$ is a constant multiple (c_2) of $f_Y(y)$ and $k(x)$ is a constant multiple (c_1) of $f_X(x)$ where c_1 and c_2 are constants.

$$\therefore f_{XY}(x,y) = f_Y(y)\, c_2\; f_X(x)\, c_1 = f_Y(y)\, f_X(x)$$

which proves independence.

⑦ X and Y are continuous random variables with joint density

$$f_{XY}(x,y) = \begin{cases} 6(x+y)^2/7 & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X > Y)$, $P(X+Y \le 1)$ and the marginal densities of X and Y.

$$P(X+Y\le 1) = \int_0^1 \int_0^{1-x} \frac{6}{7}(x+y)^2\,dx\,dy$$

$$= \frac{6}{7} \int_0^1\int_0^{1-x} (x^2 + 2xy + y^2)\,dx\,dy$$

$$= \frac{6}{7}\left[\int_0^1 \left(x^2 y\right)_0^{1-x} dx + \int_0^1 2x\left(\frac{y^2}{2}\right)_0^{1-x} dx + \int_0^1 \left(\frac{y^3}{3}\right)_0^{1-x} dx\right]$$

$$= \frac{6}{7}\int_0^1 \left(\frac{(x+y)^3}{3}\right)_0^{1-x} dx = \frac{6}{7}\frac{1}{3}\int_0^1 (1^3 - x^3)\,dx$$

$$= \frac{2}{7}\int_0^1 (1-x^3)\,dx = \frac{2}{7}\left[x - \frac{x^4}{4}\right]_0^1 = \frac{2}{7} \times \frac{3}{4} = \frac{3}{14}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^1 \frac{6}{7}(x+y)^2\,dy = \frac{6}{7}\left(\frac{(x+y)^3}{3}\right)_0^1$$

$$= \frac{6}{7}\left(\frac{(x+1)^3}{3} - \frac{x^3}{3}\right)$$