CS207 (Discrete Structures)
Exercise Problem Set 2

August 1, 2015

Instructions:

- Attempt *all* questions.
- Some of the answers will be discussed during the tutorial sessions, but again you are expected to have attempted *all* the questions.
- If you have any doubts or you find any typos in the questions, post them on piazza at once!
- In the following, “disprove” means you have to give a counterexample.

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1. Let A be any infinite set. Prove carefully that there is a surjection from A to $\mathbb{N}$. We said in class that this implies that the natural numbers are the “smallest” infinite set! Do you agree? What about the set of even numbers or set of all primes? Discuss.
2. Prove or disprove the following (with complete justifications): Let A and B be two non-empty sets.
   (a) If there is a bijection from A to B then there is a bijection from $A \times A$ to $B \times B$.
   (b) If there is an injection from A to B then there is a surjection from B to A.
   (c) There is a surjection but no injection from $\mathbb{Q} \cap [0, 1]$ to $\mathbb{N} \times \mathbb{N}$.
   (d) The countable union of countable sets is countable (note that in class we only showed that a finite union of countable sets is countable).
3. In class we showed that there is no bijection from $\mathbb{N}$ to the set of subsets of $\mathbb{N}$. Prove that for *any* non-empty set S, there is no bijection from S to the set of all subsets of S.
4. Construct a bijection
   (a) from $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$
   (b) from $\mathbb{R}$ to set of all subsets of $\mathbb{N}$. Can you conclude whether $\mathbb{R}$ is countable or uncountable from this?
5. Prove that the set of real numbers in the open interval $(0, 1)$ is uncountable.

Reading assignment

6. Read the proof of the Schröder-Bernstein Theorem: For two sets A and B, if there is a surjection from A to B and another surjection from B to A, then there exists a bijection from A to B. What happens if we replace the surjections in the above statement by injections?