CS 207: Discrete Structures

Lecture 19 – Counting and Combinatorics
Pigeon-Hole Principle

Aug 27 2015

Topics in Combinatorics

Basic counting techniques and applications

1. Basic counting techniques, double counting
2. Binomial theorem, permutations and combinations, Estimating $n!$
3. Recurrence relations and generating functions
4. Principle of Inclusion-Exclusion (PIE) and its applications.

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4. Principle of Inclusion-Exclusion (PIE) and its applications.
   - Counting the number of surjections on $[n]$.
   - Combinatorial proof of PIE.
   - Number of derangements $->\frac{1}{e}$.
   - Stirling numbers of the first kind.

Today's lecture

Pigeon-Hole Principle (PHP) and its applications

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How do you prove it?
A simple corollary

- Can a function from a set of $k + 1$ or more elements to a set with k elements be injective?

Pigeon-Hole Principle (PHP)

Another simple application

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- So among the $n+1$ integers, by PHP, at least 2 must have the same remainder.
- That is, $\exists i, j$, $k_i = pn + d$, $k_j = qn + d$.
- But then $|k_i - k_j|$ is a multiple of n and its decimal expansion only has $0's$ and $1's$. $\square$

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How many cards must be selected from a pack of 52 cards so that at least three cards of the same suit are chosen?

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- To have ≥ 3 cards from same suit, suffices $\lceil N/4 \rceil \geq 3$.
- Thus, $N = 9$. But can we do better? No. □

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1. Let $a_1, \ldots, a_{n^2+1}$ be a sequence of distinct real numbers.
2. For each $k \in \{1 \ldots n^2 + 1\}$, let (i_k, d_k) denote a pair:

$$i_k = \text{ length of longest increasing subsequence starting from } a_k$$
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$$\begin{aligned} i_k &= \text{length of longest increasing subsequence starting from } a_k \\ d_k &= \text{length of longest decreasing subsequence starting from } a_k \end{aligned}$$
3. Suppose, there are no increasing/decreasing subsequences of length $n + 1$. Then $\forall k$, $i_k \leq n$ and $d_k \leq n$.
4. $\therefore$ by PHP, $\exists \ell, m, 1 \leq \ell < m \leq n^2 + 1$ s.t. $(i_\ell, d_\ell) = (i_m, d_m)$
5. We will show that this is not possible:
    - Case 1: $a_\ell < a_m$. Then $i_m \geq i_\ell + 1$, a contradiction.
    - Case 2: $a_\ell > a_m$. Then $d_\ell \geq d_m + 1$, a contradiction.
6. All a_i's are distinct so this completes the proof. $\square$

Another variant of PHP

Theorem (PHP Variant 2)

Suppose there are $n \geq 1 + r(\ell - 1)$ objects which are colored with r different colors. Then there exist ℓ objects all with the same color.

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Theorem (PHP Variant 2)

Suppose there are $n \geq 1 + r(\ell - 1)$ objects which are colored with r different colors. Then there exist ℓ objects all with the same color.

Proof: (H.W)

- Is this coloring optimal?
- That is, if fewer than $1 + r(\ell - 1)$ objects are given, is there a way of coloring them such that no ℓ have the same color?

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The coloring game

- There are six points on board and two colored chalks.

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Lemma

Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

- 2-coloring of edges: coloring all edges of the graph using atmost 2 colors.
- monochromatic (triangle): all edges have the same color.

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Lemma

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Proof:

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Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

Proof:

- Let $1, \ldots, 6$ be the points, and red/blue the colors.
- Consider the edges $16, 26, 36, 46, 56$.
- By PHP at least 3 of them must be same color, say $16, 26, 36$ are red.
- Now there are two possibilities:
  - Either one of $12, 23, 31$ is red (then we have a red triangle).
  - Else none of them are red, implies 123 is a blue triangle. $\square$

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- What if there were 5 or lesser nodes?