CS 207: Discrete Structures

Lecture 16 – Counting and Combinatorics
Solving Recurrence relations and generating functions

Aug 24 2015

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1. $\alpha^n = \alpha^{n-1} + \alpha^{n-2}$ implies $\alpha^{n-2}(\alpha^2 - \alpha - 1) = 0$.
2. So if $\alpha^2 - \alpha - 1 = 0$, the recurrence holds for all n.
3. Solving, $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$
4. Thus, general solution is $F_n = a(\frac{1+\sqrt{5}}{2})^n + b(\frac{1-\sqrt{5}}{2})^n$.
5. Use F_0 and F_1 – initial conditions: $a = \frac{\sqrt{5}+1}{2\sqrt{5}}$, $b = \frac{\sqrt{5}-1}{2\sqrt{5}}$

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But this method may not work if we have repeated roots (but this can be fixed) and non-linear recurrences.

Proof Method 2: Using generating functions

Fibonacci recurrence relation

For $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$, $F_0 = F_1 = 1$.
Compute F_n in terms of n.

- Consider the power series... $\phi(t) = \sum_{n=0}^{\infty} F(n) t^n$.

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$$t\phi(t) = \sum_{n=0}^{\infty} F(n) t^{n+1} \qquad = \sum_{n=1}^{\infty} F(n-1) t^n$$

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(t+t^2)\phi(t) &= \sum_{n=0}^{\infty} F(n)t^n - 1 \\
(t+t^2)\phi(t) &= \phi(t) - 1
\end{aligned}
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- Now $\phi(t) = a(1 + \alpha t + \alpha^2 t^2 + \dots) + b(1 + \beta t + \beta^2 t^2 + \dots)$

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thus, as before

$$F(n) = \frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-1}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Properties of generating functions

Definition

The (ordinary) generating function for a sequence $a_0, a_1, \ldots \in \mathbb{R}$ is the infinite series $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$.

- Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then
  1. If $f(x) = g(x)$, then $a_k = b_k$ for all k.
  2. $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$,
  3. $f(x) g(x) = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) x^k$,
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- Let $u \in \mathbb{R}$, $k \in \mathbb{Z}^{\geq 0}$, Then extended binomial coefficient $\binom{u}{k}$ is defined as $\binom{u}{k} = \frac{u(u-1)\ldots(u-k+1)}{k!}$ if $k > 0$ and $= 1$ if $k = 0$.
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The extended binomial theorem

Let $u \in \mathbb{R}$, $(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$.
If you don't like this, take $x \in \mathbb{R}, |x| < 1$.

Simple examples using generating functions

Standard identities:

- $\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$
- $\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk}$
- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

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Class work:

1. Solve the recurrence $a_k = 4a_{k-1}$ with $a_0 = 3$.

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Class work:

1. Solve the recurrence $a_k = 4a_{k-1}$ with $a_0 = 3$.
2. Solve the recurrence $a_k = 8a_{k-1} + 10^{k-1}$ with $a_0 = 1, a_1 = 9$.

Solving Catalan numbers using generating functions

Catalan Numbers

$$C(n) = \sum_{i=1}^{n-1} C(i)C(n-i) \text{ for } n > 1,\ C(0) = 0, C(1) = 1.$$

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- Let $\phi(x) = \sum_{k=1}^{\infty} C(k)x^k$.
- Now consider $\phi(x)^2$.
- $\phi(x)^2 = (\sum_{k=1}^{\infty} C(k)x^k)(\sum_{k=1}^{\infty} C(k)x^k)$
  $= (\sum_{k=2}^{\infty} \sum_{i=1}^{k-1} C(i)C(k-i)x^k)$
  $= (\sum_{k=2}^{\infty} C(k)x^k) = \phi(x) - x$

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- Solving for $\phi(x)$ we get, $\phi(x) = \frac{1}{2}(1 \pm (1-4x)^{1/2})$
- But since $\phi(0) = 0$, we have
  $\phi(x) = \frac{1}{2}(1 - (1-4x)^{1/2}) = \frac{1}{2} + (-\frac{1}{2}(1-4x)^{1/2})$.

Catalan numbers

Recall: Extended binomial theorem

Let $\alpha \in \mathbb{R}, (1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$, where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}$.

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- $\sum_{k=1}^{\infty} C(k)x^k = \phi(x) = \frac{1}{2} + (-\frac{1}{2}(1-4x)^{1/2}) = \frac{1}{2} + (-\frac{1}{2}\sum_{k=0}^{\infty} \binom{1/2}{k}(-4x)^k)$.

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- The coefficient of x^k is $C(k) = -\frac{1}{2}\binom{1/2}{k}(-4)^k$
  $= -\frac{1}{2}(\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\ldots(\frac{1}{2}-k+1))\frac{(-4)^k}{k!}$
  $= -\frac{1}{2}(\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\ldots(-\frac{2k-3}{2}))\frac{(-4)^k}{k!}$

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- $C(k) = \frac{(-1)^k(-4)^k}{2^{k+1}k!} \cdot 1 \cdot 3 \cdots (2k-3)$

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- $C(k) = \frac{1 \cdot 4^k}{2^{k+1} \cdot k!} \cdot \frac{1.2.\ldots.(2k-3)(2k-2)}{2^{k-1}(k-1)!} = \frac{(2k-2)!}{k!(k-1)!}$.

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Thus, the n^{th} Catalan number is given by

$C(n) = \frac{(2n-2)!}{n!(n-1)!} = \frac{1}{n}\binom{2n-2}{n-1}$

Other examples

- (H.W) Write a recurrence for the number of derrangements. That is, no. of ways to arrange n letters into n addressed envelopes such that no letter goes to the correct envelope.
- (H.W) How many ways can a convex n-sided polygon be cut into triangles by adding non-intersecting diagonals (i.e., connecting vertices with non-crossing lines)? Write a recurrence and solve it!