CS 207: Discrete Structures

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Lecture 07 – Uncountable sets, relations, equivalence relation

Recap: Countable and countably infinite sets

Definition

- For a given set C, if there is a bijection from C to $\mathbb{N}$, then C is called countably infinite. A set is countable if it is finite or countably infinite.
- Examples: $\mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$, $\mathbb{Q}$.
- Properties: Closure under (countable) unions and (finite) products.

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Are all sets countable?

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

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Proof by contradiction: Suppose there is such a bijection, say f. This would imply that each $i \in \mathbb{N}$ maps to some set $f(i) \subseteq \mathbb{N}$.

	0	1	2	3	...
$f(0)$	✓	×	×	×	...
$f(1)$	✓	×	✓	✓	...
$f(2)$	×	×	×	×	...
$f(3)$	×	✓	×	✓	...

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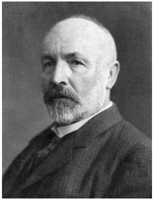
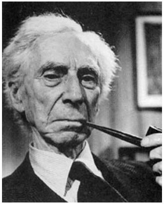
- As f is bij, $\exists j \in \mathbb{N}, f(j) = S$.
- But S and $f(j)$ differ at position j.
- Thus, $S \neq f(j)$ which is a contradiction! $\square$

Cantor's diagonalization

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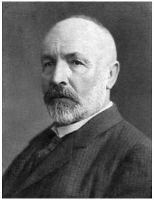 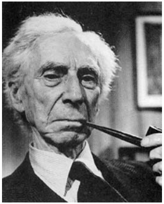

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- $S = \{i \in \mathbb{N} \mid i \notin f(i)\}$ is like the one from Russell's paradox.
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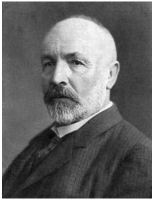 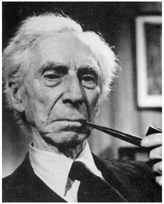

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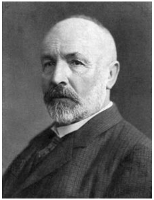 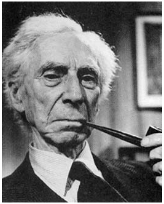

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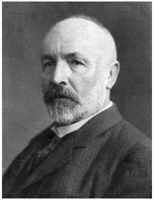

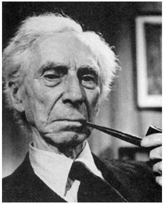

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- There is no bijection from $\mathbb{R}$ to $\mathbb{N}$ (H.W). Moreover, there is a bijection from $\mathbb{R}$ to set of subsets of $\mathbb{N}$.
- What about the set of all sets??

One infinity is “strictly” bigger than another!

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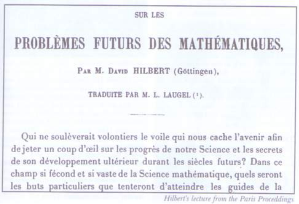
Cantor’s Continuum hypothesis

There is no set whose “cardinality” is strictly between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ (i.e., between naturals and reals).



SUR LES

PROBLÈMES FUTURS DES MATHÉMATIQUES,

PAR M. DAVID HILBERT (Göttingen),

TRADUITE PAR M. L. LAUGEL [1].

Qui ne soulèverait volontiers le voile qui nous cache l’avenir afin de jeter un coup d’œil sur les progrès de notre Science et les secrets de son développement ultérieur durant les siècles futurs? Dans ce champ si fécond et si vaste de la Science mathématique, quels seront les buts particuliers que tenteront d’atteindre les guides de la

Hilbert’s lecture from the Paris Proceedings

Figure : 1st of Hilbert’s 23 problems for the 20th century in 1900.

What did the world think about these proofs (in 1890s?)

(a) Kronecker

(b) Poincare

(c) Theologians

- Kronecker: Only constructive proofs are proofs! “Scientific Charlatan”, “Corruptor of youth”!
- Poincare: Set theory is a “disease” from which mathematics will be cured.
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- Hilbert: No one can expel us from the paradise that Cantor has created for us.

Summary and moving on...

- Finite and infinite sets.
- Using functions to compare sets: focus on bijections.
- Countable, countably infinite and uncountable sets.
- Cantor's diagonalization argument (A new powerful proof technique!).

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Next: Basic Mathematical Structures – Relations

Relations

Definition: Function

Let A, B be two sets. A function f from A to B is a subset R of $A \times B$ such that

(i) $\forall a \in A, \exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

- Now, suppose A is the set of all Btech students and B is the set of all courses. Clearly, we can assign to each student the set of courses he/she is taking. Is this a function?

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- By removing the two extra assumptions in the defn, we get:

Definition: Relation

- A relation R from A to B is a subset of $A \times B$. If $(a, b) \in R$, we also write this as $a \ R \ b$.
- Thus, a relation is a way to relate the elements of two (not necessarily different) sets.

Examples and representations of relations

We write $R(A, B)$ for a relation from A to B and just $R(A)$ if $A = B$. Also if A is clear from context, we just write R.

Examples of relations

- All functions are relations.
- $R_1(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z}, a - b \text{ is even } \}$.
- $R_2(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z}, a \leq b\}$.
- Let S be a set, $R_3(\mathcal{P}(S)) = \{(A, B) \mid A, B \subseteq S, A \subseteq B\}$.
- Relational databases are practical examples.

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Representations of a relation from A to B.

- As a set of ordered pairs of elements, i.e., subset of $A \times B$.
- As a directed graph.
- As a (database) table.

Use of relations

Practical application in relational databases: IMDB, university records, etc.

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  - Equivalence relations
  - Partial orders

Partitions of a set – grouping "like" elements

Examples

- Natural numbers are partitioned into even and odd.
- This class is partitioned into set of all those who are taking the same set of courses.

How do you define a partition?

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- if $S' \in P$, then $S' \neq \emptyset$.
- $\bigcup_{S' \in P} S' = S$: its union covers entire set S.
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Can you think of two trivial partitions that any set must have?

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What properties does this relation have?

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Definition

A relation which satisfies all these three properties is called an equivalence relation.

Examples

- Reflexive: $\forall a \in S$, aRa.
- Symmetric: $\forall a, b \in S$, aRb implies bRa.
- Transitive: $\forall a, b, c \in S$, aRb, bRc implies aRc.
- Equivalence: Reflexive, Symmetric and Transitive.

Relation	Refl.	Symm.	Trans.	Equiv.
aRb if students a and b take same set of courses	✓	✓	✓	✓
$\{(a,b) \mid a, b \in \mathbb{Z}, (a-b) \bmod 2 = 0\}$	✓	✓	✓	✓
$\{(a,b) \mid a, b \in \mathbb{Z}, a \leq b\}$				
$\{(a,b) \mid a, b \in \mathbb{Z}, a < b\}$				
$\{(a,b) \mid a, b \in \mathbb{Z}, a \mid b\}$				
$\{(a,b) \mid a, b \in \mathbb{R}, \lvert a-b \rvert < 1\}$				
$\{((a,b),(c,d)) \mid (a,b),(c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), (ad = bc)\}$				