Quiz 1: CS 215

Name: ______________________________ **Roll Number:** ____________________

Attempt all four questions. Each question carries 10 points for a total of 40.

Useful Information

1. Binomial theorem: $(x+y)^n = \sum_{k=0}^{n} C(n,k)x^k y^{n-k}$
2. The empirical mean of n independent and identically distributed random variables is approximately Gaussian distributed. The approximation accuracy is better when n is larger.
3. Defining $\Phi(x) = \int_{-\infty}^{x} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$, we have the following table:

n	$\Phi(n) - \Phi(-n)$
1	68.2%
2	95.4%
2.6	99%
2.8	99.49%
3	99.73%

4. Integration by parts: $\int u dv = uv - \int v du$.
5. Gaussian pdf: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$
6. Poisson pmf: $P(X=i) = \frac{e^{-\lambda}\lambda^i}{i!}$

1. Suppose I gather some n independent measurements of a quantity. Let us treat the measurements as independent random variables with mean μ and standard deviation σ. If I want to be 99% certain that the average of these measurements is accurate to within $\pm\frac{\sigma}{4}$ units, how many measurements must I take, i.e. what is the value of n? [10 points]
Solution:
We know that $\bar{X}$, the average of the measurements, will have mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$ and has an approximately Gaussian distribution. Hence $\sqrt{n}\frac{\bar{X}-\mu}{\sigma}$ is a standard normal random variable. We want $\mu-\frac{\sigma}{4}\leq\bar{X}\leq\mu+\frac{\sigma}{4}$, i.e. $-\frac{\sigma}{4}\leq\bar{X}-\mu\leq\frac{\sigma}{4}$, i.e. $-\frac{\sqrt{n}}{4}\leq\sqrt{n}(\frac{\bar{X}-\mu}{\sigma})\leq\frac{\sqrt{n}}{4}$. Now $P(-\frac{\sqrt{n}}{4}\leq\sqrt{n}(\frac{\bar{X}-\mu}{\sigma})\leq\frac{\sqrt{n}}{4})\geq 0.99$. Hence from the given table, we must have $\sqrt{n}/4\geq 2.7$, i.e. $\sqrt{n}\geq 10.8$, i.e. $n\geq 117$.

2. The joint density of X and Y is given as $f_{XY}(x,y)=xe^{-(x+y)}$ when $x>0, y>0$ and 0 otherwise. Deduce whether X and Y are independent and whether they are uncorrelated. Show all steps clearly. [10 points]
Solution: We have $f_X(x)=\int_0^\infty f_{XY}(x,y)dy=xe^{-x}$ and $f_Y(y)=\int_0^\infty f_{XY}(x,y)dx=e^{-y}$. Hence $f_{XY}(x,y)=f_X(x)f_Y(y)$ and the random variables are independent, and hence uncorrelated.

3. Let $X_1, X_2, ..., X_n$ be n indpendent random variables having a $[0,1]$ uniform random distribution. Define the random variable $Y=\max_{1\leq i\leq n} X_i$. Write an expression for the CDF, PDF and expected value of Y. [4+3+3=10 points]
Solution: $F_Y(y)=P(Y\leq y)=P(X_1\leq y, X_2\leq y, ..., X_n\leq y)=P(X_1\leq y)P(X_2\leq y)...P(X_n\leq y)=y^n$ for $0\leq y\leq 1$, otherwise $F_y(y)=1$ for $y>1$ and $F_Y(y)=0$ for $y<0$. The probability density function is given as $f_Y(y)=ny^{n-1}$ for $0\leq y\leq 1$ otherwise 0.
$E(Y)=\int_0^1 yny^{n-1}dy=\frac{n}{n+1}$.

4. Show that the sum of two independent Poisson random variables with mean λ_1 and λ_2 respectively is another Poisson random variable. What is its mean? Show all steps clearly. [10 points]
Solution:

$$
\begin{aligned}
P(Z=k) &= \sum_{l=0}^{k} P(X=l)P(Y=k-l) && (1)\\
&= \sum_{l=0}^{k} \frac{\lambda_1^l e^{-\lambda_1}}{l!}\frac{\lambda_2^{k-l}e^{-\lambda_2}}{(k-l)!} && (2)\\
&= e^{-\lambda_1-\lambda_2}\sum_{l=0}^{k}\frac{\lambda_1^l}{l!}\frac{\lambda_2^{k-l}}{(k-l)!} && (3)\\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}\sum_{l=0}^{k} k!\frac{\lambda_1^l}{l!}\frac{\lambda_2^{k-l}}{(k-l)!} && (4)\\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}\sum_{l=0}^{k} C(k,l)\lambda_1^l\lambda_2^{k-l} && (5)\\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}(\lambda_1+\lambda_2)^k && (6)\\
& && (7)
\end{aligned}
$$

which is a Poisson pmf with parameter $\lambda_1+\lambda_2$.