CS 207: Discrete Structures

Graph theory

Bipartite subgraphs, graph isomorphism

Lecture 26
Sept 22 2015

Topic 3: Graph theory

Topics covered

- Eulerian graphs and a characterization
- Characterization of bipartite graphs using odd cycles.
- Subgraphs and degree sum formula.
- Cliques and independent sets.

Today

- Finding a large bipartite subgraph of a given graph
- Graph representation as a matrix.
- Comparing graphs: isomorphism

Bipartite graphs

Definition

A graph is called bipartite, if the vertices of the graph can be partitioned into $V = X \cup Y$, $X \cap Y = \emptyset$ s.t., $\forall e = (u, v) \in E$,

- either $u \in X$ and $v \in Y$
- or $v \in X$ and $u \in Y$

Example: m jobs and n people, k courses and ℓ students.

- How can we check if a graph is bipartite?
- Can we characterize bipartite graphs?

Some basic stuff that we have already seen

Degree-Sum Formula (also called Handshake Lemma!)

For any graph G with vertex set V and edge set E:

$$\sum_{v \in V} d(v) = 2|E|$$

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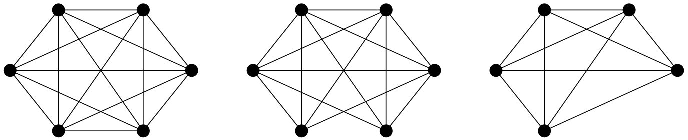
Subgraphs of a graph G

A subgraph H of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ (and the assignment of endpoints to edges in H is same as in G).



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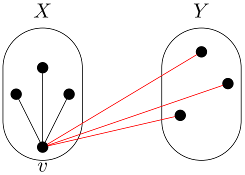

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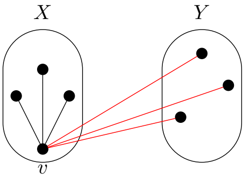
- Either, $\forall v \in V(G), d_H(v) \geq \frac{d_G(v)}{2}$, then $|E(H)| \geq \frac{|E(G)|}{2}$. $d_H(v)$ denotes degree of v in H and $d_G(v)$ its degree in G.



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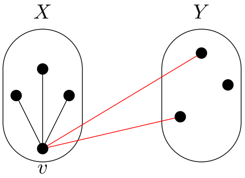

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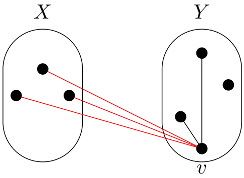

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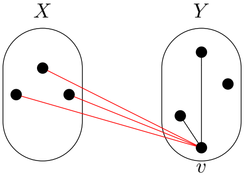
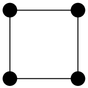


- Can we make this into an algorithm to produce such H?

Representing and comparing graphs

We start with simple graphs...



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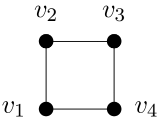

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- As an adjacency list:

v_1	v_2, v_4
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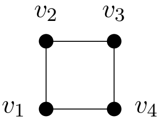

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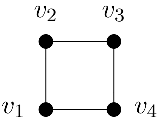

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- Are two given graphs the "same", wrt these properties?

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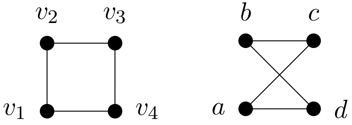

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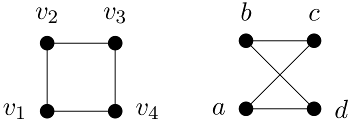

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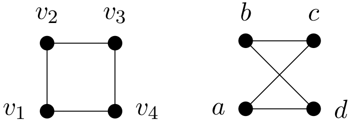

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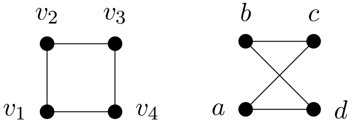

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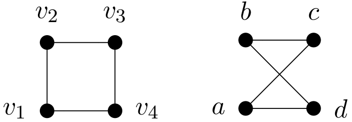

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- How do we formalize this?

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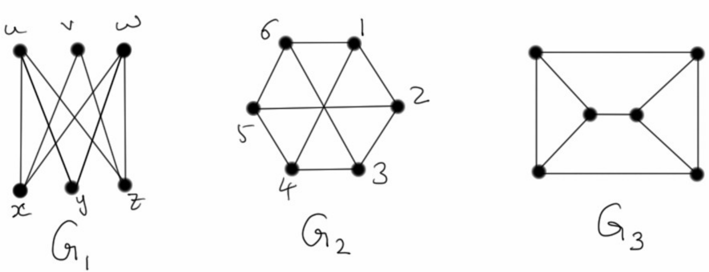
- When we talked about an "unlabeled" graph till now, we actually meant the isomorphism class of that graph!

Graph isomorphism



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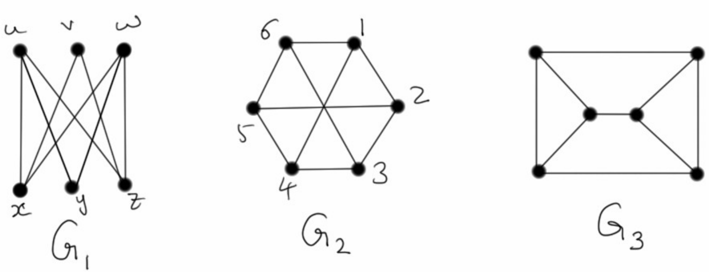
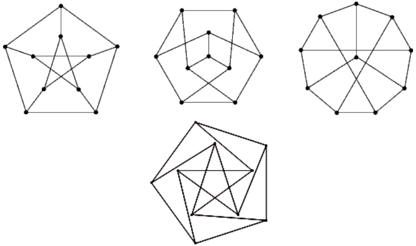


- To show that two graphs are isomorphic, you have to
  1. give names to vertices
  2. specify a bijection
  3. check that it preserves the adjacency relation
- To show that two graphs are non-isomorphic, find a structural property that is different.

Is checking graph isomorphism easy?



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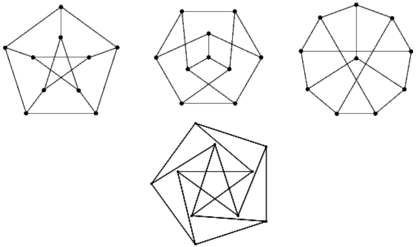

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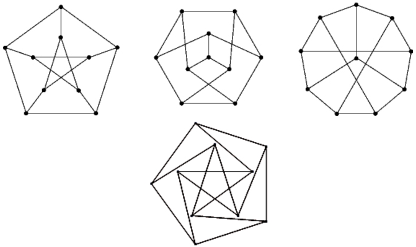

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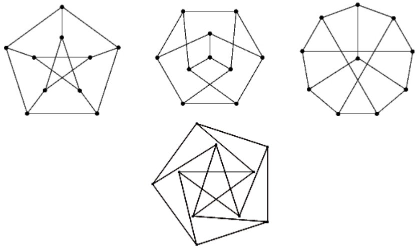

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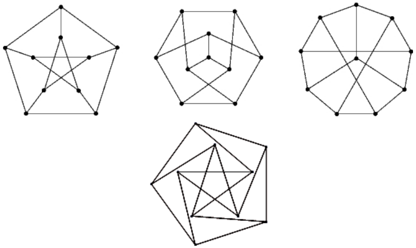

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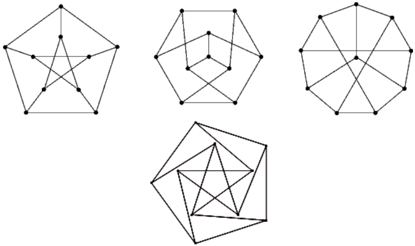

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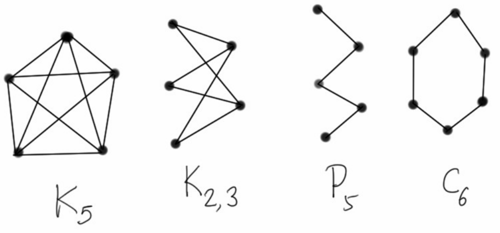
Further reading: Graph and sub-graph isomorphism problems.

Some special graphs and notations



- Complete graphs K_n
- Complete bipartite graphs $K_{i,j}$
- Paths P_n
- Cycles C_n

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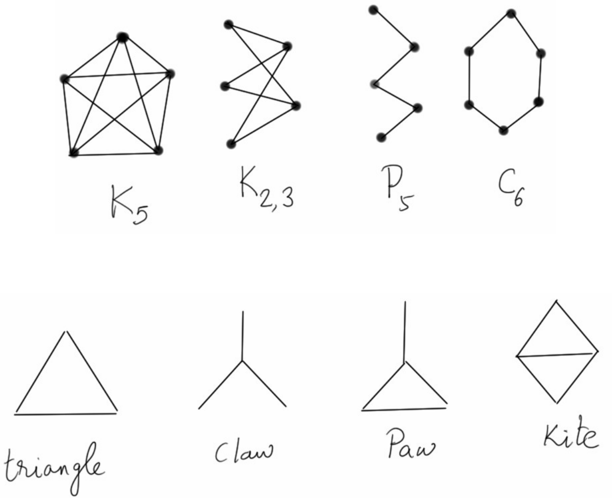

Figure: A whole graph zoo!

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- Are C_5 and $P_5 \cup \{e\}$ isomorphic?

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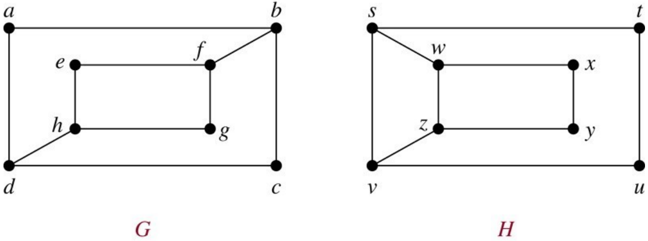

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4. G has k paths/cycles of length r iff H has k paths/cycles of length r.
5. G is bipartite iff H is bipartite.
6. G contains K_n as a subgraph iff H does.
7. …