CS207 Discrete Structures: Induction, proofs
Exercise Problem Set 1

1. Prove (by induction) or disprove: For every positive integer n,
   (a) $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$.
   (b) if $h > -1$, then $1 + nh \leq (1+h)^n$.
   (c) 12 divides $n^4 - n^2$.
2. Use the well-ordering property to show that any two positive integers a, b have a unique greatest common divisor (hint: consider the set of numbers of the form $ax + by$).
3. Consider the following game:
   - There are two piles of matches.
   - Two players take turns removing any positive (i.e., non-zero) number of matches they want from one of the two piles.
   - The player who removes the last match wins.

   Show that, if the two piles contain the same number of matches initially, then the second player can always win the game.
4. Prove that there does not exist an input-free C-program which will always determine whether an arbitrary input-free C-program will halt.
5. Use the Well-Ordering-Principle to prove that the equation $4a^3 + 2b^3 = c^3$ does not have any solutions over $\mathbb{N}$. What about the equation $a^4+b^4+c^4 = d^4$ over $\mathbb{Z}$?
6. For any $n \in \mathbb{N}$, $n \geq 2$, prove that
$$\sqrt{2\sqrt{3\sqrt{4 \ldots \sqrt{(n-1)\sqrt{n}}}}} < 3$$
7. Suppose that for every pair of cities in a country there is a one-way road connecting them in one direction or the other. Use induction to show that there is a city that can be reached from every other city either directly or via exactly one other city.