CS 207: Discrete Structures

Lecture 22 – Bounds on Ramsey Numbers

Sept 3 2015

Ramsey theory: A search for order in disorder

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Ramsey theory: A search for order in disorder

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Recall: Definition

For $k, \ell \in \mathbb{N}$, $R(k, \ell)$ denotes the minimum number of nodes such that any 2-coloring of a (complete) graph on $R(k, \ell)$ nodes has

- either, a complete graph on k-nodes with all red edges
- or, a complete graph on ℓ-nodes with all blue edges

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Ramsey's theorem (2-color simple version)

For all $k, \ell \in \mathbb{N}$, $R(k, \ell)$ exists, i.e., it is finite. In fact,

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}$$

Bounds on Ramsey numbers

Upper bounds

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&\le \frac{2^{2k-2}}{\sqrt{2(k-1)}} \qquad \text{(H.W)}
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So what about lower bounds?

Next: Lower bound for Ramsey numbers

$$R(k,k) \ge 2^{\frac{k-2}{2}}$$

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So what about lower bounds?

- For what value of n can we be sure that in a complete graph with n nodes, there exists a 2-coloring of edges such that there is no monochromatic k-complete set of nodes.

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   - blue with probability $1/2$.
   - red with probability $1/2$.

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2. Let X be a collection of k nodes.

   Prob [all edges in X are red] $=$

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3. $\therefore$ Prob $[\exists X, |X| = k$ such that: (all edges in X are red) or (all edges in X are blue)$] =$

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- That is, $\exists$ a 2-coloring of edges such that there is no monochromatic k-complete subset of nodes.

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Proof by probabilistic method (pick random coloring of edges):

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   - $\Longrightarrow$ Prob $[\forall X, |X| = k$, neither (all edges in X are red) nor (all edges in X are blue)$] > 0$,
   - $\Longrightarrow$ $\exists$ a 2-coloring of edges such that there is no monochromatic k-complete set of nodes.
6. Thus, $R(k,k) \geq 2^{(k-2)/2}$. $\square$

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Thus we have

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- Is this satisfactory? One may ask if the probability space well-defined/natural?
- Same proof can be done by counting (using PIE!).(H.W)
- In effect, we are counting the number of bad objects and trying to prove that it is less than the number of all objects, so the set of good objects must be non-empty.

More about the probabilistic method

- This general approach of using probabilistic reasoning to show existence of discrete structures is called the probabilistic method.
- Use probabilities to show something with certainity!
- Very powerful
  - not all proofs can be converted back into counting arguments.
  - can utilize advanced results from probability theory.
  - Application in graph theory, number theory, real analysis, CS.
- Pioneered by Paul Erdös.
- Further reading: Book by Noga Alon and Joel Spencer. The probabilistic method (2ed). New York: Wiley.

An exercise...

Application of Ramsey's theorem

Prove that there exists a function $f(m,n)$ such that:
if $x_1, \ldots x_N$ is any sequence of distinct real numbers with $N > f(m,n)$, then there is either

- a monotonic decreasing sequence of length $> m$, or,
- a monotonic increasing sequence of length $> n$.

Summary: End of section on combinatorics

Two broad topics covered till date - Half-time!

- Mathematical proofs and structures
- Counting and Combinatorics

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  - Propositions, proof techniques: contradiction, contrapositive
  - Induction: strong induction, well-ordering principle
  - Sets: finite and infinite sets, countable and uncountable sets
  - Functions: bijections (from e.g., $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$), injections and surjections, Cantor's diagonalization technique
  - Relations: equivalence relations and partitions; partial orders, chains, anti-chains, lattices
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- Counting and Combinatorics
  - Basic counting principles, double counting
  - Binomial theorem, permutations and combinations, Estimating $n!$
  - Recurrence relations and generating functions
  - Principle of Inclusion-Exclusion (PIE) and its applications.
  - Pigeon-Hole Principle (PHP) and its applications.
  - Introduction to Ramsey theory

Next topic (post mid-sem)

Graph theory