CS 207: Discrete Structures

Graph theory

Graph automorphisms, Connected components, cut edges.

Lecture 27

Sept 24 2015

Topic 3: Graph theory

Recap of last four lectures:

1. Basics: graphs, paths, cycles, walks, trails; connected graphs.
2. Eulerian graphs and a characterization in terms of degrees of vertices.
3. Bipartite graphs and a characterization in terms of odd length cycles.
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2. Eulerian graphs and a characterization in terms of degrees of vertices.
3. Bipartite graphs and a characterization in terms of odd length cycles.
4. Cliques and independent sets.
5. A proof and algo for large bipartite subgraphs of a graph.
6. Graph representation (as matrices) and isomorphism

Reference: Sections 1.1-1.3 of Chapter 1 from Douglas West.

This lecture

- How to check if two graphs are isomorphic/non-isomorphic?
- Graph automorphisms
- Relations between vertices

Graph isomorphism

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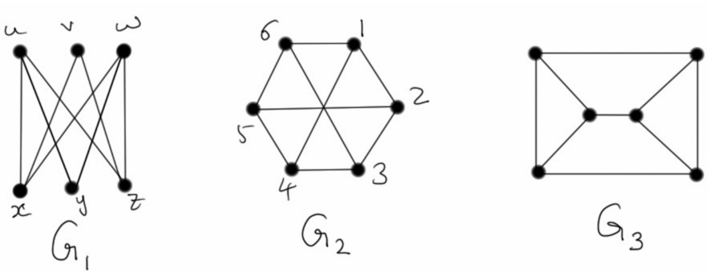
- Thus, it is a bijection that “preserves” the edge relation.
- Can be checked using adjacency matrix by reordering/renaming.
- The isomorphism relation is an equivalence relation on simple graphs.
- By an “unlabeled” graph, we mean the isomorphism class of that graph.

Graph isomorphism



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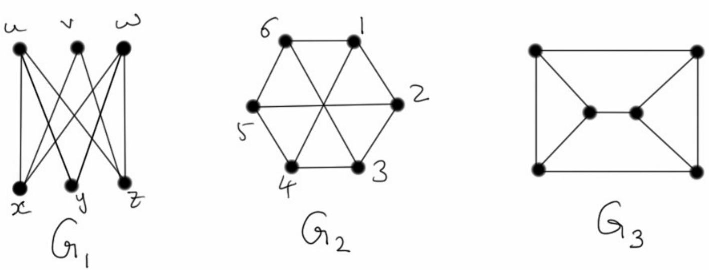

- To show that two graphs are isomorphic, you have to
    1. give names to vertices
    2. specify a bijection & check that it preserves the adjacency relation
       i.e., check that adjacency matrices become identical by permuting rows and columns.
- To show that two graphs are non-isomorphic, find a structural property that is different.

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- Are C_5 and $P_5 \cup \{e\}$ isomorphic?

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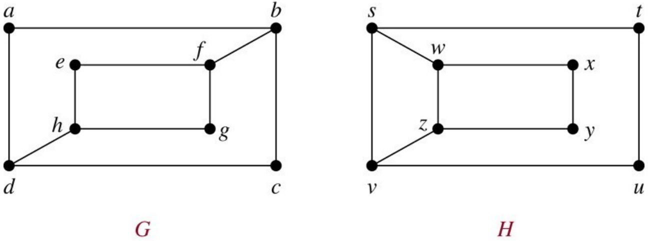

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4. G has k paths/cycles of length r iff H has k paths/cycles of length r.
5. G is bipartite iff H is bipartite.
6. G contains K_n as a subgraph iff H does.
7. $\ldots$

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- How many automorphisms does $K_{r,s}$ have?

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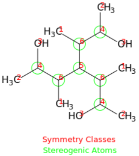
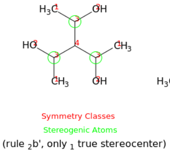
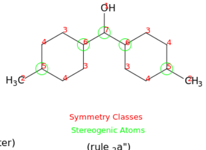
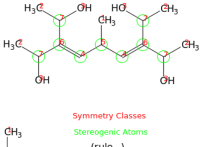
Automorphisms are a measure of symmetry.

Practical applications in graph drawing, visualization, molecular symmetry, structured boolean satisfiability, formal verification









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Definition

A maximal connected subgraph of G is a subgraph that is connected and is not contained in any other connected subgraph of G.
The components of G are its maximal connected subgraphs.

Thus, equivalence classes of P are the vertex sets of the components of G.

Difference between maximal and maximum

- Is every maximal path maximum, i.e., have maximum length?
- A maximal structure is a structure that is not contained in a larger structure, i.e., increasing the structure will violate some property.
- Maximum just means that size is the greatest among all possible.

Exercises

1. Give a path which is maximal but not maximum.
2. Give a subgraph of a graph which is maximally connected, but not maximum (i.e., does not have maximum # edges).
3. How many maximal/maximum independent sets does $K_{r,s}$ have?

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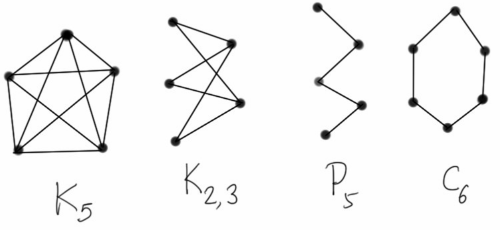

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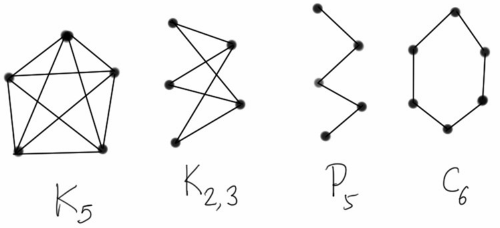

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Exercise!

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Exercise! An edge is a cut-edge iff it belongs to no cycle.