CS207 (Discrete Structures)
Problem set 4

Aug 21 2015

- Attempt *all* questions.
- Apart from things proved in lecture, you cannot assume anything as "obvious". Either quote previously proved results or provide clear justification for each statement.

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1 Lattices

Let L be a lattice. For any two elements $x, y \in L$ we use $x \vee y$ to denote the least upper bound of $\{x, y\}$ and $x \wedge y$ to denote the greatest lower bound of $\{x, y\}$ (note that both these elements exist by the definition of a lattice).

1. Show the following properties for all $x, y, z \in L$,

   (a) (commutative laws) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$

   (b) (associative laws) $((x \vee y) \vee z) = (x \vee (y \vee z))$ and $((x \wedge y) \wedge z) = (x \wedge (y \wedge z))$.

   (c) (absorption laws) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$

   (d) (idempotency laws) $x \vee x = x$ and $x \wedge x = x$.

2. Use the above to prove that every finite subset of a lattice must have a greatest lower bound and a least upper bound.

3. Conclude that every finite lattice must be complete.

2 Counting

1. Prove the following identities by counting the size of a suitably designed set in two different ways.

   (a) $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$

(b) $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$

2. Find the number of (i) all words (ii) four letter words that can be formed using the letters from the word "MISSISSIPPI".

3. Using ideas used for estimating $n!$, approximate the quantity $\sum_{i=1}^{n} \sqrt{i}$. Prove that $\frac{2}{3}(n^{3/2}-1)+1 \leq \sum_{i=1}^{n} \sqrt{i} \leq \frac{2}{3}(n^{3/2}-1)+\sqrt{n}$.

4. Let $B(n)$ be the no. of bit strings of length n that contain the string 01.

   (a) Write a recurrence relation for $B(n)$.

   (b) Determine the initial conditions.

   (c) How many strings are there of length 7 are there that contain 01?

   (d) Solve the recurrence to obtain an expression for $B(n)$ in terms of n.

5. Consider a grid from $(0,0)$ to (n,n). Starting from the point $(0,0)$, we wish to take units steps ONLY in the direction of the positive X and Y axes (i.e., right and up), and reach the point (n,n). Find a recurrence relation for the number of ways of doing so, if we are not allowed to go above the line joining $(0,0)$ and (n,n), i.e, the diagonal.

6. Find the number of integral solutions for $x_1+x_2+x_3+x_4+x_5 = 21$ with each $x_i \geq 2$. Next, generalize this to find the number of solutions for $\sum_{i=1}^{k} x_i = n$ with each $x_i \geq t$ and express it in a closed form involving n, t, and k.