CS 207: Discrete Structures

Lecture 15 – Counting and Combinatorics
Solving Recurrence relations and generating functions

Aug 20 2015

Find the Fibonacci sequence!

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 200 252 200 120 45 10 1

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- $1, 1, 2, 3, 5, 8, 13, ...$

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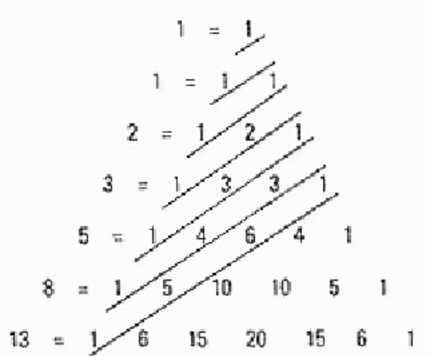

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Last few classes

Basic counting techniques and applications

1. Sum and product, bijection, double counting principles
2. Binomial coefficients and binomial theorem, Pascal's triangle
3. Permutations and combinations with/without repetitions
4. Counting subsets, relations, partitions, Handshake lemma
5. Stirling's approximation: Estimating $n!$
6. Recurrence relations

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Today

Solving recurrence relations and generating functions.

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By solving, we mean give a closed-form expression for n^{th} term.

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2. i.e., if $F_n = F_{n-1} - F_{n-2}$, $G_n = G_{n-1} - G_{n-2}$ and $H_n = aF_n + bG_n$, then what about H_n?.

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6. How do we get a and b?

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Thus, $F_n = \frac{\sqrt{5}+1}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{\sqrt{5}-1}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$.

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- Recall the recurrence for Catalan Numbers:
  $C(n) = \sum_{i=1}^{n-1} C(i)C(n-i)$ for $n > 1$, $C(0) = C(1) = 1$.
  No. of ways to bracket a sum of n terms s.t. it can be computed by adding two numbers at a time?

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Reading assignment

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We next consider a method of much wider applicability...

Proof Method 2: Using generating functions

Yet another way to count number of subsets of a set.

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- $\forall t$ with $|t| < 1/2$, power series does converge by analysis.