CS 207: Discrete Structures

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July 23, 2015
Lecture 03 – WOP, Strong Induction

Fallacies in using Induction

Conjecture: All horses have the same colour.

"Proof" by induction:

- The case with one horse is trivial.
- Assume for $n = k$ and now we have $k + 1$ horses, say $1, \ldots, k + 1$.
  (A) First, consider horses $1, \ldots, k$. By induction hypothesis, they have same color.
  (B) Next, consider horses $2, \ldots, k + 1$. By induction hypothesis, they have same color.
  (C) Therefore, 1 has same color as 2 (by A) and 2 has same color as $k + 1$ (by B), implies all $k + 1$ have same color.
- Thus all collections of horses have same color. $\square$

Where is the bug?

The Well Ordering Principle and Induction

Well Ordering Principle

Every nonempty set of non-negative integers has a smallest element.

Induction

Let $P(n)$ be a property of non-negative integers. If

- $P(0)$ is true (Base case)
- for all $k \geq 0$, $P(k) \implies P(k+1)$ (Induction step)

  then $P(n)$ is true for all $n \in \mathbb{N}$.

Theorem: Well-ordering principle iff Induction

Direct application of WOP to prove theorems

– Proving one part of the fundamental theorem of arithmetic.

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- Call this least number n. First, n can't be a prime (why?).
- So $n = a \cdot b$, where $n > a, b > 1$.

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- So $n = a \cdot b$, where $n > a, b > 1$.
- Since a and b are smaller than the smallest number in S, they can be written as product of primes.
- Let $a = p_1 \ldots p_k$ and $b = q_1 \cdots q_l$. But then $n = p_1 \cdots p_k \cdot q_1 \cdots q_l$, which is a contradiction. $\square$

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Qn: How do you show uniqueness?

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Proof by induction:

- Base case: $n = 2$, done.
- Assume induction hypothesis for $n = k$, i.e., $k = p_1 \cdots p_n$.
- Consider $n = k + 1$.
- If $k + 1$ is a prime, then done. Else $k + 1 = p \cdot q, p, q > 1$.

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- If $k + 1$ is a prime, then done. Else $k + 1 = p \cdot q, p, q > 1$.
- But now it may be that $p, q \neq k$, so we can't use induction hypothesis.
- Let us strengthen our induction hypothesis. That is...

Direct proof by induction

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Theorem: Any integer > 1 is a prime or can be written as a product of primes

Proof by induction:

- Base case: $n = 2$, done.
- Assume strong induction hypothesis, i.e., for all $1 \leq r \leq k$, $k = p_1 \cdots p_m$.
- Consider $n = k + 1$.
- If $k + 1$ is a prime, then done. Else $k + 1 = p \cdot q, p, q > 1$.

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- Consider $n = k + 1$.
- If $k + 1$ is a prime, then done. Else $k + 1 = p \cdot q, p, q > 1$.
- By the stronger hypothesis, we can write $p = p_1 \ldots p_k$ and $q = q_1 \cdots q_l$.
- Therefore $k + 1 = p_1 \cdots p_k \cdot q_1 \cdots q_k$.
- Thus, the statement holds for all $n > 1$. $\square$

Strong Induction

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Let $P(n)$ be a property of non-negative integers. If

- $P(0)$ is true (Base case)
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Theorem: Strong Induction iff Induction iff WOP

Another proof by induction

There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from the other cars on its way around.

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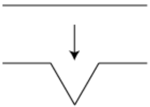
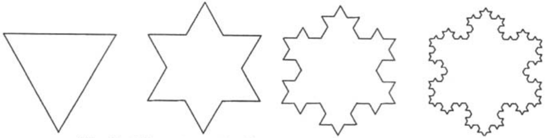
- Car A starts and when it reaches B it takes all fuel of B. Now, if we remove B from track, we have k cars and among them enough fuel to complete lap.
- So by induction hypothesis, there is a car that can complete the lap. On track with $k + 1$ cars, from A to B there is enough gas (from A) and for remaining road, the car has same amt of gas as in k car case. $\square$

Pop Quiz

Define a sequence of shapes as follows:

- $K(0)$ is an equilateral triangle.
- For $n > 0$, $K(n)$ is formed by replacing each line segment of $K(n-1)$ by the shape shown in bottom of left fig., such that the central vertex points outwards.





The first four stages in the construction of the Koch snowflake

1. Show that the no. of line segments in $K(n)$ is $4^n \cdot 3$.
2. If the original equilateral triangle $K(0)$ has side length 1,
   2.1 what is the perimeter of $K(n)$ as a function of n?
   2.2 what is the area of $K(n)$? (Prove both by induction)
3. *Prove using WOP or otherwise that the equation $4a^3 + 2b^3 = c^3$ does not have any solutions over $\mathbb{N}$.