CS 207: Discrete Structures

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July 28, 2015
Lecture 05 – Comparing infinite sets via functions

Chapter 2: Basic mathematical structures

Sets and Functions

- Definition of a set, Russel's paradox, axioms of ZFC.
- Infinite sets and using functions to compare them.

Hilbert's hotel

- Suppose there is a hotel with infinitely many rooms.
- And suppose they are all full (like in IIT guest house).

1. Can you accomodate 1 or finitely many more guests, by shifting around the existing guests?
2. What if infinitely many more guests arrive?
3. What if infinitely many more trains with infinitely many more guests arrive and no room should be empty? (H.W)

Functions

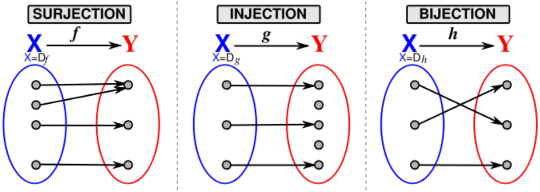
Definition

Let A, B be two sets. A function f from A to B is an assignment of exactly one element of B to each element of A. i.e., $f : A \to B$ is a subset R of $A \times B$ such that

(i) $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

Comparing (finite and infinite) sets



- Surjective or onto: $f : A \to B$ is surjective if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$.
- Injective or 1-1: $f : A \to B$ is injective if $\forall x, y \in A$, if $f(x) = f(y)$, then $x = y$.
- Bijective or 1-1 correspondence: A function is bijective if it is surjective and injective.

If f is a bijection, then its inverse function exists and $f \circ f^{-1} = f^{-1} \circ f = \mathrm{id}$

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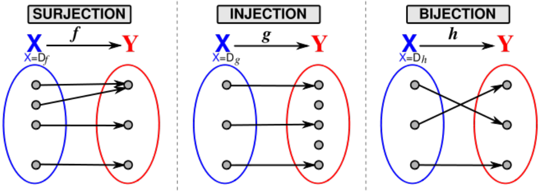

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1. $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = x^2$.
2. $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that $f(x) = x^2$.

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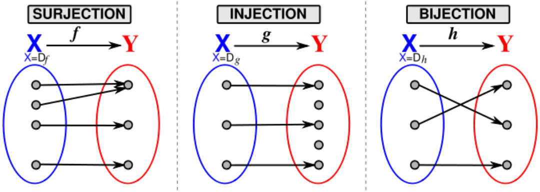

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Properties of finite and infinite sets

Relative notion of "size"

Thus, two finite/infinite sets have the same "size" iff there is a bijection between them.

Properties of finite and infinite sets

Similarities between finite and infinite sets

- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, then $\exists$ **bij** from B to A.
- $\exists$ **surj** from A to B and $\exists$ **surj** B to A, implies $\exists$ **bij** from A to B.

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Proof: essentially Hilbert's hotel but be careful...

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There is a bijection from $\mathbb{Z}$ to $\mathbb{N}$.

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$$f(x) = \begin{cases} -2x & \text{if } x \leq 0 \\ 2x - 1 & \text{else} \end{cases}$$

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There is a bijection from $\mathbb{Z}$ to $\mathbb{N}$.

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Some questions...

- Is there a bijection between $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$?
- Is there a bijection between $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$?
- Is there a bijection from $\mathbb{Q}$ to $\mathbb{N}$?
- Is there a bijection from the set of all subsets of $\mathbb{N}$ to $\mathbb{N}$?
- Is there a bijection from $\mathbb{R}$ to $\mathbb{N}$?