EE101: RL, RC, RLC Circuit Analysis

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References: L. Bobrow

Capacitors and Inductors

- Voltage-current has differential/integral (w.r.t time) relationship
  - Currents and voltages that vary with time
- Elements that store energy
- Resistor + L/C → First-order circuit
  - First order linear diff eqn
  - Natural response (no source); Complete response (with independent source)
- Two energy storage elements → Second order circuit
  - Second order, linear diff eqns
  - Natural and complete response

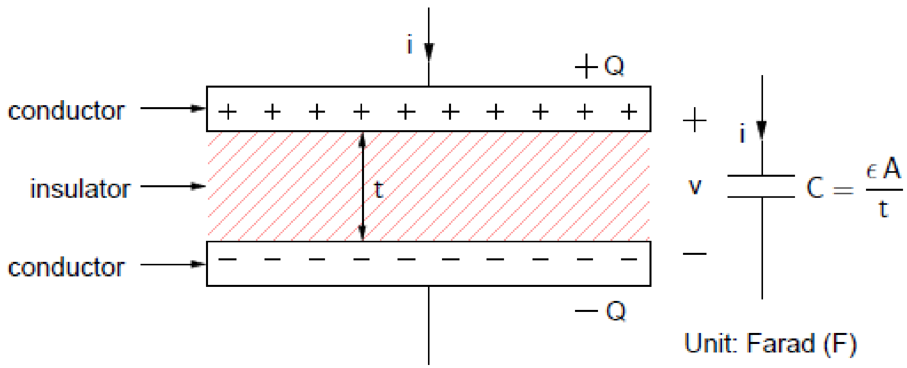
Capacitors

$$i(t) = C\frac{dv}{dt}$$

$$v(t) = \frac{1}{C}\int i(t)\,dt$$

$$p(t) = v(t) \times i(t)$$

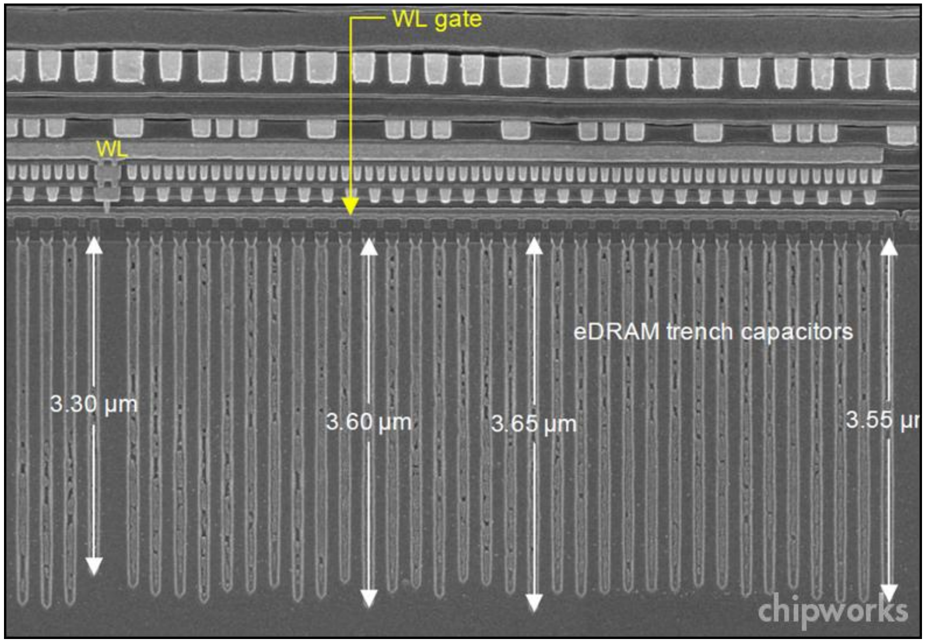
$$W(t) = \int p(t)\,dt$$

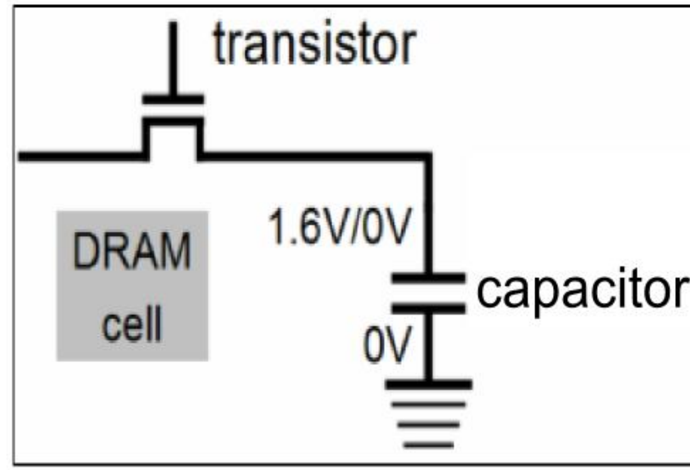


$$\begin{aligned} W(t) &= \int p(t)\,dt \\ &= C\int v\frac{dv}{dt}\,dt \\ &= C\int v\,dv \\ &= \frac{1}{2}C\,v^2 \end{aligned}$$

- Available as discrete components as well as in integrated circuits
- Various shapes and sizes, can vary from pF to 10’s of uF
  - glass, ceramic, plastic film, air, paper, mica, etc. as dielectrics
- For constant V (dc condition), i=0 → open circuit
- Extensive use in
  - Memory applications, power stabilization, analog filters, resonant circuits for frequency tuning etc.

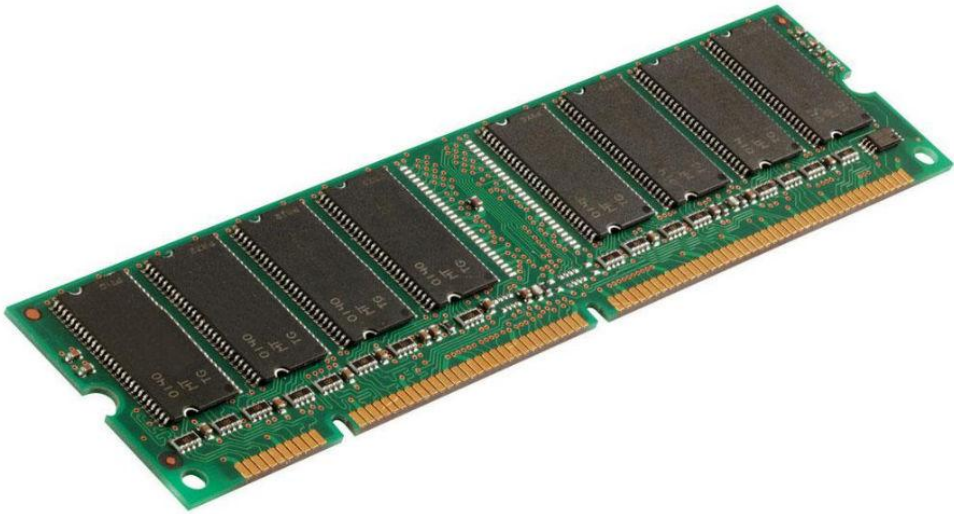
DRAM capacitors in Si





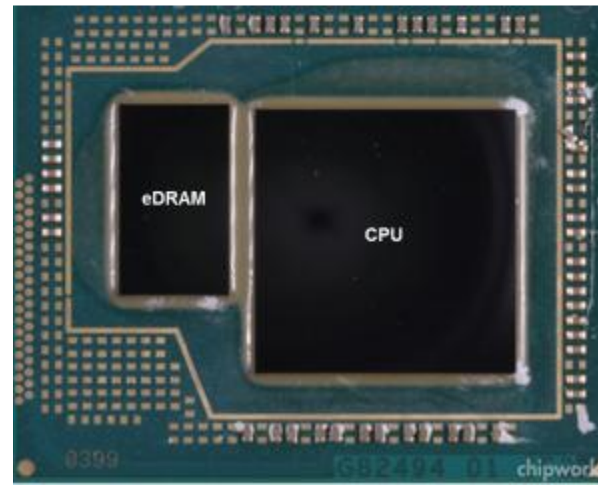
- Charged and Discharged Capacitor → stores 1 or 0
- Charge can leak away → Volatile memory

Embedded or non-embedded DRAM



Off-chip (non-embedded)

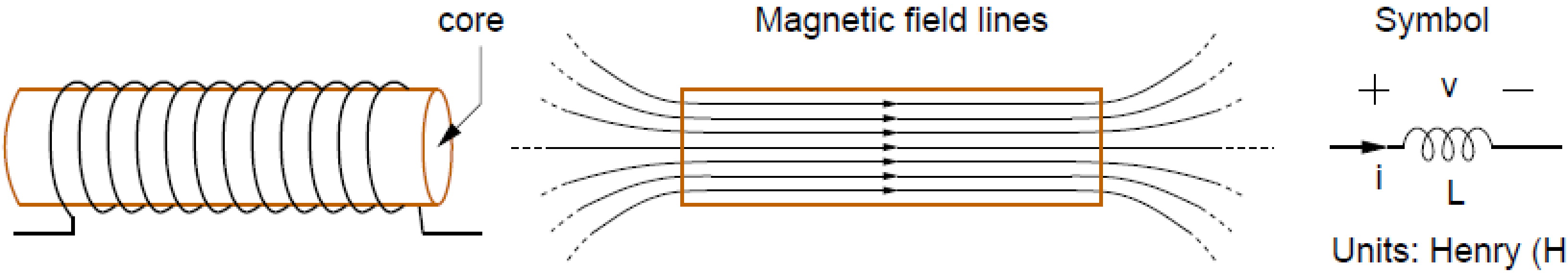
Intel Haswell 22nm CPU



On-chip (embedded)

- State-of-the-art DRAM capacitor technology is integrated with the CPU technology

Inductors



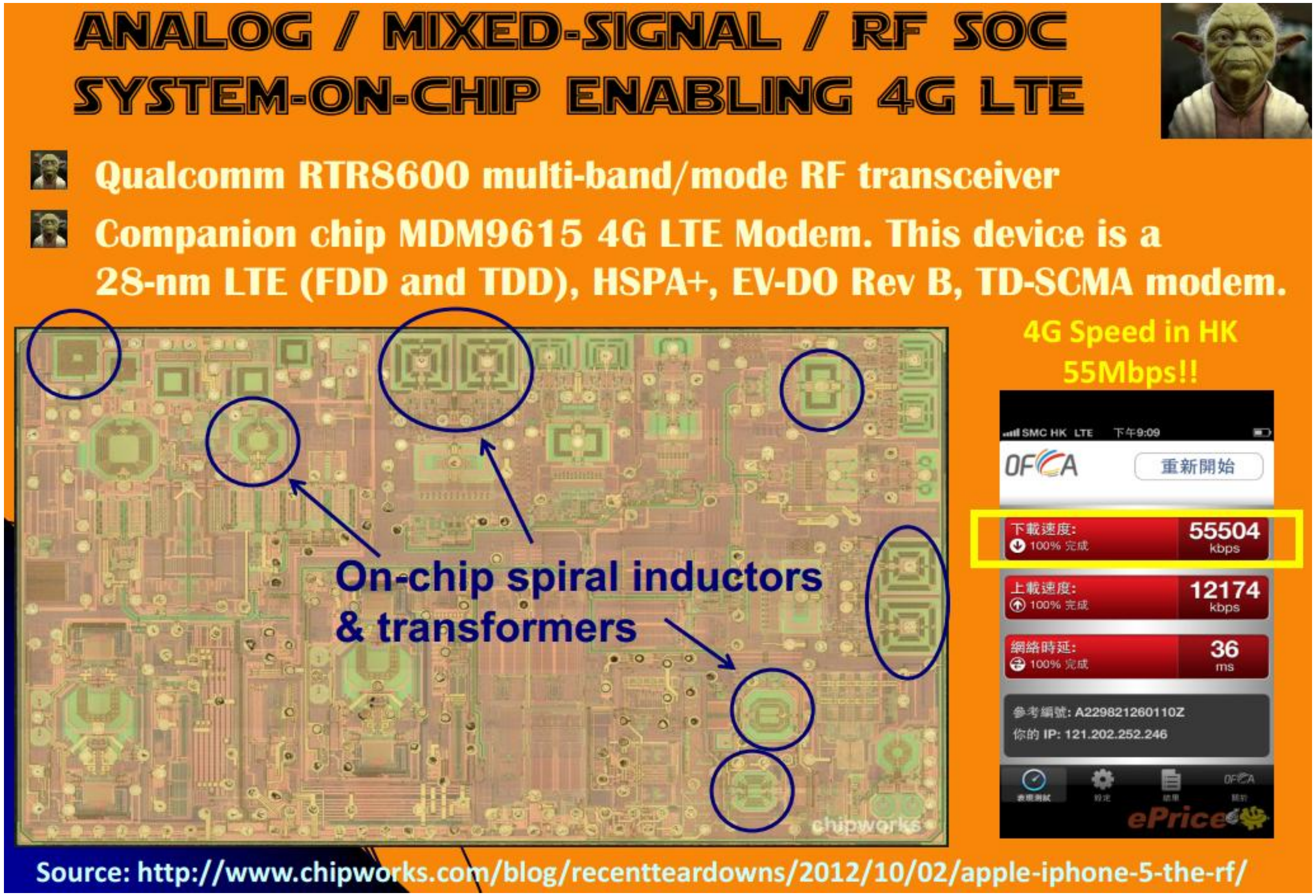
$$v = L\frac{di}{dt} \qquad L = \mu\, N^2\, A = \mu_r \mu_0\, N^2\, A$$

- u_r can vary from 5000 (Fe) to 10^6 (supermalloy: Ni- 79%, Mo- 5%, Fe)
- For constant I (dc condition), V=0 → short circuit
- Extensive use in
  - Analog/RF electronics, power supplies and systems, analog filters, resonant circuits etc.

Applications

- Large inductor used in a power station

Applications



- RF communication chip in iphone 5

Basic relationships

Capacitors

$$i(t) = C\frac{dv(t)}{dt}$$

$$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(t)dt$$

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$

$$w_C(t) = \frac{1}{2}Cv^2(t)$$

Inductors

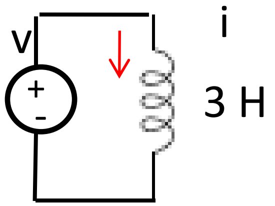
$$v(t) = L\frac{di(t)}{dt}$$

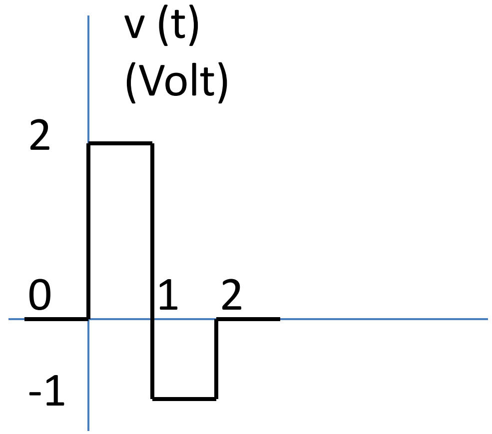
$$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(t)dt$$

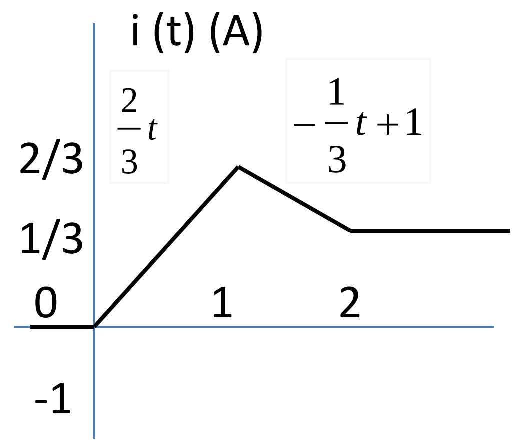
$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(t)dt$$

$$w_L(t) = \frac{1}{2}Li^2(t)$$

Simple RL circuit







- What is i(t)?

For t≤0 s, v(t)=0 V

$$i(t)=\frac{1}{L}\int_{-\infty}^{t} v(t)dt = 0A$$

For 1<t≤2 s, v(t)=-1 V

$$i(t)=i(t_0)+\frac{1}{L}\int_{t_0}^{t} v(t)dt = i(1)+\frac{1}{L}\int_{1}^{t} v(t)dt = \frac{2}{3}(1)+\frac{1}{3}\int_{1}^{t}(-1)dt = -\frac{1}{3}t+1$$

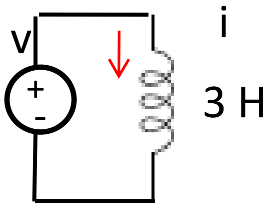
For 0<t≤1 s, v(t)=2 V

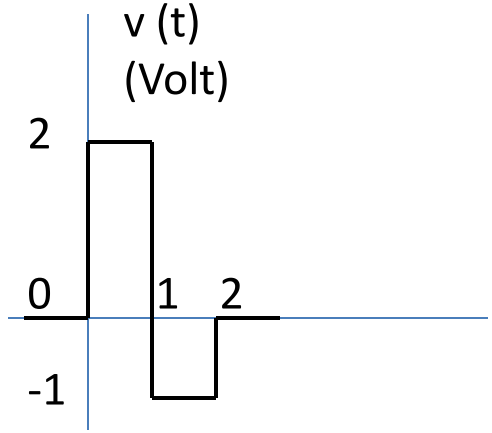
$$i(t)=i(t_0)+\frac{1}{L}\int_{t_0}^{t} v(t)dt = i(0)+\frac{1}{L}\int_{0}^{t} v(t)dt = 0+\frac{1}{3}\int_{0}^{t} 2dt = \frac{2}{3}t$$

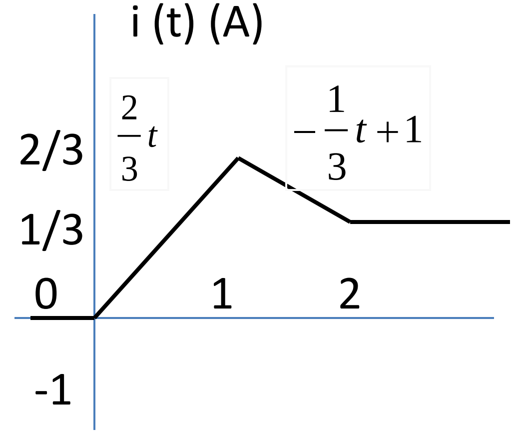
Similarly for t>2 s

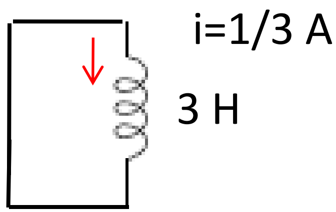
i(t)=1/3 A

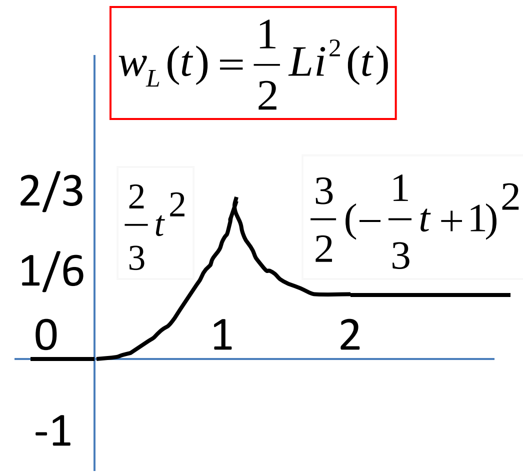
Simple RL circuit





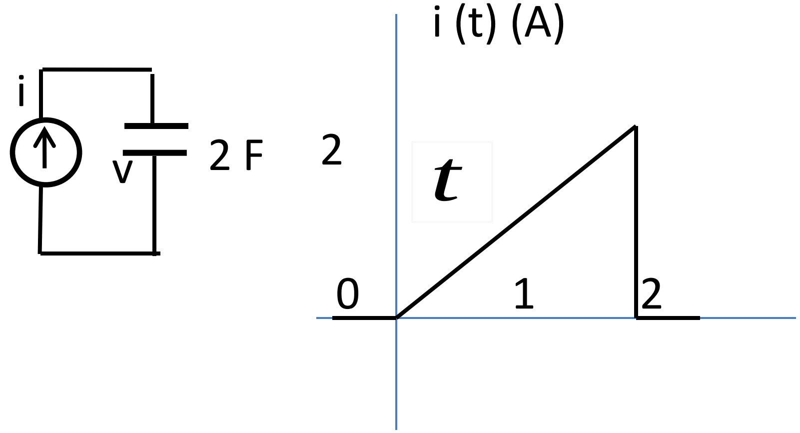






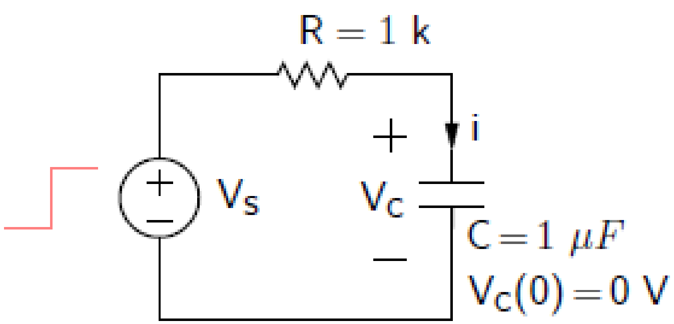
- For t>2 s
- Inductor has 0 volts across it but finite current
- Voltage source and inductor are ideal and energy stays constant for t>2 s

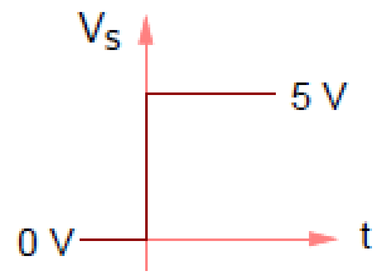
Simple RC circuit- HW



- What is v(t) and $w_C(t)$?
- You will see that even when the current is zero there will be a non-zero voltage across the capacitor

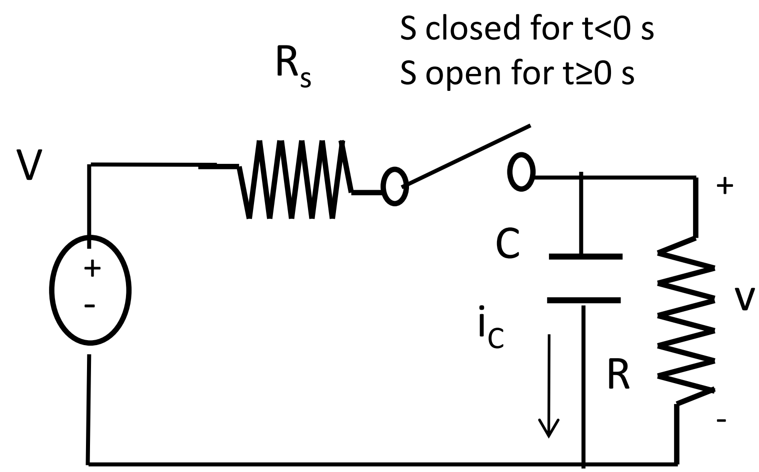
Can capacitor V change suddenly?





- How fast can V_c change?
- Suppose V_c changes by 1 V in 1 μs
  - $dV_C/dt = 10^6$ V/s
  - i= C dV_C/dt=1 A
  - Voltage drop across R = 1000V!
  - Violates KVL
  - Hence $V_c(0^+)=V_c(0^-)$ → A capacitor does not allow abrupt changes in the ckt if there is a finite resistance in the ckt
  - Similarly an inductor does not allow abrupt changes in i_L

First order RC circuit- natural response



$$v(t)=\frac{RV}{R+R_s} \qquad \text{t<0 s}$$

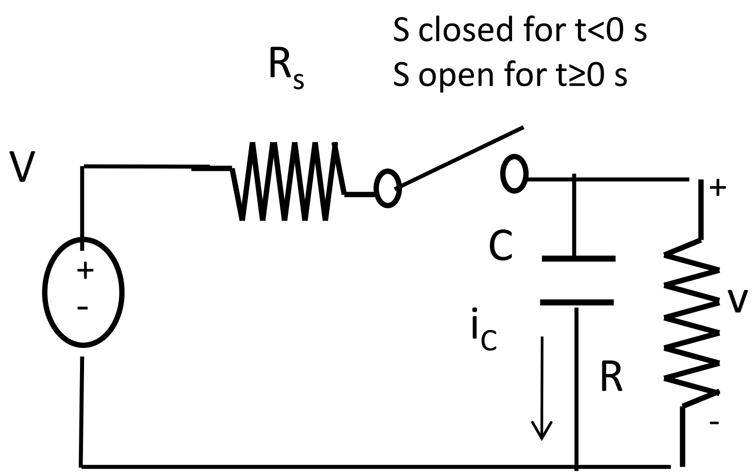
$$v(0)=\frac{RV}{R+R_s} \qquad \text{t=0 s}$$

$$i_C+i_R=0$$

$$C\frac{dv}{dt}+\frac{v}{R}=0 \qquad \text{t>0 s}$$

- For t<0 s, this is a dc circuit
- Capacitor behaves as open circuit
- At t=0, S opens, voltage cannot change instantaneously
- First order (v(t), dv(t)/dt) differential eqn → first order circuit
- Homogenous linear differential eqn (every non-zero term is of first degree)
  - Solve for v(t) with the initial condition v(0)

First order RC ckt- natural response



$$C\frac{dv}{dt}+\frac{v}{R}=0 \qquad t\geq 0\text{ s}$$

$$v(0)=\frac{RV}{R+R_s}$$

$$\boxed{v(t)=v(0)e^{-\frac{t}{RC}} \qquad t\geq 0\text{ s}}$$

$$i_C(t)=-i_R(t)=-\frac{v(0)}{R}e^{-\frac{t}{RC}}$$

$$w_C(0)=\frac{1}{2}Cv^2(0)$$

$$w_R=\int_0^{\infty} i_R^2(t)R\,dt=\frac{1}{2}Cv^2(0)=w_C(0)$$

Energy stored in capacitor is dissipated by the resistor

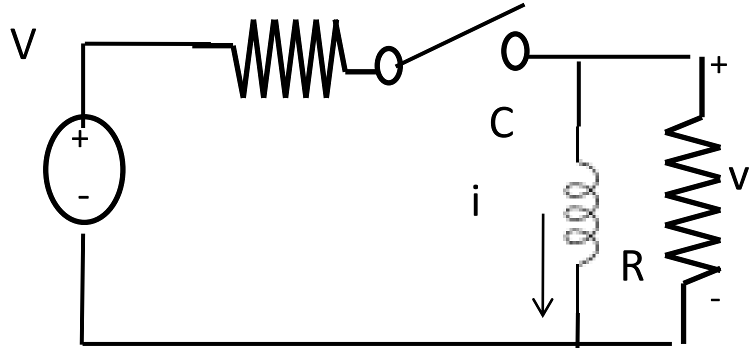
In general,

$$\frac{dx(t)}{dt}+ax=0$$

$$\boxed{x(t)=x(0)e^{-at} \qquad t\geq 0\text{ s}}$$

Only the initial condition v(0) has an effect on the circuit for t≥0 s→ Natural response

First order RL circuit- natural response



$$L\frac{di}{dt} + Ri = 0$$ t≥0 s $$i(0) = \frac{V}{R_s}$$

$$\boxed{i(t) = i(0)e^{-\frac{Rt}{L}} \quad \text{t≥0 s}}$$

$$v(t) = -Ri(t) = -Ri(0)e^{-\frac{Rt}{L}}$$

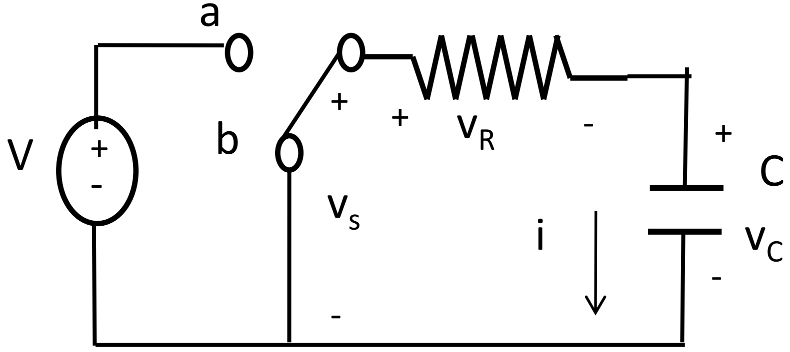
$$w_L(0) = \frac{1}{2}Li^2(0)$$

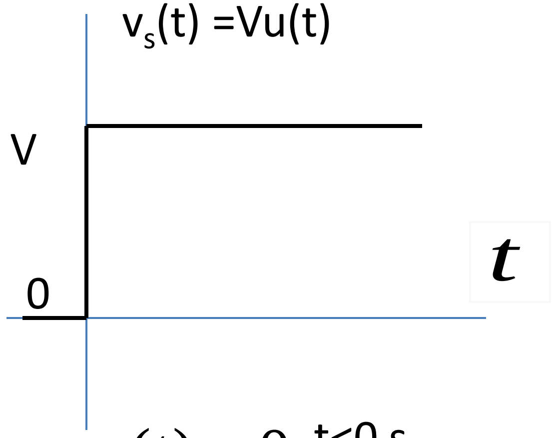
$$w_R = \int_0^\infty i^2(t)R\,dt = \frac{1}{2}Li^2(0) = w_L(0)$$

- Similar to the first-order RC ckt

First order RC circuit- complete response

S moved from b to a at t=0





$u(t) = 0$ t<0 s

$u(t) = 1$ t≥0 s

By KCL
$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{Vu(t)}{RC}$$

$v_c(t)$=0 for t<0

$$\boxed{\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{V}{RC} \quad t>0}$$

Non-homogenous first order diff eqn.

$$\frac{dx(t)}{dt} + ax(t) = f(t)$$

f(t) → forcing function

$$x(t) = e^{-at}\int e^{at} f(t)dt + Ae^{-at} = x_f(t) + x_n(t)$$

Forced response
Steady-state response

Natural response
Transient response

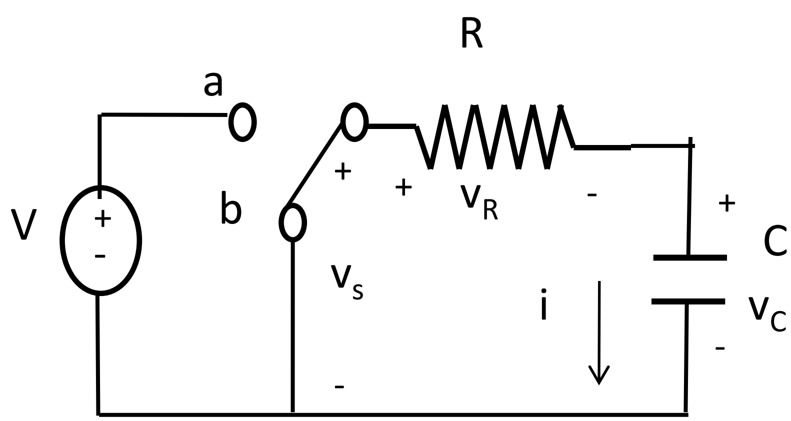
If f(t)=b (constant)

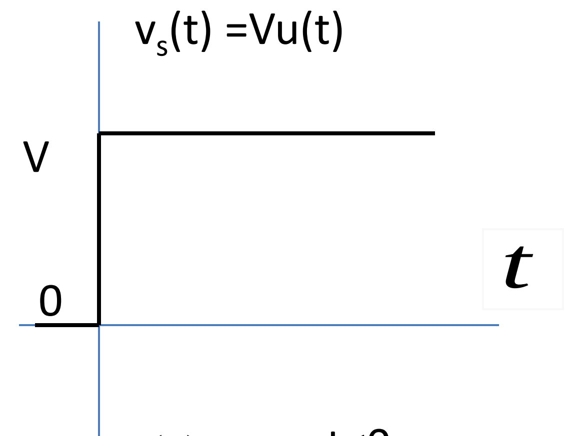
$$x_f(t) = \frac{b}{a}$$

$$x(t) = \frac{b}{a} + Ae^{-at}$$

First order RC circuit- complete response

S moved from b to a at t=0





$$u(t) = 0 \quad t<0 \text{ s}$$

$$u(t) = 1 \quad t \geq 0 \text{ s}$$

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{V}{RC} \qquad t>0$$

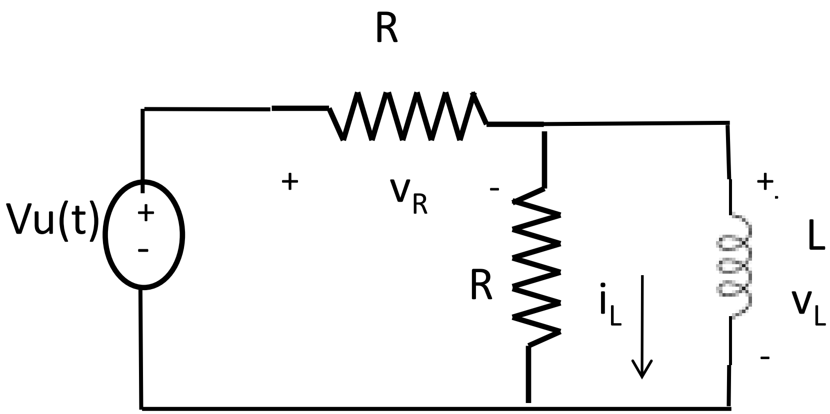
$$v_C(t) = V + Ae^{-\frac{t}{RC}}$$

$v_C(0)$=0 V → $A = -V$

$$\boxed{v_C(t) = V(1 - e^{-\frac{t}{RC}})}$$

$$\boxed{\begin{aligned} v_C(t) &= V(1 - e^{-\frac{t}{RC}})u(t) \\ i_C(t) &= \frac{V}{R} e^{-\frac{t}{RC}} u(t) \end{aligned}}$$

First order RL circuit with forcing function - HW

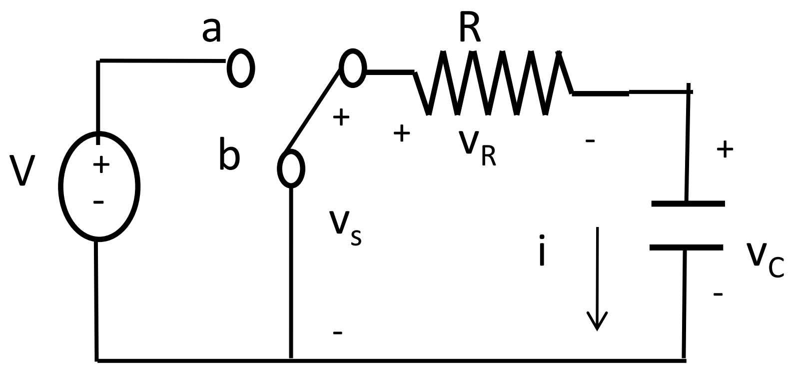


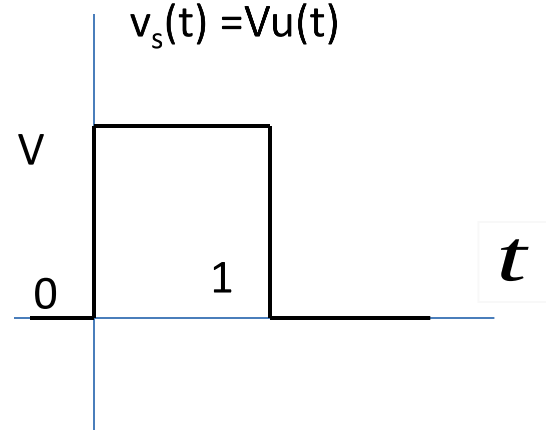
If a parallel R is connected then find out the new $v_L(t)$ and $i_L(t)$.

- Show that

$$\boxed{\begin{aligned} v_L(t) &= Ve^{-\frac{Rt}{L}}u(t) \\ i_L(t) &= \frac{V}{R}(1-e^{-\frac{Rt}{L}})u(t) \end{aligned}}$$

Linearity and time-invariance





For 0≤t<1 s, complete response

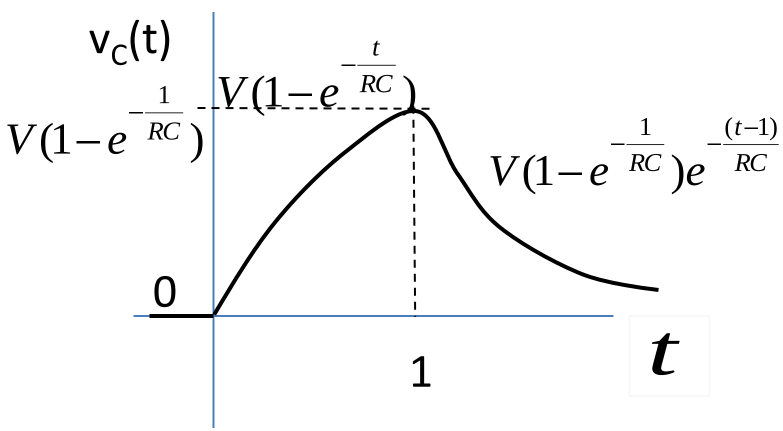
$$\boxed{v_C(t) = V(1 - e^{-\frac{t}{RC}})}$$

For t>1 s, natural response

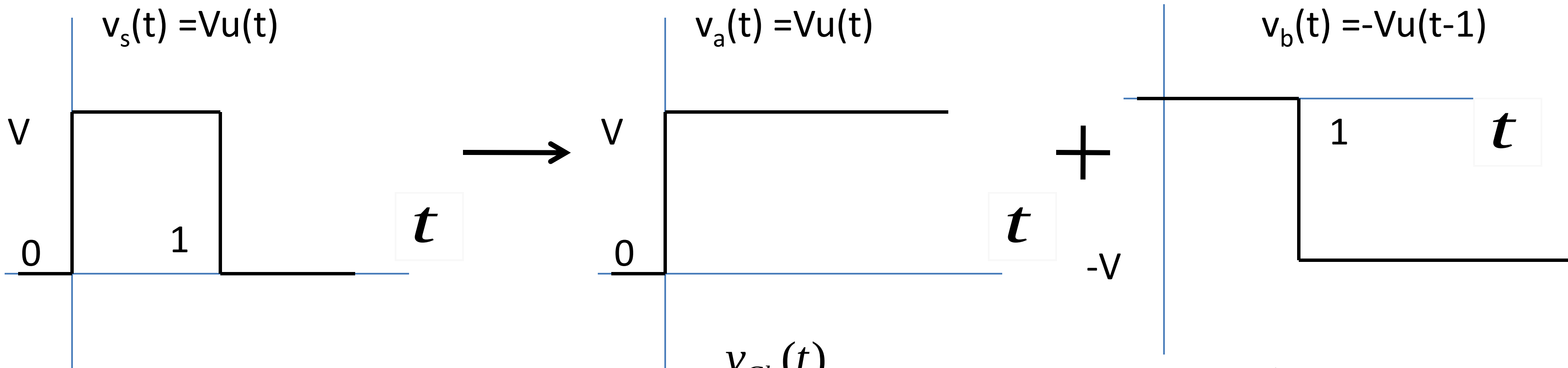
$$v_C(1) = V(1 - e^{-\frac{1}{RC}})$$

$$v_C(t) = v(1)e^{-\frac{(t-1)}{RC}}$$

$$\boxed{v_C(t) = V(1 - e^{-\frac{1}{RC}})e^{-\frac{(t-1)}{RC}}}$$



Linearity and time-invariance



$$v_s(t) = v_a(t) + v_b(t)$$

$$v_C(t) = v_{Ca}(t) + v_{Cb}(t)$$

$$v_C(t) = V(1 - e^{-\frac{t}{RC}})u(t) + v_{Cb}(t)$$

$$\boxed{v_C(t) = V(1 - e^{-\frac{t}{RC}})u(t) - V(1 - e^{-\frac{t-1}{RC}})u(t-1)}$$

$v_{Cb}(t)$

Response to –Vu(t) is $-V(1 - e^{-\frac{t}{RC}})u(t)$

Response to –Vu(t-1) is $-V(1 - e^{-\frac{t-1}{RC}})u(t-1)$

- This is due to “time invariance” → excitation delayed by time t → response delayed by the same time t
- Differential equation has constant coefficients

Solution Check

$$v_C(t) = 0$$ t<0 s

$$v_C(t) = V(1 - e^{-\frac{t}{RC}})$$ 0≤t<1 s

Complete and natural response

$$v_C(t) = V(1 - e^{-\frac{1}{RC}})e^{-\frac{(t-1)}{RC}}$$ t≥1 s

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$$v_C(t) = V(1 - e^{-\frac{t}{RC}})u(t) - V(1 - e^{-\frac{t-1}{RC}})u(t-1)$$

Linearity and time-invariance

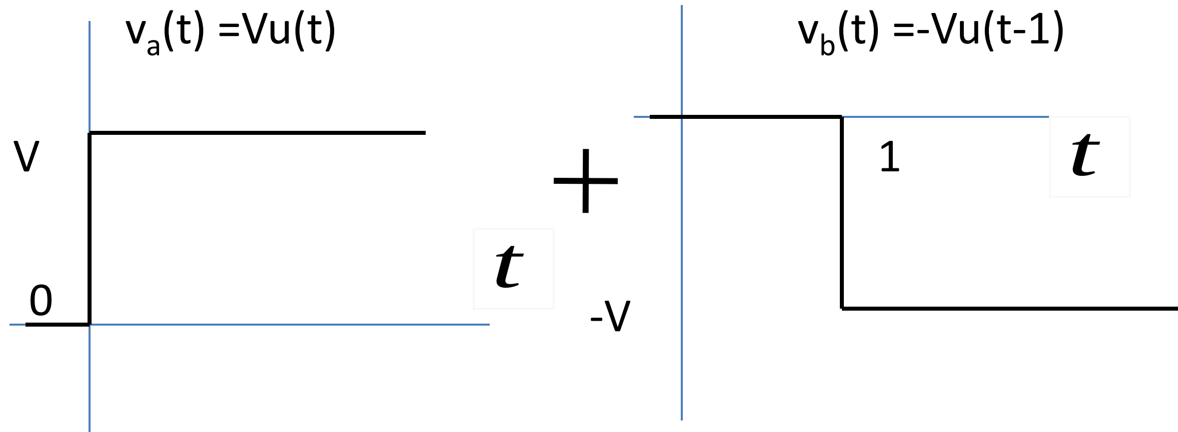
$$v_C(t) = 0$$ t<0 s

$$v_C(t) = V(1 - e^{-\frac{t}{RC}})$$ 0≤t<1 s

$$v_C(t) = V(1 - e^{-\frac{t}{RC}}) - V(1 - e^{-\frac{t-1}{RC}}) = Ve^{-\frac{(t-1)}{RC}}(1 - e^{-\frac{1}{RC}})$$ t≥1 s

Current

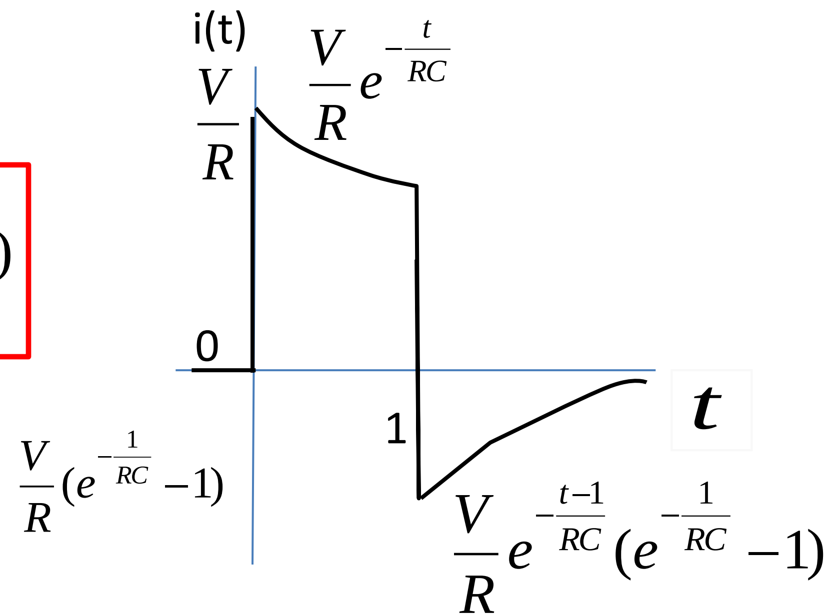
$$v_s(t) = v_a(t) + v_b(t)$$



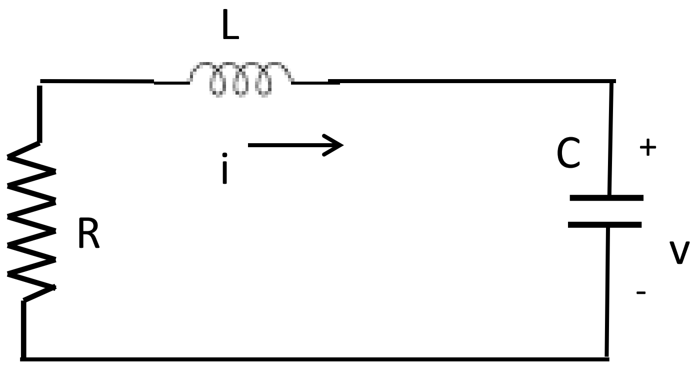
$$i_a(t) = \frac{V}{R} e^{-\frac{t}{RC}} u(t)$$

$$i_b(t) = -\frac{V}{R} e^{-\frac{t-1}{RC}} u(t-1)$$

$$\boxed{i(t) = i_a(t) + i_b(t) = \frac{V}{R} e^{-\frac{t}{RC}} u(t) - \frac{V}{R} e^{-\frac{t-1}{RC}} u(t-1)}$$



Series RLC circuit: Natural response



$$v + Ri + L\frac{di}{dt} = 0$$

$$v + RC\frac{dv}{dt} + LC\frac{d^2v}{dt^2} = 0$$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$$\boxed{\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \omega_n^2 v = 0}$$

$Assume$

$$v(t) = Ae^{st}$$

$$s^2 + 2\alpha s + \omega_n^2 = 0$$

$$\boxed{\begin{aligned} s_1 &= -\alpha - \sqrt{\alpha^2 - \omega_n^2} \\ s_2 &= -\alpha + \sqrt{\alpha^2 - \omega_n^2} \\ v(t) &= A_1 e^{s_1 t} + A_2 e^{s_2 t} \end{aligned}}$$

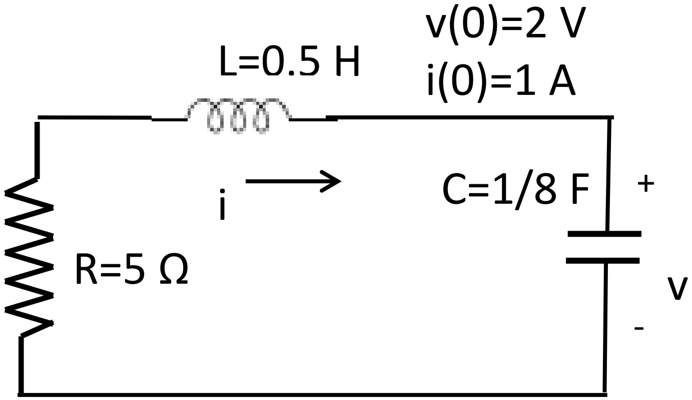
A_1 and A_2 are obtained from initial conditions

Three cases are possible:

1. $\alpha > \omega_n$ s_1 and s_2 are real → **overdamped case**

2. $\alpha < \omega_n$ s_1 and s_2 are complex →
   $$s_1 = -\alpha - \sqrt{-(\omega_n^2 - \alpha^2)} = -\alpha - j\omega_d$$
   $$s_2 = -\alpha + \sqrt{-(\omega_n^2 - \alpha^2)} = -\alpha + j\omega_d$$
   underdamped case

3. $\alpha = \omega_n$ $s_1 = s_2 = \alpha$ → **critically damped case**



Case 1: Overdamped case

$$\boxed{\alpha = \frac{R}{2L} = 5 > \omega_n = \frac{1}{\sqrt{LC}} = 4}$$

$$s_1 = -\alpha - \sqrt{\alpha^2 - \omega_n^2} = -8$$

$$s_2 = -\alpha + \sqrt{\alpha^2 - \omega_n^2} = -2$$

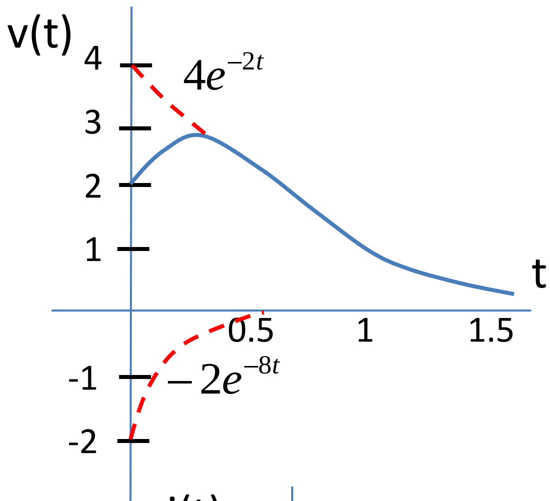
$$v(t) = A_1 e^{-8t} + A_2 e^{-2t}$$

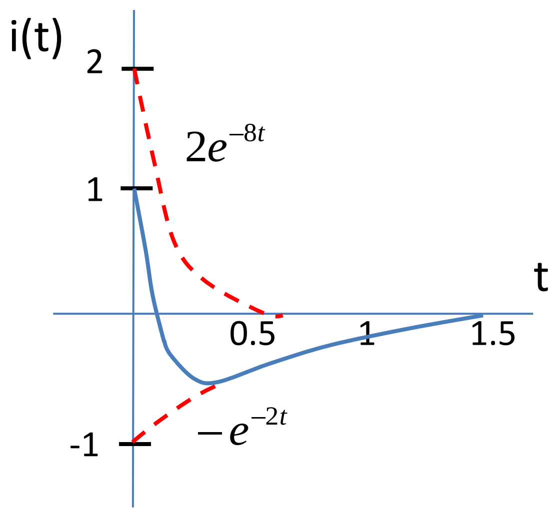
$$i(t) = C\frac{dV}{dt} = -A_1 e^{-8t} - \frac{A_2}{4} e^{-2t}$$

$$i(0) = 1A$$

$$v(0) = 2V$$

$$\boxed{\begin{aligned} v(t) &= -2e^{-8t} + 4e^{-2t} \\ i(t) &= 2e^{-8t} - e^{-2t} \end{aligned}}$$





Case 2: Underdamped case

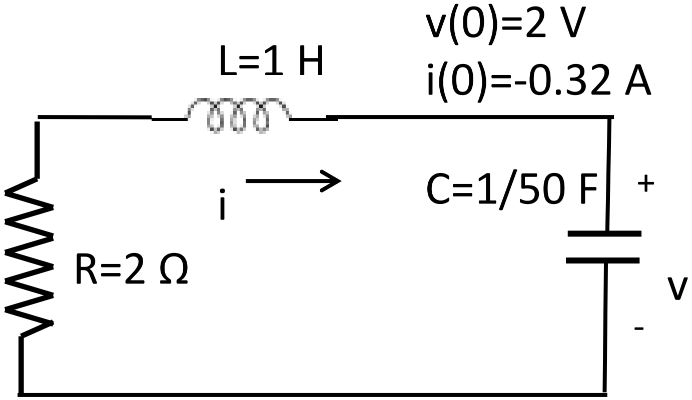
$$\boxed{\alpha = \frac{R}{2L} = 1 < \omega_n = \frac{1}{\sqrt{LC}} = \sqrt{50}}$$

$$s_1 = -\alpha - \sqrt{\alpha^2 - \omega_n^2} = -1 - j7$$

$$\omega_d = \sqrt{\omega_n{}^2 - \alpha^2} = 7$$

$$s_2 = -\alpha + \sqrt{\alpha^2 - \omega_n^2} = -1 + j7$$

$$v(t) = e^{-t}(A_1 e^{-j7t} + A_2 e^{j7t}) = e^{-t}(B_1 \cos 7t + B_2 \sin 7t)$$



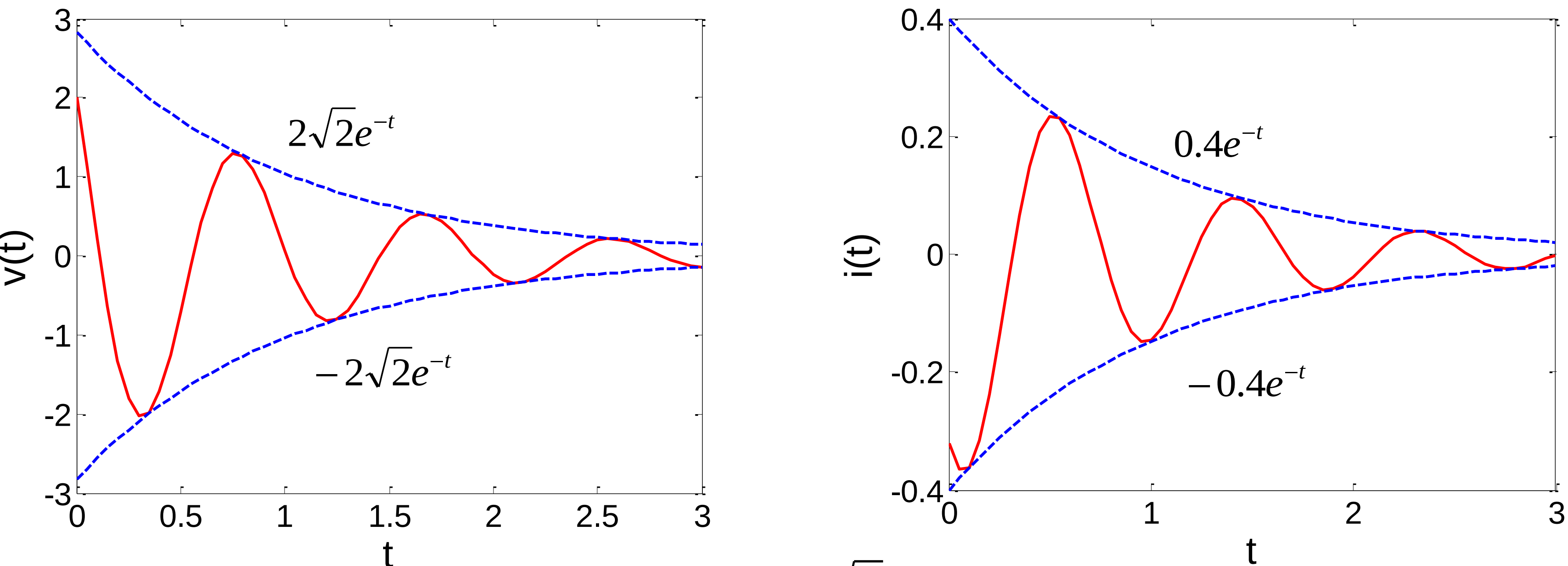
$$i(t) = C\frac{dV}{dt}$$

$$i(0) = -0.32A$$

$$v(0) = 2V$$

$$\boxed{\begin{aligned} v(t) &= 2e^{-t}(\cos 7t - \sin 7t) \\ i(t) &= e^{-t}(-0.32\cos 7t - 0.24\sin 7t) \end{aligned}}$$

Underdamped case

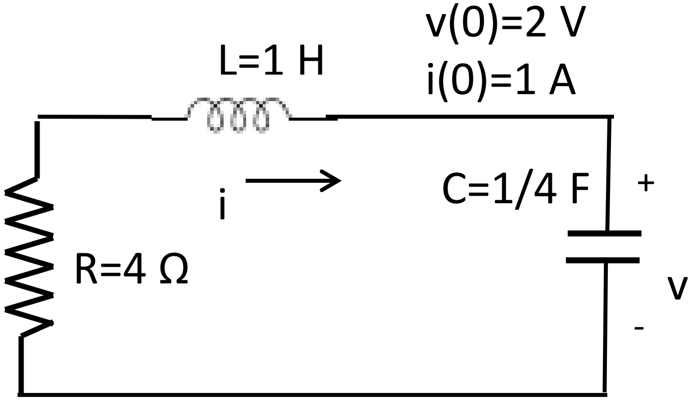


$$v(t) = 2e^{-t}(\cos 7t - \sin 7t) = 2\sqrt{2}e^{-t}\cos(7t + \pi/4)$$

$$i(t) = e^{-t}(-0.32\cos 7t - 0.24\sin 7t) = 0.4e^{-t}\cos(7t + 2.5)$$

- Product of a real exponent and a sinusoid→ damped sinusoid
- Energy transferred back and forth between L and C and dissipated in R
- α is the damping factor, smaller the α more the oscillations → more underdamped
- What if α=0? → perfect sinusoid→ oscillator
- ω_d is the damped natural frequency or damped frequency
- ω_n is the undamped natural frequency or undamped frequency

Case 3. Critically Damped Case



$$v + RC\frac{dv}{dt} + LC\frac{d^2v}{dt^2} = 0$$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \omega_n^2 v = 0$$

$$\alpha = \omega_n$$

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \alpha^2 v = 0$$

$$\boxed{v(t) = (A_1 t + A_2)e^{-\alpha t}}$$

$$\alpha = \frac{R}{2L} = 2 = \omega_n = \frac{1}{\sqrt{LC}} = 2$$

$$v(t) = A_1 t e^{-2t} + A_2 e^{-2t}$$

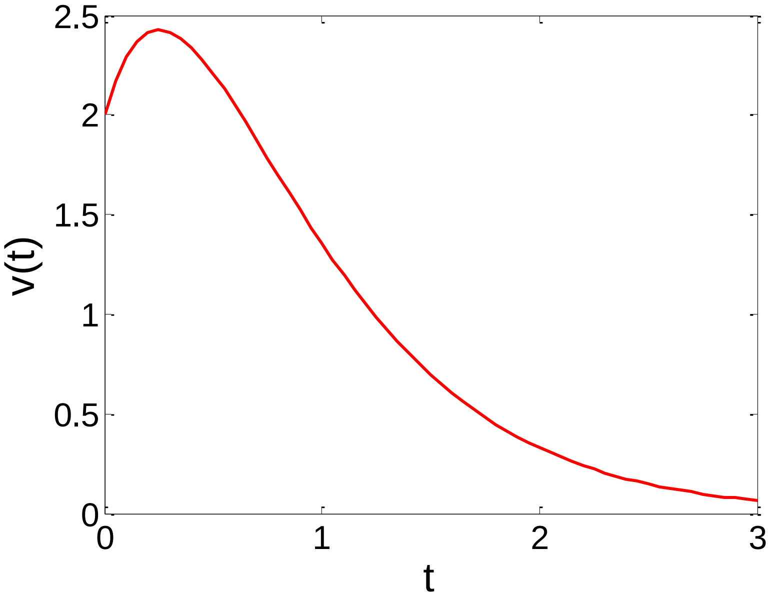
$$i(t) = C\frac{dV}{dt}$$

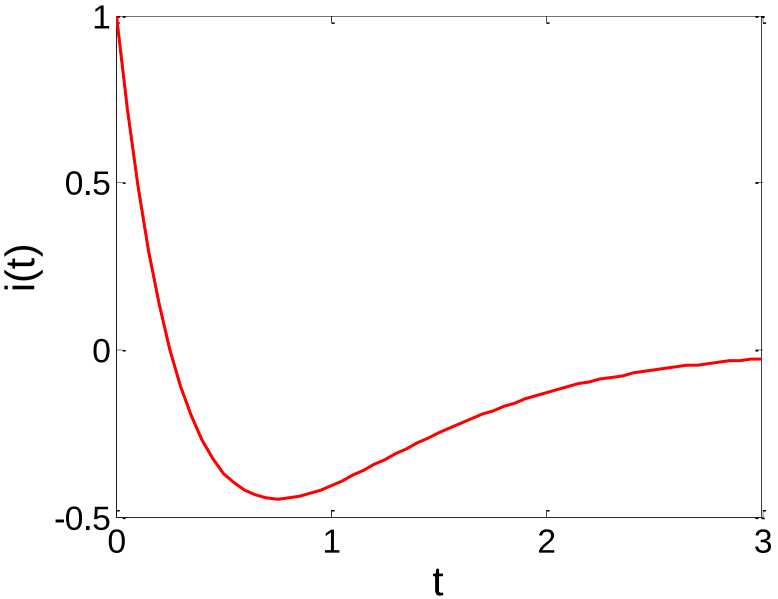
$$i(0) = 1A$$

$$v(0) = 2V$$

$$\boxed{\begin{aligned} v(t) &= (8t+2)e^{-2t} \\ i(t) &= (-4t+1)e^{-2t} \end{aligned}}$$

Case 3. Critically Damped Case

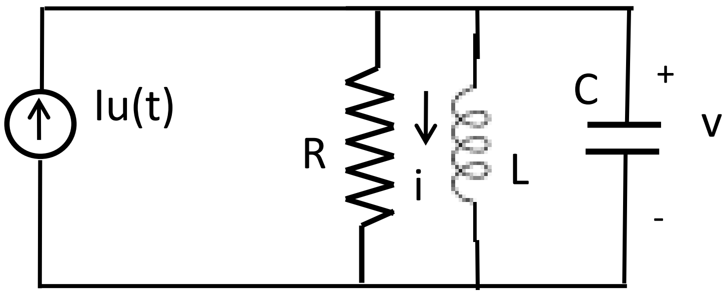




$$v(t) = (8t + 2)e^{-2t}$$

$$i(t) = (-4t + 1)e^{-2t}$$

RLC Circuits: Complete response



$$Iu(t) = \frac{v}{R} + i + C\frac{dv}{dt}$$

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{LC} i = \frac{I}{LC} u(t)$$

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{LC} i = \frac{I}{LC} \quad t>0s$$

$$i(t) = i_f(t) + i_n(t)$$

Forced response
Steady-state response

Natural response
Transient response

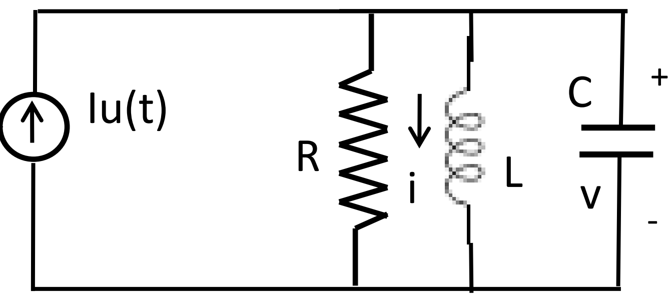
$$i_f(t) = K$$

$$K/LC = I/LC \Rightarrow K = I$$

$$\frac{d^2 i_n}{dt^2} + 2\alpha\frac{di_n}{dt} + \omega_n^2 i_n = 0$$

$$\alpha = \frac{1}{2RC}, \omega_n = \frac{1}{\sqrt{LC}}$$

RLC Circuits: Complete response



$$i(t) = i_f(t) + i_n(t)$$

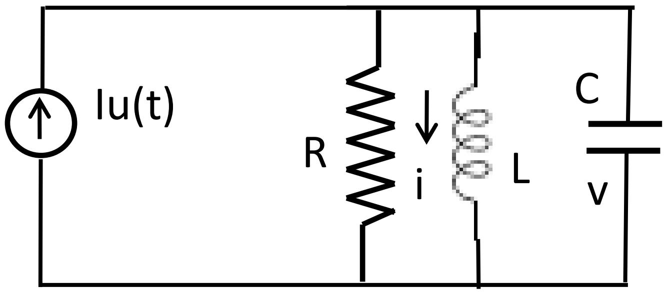
$$i(t) = I + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$ Overdamped

$$i(t) = I + e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$ Underdamped

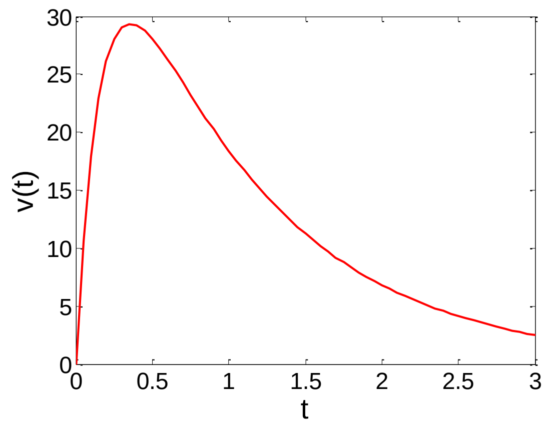
$$i(t) = I + A_1 t e^{-\alpha t} + A_2 e^{-\alpha t}$$ Critically damped

- A_1 and A_2 are determined from initial conditions
- When the current is switched on, i(0)=0 and v(0)=0 since these cannot change instantaneously
- After a long time, similar to dc→ inductor carries I amps, voltage across C=0 V.
- Shapes of i and v waveforms between initial and final values depends on the case

Case 1. Overdamped case



R=6 Ω, L=7 H, C=1/42 F, Iu(t)=6u(t)



For t>0s

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{LC} i = \frac{I}{LC}$$

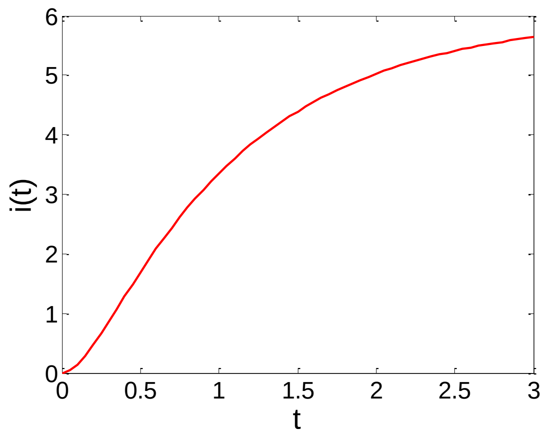
$$\frac{d^2 i}{dt^2} + 7\frac{di}{dt} + 6i = 36$$

$$\alpha = 7/2 > \omega_n = \sqrt{6}$$ Overdamped

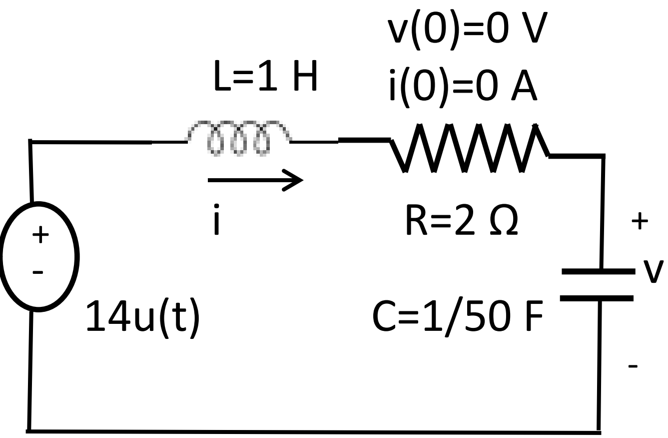
$$i(t) = 6 + A_1 e^{-6t} + A_2 e^{-t}$$

$$i(0) = v(0) = 0$$

$$\boxed{\begin{aligned} i(t) &= (6 + 1.2e^{-6t} - 7.2e^{-t})u(t) \\ v(t) &= (-50.4e^{-6t} + 50.4e^{-t})u(t) \end{aligned}}$$



Case 2. Underdamped Case



$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{LC}v=\frac{V}{LC}$$

$$\frac{d^2v}{dt^2}+2\frac{dv}{dt}+50v=50\times 14$$

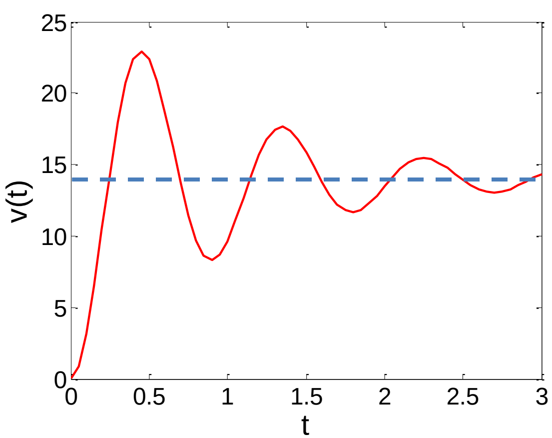
$$\alpha=1<\omega_n=\sqrt{50}$$

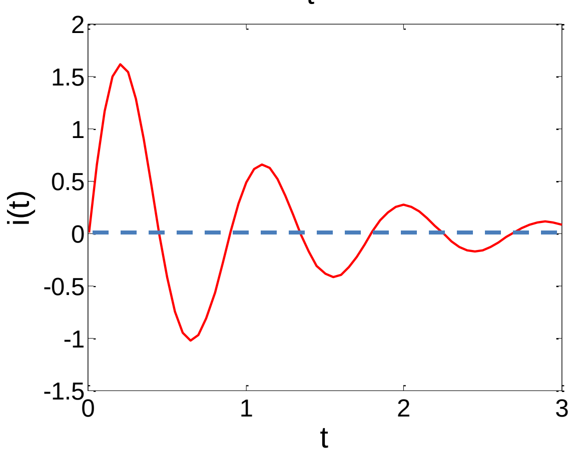
$$\omega_d=\sqrt{50-1}=7$$

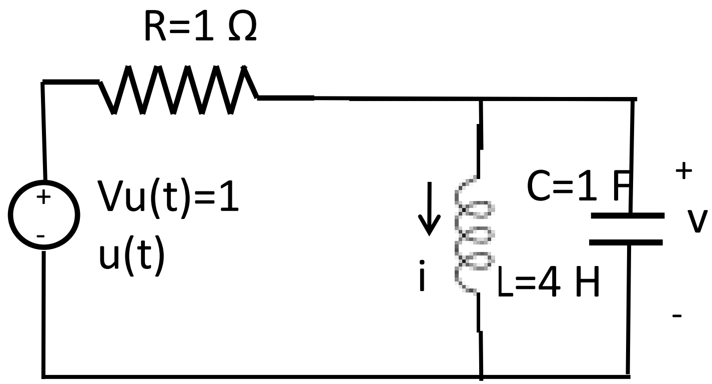
$$v(t)=14+e^{-t}(A_1\cos 7t+A_2\sin 7t)$$

$$i(0)=v(0)=0$$

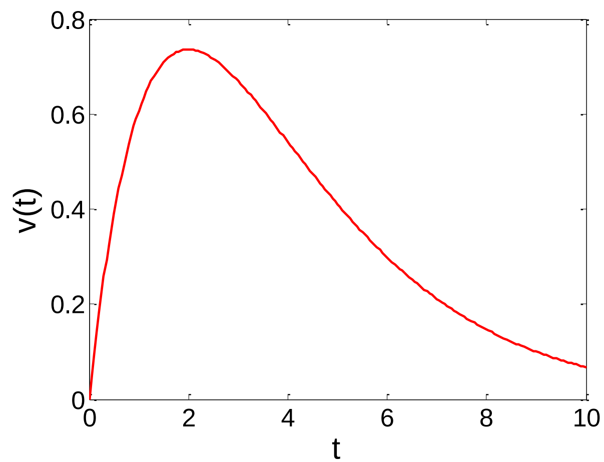
$$\boxed{\begin{aligned}v(t)&=14+e^{-t}(-14\cos 7t-2\sin 7t)=14-10\sqrt{2}e^{-t}\cos(7t-0.142)\\ i(t)&=Cdv(t)/dt=2e^{-t}\sin 7t\end{aligned}}$$

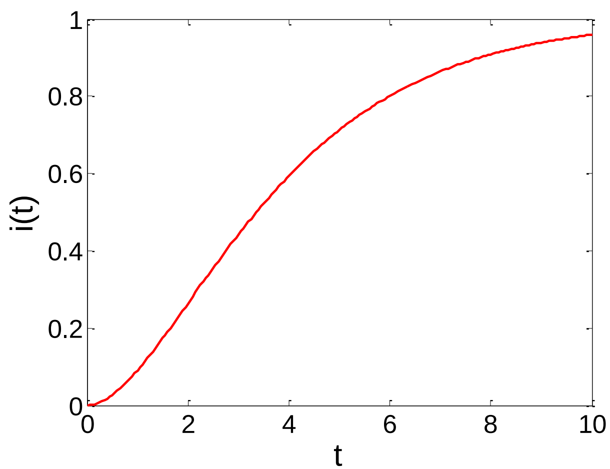






Case 3. Critically Damped





$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{LC} i = \frac{V}{RLC}$$

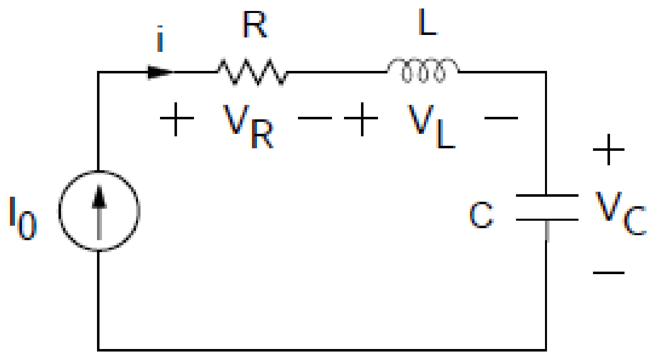
$$\alpha = 1/2 = \omega_n = 1/2$$

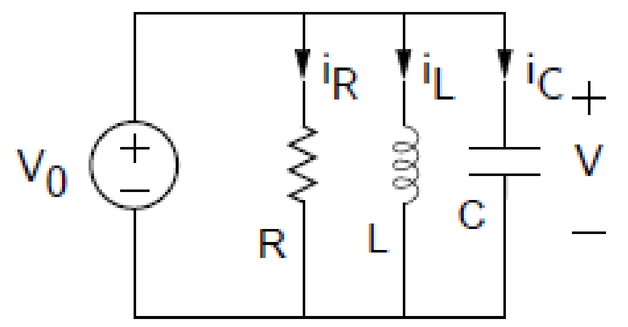
$$i(t) = 1 + A_1 t e^{-t/2} + A_2 e^{-t/2}$$

$$i(0) = v(0) = 0$$

$$\boxed{\begin{aligned} i(t) &= (1 - \frac{1}{2} t e^{-t/2} - e^{-t/2}) u(t) \\ v(t) &= L di(t)/dt = t e^{-t/2} u(t) \end{aligned}}$$

Trivial circuits





$$V_R = i\,R,\ V_L = L\,\frac{di}{dt},\ V_C = \frac{1}{C}\int i\,dt\,.$$

$$i_R = V/R,\ i_C = C\,\frac{dV}{dt},\ i_L = \frac{1}{L}\int V\,dt\,.$$