CS 207: Discrete Structures

Lecture 18 – Counting and Combinatorics
Stirling Numbers, Bell numbers and their properties

Aug 26 2015

Topics in Combinatorics

Basic counting techniques and applications

1. Basic counting techniques, double counting
2. Binomial coefficients and binomial theorem, permutations and combinations.
3. Estimating $n!$
4. Recurrence relations
5. Generating functions and their applications
6. Principle of Inclusion-Exclusion and its applications.
7. Special numbers: Fibonacci sequence, Catalan numbers

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Today

Applying all the above techniques: Stirling numbers

We start with an application of PIE

Let a set of n distinct elements be denoted by $[n] = \{1, \ldots, n\}$.

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- What about the converse?

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This is called Bell's number. Recall its recurrence...

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Can you give a recurrence relation?

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- Using this we can form a Stirling numbers triangle (just like Pascal's).

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- Fix k and let n vary to get: $\sum_{n=k}^{\infty}\frac{S(n,k)}{n!}x^n = \frac{1}{k!}\sum_{n=k}^{\infty}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n\frac{x^n}{n!}$.

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Applying Binomial theorem!

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Applying Binomial theorem!

$$\sum_{n=0}^{\infty}\frac{S(n,k)}{n!}x^n = \frac{1}{k!}(e^x-1)^k.$$

These are called exponential generating functions.

Another exercise in Double Counting

For all $x, n \in \mathbb{N}$, $n > 0$,

$x^n = \sum_{k=1}^{n} S(n,k) \cdot x(x-1)\ldots(x-k+1)$

That is, we can express the powers of x in terms of Stirling numbers (of the second kind)!

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Stirling numbers of the first kind

The number of ways to arrange n objects into k cycles.