CS 207: Discrete Structures

Graph theory

Basic terminology, Eulerian graphs and a characterization

Lecture 23
Sept 14 2014

Topic 3: Graph theory

Textbook Reference

- Introduction to Graph Theory, 2^{nd} Ed., by Douglas West.
- Low cost Indian edition available, published by PHI Learning Private Ltd.

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Topics covered in the first two lectures:

- What is a Graph?
- Paths, cycles, walks and trails; connected graphs.
- Eulerian graphs and a characterization in terms of degrees of vertices.
- Bipartite graphs and a characterization in terms of odd length cycles.

Reference: Section 1.1, 1.2 of Chapter 1 from Douglas West.

What are graphs

Recall:

Definition

A simple graph G is a pair (V, E) of a set of vertices/nodes V and edges E between unordered pairs of vertices called end-points: $e = vu$ means that e is an edge between v and u $(u \neq v)$.

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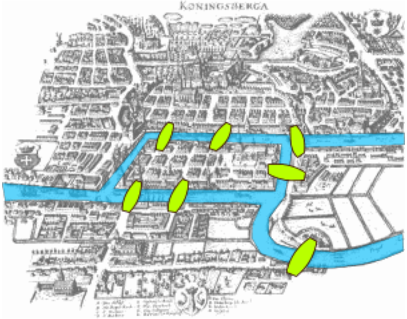
General Definition

A graph G is a triple V, E, R where V is a set of vertices, E is a set of edges and $R \subseteq E \times V \times V$ is a relation that associates each edge with two vertices called its end-points.

We will consider only finite graphs (i.e., $|V|, |E|$ are finite) and often deal with simple graphs.

Königsberg Bridge problem



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- Question was to find a walk from home, crossing every bridge exactly once and returning home.

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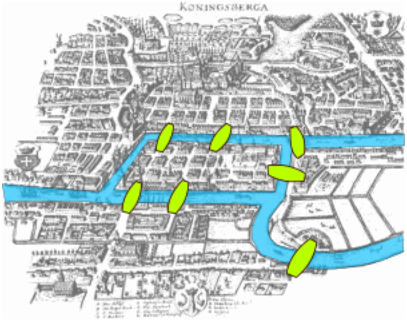

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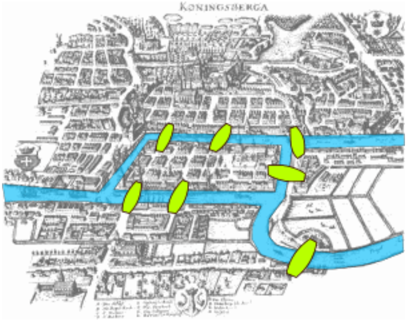

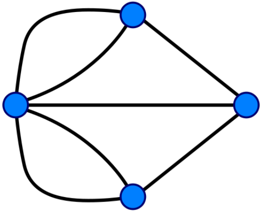

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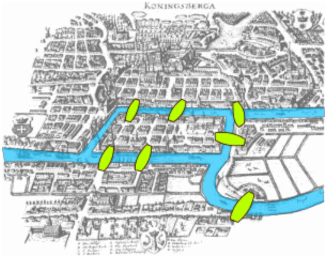

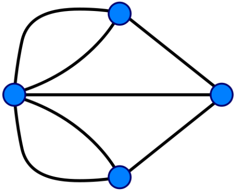

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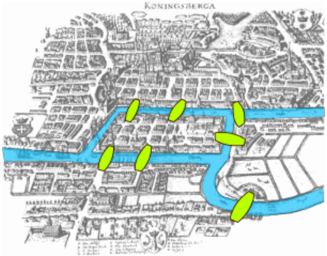

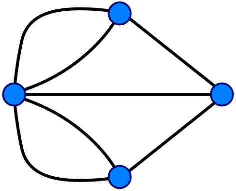

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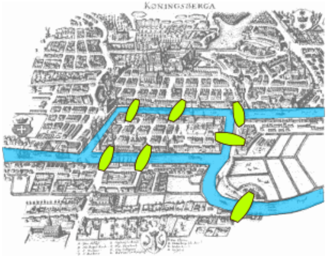

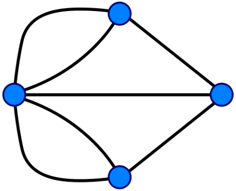

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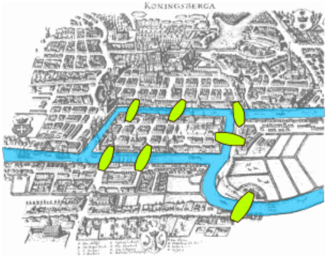

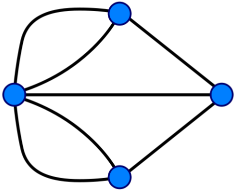

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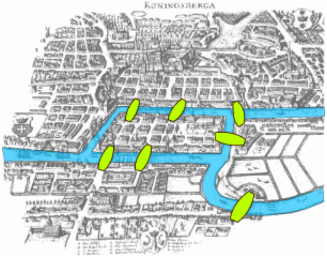

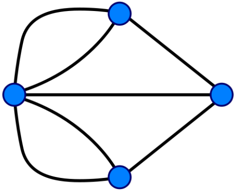

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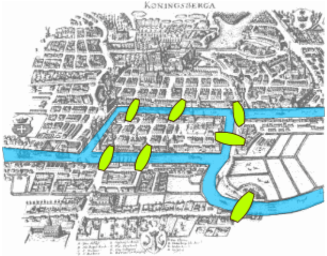

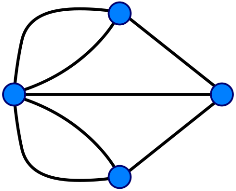

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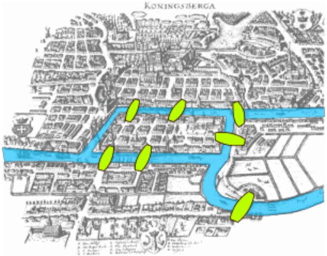 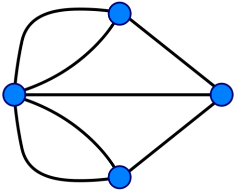

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- Clearly, this is a sufficient condition, but is it necessary?
- If every vertex is connected to an even no. of vertices in a graph, is there such a walk? This is called Eulerian walk.

Basic terminology

The degree $d(v)$ of a vertex v (in an undirected loopless graph) is the number of edges incident to it, i.e., $|\{vw \in E \mid w \in V\}|$. A vertex of degree 0 is called an isolated vertex.

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A graph is called connected if there is a path (or walk) between any two of its vertices.

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- Any two edges are in the same walk implies graph is connected (unless it has isolated vertices).

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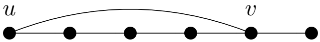

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No! Consider $V = \mathbb{Z}$, $E = \{ij : |i - j| = 1\}$.