CS 207: Discrete Structures

Instructor : S. Akshay

July 30, 2015

Lecture 06 – Countable and uncountable sets

Chapter 2: Basic mathematical structures

Sets and Functions

- Finite and infinite sets, Russell's paradox, axioms of ZFC.
- Functions, their properties, associativity, inverse.
- Types of functions: surjective, injective and bijective.
- Two sets have the same "size" or cardinality iff there is a bijection between them.

Chapter 2: Basic mathematical structures

Sets and Functions

- Finite and infinite sets, Russell's paradox, axioms of ZFC.
- Functions, their properties, associativity, inverse.
- Types of functions: surjective, injective and bijective.
- Two sets have the same "size" or cardinality iff there is a bijection between them.

Today's class

- Countable, countably infinite and uncountable sets.
- A new proof technique.

Recap: properties of functions on finite and infinite sets

Some important properties (H.W.: Prove them!)

- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, implies $\exists$ **bij** from B to A.
- $\exists$ **inj** from A to B, implies $\exists$ **surj** from B to A (& vice-versa)
- Schröder-Bernstein Theorem: $\exists$ **surj** from A to B and $\exists$ **surj** B to A, implies $\exists$ **bij** from A to B.

Recap: properties of functions on finite and infinite sets

Some important properties (H.W.: Prove them!)

- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, implies $\exists$ **bij** from B to A.
- $\exists$ **inj** from A to B, implies $\exists$ **surj** from B to A (& vice-versa)
- Schröder-Bernstein Theorem: $\exists$ **surj** from A to B and $\exists$ **surj** B to A, implies $\exists$ **bij** from A to B.

Theorem

Let A be a set and $b \notin A$. Then A is infinite iff there is a bijection from A to $A \cup \{b\}$.

Recap: properties of functions on finite and infinite sets

Some important properties (H.W.: Prove them!)

- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, implies $\exists$ **bij** from B to A.
- $\exists$ **inj** from A to B, implies $\exists$ **surj** from B to A (& vice-versa)
- Schröder-Bernstein Theorem: $\exists$ **surj** from A to B and $\exists$ **surj** B to A, implies $\exists$ **bij** from A to B.

Theorem

Let A be a set and $b \notin A$. Then A is infinite iff there is a bijection from A to $A \cup \{b\}$.

Corollary

For any infinite set A, there is a surjection from A to $\mathbb{N}$.

Recap: properties of functions on finite and infinite sets

Some important properties (H.W.: Prove them!)

- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, implies $\exists$ **bij** from B to A.
- $\exists$ **inj** from A to B, implies $\exists$ **surj** from B to A (& vice-versa)
- Schröder-Bernstein Theorem: $\exists$ **surj** from A to B and $\exists$ **surj** B to A, implies $\exists$ **bij** from A to B.

Theorem

Let A be a set and $b \notin A$. Then A is infinite iff there is a bijection from A to $A \cup \{b\}$.

Corollary

For any infinite set A, there is a surjection from A to $\mathbb{N}$.

Is there also a injection? Are all (infinite) sets bijective to $\mathbb{N}$?

Countable and countably infinite sets

Definition

- For a given set C, if there is a bijection from C to $\mathbb{N}$, then C is called countably infinite.
- A set is countable if it is finite or countably infinite.

Examples: even numbers, number of horses,...

By previous corollary

Countably infinite sets are the "smallest" infinite sets.

Countable and countably infinite sets

Definition

- For a given set C, if there is a bijection from C to $\mathbb{N}$, then C is called countably infinite.
- A set is countable if it is finite or countably infinite.

Examples: even numbers, number of horses,...

By previous corollary

Countably infinite sets are the “smallest” infinite sets.

What are the other properties of countable sets?

Some questions...

Are the following sets countable?

That is, is there a bijection from these sets to $\mathbb{N}$?

- the set of all integers $\mathbb{Z}$
- $\mathbb{N} \times \mathbb{N}$
- $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$
- the set of rationals $\mathbb{Q}$
- the set of all (finite and infinite) subsets of $\mathbb{N}$
- the set of all real numbers $\mathbb{R}$

Some questions...

Are the following sets countable?

That is, is there a bijection from these sets to $\mathbb{N}$?

- the set of all integers $\mathbb{Z}$
- $\mathbb{N} \times \mathbb{N}$
- $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$
- the set of rationals $\mathbb{Q}$
- the set of all (finite and infinite) subsets of $\mathbb{N}$
- the set of all real numbers $\mathbb{R}$

To show these it suffices to show that

- there is an injection from these sets to $\mathbb{N}$

Some questions...

Are the following sets countable?

That is, is there a bijection from these sets to $\mathbb{N}$?

- the set of all integers $\mathbb{Z}$
- $\mathbb{N} \times \mathbb{N}$
- $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$
- the set of rationals $\mathbb{Q}$
- the set of all (finite and infinite) subsets of $\mathbb{N}$
- the set of all real numbers $\mathbb{R}$

To show these it suffices to show that

- there is an injection from these sets to $\mathbb{N}$
- or there is a surjection from $\mathbb{N}$ (or any countable set) to these sets.

Union of countable sets is countable

Let $A = \{a_0, \ldots,\}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

1. $B = \{b_0\}$ is a singleton
2. $B = \{b_0, \ldots, b_n\}$ is a finite set
3. $B = \{b_0, \ldots\}$ is a countably infinite set

Union of countable sets is countable

Let $A = \{a_0, \ldots,\}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

1. $B = \{b_0\}$ is a singleton
2. $B = \{b_0, \ldots, b_n\}$ is a finite set
3. $B = \{b_0, \ldots\}$ is a countably infinite set
   Can we say $\{a_0, \ldots, b_0, \ldots\}$ is a countably infinite set?

Union of countable sets is countable

Let $A = \{a_0, \ldots,\}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

1. $B = \{b_0\}$ is a singleton
2. $B = \{b_0, \ldots, b_n\}$ is a finite set
3. $B = \{b_0, \ldots\}$ is a countably infinite set
   Can we say $\{a_0, \ldots, b_0, \ldots\}$ is a countably infinite set?

- But then what is the position of b_i (i.e., natural number corresponding to it)?

Union of countable sets is countable

Let $A = \{a_0, \ldots,\}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

1. $B = \{b_0\}$ is a singleton
2. $B = \{b_0, \ldots, b_n\}$ is a finite set
3. $B = \{b_0, \ldots\}$ is a countably infinite set
   Can we say $\{a_0, \ldots, b_0, \ldots\}$ is a countably infinite set?

- But then what is the position of b_i (i.e., natural number corresponding to it)?
- Rather, choose $\{a_0, b_0, a_1, b_1, \ldots\}$, then b_i is at $(2i + 1)^{th}$ position.

Union of countable sets is countable

Let $A = \{a_0, \ldots, \}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

1. $B = \{b_0\}$ is a singleton
2. $B = \{b_0, \ldots, b_n\}$ is a finite set
3. $B = \{b_0, \ldots\}$ is a countably infinite set
   Can we say $\{a_0, \ldots, b_0, \ldots\}$ is a countably infinite set?

- But then what is the position of b_i (i.e., natural number corresponding to it)?
- Rather, choose $\{a_0, b_0, a_1, b_1, \ldots\}$, then b_i is at $(2i+1)^{th}$ position.
- Formally, define a bijection $f : (A \cup B) \to \mathbb{N}$ by $f(a_i) = 2i$ and $f(b_i) = 2i + 1$

Union of countable sets is countable

Let $A = \{a_0, \ldots,\}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

1. $B = \{b_0\}$ is a singleton
2. $B = \{b_0, \ldots, b_n\}$ is a finite set
3. $B = \{b_0, \ldots\}$ is a countably infinite set
   Can we say $\{a_0, \ldots, b_0, \ldots\}$ is a countably infinite set?

- But then what is the position of b_i (i.e., natural number corresponding to it)?
- Rather, choose $\{a_0, b_0, a_1, b_1, \ldots\}$, then b_i is at $(2i+1)^{th}$ position.
- Formally, define a bijection $f : (A \cup B) \to \mathbb{N}$ by $f(a_i) = 2i$ and $f(b_i) = 2i + 1$
- Are we done? What is missing?

Products of countable sets are countable

Theorem: The cartesian product of two countably infinite sets is countably infinite

Proof: Let A, B be countably infinite. Find a way to “number” the elements in $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

- That is, define a bijection from $A \times B$ to $\mathbb{N}$.

Products of countable sets are countable

Theorem: The cartesian product of two countably infinite sets is countably infinite

Proof: Let A, B be countably infinite. Find a way to "number" the elements in $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

- That is, define a bijection from $A \times B$ to $\mathbb{N}$.

$$f(a_i, b_j) = \Big(\sum_{k=1}^{i+j} k\Big) + j + 1$$

Products of countable sets are countable

Theorem: The cartesian product of two countably infinite sets is countably infinite

Proof: Let A, B be countably infinite. Find a way to "number" the elements in $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

- That is, define a bijection from $A \times B$ to $\mathbb{N}$.

$$f(a_i, b_j) = \Big(\sum_{k=1}^{i+j} k\Big) + j + 1$$

Corollaries

- $\mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{Z} \times \mathbb{N}$ are countable.

Products of countable sets are countable

Theorem: The cartesian product of two countably infinite sets is countably infinite

Proof: Let A, B be countably infinite. Find a way to "number" the elements in $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

- That is, define a bijection from $A \times B$ to $\mathbb{N}$.

$$f(a_i, b_j) = \Big(\sum_{k=1}^{i+j} k\Big) + j + 1$$

Corollaries

- $\mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{Z} \times \mathbb{N}$ are countable.
- The set of (positive) rationals is countable.

Products of countable sets are countable

Theorem: The cartesian product of two countably infinite sets is countably infinite

Proof: Let A, B be countably infinite. Find a way to "number" the elements in $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

- That is, define a bijection from $A \times B$ to $\mathbb{N}$.

$$f(a_i, b_j) = \Big(\sum_{k=1}^{i+j} k\Big) + j + 1$$

Corollaries

- $\mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{Z} \times \mathbb{N}$ are countable.
- The set of (positive) rationals is countable.

Hint: Show that $f(a, b) = a/b$ is a surjection. How does the result follow?

Countable sets and functions

Are the following sets countable?

- the set of all integers $\mathbb{Z}$
- $\mathbb{N} \times \mathbb{N}$
- $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$
- the set of rationals $\mathbb{Q}$
- the set of all (finite and infinite) subsets of $\mathbb{N}$
- the set of all real numbers $\mathbb{R}$

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

But, there is a surjection from set of all subsets of $\mathbb{N}$ to $\mathbb{N}$.

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

But, there is a surjection from set of all subsets of $\mathbb{N}$ to $\mathbb{N}$.

Thus, the "size" of this infinity (i.e., set of all subsets of $\mathbb{N}$) must be greater than the other infinity (i.e., $\mathbb{N}$)!

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

Proof by contradiction: Suppose there is such a bijection, say f. This would imply that each $i \in \mathbb{N}$ maps to some set $f(i) \subseteq \mathbb{N}$.

	0	1	2	3	...
$f(0)$	✓	×	×	×	...
$f(1)$	✓	×	✓	✓	...
$f(2)$	×	×	×	×	...
$f(3)$	×	✓	×	✓	...

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

Proof by contradiction: Suppose there is such a bijection, say f. This would imply that each $i \in \mathbb{N}$ maps to some set $f(i) \subseteq \mathbb{N}$.

	0	1	2	3	...
$f(0)$	$\cancel{\checkmark}$ $\times$	$\times$	$\times$	$\times$	...
$f(1)$	$\checkmark$	$\cancel{\times}$ $\checkmark$	$\checkmark$	$\checkmark$	...
$f(2)$	$\times$	$\times$	$\cancel{\times}$ $\checkmark$	$\times$	...
$f(3)$	$\times$	$\checkmark$	$\times$	$\cancel{\checkmark}$ $\times$	...

- Consider the set $S \subseteq \mathbb{N}$ obtained by switching the diagonal elements.

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

Proof by contradiction: Suppose there is such a bijection, say f. This would imply that each $i \in \mathbb{N}$ maps to some set $f(i) \subseteq \mathbb{N}$.

	0	1	2	3	...
$f(0)$	$\cancel{\checkmark}$ $\times$	$\times$	$\times$	$\times$	...
$f(1)$	$\checkmark$	$\cancel{\times}$ $\checkmark$	$\checkmark$	$\checkmark$	...
$f(2)$	$\times$	$\times$	$\cancel{\times}$ $\checkmark$	$\times$	...
$f(3)$	$\times$	$\checkmark$	$\times$	$\cancel{\checkmark}$ $\times$	...

- Consider the set $S \subseteq \mathbb{N}$ obtained by switching the diagonal elements. As f is bij, $\exists j \in \mathbb{N}, f(j) = S$. But,

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

Proof by contradiction: Suppose there is such a bijection, say f. This would imply that each $i \in \mathbb{N}$ maps to some set $f(i) \subseteq \mathbb{N}$.

	0	1	2	3	...
$f(0)$	$\cancel{\checkmark}$ $\times$	$\times$	$\times$	$\times$	...
$f(1)$	$\checkmark$	$\cancel{\times}$ $\checkmark$	$\checkmark$	$\checkmark$	...
$f(2)$	$\times$	$\times$	$\cancel{\times}$ $\checkmark$	$\times$	...
$f(3)$	$\times$	$\checkmark$	$\times$	$\cancel{\checkmark}$ $\times$	...

- Consider the set $S \subseteq \mathbb{N}$ obtained by switching the diagonal elements. As f is bij, $\exists j \in \mathbb{N}, f(j) = S$. But,
- $S \neq f(0)$ as they differ at position 0.
- $S \neq f(1)$ as they differ at position 1.
- $S \neq f(i)$ as they differ at position i.

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

Proof by contradiction: Suppose there is such a bijection, say f. This would imply that each $i \in \mathbb{N}$ maps to some set $f(i) \subseteq \mathbb{N}$.

	0	1	2	3	...
$f(0)$	$\cancel{\checkmark}$ $\times$	$\times$	$\times$	$\times$	...
$f(1)$	$\checkmark$	$\cancel{\times}$ $\checkmark$	$\checkmark$	$\checkmark$	...
$f(2)$	$\times$	$\times$	$\cancel{\times}$ $\checkmark$	$\times$	...
$f(3)$	$\times$	$\checkmark$	$\times$	$\cancel{\checkmark}$ $\times$	...

- Consider the set $S \subseteq \mathbb{N}$ obtained by switching the diagonal elements. As f is bij, $\exists j \in \mathbb{N}, f(j) = S$. But,
- $S \neq f(0)$ as they differ at position 0.
- $S \neq f(1)$ as they differ at position 1.
- $S \neq f(i)$ as they differ at position i.
- Thus, $S \neq f(j)$ for any $j \in \mathbb{N}$ which is a contradiction! $\square$

Cantor's diagonalization

Does this proof look familiar??

Cantor's diagonalization

Does this proof look familiar??

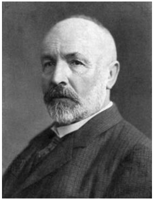
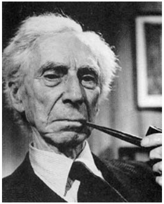
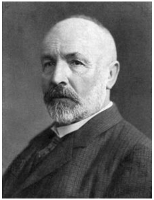
 

Figure: Cantor and Russell

Cantor's diagonalization

Does this proof look familiar??

- In fact, $S = \{i \in \mathbb{N} \mid i \notin f(i)\}$ is like the one from Russell's paradox!



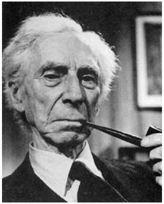


Figure: Cantor and Russell

Cantor's diagonalization

Does this proof look familiar??

- In fact, $S = \{i \in \mathbb{N} \mid i \notin f(i)\}$ is like the one from Russell's paradox!
- Thus, if $\exists j \in \mathbb{N}$ such that $f(j) = S$, then we have a contradiction.
  - If $j \in S$, then $j \notin f(j) = S$.
  - If $j \notin S$, then $j \in f(j) = S$.

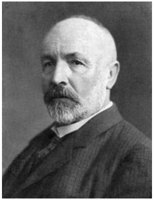
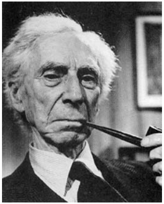
 

Figure: Cantor and Russell

Cantor's diagonalization

Does this proof look familiar??

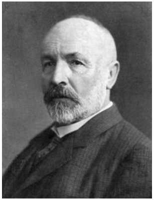
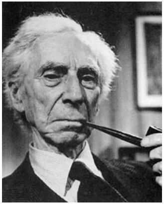
 

Figure: Cantor and Russell

In fact, using diagonalization Cantor showed that...

- There cannot be a bijection between any set and its power set (i.e., its set of subsets).(H.W)
- So there is an infinite hierarchy of "larger" infinities...

Cantor's diagonalization

Does this proof look familiar??

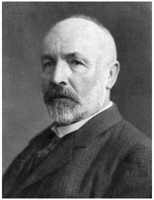
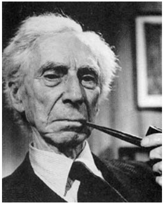
 

Figure: Cantor and Russell

In fact, using diagonalization Cantor showed that...

- There cannot be a bijection between any set and its power set (i.e., its set of subsets).(H.W)
- So there is an infinite hierarchy of "larger" infinities...
- There is no bijection from $\mathbb{R}$ to $\mathbb{N}$ (H.W). Moreover, there is a bijection from $\mathbb{R}$ to set of subsets of $\mathbb{N}$.

Cantor's diagonalization

Does this proof look familiar??

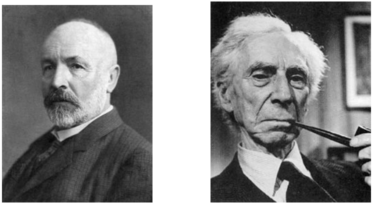


Figure: Cantor and Russell

In fact, using diagonalization Cantor showed that...

- There cannot be a bijection between any set and its power set (i.e., its set of subsets).(H.W)
- So there is an infinite hierarchy of "larger" infinities...
- There is no bijection from $\mathbb{R}$ to $\mathbb{N}$ (H.W). Moreover, there is a bijection from $\mathbb{R}$ to set of subsets of $\mathbb{N}$.
- What about the set of all sets??