# CS 207: Discrete Structures

## Lecture 13 – Counting and Combinatorics

Aug 17 2015

# Last class

**Basic counting techniques**

- Sum and product principles
- Bijection principle
- Binomial coefficients, permutations and combinations
- Double counting

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## Applications

- No. of subsets of a set, no. of subsets of fixed size, no. of ordered subsets of a fixed size.
- No. of reflexive relations, no. of symmetric relations.
- Proving identities on binomial coefficients.
- Handshake lemma: Number of people who shake hands an odd number of times is even.
- No. of equivalence relations or partitions of a set.

# What about partitions?

We have looked at:

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- number of ordered subsets of a set (permutations)

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- What about $B_n$ in general?

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- there is no closed-form answer to this question!
- can we get a recurrence?
- Prove by counting: $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$

# Next, we will see

**Basic counting techniques... (contd.)**

1. Binomial coefficients and Binomial theorem
2. Pascal's triangle
3. Permutations and combinations with repetitions
4. Estimating $n!$

# Binomial theorem

Recall: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

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## Binomial Theorem

Let $x, y$ be variables and $n \in \mathbb{Z}^{\geq 0}$. Then,

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$(x+y)^1 = x+y$$

$$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$$

$$(x+y)^3 = (x+y)(x+y)^2 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = (x+y)(x+y)^3 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

...

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(H.W-1) Prove this by induction.

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## Proof (combinatorial):

1. Consider any term $x^i y^j$, where $i + j = n$.
2. To get $x^i y^j$ term in

$$(x+y)(x+y)\cdots(x+y) \qquad (n \text{ times})$$

we need to pick $j$ $y's$ from $n$ sums and remaining $x$'s.

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3. Thus, the coefficient of this term = number of ways to get this term = number of ways to pick $j$ $y$'s from $n$ elts = $\binom{n}{j}$.

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## Corollaries:

1. $\binom{n}{j} = \binom{n}{n-j}$,
2. $\displaystyle\sum_{j=0}^{n} \binom{n}{j} 2^j = 3^n$.
3. No. of subsets of $n$-element set having even cardinality = ? (H.W-2)

# Pascal's Triangle

**A recursive way to compute binomial coefficients**

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

# Fun with Pascal's triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 200 252 200 120 45 10 1

**Some simple observations. Recall: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$**

1. Row $i$ adds up to $2^i$, Row $i+1$ adds up to twice of row $i$.
2. Sequence of numbers, squares, cubes?

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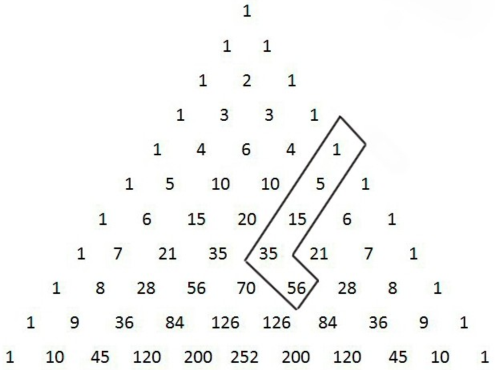


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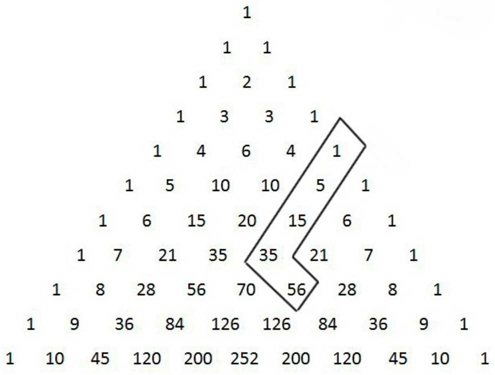


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1. Row $i$ adds up to $2^i$, Row $i+1$ adds up to twice of row $i$.
2. Sequence of numbers, squares, cubes?
3. Hockey stick patterns: (H.W-3)
   $\binom{n+1}{m} = \binom{n}{m} + \binom{n-1}{m-1} \ldots + \binom{n-m}{0}$

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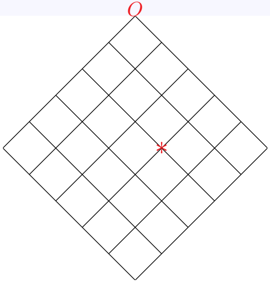
- In fact, for all prime rows? why should $p$ divide $\binom{p}{r}$, $r < p$?
- Corollary: $2^p - 2$ is a multiple of $p$, for any prime $p$.
- Interesting Ex.: Count no. of odd numbers in each row...

# An application to path counting

## Map problems

From the top corner, how many shortest routes lead to a particular junction?



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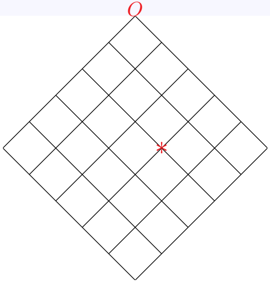


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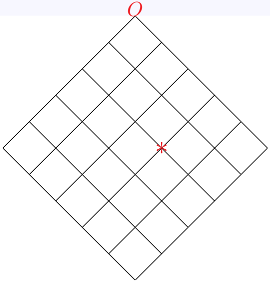


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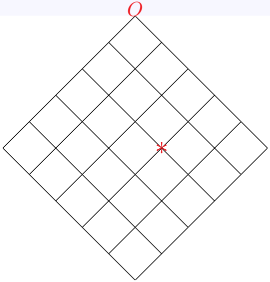


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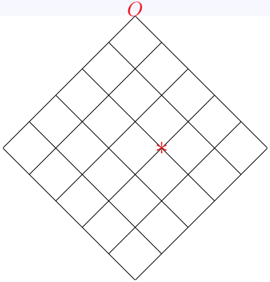


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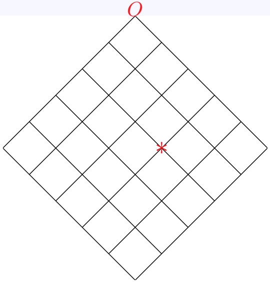


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H.W-4: Prove/verify this formally.

# Permutations and Combinations with repetitions

**How many ways can you select $k$ objects from a set of $n$ elements?**

- Depends on whether order is significant: If yes permutations, else combinations.
- What if repetitions are allowed?

| | Order significant | Order not significant |
|---|---|---|
| Repetitions not allowed | $\frac{n!}{(n-k)!}$ | $\binom{n}{k}$ |
| Repetitions are allowed | $n^k$ | ?? |

# Combinations with repetitions

**Theorem**

The no. of ways $k$ elements can be chosen from $n$-elements, when repetition is allowed is $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$.

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- H.W-5: How many solutions does the equation $x_1 + x_2 + x_3 + x_4 = 17$ have such that $x_1, x_2, x_3, x_4 \in \mathbb{Z}^{\geq 0}$ ?