CS 207: Discrete Structures

Graph theory

Perfect matchings in bipartite graphs: Hall's theorem

Lecture 30
Oct 05 2015

Topic 3: Graph theory

Basic definitions and concepts

- Basics: graphs, paths, cycles, walks, trails, ...
- Cliques and independent sets.
- Graph representations, isomorphisms and automorphisms.
- Matchings: perfect, maximal and maximum.

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4. Maximum matchings: Using augmenting paths.
5. Perfect matchings and Hall's condition.

Recap: Matchings

Definitions

- A matching in a graph G is a set of (non-loop) edges with no shared end-points. The vertices incident to edges in a matching are called matched or saturated. Others are unsaturated.
- A perfect matching in a graph is a matching that saturates every vertex.
- A maximal matching in a graph is a matching that cannot be enlarged by adding an edge.
- A maximum matching is a matching of maximum size (# edges) among all matchings in a graph.

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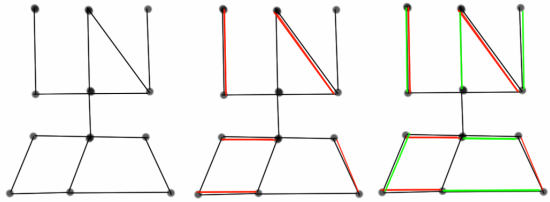
Perfect matching $\implies$ maximum matching $\implies$ maximal matching

Recap: Alternating and Augmenting paths

Definition

- Given a matching M, an M-alternating path is a path that alternates between edges in M and edges not in M.
- An M-alternating path whose endpoints are unmatched by M is an M-augmenting path.



Theorem

A matching M in G is a maximum matching iff G has no M-augmenting path.

Recap: Characterizing maximum matchings

A definition and a lemma

- For matchings M, M' of graph G, the symmetric difference $M \triangle M' = (M \setminus M') \cup (M' \setminus M)$.
- Every component of the symmetric difference of two matchings is either a path or an even cycle.

Recap: Characterizing maximum matchings

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Proof of Lemma:

- Let $F = M \triangle M'$. F has at most 2 edges at each vertex, hence every component is a path or a cycle.
- Further every path/cycle alternates between edges of $M \setminus M'$ and $M' \setminus M$.
- Thus, each cycle has even length with equal edges from M and M'. $\square$

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- ($\implies$) Forward direction is trivial.

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- ($\Longleftarrow$) We will show if $\exists$ matching M' larger than M, we will construct an M-augmenting path.
- Let $F = M \triangle M'$. By Lemma, F has only paths and even cycles with equal no. of edges from M and M'.
- But then as $|M'| > |M|$, F must have a component with more edges in M' than M, which is a path that starts and ends with an edge of M'; i.e., an M-augmenting path. $\square$

Perfect matchings in bipartite graphs

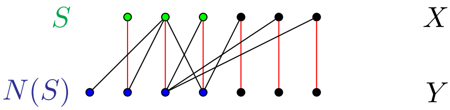
- If there are n women and n men, and each woman is compatible with exactly k men and each man compatible with exactly k women, can they be perfectly matched?

Perfect matchings in bipartite graphs

- If there are n women and n men, and each woman is compatible with exactly k men and each man compatible with exactly k women, can they be perfectly matched?
- If there are m jobs and n applicants, when can we find a perfect matching where all m jobs are saturated?

Perfect matchings in bipartite graphs

- Consider a bipartite graph with X, Y as partitions.
- If a matching M saturates X, then for every $S \subseteq X$, there must exist at least $|S|$ edges that have neighbours in S.



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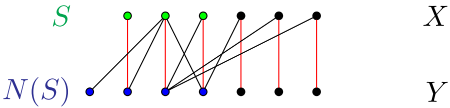

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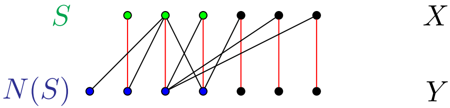

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A bipartite graph G with bipartitions X, Y has a matching that saturates X iff for all $S \subseteq X$, $|N(S)| \geq |S|$.

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Proof: ($\Longrightarrow$) is straightforward:

- Let M be a matching.
- Then for any $S \subseteq X$, each vertex of S is matched to a distinct vertex in $N(S)$
- So $|N(S)| \geq |S|$.

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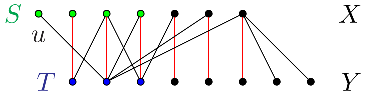
- Let $u \in X$ be any unsaturated vertex of M.
- Consider vertices V_u from u by M-alternating paths in G and let $S = V_u \cap X$ and $T = V_u \cap Y$.



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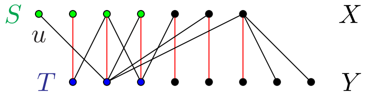

Claim: M matches T with $S \setminus \{u\}$ and $|N(S)| = |T|$.

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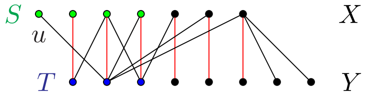

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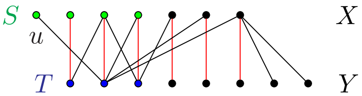

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- Every vertex of $S \setminus \{u\}$ has an edge in M to a vertex in T.
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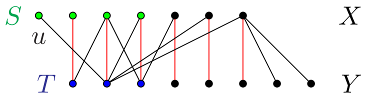

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- Every vertex of $S \setminus \{u\}$ has an edge in M to a vertex in T.
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- Thus, there is a bijection between T and $S \setminus \{u\}$.

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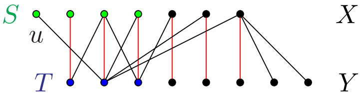

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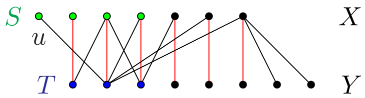

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- $T \subseteq N(S)$ (from T any M-alternating path will reach S).
- Conversely, if $v \in S$ has edge to $y \in Y \setminus T$, then path from u to v via M to y is an M-alternating path, implies $y \in T$.

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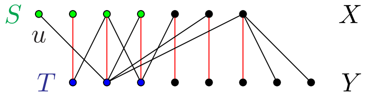

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Thus, $|N(S)| = |T| = |S| - 1 < |S|$ $\square$.