CS 207: Discrete Structures

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Lecture 08 – Basic mathematical structures

Equivalence relations and partitions, posets

Recap: Relations

Definition: Relation

- A relation R from A to B is a subset of $A \times B$. If $(a, b) \in R$, we also write this as $a \; R \; b$.

We write $R(A, B)$ for a relation from A to B and just $R(A)$ if $A = B$. Also if A is clear from context, we just write R.

- All functions are relations.
- $R_1(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z}, a - b \text{ is even } \}$.
- Relational databases are practical examples.

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Representations of a relation from A to B.

- As a set of ordered pairs of elements, i.e., subset of $A \times B$, as a directed graph, as a (database) table.

Special types of relations

- Reflexive: $\forall a \in S$, aRa.
- Symmetric: $\forall a, b \in S$, aRb implies bRa.
- Transitive: $\forall a, b, c \in S$, aRb, bRc implies aRc.
- Equivalence: Reflexive, Symmetric and Transitive.

Relation	Refl.	Symm.	Trans.	Equiv.
aRb if students a and b take same set of courses	✓	✓	✓	✓
$\{(a,b) \mid a, b \in \mathbb{Z}, (a-b) \bmod 2 = 0\}$	✓	✓	✓	✓
$\{(a,b) \mid a, b \in \mathbb{Z}, a \leq b\}$				
$\{(a,b) \mid a, b \in \mathbb{Z}, a < b\}$				
$\{(a,b) \mid a, b \in \mathbb{Z}, a \mid b\}$				
$\{(a,b) \mid a, b \in \mathbb{R}, \lvert a-b \rvert < 1\}$				
$\{((a,b),(c,d)) \mid (a,b),(c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), (ad = bc)\}$				

Partitions of a set – grouping "like" elements

Definition

A partition of a set S is a set P of its subsets such that

- if $S' \in P$, then $S' \neq \emptyset$.
- $\bigcup_{S' \in P} S' = S$: its union covers entire set S.
- If $S_1, S_2 \in P$, then $S_1 \cap S_2 = \emptyset$: sets are disjoint.

Example: natural numbers partitioned into even and odd...

Theorem

Every partition of set S gives rise to a canonical equivalence relation R on S, namely,

- aRb if a and b belong to the same set in the partition of S.

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Is the converse true? Can we generate a partition from every equivalence relation?

Equivalence classes

Definition

- Let R be an equivalence relation on set S, and let $a \in S$.
- Then the equivalence class of a, denoted $[a]$, is the set of all elements related to it, i.e., $[a] = \{b \in S \mid (a, b) \in R\}$.

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Let R be an equivalence relation on S. Let $a, b \in S$. Then, the following statements are equivalent:

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Proof Sketch: (1) to (2) symm and trans, (2) to (3) refl, (3) to (1) symm and trans. (H.W.: Do the proof formally.)

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Proof sketch of (1): Union, non-emptiness follows from reflexivity. The rest (pairwise disjointness) follows from the previous lemma.

(H.W.): Write the formal proofs of (1) and (2).

More “applications” of equivalence relations

Defining new objects using equivalence relations

Consider
$R = \{((a, b), (c, d)) \mid (a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), (ad = bc)\}$.

- Then the equivalence classes of R define the rational numbers.
- e.g., $\left[\frac{1}{2}\right] = \left[\frac{2}{4}\right]$ are two names for the same rational number.
- Indeed, when we write $\frac{p}{q}$ we implicitly mean $\left[\frac{p}{q}\right]$.

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- With this definition, why are addition and multiplication "well-defined"?

Can we define integers and real numbers starting from naturals by using equivalence classes?

Geometrical objects using equivalence relations

Cut-and-paste

Consider the relation $R([0,1]) = \{aRb \mid a, b \in [0,1]$, either $a = b$ or $a = 1, b = 0$, or $a = 0, b = 1\}$.

- Is R an equivalence relation? What does it define?

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- Is R an equivalence relation? What does it define?
- This is $[0,1]$ in which the end-points have been related to each other.
- So the equivalence classes form a "loop", since end-points are joined. If we imagine $[0,1]$ as a 1-length string, we have glued its ends!

Geometrical objects using equivalence relations

Forming 2D objects

Consider a rectangular piece of the real plane, $[0, 1] \times [0, 1]$.

- Define $R_1([0, 1] \times [0, 1])$ by $(a, b)R_1(c, d)$ if
  - $(a, b) = (c, d)$ or
  - $b = d$, $a = 0$, $c = 1$ or
  - $b = d$, $c = 0$, $a = 1$.

  Is R_1 an equivalence relation? What do its equivalence classes define?

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  - $b = d$, $c = 0$, $a = 1$.

  Is R_1 an equivalence relation? What do its equivalence classes define?
- Define $R_2([0,1] \times [0,1])$ by $(a,b)R_2(c,d)$ if
  - $(a,b) = (c,d)$ or
  - $a, b, c, d \in \{0,1\}$.

  Is R_2 an equivalence relation? What does it define?

Can you build even more interesting "shapes"? Torus? Mobius strip?!

Partial Orders

Consider $\{(a, b) \mid a, b \in \mathbb{Z}, a \leq b\}$.

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Anti-symmetric

A relation R on S is anti-symmetric if for all $a, b \in S$ aRb and bRa implies $a = b$.

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Examples:

- $R_1(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z}, a \leq b\}$.
- $R_2(\mathcal{P}(S)) = \{(A, B) \mid A, B \in \mathcal{P}(S), A \subseteq B\}$.

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Definition

A partial order is a relation which is reflexive, transitive and anti-symmetric.

Partial orders and equivalences relations

- Reflexive: $\forall a \in S$, aRa.
- Symmetric: $\forall a, b \in S$, aRb implies bRa.
- Anti-symmetric: $\forall a, b \in S$, aRb, bRa implies $a = b$.
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	Reflexive	Transitive	Symmetric	Anti-symmetric
Equivalence relation	✓	✓	✓	
Partial order	✓	✓		✓

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	Refl.	Anti-Sym	Trans.	PO
$\{(a, b) \mid a, b \in \mathbb{Z}, a \leq b\}$	✓	✓	✓	✓
$\{(A, B) \mid A, B \in \mathcal{P}(S), A \subseteq B\}$	✓	✓	✓	✓
$\{(a, b) \mid a, b \in \mathbb{Z}, a < b\}$				
$\{(a, b) \mid a, b \in \mathbb{Z}^+, a \mid b\}$				
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  - i.e., $\forall a, b \in S$, either $a \preceq b$ or $b \preceq a$.

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- Qn: Can a relation be symmetric and anti-symmetric?
- (H.W): Can a relation be neither symmetric nor anti-symmetric?

Partially ordered sets (Posets)

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Examples

- $(\mathbb{Z}, \leq)$: integers with the usual less than or equal to relation.
- $(\mathcal{P}(S), \subseteq)$: powerset of any set with the subset relation.
- $(\mathbb{Z}^+, \mid\ )$: positive integers with divisibility relation.