Sinusoidal steady state analysis (Phasors)

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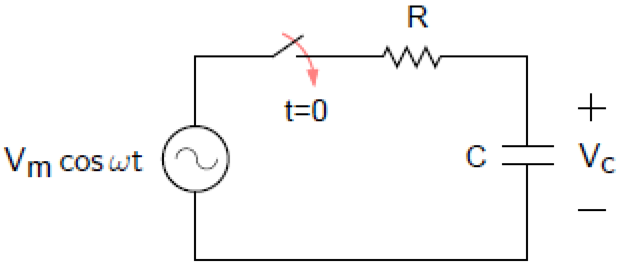
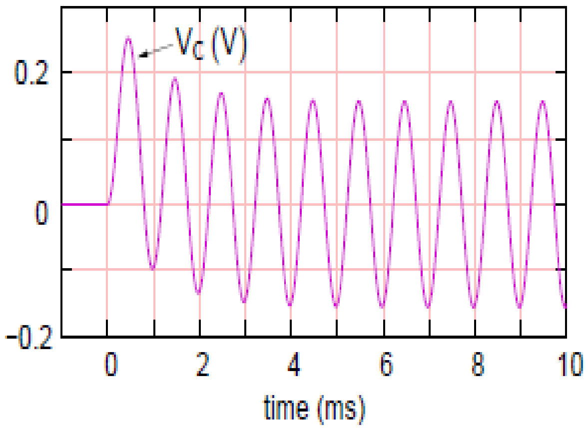
Reference: L. Bobrow

Sinusoidal signals (AC signals)

- Response to repetitive, sinusoidal (AC) signal
  - Ubiquitous
  - Ordinary household voltage
  - Radio (AM/FM)
  - Television etc.
- In a lot of circuit applications
  - Power consumed/supplied is of importance
  - Instantaneous power (ratings in circuits)

Sinusoidal analysis (time-domain, steady state)





$$RCV_c' + V_c = V_m \cos \omega t$$

$$V_c(t) = V_f(t) + V_n(t)$$

$$V_n(t) = Ae^{-\frac{t}{RC}}$$

$$V_f(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

Substituting

$$\omega RC(-C_1 \sin \omega t + C_2 \cos \omega t) + C_1 \cos \omega t + C_2 \sin \omega t = V_m \cos \omega t$$

C_1 an C_2 can be found by equating coefficients on both sides

$$V_c(t) = Ae^{-\frac{t}{RC}} + C_1 \cos \omega t + C_2 \sin \omega t$$

$$t \to \infty$$

$$\boxed{V_c(t) = C_1 \cos \omega t + C_2 \sin \omega t}$$

Sinusoidal steady-state response

- We assume the forced response of RLC ckts to a sinusoidal input to be a sinusoid
  - Each of the elements – R, L, C will respond with a sinusoid to a sinusoidal input; no change in frequency ω
  - Sum of sinusoidal outputs is also a sinusoid

- Any circuit containing resistors, capacitors, inductors, sinusoidal voltage and current sources (of the same frequency), dependent (linear) sources behaves in a similar manner, viz., each current and voltage in the circuit becomes purely sinusoidal as t →infinity

Real sinusoid → Complex Sinusoid

RLC ckt	Input	Output
	$A\cos(\omega t+\theta)$	$B\cos(\omega t+\phi)$
Shift by 90deg	$A\sin(\omega t+\theta)$	$B\sin(\omega t+\phi)$
Scale by K, linearity	$KA\sin(\omega t+\theta)$	$KB\sin(\omega t+\phi)$
Superposition	$A\cos(\omega t+\theta)+KA\sin(\omega t+\theta)$	$B\cos(\omega t+\phi)+KB\sin(\omega t+\phi)$
K=j	$A\cos(\omega t+\theta)+jA\sin(\omega t+\theta)$	$B\cos(\omega t+\phi)+jB\sin(\omega t+\phi)$
Euler's formula	$Ae^{j(\omega t+\theta)}$	$Be^{j(\omega t+\varphi)}$

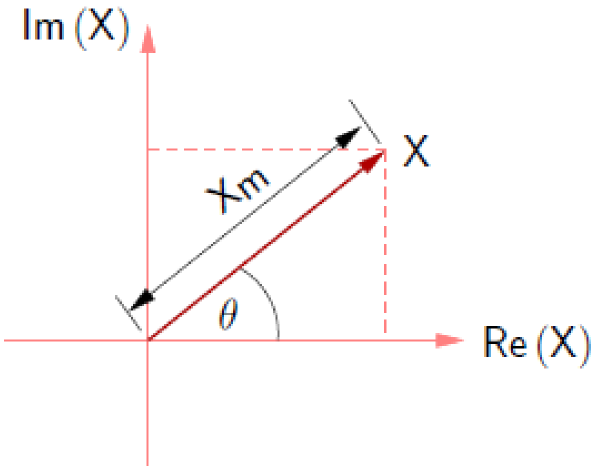
- Finding soln (B, ϕ) to real sinusoid is equivalent to finding the soln (B, ϕ) to the complex sinusoid (B, ϕ)
- Using complex numbers makes differential eqn soln easier

Phasors: Definition

$$\mathbf{X} = X_m e^{j\theta} = X_m \angle \theta$$

$$x(t) = \mathrm{Re}[\mathbf{X} e^{j\omega t}] = \mathrm{Re}[X_m e^{j\theta} e^{j\omega t}] = X_m \cos(\omega t + \theta)$$

$$X_m \angle \theta = X_m \cos\theta + X_m j \sin\theta$$



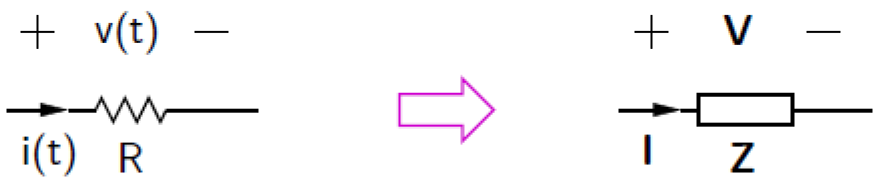
- Phasor is a complex number
- The term ωt is implicit
- Can also be written in the polar or rectangular form
- Also called frequency domain representation
- Significant simplification of sinusoidal steady state analysis

Phasor: Examples

Time domain	Frequency domain
$v_1(t) = 3.2\cos(\omega t + 30^\circ)$ V	$V_1 = 3.2\angle 30^\circ = 3.2\exp(j\pi/6)$ V
$i(t) = -1.5\cos(\omega t + 60^\circ)$ A $= 1.5\cos(\omega t + \pi/3 - \pi)$ A $= 1.5\cos(\omega t - 2\pi/3)$ A	$I = 1.5\angle(-2\pi/3)$ A
$v_2(t) = -0.1\cos(\omega t)$ V $= 0.1\cos(\omega t + \pi)$ V	$V_2 = 0.1\angle\pi$ V
$i_2(t) = 0.18\sin(\omega t)$ A $= 0.18\cos(\omega t - \pi/2)$ A	$I_2 = 0.18\angle(-\pi/2)$ A
$i_3(t) = \sqrt{2}\cos(\omega t + 45^\circ)$ A	$I_3 = 1 + j1$ A $= \sqrt{2}\angle 45^\circ$ A

Phasor representation/Impedance of resistor

Time domain　　　　　　　　　　　　Phasor



$$i(t) = I_m \cos(\omega t + \theta)$$

$$v(t) = Ri(t) = RI_m \cos(\omega t + \theta) = V_m \cos(\omega t + \theta)$$

$$\mathrm{Re}[V_m e^{j(\omega t + \theta)}] = \mathrm{Re}[RI_m e^{j(\omega t + \theta)}]$$

$$\boxed{\mathbf{V} = R\mathbf{I}}$$

Phasor representation

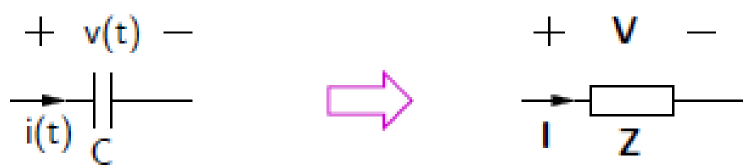
$$\boxed{\mathbf{Z} = \mathbf{V}/\mathbf{I} = R + j0}$$

Z is the impedance of the resistor

Phasor representation/Impedance of capacitor

Time domain

Phasor



$$v(t) = V_m \cos(\omega t + \theta)$$

$$i(t) = C\frac{dv(t)}{dt} = -C\omega V_m \sin(\omega t + \theta) = C\omega V_m \cos(\omega t + \theta + \frac{\pi}{2})$$

$$\mathbf{V} = V_m \angle \theta$$

$$\boxed{\mathbf{I} = C\omega V_m \angle(\theta + \frac{\pi}{2}) = \omega C V_m e^{j\theta} e^{j\frac{\pi}{2}} = j\omega C\mathbf{V}}$$

Phasor representation

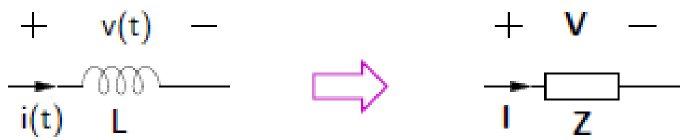
$$\boxed{\begin{aligned} \mathbf{Z} &= \mathbf{V/I} = 0 + \frac{1}{j\omega C} \\ \mathbf{Y} &= \mathbf{I/V} = 0 + j\omega C \end{aligned}}$$

Z is the impedance of the capacitor

Y is the admittance of the capacitor

Phasor representation/Impedance of inductor

Time domain | Phasor



$$i(t) = I_m \cos(\omega t + \theta)$$

$$v(t) = L\frac{di(t)}{dt} = -L\omega I_m \sin(\omega t + \theta) = L\omega I_m \cos(\omega t + \theta + \frac{\pi}{2})$$

$$\mathbf{I} = I_m \angle \theta$$

$$\mathbf{V} = L\omega I_m \angle(\theta + \frac{\pi}{2}) = \omega L I_m e^{j\theta} e^{j\frac{\pi}{2}} = j\omega L \mathbf{I}$$

Phasor representation

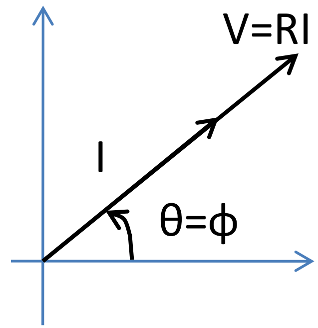
$$\mathbf{Z} = \mathbf{V/I} = 0 + j\omega L$$

$$\mathbf{Y} = \mathbf{I/V} = 0 + \frac{1}{j\omega L}$$

Z is the impedance of the capacitor

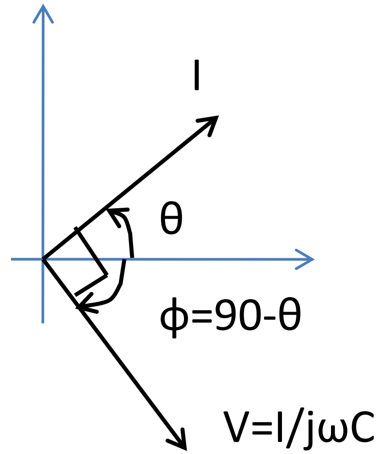
Y is the admittance of the capacitor

Lead and Lag

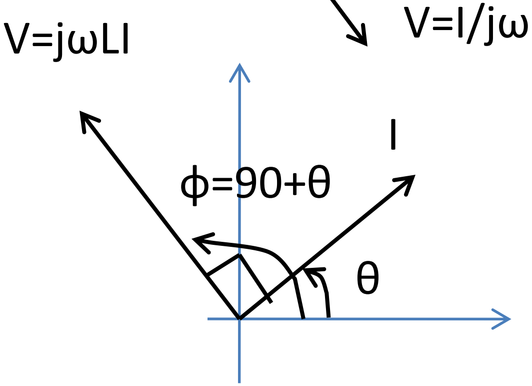


$$\mathbf{V} = R\mathbf{I}$$

$$\mathbf{I} = C\omega V_m \angle(\theta + \frac{\pi}{2}) = \omega C V_m e^{j\theta} e^{j\frac{\pi}{2}} = j\omega C\mathbf{V}$$

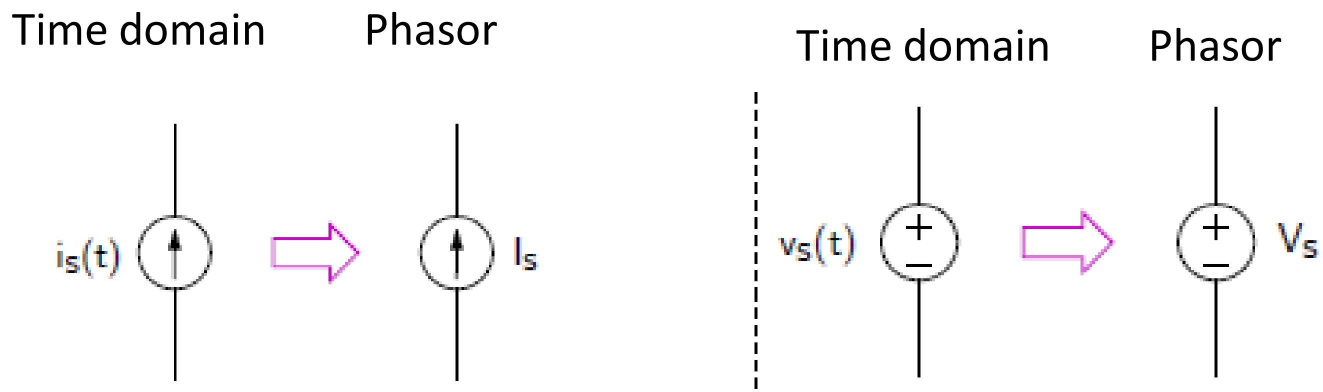


$$\mathbf{V} = L\omega I_m \angle(\theta + \frac{\pi}{2}) = \omega L I_m e^{j\theta} e^{j\frac{\pi}{2}} = j\omega L\mathbf{I}$$



- Resistor→ in phase
- Capacitor→ current ***leads*** V by 90deg
- Inductor→voltage ***leads*** I by 90deg

Sources



- An independent sinusoidal current source, $i_s(t) = I_m\cos(\omega t + \theta)$, can be represented by the phasor $I_m < \theta$ (i.e., a constant complex number)

- An independent sinusoidal voltage source, $v_s(t) = V_m\cos(\omega t + \theta)$, can be represented by the phasor $V_m < \theta$ (i.e., a constant complex number)

- Dependent (linear) sources can be treated in the sinusoidal steady state in the same manner as a resistor, i.e., by the corresponding phasor relationship. For example, for a CCVS, we have,

  $v(t) = r\, i_c\,(t)$ in the time domain

  $\mathbf{V} = r\,\mathbf{I_c}$ in the frequency domain

Phasor addition

$$v(t) = v_1(t) + v_2(t) = V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2)$$

$$\mathbf{V} = \mathbf{V_1} + \mathbf{V_2} = V_{m1}e^{j\theta_1} + V_{m2}e^{j\theta_2}$$

$$\tilde{v}(t) = \mathrm{Re}[(V_{m1}e^{j\theta_1} + V_{m2}e^{j\theta_2})e^{j\omega t}] = V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) = v(t)$$

- Addition of sinusoidal quantities in the time domain can be replaced by addition of the corresponding phasors in the sinusoidal steady state.
- KCL and KVL in time domain hold for phasors also

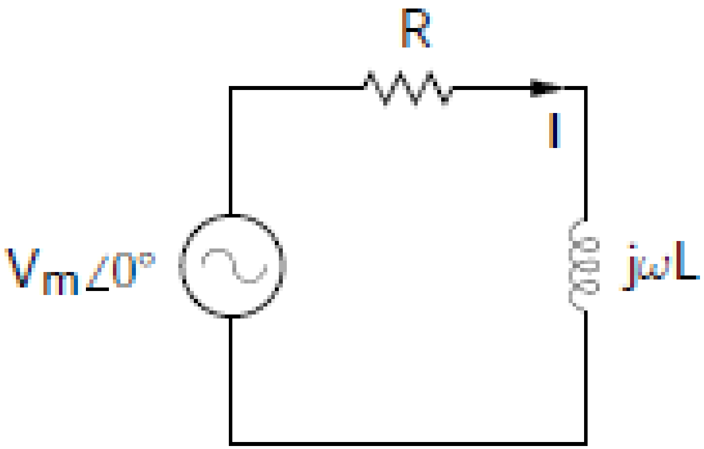
For meshes $\sum v_k(t) = 0$ $\qquad \sum \mathbf{V_k} = 0$

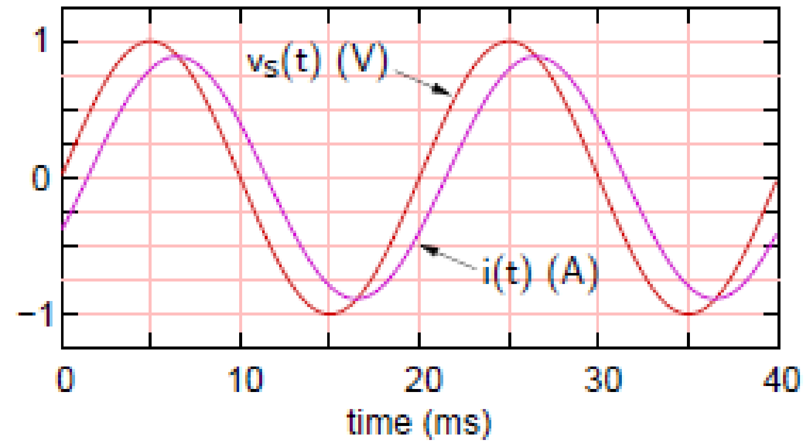
For nodes $\sum i_k(t) = 0$ $\qquad \sum \mathbf{I_k} = 0$

Sinusoidal Steady State Analysis

- Resistors, capacitors, and inductors can be described by **V = Z I** in the frequency domain, which is similar to V = R I in DC conditions (except that we are dealing with complex numbers in the frequency domain)
- An independent sinusoidal source in the frequency domain behaves like a DC source, e.g., V_s=constant (a complex number)
- For dependent sources, a time-domain relationship such as $i(t)=\beta i_c(t)$ translates to $\mathbf{I}=\beta \mathbf{I_c}$ in the frequency domain
- **Circuit analysis in the sinusoidal steady state using phasors is therefore very similar to DC circuits with independent and dependent sources, and resistors**
- **Series/parallel formulas for resistors, nodal analysis, mesh analysis, Thevenin's and Norton's theorems can be directly applied**
- No differential equations!
- This is called the **frequency domain approach**

Example: RL circuit





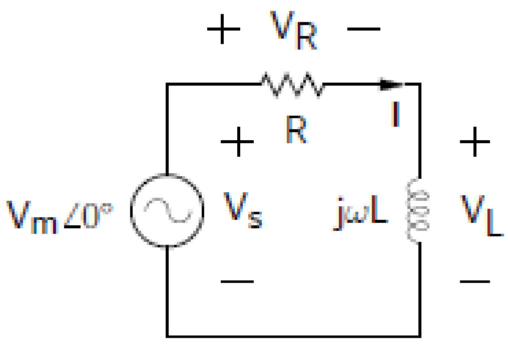
$R = 1\,\Omega$
$L = 1.6\,\text{mH}$

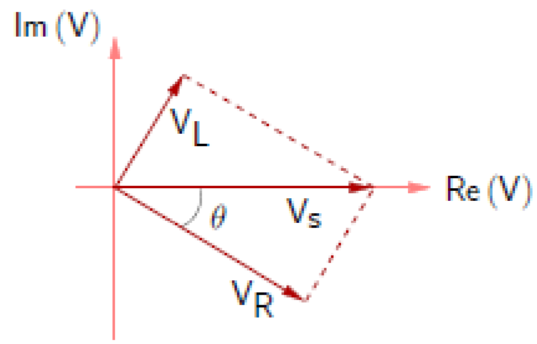
$$\mathrm{I} = \frac{V_m\angle 0}{R + j\omega L} \equiv I_m\angle(-\theta),$$

$$\text{where } I_m = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}, \text{ and } \theta = \tan^{-1}(\omega L/R).$$

- In the time domain, $i(t) = I_m \cos(\omega t - \theta)$, which lags the source voltage since the peak (or zero) of i(t) occurs $t = \theta/\omega$ seconds after that of the source voltage.
- For R = 1Ω , L = 1.6 mH, f = 50 Hz, $\theta = 26.6^\circ$, t_{lag}= 1.48 ms.

Example: RL Circuit (contd)





$$\mathbf{I} = \frac{V_m\angle 0}{R + j\omega L} \equiv I_m\angle(-\theta),$$

where $I_m = \dfrac{V_m}{\sqrt{R^2+\omega^2L^2}}$, and $\theta = \tan^{-1}(\omega L/R)$.

KVL: $V_s = V_R + V_L$

$$\mathbf{V_R} = \mathbf{I}\times R = R\,I_m\,\angle(-\theta)\,,$$
$$\mathbf{V_L} = \mathbf{I}\times j\omega L = \omega I_m L\,\angle(-\theta+\pi/2)\,,$$

- If R >> jωL, θ → 0, $|V_R| \approx |V_s| = V_m$.
- If R << jωL, θ → π/2, $|V_L| \approx |V_s| = V_m$.