CS 207: Discrete Structures

Graph theory

Eulerian graphs, Bipartite graphs

Lecture 24

Sept 15 2014

Topic 3: Graph theory

Textbook Reference

- Introduction to Graph Theory, 2^{nd} Ed., by Douglas West.
- Low cost Indian edition available, published by PHI Learning Private Ltd.

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Topics covered in the first two lectures:

- What is a Graph?
- Paths, cycles, walks and trails; connected graphs.
- Eulerian graphs and a characterization
- Bipartite graphs and a characterization

Reference: Section 1.1, 1.2 of Chapter 1 from Douglas West.

Recall: Basic terminology

Definition

A simple graph G is a pair (V, E) of a set of vertices/nodes V and edges E between unordered pairs of vertices called end-points: $e = vu$ means that e is an edge between v and u $(u \neq v)$.

Recall: Basic terminology

- The degree $d(v)$ of a vertex v (in an undirected loopless graph) is the number of edges incident to it, i.e., $|\{vw \in E \mid w \in V\}|$. A vertex of degree 0 is called an isolated vertex.
- A walk is a sequence of vertices $v_1, \ldots v_k$ such that $\forall i \in \{1, \ldots k-1\}$, $(v_i, v_{i+1}) \in E$. The vertices v_1 and v_k are called the end-points and others are called internal vertices.

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- A walk is called closed if it starts and ends with the same vertex, i.e., its endpoints are the same.
- A closed walk is called a cycle if its internal vertices are all distinct from each other and from the end-point.
- A graph is called connected if there is a path (or walk) between any two of its vertices.

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- Any two edges are in the same walk implies graph is connected (unless it has isolated vertices).

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If every vertex of a graph G has degree at least 2, then G contains a cycle.

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- Thus we have the cycle $u \dots vu$. $\square$

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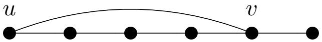

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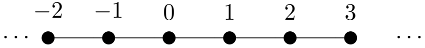
No! Consider $V = \mathbb{Z}$, $E = \{ij : |i - j| = 1\}$.



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Proof ($\Longleftarrow$): By induction on number of edges m.

- Base case: $m = 1$.

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  - Get $G_1, \ldots, G_k$. Each G_i is
    - connected
    - has $< m$ edges.
    - all its vertices have even degree (why? – degree of any vertex was even and removing C, reduces each vertex degree by 0 or 2.)

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    - Traverse along cycle C in G and when some G_i is entered for first time, detour along an Eulerian walk of G_i.
    - This walk ends at vertex where we started detour.
    - When we complete traversal of C in this way, we have completed an Eulerian walk on G. □

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- E.g., Above, since a maximal path could not be extended, we got that every neighbour of an endpoint of P is in P.
- (H.W) Can you show the theorem directly from extremality without using induction?

Applications of Eulerian graphs

Corollary

Every graph with all vertices having even degree decomposes into cycles

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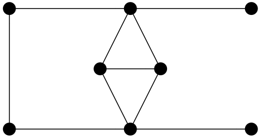

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- Each trail has only 2 ends implies we use at least k trails to satisfy $2k$ odd edges.
- We need at least one trail since G has an edge.
- Thus, we have shown that at least $\max\{k, 1\}$ trails are required.

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- If $k = 0$, one trail suffices (i.e., an Eulerian walk by previous Thm)
- If $k > 0$ we need to prove that k trails suffice.
  - Pair up odd vertices in G (in any order) and form G' by adding an edge between them.
  - G' is connected, by previous Thm has an Eulerian walk C.
  - Traverse C in G' and for each time we cross an edge of G' not in G, start a new trail (lift pen!).
  - Thus, we get k trails decomposing G. $\square$

Some simple types of Graphs

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Some simple types of Graphs

- We have already seen some: connected graphs.
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- Are there other interesting classes of graphs?

Bipartite graphs

Definition

A graph is called bipartite, if the vertices of the graph can be partitioned into $V = X \cup Y$, $X \cap Y = \emptyset$ s.t., $\forall e = (u, v) \in E$,

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- How can we check if a graph is bi-partite?
- Can we characterize bi-partite graphs?

Bipartite graphs

We will use cycles to characterize them.

Bipartite graphs

- Recall: A path or a cycle has length n if the number of edges in it is n.
- A path (or cycle) is call odd (or even) if its length is odd (or even, respectively).

Bipartite graphs

Lemma

Every closed odd walk contains an odd cycle.

Proof: By induction on the length of the given closed odd walk. (Class work; see section 1.2 of Douglas West).

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What about closed even walks?

Characterizing Bipartite graphs

Theorem, Konig, 1936

A graph is bipartite iff it has no odd cycle.