CS207 (Discrete Structures)
Exercise problem set 10 – Abstract algebra

October 21, 2015

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1. Are the following statements true or false? If true, prove them. If false, specify which defining property fails as well as give a counterexample. G be a group and H and K are subgroups of G. Then,

   (a) $H \cap K$ is always a subgroup of G.

   (b) $H \cup K$ is always a subgroup of G.

   (c) $G \setminus H$ is always a subgroup of G.

   (d) Let $G = (\mathbb{R} \setminus \{0\}, \times)$ and $G' = \{1, -1\}$, then G' is a subgroup of G.

2. Suppose G is a group wth the property that $g^2 = 1$ for all $g \in G$. Prove that G is an Abelian (commutative) group.

3. Consider the rigid transformations of a square, also called its symmetries (just as we did for equilateral triangles in class).

   (a) How many of them are there? Prove that they form a group (by writing the table for composition).

   (b) Enumerate all the subgroups. How many elements does each subgroup have?

   Answer the above questions for a rectangle and a regular pentagon.

4. For any group G, let $Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}$. $Z(G)$ is called the center of G.

   (a) Is $Z(G)$ always a subgroup of G?

   (b) What is $Z(GL_2(\mathbb{R}))$?

   (c) Prove that G is abelian iff $Z(G) = G$.

   (d) * If G is a finite non-abelian group, show that $|Z(G)| \leq \frac{1}{4}|G|$.

5. Prove that an integer $n > 1$ is prime iff $(n-1)! \equiv -1 \mod n$. This is called Wilson's theorem. Can this be used as an efficient test for primality? Why or why not?