CS 228 : Logic in Computer Science

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- Given φ, write an algorithm to check $L(\varphi) = \emptyset$?

First-Order Logic over Words

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  - Given a FO formula φ over words, is $L(\varphi)$ non-empty?

A Primer for Words

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- By convention, $\{\}^* = \{\epsilon\}$

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  - $Pref(aaba) = \{\epsilon, a, aa, aab, aaba\}$
  - Proper prefixes = $\{a, aa, aab\}$
  - ϵ, $aaba$ improper prefixes

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  - $A = \{a, ba\}$, $B = \{\epsilon, aa, bb\}$
  - $AB = \{a, a^3, abb, ba, ba^3, babb\}$
  - $BA = \{a, ba, a^3, aaba, bba, bbba\}$

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- Concatenation does not distribute over interesection
  - $A = \{a, ab\}, B = \{b\}, C = \{\epsilon\}$
  - $A(B \cap C) \neq AB \cap AC$

FO for Languages

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- L_5 = Words in which whenever an a occurs, it is followed eventually by a b, and no c occurs in between the a and the b
  $aabbabab, aabbcbccaab \in L_5$, $aacaab \notin L_5$.

Satisfiability of FO over Words

- Recall : Given an FO sentence φ over words, is $L(\varphi) = \emptyset$?

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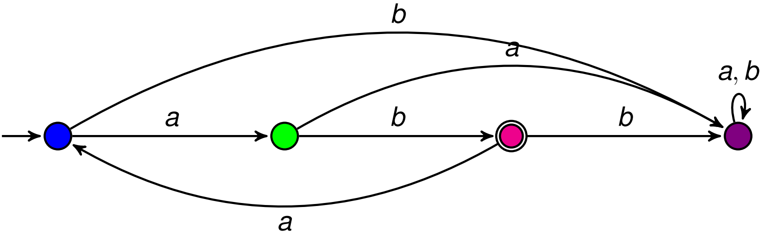
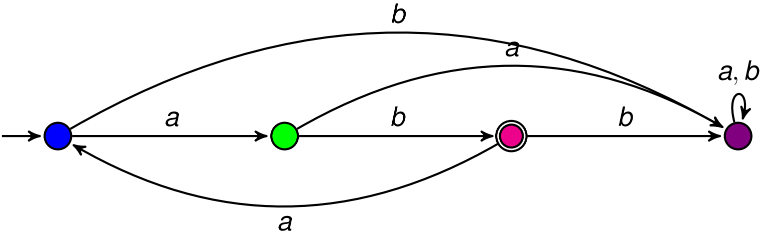
- Algorithm?
- Given φ, can we easily convert φ into some other mechanism M, which we know how to deal with?

In Search of a Mechanism

A First Machine A

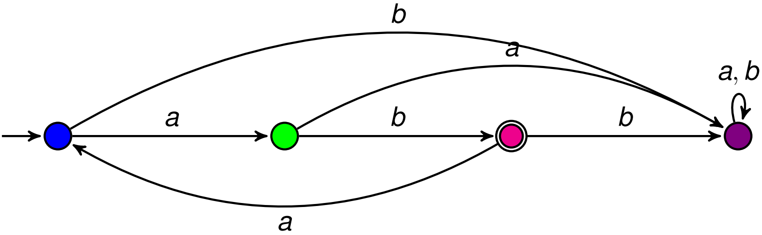


A First Machine A



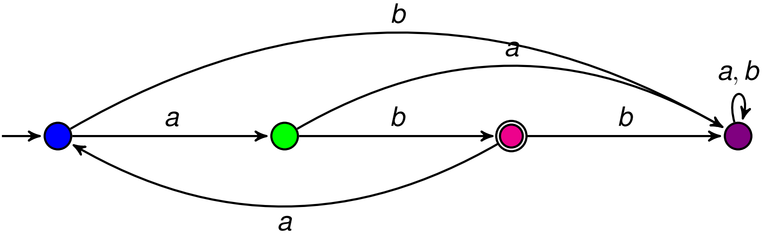
- Colored circles called states

A First Machine *A*



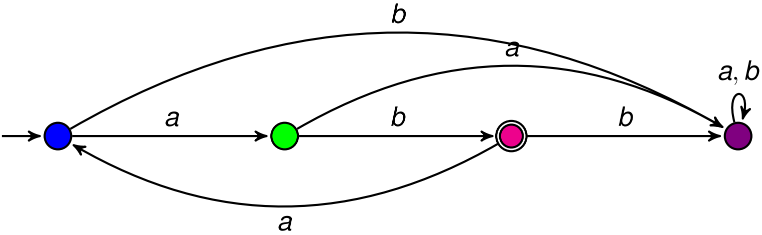
- Colored circles called states
- Arrows between circles called transitions

A First Machine A



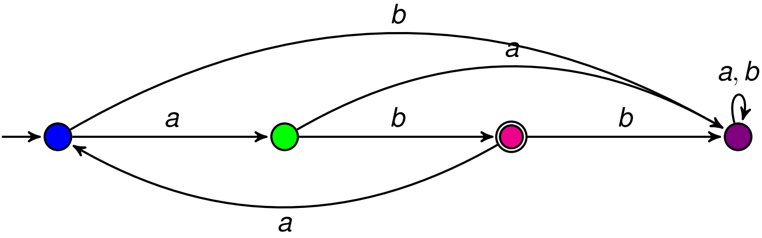
- Colored circles called states
- Arrows between circles called transitions
- Blue state called an initial state

A First Machine A



- Colored circles called states
- Arrows between circles called transitions
- Blue state called an initial state
- Doubly circled state called a final state

A First Machine A



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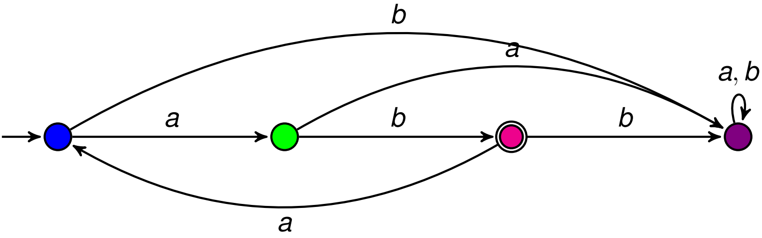

- A path from one state to another gives a word over $\Sigma = \{a, b, c\}$

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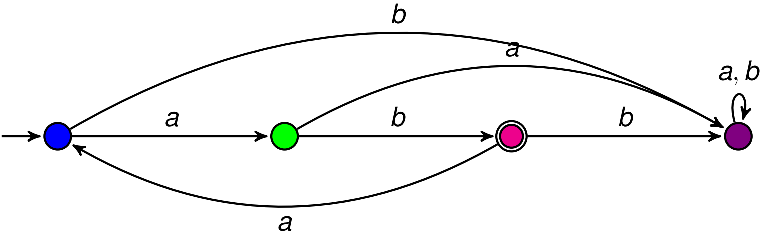

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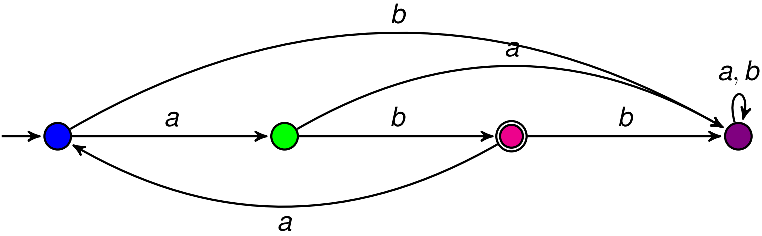
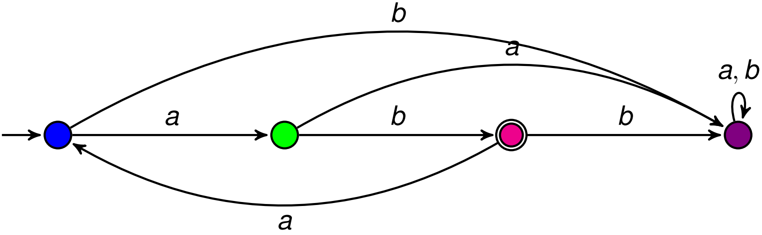
- The machine accepts words along paths from an initial state to a final state

A First Machine A



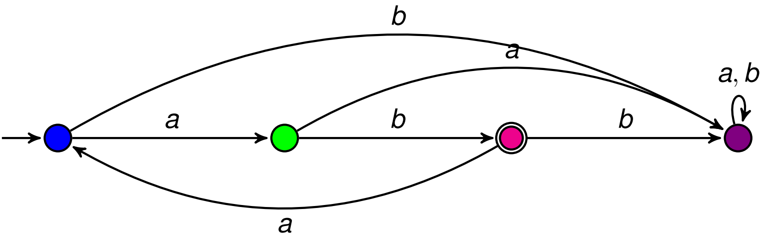
- A path from one state to another gives a word over $\Sigma = \{a, b, c\}$
- The machine accepts words along paths from an initial state to a final state
- The set of words accepted by the machine is called the language accepted by the machine

A First Machine A



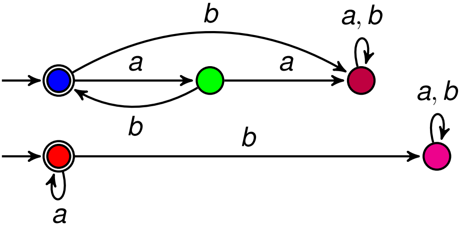
- What is the language L accepted by this machine, $L(A)$?

A First Machine A



- What is the language L accepted by this machine, $L(A)$?
- Write an FO formula φ such that $L(\varphi) = L(A)$

A Second and a Third Machine B, C



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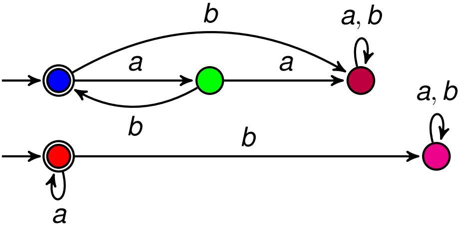

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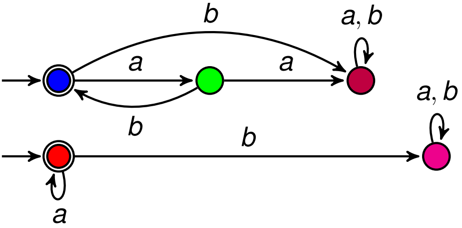

- What are $L(B), L(C)$?
- Give an FO formula φ such that $L(\varphi) = L(B) \cup L(C)$

Finite State Machines

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- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final states
- $L(A)$=all words leading from q_0 to some $f \in F$

Languages, Machines and Logic

A language $L \subseteq \Sigma^*$ is called regular iff there exists some DFA A such that $L = L(A)$.

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A language $L \subseteq \Sigma^*$ is called FO-definable iff there exists an FO formula φ such that $L = L(\varphi)$.