# CS 228 : Logic in Computer Science

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- ▶ Given  $p \rightarrow q$ , let us assume  $\neg q$ . Can we then prove  $\neg p$ ?
- ► Yes, using MT.

### The implies introduction rule $\rightarrow i$

۱.	extstyle p  ightarrow q	premise
2.	$\neg q$	assumption

3. 
$$\neg p$$
 assumption  $\neg p$  MT 1,2

$$\neg q 
ightarrow \neg p 
ightarrow i 2-3$$

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

true

2.

premise

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

- 1. true premise
- 2.  $q \rightarrow r$  assumption 3.

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q  o r	assumption
3.	eg q  o  eg p	assumption
4.		

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1.	true	premise
2.	$q \rightarrow r$	assumption
3.	eg q  o  eg p	assumption
4.	p	assumption
5.		

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q  o r	assumption
3.	eg q  ightarrow  eg p	assumption
4.	p	assumption
5.	$  \   \ \neg \neg p$	¬¬ <i>i</i> 4
6.		

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q  o r	assumption
3.	eg q  o  eg p	assumption
4.	p	assumption
5.	$  \   \ \neg \neg p$	¬¬ <i>i</i> 4
6.	$  \   \   \ \neg \neg q$	MT 3,5
7.		

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	$q \rightarrow r$	assumption
3.	eg q  ightarrow  eg p	assumption
4.	p	assumption
5.	$    \neg \neg \rho$	¬¬ <i>i</i> 4
6.		MT 3,5
7.	q	¬¬e 6
8.		

1 truo

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

١.	uue	premise
2.	$q \rightarrow r$	assumption
3.	eg q  ightarrow  eg p	assumption
4.	p	assumption
5.		¬¬ <i>i</i> 4
6.		MT 3,5
7.	q	¬¬ <i>e</i> 6
8.	$  \cdot  $ r	MP 2.7

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$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	$q \rightarrow r$	assumption
3.	eg q  ightarrow  eg p	assumption
4.	p	assumption
5.		¬¬ <i>i</i> 4
6.	$  \neg \neg q$	MT 3,5
7.	q	¬¬ <i>e</i> 6
8.	r	MP 2,7
9.	$n \rightarrow r$	→ <i>i</i> 4-8

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q  o r	assumption
3.	eg q  o  eg p	assumption
4.	p	assumption
5.	$  \cdot   \cdot   \neg \neg p$	¬¬ <i>i</i> 4
6.	$  \cdot   \cdot   \neg \neg q$	MT 3,5
7.		¬¬ <i>e</i> 6
8.	r	MP 2,7
9.	ho  ightarrow r	→ <i>i</i> 4-8
10.	$(\neg q  ightarrow  eg p)  ightarrow (p  ightarrow r)$	→ <i>i</i> 3-9

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11.

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q  o r	assumption
3.	eg q  ightarrow  eg p	assumption
4.	P	assumption
5.	$      \neg \neg p$	¬¬ <i>i</i> 4
6.	$  \cdot   \cdot   \neg \neg q$	MT 3,5
7.		¬¬ <i>e</i> 6
8.	r	MP 2,7
9.	$p \rightarrow r$	→ <i>i</i> 4-8
10.	$(\lnot q  ightarrow \lnot p)  ightarrow (p  ightarrow r)$	→ <i>i</i> 3-9
11.	$(q  ightarrow r)  ightarrow [(\lnot q  ightarrow \lnot p)  ightarrow (p  ightarrow r)]$	→ <i>i</i> 2-10

## **Transforming Proofs**

- $ightharpoonup (q 
  ightarrow r), (\neg q 
  ightarrow \neg p), p \vdash r$
- ► Transform any proof  $\varphi_1, \ldots, \varphi_n \vdash \psi$  to  $\vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi) \ldots))$  by adding n lines of the rule  $\rightarrow i$

$$\begin{array}{ccc} \blacktriangleright & p \to (q \to r) \vdash (p \land q) \to r \\ & 1. & p \to (q \to r) & \mathsf{premise} \\ & 2. & \end{array}$$

$$\begin{array}{c|cccc} \blacktriangleright & p \rightarrow (q \rightarrow r) \vdash (p \land q) \rightarrow r \\ \hline & 1. & p \rightarrow (q \rightarrow r) & \text{premise} \\ \hline & 2. & p \land q & \text{assumption} \\ \hline & 3. & p & \land e_1 \ 2 \\ \hline & 4. & q & \land e_2 \ 2 \\ \hline & 5. & q \rightarrow r & \text{MP 1,3} \\ \hline & 6. & \end{array}$$

$$\begin{array}{c|cccc} \blacktriangleright & p \rightarrow (q \rightarrow r) \vdash (p \land q) \rightarrow r \\ & 1. & p \rightarrow (q \rightarrow r) & \text{premise} \\ & 2. & p \land q & \text{assumption} \\ & 3. & p & \land e_1 \ 2 \\ & 4. & q & \land e_2 \ 2 \\ & 5. & q \rightarrow r & \text{MP 1,3} \\ & 6. & r & \text{MP 4,5} \\ & 7. & \end{array}$$

#### The or introduction rule $\vee i_1$

$$\frac{\varphi}{\varphi\vee\psi}$$

#### The or introduction rule $\vee i_2$

$$\frac{\psi}{\varphi \vee \psi}$$

#### The or elimination rule $\vee e$

$$\frac{\varphi \vee \psi \qquad \varphi \vdash \chi \qquad \psi \vdash \chi}{\chi}$$

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

- 1.  $q \rightarrow r$
- 2

premise

$$P q \rightarrow r \vdash (p \lor q) \rightarrow (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	$p \lor q$	assumption
^		

3.

$$P q \rightarrow r \vdash (p \lor q) \rightarrow (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	$p \lor q$	assumption
3.	р	assumption
4.		

$$P q \rightarrow r \vdash (p \lor q) \rightarrow (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	p∨r	∨ <i>i</i> <sub>1</sub> 3
5.		

$$P q \rightarrow r \vdash (p \lor q) \rightarrow (p \lor r)$$

	q  o r	premise
2.	$p \lor q$	assumption
3.	p	assumption
ŀ.	p∨r	∨ <i>i</i> <sub>1</sub> 3
5.	q	assumption
<b>3</b> .		

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor r$	∨ <i>i</i> <sub>1</sub> 3
5.	q	assumption
6.	r	MP 1,5
7.		

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1.	q  o r	premise
2.	$p \lor q$	assumption
3.	р	assumption
4.	p∨r	∨ <i>i</i> ₁ 3
5.	q	assumption
6.	r	MP 1,5
7.	<i>p</i> ∨r	∨ <i>i</i> <sub>2</sub> 6

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	p∨q	assumption
3.	р	assumption
4.	p∨r	∨ <i>i</i> <sub>1</sub> 3
5.	q	assumption
6.	r	MP 1,5
7.	p∨r	∨ <i>i</i> <sub>2</sub> 6
8.	p∨r	∨ <i>e</i> 2, 3-4, 5-7

## **Or Elimination Example**

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor r$	∨ <i>i</i> <sub>1</sub> 3
5.	q	assumption
6.	r	MP 1,5
7.	$p \lor r$	∨ <i>i</i> <sub>2</sub> 6
8.	$p \lor r$	∨ <i>e</i> 2, 3-4, 5-7
9.		

### Or Elimination Example

$$P q \rightarrow r \vdash (p \lor q) \rightarrow (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	$p \lor q$	assumption
3.	р	assumption
4.	<i>p</i> ∨ <i>r</i>	∨ <i>i</i> <sub>1</sub> 3
5.	q	assumption
6.	r	MP 1,5
7.	p∨r	∨ <i>i</i> <sub>2</sub> 6
8.	p∨r	∨ <i>e</i> 2, 3-4, 5-7
9	$(p \lor q) \to (p \lor r)$	→ <i>i</i> 2-8

► 
$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$

1.  $(p \lor q) \lor r$  premise

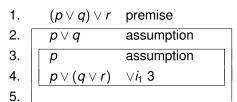
$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$

- 1.  $(p \lor q) \lor r$  premise
- 2.  $p \lor q$  assumption 3.

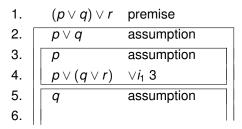
$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$

- 1.  $(p \lor q) \lor r$  premise
- 2.  $p \lor q$  assumption assumption
  - B. p assumption
    B.

$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$



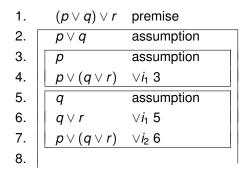
$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$



 $(p \lor q) \lor r \vdash p \lor (q \lor r)$ 

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	р	assumption
4.	$p \lor (q \lor r)$	√ <i>i</i> <sub>1</sub> 3
5.	q	assumption
6.	$q \vee r$	∨ <i>i</i> <sub>1</sub> 5
7.		

$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$



 $\blacktriangleright (p \lor q) \lor r \vdash p \lor (q \lor r)$ 

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor (q \lor r)$	√ <i>i</i> <sub>1</sub> 3
5.	q	assumption
6.	$  q \lor r$	∨ <i>i</i> <sub>1</sub> 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>2</sub> 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.		

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$ 

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>1</sub> 3
5.	q	assumption
6.	$q \vee r$	∨ <i>i</i> <sub>1</sub> 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>2</sub> 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	assumption
١0.		

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$ 

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	р	assumption
4.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>1</sub> 3
5.	q	assumption
6.	$q \vee r$	∨ <i>i</i> <sub>1</sub> 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>2</sub> 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	assumption
0.	$q \vee r$	√ <i>i</i> <sub>2</sub> 9
1.		

$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	р	assumption
4.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>1</sub> 3
5.	q	assumption
6.	$  q \lor r$	∨ <i>i</i> <sub>1</sub> 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>2</sub> 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	assumption
10.	$q \lor r$	√ <i>i</i> <sub>2</sub> 9
11.	$p \lor (q \lor r)$	√ <i>i</i> <sub>2</sub> 10

$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	р	assumption
4.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>1</sub> 3
5.	q	assumption
6.	$  q \lor r$	∨ <i>i</i> <sub>1</sub> 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>2</sub> 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	assumption
0.	$q \vee r$	√ <i>i</i> <sub>2</sub> 9
1.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>2</sub> 10
2.	$p \lor (q \lor r)$	∨ <i>e</i> 1, 2-8, 9-11

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1. true

premise

2.

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.		

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

	true	premise
2.	р	assumption
3.	q	assumption
ŀ.	p	copy 2

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption
4.	p	copy 2
5.	$oldsymbol{q} ightarrow oldsymbol{p}$	<i>→ i</i> 3-4

6.

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption
4.	р	copy 2
5.	q o p	<i>→ i</i> 3-4
6.	$a \rightarrow (a \rightarrow b)$	→ <i>i</i> 2-5

▶ We have seen  $\neg \neg e$  and  $\neg \neg i$ , the elimination and introduction of double negation.

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- How about introducing and eliminating single negations?
- ▶ We use the notion of contradictions, an expression of the form  $\varphi \land \neg \varphi$ , where  $\varphi$  is any propositional logic formula.

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- ▶ Any two contradictions are equivalent :  $p \land \neg p$  is equivalent to  $\neg r \land r$ .
- $ightharpoonup \perp \to \varphi$  for any formula  $\varphi$ .

#### Rules with $\bot$

#### The $\perp$ elimination rule $\perp e$

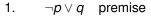
$$\frac{\perp}{\psi}$$

#### The $\perp$ introduction rule $\perp i$

$$\frac{\varphi \qquad \neg \varphi}{\bot}$$

- 1.  $\neg p \lor q$  premise
- 2.

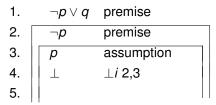
- 1.  $\neg p \lor q$  premise
- 2.  $\neg p$  premise
- 3.



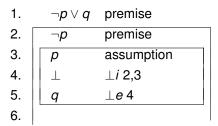
2.  $\neg p$  premise

3. 4. p assumption

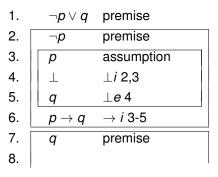
▶ 
$$\neg p \lor q \vdash p \rightarrow q$$



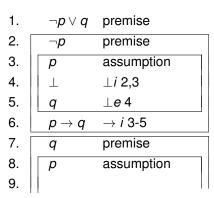
▶ 
$$\neg p \lor q \vdash p \rightarrow q$$



▶ 
$$\neg p \lor q \vdash p \rightarrow q$$



▶ 
$$\neg p \lor q \vdash p \rightarrow q$$



▶ 
$$\neg p \lor q \vdash p \rightarrow q$$

1.	$\neg p \lor q$	premise
2.	$\neg p$	premise
3.	р	assumption
4.		<i>⊥i</i> 2,3
5.	q	⊥ <i>e</i> 4
6.	p  o q	→ <i>i</i> 3-5
7.	q	premise
8.	р	assumption
9.	q	copy 7
0.	p  o q	<i>→ i</i> 8-9
1.	$oldsymbol{ ho}  ightarrow oldsymbol{q}$	∨ <i>e</i> 1, 2-6, 7-10

### **Introducing Negations**

- In the course of a proof, if you assume  $\varphi$  (by opening a box) and obtain  $\bot$  in the box, then we conclude  $\neg \varphi$
- ▶ This rule is denoted  $\neg i$  and is read as  $\neg$  introduction.

- 1.  $p \rightarrow \neg p$  premise
- 2.

# **An Example**

1.	p  ightarrow  eg p	premise
----	--------------------	---------

2. p assumption

3.

# **An Example**

1.	p  ightarrow  eg p	premise
2.	р	assumption
3.	eg p	MP 1,2
4.		

# **An Example**

1.	$oldsymbol{ ho}  ightarrow  eg eta$	premise
2.	р	assumption
3.	$\neg p$	MP 1,2
4.		<i>⊥i</i> 2,3
5.	$\neg p$	¬i 2-4

## **Natural Deduction: Summary**

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- ▶ If  $\psi$  is of the form  $\eta \to \kappa$ , then open a box assuming  $\eta$ , use the proof rules, as well as the premises and obtain  $\kappa$ .

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- ▶ If  $\psi$  is of the form  $\eta \to \kappa$ , then open a box assuming  $\eta$ , use the proof rules, as well as the premises and obtain  $\kappa$ .
- ▶ If  $\varphi$  is a disjunction of the form  $\psi_1 \vee \psi_2$ , then show that  $\psi_1 \vdash \psi$  and  $\psi_2 \vdash \psi$

#### The Proofs So Far

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- Now we show that whatever can be proved makes sense semantically too.

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- ▶ Two formulae  $\varphi$  and  $\psi$  are semantically equivalent iff  $\varphi \models \psi$  and  $\psi \models \varphi$

# **Soundness of Propositional Logic**

$$\varphi_1, \ldots, \varphi_n \vdash \psi \Rightarrow \varphi_1, \ldots, \varphi_n \models \psi$$

Whenever  $\psi$  can be proved from  $\varphi_1, \dots, \varphi_n$ , then  $\psi$  evaluates to true whenever  $\varphi_1, \dots, \varphi_n$  evaluate to true

▶ Assume  $\varphi_1, \ldots, \varphi_n \vdash \psi$ .

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- ▶ By inductive hypothesis, we have  $\varphi_1, \dots, \varphi_n \models \psi_1$  and  $\varphi_1, \dots, \varphi_n \models \psi_2$ . By semantics, we have  $\varphi_1, \dots, \varphi_n \models \psi_1 \land \psi_2$ .

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- ▶ Consider adding  $\psi_1$  in the premises along with  $\varphi_1, \ldots, \varphi_n$ . Then we will get a proof  $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$ , of length k-1. By inductive hypothesis,  $\varphi_1, \ldots, \varphi_n, \psi_1 \models \psi_2$ . By semantics, this is same as  $\varphi_1, \ldots, \varphi_n \models \psi_1 \rightarrow \psi_2$

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- ▶ The equivalence of  $\varphi_1, \ldots, \varphi_n \vdash \psi_1 \rightarrow \psi_2$  and  $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$  gives the proof.