Problem Set 4

1. Consider the formula $\varphi = \forall x \exists y R(x,y) \wedge \exists y \forall x \neg R(x,y)$. Show that φ is satisfiable over a structure whose universe is infinite and countable.

2. Let τ be a signature consisting of a binary relation P and a unary relation F. Let $\mathcal{F}$ be a structure consisting of a universe of people, $P(x,y)$ is interpreted on $\mathcal{F}$ as "x is a parent of y" and $F(x)$ is interpreted as "x is female". Given the τ-structure $\mathcal{F}$,
   – Define a formula $\varphi_B(x,y)$ which says x is a brother of y
   – Define a formula $\varphi_A(x,y)$ which says x is an aunt of y
   – Define a formula $\varphi_C(x,y)$ which says x and y are cousins
   – Define a formula $\varphi_O(x)$ which says x is an only child
   – Give an example of a family relationship that cannot be defined by a formula

3. Consider the signature τ that has the binary functions $+, \times$. Let $\mathcal{N}$ be the structure over τ having as universe the set $\mathbb{N}$ of natural numbers and which interprets $+, \times$ in the usual way. Construct FO formulae $Zero(x), One(x), Even(x), Odd(x)$ and $Prime(x)$ using τ such that
   – For any $a \in \mathbb{N}$, $\mathcal{N} \models Zero(a)$ iff a is zero.
   – For any $a \in \mathbb{N}$, $\mathcal{N} \models One(a)$ iff a is one.
   – For any $a \in \mathbb{N}$, $\mathcal{N} \models Even(a)$ iff a is even.
   – For any $a \in \mathbb{N}$, $\mathcal{N} \models Odd(a)$ iff a is odd.
   – For any $a \in \mathbb{N}$, $\mathcal{N} \models Prime(a)$ iff a is prime.
   – Goldbach's conjecture says that every even integer greater than 2 is the sum of two primes. Whether or not this is true is an open question in number theory. State Goldbach's conjecture as a FO-sentence over τ.

4. A group is a structure $(G, +, 0)$ where G is a set, $0 \in G$ is a special element called the identity and $+ : G \times G \to G$ is a binary operation such that
   (A1) The operation $+$ is associative
   (A2) The constant 0 is a right-identity for the operation $+$
   (A3) Every element in G has a right inverse: for each $x \in G$, we can find $y \in G$ such that $x + y = 0$
   (A4) For any three elements $x, y, z \in G$, if $x + z = y + z$, then $x = y$ Using a signature $\tau = (c, op)$ where c is a constant and op is a binary function symbol write (A1)-(A4) in FO.

5. Let τ be a signature consisting of the binary function symbol $+$ and a constant 0. We denote by $x + y$ the function $+(x,y)$. Consider the following sentences:

$$\varphi_1 : \forall x \forall y \forall z[(x + (y + z)) = ((x + y) + z)]$$

$$\varphi_2 : \forall x[(x + 0) = x \wedge (0 + x) = x]$$

$$\varphi_3 : \forall x[\exists y(x + y = 0) \wedge \exists z(z + x) = 0]$$

Let ψ be the conjunction of the three sentences.

- Show that ψ is satisfiable by exhibiting a τ-structure.
- Show that ψ is not valid.
- Let α be the sentence $\forall x \forall y((x + y) = (y + x))$. Does α follow as a consequence of ψ? That is, is it the case that $\psi \rightarrow \alpha$?
- Show that ψ is not equivalent to any of $\varphi_1 \wedge \varphi_2$, $\varphi_2 \wedge \varphi_3$ and $\varphi_1 \wedge \varphi_3$.

6. Explain the difference between the first order prefixes $\exists x \forall y \exists z$ and $\forall x \exists y \forall z$.

7. Show that the sentences $\forall x \exists y \forall z(E(x, y) \wedge E(x, z) \wedge E(y, z))$ and $\exists x \forall y \exists z(E(x, y) \wedge E(x, z) \wedge E(y, z))$ are not equivalent by exhibiting a graph which satisfies one but not both of the sentences.

8. For each $n \in \mathbb{N}$, $\exists^{\geq n}$ denotes a counting quantifier. Intuitively, $\exists^{\geq n}$ means that "there exist atleast n such that". FO with counting quantifiers is the logic obtained by adding these quantifiers (for each $n \in \mathbb{N}$) to the fixed symbols of FO. The syntax and semantics are as follows:

   Syntax : For any formula φ of FO with counting quantifiers, $\exists^{\geq n} x \varphi$ is also a formula.

   Semantics : $\mathcal{A} \models \exists^{\geq n} x \varphi$ iff $\mathcal{A} \models \varphi(a_i)$ for each of n distinct elements$a_1, a_2, \ldots, a_n$ from the universe $u(\mathcal{A})$.

   - Using counting quantifiers, define a sentence φ_{45} such that $\mathcal{A} \models \varphi_{45}$ iff $|u(\mathcal{A})| = 45$.
   - Define a FO sentence φ (not using counting quantifiers) that is equivalent to the sentence $\exists^{\geq n} x(x = x)$.

9. Write an FO formula that will evaluate to true only over a structure that has atleast n elements and atmost m elements.