# Sorting Lower Bounds

Abhiram Ranade

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Today: A non-trivial lower bound for sorting algorithms of a certain type.



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Example: determining if some x is present in a sorted array.  $O(\log n)$  suffices,  $\Omega(n)$  not needed.

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- ▶ We will prove that every decision tree algorithm must take time at least  $\Omega(n \log n)$ .

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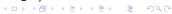
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Theorem 1: Bubble sort, ... heap sort are decision tree algorithms.

Theorem 2: Any decision tree takes time  $\Omega(n \log n)$ .



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So we can construct decision tree.

Theorem 2: Time =  $\Omega(n \log n)$ 

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Binary tree of height h can have at most  $2^h$  leaves.

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Consider a permutation  $\pi$  of  $1, \ldots, n$ .

We could always set  $x_1, \ldots, x_n$  so that final order is

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Worst case time = height.

