CS 228 : Logic in Computer Science

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- Q6: How is logic L used in computer science?
- Q7: What are the techniques needed to go about these questions?

Members of the mini-zoo

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- Linear Temporal Logic
- Their applications in CS

More if time permits!

References

- To start with, the text book of Huth and Ryan : Logic for CS. Already on Piazza.
- As we go ahead, lecture notes/monographs/other text books.
- Confirmed TAs: Karan Vaidya, Sai Sandeep Reddy, Saptarshi Sarkar, Shantanu Thakoor, Sourabh Ghurye, Vrunda Dave.
- Classes : Slot 6. Tutorial: Slot 14B (Fridays: 5.30 pm-7.00 pm)
- Who is the CR?

Propositional Logic

Syntax

- Finite set of propositional variables $p, q, \ldots$

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- Combine propositions using $\neg, \vee, \wedge, \rightarrow$
- Parantheses as required
- Example : $[p \wedge (q \vee r)] \rightarrow [\neg r \wedge p]$
- $\neg$ binds tighter than $\vee, \wedge$, which bind tighter than $\rightarrow$. In the absence of parantheses, $p \rightarrow q \rightarrow r$ is read as $p \rightarrow (q \rightarrow r)$

Natural Deduction

- If it rains, Tia is outside and does not have any raingear with her, she will get wet. $\varphi = (R \land \mathit{TiaOut} \land \neg RG) \to \mathit{TiaWet}$

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- It is raining, and Tia is outside, and is not wet. $\psi = (R \wedge \mathit{TiaOut} \wedge \neg \mathit{TiaWet})$

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- It is raining, and Tia is outside, and is not wet. $\psi = (R \wedge TiaOut \wedge \neg TiaWet)$
- So, Tia has her rain gear with her. RG
- Thus, $\chi = \varphi \wedge \psi \to RG$. You can deduce RG from $\varphi \wedge \psi$.
- Is χ valid? Is χ satisfiable?

An Application : Propositional Logic

Consider the following kid's version of Sudoku.

	2	4	
1			3
4			2
	1	3	

Rules:

- Each row must contain all numbers 1-4
- Each column must contain all numbers 1-4
- Each 2×2 block must contain all numbers 1-4
- No cell contains 2 or more numbers

1	2	3	4
4	3	2	1
3	4	1	2
2	1	4	3

Encoding as Propositional Satisfiability

- Proposition $P(i, j, n)$ is true when cell (i, j) has number n

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- Each row must contain all 4 numbers
  - Row 1: $[P(1, 1, 1) \vee P(1, 2, 1) \vee P(1, 3, 1) \vee P(1, 4, 1)] \wedge$
    $[P(1, 1, 2) \vee P(1, 2, 2) \vee P(1, 3, 2) \vee P(1, 4, 2)] \wedge$
    $[P(1, 1, 3) \vee P(1, 2, 3) \vee P(1, 3, 3) \vee P(1, 4, 3)] \wedge$
    $[P(1, 1, 4) \vee P(1, 2, 4) \vee P(1, 3, 4) \vee P(1, 4, 4)]$

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  - Row 2: $[P(2,1,1) \vee \ldots$
  - Row 3: $[P(3,1,1) \vee \ldots$
  - Row 4: $[P(4,1,1) \vee \ldots$

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- Column 4: $[P(1,4,1) \vee \ldots$

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- Upper left block contains all numbers 1-4:

$$[P(1,1,1) \vee P(1,2,1) \vee P(2,1,1) \vee P(2,2,1)] \wedge$$
$$[P(1,1,2) \vee P(1,2,2) \vee P(2,1,2) \vee P(2,2,2)] \wedge$$
$$[P(1,1,3) \vee P(1,2,3) \vee P(2,1,3) \vee P(2,2,3)] \wedge$$
$$[P(1,1,4) \vee P(1,2,4) \vee P(2,1,4) \vee P(2,2,4)]$$

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- Upper right block contains all numbers 1-4:

$$[P(1,3,1) \vee P(1,4,1) \vee P(2,3,1) \vee P(2,4,1)] \wedge \ldots$$

- Lower left block contains all numbers 1-4:

$$[P(3,1,1) \vee P(3,2,1) \vee P(4,1,1) \vee P(4,2,1)] \wedge \ldots$$

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Encoding as Propositional Satisfiability

No cell contains 2 or more numbers

- For cell(1,1):

$$P(1,1,1) \rightarrow [\neg P(1,1,2) \wedge \neg P(1,1,3) \wedge \neg P(1,1,4)] \wedge$$

$$P(1,1,2) \rightarrow [\neg P(1,1,1) \wedge \neg P(1,1,3) \wedge \neg P(1,1,4)] \wedge$$

$$P(1,1,3) \rightarrow [\neg P(1,1,1) \wedge \neg P(1,1,2) \wedge \neg P(1,1,4)] \wedge$$

$$P(1,1,4) \rightarrow [\neg P(1,1,1) \wedge \neg P(1,1,2) \wedge \neg P(1,1,3)] \wedge$$

- Similar for other cells

Encoding as Propositional Satisfiability

Encoding Initial Configuration:

$$P(1,2,2) \wedge P(1,3,4) \wedge P(2,1,1) \wedge P(2,4,3) \wedge$$

$$P(3,1,4) \wedge P(3,4,2) \wedge P(4,2,1) \wedge P(4,3,3)$$

Solving Sodoku

To solve the puzzle, just conjunct all the above formulae and find a satisfiable truth assignment!

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- $\varphi_1, \ldots, \varphi_n \vdash \psi$: This is called a sequent. Given $\varphi_1, \ldots, \varphi_n$, we can deduce or prove ψ. What was the sequent in Tia's case?

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- For example, $\neg p \to q, q \to r, \neg r \vdash p$ is a sequent. How do you prove this?

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- For example, $\neg p \rightarrow q, q \rightarrow r, \neg r \vdash p$ is a sequent. How do you prove this?
- Proof rules to be carefully chosen, for instance you shouldnt end up proving something like $p \wedge q \vdash \neg q$

Rules for Natural Deduction

The and introduction rule denoted $\wedge i$

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}$$

Rules for Natural Deduction

The and elimination rule denoted $\wedge e_1$

$$\frac{\varphi \wedge \psi}{\varphi}$$

The and elimination rule denoted $\wedge e_2$

$$\frac{\varphi \wedge \psi}{\psi}$$

A first proof using $\wedge i, \wedge e_1, \wedge e_2$

- Show that $p \wedge q, r \vdash q \wedge r$

1.	$p \wedge q$	premise
2.		

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1.	$p \wedge q$	premise
2.	r	premise
3.	q	$\wedge e_2$ 1
4.		

A first proof using $\wedge i, \wedge e_1, \wedge e_2$

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1.	$p \wedge q$	premise
2.	r	premise
3.	q	$\wedge e_2$ 1
4.	$q \wedge r$	$\wedge i$ 3,2

Rules for Natural Deduction

The rule of double negation elimination $\neg\neg e$

$$\frac{\neg\neg\varphi}{\varphi}$$

The rule of double negation introduction $\neg\neg i$

$$\frac{\varphi}{\neg\neg\varphi}$$

Rules for Natural Deduction

The implies elimination rule or Modus Ponens MP

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Another Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.		

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3.	p	premise
4.		

Another Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.		

Another Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.		

Another Proof

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1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.	$\neg\neg r$	MP 4,5
7.		

Another Proof

- Show that $p, p \rightarrow q, p \rightarrow (q \rightarrow \neg\neg r) \vdash r$

1.	$p \rightarrow (q \rightarrow \neg\neg r)$	premise
2.	$p \rightarrow q$	premise
3.	p	premise
4.	$q \rightarrow \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.	$\neg\neg r$	MP 4,5
7.	r	$\neg\neg e$ 6

Rules for Natural Deduction

Another implies elimination rule or Modus Tollens MT

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi}$$

A Proof

- Show that $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	premise
2.		

A Proof

- Show that $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	premise
2.	q	premise
3.		

A Proof

- Show that $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	premise
2.	q	premise
3.	$\neg\neg q$	$\neg\neg i$ 2
4.		

A Proof

- Show that $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	premise
2.	q	premise
3.	$\neg\neg q$	$\neg\neg i$ 2
4.	$\neg p$	MT 1,3