

## Problem Set 8

1. Write second order logic formulae to capture the following:
  - (a) There is a path from node  $s$  to node  $t$  in the graph. The signature is  $\tau = \{E\}$ .
  - (b) Every bounded non empty set has a least upper bound. The signature is  $\tau = \{\leq\}$
  - (c) The graph is acyclic. The signature is  $\tau = \{E\}$ .
2. Let  $\Sigma$  be a finite alphabet. The atomic formulae in MSO defined over  $\Sigma^*$  are  $x = y, x < y, S(x, y), X(x)$  and  $Q_a(x)$ ,  $a \in \Sigma$ . Consider the following logic called  $\text{MSO}_0$  having atomic formulae of the following forms:

$$\text{Sing}(X), X \subseteq Y, X < Y, S(X, Y), Q_a(X)$$

where

- $\text{Sing}(X)$  means that  $X$  is a SO variable of cardinality 1;
- $X \subseteq Y$  means that every element of the SO variable  $X$  is contained in the SO variable  $Y$ ;
- $X < Y$  means that SO variables  $X, Y$  have cardinality 1, and that the element in  $Y$  is greater than the element in  $X$ ;
- $S(X, Y)$  means that SO variables  $X, Y$  have cardinality 1, and  $Y$  contains the successor of the element in  $X$ ; and,
- $Q_a(X)$  means that all positions in  $X$  are decorated by  $a \in \Sigma$ .

If  $\varphi$  is an atomic formula in MSO, then  $\varphi \wedge \varphi, \neg\varphi, \varphi \vee \varphi, \forall x \varphi$  and  $\forall X \varphi$  are formulae in MSO. Similarly, if  $\varphi$  is an atomic formula in  $\text{MSO}_0$ , then,  $\varphi \wedge \varphi, \neg\varphi, \varphi \vee \varphi$  and  $\forall X \varphi$  are formulae in  $\text{MSO}_0$ .

Compare the expressiveness of  $\text{MSO}$  and  $\text{MSO}_0$ .

3. For the formula  $\exists x \forall y (x < y \rightarrow Q_a(y))$  give an equivalent  $\text{MSO}_0$  formula.
4. Write an MSO formula for  $x < y$  without using  $<$ . You can use  $S$  in your MSO formula.
5. Consider the following NFA  $N = (\{0, 1, 2, 3\}, \{a, b\}, \Delta, \{0\}, \{1\})$  with  $\Delta(0, b) = \{1\}$ ,  $\Delta(1, a) = \{2\}$ ,  $\Delta(2, a) = \{2\}$ ,  $\Delta(2, b) = \{3\}$  and  $\Delta(3, b) = \{0\}$ . Write an MSO formula with two SO variables that characterizes  $L(N)$ .