Quicksort and Selection

Abhiram Ranade

January 10, 2016

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- Quicksort is among the fastest sorting algorithms.
- Make sure you understand what we proved: Expected time for any instance is $O(n \log n)$. Hence also the expected time for the worst instance.

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Can be done in $O(n)$ time. Next.

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$\Rightarrow$ Total time = $O(n)$.

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Lemma: Using recursion tree $T(n = O(n)$ Proof soon.

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Solving $T(n) \leq T(\lfloor n/5 \rfloor) + T(n - 3\lfloor n/10 \rfloor) + cn$

Ignoring floors: Subproblem sizes add up to $9n/10$.
$\Rightarrow$ Work reduces by factor $9/10$ each level of recursion tree.
Work decreases geometrically by levels, so work = O(work at top level) $= O(n)$.

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- Exercise: Will the algorithm work if we took medians of 3-tuples or of 7-tuples?
- We can improve constants, e.g. 24/25. Key emphasis in this course: do not worry about the constant, but get a simple clean argument.

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$o(g) \subset O(g)$, $\quad \omega(g) \subset \Omega(g)$