Quicksort and Selection

Abhiram Ranade

January 10, 2016

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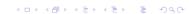
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$$\sum_{i < j} E[x_{ij}] = \sum_{j=2}^{j=n} \sum_{i=1}^{i=j-1} \frac{2}{j-i+1} \le 2 \sum_{j=2}^{j} j = n \ln j$$

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- Quicksort is among the fastest sorting algorithms.
- ► Make sure you understand what we proved: Expected time for any instance is O(n log n). Hence also the expected time for the worst instance.

Input: Keys x_1, \ldots, x_n , Integer r.

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Can be done in O(n) time.

Next.

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Adapt Quicksort analysis to give O(n) time

Homework

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Deterministic way to ensure p = approximate median.

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Proof soon.

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Lemma: Using recursion tree T(n = O(n))
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p = median of |n/5| medians of 5 tuples.

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Similarly |S|.

Solving $T(n) \le T(|n/5|) + T(n-3|n/10|) + cn$

Ignoring floors: Subproblem sizes add up to 9n/10.

Solving
$$T(n) \leq T(\lfloor n/5 \rfloor) + T(n-3\lfloor n/10 \rfloor) + cn$$

 \Rightarrow Work reduces by factor 9/10 each level of recursion tree.

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We stop recursion when $n < 50 \implies$ In the recurrence $n \ge 50$

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We stop recursion when n < 50 \Rightarrow In the recurrence $n \ge 50$

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Overall: O(n)



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- ▶ We can improve constants, e.g. 24/25. Key emphasis in this course: do not worry about the constant, but get a simple clean argument.

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$$o(g)\subset O(g), \quad \omega(g)\subset \Omega(g)$$

