CS 228 : Logic in Computer Science

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So Far

- Proof Rules
- Soundness : If $\varphi \vdash \psi$, then $\varphi \models \psi$.
- Completeness : If $\varphi \models \psi$, then $\varphi \vdash \psi$.
- Normal Forms
- Horn Formulae
- Resolution
- Quiz 1 on Jan 27

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- Let C_1, C_2 be two clauses. Assume $a \in C_1$ and $\neg a \in C_2$ for some atomic formula a. Then the clause $R = (C_1 - \{a\}) \cup (C_2 - \{\neg a\})$ is a resolvent of C_1 and C_2.

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- Let $C_1 = \{a_1, \neg a_2, a_3\}$ and $C_2 = \{a_2, \neg a_3, a_4\}$. As $a_3 \in C_1$ and $\neg a_3 \in C_2$, we can find the resolvent. A resolvent is $\{a_1, a_2, \neg a_2, a_4\}$.

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- Resolvent not unique : $\{a_1, a_3, \neg a_3, a_4\}$ is also a resolvent.

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Show that resolution can be used to determine whether any given formula is satisfiable.

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- There is some m such that $Res^m(F) = Res^{m+1}(F)$. Denote it by $Res^*(F)$.

Example

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- Since $\emptyset \notin Res^0(F)$ ($\emptyset$ is not a clause), there is an $m > 0$ such that $\emptyset \notin Res^m(F)$ and $\emptyset \in Res^{m+1}(F)$.
- Then $\{A\}, \{\neg A\} \in Res^m(F)$. By the rules of resolution, we have $F \vdash A, \neg A$, and thus $F \vdash \bot$. Hence, F is unsatisfiable.

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- Hence, $F = \{\{p\}, \{\neg p\}\}$. Clearly, $\emptyset \in Res(F)$.

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- Let $F_0 = \{C_i - \{p_{n+1}\} \mid C_i \in G_0\}$
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- Hence $F \equiv F_0 \vee F_1$.

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- If $p_{n+1} = \textit{false}$ in F, then F is equivalent to F_0
- If $p_{n+1} = \textit{true}$ in F, then F is equivalent to F_1
- Hence $F \equiv F_0 \vee F_1$.
- As F is unsatisfiable, F_0 and F_1 are both unsatisfiable.

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- Hence, $\emptyset \in Res^*(G_0)$ or $\{p_{n+1}\} \in Res^*(G_0)$, and $\emptyset \in Res^*(G_1)$ or $\{\neg p_{n+1}\} \in Res^*(G_1)$.

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- Hence $\emptyset \in \mathit{Res}^*(F)$.

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Given a formula ψ, convert it into CNF, say ζ. ψ is satisfiable iff $\emptyset \notin \mathit{Res}^*(\zeta)$.

- If ψ is unsat, we might get $\emptyset$ before reaching $\mathit{Res}^*(\zeta)$.

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- Time to move on!