CS 228 : Logic in Computer Science

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Summary

- Started looking at FO nondefinability
- Defined quantifier depth or quantifier rank of a formula
- Showed that there are finitely many FO formulae of quantifier rank r
- Introduced some new notations for words, mimicking assignments of values to free variables

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- If φ is atomic ($x = y, x < y, S(x, y), Q_a(x)$) then $c(\varphi) = 0$

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- Quantifier free formulae written in DNF : $C_1 \vee C_2 \vee \cdots \vee C_n$

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- $c(\exists\varphi) = c(\varphi) + 1$
- Quantifier free formulae written in DNF : $C_1 \vee C_2 \vee \cdots \vee C_n$
- Formulae of quantifier rank $c + 1$ written as a disjunction of the conjunction of formulae, each formula of the form $\exists x\varphi, \neg\exists x\varphi$ or φ, with $c(\varphi) \leqslant c$. Eliminate repeated disjuncts/conjunts

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- If $\mathcal{V}$ has 2 variables x, y, and τ has $Q_a, S, <$.
- Atomic formulae : $\{Q_a(x), Q_a(y), S(x,y), x < y\}$
- $G = \{Q_a(x), \neg Q_a(x), Q_a(y), \neg Q_a(y), S(x,y), \neg S(x,y), x < y, \neg(x < y)\}$
- Each subset of G is a possible conjunct C_i.
- All possible disjuncts using each C_i : formulae in DNF of rank 0

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- $2m$ atomic/negated atomic formulae
- Number of conjunctions C_i possible $\leqslant 2^{2m}$
- Number of formulae in DNF $\leqslant 2^{2^{2m}}$ ($c = 0$)

Rank 1

Let there be p formulae φ of rank 0.

- $2p$ formulae of the form $\exists x\varphi, \neg\exists x\varphi$
- 2^{2p} conjunctions of rank 1
- Conjuncting any one of the p formulae of rank 0 gives all conjuncts of rank $\leqslant 1$: $p2^{2p}$ more
- Possible conjuncts of rank $\leqslant 1$ is $q = (p+1)2^{2p}$
- Possible disjuncts of these : 2^q

Number of FO formulae of rank c

Let $\mathcal{V}$ be a finite set of first order variables, and let $c \geqslant 0$. There are finitely many FO formulae in DNF with rank c over $\mathcal{V}$.

Some Notation

Given a word $w = a_1 \dots a_n$, and a finite set of variables $\mathcal{V}$, define a $\mathcal{V}$-structure with respect to w as

- $(a_1, U_1)(a_2, U_2) \dots (a_n, U_n)$ where
- $\bigcup_i U_i = \mathcal{V}$
- $U_i \cap U_j = \emptyset$

- Think of a $\mathcal{V}$-structure as a word over the alphabet $\Sigma \times 2^{\mathcal{V}}$
- $(a, \{x\})(b, \{y, z\})(c, \emptyset)(d, \{u, v\})$ is a $\{x, y, z, u, v\}$-structure with respect to the word *abcd*.

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- $w \models (x = y)$ iff there exists j such that $x, y \in S_j$
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- $w \models (x = y)$ iff there exists j such that $x, y \in S_j$
  - $(a, \{x\})(b, \{y, z\})(c, \emptyset) \not\models (x = y)$
- $w \models x < y$ iff there exists $i < j$ such that $x \in S_i, y \in S_j$
  - $(a, \{x\})(b, \{y, z\})(c, \emptyset) \models x < y$

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  - $(a, \{x\})(b, \{y, z\})(c, \emptyset) \models x < y$
- $w \models \exists x Q_a(x)$ iff there exists i such that $(a_1, S_1) \dots (a_i, S_i \cup \{x\}) \dots (a_n, S_n) \models Q_a(x)$
  - $(b, \{y, z\})(a, \{u\})(c, \emptyset) \models \exists x Q_a(x)$ since $(b, \{y, z\})(a, \{x, u\})(c, \emptyset) \models Q_a(x)$

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Similarly, $(a,\emptyset)(a,\{x\})(b,\{y\}) \models (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])$ and $(a,\emptyset)(a,\emptyset)(b,\{x,y\}) \models (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])$

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- For a formula $\varphi(x_1, \ldots, x_m)$, $L(\varphi)$ is the set of all $\{x_1, \ldots, x_m\}$ structures satisfying φ
- For a sentence φ, $L(\varphi)$ is the set of all $\emptyset$ structures satisfying φ

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- $\sim_r$ is an equivalence relation
- Finitely many equivalence classes : each class consists of words that behave the same way on formulae of rank $\leqslant r$

Non-Expressibility in FO : The Game Begins

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- Duplicator wants to show that they are same ($w_1 \sim_r w_2$)
- Each player has r pebbles $z_1, \ldots, z_r$

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- A pebble once placed, cannot be removed
- The game ends after r rounds, when both players have used all their pebbles

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- Round 2:

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  - After round 1, we have two $\{z_1\}$ structures (w_1', w_2')
- Round 2:
  - Spoiler continues on the structure w_2'
  - Duplicator gets w_1' to play

A Play

- $w_1 = (a, \emptyset)(b, \emptyset)$ and $w_2 = (a, \emptyset)(b, \emptyset)(a, \emptyset)$
- 2 rounds, so 2 pebbles : z_1, z_2
- Spoiler picks w_2, duplicator picks w_1
- Round 1:
  - Spoiler : $(a, \{z_1\})(b, \emptyset)(a, \emptyset)$
  - Duplicator : $(a, \{z_1\})(b, \emptyset)$
  - After round 1, we have two $\{z_1\}$ structures (w_1', w_2')
- Round 2:
  - Spoiler continues on the structure w_2'
  - Duplicator gets w_1' to play
  - Spoiler : $(a, \{z_1\})(b, \emptyset)(a, \{z_2\})$

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  - Spoiler : $(a, \{z_1\})(b, \emptyset)(a, \{z_2\})$
  - Duplicator : $(a, \{z_1, z_2\})(b, \emptyset)$ or $(a, \{z_1\})(b, \{z_2\})$

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- That is, $w_1' \sim_0 w_2'$
- Spoiler wins otherwise.

Winner

Given two word structures (w_1, w_2), duplicator wins on (w_1, w_2) if for every atomic formula α, $w_1 \models \alpha$ iff $w_2 \models \alpha$

Play continues

- Who won in the earlier play?
- We had
  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\})$ and $(a, \{z_1, z_2\})(b, \emptyset)$
  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\}) \models (z_1 < z_2)$
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  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\}) \models Q_a(z_2)$
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- Spoiler wins in two rounds
- If the game was played only for one round, who will win?

Unique Winner

Given structures w_1, w_2, and a number of rounds r, exactly one of the players win.

Logical Equivalence and Winning

Let w_1, w_2 be $\mathcal{V}$-structures and let $r \geqslant 0$. Then $w_1 \sim_r w_2$ iff Duplicator has a winning strategy in the r-round game on (w_1, w_2).

Logical Equivalence and Winning

Assume $w_1 \sim_r w_2$, and induct on r

- Base : $r = 0$ and $w_1 \sim_0 w_2$. Duplicator wins, since by assumption, w_1, w_2 agree on all atomic formulae.

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- Assume for $r - 1$: $w_1 \sim_{r-1} w_2 \Rightarrow$ Duplicator has a winning strategy in a $r - 1$ round game

Logical Equivalence and Winning

- Now, let $w_1 \sim_r w_2$, and assume spoiler wins the r-round game on (w_1, w_2).
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  - By assumption, spoiler wins the $r-1$ round game on (w_1', w_2')
  - By inductive hypothesis, $w_1' \nsim_{r-1} w_2'$
  - Let ψ be the conjunction of all formulae of rank $\leqslant r-1$ in normal form that are satisfied by w_1'
  - Then $w_1' \models \psi$, $w_2' \not\models \psi$
  - We thus have

$$w_1 \models \exists z_1 \psi, w_2 \not\models \exists z_1 \psi$$

  contradicting $w_1 \sim_r w_2$