Problem Set 7

1. Let $L_n = \{x \mid x \text{ is a binary number that is a multiple of } n\}$. Show that for all $n \geq 1$, L_n is regular.

2. Recall that we defined an angelic acceptance condition for NFAs in class : a word w is accepted whenever it has atleast one accepting run. Under this, we showed that the languages accepted by NFAs are regular. Consider the following *devilish* acceptance condition, which says that an NFA M accepts a word x iff every possible computation of M on x ends in an accept state. Show that NFAs with the devilish acceptance condition recognize the class of regular languages.

3. Let L be a regular language. Consider the language L' defined as

$$\{w \in L \mid \text{no proper prefix of } w \text{ is in } L\}$$

Show that L' is regular.

4. Show that any DFA accepting the language

$$L_n = \{x \in \{0,1\}^* \mid \text{ the } n \text{ th bit from the right is a 1 }\}$$

has $\geq 2^{n-1}$ states.

5. Find the smallest r such that the following pairs of words are r-distinguishable ($w_1 \nsim_r w_2$) with respect to $FO[<]$ by playing a game:
(a) $((ab)^n b(ab)^n, (ab)^m (ab)^m)$
(b) $((aa)^n b(aaa)^n, (aaa)^n b(aa)^n)$
(c) $((aabbacb)^n, (aabbacb)^m aab(aabbacb)^k)$
In each case, write the $FO[<]$ that distinguishes the two words.

6. Say that S is a star-free expression over an alphabet Σ if S is
(1) a for some $a \in \Sigma$
(2) ϵ,
(3) $\emptyset$,
(4) $S_1 + S_2$ where S_1, S_2 are both star-free expressions,
(5) $S_1.S_2$ where S_1, S_2 are both star-free expressions,
(6) $\neg S$ where S is a star-free expression.
Given a star-free expression S, the language of S denoted $L(S)$ is defined inductively as follows:
(1) $L(a) = \{a\}$
(2) $L(\epsilon) = \{\epsilon\}$
(3) $L(\emptyset) = \emptyset$
(4) $L(S_1 + S_2) = L(S_1) \cup L(S_2)$

(5) $L(S_1.S_2) = L(S_1).L(S_2)$
(6) $L(\neg S) = \overline{L(S)}$
The language $L(S)$ defined by a star-free expression S is also called star-free. For example, with $\Sigma = \{a, b, c\}$,
(a) Σ^* is star-free, denoted by $\neg\emptyset$;
(b) a^* is star-free, denoted by $\neg(\Sigma^*.(b + c).\Sigma^*)$.

Consider the following languages/expressions. In each case, either show that it is star-free, or show that it is not star-free by playing a game where duplicator always wins.
(a) $(ab)^*$
(b) a^+b^*
(c) aab^*aa
(d) there are atleast 3 occurrences of b, and before the first b there are atmost 2 occurrences of a
(e) Equal number of a's and b's
(f) $b(a^*bb)^*$