CS 228 : Logic in Computer Science

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Summary

- DFA, NFA, ϵ-NFA as formalisms for regular languages
- Emptiness checking of automata : easy
- Given FO formula φ, build an automaton A_φ preserving the language
- Satisfiability of FO reduces to non-emptiness of underlying automaton

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- Quantifier free formulae written in DNF : $C_1 \vee C_2 \vee \cdots \vee C_n$

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- $c(\exists\varphi) = c(\varphi) + 1$
- Quantifier free formulae written in DNF : $C_1 \vee C_2 \vee \cdots \vee C_n$
- Formulae of quantifier rank $c + 1$ written as a disjunction of the conjunction of formulae, each formula of the form $\exists x\varphi, \neg\exists x\varphi$ or φ, with $c(\varphi) \leqslant c$. Eliminate repeated disjuncts/conjunts

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- If $\mathcal{V}$ has 2 variables x, y, and τ has $Q_a, S, <$.
- Atomic formulae : $\{Q_a(x), Q_a(y), S(x, y), x < y\}$
- $G = \{Q_a(x), \neg Q_a(x), Q_a(y), \neg Q_a(y), S(x, y), \neg S(x, y), x < y, \neg(x < y)\}$
- Each subset of G is a possible conjunct C_i.
- All possible disjuncts using each C_i : formulae in DNF of rank 0

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- Number of conjunctions C_i possible $\leqslant 2^{2m}$
- Number of formulae in DNF $\leqslant 2^{2^{2m}}$ ($c = 0$)

Rank 1

Let there be p formulae φ of rank 0.

- $2p$ formulae of the form $\exists x\varphi, \neg\exists x\varphi$
- 2^{2p} conjunctions of rank 1
- Conjuncting any one of the p formulae of rank 0 gives all conjuncts of rank $\leqslant 1$: $p2^{2p}$ more
- Possible conjuncts of rank $\leqslant 1$ is $q = (p+1)2^{2p}$
- Possible disjuncts of these : 2^q

Number of FO formulae of rank c

Let $\mathcal{V}$ be a finite set of first order variables, and let $c \geqslant 0$. There are finitely many FO formulae in DNF with rank c over $\mathcal{V}$.

Some Notation

Given a word $w = a_1 \dots a_n$, and a finite set of variables $\mathcal{V}$, define a $\mathcal{V}$-structure with respect to w as

- $(a_1, U_1)(a_2, U_2)\dots(a_n, U_n)$ where
- $\bigcup_i U_i = \mathcal{V}$
- $U_i \cap U_j = \emptyset$

- Think of a $\mathcal{V}$-structure as a word over the alphabet $\Sigma \times 2^{\mathcal{V}}$
- $(a, \{x\})(b, \{y, z\})(c, \emptyset)(d, \{u, v\})$ is a $\{x, y, z, u, v\}$-structure with respect to the word *abcd*.

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- $w \models x < y$ iff there exists $i < j$ such that $x \in S_i, y \in S_j$
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  - $(a, \{x\})(b, \{y, z\})(c, \emptyset) \models x < y$
- $w \models \exists x Q_a(x)$ iff there exists i such that $(a_1, S_1) \dots (a_i, S_i \cup \{x\}) \dots (a_n, S_n) \models Q_a(x)$
  - $(b, \{y, z\})(a, \{u\})(c, \emptyset) \models \exists x Q_a(x)$ since $(b, \{y, z\})(a, \{x, u\})(c, \emptyset) \models Q_a(x)$

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Similarly, $(a, \emptyset)(a, \{x\})(b, \{y\}) \models (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])$ and $(a, \emptyset)(a, \emptyset)(b, \{x, y\}) \models (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])$