CS 228 : Logic in Computer Science

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So Far

- Syntax, semantics of LTL
- Temporal operators of LTL $\bigcirc$, U, $\Diamond$, $\Box$

Syntax of LTL

Given *AP*, a set of propositions,

- Propositional logic formulae over *AP*
  - $a \in AP$ (atomic propositions)
  - $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$
- Temporal Operators
  - $\bigcirc\varphi$ (Next φ)
  - $\varphi\, \mathsf{U} \psi$ (φ holds until a ψ-state is reached)

$$\varphi ::= a \mid \varphi \vee \varphi \mid \neg\varphi \mid \varphi \wedge \varphi \mid \bigcirc \varphi \mid \varphi\, \mathsf{U} \varphi$$

Semantics over Infinite Words

Given LTL formula φ over AP,

$$L(\varphi) = \{\sigma \in (2^{AP})^\omega \mid \sigma \models \varphi\}$$

Let $\sigma = A_0 A_1 A_2 \ldots$.

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- $\sigma \models \varphi \,\mathsf{U} \psi$ iff
  $\exists j \geqslant 0$ such that $A_j A_{j+1} \ldots \models \psi \wedge \forall 0 \leqslant i < j, A_i A_{i+1} \ldots \models \varphi$

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If $\sigma = A_0A_1A_2\ldots$, $\sigma \models \varphi$ is also written as $\sigma, 0 \models \varphi$. This simply means $A_0A_1A_2\ldots \models \varphi$. One can also define $\sigma, i \models \varphi$ to mean $A_iA_{i+1}A_{i+2}\ldots \models \varphi$ to talk about a suffix of the word σ satisfying a property.

Paths and States: Semantics

Let $TS = (S, S_0, \rightarrow, AP, L)$ be a transition system, and φ an LTL formula over AP

- For an infinite path π of TS,

$$\pi \models \varphi \text{ iff } \mathit{trace}(\pi) \models \varphi$$

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- For $s \in S$,

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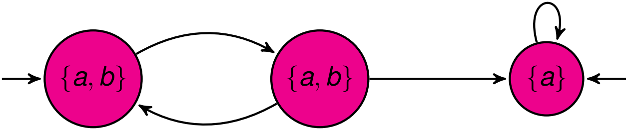
- $TS \models \varphi$ iff $\mathit{Traces}(TS) \subseteq L(\varphi)$

Paths and States: Semantics

Assume all states in TS are reachable from S_0.

- $TS \models \varphi$ iff $\pi \models \varphi \; \forall \pi \in Paths(TS)$
- $\pi \models \varphi \; \forall \pi \in Paths(TS)$ iff $s_0 \models \varphi \; \forall s_0 \in S_0$

Example



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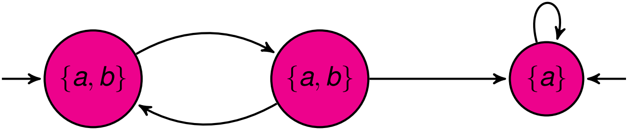

- $TS \models \Box a$,
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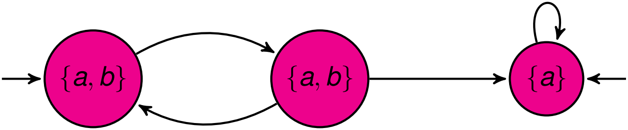

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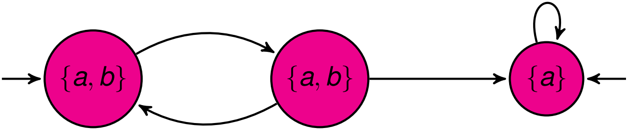

- $TS \models \Box a$,
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- $TS \not\models (b \, \mathsf{U} (a \wedge \neg b))$
- $TS \models \Box(\neg b \rightarrow \Box(a \wedge \neg b))$

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  $\mathit{trace}(\pi) \in L(\varphi)$ iff $\mathit{trace}(\pi) \notin L(\neg\varphi) = \overline{L(\varphi)}$
- $TS \not\models \varphi$ iff $TS \models \neg\varphi$?
  - $TS \models \neg\varphi \rightarrow \forall$ paths π of TS, $\pi \models \neg\varphi$
  - Thus, $\forall\pi$, $\pi \not\models \varphi$. Hence, $TS \not\models \varphi$

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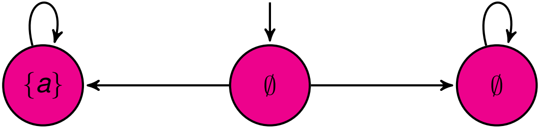
  - Thus, $\forall\pi$, $\pi \not\models \varphi$. Hence, $TS \not\models \varphi$
  - Now assume $TS \not\models \varphi$
  - Then $\exists$ some path π in TS such that $\pi \models \neg\varphi$
  - However, there could be another path π' such that $\pi' \models \varphi$
  - Then $TS \not\models \neg\varphi$ as well
- Thus, $TS \not\models \varphi \not\equiv TS \models \neg\varphi$.

An Example



$TS \not\models \Diamond a$ and $TS \not\models \Box\neg a$

Equivalence of LTL Formulae

Equivalence

φ and ψ are equivalent ($\varphi \equiv \psi$) iff $L(\varphi) = L(\psi)$.

Expansion Laws

- $\varphi \, \mathsf{U} \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \psi))$
- $\Diamond\varphi \equiv \varphi \vee \bigcirc\Diamond\varphi$
- $\Box\varphi \equiv \varphi \wedge \bigcirc\Box\varphi$

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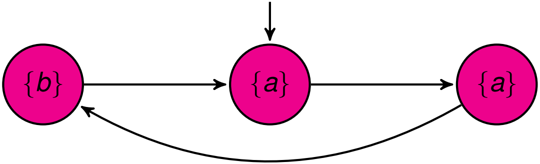

$$TS \models \Diamond a \wedge \Diamond b, TS \not\models \Diamond(a \wedge b)$$

$$TS \models \Box(a \vee b), TS \not\models \Box a \vee \Box b$$

Satisfiability, Model Checking LTL

Two Questions

1. Given transition system *TS*, and an LTL formula φ. Does *TS* $\models \varphi$?
2. Given an LTL formula φ, is $L(\varphi) = \emptyset$?

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How we go about this:

- Translate φ into an automaton A_φ that accepts infinite words such that $L(A_\varphi) = L(\varphi)$. Check (somehow) for emptiness of A_φ to check satisfiability of φ.

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- Check (somehow) $TS \cap \overline{A_\varphi}$ is empty, to answer the model-checking problem.

Notations for Infinite Words

- Σ is a finite alphabet
- Σ^* set of finite words over Σ
- Σ^ω set of infinite words over Σ
- An infinite word is written as $\alpha = \alpha(0)\alpha(1)\alpha(2)\ldots$, where $\alpha(i) \in \Sigma$
- Such words are called ω-words
- $\mathit{Inf}(\alpha) = \{a \in \Sigma \mid \alpha(i) = a \text{ for infinitely many } i\}$. $\mathit{Inf}(\alpha)$ gives the set of symbols occurring infinitely often in α.

ω-automata

An ω-automaton is a tuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, Acc)$ where

- Q is a finite set of states
- Σ is a finite alphabet
- $\delta : Q \times \Sigma \to 2^Q$ is a state transition function (if non-deterministic, otherwise, $\delta : Q \times \Sigma \to Q$)
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Büchi Acceptance

For Büchi Acceptance, Acc is specified as a set of states, $G \subseteq Q$. The ω-word α is accepted if there is a run ρ of α such that $Inf(\rho) \cap G \neq \emptyset$.

Comparing NFA and NBA

(Non)deterministic Büchi Automata

$L(\mathcal{A}) = \{\alpha \in \Sigma^\omega \mid \alpha$ has a run ρ such that $\mathit{Inf}(\rho) \cap G \neq \emptyset\}$

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$L(\mathcal{A}) = \{\alpha \in \Sigma^* \mid \alpha$ has a run ρ ending in some final state $\}$

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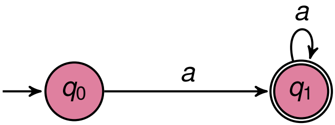

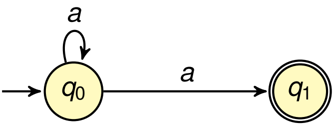

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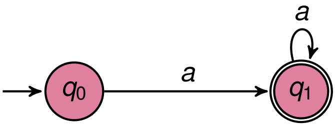
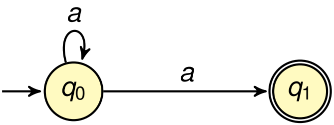
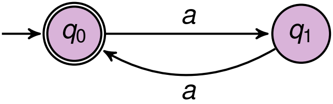
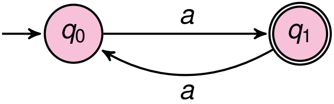
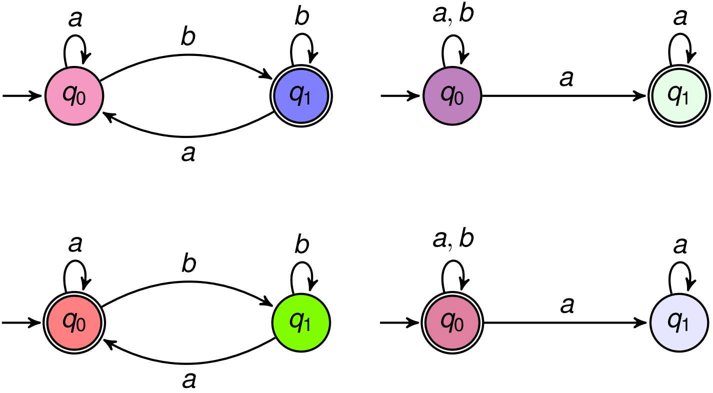








ω-Automata with Büchi Acceptance



- Left (T-B): Inf many b's, Inf many a's
- Right (T-B): Finitely many b's, $(a+b)^\omega$

Büchi Acceptance

A language $L \subseteq \Sigma^\omega$ is called ω-regular if there exists a NBA $\mathcal{A}$ such that $L = L(\mathcal{A})$. Recall definition of regular languages and NFA/DFA acceptance.

NBA and DBA

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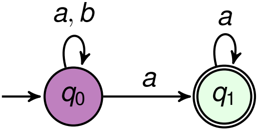
- Can we do subset construction on NBA and obtain DBA?



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