CS 228 : Logic in Computer Science

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Summary

- On the way to FO nondefinability
- Defined quantifier depth or quantifier rank of a formula
- Showed that there are finitely many FO formulae of quantifier rank r
- Introduced some new notations for words, mimicking assignments of values to free variables
- Started playing a game on word structures, and relating game equivalence and formula equivalence

Non-Expressibility in FO : The Game Begins

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- Duplicator wants to show that they are same ($w_1 \sim_r w_2$)
- Each player has r pebbles $z_1, \ldots, z_r$

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- The game ends after r rounds, when both players have used all their pebbles

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  - Spoiler : $(a, \{z_1\})(b, \emptyset)(a, \{z_2\})$
  - Duplicator : $(a, \{z_1, z_2\})(b, \emptyset)$ or $(a, \{z_1\})(b, \{z_2\})$

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- That is, $w_1' \sim_0 w_2'$
- Spoiler wins otherwise.

Winner

Given two word structures (w_1, w_2), duplicator wins on (w_1, w_2) if for every atomic formula α, $w_1 \models \alpha$ iff $w_2 \models \alpha$

Play continues

- Who won in the earlier play?
- We had
  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\})$ and $(a, \{z_1, z_2\})(b, \emptyset)$
  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\}) \models (z_1 < z_2)$
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- If the game was played only for one round, who will win?

Unique Winner

Given structures w_1, w_2, and a number of rounds r, exactly one of the players win.

Logical Equivalence and Winning

Let w_1, w_2 be $\mathcal{V}$-structures and let $r \geqslant 0$. At the end of r rounds, we are left with $\mathcal{V} \cup \{z_1, \ldots, z_r\}$-structures (w_1', w_2'). Then $w_1 \sim_r w_2$ iff $w_1' \sim_0 w_2'$ iff Duplicator has a winning strategy in the r-round game on (w_1, w_2).

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Assume $w_1 \sim_r w_2$, and induct on r

- Base : $r = 0$ and $w_1 \sim_0 w_2$. Duplicator wins, since by assumption, w_1, w_2 agree on all atomic formulae.

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- Now, let $w_1 \sim_r w_2$, and assume spoiler wins the r-round game on (w_1, w_2).
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  - Then $w_1' \models \psi$, $w_2' \not\models \psi$
  - We thus have

$$w_1 \models \exists z_1 \psi, w_2 \not\models \exists z_1 \psi$$

contradicting $w_1 \sim_r w_2$

Logical Equivalence and Winning : Converse

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- Assume for $r - 1$: Duplicator has a winning strategy in a $r - 1$ round game $\Rightarrow w_1 \sim_{r-1} w_2$

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- Now, let duplicator win in the r round game, but $w_1 \not\sim_r w_2$.
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  - In reply, duplicator keeps pebble z_1 on w_2 obtaining w_2'
  - By assumption, $w_2' \nvDash \varphi$
  - Also, by assumption, duplicator wins the $r - 1$ round game on (w_1', w_2') : this by inductive hypothesis says that $w_1' \sim_{r-1} w_2'$

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  - By assumption, $w_2' \not\models \varphi$
  - Also, by assumption, duplicator wins the $r - 1$ round game on (w_1', w_2') : this by inductive hypothesis says that $w_1' \sim_{r-1} w_2'$
  - That is, either both w_1', w_2' satisfy φ, or both dont, a contradiction.

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- We know $w_1 \models \varphi$ and $w_2 \not\models \varphi$ (hence, $w_1 \not\approx_r w_2$)
- Play a r-round game on (w_1, w_2). Then spoiler will win the r-round game.

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FO Definability

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Non FO Definability

For all $r \geqslant 0$, there exists a (w_1, w_2) pair with $w_1 \notin L, w_2 \in L$, duplicator wins in r rounds $\Rightarrow$ L is not FO definable

Non FO$[<]$ definability

- $FO[<, S] \subseteq FO[<]$
- Non definability in $FO[<]$ implies non definability in $FO[S, <]$

$(aa)^*$ is not $FO[<]$ Definable

- Assume that there is a sentence φ that defines words of even length, with $c(\varphi) = r$.
- Then, $a^i \models \varphi$ iff i is even
- Show that for all $r > 0$, $a^{2^r} \sim_r a^{2^r - 1}$

$(aa)^*$ is not $FO[<]$ Definable

- Base case : $(a, \emptyset)(a, \emptyset)$ and $(a, \emptyset)$ for $r = 1$
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- Consider $(aaaa, aaa)$ for $r = 3$. Who wins?
- Consider $(aaaa, aaa)$ for $r = 2$. Who wins?

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- Show that for all $k \geqslant 2^r - 1$, duplicator has a winning strategy for the r-round game in (a^k, a^{k+1}), for all $r \geqslant 0$
- Induct on r
- If $r = 1$, then on (a, aa) duplicator wins in one round
- Assume now that the claim is true for $\leqslant r - 1$

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- $s \leqslant \frac{k-1}{2}$ or $t \leqslant \frac{k-1}{2}$

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- Assume $s \leqslant \frac{k-1}{2}$. Duplicator puts her pebble z_1 on the $(s+1)$th letter of the other word obtaining

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- The structures after round 1 are thus

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- We have $2^r - 1 \leqslant k = min(t, t') + s + 1 \leqslant min(t, t') + \frac{k-1}{2} + 1$
- Hence $min(t, t') \geqslant \frac{k-1}{2} \geqslant 2^{r-1} - 1$
- By inductive hypothesis, duplicator has a winning strategy for the $r - 1$ round game on $(a^t, a^{t'})$.

Duplicator's Win

- Use the duplicator's winning strategy for the $r-1$ round game on $(a^t, a^{t'})$, to obtain a winning strategy in $r-1$ rounds on

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- Whenever spoiler plays on a structure on letter $i \leqslant s+1$, duplicator plays on the same position on the other structure
- When spoiler plays at a position $i > s+1$ in either word, duplicator plays in the part of the other word $> s+1$ using her winning strategy in $(a^t, a^{t'})$

Duplicator's Win

- At the end of r rounds, we have structures w'_1, w'_2.
- For $i \leqslant s+1$, pebble z_j appears at position i of w'_1 iff pebble z_j appears at position i of w'_2
- Lets erase the first $s+1$ letters in w'_1, w'_2, obtaining v'_1, v'_2
- v'_1, v'_2 are the words that result after $r' \leqslant r-1$ rounds of play on $(a^t, a^{t'})$. Recall that duplicator won this.
- Show that w'_1, w'_2 satisfy the same atomic formulae

Duplicator's Win

- Atomic Formulae : $Q_a(z_j)$: Both w'_1, w'_2 satisfy this.
- $w'_1 \models z_i < z_j$. If z_i, z_j are in the first $s+1$ letters, then $w'_2 \models z_i < z_j$.
- If z_i, z_j occur in the last $|w'_1| - s - 1$ positions, then $v'_1 \models z_i < z_j$. By duplicator's win in $(a^t, a^{t'})$, $v'_2 \models z_i < z_j$
- If z_i appears among the first $s+1$ letters and z_j after the first $s+1$ letters of w'_1, same is true in w'_2.

Ehrenfeucht and Fraïssé Games

The games that we saw are due to Ehrenfeucht and Fraïssé

Reference: Finite Automata, Formal Logic and Circuit Complexity, by Howard Straubing.

Summary

- Propositional Logic : syntax, semantics, proof rules, solving puzzles, soundness, completeness, resolution for satisfiability, normal forms
- FO : syntax, semantics using structures in general, satisfiability and validity, focus on word structures, satisfiability of FO on words using automata, FO-definability and FO non-definability, regular languages, closure properties