CS 228 : Logic in Computer Science

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MSO on Words : Satisfiability

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- MSO over words: Atomic formulae

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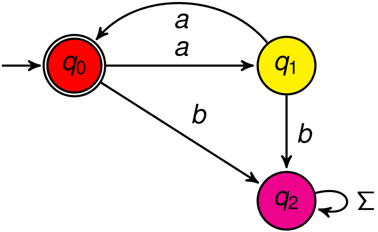
- Given a MSO sentence φ, $L(\varphi)$ defined as usual
- A language $L \subseteq \Sigma^*$ is MSO definable iff there is an MSO formula φ such that $L = L(\varphi)$
- Given an MSO sentence φ, is it satisfiable/valid?

Regular Languages to MSO



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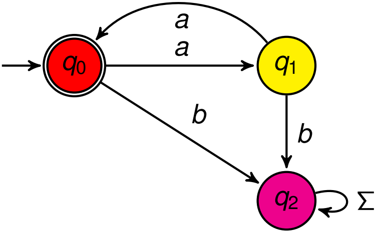

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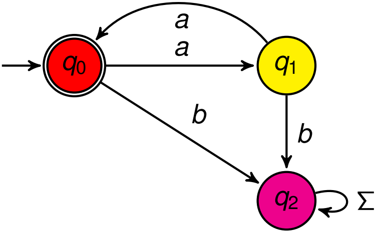

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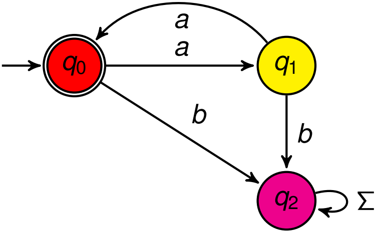

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- The initial position of any word must belong to X_{q_0} : $0 \in X_{q_0}$

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- If a word wa is accepted, then
  - The last position x of the word satisfies $Q_a(x)$
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- $X_{q_0}(0)$, $X_{q_1}(1)$ and $Q_a(0)$. $\delta(q_0, a) = q_1$.
- $X_{q_1}(1)$, $X_{q_0}(2)$ and $Q_a(1)$. $\delta(q_1, a) = q_0$.

Regular Languages to MSO

Given a DFA $A = (Q, \Sigma, \delta, q_0, F)$, a word w is accepted iff it satisfies

$$\exists X_0 \exists X_1 \dots X_n \{[\forall x (X_0(x) \vee \cdots \vee X_n(x)) \wedge \forall x \bigwedge_{i \neq j} \neg(X_i(x) \wedge X_j(x))] \wedge$$

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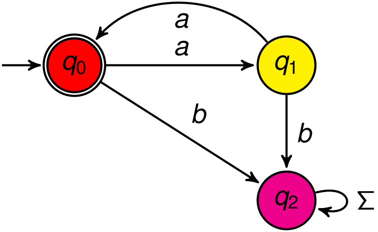
$$\exists x [\mathit{last}(x) \wedge \bigvee_{\delta(i,a)=j \in F} [X_i(x) \wedge Q_a(x)]]\}$$

- $w \in L(A)$ iff $w \models \varphi$

Example : Regular to MSO



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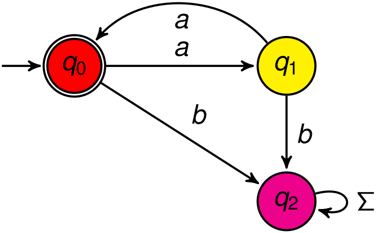

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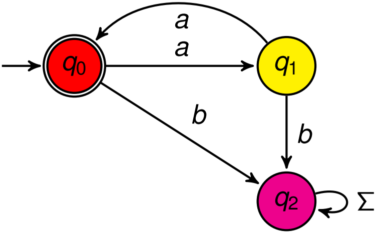

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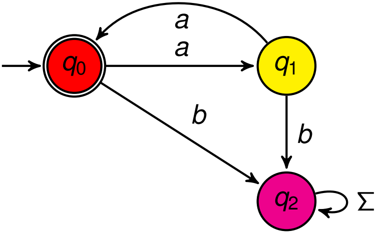

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$$\wedge \exists x [last(x) \wedge (X_1(x) \wedge Q_a(x))] \}$$

MSO to Regular Languages

- Every MSO sentence φ over words can be converted into a DFA A_φ such that $L(\varphi) = L(A_\varphi)$.
- Start with atomic formulae, construct DFA for each of them.
- Conjunctions, Disjunctions, Negation easily handled via union, intersection and complementation of respective DFA
- Handling quantifiers?

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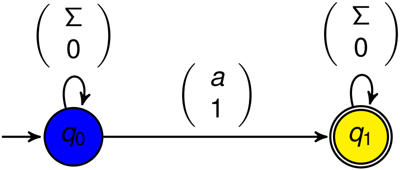
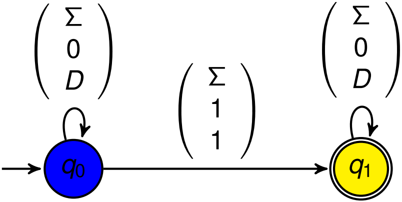
- Deterministic, not complete.



Simple Formulae to DFA

- $X(x)$



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- $Q_a(x) \wedge X(x)$ means that the position x is in the set X, and letter a is true when $x = 1$.

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- Think of a word *baab* which satisfies $Q_a(x) \wedge X(x)$ as

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$DD1D$		$D1DD$

  where D stands for *dont care*. X can have value 0 or 1 at D.

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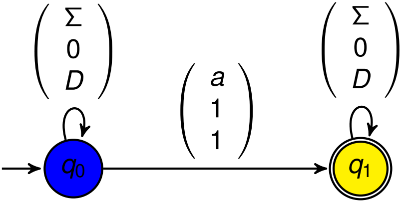
  where D stands for *dont care*. X can have value 0 or 1 at D.
- However, the position where $x = 1$ must belong to X.

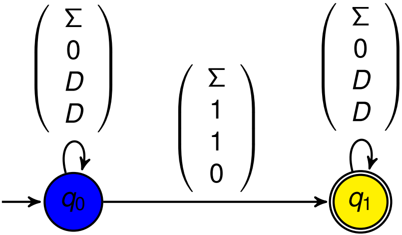
Simple Formulae to DFA

- The first row is over Σ, and the second row captures a possible assignment to x, and the third row captures a possible assignment to X.
- Think of an extended alphabet $\Sigma' = \Sigma \times \{0,1\} \times \{0,1\}$, and construct an automaton over Σ'.
- $Q_a(x) \wedge X(x)$: deterministic, not complete



Simple Formulae to DFA

- $X(x) \land \neg Y(x)$
- $\Sigma' = \Sigma \times \{0,1\} \times \{0,1\} \times \{0,1\}$



Formulae to DFA

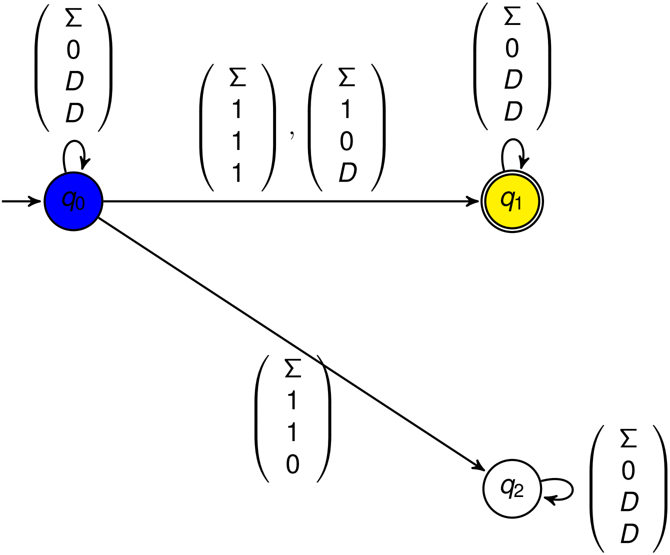
- Given $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m)$, an MSO formula over Σ, consider the extended alphabet

$$\Sigma' = \Sigma \times \{0, 1\}^{m+n}$$

- Assign values to x_i, X_j at every position as seen in the cases of atomic formulae
- Keep in mind that every x_i can be assigned 1 at a unique position

Handling Quantifiers

- $\exists X \exists Y \forall x [X(x) \rightarrow Y(x)]$



Handling Quantifiers : Summary

Quantifier Lemma

Let $L \subseteq (\Sigma \times \{0,1\}^{n+m})^*$ be defined by $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m)$. Let $f : (\Sigma \times \{0,1\}^{n+m})^* \to (\Sigma \times \{0,1\}^{n+m-1})^*$ be the projection $f(a, c_1, \ldots, c_n, c_{n+1}, \ldots, c_{n+m}) = (a, c_1, \ldots, c_{n-1}, c_{n+1}, \ldots, c_{n+m})$. Then $\exists x_n \varphi(x_1, \ldots, x_n, X_1, \ldots, X_m)$ defines $f(L)$.

Points to Remember

- Given $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m)$, construct automaton for atomic MSO formulae over the extended alphabet $\Sigma \times \{0, 1\}^{m+n}$
- Intersect with the regular language where every x_i is assigned 1 exactly at one position
- Given a sentence $Q_{x_1} \ldots Q_{x_n} Q_{X_1} \ldots Q_{X_m} \varphi$, first construct the automaton for the formula $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m)$
- Replace $\forall$ in terms of $\exists$

Points to Remember

- Given the automaton for $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_n)$, the automaton for $\exists X_i \varphi(x_1, \ldots, x_n, X_1, \ldots, X_n)$ is obtained by projecting out the row of X_i
- This may result in an NFA
- Determinize it and complement it to get a DFA for $\neg\exists X_i \varphi(x_1, \ldots, x_n, X_1, \ldots, X_n)$

The Automaton-Logic Connection

Given any MSO sentence φ, one can construct a DFA A_φ such that $L(\varphi) = L(A_\varphi)$. If a language L is regular, one can construct an MSO sentence φ such that $L = L(\varphi)$.

Reference for MSO : wolfgangaat.pdf posted online.