CS 228 : Logic in Computer Science

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Monadic Second Order Logic (MSO)

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- The symbols (and) called paranthesis

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- In a wff $\varphi = \forall X \psi$, every occurrence of X in ψ is bound
- A sentence is a formula with no free first order and second order variables

Assignments on τ-structures

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For a τ-structure $\mathcal{A}$, an assignment over $\mathcal{A}$ is a pair of functions (α_1, α_2), where

- $\alpha_1 : \mathcal{V}_1 \to u(\mathcal{A})$ assigns every first order variable $x \in \mathcal{V}_1$ a value $\alpha_1(x) \in u(\mathcal{A})$. If t is a constant symbol c, then $\alpha_1(t)$ is $c^{\mathcal{A}}$.

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Binding on a Variable

For an assignment $\alpha = (\alpha_1, \alpha_2)$ over $\mathcal{A}$, and $x \in \mathcal{V}_i$, $i = 1, 2$, $\alpha_i[x \mapsto a]$ is the assignment $\alpha_i[x \mapsto a](y) = \begin{cases} \alpha_i(y), y \neq x, \\ a, y = x \end{cases}$

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- $\mathcal{A} \models_\alpha (\forall X)\varphi$ iff for every $S \subseteq u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[X \mapsto S]} \varphi$

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$$\exists X \exists Y \exists Z (\forall x [X(x) \vee Y(x) \vee Z(x)] \wedge$$

$$\forall x \forall y [E(x,y) \rightarrow \{\neg(X(x) \wedge X(y)) \wedge \neg(Y(x) \wedge Y(y)) \wedge \neg(Z(x) \wedge Z(y))\}])$$

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$$\exists I\{\forall x \forall y[(\neg(x = y) \wedge I(x) \wedge I(y)) \rightarrow \neg E(x, y)] \wedge$$

$$\exists x_1 \dots x_k[\bigwedge_{i \neq j} \neg(x_i = x_j) \wedge \bigwedge_i I(x_i)]\}$$

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$$\wedge \forall x \forall y[S(x, y) \wedge O(x) \rightarrow E(y)]$$

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