CS 228 : Logic in Computer Science

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More Rules

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- Given $p \to q$, let us assume $\neg q$. Can we then prove $\neg p$?

More Rules

- Thanks to MT, we have $p \rightarrow q, \neg q \vdash \neg p$.
- Can we prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$?
- So far, no proof rule that can do this.
- Given $p \rightarrow q$, let us assume $\neg q$. Can we then prove $\neg p$?
- Yes, using MT.

The implies introduction rule $\rightarrow i$

- $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1.	$p \rightarrow q$	premise
2.	$\neg q$	assumption
3.	$\neg p$	MT 1,2
4.	$\neg q \rightarrow \neg p$	$\rightarrow i$ 2-3

More on $\to i$

- $\vdash (q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$

1.	*true*	premise
2.		

More on $\rightarrow i$

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1.	$true$	premise
2.	$q \rightarrow r$	assumption
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3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.		

More on $\rightarrow i$

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1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.		

More on $\rightarrow i$

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1.	*true*	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.		

More on $\rightarrow i$

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1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7

More on $\rightarrow i$

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1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7
9.	$p \rightarrow r$	$\rightarrow i$ 4-8

More on $\rightarrow i$

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1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7
9.	$p \rightarrow r$	$\rightarrow i$ 4-8
10.	$(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$	$\rightarrow i$ 3-9
11.		

More on $\rightarrow i$

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1.	*true*	premise
2.	$q \rightarrow r$	assumption
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8.	r	MP 2,7
9.	$p \rightarrow r$	$\rightarrow i$ 4-8
10.	$(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$	$\rightarrow i$ 3-9
11.	$(q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$	$\rightarrow i$ 2-10

Transforming Proofs

- $(q \to r), (\neg q \to \neg p), p \vdash r$
- Transform any proof $\varphi_1, \ldots, \varphi_n \vdash \psi$ to $\vdash \varphi_1 \to (\varphi_2 \to \ldots (\varphi_n \to \psi) \ldots))$ by adding n lines of the rule $\to i$

More Examples

- $p \to (q \to r) \vdash (p \land q) \to r$

1.	$p \to (q \to r)$	premise
2.		

More Examples

- $p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$

1.	$p \rightarrow (q \rightarrow r)$	premise
2.	$p \wedge q$	assumption
3.		

More Examples

- $p \to (q \to r) \vdash (p \land q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \land q$	assumption
3.	p	$\land e_1$ 2
4.		

More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge e_1$ 2
4.	q	$\wedge e_2$ 2
5.		

More Examples

- $p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$

1.	$p \rightarrow (q \rightarrow r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge e_1$ 2
4.	q	$\wedge e_2$ 2
5.	$q \rightarrow r$	MP 1,3
6.		

More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge e_1$ 2
4.	q	$\wedge e_2$ 2
5.	$q \to r$	MP 1,3
6.	r	MP 4,5
7.		

More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge e_1$ 2
4.	q	$\wedge e_2$ 2
5.	$q \to r$	MP 1,3
6.	r	MP 4,5
7.	$p \wedge q \to r$	$\to i$ 2-6

More Rules

The or introduction rule $\vee i_1$

$$\frac{\varphi}{\varphi \vee \psi}$$

The or introduction rule $\vee i_2$

$$\frac{\psi}{\varphi \vee \psi}$$

More Rules

The or elimination rule $\vee e$

$$\frac{\varphi \vee \psi \qquad \varphi \vdash \chi \qquad \psi \vdash \chi}{\chi}$$

Or Elimination Example

- $q \to r \vdash (p \lor q) \to (p \lor r)$

1.	$q \to r$	premise
2.		

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1.	$q \to r$	premise
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- $q \to r \vdash (p \lor q) \to (p \lor r)$

1.	$q \to r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.		

Or Elimination Example

- $q \to r \vdash (p \lor q) \to (p \lor r)$

1.	$q \to r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor r$	$\lor i_1$ 3
5.		

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$p \vee r$	$\vee i_1$ 3
5.	q	assumption
6.		

Or Elimination Example

- $q \to r \vdash (p \lor q) \to (p \lor r)$

1.	$q \to r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor r$	$\lor i_1$ 3
5.	q	assumption
6.	r	MP 1,5
7.		

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$p \vee r$	$\vee i_1$ 3
5.	q	assumption
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6

Or Elimination Example

- $q \to r \vdash (p \lor q) \to (p \lor r)$

1.	$q \to r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor r$	$\lor i_1$ 3
5.	q	assumption
6.	r	MP 1,5
7.	$p \lor r$	$\lor i_2$ 6
8.	$p \lor r$	$\lor e$ 2, 3-4, 5-7

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$p \vee r$	$\vee i_1$ 3
5.	q	assumption
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6
8.	$p \vee r$	$\vee e$ 2, 3-4, 5-7
9.		

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$p \vee r$	$\vee i_1$ 3
5.	q	assumption
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6
8.	$p \vee r$	$\vee e$ 2, 3-4, 5-7
9.	$(p \vee q) \to (p \vee r)$	$\to i$ 2-8

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.		

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2.	$p \vee q$	assumption
3.	p	assumption
4.		

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- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.		

Associativity Using Or Elimination

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1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	assumption
6.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	assumption
6.	$q \vee r$	$\vee i_1$ 5
7.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	assumption
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.		

Associativity Using Or Elimination

- $(p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor (q \lor r)$	$\lor i_1$ 3
5.	q	assumption
6.	$q \lor r$	$\lor i_1$ 5
7.	$p \lor (q \lor r)$	$\lor i_2$ 6
8.	$p \lor (q \lor r)$	$\lor e$ 2, 3-4, 5-7
9.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	assumption
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.	$p \vee (q \vee r)$	$\vee e$ 2, 3-4, 5-7
9.	r	assumption
10.		

Associativity Using Or Elimination

- $(p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor (q \lor r)$	$\lor i_1$ 3
5.	q	assumption
6.	$q \lor r$	$\lor i_1$ 5
7.	$p \lor (q \lor r)$	$\lor i_2$ 6
8.	$p \lor (q \lor r)$	$\lor e$ 2, 3-4, 5-7
9.	r	assumption
10.	$q \lor r$	$\lor i_2$ 9
11.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	assumption
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.	$p \vee (q \vee r)$	$\vee e$ 2, 3-4, 5-7
9.	r	assumption
10.	$q \vee r$	$\vee i_2$ 9
11.	$p \vee (q \vee r)$	$\vee i_2$ 10

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	assumption
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.	$p \vee (q \vee r)$	$\vee e$ 2, 3-4, 5-7
9.	r	assumption
10.	$q \vee r$	$\vee i_2$ 9
11.	$p \vee (q \vee r)$	$\vee i_2$ 10
12.	$p \vee (q \vee r)$	$\vee e$ 1, 2-8, 9-11

The Copy Rule

- $\vdash p \to (q \to p)$

1.	*true*	premise
2.		

The Copy Rule

- $\vdash p \rightarrow (q \rightarrow p)$

1.	*true*	premise
2.	p	assumption
3.		

The Copy Rule

- $\vdash p \to (q \to p)$

1.	*true*	premise
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3.	q	assumption
4.		

The Copy Rule

- $\vdash p \to (q \to p)$

1.	*true*	premise
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3.	q	assumption
4.	p	copy 2
5.		

The Copy Rule

- $\vdash p \to (q \to p)$

1.	*true*	premise
2.	p	assumption
3.	q	assumption
4.	p	copy 2
5.	$q \to p$	$\to i$ 3-4
6.		

The Copy Rule

- $\vdash p \rightarrow (q \rightarrow p)$

1.	*true*	premise
2.	p	assumption
3.	q	assumption
4.	p	copy 2
5.	$q \rightarrow p$	$\rightarrow i$ 3-4
6.	$p \rightarrow (q \rightarrow p)$	$\rightarrow i$ 2-5

The Rules of Single Negtion

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- How about introducing and eliminating single negations?
- We use the notion of contradictions, an expression of the form $\varphi \wedge \neg\varphi$, where φ is any propositional logic formula.
- Any two contradictions are equivalent : $p \wedge \neg p$ is equivalent to $\neg r \wedge r$.
- $\perp \rightarrow \varphi$ for any formula φ.

Rules with $\perp$

The $\perp$ elimination rule $\perp e$

$$\frac{\perp}{\psi}$$

The $\perp$ introduction rule $\perp i$

$$\frac{\varphi \qquad \neg\varphi}{\perp}$$

An Example

- $\neg p \vee q \vdash p \rightarrow q$

1.	$\neg p \vee q$	premise
2.		

An Example

- $\neg p \lor q \vdash p \to q$

1.	$\neg p \lor q$	premise
2.	$\neg p$	premise
3.		

An Example

- $\neg p \vee q \vdash p \rightarrow q$

1.	$\neg p \vee q$	premise
2.	$\neg p$	premise
3.	p	assumption
4.		

An Example

- $\neg p \vee q \vdash p \rightarrow q$

1.	$\neg p \vee q$	premise
2.	$\neg p$	premise
3.	p	assumption
4.	$\perp$	$\perp i$ 2,3
5.		

An Example

- $\neg p \lor q \vdash p \rightarrow q$

1.	$\neg p \lor q$	premise
2.	$\neg p$	premise
3.	p	assumption
4.	$\bot$	$\bot i$ 2,3
5.	q	$\bot e$ 4
6.		

An Example

- $\neg p \vee q \vdash p \rightarrow q$

1.	$\neg p \vee q$	premise
2.	$\neg p$	premise
3.	p	assumption
4.	$\bot$	$\bot i$ 2,3
5.	q	$\bot e$ 4
6.	$p \rightarrow q$	$\rightarrow i$ 3-5
7.	q	premise
8.		

An Example

- $\neg p \vee q \vdash p \rightarrow q$

1.	$\neg p \vee q$	premise
2.	$\neg p$	premise
3.	p	assumption
4.	$\bot$	$\bot i$ 2,3
5.	q	$\bot e$ 4
6.	$p \rightarrow q$	$\rightarrow i$ 3-5
7.	q	premise
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An Example

- $\neg p \lor q \vdash p \to q$

1.	$\neg p \lor q$	premise
2.	$\neg p$	premise
3.	p	assumption
4.	$\bot$	$\bot i$ 2,3
5.	q	$\bot e$ 4
6.	$p \to q$	$\to i$ 3-5
7.	q	premise
8.	p	assumption
9.	q	copy 7
10.	$p \to q$	$\to i$ 8-9
11.	$p \to q$	$\lor e$ 1, 2-6, 7-10

Introducing Negations

- In the course of a proof, if you assume φ (by opening a box) and obtain $\perp$ in the box, then we conclude $\neg\varphi$
- This rule is denoted $\neg i$ and is read as $\neg$ introduction.

An Example

- $p \rightarrow \neg p \vdash \neg p$

1.	$p \rightarrow \neg p$	premise
2.		

An Example

- $p \to \neg p \vdash \neg p$

1.	$p \to \neg p$	premise
2.	p	assumption
3.		

An Example

- $p \to \neg p \vdash \neg p$

1.	$p \to \neg p$	premise
2.	p	assumption
3.	$\neg p$	MP 1,2
4.		

An Example

- $p \to \neg p \vdash \neg p$

1.	$p \to \neg p$	premise
2.	p	assumption
3.	$\neg p$	MP 1,2
4.	$\bot$	$\bot i$ 2,3
5.	$\neg p$	$\neg i$ 2-4

Natural Deduction : Summary

- Given a sequent $\varphi \vdash \psi$, start writing all the premises on the top (think of φ as a conjunction of several formulae)

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- Assume (opening a box) $\neg\psi$, then obtain using the premises $\perp$. Infer henceforth that ψ holds.

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- Assume (opening a box) $\neg\psi$, then obtain using the premises $\perp$. Infer henceforth that ψ holds.
- If ψ is of the form $\eta \rightarrow \kappa$, then open a box assuming η, use the proof rules, as well as the premises and obtain κ.

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- Assume (opening a box) $\neg\psi$, then obtain using the premises $\perp$. Infer henceforth that ψ holds.
- If ψ is of the form $\eta \rightarrow \kappa$, then open a box assuming η, use the proof rules, as well as the premises and obtain κ.
- If φ is a disjunction of the form $\psi_1 \vee \psi_2$, then show that $\psi_1 \vdash \psi$ and $\psi_2 \vdash \psi$

The Proofs So Far

- So far, the “proof” we have seen is purely syntactic, no true/false meanings were attached

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- Now we show that whatever can be proved makes sense semantically too.

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Soundness of Propositional Logic

$\varphi_1, \ldots, \varphi_n \vdash \psi \Rightarrow \varphi_1, \ldots, \varphi_n \models \psi$

Whenever ψ can be proved from $\varphi_1, \ldots, \varphi_n$, then ψ evaluates to true whenever $\varphi_1, \ldots, \varphi_n$ evauate to true

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- Consider now a proof with k lines.

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- By inductive hypothesis, we have $\varphi_1, \ldots, \varphi_n \models \psi_1$ and $\varphi_1, \ldots, \varphi_n \models \psi_2$. By semantics, we have $\varphi_1, \ldots, \varphi_n \models \psi_1 \wedge \psi_2$.

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- Consider adding ψ_1 in the premises along with $\varphi_1, \ldots, \varphi_n$. Then we will get a proof $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$, of length $k - 1$. By inductive hypothesis, $\varphi_1, \ldots, \varphi_n, \psi_1 \models \psi_2$. By semantics, this is same as $\varphi_1, \ldots, \varphi_n \models \psi_1 \rightarrow \psi_2$

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- The equivalence of $\varphi_1, \ldots, \varphi_n \vdash \psi_1 \rightarrow \psi_2$ and $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$ gives the proof.