CS 228 : Logic in Computer Science

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- Given φ, denote by $\varphi(x_1, \ldots, x_n)$, that $x_1, \ldots, x_n$ are the free variables of φ, also $free(\varphi)$
- A sentence is a formula φ none of whose variables are free

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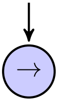

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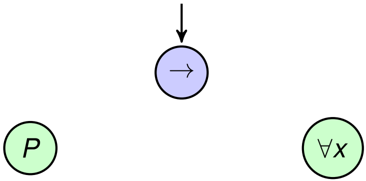

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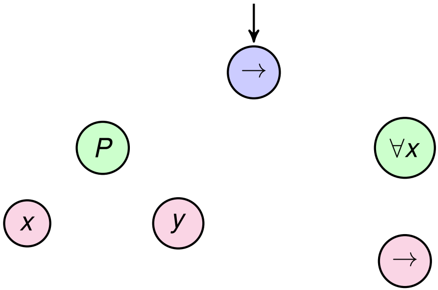

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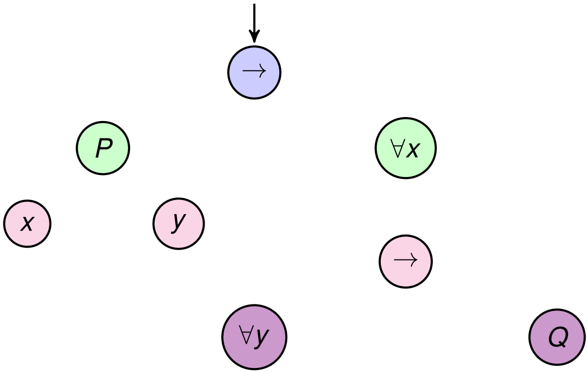

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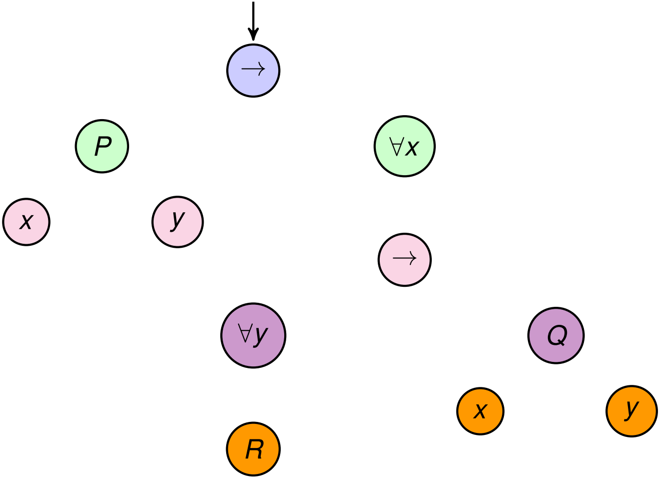

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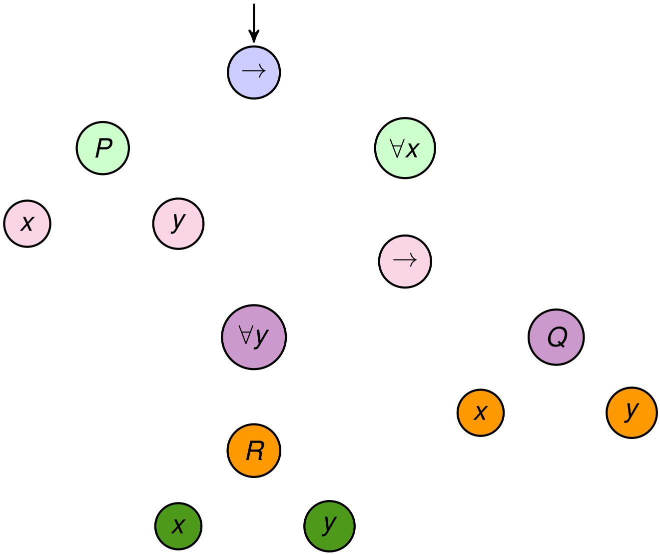

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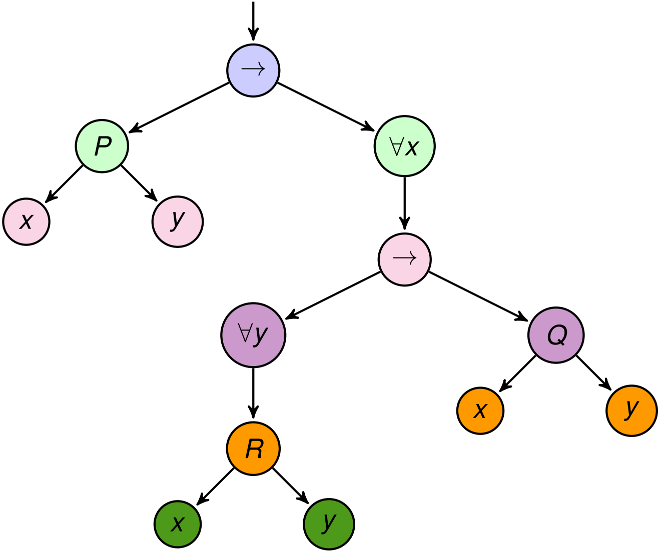

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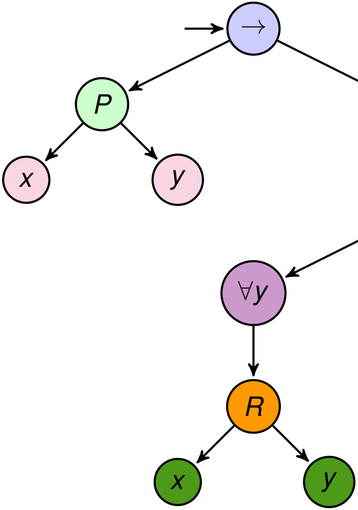

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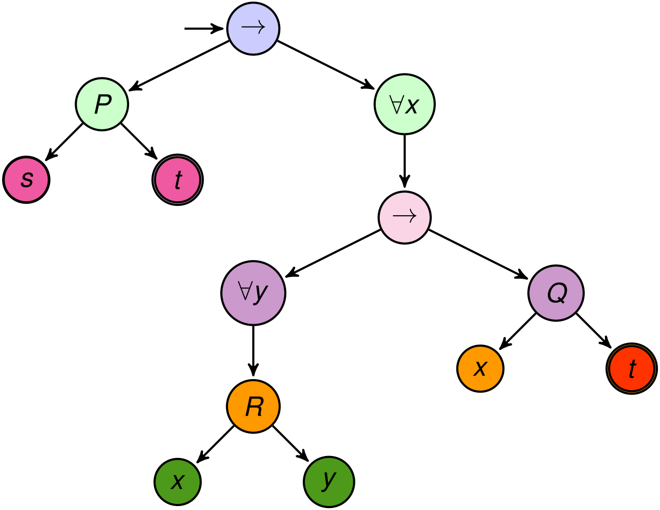

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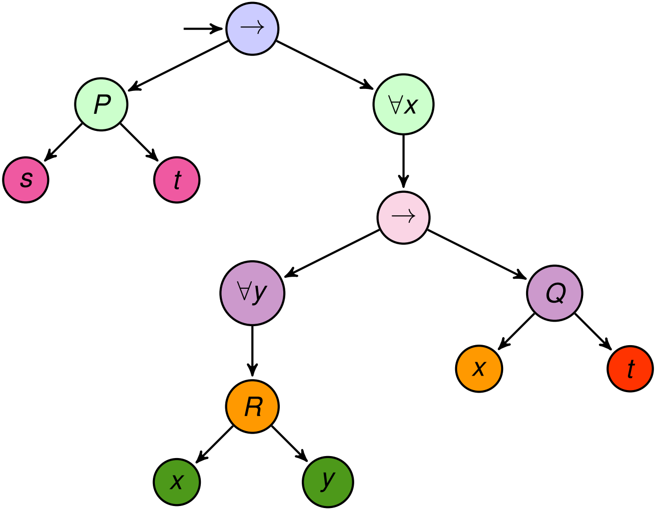

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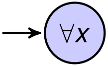

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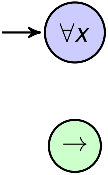

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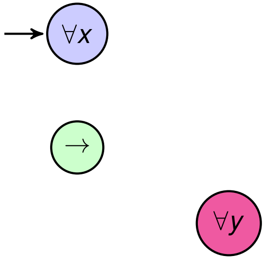

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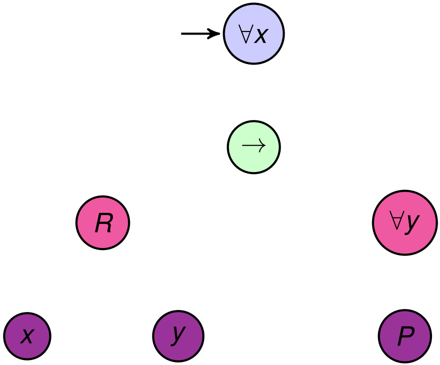

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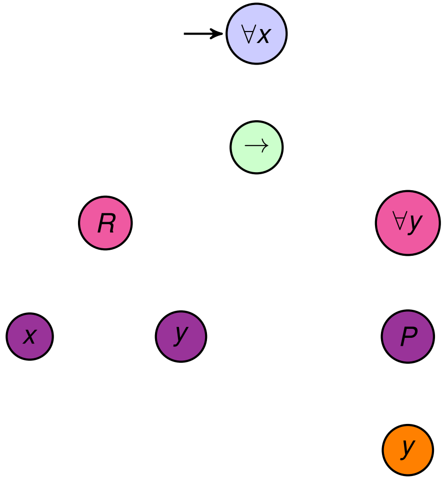

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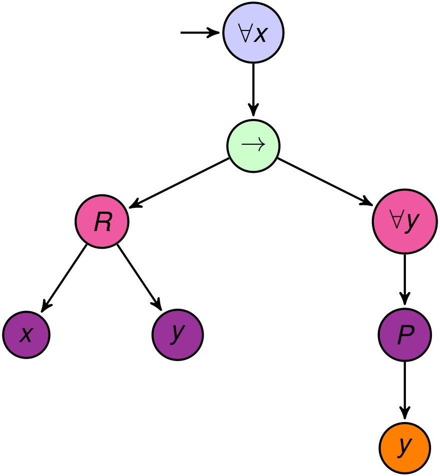

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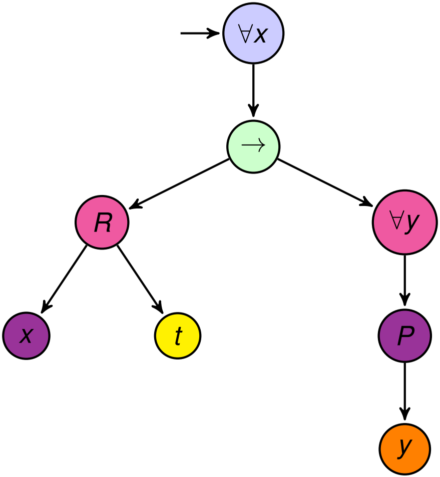

$\varphi(t) = \forall x(R(x,t) \to \forall y P(y))$

Assignments on τ-structures

Assignments

For a τ-structure $\mathcal{A}$, an assignment over $\mathcal{A}$ is a function $\alpha : \mathcal{V} \to u(\mathcal{A})$ that assigns every variable $x \in \mathcal{V}$ a value $\alpha(x) \in u(\mathcal{A})$. If t is a constant symbol c, then $\alpha(t)$ is $c^{\mathcal{A}}$

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Binding on a Variable

For an assignment α over $\mathcal{A}$, $\alpha[x \mapsto a]$ is the assignment

$$\alpha[x \mapsto a](y) = \begin{cases} \alpha(y), y \neq x, \\ a, y = x \end{cases}$$

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- $\mathcal{A} \models_\alpha (\exists x)\varphi$ iff there is some $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$

Last two cases, α has no effect on the value of x. Thus, assignments matter to free variables.

Example of Satisfaction

- $\mathcal{G} = (\{1, 2, 3\}, E^{\mathcal{G}} = \{(1, 2), (2, 1), (2, 3), (3, 2)\})$
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- Formulae φ and ψ are equivalent denoted $\varphi \equiv \psi$ iff for every $\mathcal{A}$ and α, $\mathcal{A} \models_\alpha \varphi$ iff $\mathcal{A} \models_\alpha \psi$

True or False?

For a formula φ and assignments α_1 and α_2 such that for every $x \in \mathit{free}(\varphi)$, $\alpha_1(x) = \alpha_2(x)$, $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

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No free variables!