Problem Set 2

1. Let $\mathcal{H}$ be a given set of premises. If $\mathcal{H} \vdash (A \to B)$ and $\mathcal{H} \vdash (C \vee A)$, then show that $\mathcal{H} \vdash (B \vee C)$ where A, B, C are wffs.

2. Let $\mathcal{H}$ be a given set of premises. If $\mathcal{H} \vdash (A \to C)$ and $\mathcal{H} \vdash (B \to C)$, then show that $\mathcal{H} \vdash ((A \vee B) \to C)$. Here, A, B and C are wffs.

3. Let $\mathcal{L}$ be a formulation of propositional logic in which the sole connectives are negation and disjunction. The rules of natural deduction corresponding to disjunction and negation (also includes double negation) are available. For any wffs A, B and C, let $\neg(A \vee B) \vee (B \vee C)$ be an axiom of $\mathcal{L}$. Show that any wff of $\mathcal{L}$ is a theorem of $\mathcal{L}$.

4. Let $\mathcal{P}$ denote propositional logic. Suppose we add to $\mathcal{P}$ the axiom schema $(A \to B)$ for wffs A, B of $\mathcal{P}$. Comment on the consistency of the resulting logical system obtained. A logic system $\mathcal{P}$ is inconsistent if it is capable of producing $\perp$ using the rules of natural deduction.

5. An adequate set of connectives is a set such that for every formula there is an equivalent formula with only connectives from that set. For example, $\{\neg, \vee\}$ is adequate for propositional logic since any occurrence of $\wedge$ and $\to$ can be removed using the equivalences

$$\varphi \to \psi \equiv \neg\varphi \vee \psi$$

$$\varphi \wedge \psi \equiv \neg(\neg\varphi \vee \neg\psi)$$

   - Show that $\{\neg, \wedge\}$, $\{\neg, \to\}$ and $\{\to, \perp\}$ are adequate sets of connectives. ($\perp$ treated as a nullary conenctive).
   - Show that if $C \subseteq \{\neg, \wedge, \vee, \to, \perp\}$ is adequate, then $\neg \in C$ or $\perp \in C$.

6. Consider a formula φ which is of the form $C_1 \wedge C_2 \wedge \dots C_n$ where each clause C_i is of the form $(\top \to \alpha)$ or $(\alpha_1 \wedge \dots \alpha_n \to \beta)$ or $(\gamma \to \perp)$ where $\alpha, \alpha_i, \beta, \gamma$ are literals. A logician wishes to apply *HornSAT* to this formula φ by renaming negative literals (if any) with fresh positive literals. Thus, if any $\alpha, \alpha_i, \beta, \gamma$ was of the form $\neg p$, the logician will replace that $\neg p$ with a fresh variable p'. The logician claims that he can check satisfiability of φ correctly by applying *HornSAT* on the new formula (call it φ') in the following way: φ is satisfiable iff *HornSAT* concludes that φ' is satisfiable, and φ is unsatisfiable iff *HornSAT* concludes that φ' is unsatisfiable. Do you agree with the logician?

7. If a contradiction can be derived from a set of formulae, then the set of formulae is said to be inconsistent. Otherwise, the set of formulae is consistent. Let $\mathcal{F}$ be a set of formulae. Show that $\mathcal{F}$ is consistent iff it is satisfiable.

8. Suppose $\mathcal{F}$ is an inconsistent set of formulae. For each $G \in \mathcal{F}$, let $\mathcal{F}_G$ be the set obtained by removing G from $\mathcal{F}$.
   (a) Prove that for any $G \in \mathcal{F}$, $\mathcal{F}_G \vdash \neg G$, using the previous question.
   (b) Prove this using a formal proof.

9. Let p and q be propositional variables. Consider the following two formulae φ_1, φ_2 where $\varphi_1 \equiv (p \to \neg\varphi_2)$ and $\varphi_2 \equiv (q \to \neg\varphi_1)$. Give two pairs of propositional logic formulae (φ_1, φ_2) that satisfy the above definitions.