Computation of Eigenvectors

Let *A* be a square matrix of order n and λ one of its eigenvalues. Let *X* be an eigenvector of *A* associated to λ. We must have

$$AX = \lambda X \quad \text{or} \quad \left(A - \lambda I_n\right)X = 0.$$

This is a linear system for which the matrix coefficient is $A - \lambda I_n$. Since the zero-vector is a solution, the system is consistent. In fact, we will in a different page that the structure of the solution set of this system is very rich. In this page, we will basically discuss how to find the solutions.

Remark. It is quite easy to notice that if *X* is a vector which satisfies $AX = \lambda X$, then the vector *Y* = *c X* (for any arbitrary number *c*) satisfies the same equation, i.e. $AY = \lambda Y$. In other words, if we know that *X* is an eigenvector, then *cX* is also an eigenvector associated to the same eigenvalue.

Let us start with an example.

Example. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}.$$

First we look for the eigenvalues of *A*. These are given by the characteristic equation $\det(A - \lambda I_3) = 0$, i.e.

$$\begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = 0.$$

If we develop this determinant using the third column, we obtain

$$\begin{vmatrix} 6 & -1-\lambda \\ -1 & -2 \end{vmatrix} + (-1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 6 & -1-\lambda \end{vmatrix} = 0.$$

Using easy algebraic manipulations, we get

$$-\lambda(\lambda+4)(\lambda-3) = 0$$

which implies that the eigenvalues of *A* are 0, -4, and 3.
Next we look for the eigenvectors.

1.

Case $\lambda = 0$: The associated eigenvectors are given by the linear system

$$AX = \mathcal{O}$$

which may be rewritten by

$$\begin{cases} x + 2y + z & = & 0 \\ 6x - y & = & 0 \\ -x - 2y - z & = & 0 \end{cases}$$

Many ways may be used to solve this system. The third equation is identical to the first. Since, from the second equations, we have *y* = 6*x*, the first equation reduces to 13*x* + *z* = 0. So this system is equivalent to

$$\begin{cases} y & = & 6x \\ z & = & -13x \end{cases}$$

So the unknown vector *X* is given by

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 6x \\ -13x \end{pmatrix} = x\begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}.$$

Therefore, any eigenvector *X* of *A* associated to the eigenvalue 0 is given by

$$X = c \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix},$$

where *c* is an arbitrary number.

2.

Case $\lambda = -4$: The associated eigenvectors are given by the linear system

$$AX = -4X \ \text{ or } \ \left(A + 4I_3\right)X = \mathcal{O}$$

which may be rewritten by

$$\begin{cases} 5x + 2y + z & = \ 0 \\ 6x + 3y & = \ 0 \\ -x - 2y + 3z & = \ 0 \end{cases}$$

In this case, we will use elementary operations to solve it. First we consider the augmented matrix $[A + 4I_3 | \mathcal{O}]$, i.e.

$$\begin{pmatrix} 5 & 2 & 1 & | & 0 \\ 6 & 3 & 0 & | & 0 \\ -1 & -2 & 3 & | & 0 \end{pmatrix}.$$

Then we use elementary row operations to reduce it to a upper-triangular form. First we interchange the first row with the first one to get

$$\begin{pmatrix} -1 & -2 & 3 & | & 0 \\ 5 & 2 & 1 & | & 0 \\ 6 & 3 & 0 & | & 0 \end{pmatrix}.$$

Next, we use the first row to eliminate the 5 and 6 on the first column. We obtain

$$\begin{pmatrix} -1 & -2 & 3 & | & 0 \\ 0 & -8 & 16 & | & 0 \\ 0 & -9 & 18 & | & 0 \end{pmatrix}.$$

If we cancel the 8 and 9 from the second and third row, we obtain

$$\begin{pmatrix} -1 & -2 & 3 & \bigm| & 0 \\ 0 & -1 & 2 & \bigm| & 0 \\ 0 & -1 & 2 & \bigm| & 0 \end{pmatrix}.$$

Finally, we subtract the second row from the third to get

$$\begin{pmatrix} -1 & -2 & 3 & \bigm| & 0 \\ 0 & -1 & 2 & \bigm| & 0 \\ 0 & 0 & 0 & \bigm| & 0 \end{pmatrix}.$$

Next, we set $z = c$. From the second row, we get $y = 2z = 2c$. The first row will imply $x = -2y+3z = -c$. Hence

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -c \\ 2c \\ c \end{pmatrix} = c \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

Therefore, any eigenvector X of A associated to the eigenvalue -4 is given by

$$X = c \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix},$$

where c is an arbitrary number.

2.

Case $\lambda = 3$**:** The details for this case will be left to the reader. Using similar ideas as the one described above, one may easily show that any eigenvector X of A associated to the eigenvalue 3 is given by

$$X = c \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix},$$

where c is an arbitrary number.

Remark. In general, the eigenvalues of a matrix are not all distinct from each other (see the page on the eigenvalues for more details). In the next two examples, we discuss this problem.

Example. Consider the matrix

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}.$$

The characteristic equation of A is given by

$$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & 0-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = -(\lambda+1)(\lambda+1)(\lambda-8) = -(\lambda+1)^2(\lambda-8) = 0.$$

Hence the eigenvalues of A are -1 and 8. For the eigenvalue 8, it is easy to show that any eigenvector X is given by

$$X = c \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix},$$

where c is an arbitrary number. Let us focus on the eigenvalue -1. The associated eigenvectors are given by the linear system

$$AX = (-1)X \quad \text{or} \quad \left(A + I_3\right)X = \mathcal{O}$$

which may be rewritten by

$$\begin{cases} 4x + 2y + 4z & = & 0 \\ 2x + y + 2z & = & 0 \\ 4x + 2y + 4z & = & 0 \end{cases}$$

Clearly, the third equation is identical to the first one which is also a multiple of the second equation. In other words, this system is equivalent to the system reduced to one equation

$$2x + y + 2z = 0.$$

To solve it, we need to fix two of the unknowns and deduce the third one. For example, if we set $x = \alpha$ and $z = \beta$, we obtain $y = -2\alpha - 2\beta$. Therefore, any eigenvector X of A associated to the eigenvalue -1 is given by

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ -2\alpha - 2\beta \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

In other words, any eigenvector X of A associated to the eigenvalue -1 is a linear combination of the two eigenvectors

$$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

Example. Consider the matrix

$$\begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}.$$

The characteristic equation is given by

$$\begin{vmatrix} 1 - \lambda & -4 \\ 4 & -7 - \lambda \end{vmatrix} = (\lambda + 3)^2 = 0.$$

Hence the matrix A has one eigenvalue, i.e. -3. Let us find the associated eigenvectors. These are given by the linear system

$$AX = (-3)X \quad \text{or} \quad (A + 3I_2)X = \mathcal{O}$$

which may be rewritten by

$$\begin{cases} 4x - 4y &= 0 \\ 4x - 4y &= 0 \end{cases}$$

This system is equivalent to the one equation-system

$$x - y = 0.$$

So if we set $x = c$, then any eigenvector X of A associated to the eigenvalue -3 is given by

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let us summarize what we did in the above examples.

Summary: *Let A be a square matrix. Assume λ is an eigenvalue of A. In order to find the associated eigenvectors, we do the following steps:*

1.

Write down the associated linear system

$$AX = \lambda X \quad \text{or} \quad \left(A - \lambda I_n\right)X = \mathcal{O}.$$

2.

Solve the system.

3.

Rewrite the unknown vector X as a linear combination of known vectors.

The above examples assume that the eigenvalue λ is real number. So one may wonder whether any eigenvalue is always real. In general, this is not the case except for symmetric matrices. The proof of this is very complicated. For square matrices of order 2, the proof is quite easy. Let us give it here for the sake of being little complete.
Consider the symmetric square matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Its characteristic equation is given by

$$\det(A - \lambda I_2) = \begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = \lambda^2 - (a + c)\lambda + ac - b^2 = 0.$$

This is a quadratic equation. The nature of its roots (which are the eigenvalues of *A*) depends on the sign of the discriminant

$$\Delta = (a + c)^2 - 4(ac - b^2).$$

Using algebraic manipulations, we get

$$\Delta = (a - c)^2 + 4b^2.$$

Therefore, Δ is a positive number which implies that the eigenvalues of *A* are real numbers.

Remark. Note that the matrix *A* will have one eigenvalue, i.e. one double root, if and only if $\Delta = 0$. But this is possible only if *a=c* and *b*=0. In other words, we have

A = a I$_2$.

In the next page, we will discuss the case of complex eigenvalues.

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