Property P and $\mathcal{L}_P$

Definition

A property P is simply a subset of Turing recognizable languages. We say that a language L satisfies a property P, if $L \in P$.

For any property P, let $\mathcal{L}_P = \{M \mid L(M) \in P\}$, i.e. the set of all Turing machine such that $L(M) \in P$.

We say that a property P is trivial if either $\mathcal{L}_P = \emptyset$ or $\mathcal{L}_P$ is the set of all the Turing recognizable languages.

Examples of properties

Examples

1. $\mathcal{L}_P = \{M \mid L(M) \text{ is regular}\}$.

   $\mathcal{L}_P$ is collection of TMs M such that $L(M)$ is regular.

   Is $\mathcal{L}_P = \varnothing$? No. For example, a TM accepting $a^* b^*$ is in $\mathcal{L}_P$.

   Is $\mathcal{L}_P = \Sigma^*$? No. For example, a TM accepting $\{a^n b^n \mid n \geq 0\}$ is not in $\mathcal{L}_P$.

   Therefore, P is not trivial.

Examples of properties

Examples

2 $\mathcal{L}_P = \{M \mid L(M) = \varnothing\}$.

Here $\mathcal{L}_P$ is a collection of TMs M such that $L(M) = \varnothing$.

Is $\mathcal{L}_P = \varnothing$? No. For example, a TM M that rejects any string is in $\mathcal{L}_P$.

Is $\mathcal{L}_P = \Sigma^*$? No. For example, a TM M that accepts a single string $\{a\}$ is not in $\mathcal{L}_P$.

Example of a trivial property

Examples

3. $\mathcal{L}_P = \left\{ M \;\middle|\; \begin{array}{c} M \text{ is a TM and } L(M) \text{ is accepted by} \\ \text{a TM that has even number of states} \end{array} \right\}.$

Here P is a property of Turing recognizable languages.

But any TM can be converted into another one that has even number of states.

Therefore, any Turing recognizable language has property P.

Therefore, P is in fact all Turing recognizable languages.

Rice's theorem

Theorem

Let P be a property such that it is not trivial. Recall that $\mathcal{L}_P = \{M \mid L(M) \in P\}$. Then $\mathcal{L}_P$ is undecidable.

When is the theorem NOT applicable?

$\{\langle M \rangle \mid M$ has at least ten states$\}$.

$\{\langle M \rangle \mid M$ never moves left on any input string $\}$.

$\{\langle M \rangle \mid M$ has no useless state $\}$.

To prove non-recognizability of a property of languages.

Rice's theorem cannot be used to prove non-recognizability of languages.

It is only used to prove undecidability.

Applications of Rice's theorem

We now learn how to apply Rice's theorem

$\{\langle M \rangle \mid M$ runs for atmost 10 steps on $aab\}$.

Not applicable.

Applications of Rice's theorem

We now learn how to apply Rice's theorem

$\{\langle M \rangle \mid L(M)$ is recognized a TM with at least 10 states$\}$.

Applicable, but property is trivial.

Applications of Rice's theorem

We now learn how to apply Rice's theorem

$\{\langle M \rangle \mid M \text{ has at most 10 states}\}.$

Not applicable, but the language is decidable.

Applications of Rice's theorem

We now learn how to apply Rice's theorem

$\{M \mid L(M) \text{ contains } \langle M \rangle\}$.

Applicable, the property is not trivial, therefore undecidable.