Correctness of subset construction

We will prove the following first.

Lemma

For any NFA A that has no ϵ transitions, there is an equivalent DFA B such that $L(A) = L(B)$.

Recall the definition of $\hat{\delta}$

Definition

Let $A = (Q, \Sigma, \delta, q_0, F)$ be an NFA with no ϵ moves.
Let $\hat{\delta} : 2^Q \times \Sigma^* \to 2^Q$ be defined as follows:
Let $S \subseteq Q$

$\hat{\delta}(S, \epsilon) := S$ If A has epsilon transitions, then $\hat{\delta}(S, \epsilon)$ will be defined accordingly

$\hat{\delta}(S, xa) := \bigcup_{q \in \hat{\delta}(S,x)} \delta(q, a)$

Properties of $\hat{\delta}$

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(P_1) For any $a \in \Sigma$, $S \subseteq Q$, $\hat{\delta}(S, a) = \bigcup_{q \in \hat{\delta}(S,\epsilon)} \delta(q, a)$.

(P_2) For any $x, y \in \Sigma^*$, $S \subseteq Q$, $\hat{\delta}(S, xy) = \hat{\delta}\left(\hat{\delta}(S, x), y\right)$.

(P_3) For any $S_1, S_2, \ldots, S_k \subseteq Q$, $x \in \Sigma^*$, $\hat{\delta}(\bigcup_i S_i, x) = \bigcup_i \hat{\delta}(S_i, x)$.

Definition (of $\hat{\delta}$)

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Let $\hat{\delta} : 2^Q \times \Sigma^* \to 2^Q$ be defined as follows:
Let $S \subseteq Q$

$$\hat{\delta}(S, \epsilon) := S$$
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Lemma

For all $w \in \Sigma^*$ and for all $S \subseteq Q$, $\hat{\delta}'(S, w) = \hat{\delta}(S, w)$.

Proof of correctness

Lemma

For all $w \in \Sigma^*$ and for all $S \subseteq Q$, $\hat{\delta}'(S, w) = \hat{\delta}(S, w)$.

Proof.

Base case: Let $w = \epsilon$.

$$\hat{\delta}'(S, \epsilon) = \hat{\delta}(S, \epsilon)$$

Induction step: Let $w = x \cdot a$

$$\begin{aligned} \hat{\delta}'(S, w) &= \hat{\delta}'\left(\hat{\delta}'(S, x), a\right) && \text{By } (P_2) \\ &= \hat{\delta}'\left(\hat{\delta}(S, x), a\right) && \text{By Induction hypothesis} \\ &= \hat{\delta}\left(\hat{\delta}(S, x), a\right) && \text{By def'n of } \hat{\delta} \\ &= \hat{\delta}(S, w) && \text{By } (P_2) \end{aligned}$$

Hanlding the ϵ moves

Lemma

For any NFA A with ϵ transitions, there is another NFA, say B, such that B has no ϵ transitions and $L(A) = L(B)$.

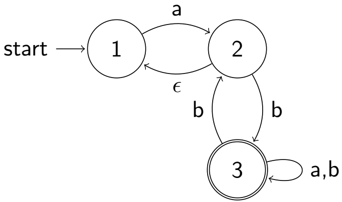
Proof Idea

Let $S \subseteq Q$.

Let

$$E(S) = \left\{ q \;\middle|\; \begin{array}{l} q \text{ is reachable from some state in } S \\ \text{with zero or more } \epsilon \text{ transitions} \end{array} \right\}$$

Example



$$\begin{aligned} \delta'(1, a) &= E(\delta(1, a)) \\ &= E(\{2\}) \\ &= \cancel{\{1\}}\ \{1, 2\} \end{aligned}$$