Tutorial 7

1. Describe the languages recognized by the following CFGs.
   (a) $S \to aSa \mid bSb \mid SS \mid \varepsilon$
   (b) $S \to aSb \mid bY \mid Ya$;
   $Y \to bY \mid aY \mid \varepsilon$
   (c) $E \to 0 \mid 1 \mid E + E \mid E \times E \mid (E)$

2. Give CFG and PDA for the following context-free languages.
   (a) $\{a^n b^m c^k \mid m = n + k\}$
   (b) $\{a^n b^m c^k \mid k \neq n + m\}$
   (c) $\{w \mid w \in \Sigma^*, w$ has odd length and the middle letter is $a\}$
   (d) $\{w\#x \mid w, x \in \Sigma^*, w^R$ is a substring of $x\}$
   (e) $\{w \mid \#_a(w) \leq 2 \cdot \#_b(w)\}$
   (f) $\{w\#w' \mid w, w' \in \Sigma^*$ and $w \neq w'\}$

3. Use pumping lemma for Context Free languages to show that the following languages are not context free.
   (a) $\{0^n \mid$ n is a prime$\}$
   (b) $\{0^i 1^j \mid j = i^2\}$
   (c) $\{a^n b^n c^i \mid n \leq i \leq 2n\}$

4. Recall the following problem from the midsem.

   A finite state transducer is a finite state machine that reads the given input and outputs a string. Formally, it is a machine A given by the following components $(Q, \Sigma, \Gamma, \delta, q_0, F)$. Here Q, Σ, q_0, and F are as in a usual DFA, i.e. Q is a set of states, Σ is the input alphabet, q_0 is the initial state, and F is a set of final states. The set Γ is the output alphabet.

   Finally, the transition function is given by: $\delta \subseteq Q \times \Sigma \cup \{\varepsilon\} \times \Gamma \cup \{\varepsilon\} \times Q$. If $(q, a, b, r) \in \delta$ then it means that from state q reading the letter a the machine outputs a letter b and goes to state r.

   We define $\hat{\delta}$ as follows:

   - $\forall q \in Q \ (q, \varepsilon, \varepsilon, q) \in \hat{\delta}$,
   - $\delta \subseteq \hat{\delta}$, and
   - if $(q, x, y, r) \in \hat{\delta}$ and $(r, a, b, s) \in \delta$ then $(q, x \cdot a, y \cdot b, s) \in \hat{\delta}$.

   Definition 0.1. *We say that two strings $x \in \Sigma^*$ and $y \in \Gamma^*$ are related by A, denoted as $x \sim_A y$ if and only if $\exists f \in F$ such that $(q_0, x, y, f) \in \hat{\delta}$.*

   Let $L(\Sigma \to^A \Gamma) = \{(x, y) \mid x \in \Sigma^*, y \in \Gamma^*,$ and $x \sim_A y\}$. We will call this the transduction *of A.*

   (a) Let Σ, Γ be finite alphabets. Give a construction of a finite state transducer A such that
   $$L(\Sigma \to_A \Gamma) = \{(a^n, b^n) \mid n \geq 0\}.$$

(b) Let A be an FST and let Σ, Γ be finite alphabets. Prove that there is a another FST B such that the transduction of B is the following set: $\{(y, x) \mid y \in \Gamma^*, x \in \Sigma^*, x \sim_A y\}$.

(c) Prove that FSTs are closed under the union operation. That is, if A and B are two FSTs and L_1 and L_2 are the two transduction realized by them, respectively then there is another transducer C such that the transduction of C is equal to $L_1 \cup L_2$.

(d) Let L, L' be two regular languages. Prove that there is a an FST A such that the transduction of A is the following set: $\{(x, y) \mid x \in L, y \in L')\}$.

(e) Let $L_A = \{x \cdot y^R \mid x \in \Sigma^*, y \in \Gamma^*, x \sim_A y\}$. Prove that if A is a finite state transducer and $\sim_A$ is as defined above then L_A is a context-free language. (Prove this without assuming that $\Sigma \cap \Gamma = \emptyset$.

(f) Let $L = \{(a^n b^n, c^n) \mid n \geq 0\}$. Using the part (b) above and results proved in class, show that there does not exist a finite state trasducer A such that $L_A = L$.

(g) Using the part (d) above prove that FSTs are not closed under the intersection operation.

(h) Give a polynomial time algorithm for the following problem.

Given: FST A, $x \in \Sigma^*$, $y \in \Gamma^*$
Check: is $x \sim_A y$?