CS310 Automata Theory – 2016-2017

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Lecture 35: Effective computation
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Polynomial time reductions and NP-hardness

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A function $f : \Sigma^* \to \Sigma^*$ is polynomial time computable if there is a polynomial time Turing machine TM, say M, such that on any input $w \in \Sigma^*$, M stops with only $f(w)$ on its tape.

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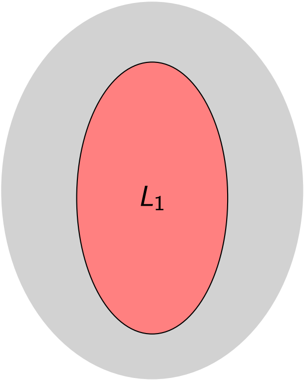

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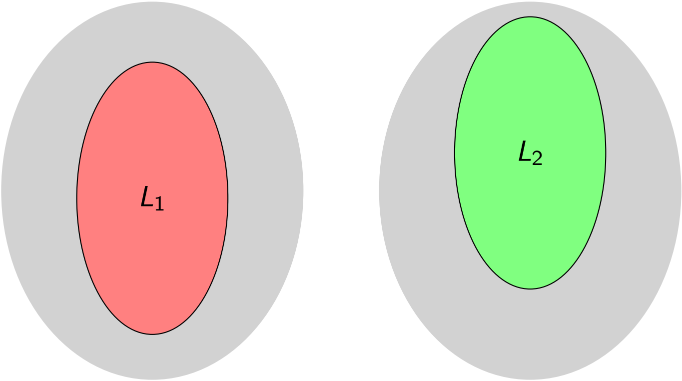

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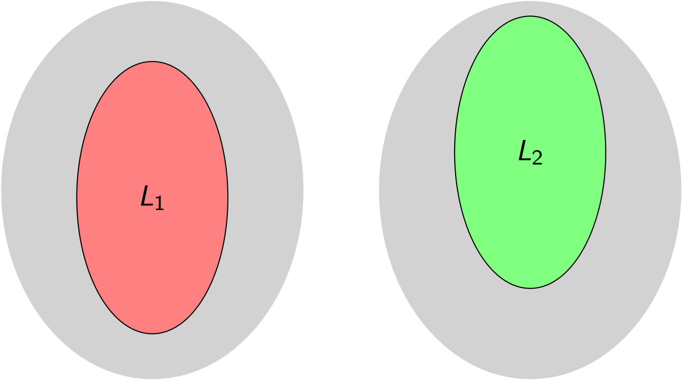

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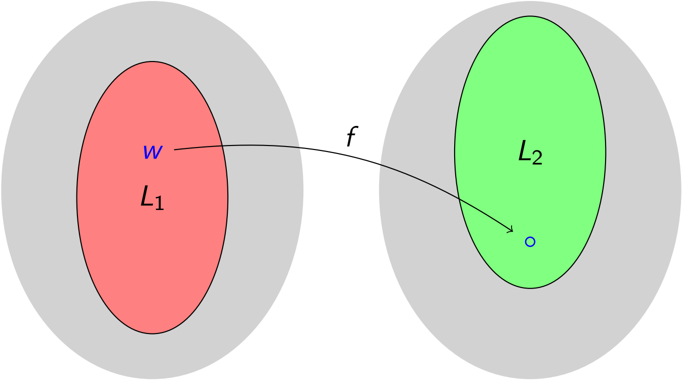

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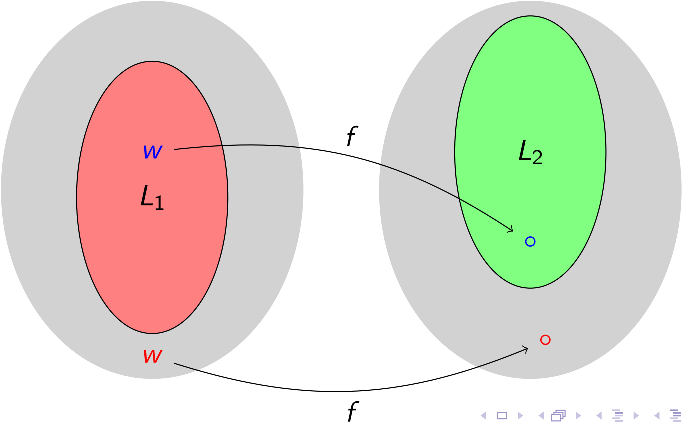

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Theorem ([Cook-Levin, 1970])

SAT is NP-complete.

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