Turing recognizability for $L, \overline{L} \Rightarrow$ Turing decidibility for L

We design 2-tape TM M, using TMs M_1, M_2 as follows:

M copies input from tape 1 to tape 2.

It acts as M_1 on tape 1 and as M_2 on tape 2.

M keeps track of the state control of M_1, M_2 in $Q_1 \times Q_2$.

Can you give a full decsription of M? DIY!

Turing machines as strings

Every TM represented as a string in $\{0,1\}^*$ with the following properties:

Every string over $\{0,1\}^*$ represents some TM.

Every TM is represented by infinitely many strings.

Notation

$M \longrightarrow \langle M \rangle$, a string representation of M.

$\alpha \longrightarrow M_\alpha$, a machine corresponding to α.

Turing machines as strings

Lemma

There exists a language which is not Turing recognizable.

Proof.

Fix an alphabet Σ.

$$\text{languages over } \Sigma^* \xrightarrow{\text{bijection}} 2^{\mathbb{N}}$$

Let L be a language, i.e. $L \subseteq \Sigma^*$, $w \in \Sigma^*$.

$$\chi_L(w) = \begin{cases} 1 & \text{if } w \in L \\ 0 & \text{otherwise} \end{cases}$$

Therefore, set of all languages is uncountable. However, the set of all TMs is countable.

There must be a language which is not Turing recognizable.

A decision problem about TMs

$$A_{TM} = \{(M, w) \mid M \text{ accepts } w\}$$

Lemma

A_{TM} is Turing recognizable.

Proof sketch

Design a TM, say N such that,

N behaves like M on w at each step,

if M reaches q_{acc} then N also accepts.

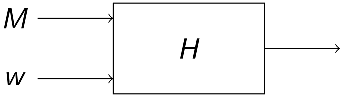
Is A_{TM} decidable?

A decision problem about TMs

Lemma

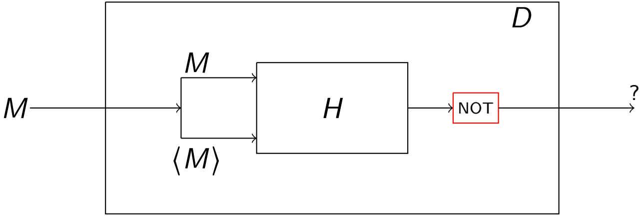
A_{TM} is not Turing decidable.

Assume that there exists M such that M decides A_{TM}.



q_{acc} if M accepts w

q_{rej} if M does not accepts w



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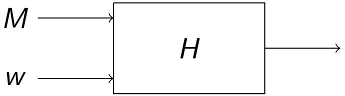

q_{acc} if M accepts w

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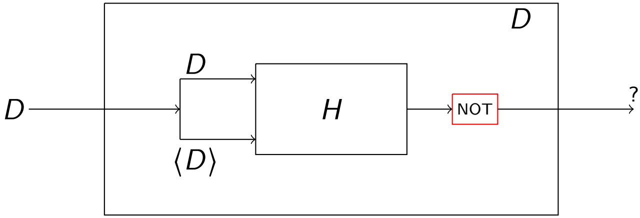
What happens if we give D as input to itself?



A decision problem about TMs

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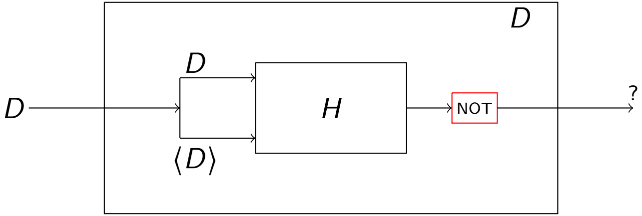

If D accepts $\langle D \rangle$ then D rejects $\langle D \rangle$.

If D rejects $\langle D \rangle$ then D accepts $\langle D \rangle$. ☹