CS310 Automata Theory – 2016-2017

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Lecture 23: Turing machines, computability
March 14, 2017

Last three classes

Introduction to Turing machines

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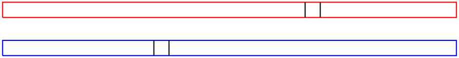

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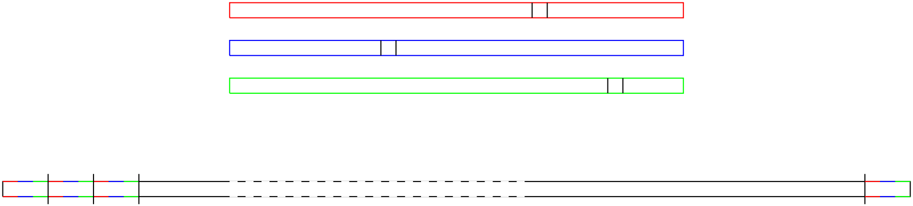

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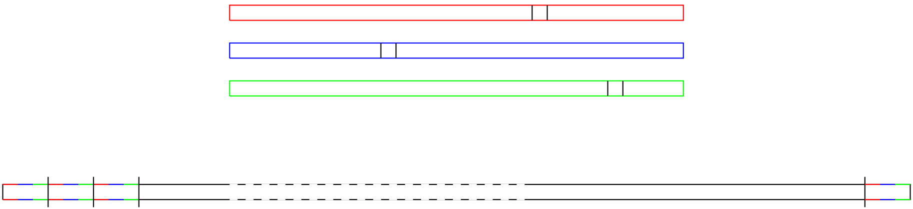

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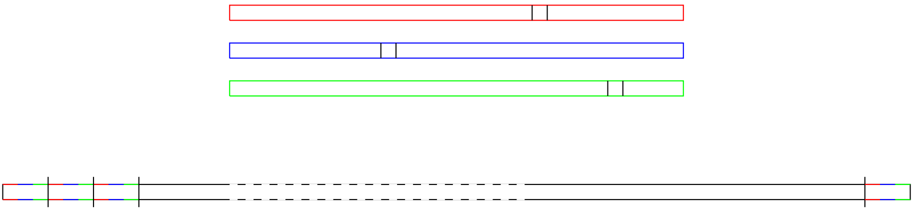

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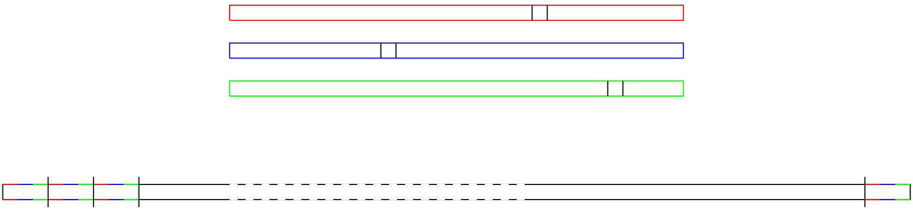

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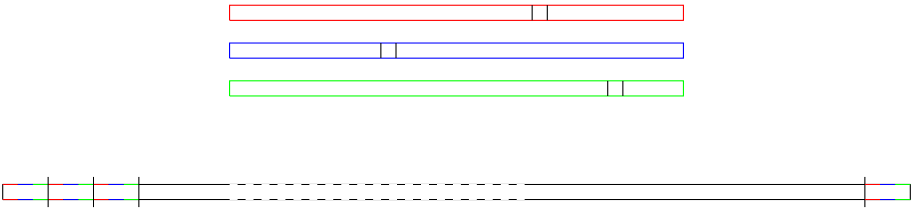

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$\overline{\Gamma}$ symbols used to denote tape head positions.

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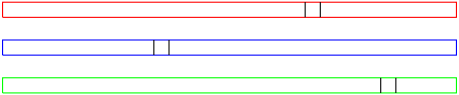

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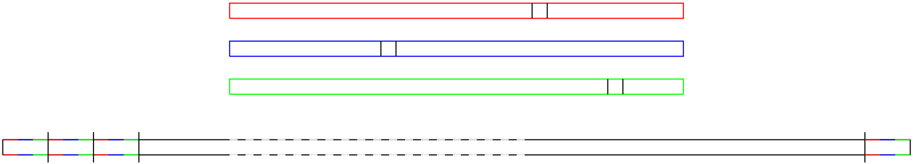

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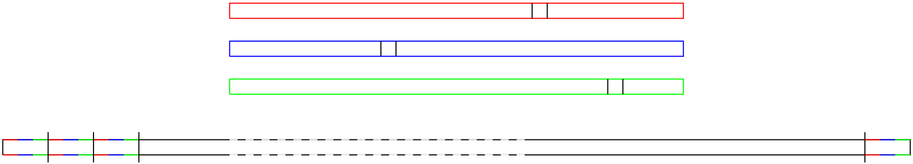

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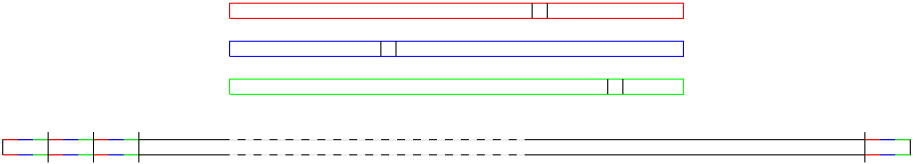

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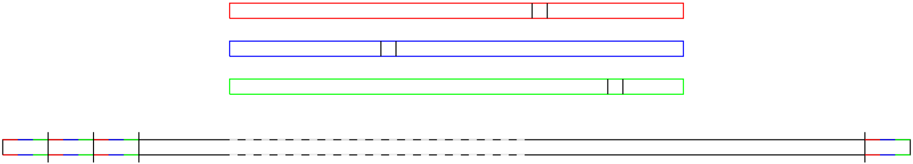

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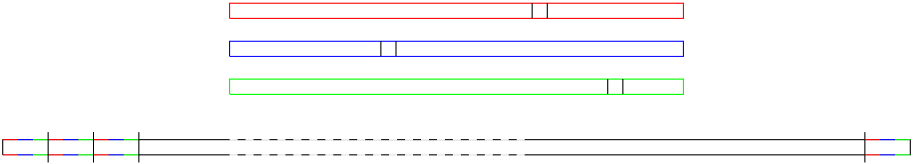

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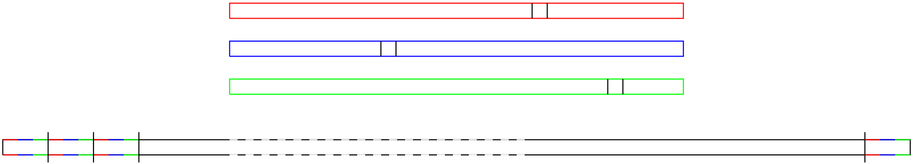

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If L is Turing decidable then L is also Turing recognizable

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Therefore, $\overline{L}$ is also Turing recognizable.

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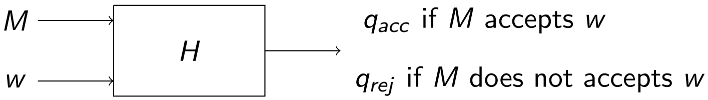

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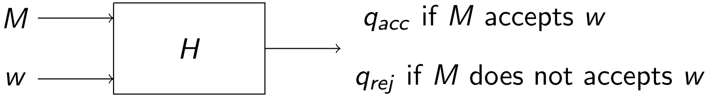

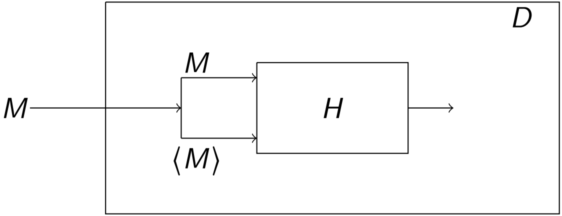

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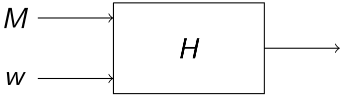

q_{acc} if M accepts w

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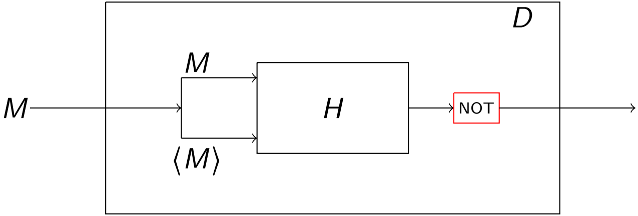

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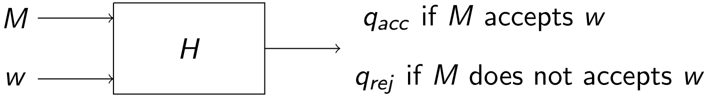

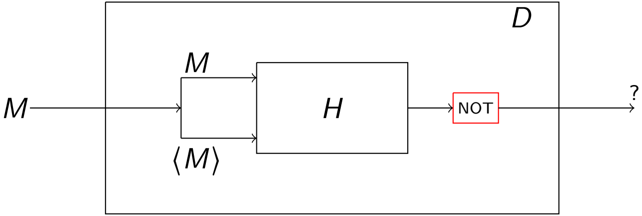

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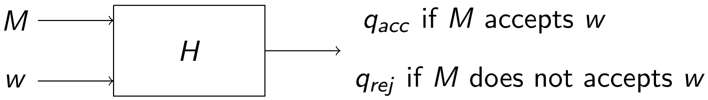

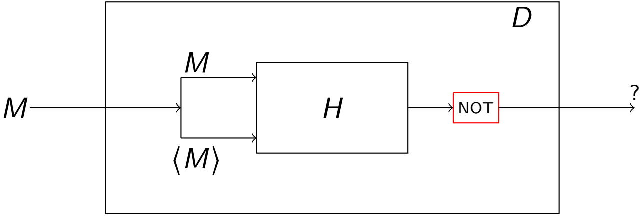

A decision problem about TMs

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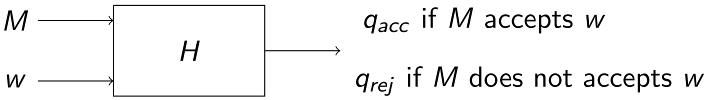
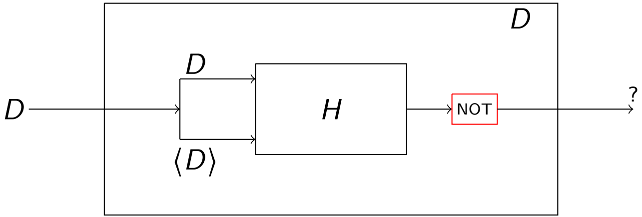


What happens if we give D as input to itself?



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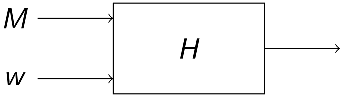

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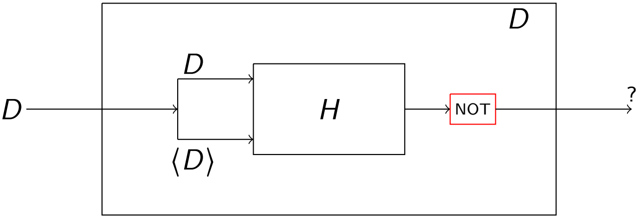

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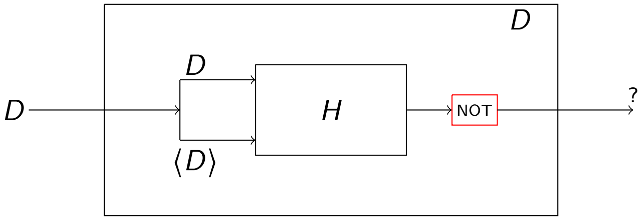

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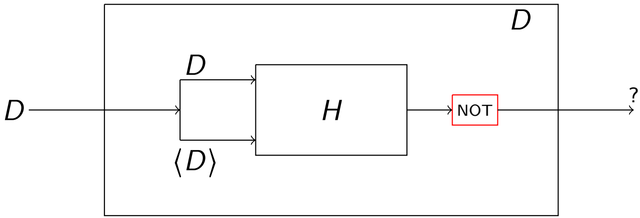

If D accepts $\langle D \rangle$

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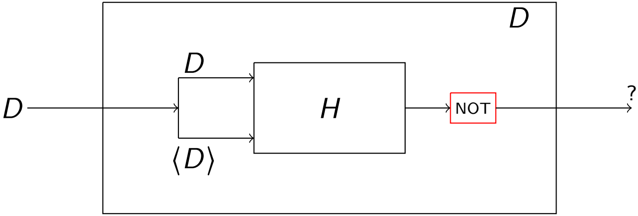

If D accepts $\langle D \rangle$ then D rejects $\langle D \rangle$.

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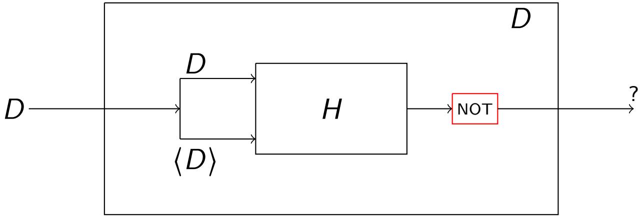

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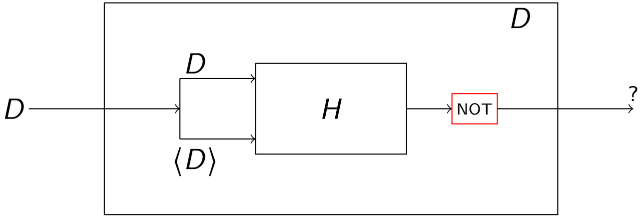

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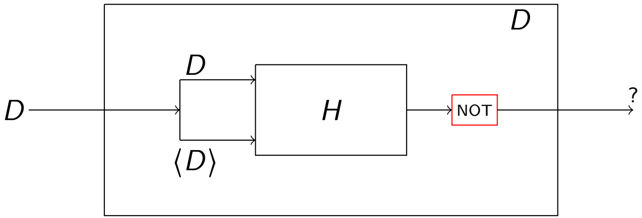

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