CS310 Automata Theory – 2016-2017

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Lecture 21: Turing machines, computability
March 07, 2017

Last two modules

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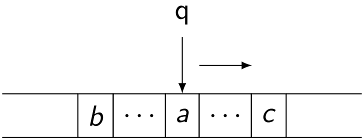

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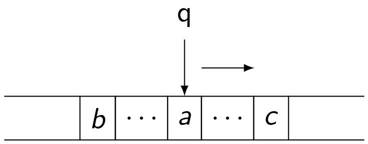

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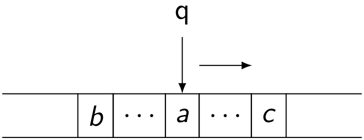

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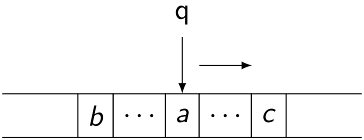

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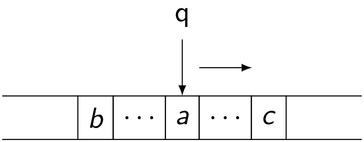

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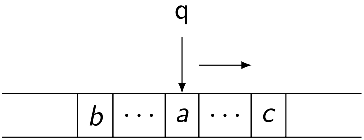

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Special states for accepting and rejecting.

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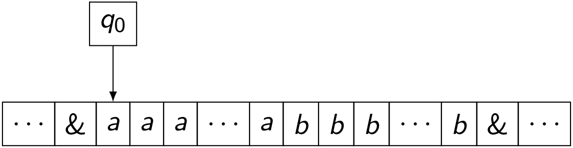

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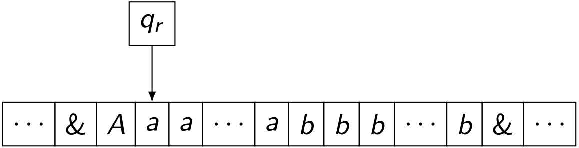

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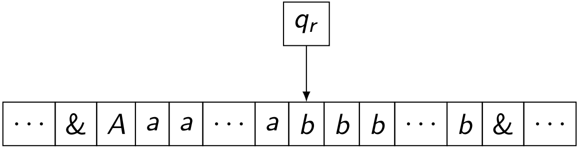

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q_I

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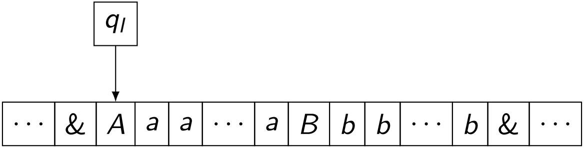

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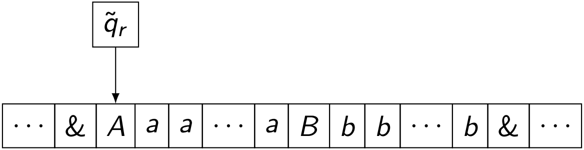

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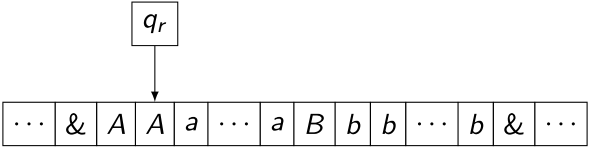

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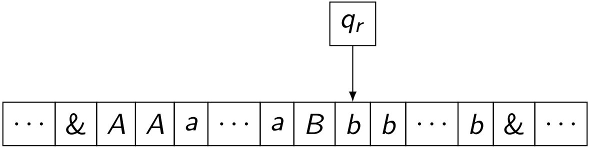

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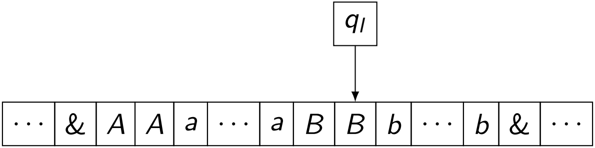

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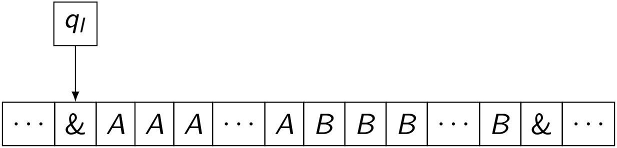

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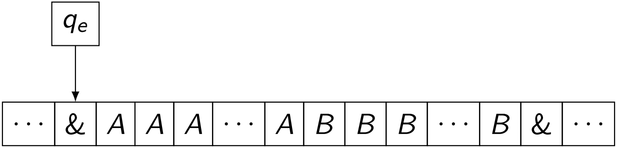

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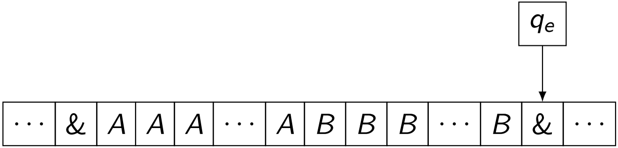

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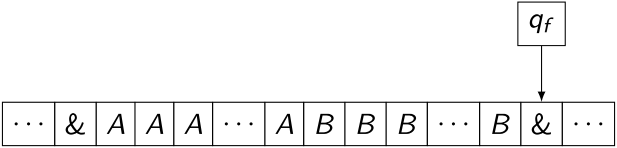

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Turing machine for a non-context free language

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Give a full description of a Turing machine for the above language.

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We denote it by $u \cdot a \cdot q \cdot b \cdot v \mapsto u \cdot q' \cdot a \cdot c \cdot v$.

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A TM may not halt!

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The notion of rejection by TM is not as straightforward!

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Turing decidable languages form a subclass of Turing recognizable languages.

Comparing decidability and recognizability

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