CS310 Automata Theory – 2016-2017

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Lecture 22: Turing machines, computability
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Last two classes

Introduction to Turing machines

What are Turing machines? Informal and formal definitions.

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Turing recognizable and Turing decidable languages.

Turing machines

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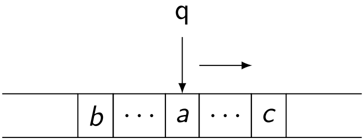


Read and write on the input tape. Head moves left/right.

The tape is infinite.

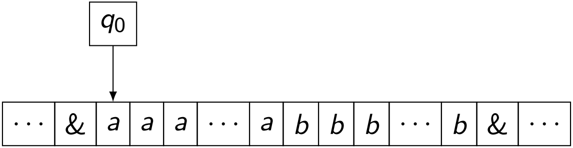
A special symbol & to indicate blank cells.

Initially all cells blank except the part where the input is written.

Special states for accepting and rejecting.

Example

$L_{a,b} = \{a^n b^n \mid n \geq 0\}.$



Formal definition

Definition

A Turing machine (TM) is given by $M = (Q, \Sigma, \Gamma, \delta, q_0, q_f, q_{rej})$

Q:	set of states	Σ:	input alphabet
q_0:	start state	Γ:	tape alphabet, $\Sigma \subseteq \Gamma$, & $\in \Gamma$
q_{acc}:	accept state	q_{rej}:	reject state

$\delta \subseteq Q \times \Gamma \times Q \times \Gamma \times \{L, R, S\}$.

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Understanding δ

For a $q \in Q, a \in \Gamma$ if $\delta(q, a) = (p, b, L)$,

then p is the new state of the machine,

b is the letter with which a gets overwritten,

the head moves to the left of the current position.

Turing machine for a non-context free language

Example

Turing machine for a non-context free language

Example

$$\mathrm{EQ} = \{w \cdot \# \cdot w \mid w \in \Sigma^*\}.$$

Give a full description of a Turing machine for the above language.

Configuration

Definition

The configuration of a TM $M = (Q, \Sigma, \Gamma, \delta, q_0, q_f, q_{rej})$ is given by

$$\Gamma^* \times Q \times \Gamma^*$$

Let $u, v \in \Gamma^*$, $a, b, c \in \Gamma$ and $q, q' \in Q$.

Suppose $(q', c, L) \in \delta(q, b)$ is a transition in M,
then starting from $u \cdot a \cdot q \cdot b \cdot v$ in one step we get $u \cdot q' \cdot a \cdot c \cdot v$.

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then starting from $u \cdot a \cdot q \cdot b \cdot v$ in one step we get $u \cdot q' \cdot a \cdot c \cdot v$.

We say that $u \cdot a \cdot q \cdot b \cdot v$ **yields** $u \cdot q' \cdot a \cdot c \cdot v$.
We denote it by $u \cdot a \cdot q \cdot b \cdot v \mapsto u \cdot q' \cdot a \cdot c \cdot v$.

Special configurations

Start configuration

We assume that the head is on the left of the input in the beginning.

Therefore, $q_0 \cdot w$ is the start configuration.

Accepting configuration

Any configulation that contains q_{acc} is an accepting configuration.

Rejecting configuration

Any configulation that contains q_{rej} is a rejecting configuration.

Halting configurations: if a configuration is accepting or rejecting then it is called a halting configuration.

A TM may not halt!

Acceptance by a TM

A TM M is said to accept a word $w \in \Sigma^*$ if there exists a sequence of configurations $C_0, C_1, \ldots, C_k$ such that

C_0 is a start configuration,

$C_i \mapsto C_{i+1}$ for all $0 \leq i \leq k-1$,

C_k is an accepting configuration.

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The notion of rejection by TM is not as straightforward!

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the machine may run forever,

or may reach q_{rej},

both are valid outcomes,

and the machine is allowed to do either of the two.

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Turing decidable languages form a subclass of Turing recognizable languages.

Comparing decidability and recognizability

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Else M_2 will reach the accepting configuraion. In that case, reject.

□

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k-tape Turing machines

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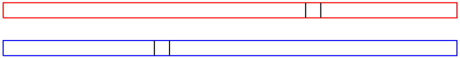

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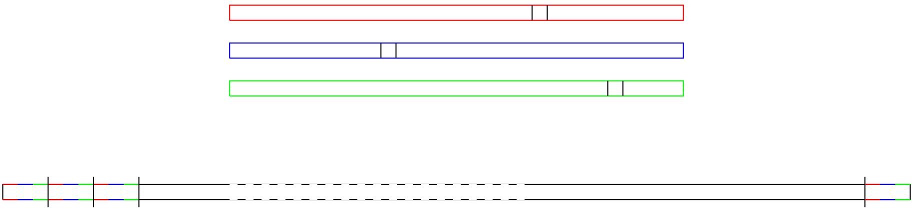

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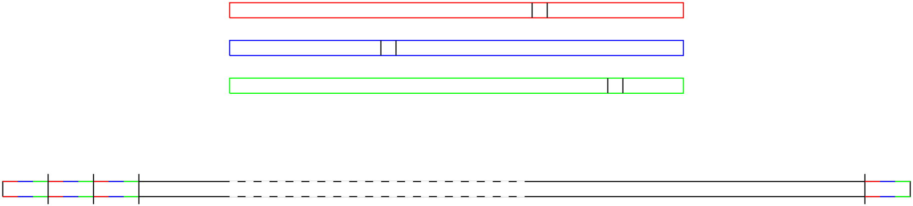

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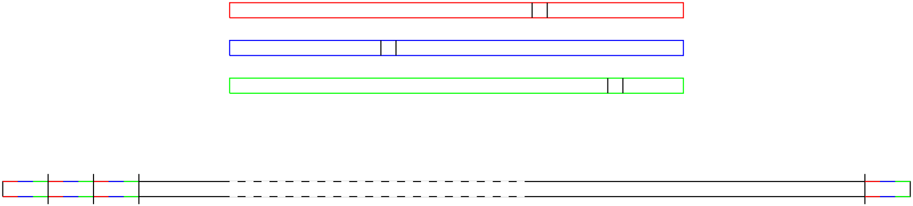

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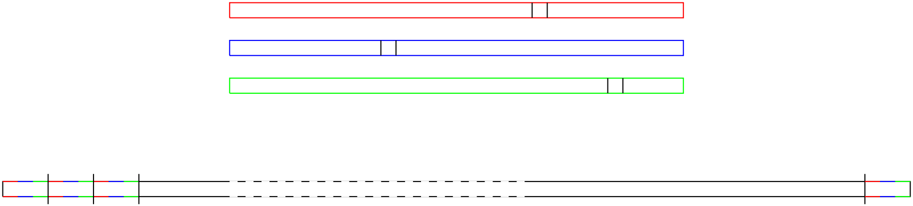

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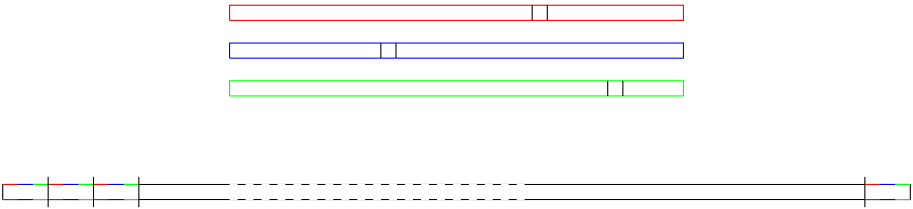

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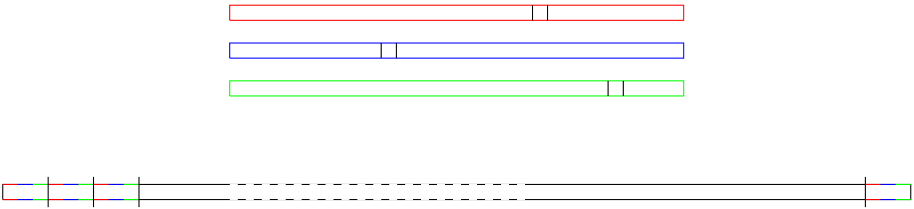

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$\overline{\Gamma}$ symbols used to denote tape head positions.

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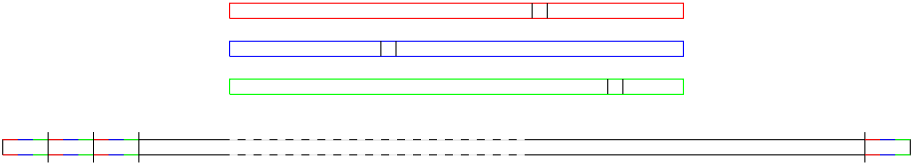

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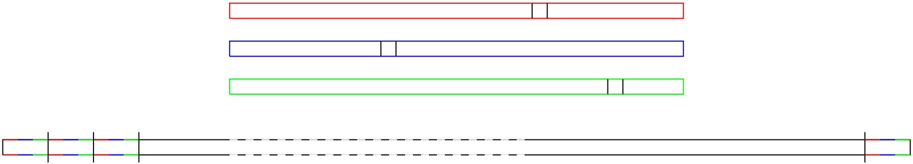

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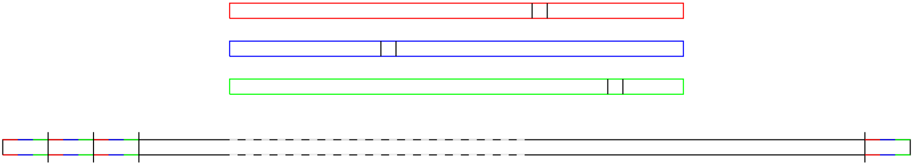

To simulate 1 step of M, M' works follows:

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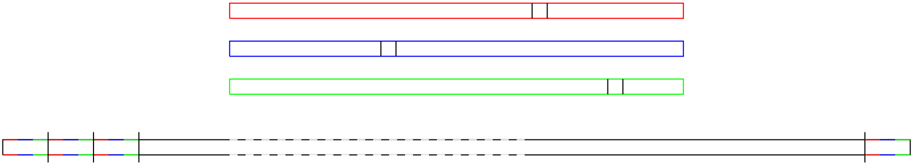

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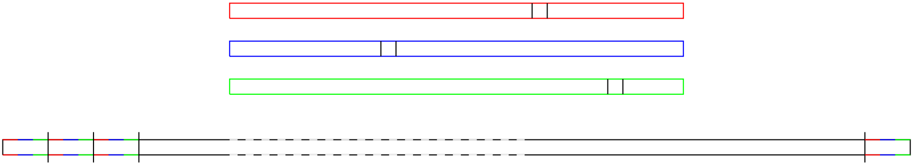

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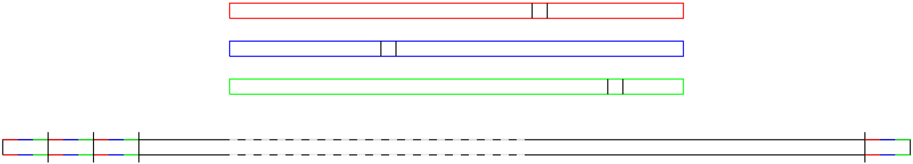

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