Tutorial 5

1. Prove that for any $k \in \mathbb{N}$, there exists a language L_k such that L_k is accepted by a DFA with k states but not accepted by a DFA with $k-1$ states.

2. Let us assume that we are working over $\Sigma = \{a, b\}$.

   (a) Consider the following language.

   $$L_i = \{w \mid i\text{th letter from the end in } w \text{ is an } a\}.$$

   Let s_i denote the number of states in an NFA that recognizes this language. Show that $s_i = O(i)$.

   (b) Construct a DFA for L_i with $2^{O(s_i)}$ states. Can you give an exact constant hidden under the $O(\cdot)$ in the exponent?

   (c) Argue using the minimization algorithm (or by any other method) that any DFA for L_i must have $2^{\Omega(s_i)}$ states.

   (d) Give a 2DFA for L_i with $O(s_i)$ states. This shows that 2DFAs are exponentially more powerful than DFAs (in terms of the number of states).

3. Prove that minimal NFA are not unique.

4. Let $\Sigma_2 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. For any word over Σ_2^*, think of the bottom and top rows as a string over $\{0,1\}^*$. For example, if $w = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then the top row is a string $0 \cdot 0 \cdot 1$ and the bottom row is a string equal to $0 \cdot 1 \cdot 0$.

   Prove that the following language is not regular.

   $$L = \{w \in \Sigma_2^* \mid \text{bottom row of } w \text{ is reverse of top row of } w\}.$$

5. Let $\Sigma = \{0, 1, +, =\}$. Prove that the following language is not regular.

   $$\text{ADD} = \{x = y + z \mid x, y, z \text{ are binary integers and } x \text{ is a sum of } y, z\}.$$

6. Let L be a regular language. Consider the following language:

   $$\text{MID}_L = \{x \cdot z \mid \exists y \text{ s.t. } |x| = |y| = |z| \;\&\; x \cdot y \cdot z \in L\}.$$

   Prove or disprove that if L is regular then MID_L is also regular.

7. Let L be a context-free language. Prove that the following languages are context-free.

   (a) $\{x \cdot \# \cdot y \mid x \neq y\}$.

   (b) $\{a^i \cdot b^j \mid i \neq j \text{ and } 2i \neq j\}$.

   (c) $\{y \mid \exists x \text{ s.t. } x \cdot y \in L\}$.

   (d) $a^* b^* c^* \setminus \{a^n b^n c^n \mid n \geq 0\}$.