CS310 Automata Theory – 2016-2017

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Last class

Regular expressions

Existence of non-regular languages.

Using pigeon hole principle to prove that a certain language is not regular.

The statement of the pumping lemma and one application.

Pumping lemma

A recipe for proving that a given language is non-regular.

Lemma (Pumping Lemma)

If L is a regular language, then $\exists p \in \mathbb{N}$ such that for any strings x, y, z with $x \cdot y \cdot z \in L$ and $|y| \geq p$,

1. *there exist strings u, v, w, s.t. y can be written as $y = u \cdot v \cdot w$,*
2. *$\forall i \geq 0 \ x \cdot u \cdot v^i \cdot w \cdot z \in L$,*
3. *$|v| > 0$.*

To prove that a given language L is not regular, the contrapositive of the above statement is useful.

Contrapositive of the pumping lemma

Lemma

We say that a language L satisfies **Property-NR** *if the following conditions hold:*

$\forall p \geq 0$,

$\exists x, y, z$ *such that* $x \cdot y \cdot z \in L$ *and* $|y| \geq p$,

$\forall u, v, y$ *such that* $|v| > 0$, $y = u \cdot v \cdot w$,

$\exists i \ x \cdot u \cdot v^i \cdot w \cdot z \notin L$.

If L satisfies Property-NR then L is not regular.

Using the pumping lemma

We say that a language L satisfies **Property-NR** if the following conditions hold:

☺ $\forall p \geq 0$,

☺ $\exists x, y, z$ such that $x \cdot y \cdot z \in L$ and $|y| \geq p$,

☹ $\forall u, v, y$ such that $|v| > 0$, $y = u \cdot v \cdot w$,

☺ $\exists i \ x \cdot u \cdot v^i \cdot w \cdot z \notin L$.

If L satisfies Property-NR then L is not regular.

We will now use the lemma to prove that $L_{a,b} = \{a^n b^n \mid n \geq n\}$ is not regular.

For any chosen $p \geq 0$, let $x := a^p$, $y := b^p$, $z = \epsilon$.

For any split of y as $u \cdot v \cdot w$, if we take $x \cdot u \cdot v^i \cdot w = 0^p 1^q$, where $q > p$ as long as $i > 0$.

In particular, $x \cdot u \cdot v^2 \cdot w \cdot z \notin L$.

Applications of pumping lemma

Let $L = \{ ww^R \mid w \in \Sigma^* \}$

- ☹ For any chosen p,

- ☺ let $x = \epsilon$, $y = 0^p$, $z = 110^p$.

- ☹ For any split of y into u, v, w

- ☺ $xuv^i wz = 0^q 110^p$, as long as $i > 0$.
  In particular, $xuv^2 wz \notin L$.

We say that a language L satisfies
Property-NR
if the following conditions hold:

- ☹ $\forall p \geq 0$,

- ☺ $\exists x, y, z$ such that $x \cdot y \cdot z \in L$
  and $|y| \geq p$,

- ☹ $\forall u, v, y$ such that $|v| > 0$,
  $y = u \cdot v \cdot w$,

- ☺ $\exists i \; x \cdot u \cdot v^i \cdot w \cdot z \notin L$.

If L satisfies Property-NR then L is not regular.

Applications of pumping lemma

$L = \{a^q \mid q \text{ is a prime number }\}$

☺ For any chosen p,

☺ let $x, z = \epsilon$, $y = a^n$, $n \geq p$ and a prime.

☺ For any split of y into u, v, w

☺ $xuv^{n+1}wz = a^{n(k+1)}$, where $k := |v|$.
That is, $xuv^{n+1}wz = a^{n(k+1)} \notin L$.

We say that a language L satisfies **Property-NR**
if the following conditions hold:

☺ $\forall p \geq 0$,

☺ $\exists x, y, z$ such that $x \cdot y \cdot z \in L$ and $|y| \geq p$,

☺ $\forall u, v, y$ such that $|v| > 0$, $y = u \cdot v \cdot w$,

☺ $\exists i \ x \cdot u \cdot v^i \cdot w \cdot z \notin L$.

If L satisfies Property-NR then L is not regular.

Building on pumping lemma

The following language is not regular:

$$EQ = \{w \in \{a, b\}^* \mid \#_a(w) = \#_b(w)\}$$

Suppose D is regular.

$D \cap L(a^*b^*)$ is also regular, as the intersection of two regular languages is regular and any regular expression defines a regular language.

But $D \cap L(a^*b^*) = \{a^n b^n \mid n \geq 0\}$ is not regular, which we proved using the pumping lemma.

Pumping down

Let $L = \{0^i 1^j \mid i, j \in \mathbb{N} \text{ and } i > j\}$.

   For any choice of $p \geq 0$,

   Let $x = \epsilon$, $y = 0^{p+1}$, $z = 1^p$.
   Then $x \cdot y \cdot z \in L$.

   Now for any choice of u, v, w, s.t $u \cdot v \cdot w = y$ and $|v| > 0$
   $x \cdot u \cdot v^0 \cdot w \cdot z = 0^{p'} 1^p$, where $p' \leq p$.

   $\therefore x \cdot u \cdot v^0 \cdot w \cdot z \notin L$.

Relations on Σ

Let R be an equivalence relation on the set Σ^*, i.e. $R \subseteq \Sigma^* \times \Sigma^*$ such that

REFLEXIVE $\quad \forall x \in \Sigma^* \ R(x, x)$ holds.

SYMMETRIC $\quad \forall x, y \in \Sigma^* \ R(x, y) = R(y, x)$ hold.

TRANSITIVE $\quad \forall x, y, z \in \Sigma^*$ if $R(x, y), R(y, z)$ hold then $R(x, z)$ also holds.

Relation of Σ^*

Let L be a regular language recognized by a DFA $A = (Q, \Sigma, \delta, q_0, F)$.

We say that $\forall x, y \in \Sigma^*$

$$x \equiv_A y \quad \text{iff} \quad \tilde{\delta}(q_0, x) = \tilde{\delta}(q_0, y)$$

state
reached
on x
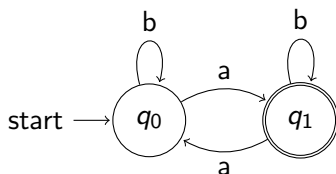from q_0

state
reached
on y
from q_0

Assume that the auomaton is complete.

Observe that $\equiv_A$ is an equivalence relation.

Example

Example of an equivalence relation.

Consider the following automaton, say A.

$aab \equiv_A ababab a$.

$aabaaa \equiv_A a$.

The words with even number of a's form one equivalence class.

The words with odd number of a's form the other equivalence class.

There are no other equivalence classes.