Tutorial 8

1. Give Turing machines (with full formal descriptions) for the following languages. Once you design a TM for one of the parts, you may use that as a subroutine in the subsequent parts of the question. (See for example the description of how to use TMs as subroutines in the textbook by Hopcroft and Ullman.)

   (a) Given a word w on the first tape, copy the word on the second tape and make the first tape completely blank.

   (b) Given a word $w \in \{a, b\}^*$ on the first tape, keep only every alternate letter of w on that tape. That is, suppose $abbab$ is written on the tape, you should finally end up with abb on the same tape. You may design a 2-tape TM for this.

   (c) Given a word $w = w_1 w_2 \ldots w_{2n}$ over the alphabet $\{a, b\}$ on the tape, output the word $w_1 w_3 \ldots w_{2n-1} w_2 w_4 \ldots w_{2n}$ on the same tape. You may design a 2-tape TM for this.

   (d) Given a word $w \in \{a, b, \overline{a}, \overline{b}\}^*$ design a single tape TM that accepts if and only if the following two conditions are satisfied:

   - exactly 3 positions in w come from $\{\overline{a}, \overline{b}\}$ and all the others are from $\{a, b\}$,
   - and the values at those positions are either $\overline{b}\overline{a}\overline{a}$ or $\overline{a}\overline{b}\overline{a}$.

   (e) Given a word $w \in \{a, b, \overline{a}, \overline{b}, \#\}^*$ design a single tape TM that accepts if and only if the word satisfies all the following three conditions:

   - exactly 3 positions in w come from $\{\overline{a}, \overline{b}\}$ and all the others are from $\{a, b, \#\}$,
   - every 4th letter in the word w is #, i.e. $w = x_1 \# x_2 \# \ldots x_n \#$, where $x_i = x_{i_1} x_{i_2} x_{i_3}$ (i.e. $|x_i| = 3$),
   - and if a letter with overline appears in a certain block, say x_i, at a position j, then in no other block, say $x_{i'}$ where $i' \neq i$, does the letter with overline appear in that position, i.e. if there exist $i \in [n]$ and $j \in \{1, 2, 3\}$ such that $x_{i_j} \in \{\overline{a}, \overline{b}\}$ then for any $i \neq i'$ $x_{i'_j} \in \{a, b\}$.

   (f) Given a word $w \in \{a, b, \overline{a}, \overline{b}, \#\}^*$ design a single tape TM that does the following:

   - checks that the three conditions in part (1e) above are satisfied,
   - if they are not satisfied then rejects and halts,
   - if they are satisfied then updates the word as follows: (i) if $x_{i_j} = \overline{a}$ then changes $x_{i_j} = a$, (ii) if $\left(x_{i_j} = \overline{b} \text{ AND } i > 1\right)$ then overwrites $x_{(i-1)_j}$ with $\overline{b}$ and x_{i_j} with b, (iii) makes no updates in all the other cases.

   (g) Given 1^n on the first tape, output n in binary on the second tape.

   (h) Given $w \in \{0, 1\}^n$ on the first tape, output the number represented by w in unary on the second tape.

2. Prove that for any 3-tape TM there is an equivalent 1-tape TM. You may use various subparts from the Question 1 above as subroutines.