Closure under star

Lemma

For every regular language L, L^* is also regular, where $L^i = L \circ L \circ \ldots \circ L$ (i times) and $L^* = \cup_{i \geq 0} L^i$.

Proof.

Construction: Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA for L. Let B be an NFA $(Q', \Sigma', \delta', q_0', F')$ defined as follows:

$Q' = Q \cup \{\tilde{q}_0\}$,

Σ' same as Σ but with additional ϵ,

$\delta' =$ if $q \in F$ then $\delta'(q, \epsilon) = q_0$
also, $\delta'(\tilde{q}_0, \epsilon) = q_0$
contains all transitions from δ as well.

$q_0' = \tilde{q}_0$

$F' = F \cup \{\tilde{q}_0\}$

Correctness: DIY. □

Regular expressions

Various expressions formed by $\cup, \circ, *$ *operators on* Σ.

Definition (Regular expression)

The following are regular expressions:

1. ϵ,
2. $a, \forall a \in \Sigma$,
3. $\varnothing$,
4. $R_1 \cup R_2$,
5. $R_1 \circ R_2$,
6. R_1^*,

where, R_1, R_2 are regular expressions.

Example

$$\Sigma^* a \Sigma^* = \{w \mid w \text{ contains at least one } a\}$$

$$(\Sigma\Sigma)^* = w \mid |w| \equiv 0(mod2)$$

Language defined by a regular expression

Definition (Language defined by regular expression)

The language defined by a regular expression is:

1. $L(\epsilon) = \epsilon$,
2. $L(a) = \{a\}, \forall a \in \Sigma$,
3. $L(\emptyset) = \emptyset$,
4. $L(R_1 \cup R_2) = L(R_1) \cup L(R_2)$
5. $L(R_1 \circ R_2) = L(R_1) \circ L(R_2)$,
6. $L(R_1^*) = (L(R_1))^*$,

where, R_1, R_2 are regular expressions.

Lemma

The language defined by any regular expression is regular.

Language defined by regular expression

Lemma

The language defined by any regular expression is regular.

Proof idea

It is easy to construct NFAs for 1.,2.,3.

If we inductively have NFAs for $L(R_1), L(R_2)$ then we can create an NFA for $L(R_1 \cup R_2)$ and $L(R_1 \circ R_2)$.

Similarly, if we inductively have NFAs for $L(R_1)$ then we can create an NFA for $(L(R_1))^*$