Applications of Rice's theorem

We now learn how to apply Rice's theorem

$$\{\langle M\rangle \mid L(M) \text{ is recognized a TM with atmost 10 states}\}.$$

Applicable.
If we can simulate any TM with another with less than 10 states, then the property will be trivial.

This is doable if we allow for the tape alphabet size to grow.

In that case, the property is trivial.

Textbooks usually consider this property to be not trivial.

This is because the usual assumption is that you always fix the tape alphabet.
In that case, Rice's theorem is applicable and the property is not trivial, therefore undecidable.

Applications of Rice's theorem

We now learn how to apply Rice's theorem

$\{M \mid L(M) \text{ is finite }\}$.

Applicable, the property is not trivial, therefore undecidable.

Applications of Rice's theorem

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$$\{M \mid L(M) = \Sigma^*\}.$$

Applicable, the property is not trivial, therefore undecidable.

Applications of Rice's theorem

We now learn how to apply Rice's theorem

$\{M \mid M$ has a useless state $\}$.

Not applicable, but the language is in fact undecidable.

Rice's theorem cannot be used to prove the undecidability of this language!

Applications of Rice's theorem

We now learn how to apply Rice's theorem

$$\{(M, w) \mid M \text{ writes a symbol a on the tape on input } w\}.$$

Not applicable, but the language is in fact undecidable.

Rice's theorem cannot be used to prove the undecidability of this language!

Applications of Rice's theorem

We now learn how to apply Rice's theorem

$$\left\{ M \;\middle|\; \begin{array}{l} M \text{ tries to write on the left of the cell when it} \\ \text{is at the leftmost bit of the input} \end{array} \right\}.$$

Not applicable, but the language is in fact undecidable.

Rice's theorem cannot be used to prove the undecidability of this language!

Proof of Rice's theorem

Theorem

Let P be a property such that it is not trivial. Recall that $\mathcal{L}_P = \{M \mid L(M) \in P\}$. Then $\mathcal{L}_P$ is undecidable.

Proof Idea:

Let P be a non-trivial property.

Assume that $\mathcal{L}_P$ is decidable.

Using this assumption prove that A_{TM} is decidable.

More specifically:

$$(M, w) \longrightarrow N$$

$$\begin{array}{lcl} \text{if } w \in L(M) & \longrightarrow & \langle N \rangle \in \mathcal{L}_P \\ \text{if } w \notin L(M) & \longrightarrow & \langle N \rangle \notin \mathcal{L}_P \end{array}$$