Proving that $L_{a,b}$ is not a regular language

Lemma

There is no finite state automaton accepting $L_{a,b}$.

Proof.

By Pigeon Hole Principle.

Suppose $\exists i, j \in \mathbb{N}$ such that $i \neq j$,

automaton reaches the same state after reading both a^i, a^j.

Then $a^i \cdot b^j$ and $a^j \cdot b^j$ are both accepted or both rejected,

which is a contradiction.

$\square$

Pumping lemma

A recipe for proving that a given language is non-regular.

Lemma (Pumping Lemma)

If L is a regular language, then $\exists p \in \mathbb{N}$ such that for any strings x, y, z with $x \cdot y \cdot z \in L$ and $|y| \geq p$,

1. *there exist strings u, v, w, s.t. y can be written as $y = u \cdot v \cdot w$,*
2. $\forall i \geq 0 \; x \cdot u \cdot v^i \cdot w \cdot z \in L$,
3. $|v| > 0$.

To prove that a given language L is not regular, the contrapositive of the above statement is useful.

Contrapositive of the pumping lemma

Lemma

We say that a language L satisfies **Property-NR** *if the following conditions hold:*

$$\forall p \geq 0,$$

$$\exists x, y, z \text{ such that } x \cdot y \cdot z \in L \text{ and } |y| \geq p,$$

$$\forall u, v, y \text{ such that } |v| > 0,\ y = u \cdot v \cdot w,$$

$$\exists i\ x \cdot u \cdot v^i \cdot w \cdot z \notin L.$$

If L satisfies Property-NR then L is not regular.

Using the pumping lemma

We say that a language L satisfies **Property-NR** if the following conditions hold:

- ☹ $\forall p \geq 0,$
- ☺ $\exists x, y, z$ such that $x \cdot y \cdot z \in L$ and $|y| \geq p,$
- ☹ $\forall u, v, y$ such that $|v| > 0,\ y = u \cdot v \cdot w,$
- ☺ $\exists i\ x \cdot u \cdot v^i \cdot w \cdot z \notin L.$

If L satisfies Property-NR then L is not regular.

We will now use the lemma to prove that $L_{a,b} = \{a^n b^n \mid n \geq n\}$ is not regular.

For any chosen $p \geq 0$, let $x := a^p$, $y := b^p$, $z = \epsilon$.

For any split of y as $u \cdot v \cdot w$, if we take $x \cdot u \cdot v^i \cdot w = 0^p 1^q$, where $q > p$ as long as $i > 0$.

In particular, $x \cdot u \cdot v^2 \cdot w \cdot z \notin L$.