Tutorial 10

- You may benefit from solving the problems on your own or in collaboration with others rather than reading solutions given in books or sources online.
- For the sake of getting better at solving problems, start writing down your thoughts and go over them repeatedly, even if you do not get the full solution.
- No credit is attached to solving this tutorial. However, solving these problems may help in being able to perform better in quizzes and exams which count towards the overall credits.

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1. Let M be a single tape TM given by $(Q, \Sigma = \{a, b\}, \Gamma, \delta, q_0, q_{acc}, q_{rej})$.

   (a) Give an NFA which does the following:
   Given: $w \in (\Gamma \cup Q)^*$
   Check: is w a valid configuration of M?

   (b) Give a PDA which does the following:
   Given: $C_1\#C_2$, where C_1, C_2 are two valid configurations of M and $\#$ is a new symbol which is not in Γ
   Check: does C_1 yield C_2^R in one step in M?

   (c) Give a PDA which does the following:
   Given: $C_1\#C_2^R\#C_3$, where C_1, C_2, C_3 are valid configurations of M and $\#$ is a new symbol which is not in Γ
   Check: does (C_1 yield C_2^R in one step in M) AND (C_2^R yield C_3 in one step in M)?

2. In this problem we will define the Post Correspondence Problem (PCP) and prove that it is undecidable. This problem and its variants are very useful in proving undecidibility of many interesting languages.

   Definition 0.1. *A domino over Σ consists of two strings $\left[\dfrac{u}{\ell}\right]$, where both $u, v \in \Sigma^*$. Given a domino $w = \left[\dfrac{u}{\ell}\right]$, u is called the upper(w) and ℓ is called the lower(w).*

   Definition 0.2. *Given a collection of dominos $d_1, d_2, \ldots, d_t$ over Σ. For any n $x_1x_2\ldots x_n \in \{d_1, d_2, \ldots, d_t\}^n$ is said to be a match if upper$(x_1) \cdot$ upper$(x_2) \ldots \cdot$ upper(x_n) equals lower$(x_1) \cdot$ lower$(x_2) \ldots \cdot$ lower(x_n).*

   Suppose $d_1 = \left[\dfrac{a}{ab}\right]$ and $d_2 = \left[\dfrac{ba}{a}\right]$.

   (a) Give an $x \in \{d_1, d_2\}^*$ such that x is a match.

   (b) Give an $x \in \{d_1, d_2\}^*$ such that x is not a match.

   Definition 0.3. *The Post Correspondence Problem can be stated as follows:*
   Given: *a collection of dominos $w_1, w_2, \ldots, w_t$*
   Output: *$x \in w_1 \cdot \{w_1, w_2, \ldots, w_t\}^*$ such that x is a match.*

   Prove that PCP is undecidable by filling in the details below. The overall proof strategy is as follows: we will create dominos over Γ to encode one valid step of TM M on input $w \in \Sigma^*$. We will then prove that there is a match if and only if M accepts w.

(a) Let d_1 be the domino which checks the initial configuration.

[Hint: $d_1 = \left[\frac{\#}{\#q_0w_1w_2\cdots w_n\#}\right]$.]

(b) Design a domino d_2 to check the correctness of the following move: $(p, b, R) \in \delta(q, a)$.

[Hint: Recall the notion of a configuration and how it changes due to such a transition. Encode the upperd_2 so that it encodes the relevant information regarding the configuration before this transition takes place and encode the lowerd_2 so that it encodes the relevant information regarding the configuration after this transition takes place.]

(c) Design a domino d_3 to check the correctness of the following move: $(p, b, L) \in \delta(q, a)$.

(d) Add a domino d_4 which allows parts of the tape to stay the same (wherever the tape head is not present.)

(e) Add a dominos to handle halting after reaching q_{acc}.

(f) Finally, add a domino to handle parts of the tape which remain after the TM halts.

(g) Let D denote the set of dominos which we designed above. Show that if M accepts w then there exists a string $x \in D^*$ such that it is a match.

(h) Show that if M does not accept w then there is no match for any string over D^*.

(i) Observe that above parts together show that PCP is undecidable.