# Introduction to Numerical Analysis

Tutorial Sheets - Part 1 (*Pre-Midsem*)

MA 214, Spring 2016-17

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# General Information

## Syllabus

**Up to Midsem (50 Marks)**

- **Mathematical Preliminaries:** Sequences of real numbers; Continuity of a Function and Intermediate Value Theorem; Mean Value Theorems for Differentiation and Integration; Taylor's Theorem; Order of Convergence.

- **Error Analysis:** Floating-Point Representation; Types of Errors; Loss of Significance; Propagation of Error.

- **Numerical Linear Algebra:** Gaussian Elimination; Pivoting Strategy; LU factorization; Matrix Norms, Condition Number of a Matrix; Solution by Iteration: Methods of Jacobi, Gauss-Seidal, and Residual Corrector Method; Eigenvalue problems: Power Method, Gershgorin's Theorem.

**After Midsem (50 Marks)**

- **Nonlinear Equations:** Bisection Method; Secant Method; Newton-Raphson Method; Fixed-point Iteration Method; Rates of Convergence

- **Interpolation:** Lagrange and Newton Forms of Interpolating Polynomials; Divided Differences; Error in Polynomial Interpolation; Runge Phenomena; Piecewise Polynomial and Cubic Spline Interpolations.

- **Numerical Integration and Differentiation:** Rectangule rule, Trapezoidal rule, Simpson's rule; Composite Rules; Gaussian Rules; Difference Formulae.

- **Numerical Ordinary Differential Equations:** Euler's Method; Modified Euler's Methods; Runge-Kutta Methods.

## Texts:

**Text to be followed:** S. Baskar and S. Sivaji Ganesh, *Introduction to Numerical Analysis*, Lecture notes. (Will be uploaded on course webpage at the site *http://moodle.iitb.ac.in*).

**Other References:**


(1) K. Atkinson and W. Han, *Elementary Numerical Analysis* ($3^{\text{rd}}$ edition), Wiley-India, 2004.

(2) S. D. Conte and C. de Boor, *Elementary Numerical Analysis - An Algorithmic Approach* ($3^{\text{rd}}$ edition), McGraw-Hill, 1981.

(3) R. L. Burden and J. D. Faires, *Numerical Analysis: Theory and Applications*, Cengage Learning India Pvt. Ltd., 2010.

## Lectures and Tutorials

We will have two lectures and one tutorial session, each of one and a half hours duration, every week. Since the class size is quite large, it may be difficult for us to give you the kind of personal attention that is ideal. Therefore, the onus is on you to be attentive in the class and make the best use of the lectures. It may not be possible for you to take down the class notes; in fact it is not necessary as the lecture notes will be uploaded on the moodle site for the course. These notes are intended to serve as a supplement to the lectures of MA 214, spring 2016-17. From the examination point of view, it is enough for a student to study only these lecture notes. However, these lecture notes are by no means a substitute for a text book. You are strongly encouraged to read the text books for a better understanding of the subject; copies of these books are there in the central library.

As far as the tutorials are concerned, each section will be divided into 6 tutorial batches. Each batch has a "course associate". The tutorials are meant for you to practise problem solving. You are expected to try the problems from the relevant tutorial sheet before coming to the class. You should also make use of the tutorial hour to clear your doubts with the course associate. More problems are given as exercises at the end of each chapter in the lecture notes. These exercise problems may not be discussed in the tutorial classes, but may have weightage in the quizzes and the examinations.

We strongly recommend students to attend the classes regularly and study systematically from day one. You are, of course, welcome to approach the course instructor or your course associate for any guidance or assistance regarding the course.

## Policy on Attendance

**Attendance in the lectures and tutorials is compulsory. Students who do not meet 80% attendance requirement will be given an *DX* grade.** In case you miss lectures for valid (medical) reasons, get a medical certificate (issued only by the IIT hospital) and **keep it with you**. You will be asked to produce it if you fall short of attendance.

## Evaluation Plan

(1) There will be **two quizzes** common to both the sections. These quizzes will be of one hour duration and they will carry **15 marks** each. First quiz will be held on **1st February, 2017**, and date for the second quiz will be announced later.

(2) The **Mid-semester examination**, scheduled to be held during **February 20-25, 2017** will be for **35 marks**. The **End-Semester examination**, scheduled to be held during **April 15 -29, 2017** will be for **35 marks**.

(3) **Make-up examination** will be conducted after the last day of instructions for those who produce the medical certificate(s) issued by the institute hospital or with valid reasons as per the institute rules. Syllabus for make-up examination will be the **full syllabus** of the course and the difficulty level of the question paper will be **more**

than the usual examinations. This examination will be for 30 marks, and marks will be correspondingly scaled.

## Grading Scheme

(1) **AP** will be awarded if the total marks is 90 and above.

(2) Let $M = 40\%$ of the highest mark. **DD** will be awarded if the total marks lie in the interval $[M, M+5]$. **FR** will be awarded if the total marks is **strictly less than** $M$.

(3) Other grades are decided relatively.

## Schedule of Lectures and Tutorials

| Section | Students | Lectures and Venue | Tutorial and Venue(s) |
|---|---|---|---|
| S1 | 2nd year EN, EP, AE<br>3rd year CS, 1st year GP | Mon: 5:30-6:55 PM (12A)<br>Thu: 5:30-6:55 PM (12B)<br><br>Venue: **LA 302** | Wed: 5:30-6:55 PM (XC)<br><br>**LT rooms 001 to 006** |
| S2 | 2nd year ME, MM | Tue: 5:30-6:55 PM (14A)<br>Fri: 5:30-6:55 PM (14B)<br><br>Venue: **LA 302** | Wed: 5:30-6:55 PM (XC)<br><br>**LT rooms 101 to 106** |

# Tutorial Sheet 1

## Mathematical Preliminaries

(1) Let $L$ be a real number and let $\{a_n\}$ be a sequence of real numbers. If there exists a positive integer $N$ such that

$$|a_n - L| \leq \mu |a_{n-1} - L|,$$

for all $n \geq N$ and for some fixed $\mu \in (0,1)$, then show that $a_n \to L$ as $n \to \infty$.

(2) Show that the equation $\sin x + x^2 = 1$ has at least one root in the interval $[0,1]$.

(3) Let $f$ be continuous on $[a,b]$, let $x_1, \cdots, x_n$ be points in $[a,b]$, and let $g_1, \cdots, g_n$ be real numbers having same sign. Show that

$$\sum_{i=1}^{n} f(x_i) g_i = f(\xi) \sum_{i=1}^{n} g_i, \quad \text{for some } \xi \in [a,b].$$

(4) Let $f : [0,1] \to [0,1]$ be a continuous function. Prove that the equation $f(x) = x$ has at least one root lying in the interval $[0,1]$ (Note: A root of this equation is called a **fixed point** of the function $f$).

(5) Let $g$ be a continuously differentiable function ($C^1$ function) such that the equation $g(x) = 0$ has at least $n$ real roots. Show that the equation $g'(x) = 0$ has at least $n-1$ real roots.

(6) Prove the second mean value theorem for integrals. Does the theorem hold if the hypothesis $g(x) \geq 0$ for all $x \in \mathbb{R}$ is replaced by $g(x) \leq 0$ for all $x \in \mathbb{R}$.

(7) In the second mean-value theorem for integrals, let $f(x) = e^x$, $g(x) = x$, $x \in [0,1]$. Find the point $c$ specified by the theorem and verify that this point lies in the interval $(0,1)$.

(8) Obtain Taylor expansion for the function $f(x) = \sin(x)$ about the point $a = 0$ when $n = 1$ and $n = 5$. Give the remainder term in both the cases and obtain their remainder estimates when $x \in [0,1]$.

(9) Prove or disprove:
(i) $\dfrac{n+1}{n^2} = O\left(\dfrac{1}{n}\right)$ as $n \to \infty$ (ii) $\dfrac{1}{\ln n} = o\left(\dfrac{1}{n}\right)$ as $n \to \infty$

(10) Assume that $f(h) = p(h) + O(h^n)$ and $g(h) = q(h) + O(h^m)$, for some positive integers $n$ and $m$, and as $h \to 0$. Find the *order of approximation* of their sum, ie., find the largest integer $r$ such that

$$f(h) + g(h) = p(h) + q(h) + O(h^r), \quad h \to 0.$$

# Tutorial Sheet 2

Error Analysis

(1) Let $X$ be a sufficiently large number which results in an overflow of memory on a computing device. Let $x$ be a sufficiently small number which results in underflow of memory on the same computing device. Then give the output of the following operations: (i) $X \times x$ (ii) $X/X$ (iii) $x/x$ (iv) $3 \times X$ (v) $3 \times x$ (vi) $x/X$

(2) In a computing device that uses $n$-digit rounding binary floating-point arithmetic, show that $\delta = 2^{-n}$ is the machine epsilon.

(3) Let $x$, $y$, and $z$ be real numbers whose floating point approximations in a computing device coincide with $x$, $y$, and $z$, respectively. Show that the relative error in computing $x(y+z)$ equals $\epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2$, where $\epsilon_1 = E_r(\mathrm{fl}(y+z))$ and $\epsilon_2 = E_r(\mathrm{fl}(x \times \mathrm{fl}(y+z)))$.

(4) Let $\epsilon = E_r(\mathrm{fl}(x))$. Show that $|\epsilon| \leq 10^{-n+1}$ if the computing device uses $n$-digit (decimal) chopping. Can the equality hold in the above inequality?

(5) Let $x_A = 3.14$ and $y_A = 2.651$ be obtained from the numbers $x_T$ and $y_T$, respectively, using 4-digit rounding. For any such values of $x_T$ and $y_T$, find the smallest interval that contains
(i) $x_T + y_T$ (ii) $x_T/y_T$.

(6) Let $x < 0 < y$ be such that the approximate numbers $x_A$ and $y_A$ has seven and nine significant digits with $x$ and $y$ respectively. Show that $z_A := x_A - y_A$ has at least six significant digits when compared to $z := x - y$.

(7) Let $x_T$ be a real number. Let $x_A = 2.5$ be an approximate value of $x_T$ with an absolute error at most 0.01. The function $f(x) = x^3$ is evaluated at $x = x_A$ instead of $x = x_T$. Estimate the resulting absolute error.

(8) Check for stability of computing the function

$$h(x) = \frac{\sin^2 x}{1 - \cos^2 x}$$

for values of $x$ very close to 0.

Run the following statement in Matlab and see the output:

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x=10^(-7.97); h=sin(x)^2/(1-cos(x)^2)
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Also, try with some non-negative number less than $x$, for instance, $x = 10^{-8}$. Note that for any $x \in \mathbb{R}$, $h(x) = 1$.

# Tutorial Sheet 3

Numerical Linear Algebra

(Direct Methods)

(1) Solve the following system of linear equations using modified Gaussian elimination method with partial pivoting using infinite precision arithmetic:

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2, \\ 3x_1 - 3x_2 + x_3 &= -1, \\ x_1 + x_2 &= 3. \end{aligned}$$

(2) Count the number of operations involved in finding a solution using naive Gaussian elimination method to the following special class of linear systems having the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1, \\ &\cdots \\ &\cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n, \end{aligned}$$

where $a_{ij} = 0$ whenever $i - j \geq 2$. In this exercise, we assume that the naive Gaussian elimination method has been implemented successfully. You must take into account the special nature of the given system.

(3) Use Thomas method to solve the tri-diagonal system of equations

$$\begin{aligned} 2x_1 + 3x_2 = 1, &\quad x_1 + 2x_2 + 3x_3 = 4, \\ x_2 + 2x_3 + 3x_4 = 5, &\quad x_3 + 2x_4 = 2. \end{aligned}$$

(4) Prove or disprove the following statements:

(i) An invertible matrix has at most one Doolittle factorization.

(ii) If a singular matrix has a Doolittle factorization, then the matrix has at least two Doolittle factorizations.

(5) Prove that if an invertible matrix $A$ has a $LU$-factorization, then all principal minors of $A$ are non-zero. Show that the matrix

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$

is invertible but has no LU factorization. Do a suitable interchange of rows to get an invertible matrix, which has an LU factorization.

(6) Use Cholesky factorization to solve the system of equations

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 4, \\ -2x_1 + 5x_2 - 3x_3 &= -7, \\ 2x_1 - 3x_2 + 6x_3 &= 10. \end{aligned}$$

# Tutorial Sheet 4

## Numerical Linear Algebra

(Matrix Theory and Iterative Methods)

(1) Show that the norm defined on the set of all $n \times n$ matrices by

$$\|A\| := \max_{\substack{1 \le i \le n \\ 1 \le j \le n}} |a_{ij}|$$

is not subordinate to any vector norm on $\mathbb{R}^n$.

(2) Let $A$ be an invertible matrix. Show that its condition number $\kappa(A)$ satisfies $\kappa(A) \ge 1$.

(3) Let $A$ be an $n \times n$ matrix with real entries. Let $\kappa_2(A)$ and $\kappa_\infty(A)$ denote the condition numbers of the matrix $A$ that are computed using the matrix norms $\|A\|_2$ and $\|A\|_\infty$, respectively. Answer the following questions.

(i) Determine all the diagonal matrices such that $\kappa_\infty(A) = 1$.

(ii) Let $Q$ be a matrix such that $Q^T Q = I$ (such matrices are called *orthogonal matrices*). Show that $\kappa_2(Q) = 1$.

(4) In solving the system of equations $A\boldsymbol{x} = \boldsymbol{b}$ with matrix

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2.01 \end{pmatrix},$$

estimate the relative error in the solution vector $\boldsymbol{x}$ in terms of the relative error in $\boldsymbol{b}$. Test your estimate in the case when $\boldsymbol{b} = (4,4)^T$ and $\tilde{\mathbf{b}} = (3,5)^T$. Use the maximum norm for vectors in $\mathbb{R}^2$.

(5) Let $A$ and $\tilde{A}$ be non-singular square matrices, and $\boldsymbol{b} \neq 0$. If $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ are the solutions of the systems $A\boldsymbol{x} = \boldsymbol{b}$ and $\tilde{A}\tilde{\boldsymbol{x}} = \boldsymbol{b}$, respectively, then show that

$$\frac{\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|}{\|\tilde{\boldsymbol{x}}\|} \le \kappa(A) \frac{\|A - \tilde{A}\|}{\|A\|}.$$

(Note: The above inequality gives an estimate of the relative error in the solution in terms of the relative error in the coefficient matrix).

(6) Find the $n \times n$ matrix $B$ and the $n$-dimensional vector $\boldsymbol{c}$ such that the Gauss-Seidal method can be written in the form

$$\boldsymbol{x}^{(k+1)} = B\boldsymbol{x}^{(k)} + \boldsymbol{c}, \qquad k = 0, 1, 2, \cdots.$$

(7) *Spectral radius* of a square matrix $A$ is defined as $\rho(A) := \max_{j=1,\cdots,n} |\lambda_j|$, where $\lambda_j$'s are the eigenvalues of $A$.

(i) For any subordinate matrix norm $\|\cdot\|$, show that $\rho(A) \le \|A\|$.

(ii) If $A$ is invertible and if an iterative method of the form $\boldsymbol{x}^{(k+1)} = B\boldsymbol{x}^{(k)} + \boldsymbol{c}$ converges to the solution of $A\boldsymbol{x} = \boldsymbol{b}$ for any initial guess $\boldsymbol{x}^{(0)}$ and any vector $\boldsymbol{b}$, then show that $\rho(B) < 1$.

(8) Write the formula for the Jacobi iterative sequence of the system

$$\begin{aligned} 7x_1 - 15x_2 - 21x_3 &= 2, \\ 7x_1 - x_2 - 5x_3 &= -3, \\ 7x_1 + 5x_2 + x_3 &= 1. \end{aligned}$$

Without performing the iterations, show that the sequence does not converge to the exact solution of this system. Can you make a suitable interchange of rows so that the resulting system is diagonally dominants?

(9) Let $\boldsymbol{x}^{(7)}$ be the $7^{\text{th}}$ term of the Gauss-Seidel iterative sequence for the system

$$\begin{aligned} 3x_1 + 2x_2 &= 1 \\ 4x_1 + 12x_2 + 3x_3 &= -2 \\ x_1 + 3x_2 - 5x_3 &= 3 \end{aligned}$$

with $\boldsymbol{x}^{(0)} = (0, 0, 0)^T$. If $\boldsymbol{x}$ denotes the exact solution of the given system, then show that

$$\|\boldsymbol{e}^{(7)}\|_\infty \leq 0.058527664 \|\boldsymbol{x}\|_\infty.$$

# Tutorial Sheet 5

## Numerical Linear Algebra

(Power Method)

(1) The matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = 1$ and the corresponding eigenvectors may be taken as $\mathbf{v}_1 = (1, 2, 3)^T$, $\mathbf{v}_2 = (0, 2, 3)^T$, and $\mathbf{v}_3 = (0, 3, 2)^T$. Perform 2 iterations to find the eigenvalue and the corresponding eigen vector to which the power method converges when we start the iteration with the initial guess $\mathbf{x}^{(0)} = (0, 0.5, 0.75)^T$. Without performing the iteration, find the eigenvalue and the corresponding eigenvector to which the power method converges when we start the iteration with the initial guess $\mathbf{x}^{(0)} = (0.001, 0.5, 0.75)^T$. Justify your answer.

(2) The matrix

$$A = \begin{pmatrix} 4/3 & 1/3 & 1/3 \\ -4/3 & -1/3 & 11/3 \\ 2 & 2 & -2 \end{pmatrix}$$

has eigenvalues $\lambda_1 = -4$, $\lambda_2 = 2$, and $\lambda_3 = 1$ with corresponding eigenvectors $\mathbf{v}_1 = (0, -1, 1)^T$, $\mathbf{v}_2 = (1, 1, 1)^T$, and $\mathbf{v}_3 = (-1, 1, 0)^T$. To which eigenvalue and the corresponding eigenvector does the power method converge if we start with the initial guess $\mathbf{x}^{(0)} = (3, -1, 1)$? Justify your answer without performing the iterations.

(3) Use Gerschgorin circle theorem to determine the intervals in which the eigenvalues of the matrix

$$A = \begin{pmatrix} 0.5 & 0 & 0.2 \\ 0 & 3.15 & -1 \\ 0.57 & 0 & -7.43 \end{pmatrix}.$$

lie, given that all eigenvalues of $A$ are real. Show that the power method can be applied for this matrix to find the dominant eigenvalue without computing eigenvalues explicitly.

(4) Use the Gerschgorin circle theorem to find the lower and the upper bounds for the absolute values of the eigenvalues of the following matrices:

(i) $\begin{pmatrix} 6 & 2 & 1 \\ 1 & -5 & 0 \\ 2 & 1 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 7 & -1 & 4 \\ 3 & -6 & -1 \\ -1 & 1 & 5 \end{pmatrix}$

Also, find the optimum bounds in each case.