CS310 Automata Theory – 2016-2017

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Lecture 28: Turing machines, computability
March 27, 2017

At the end of last class

Introduction to Turing machines

- Undecidability of the following languages:
  - $A_{TM} = \{(M, w) \mid M \text{ accepts } w\}$.

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  - $REG_{TM} = \{\langle M \rangle \mid L(M) \text{ is regular}\}$.

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\end{array}$$

We will do the proof on Thursday (after the Tutorial 10 is solved).

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Rice's theorem: A systematic way of proving undecidability of languages.

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We say that a property P is trivial if either $\mathcal{L}_P = \varnothing$ or $\mathcal{L}_P$ is the set of all the Turing recognizable languages.

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   Therefore, P is not trivial.

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Therefore, P is in fact all Turing recognizable languages.

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