CS310 Automata Theory – 2016-2017

Nutan Limaye

Indian Institute of Technology, Bombay
nutan@cse.iitb.ac.in

Lecture 24: Turing machines, computability
March 16, 2017

Last three classes

Introduction to Turing machines

> What are Turing machines? Informal and formal definitions and examples.

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Configurations of a Turing machine.

Turing recognizable and Turing decidable languages.

k-tape TMs equivalent to 1-tape TMs.

Existence of unrecognizable languages.

Proof that A_{TM} is recognizable but not decidable.

Variants of Turing machines

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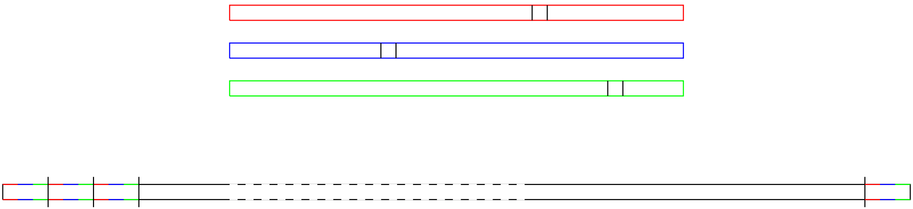

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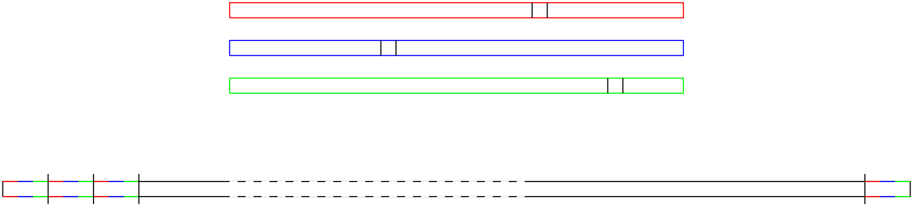

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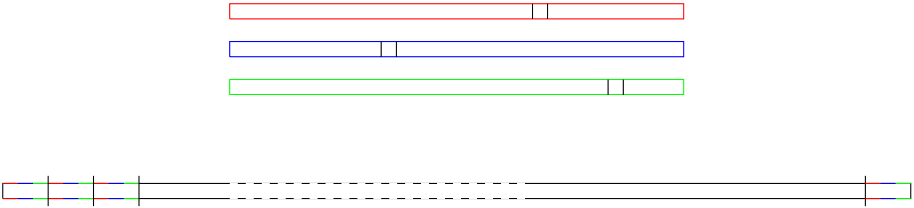

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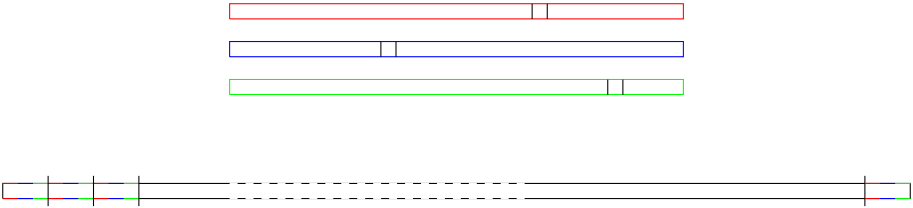

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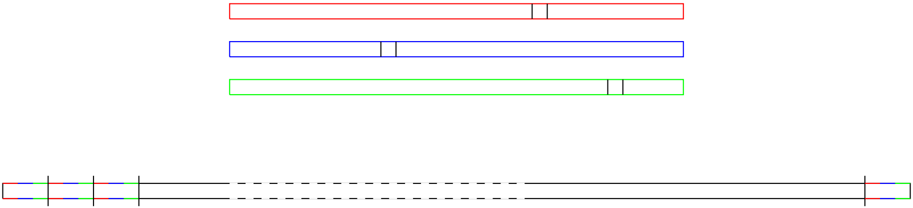

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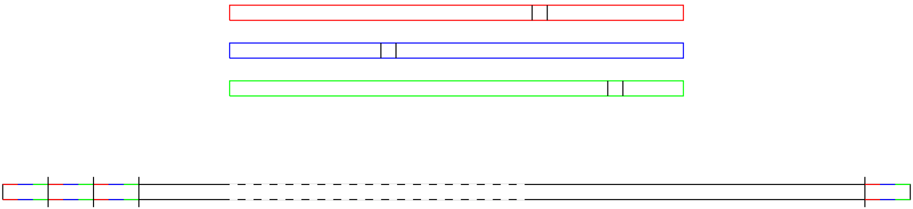

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$\overline{\Gamma}$ symbols used to denote tape head positions.

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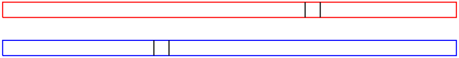

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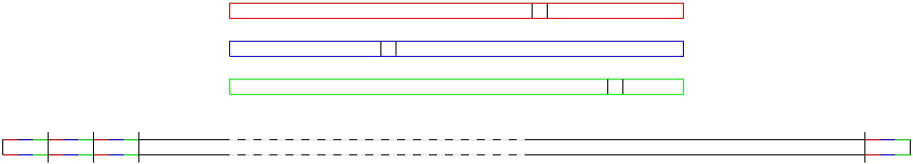

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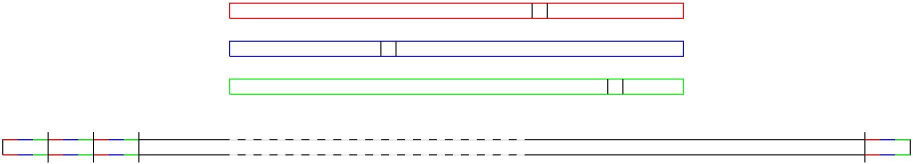

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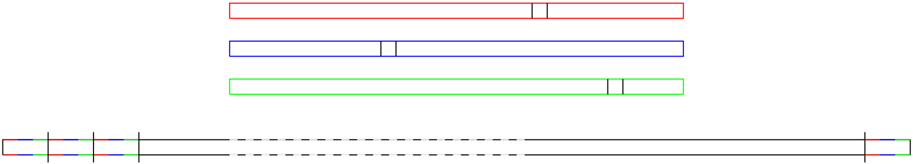

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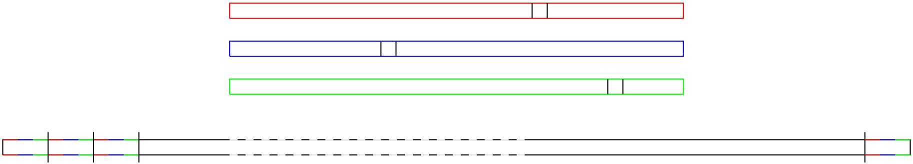

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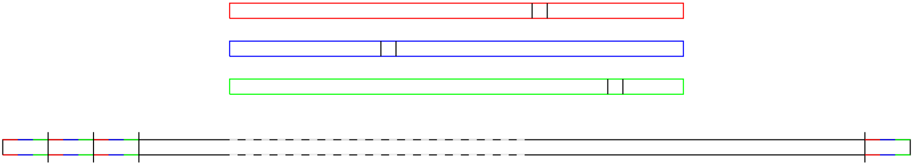

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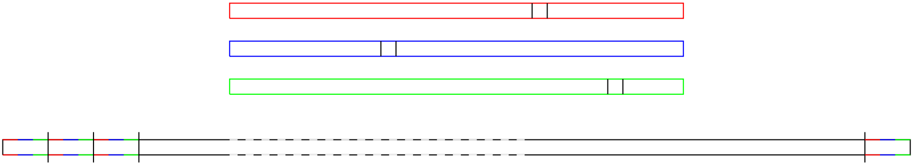

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Let M_1, M_2 be two TMs recognizing $L, \overline{L}$, respectively.

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There must be a language which is not Turing recognizable.

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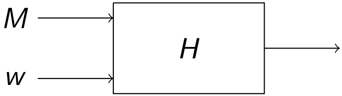

q_{acc} if M accepts w

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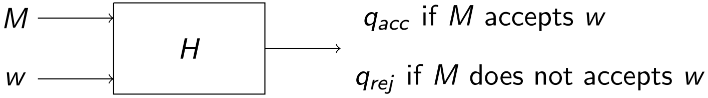

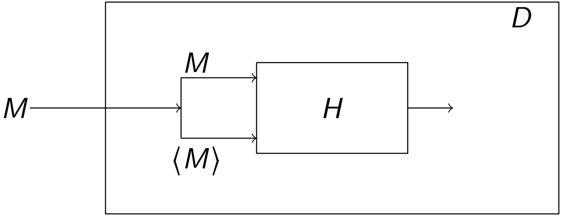

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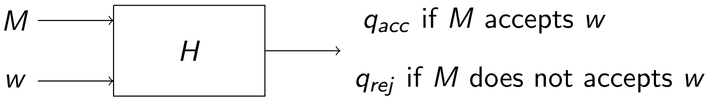

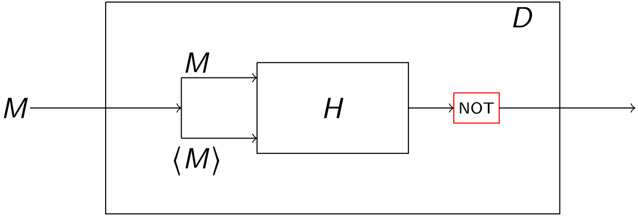

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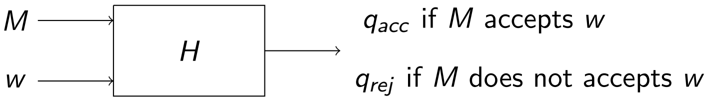

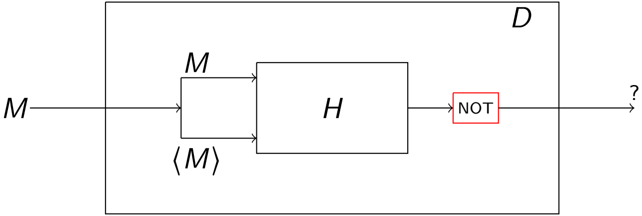

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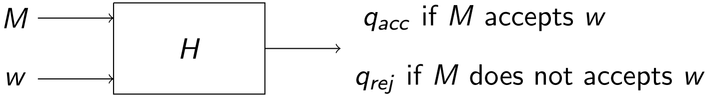

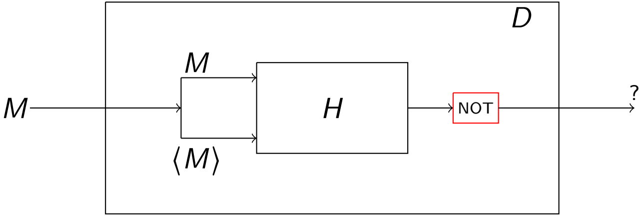

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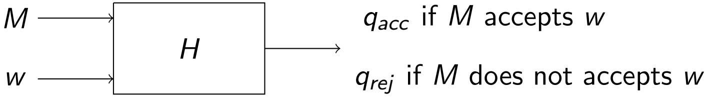

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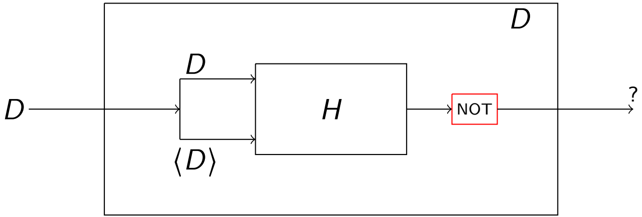

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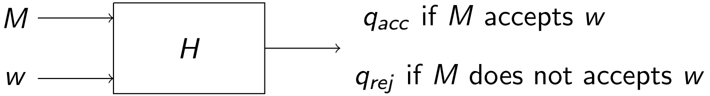

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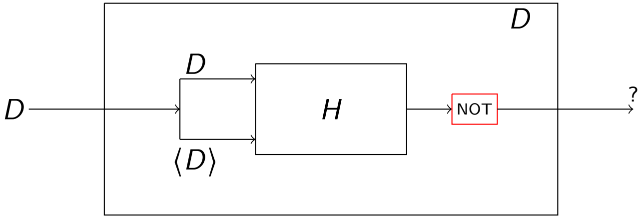

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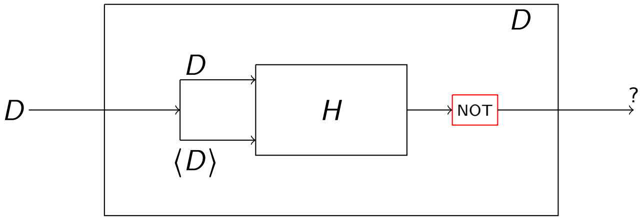

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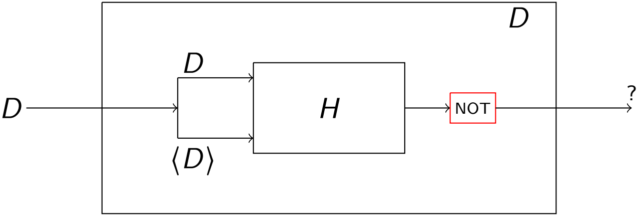

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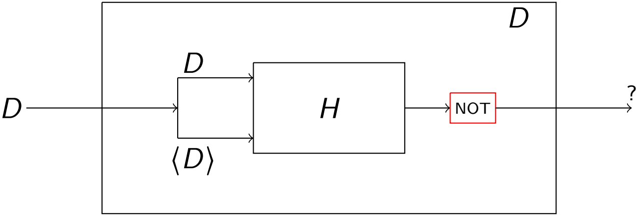

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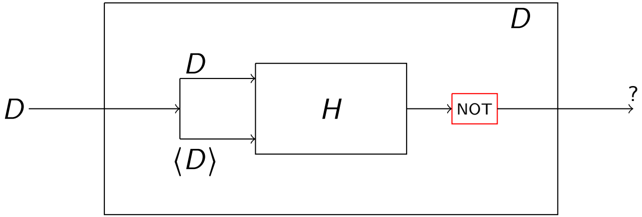

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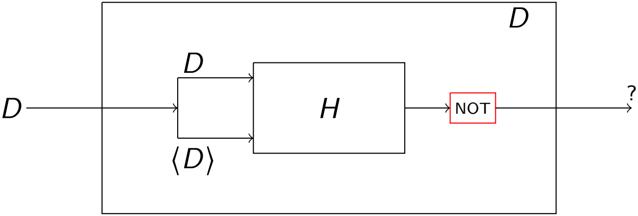

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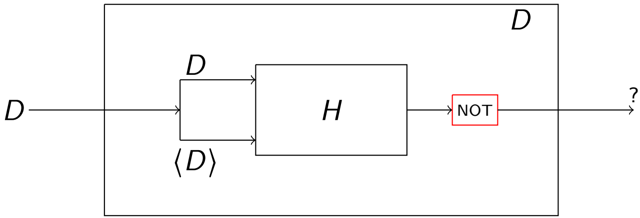

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Similarly, $A_{\text{PDA}} = \{(M, w) \mid \text{ PDA } M \text{ accepts } w\}$ is also decidable.

Diagonalization inside the proof

Behaviour of the machines.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	$\checkmark$		$\checkmark$	$\checkmark\ldots$	$\ldots\ldots$

Diagonalization inside the proof

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	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	$\checkmark$		$\checkmark$	$\checkmark \ldots$	$\ldots\ldots$
M_2	$\checkmark$	$\times$		$\times \ldots$	$\checkmark \ldots \times \checkmark \ldots$

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	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	$\checkmark$		$\checkmark$	$\checkmark\ldots$	$\ldots\ldots$
M_2	$\checkmark$	$\times$		$\times\ \ldots$	$\checkmark\ldots\times\checkmark\ \ldots$
M_3	$\times$	$\times$	$\checkmark$	$\ldots\times$	$\checkmark\ldots\ldots$
$\vdots$					
$\vdots$					

Diagonalization inside the proof

Behaviour of H.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	✓	$\times$	✓	✓ $\ldots$	$\ldots\ldots$
M_2	✓	$\times$	$\times$	$\times$ $\ldots$	✓ $\ldots$ $\times$ ✓ $\ldots$
M_3	$\times$	$\times$	✓	$\ldots$ $\times$	✓ $\ldots\ldots$
$\vdots$					
$\vdots$					

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Behaviour of H.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	✓	$\times$	✓	✓ $\ldots$	$\ldots\ldots$
M_2	✓	$\times$	$\times$	$\times$ $\ldots$	✓ $\ldots$ $\times$ ✓ $\ldots$
M_3	$\times$	$\times$	✓	$\ldots$ $\times$	✓ $\ldots\ldots$
$\vdots$					
$\vdots$					

Diagonalization inside the proof

Behaviour of D.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	$\cancel{\checkmark}$ $\times$	$\times$	$\checkmark$	$\checkmark \ldots$	$\ldots\ldots$
M_2	$\checkmark$	$\cancel{\times}$ $\checkmark$	$\times$	$\times \ldots$	$\checkmark \ldots \times \checkmark \ldots$
M_3	$\times$	$\times$	$\cancel{\checkmark}$ $\times$	$\ldots \times$	$\checkmark \ldots\ldots$
$\vdots$					
$\vdots$					

Diagonalization inside the proof

Behaviour of D on itself.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots \langle D \rangle \ldots$	$\ldots\ldots$
M_1	$\cancel{\checkmark}$ $\times$	$\times$	$\checkmark$	$\checkmark \ldots$	$\ldots\ldots$
M_2	$\checkmark$	$\cancel{\times}$ $\checkmark$	$\times$	$\times \ \ldots$	$\checkmark \ldots \times \checkmark \ \ldots$
M_3	$\times$	$\times$	$\cancel{\checkmark}$ $\times$	$\ldots \times$	$\checkmark \ldots\ldots$
$\vdots$					
$\vdots$					
D				$\ldots \quad ? \ldots$	$\ldots\ldots$

Other undecidable problems and reducibility

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$\mathcal{H}$ decides Halt if and only if $\mathcal{A}$ decides A_{TM}.

The halting problem

Lemma

The halting problem, $Halt = \{(M, w) \mid M \text{ halts on } w\}$*, is undecidable.*

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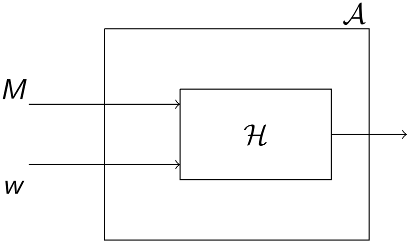

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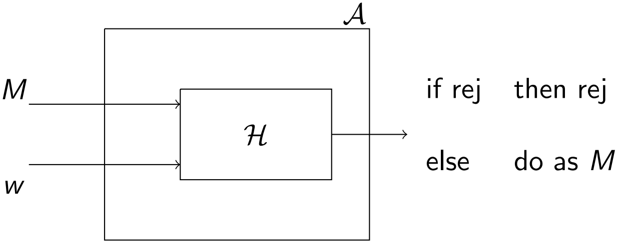

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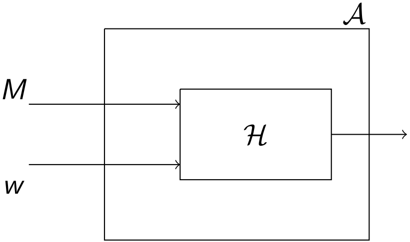

if rej then rej

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If Halt is decidable then $\mathcal{A}$ decides A_{TM}

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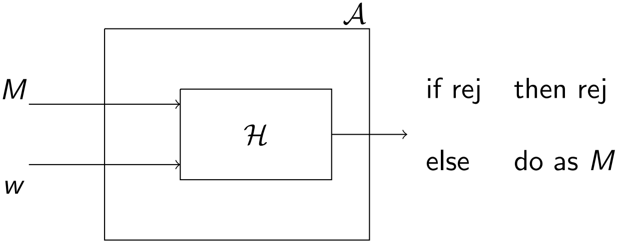

If Halt is decidable then $\mathcal{A}$ decides A_{TM}, which is a contradiction.