CS310 Automata Theory – 2016-2017

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Lecture 25: Turing machines, computability
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Module III so far...

Introduction to Turing machines

What are Turing machines? Informal and formal definitions and examples.

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Configurations of a Turing machine.

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What are Turing machines? Informal and formal definitions and examples.

Configurations of a Turing machine.

Turing recognizable and Turing decidable languages.

k-tape TMs equivalent to 1-tape TMs.

Existence of unrecognizable languages.

A_{TM} is recognizable but not decidable.

A decision problem about TMs

$$A_{TM} = \{(M, w) \mid M \text{ accepts } w\}$$

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Lemma

A_{TM} is Turing recognizable.

Proof sketch

Design a TM, say N such that,

N behaves like M on w at each step,

if M reaches q_{acc} then N also accepts.

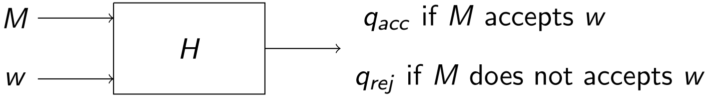
Is A_{TM} decidable?

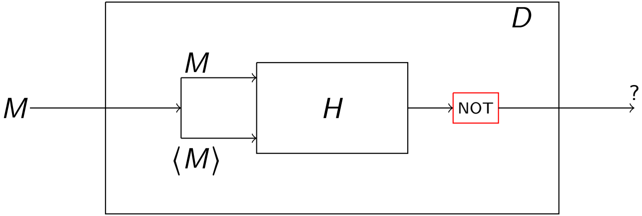
A decision problem about TMs

Lemma

A_{TM} is not Turing decidable.

Assume that there exists M such that M decides A_{TM}.





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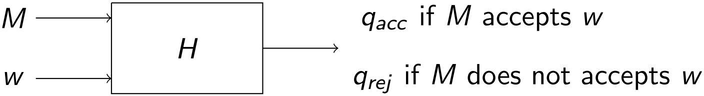
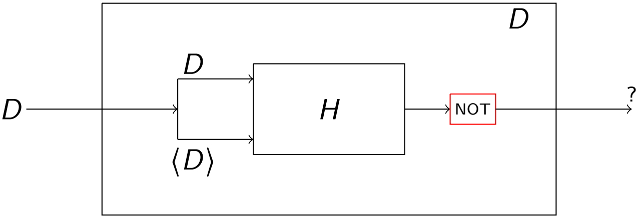


What happens if we give D as input to itself?



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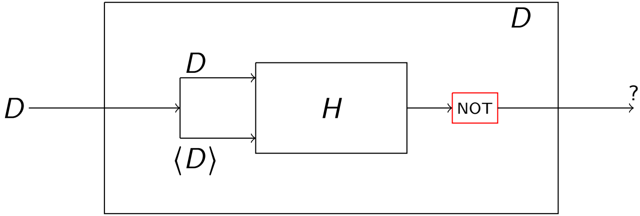

If D accepts $\langle D \rangle$ then D rejects $\langle D \rangle$.

If D rejects $\langle D \rangle$ then D accepts $\langle D \rangle$. ☹

Diagonalization inside the proof

Behaviour of the machines.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	$\checkmark$		$\checkmark$	$\checkmark\ldots$	$\ldots\ldots$

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M_1	$\checkmark$		$\checkmark$	$\checkmark \ldots$	$\ldots\ldots$
M_2	$\checkmark$	$\times$		$\times \ldots$	$\checkmark \ldots \times \checkmark \ldots$

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M_1	$\checkmark$		$\checkmark$	$\checkmark\dots$	$\dots\dots$
M_2	$\checkmark$	$\times$		$\times\ \dots$	$\checkmark\dots\times\checkmark\ \dots$
M_3	$\times$	$\times$	$\checkmark$	$\dots\times$	$\checkmark\dots\dots$
$\vdots$					
$\vdots$					

Diagonalization inside the proof

Behaviour of H.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	$\checkmark$	$\times$	$\checkmark$	$\checkmark\ldots$	$\ldots\ldots$
M_2	$\checkmark$	$\times$	$\times$	$\times\ \ldots$	$\checkmark\ldots\times\checkmark\ \ldots$
M_3	$\times$	$\times$	$\checkmark$	$\ldots\ \times$	$\checkmark\ldots\ldots$
$\vdots$					
$\vdots$					

Diagonalization inside the proof

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	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	$\checkmark$	$\times$	$\checkmark$	$\checkmark\ldots$	$\ldots\ldots$
M_2	$\checkmark$	$\times$	$\times$	$\times\ \ldots$	$\checkmark\ldots\times\checkmark\ \ldots$
M_3	$\times$	$\times$	$\checkmark$	$\ldots\times$	$\checkmark\ldots\ldots$
$\vdots$					
$\vdots$					

Diagonalization inside the proof

Behaviour of D.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	$\cancel{\checkmark}$ $\times$	$\times$	$\checkmark$	$\checkmark \ldots$	$\ldots\ldots$
M_2	$\checkmark$	$\cancel{\times}$ $\checkmark$	$\times$	$\times \ldots$	$\checkmark \ldots \times \checkmark \ldots$
M_3	$\times$	$\times$	$\cancel{\checkmark}$ $\times$	$\ldots \times$	$\checkmark \ldots\ldots$
$\vdots$					
$\vdots$					

Diagonalization inside the proof

Behaviour of D on itself.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots \langle D \rangle \ldots$	$\ldots\ldots$
M_1	$\cancel{\checkmark}$ $\times$	$\times$	$\checkmark$	$\checkmark \ldots$	$\ldots\ldots$
M_2	$\checkmark$	$\cancel{\times}$ $\checkmark$	$\times$	$\times \ldots$	$\checkmark \ldots \times \checkmark \ldots$
M_3	$\times$	$\times$	$\cancel{\checkmark}$ $\times$	$\ldots \times$	$\checkmark \ldots\ldots$
$\vdots$					
$\vdots$					
D				$\ldots$? $\ldots$	$\ldots\ldots$

Other undecidable problems and reducibility

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Reducing A_{TM} to another problem to prove undecidibility.

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Assume that Halt is decidable.

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$\mathcal{A}$: Run $\mathcal{H}$ on (M, w).

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Assume that Halt is decidable. Let $\mathcal{H}$ be the TM deciding Halt.

$\mathcal{A}$: Run $\mathcal{H}$ on (M, w). If it rejects then reject,

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$\mathcal{A}$ accepts (M, w) if M accepts w

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Assume that Halt is decidable. Let $\mathcal{H}$ be the TM deciding Halt.

$\mathcal{A}$: Run $\mathcal{H}$ on (M, w). If it rejects then reject, else do as per M on w.

$\mathcal{A}$ accepts (M, w) if M accepts w and rejects it if either M rejects w or M loops forever on w.

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$\mathcal{A}$ accepts (M, w) if M accepts w and rejects it if either M rejects w or M loops forever on w.

$\mathcal{H}$ decides Halt if and only if $\mathcal{A}$ decides A_{TM}.

The halting problem

Lemma

The halting problem, $Halt = \{(M, w) \mid M \text{ halts on } w\}$*, is undecidable.*

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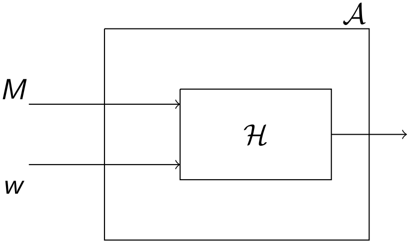

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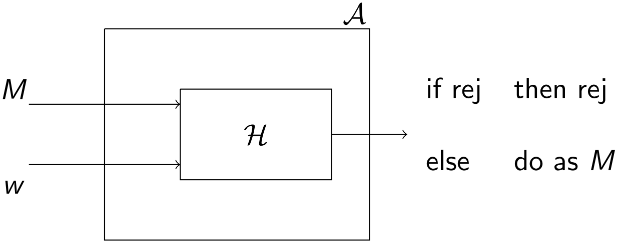

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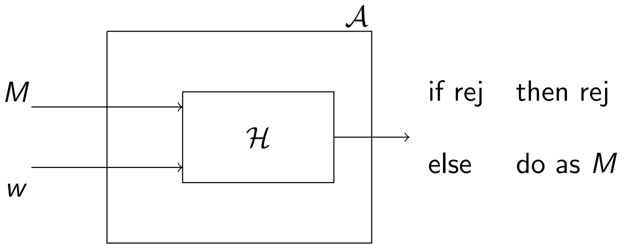

If Halt is decidable then $\mathcal{A}$ decides A_{TM}

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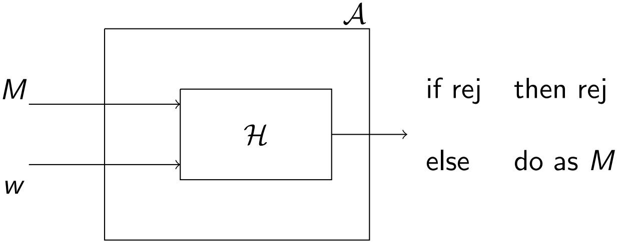

If Halt is decidable then $\mathcal{A}$ decides A_{TM}, which is a contradiction.

Emptiness problem for TM

Lemma

The emptiness problem for TMs, $E_{TM} = \{\langle M\rangle \mid L(M) = \varnothing\}$, is undecidable.

Assume for the sake of contradiction that it is decidable.

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The emptiness problem for TMs, $E_{TM} = \{\langle M \rangle \mid L(M) = \varnothing\}$, is undecidable.

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Let $T'_{M,w}$ be as follows:

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On input x

{

if $w \neq x$ then reject

else do as per M

}

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Let $T'_{M,w}$ be as follows:

On input x

{

if $w \neq x$ then reject

else do as per M

}

Let A be as follows:

On input M, w

{

Create machine $T'_{M,w}$.

If T on $\langle T'_{M,w} \rangle$ rejects

then accept

else reject

}

Equality for TM

Lemma

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Given a machine M_2 as an input, use M to check whether $L(M_2) = L(M_1)$, i.e. to check whether $L(M_2) = \emptyset$ or not.

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This implies that if EQ_{TM} is decidable then E_{TM} is decidable.

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This implies that if EQ_{TM} is decidable then E_{TM} is decidable.

But from the previous result we know that E_{TM} is undecidable.

Regularity checking

Lemma

$REG_{TM} = \{\langle M \rangle \mid L(M)$ is regular$\}$ is undecidable.

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$REG_{TM} = \{\langle M \rangle \mid L(M) \text{ is regular}\}$ is undecidable.