CS310 Automata Theory – 2016-2017

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Lecture 26: Turing machines, computability
March 23, 2017

At the end of last class

Introduction to Turing machines

Undecidability of the following languages:

$A_{TM} = \{(M, w) \mid M \text{ accepts } w\}$.

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$E_{TM} = \{\langle M\rangle \mid L(M) = \varnothing\}$.

$EQ_{TM} = \{(M_1, M_2 \mid L(M_1) = L(M_2))\}$.

Started with trying to prove $REG_{TM} = \{\langle M\rangle \mid L(M) isregular\}$.

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$\mathcal{H}$ decides Halt if and only if $\mathcal{A}$ decides A_{TM}.

The halting problem

Lemma

The halting problem, $Halt = \{(M, w) \mid M \text{ halts on } w\}$, is undecidable.

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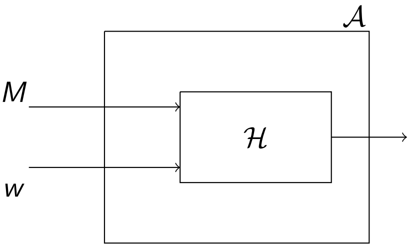

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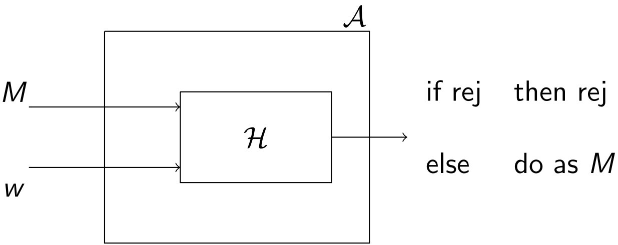

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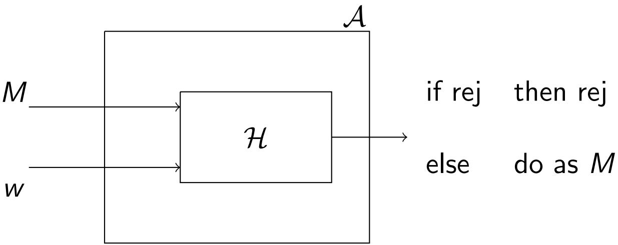

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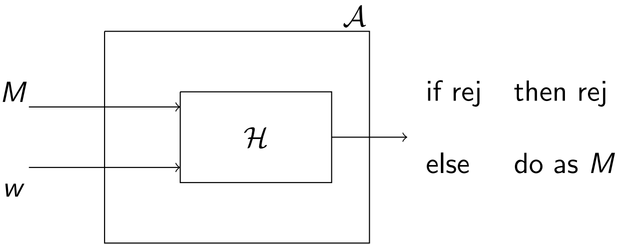

If Halt is decidable then $\mathcal{A}$ decides A_{TM}, which is a contradiction.

Emptiness problem for TM

Lemma

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Assume for the sake of contradiction that it is decidable.

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Let $T'_{M,w}$ be as follows:

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Let $T'_{M,w}$ be as follows:

On input x

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if $w \neq x$ then reject

else do as per M

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Let A be as follows:

On input M, w

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Create machine $T'_{M,w}$.

If T on $\langle T'_{M,w}\rangle$ rejects

then accept

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Equality for TM

Lemma

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Assume for the sake of contradiction that EQ_{TM} is decidable. Let M be the TM for it.

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Assume for the sake of contradiction that EQ_{TM} is decidable. Let M be the TM for it.

Let M_1 be a machine that rejects all strings. That is, $L(M_1) = \varnothing$.

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Assume for the sake of contradiction that EQ_{TM} is decidable. Let M be the TM for it.

Let M_1 be a machine that rejects all strings. That is, $L(M_1) = \emptyset$.

Given a machine M_2 as an input, use M to check whether $L(M_2) = L(M_1)$

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Given a machine M_2 as an input, use M to check whether $L(M_2) = L(M_1)$, i.e. to check whether $L(M_2) = \varnothing$ or not.

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This implies that if EQ_{TM} is decidable then E_{TM} is decidable.

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This implies that if EQ_{TM} is decidable then E_{TM} is decidable.

But from the previous result we know that E_{TM} is undecidable.

Regularity checking

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Let $R'_{M,w}$ be s.t.

$$L(R'_{M,w}) = \begin{cases} \{0^n 1^n \mid n \geq 0\} & \text{if } M \text{ rej } w \\ \Sigma^* & \text{if } M \text{ acc } w \end{cases}$$

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If we get such an $R'_{M,w}$ we can design A as a follows.

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Let A be as follows:

On input M, w

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If R on $\langle R'_{M,w}\rangle$ accepts

then accept

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$\text{valid}_{M,w}(x) \quad = 1 \quad$ if x is a valid encoding of a run of M on w

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$$L_{M,w} = \{x \mid \overline{\text{valid}_{M,w}(x)} \text{ or } M \text{ rej } w\}.$$

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If M acc w then $\exists x$, $x \notin L_{M,w}$, where x encodes the run of M on w.

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$$L_{M,w} = \{x \mid \overline{\text{valid}_{M,w}(x)} \text{ or } M \text{ rej } w\}.$$

If M acc w then $\exists x,\ x \notin L_{M,w}$, where x encodes the run of M on w.

If M rej w then $\forall x,\ x \in L_{M,w}$

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}

$$\text{valid}_{M,w}(x) \begin{cases} = 1 & \text{if } x \text{ is a valid encoding of a run of } M \text{ on } w \\ = 0 & \text{otherwise} \end{cases}$$

$$L_{M,w} = \{x \mid \overline{\text{valid}_{M,w}(x)} \text{ or } M \text{ rej } w\}.$$

If M acc w then $\exists x$, $x \notin L_{M,w}$, where x encodes the run of M on w.

If M rej w then $\forall x$, $x \in L_{M,w}$, i.e. $L_{M,w} = \Sigma^*$.

Universality of CFLs

Lemma

$ALL_{CFL} = \{\langle M \rangle \mid M$ is a PDA and $L(M) = \Sigma^*\}$ is undecidable.

Design of N

if x does not encode a run of M on w

then accept

else

{

if M accepts w
then reject
else accept

}

$$\text{valid}_{M,w}(x) \quad = 1 \quad \text{if } x \text{ is a valid encoding of a run of } M \text{ on } w$$
$$= 0 \quad \text{otherwise}$$

$L_{M,w} = \{x \mid \overline{\text{valid}_{M,w}(x)} \text{ or } M \text{ rej } w\}$.

We will design N s.t. $L(N) = L_{M,w}$.

If M acc w then $\exists x$, $x \notin L_{M,w}$, where x encodes the run of M on w.

If M rej w then $\forall x$, $x \in L_{M,w}$, i.e. $L_{M,w} = \Sigma^*$.