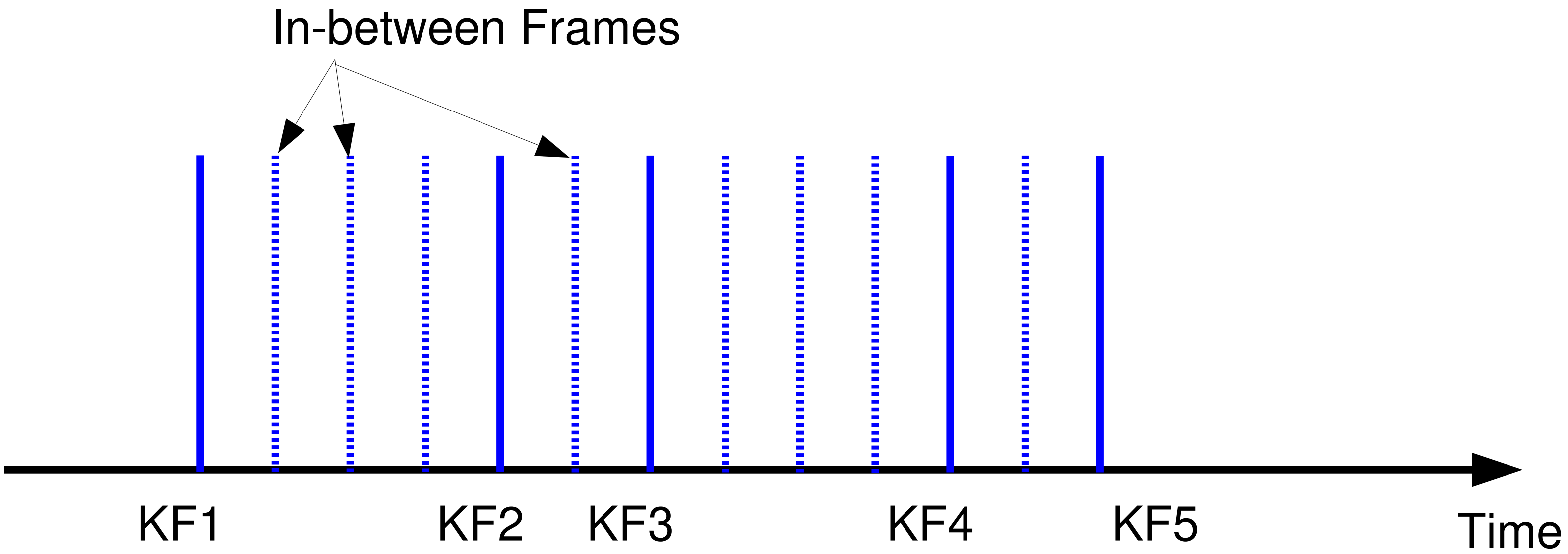
CS475/CS675
Computer Graphics

Interpolation for Animation

Animation

- Keyframing
  - Selected (key) frames are specified.
  - Interpolation of intermediate frames.
  - Simple and popular approach.
  - May give incorrect (inconsistent) results.



Animation

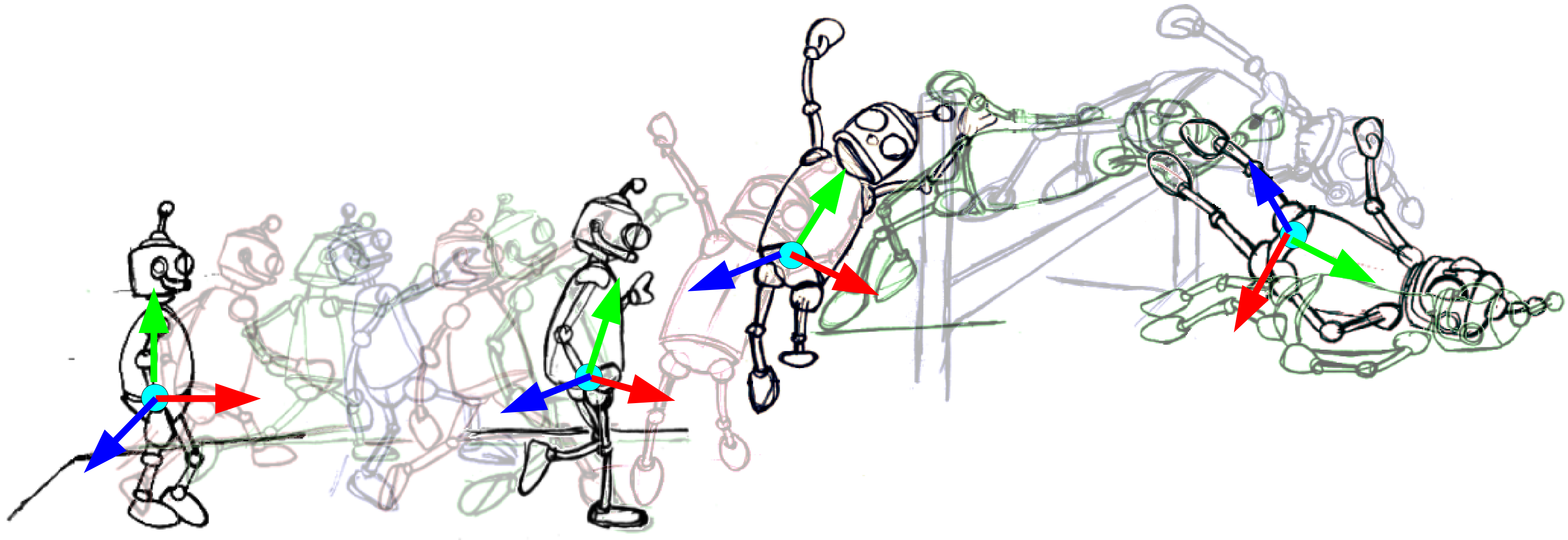
- Keyframing

Animation

- Keyframing
  - Interpolate Position

Animation

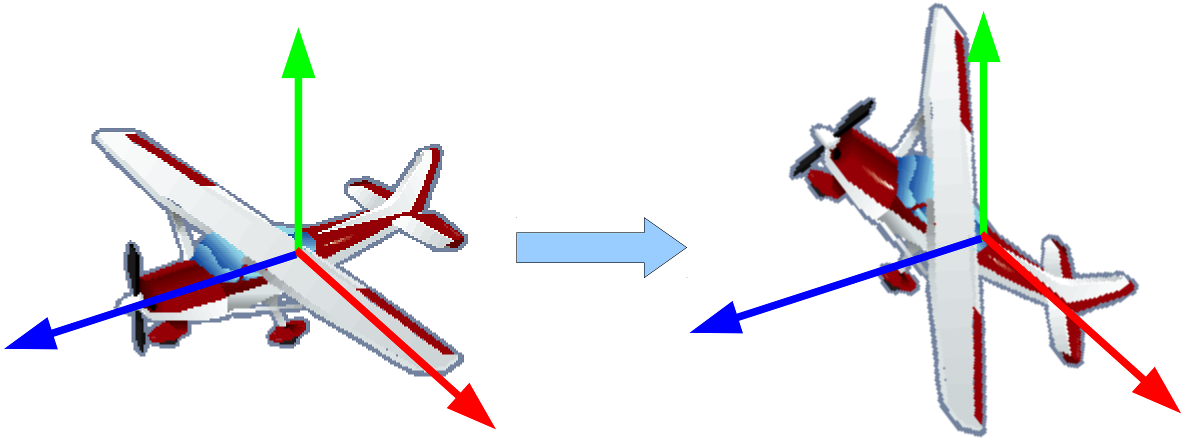
- Keyframing
  - Interpolate Orientation



Animation

- Interpolating orientation
  - **Fixed Angle Representation** - Ordered triple of rotations about ***global axes.***
  - Any triple is valid that doesn't immediately repeat an axis, e.g., x-y-z or z-x-y. But not x-x-y.
  - Let us assume a z-y-x order for now. (α, β, γ)

$$P' = R_z(\gamma) . R_y(\beta) . R_x(\alpha) . P$$



Animation

- Interpolating orientation – Fixed Angle Representation
  - Make a rotation matrix from the angles and interpolate

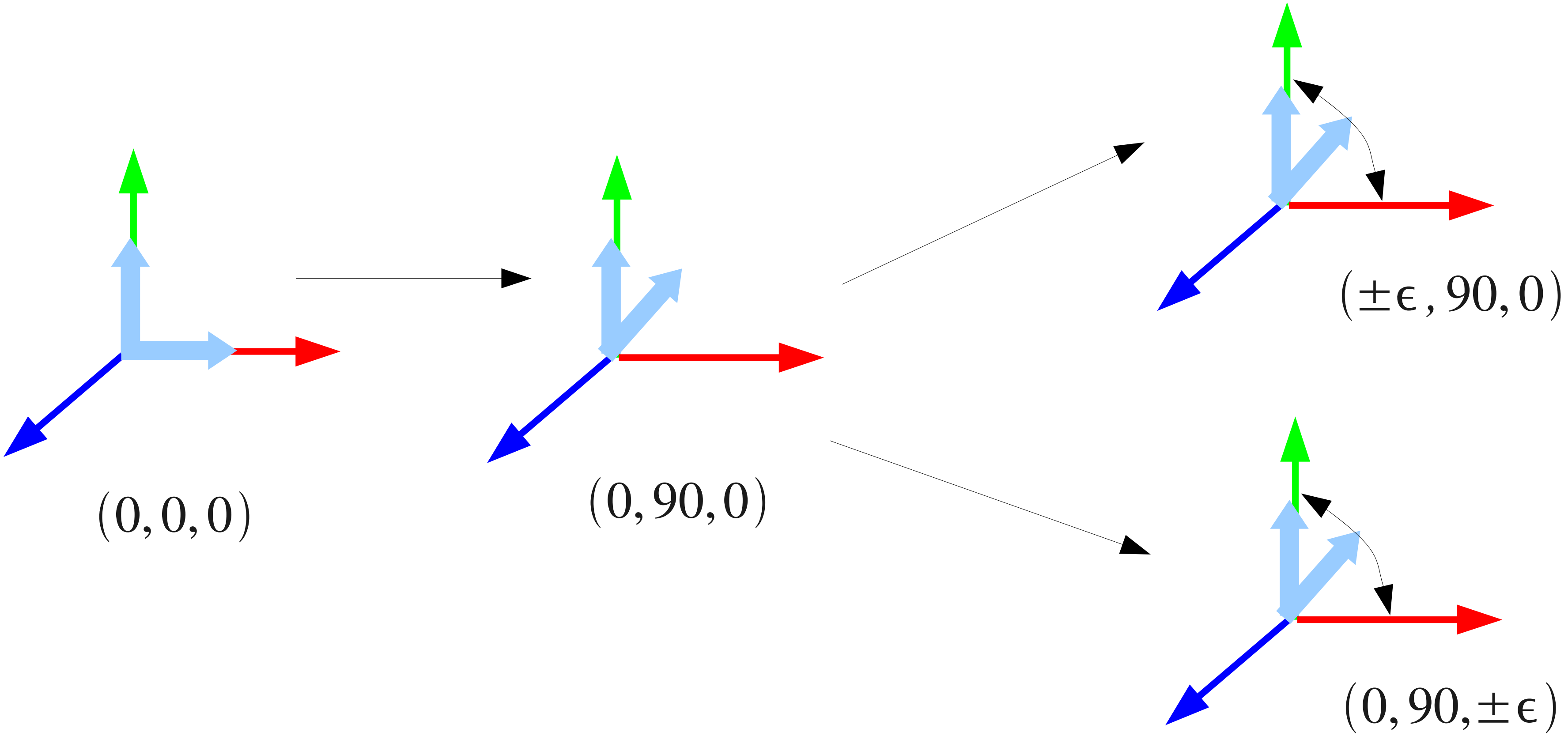
$$R_z(90) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(-90) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(?) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Animation

- Interpolating orientation – Fixed Angle Representation
  - Interpolate the angles and then form the matrix.
  - Suffers from the Gimbal lock!

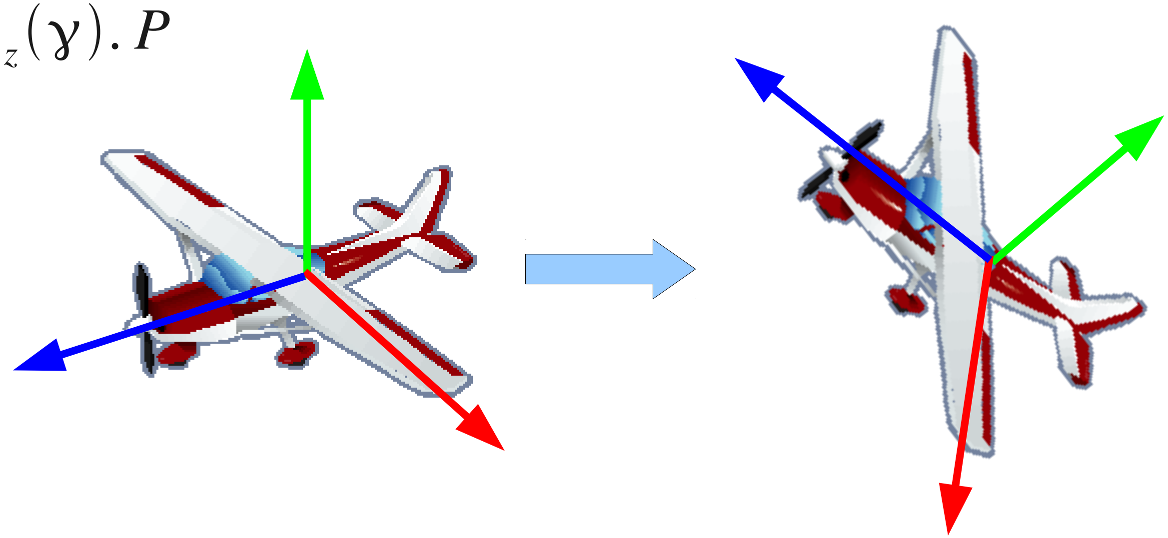


Animation

- Interpolating orientation
  - **Euler Angle Representation** - Ordered triple of rotations about ***local axes.***
  - Any triple is valid that doesn't immediately repeat an axis, e.g., x-y-z or z-x-y. But not x-x-y.
  - Let us assume a x-y-z order for now. (α, β, γ)

$$P' = R_x(\alpha).R_y(\beta).R_z(\gamma).P$$

Rotation given by a triad of Euler angles is the same as given by a triad of Fixed angles considered in **opposite** order. So it has the same Gimbal Lock problem.

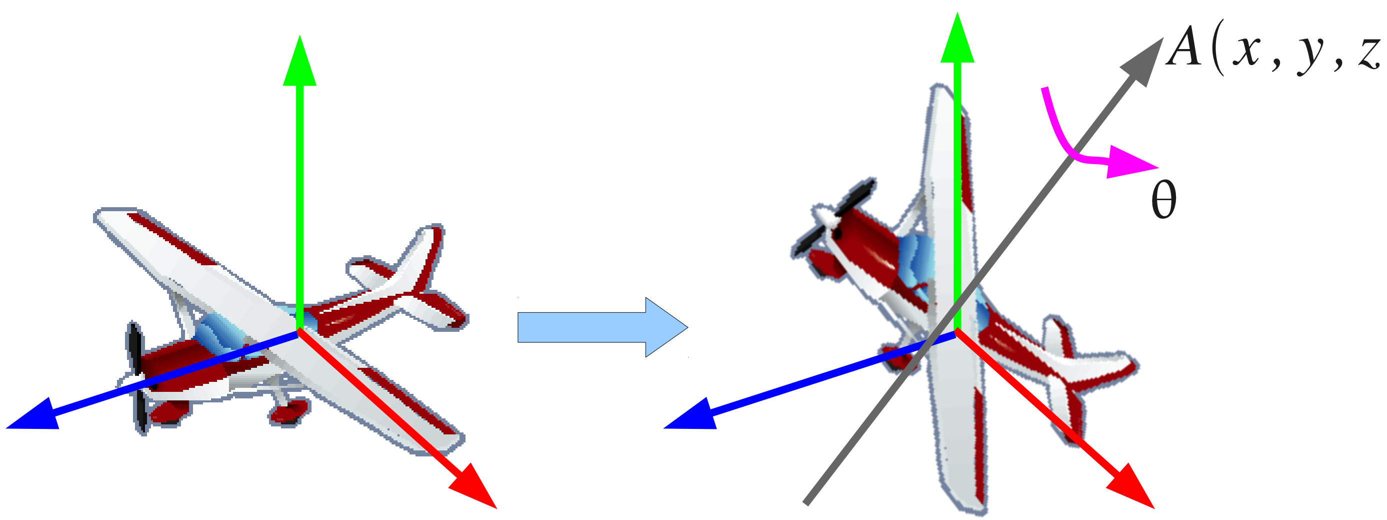


Animation

- Interpolating orientation
  - When to form and apply the matrix if rotation Θ has to be incremented by $\delta\theta$ in each frame?
    - Form a rotation matrix for $\delta\theta$ and apply repeatedly to rotated object in each frame.
    - Update the rotation matrix $R_{axis}(\Theta)$ by multiplying with $R_{axis}(\delta\theta)$ in each frame. Apply updated matrix to the object.
    - Update the rotation angle, Θ by the increment $\delta\theta$ and form the new matrix $R_{axis}(\Theta+\delta\theta)$ in each frame. Apply this matrix to the object.

Animation

- Interpolating orientation
  - **Axis Angle Representation** - Specified as an axis of rotation $A(x, y, z)$ and an angle of rotation, θ around it.
  - Euler's Theorem – Any orientation can be derived from another by a single rotation about and axis.



Animation

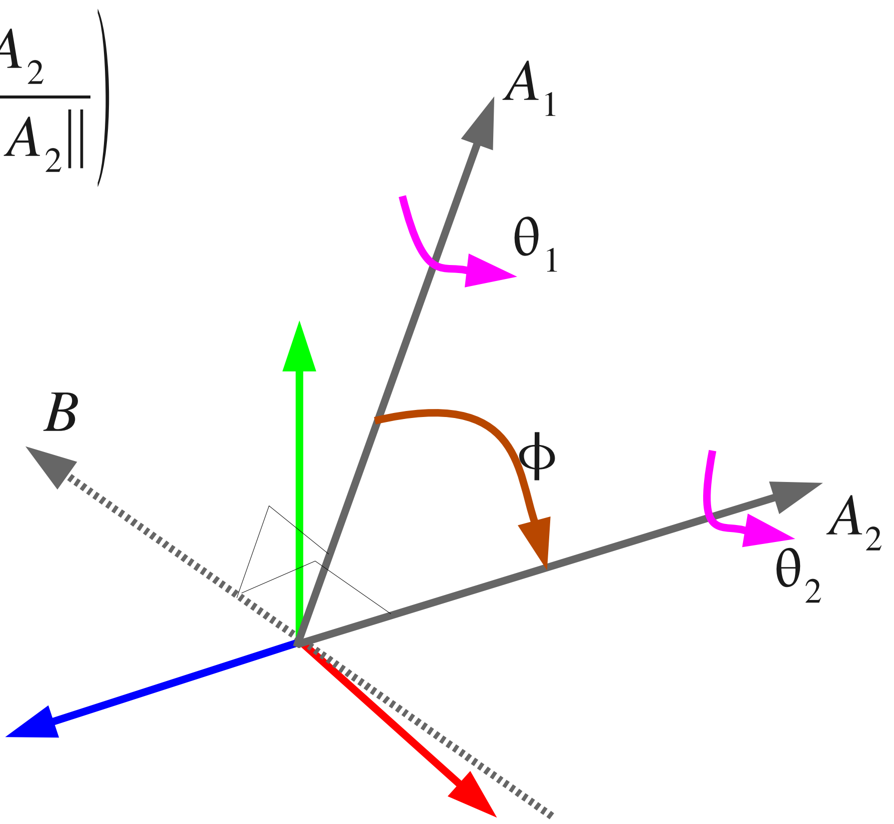
- Interpolating orientation - Axis Angle Representation
  - To interpolate between two orientations $(A_{1,}\theta_1)$ and $(A_{2,}\theta_2)$

$$B = A_1 \times A_2 \qquad \phi = \cos^{-1}\left(\frac{A_1 \cdot A_2}{\|A_1\|\|A_2\|}\right)$$

$$A_k = R_B(k.\phi)A_1$$

$$\theta_k = (1-k)\theta_1 + k\theta_2$$

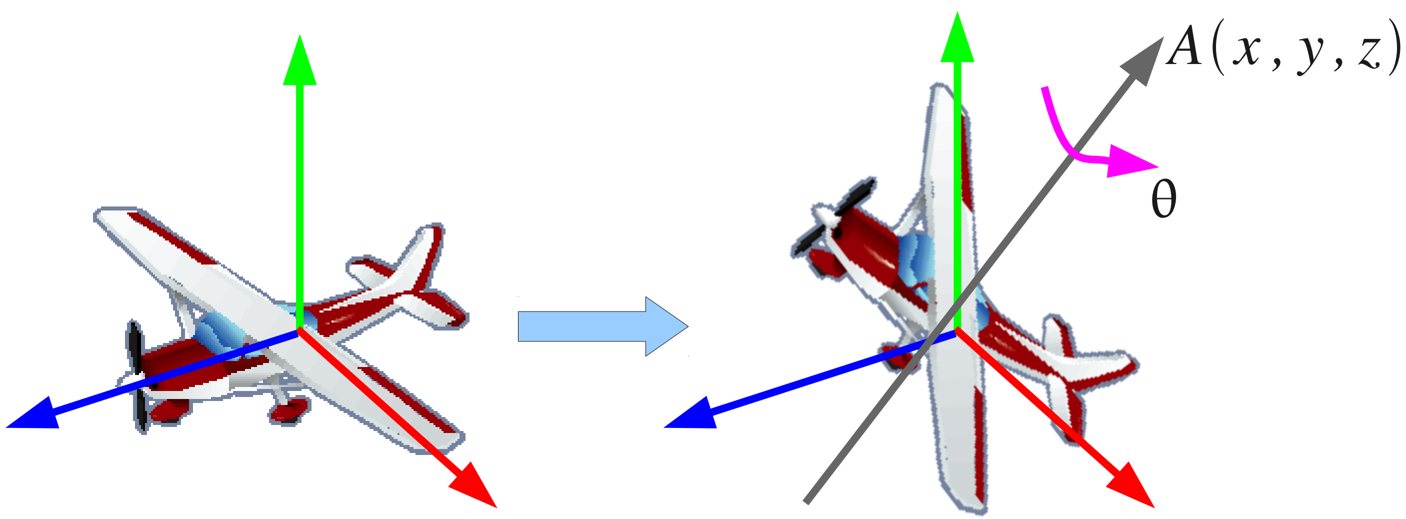
with $0 \le k \le 1$



Animation

- Interpolating orientation
  - **Unit Quaternions**
  - Have the same information as the axis-angle representation but in a more convenient form.

$$q=[\,s, x, y, z\,]=[\,s, v\,]=[\,\cos\theta/2, \sin\theta/2 * \hat{a}\,]\text{, where } \hat{a}=\frac{A}{\|A\|}$$



Animation

- Interpolating orientation – Quaternions
  - A non commutative number system that extends complex numbers
  - Defined as: $q = s + x\hat{i} + y\hat{j} + z\hat{k} = [s, v]$
  - where $1, \hat{i}, \hat{j}, \hat{k}$ are called the Hamilton basis
  - The product of the basis elements is defined as:

$$\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = \hat{i}\hat{j}\hat{k} = -1$$

$$\Rightarrow \hat{i}\hat{j} = \hat{k},\ \hat{j}\hat{k} = \hat{i},\ \hat{k}\hat{i} = \hat{j}$$
$$\Rightarrow \hat{j}\hat{i} = -\hat{k},\ \hat{k}\hat{j} = -\hat{i},\ \hat{i}\hat{k} = -\hat{j}$$

Quaternions form a four dimensional normed division algebra, $\boldsymbol{H}$ over the real numbers.

Animation

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$$\Rightarrow \hat{j}\hat{i} = -\hat{k}, \hat{k}\hat{j} = -\hat{i}, \hat{i}\hat{k} = -\hat{j}$$

Note that the multiplication being defined here is a quaternion multiplication and not the inner or out product of vectors.

It is **not commutative.**

Animation

- Interpolating orientation – Quaternions
  - Quaternion Arithmetic
  - Addition: $q_1+q_2=[s_1+s_{2,}v_1+v_2]$
  - Scalar Multiplication: $kq=[ks,kv]=ks+kx\hat{i}+ky\hat{j}+kz\hat{k}$
  - Quaternion Multiplication:

$$\begin{aligned}
q_1q_2 &= (a_1+b_1\hat{i}+c_1\hat{j}+d_1\hat{k})(a_2+b_2\hat{i}+c_2\hat{j}+d_2\hat{k}) \\
&= a_1(a_2+b_2\hat{i}+c_2\hat{j}+d_2\hat{k})+b_1\hat{i}(a_2+b_2\hat{i}+c_2\hat{j}+d_2\hat{k})+ \\
&\qquad c_1\hat{j}(a_2+b_2\hat{i}+c_2\hat{j}+d_2\hat{k})+d_1\hat{k}(a_2+b_2\hat{i}+c_2\hat{j}+d_2\hat{k}) \\
&= (a_1a_2)-(b_1b_2+c_1c_2+d_1d_2)+(a_1b_2+b_1a_2+c_1d_2-d_1c_2)\hat{i}+ \\
&\qquad (a_1c_2-b_1d_2+c_1a_2+d_1b_2)\hat{j}+(a_1d_2+b_1c_2-c_1b_2+d_1a_2)\hat{k}
\end{aligned}$$

Animation

- Interpolating orientation – Quaternions
  - Conjugate Quaternion: $q^{*} = s - x\hat{i} - y\hat{j} - z\hat{k}$

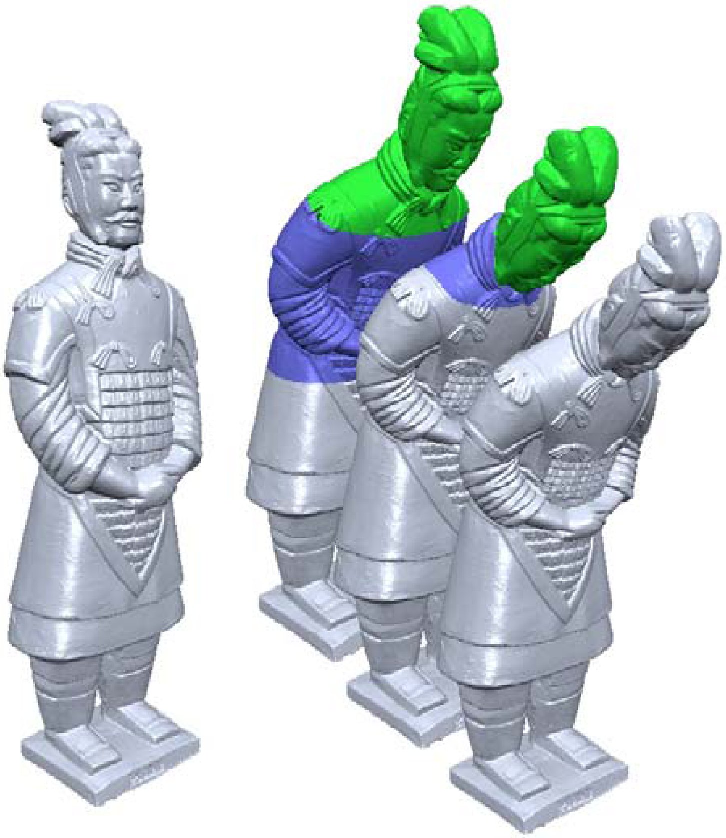
  - Quaternion Norm: $\|q\| = \sqrt{q\,q^{*}} = \sqrt{q^{*}q} = \sqrt{s^2 + x^2 + y^2 + z^2}$
  - If α is real then, $\|\alpha q\| = |\alpha|\,\|q\|$
  - The norm is multiplicative: $\|p\,q\| = \|p\|\,\|q\|$

  - Unit Quaternion: $\hat{q} = \dfrac{q}{\|q\|}$

  - Quaternion Inverse: $q^{-1} = \dfrac{q^{*}}{\|q\|^2}$

Animation

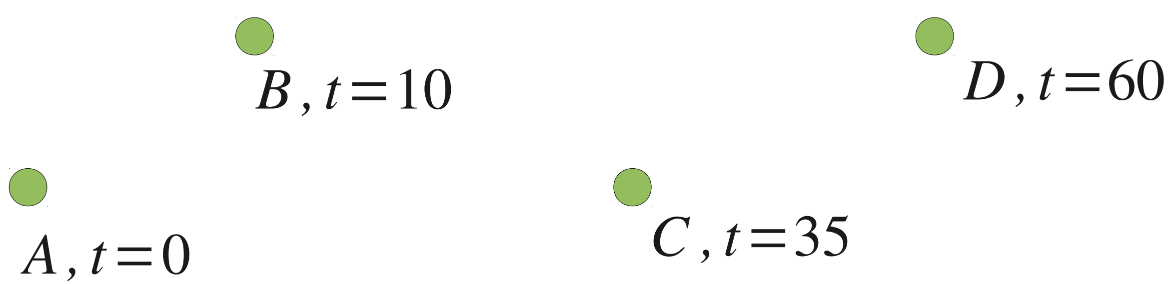
- Keyframing
  - Interpolate Orientation
  - Interpolate Position
  - Interpolate Shape
  - Interpolate Colour
  - Light Intensity
  - Camera Zoom
  - Any other parameter



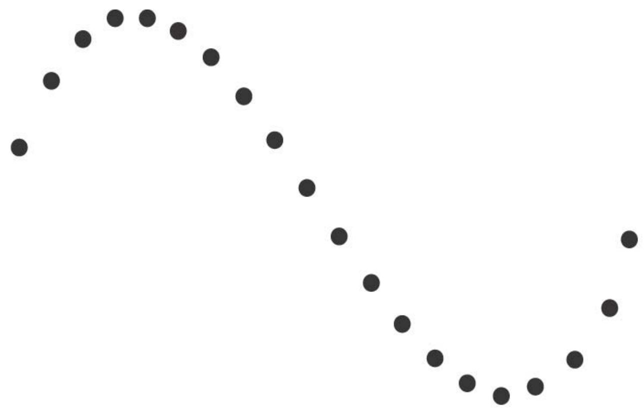
Real-Time Shape Editing using Radial Basis Functions. Mario Botsch, Leif Kobbelt. Computer Graphics Forum 24(3), Proc. Eurographics 2005

Animation

- Keyframing
  - Moving on curves.
  - Specify spatial position to fix the curve
  - In addition, we specify the speed at which we travel along the curve



Animation



- Controlling speed on curves
  - Typically parametrization is not arc length.
  - Arc length is the distance along the curve.
  - Arc length parametrization can be computed using
  - Analytical Computation
  - Table-based
    - Summed linear distances (forward differencing)
    - Gaussian quadrature (numerical integration)

Animation

- Controlling speed on curves
  - Given a parametric curve, $P(u) = (x(u), y(u), z(u))$
  - We may have to solve two versions of the problem:
    - Given parameters u_1 and u_2, find arc length, $\mathrm{LENGTH}(u_1, u_2)$
    - Given an arc length s and parameter u_1, find u_2 so that $\mathrm{LENGTH}(u_1, u_2) = s$

Animation

- Controlling speed on curves
  - Generally, neither of the two forms of the problem admit analytical solutions.

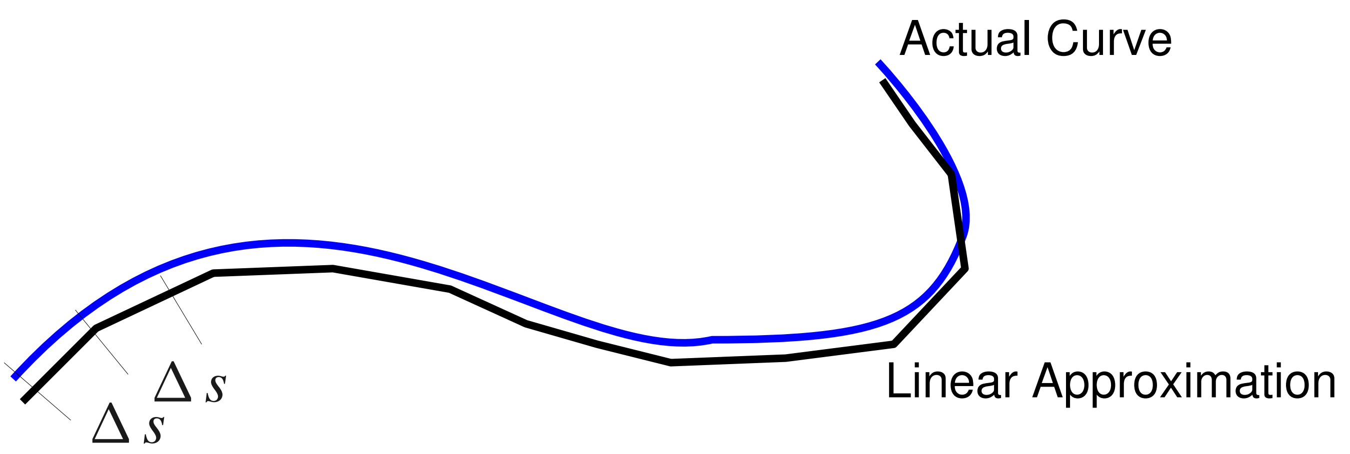
- Arc Length

$$\mathrm{LENGTH}(u_{1,} u_2) = s = \int_{u_1}^{u_2} \left\| \frac{dP(u)}{du} \right\| du = \int_{u_1}^{u_2} \sqrt{\left(\frac{dP(u)}{du} \right)^2} du$$

$$\sqrt{\left(\frac{dP(u)}{du} \right)^2} = \sqrt{\left(\frac{dx(u)}{du} \right)^2 + \left(\frac{dy(u)}{du} \right)^2 + \left(\frac{dz(u)}{du} \right)^2}$$

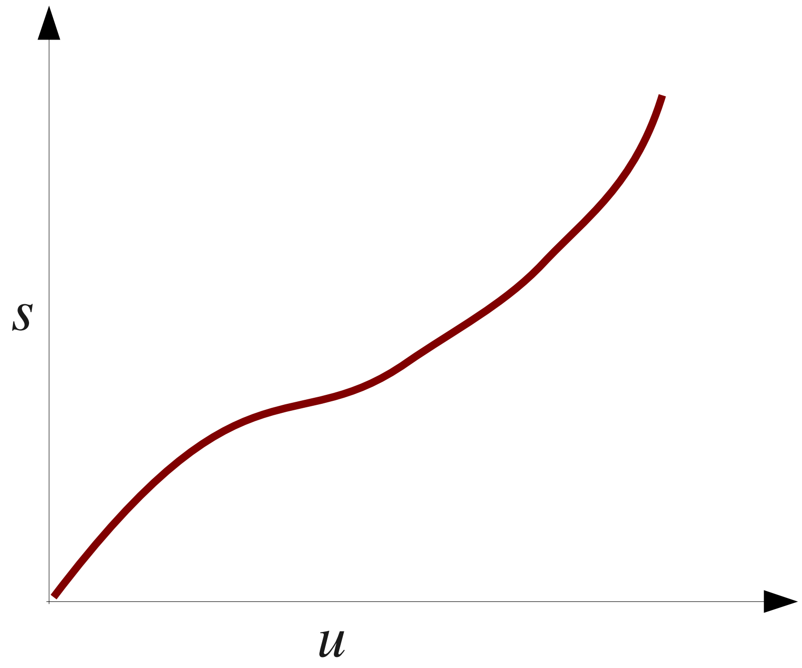
Animation

- Controlling speed on curves
  - The arc length integral can be approximated using a forward differencing method.
  - Create a piece wise linear approximation of the curve from many parameter evaluations and sum these to form the arc length.
  - Store these values into a table.



Animation

- Controlling speed on curves
  - The inversion can then be calculated using bisection



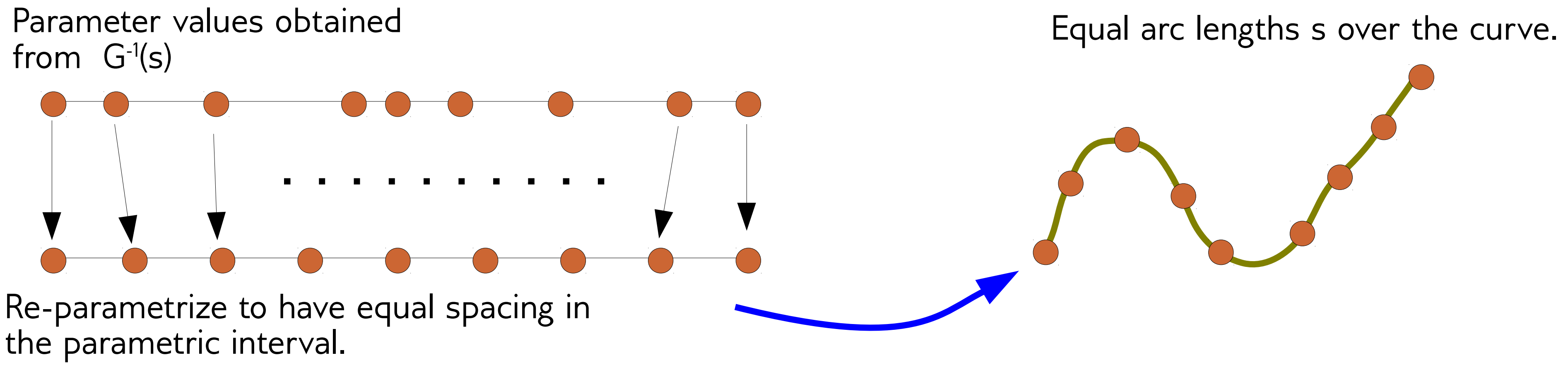
$s = G(u)$

is a monotonically increasing function.
i.e., if $u_1 < u_2$ then $s_1 < s_2$

So we can do a bisection or a binary search for u, given a value of s.

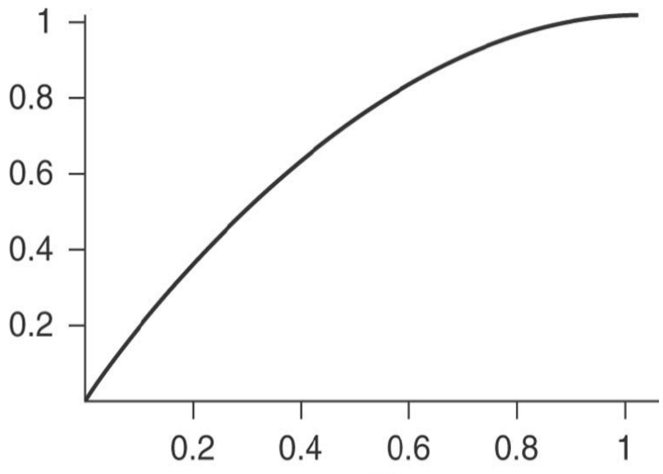
Animation

- Controlling speed on curves
  - Given parameters u_1 and u_2, find arc length, LENGTH(u_1, u_2)
    - Can we compute $s = G(u)$ = distance from start of curve to point at u?
    - With G, arc length parametrization can be obtained by inversion as $P(G^{-1}(s))$, where $G^{-1}(s)$ gives the parameter u up to which distance travelled on the curve is s.



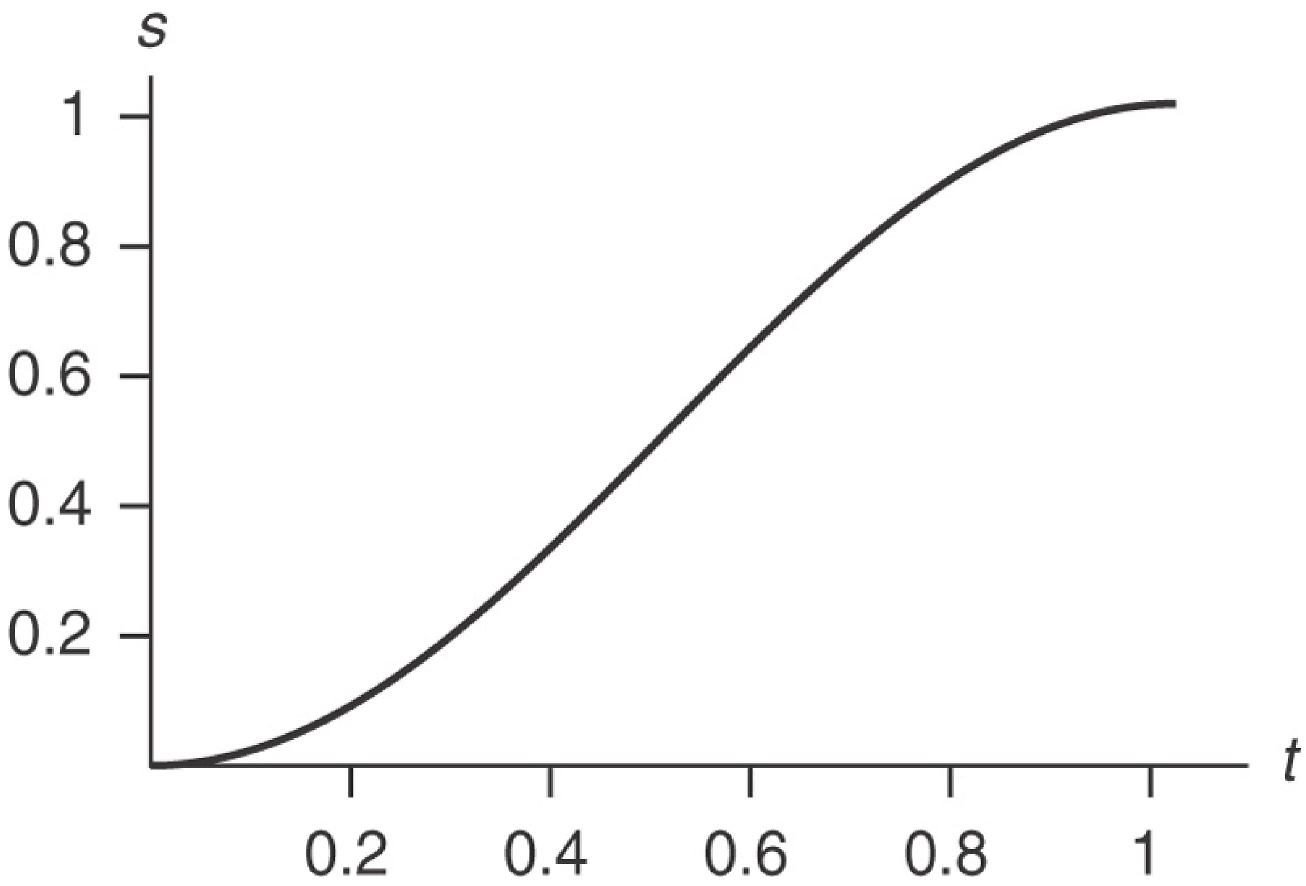
Animation

- Controlling speed on curves
  - Space curves we have seen till now give the path.
  - What if we want to control the speeds
    - Accelerates from stop position
    - Reaches maximum speed
    - Decelerates to a stop.
  - Given the speed as a plot of s vs. t.
    - v = ds/dt



Animation

- Controlling speed on curves
  - A slow in – slow out curve may look like:



Animation

- Controlling speed on curves
  - Given next time instant t
  - Distance-time curve gives total distance s travelled up to time t.
  - $P(G^{-1}(s))$ gives the position on the path space curve.
- Or solve a space-time optimization for the whole path.

Animation

- Interpolating orientation – Quaternions
  - Quaternion and the Geometry of R^3

    $\hat{i}, \hat{j}, \hat{k}$ denote both the basis vector of $\boldsymbol{H}$ and a basis for $\boldsymbol{R}^3$
  - Vectors in $\boldsymbol{R}^3$ can be written as pure imaginary quaternions

    $$v = 0 + x\hat{i} + y\hat{j} + z\hat{k} = [0, u]$$
  - Inner product of vectors in $\boldsymbol{R}^3$

    $$v_1 \cdot v_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 = \frac{1}{2}(v_1^* v_2 + v_2^* v_1) = \frac{1}{2}(v_1 v_2^* + v_2 v_1^*)$$
  - Cross product of vectors in

    $$v_1 \times v_2 = \frac{1}{2}(v_1 v_2 - v_2^* v_1^*)$$
  - Quaternion multiplication can be written as:

    $$q_1 q_2 = [\ s_1 s_2 - v_1 \cdot v_{2,} \quad s_2 v_1 + s_1 v_2 + v_1 \times v_2]$$

Animation

- Interpolating orientation – **Unit** Quaternions
  - A unit quaternion denotes a rotation by an angle θ about an axis A
  - $q=[s,x,y,z]=[s,v]=[\cos\theta/2, \sin\theta/2*\hat{a}]$, where $\hat{a}=\frac{A}{||A||}$
  - Multiplication with a unit quaternion q can be used to rotate a vector v $$R_q(v)=q\,v\,q^{-1}=q[0,v]q^{-1}$$
  - Composition of rotations is equivalent to quaternion multiplication $$R_{q_1}(R_{q_2}(\vec{v}))=R_{q_1}(q_2\vec{v}\,q_2^{-1})=q_1q_2[\ 0,v]\ q_2^{-1}q_1^{-1}$$ $$=(q_1q_2)[\ 0,v]\ (q_1q_2)^{-1}=R_{q_1q_2}(\vec{v})$$
  - Rotating by a scalar multiple of a unit quaternion is the same as rotating by the unit quaternion $R_q(\vec{v})=R_{kq}(\vec{v})$

  Only Unit Quaternions represent rotations!

Animation

- Interpolating orientation – **Unit** Quaternions
  - Antipodal Unit Quaternions
  - $q = [\cos(\theta/2), \hat{a}\sin(\theta/2)]$
  - If we rotate by θ-2π instead of θ

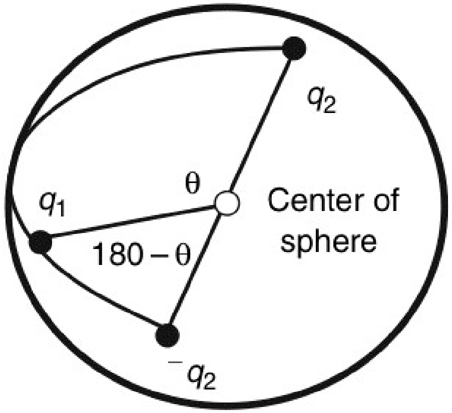
    $$[\cos((\theta-2\pi)/2), \hat{a}\sin((\theta-2\pi)/2)]$$
    $$= [\cos(\theta/2-\pi), \hat{a}\sin(\theta/2-\pi)] = [-\cos(\theta/2), -\hat{a}\sin(\theta/2)] = -q$$
  - So both q and $-q$ represent the same rotation and are called antipodal points. $R_q(v) = R_{-q}(v)$
  - If 0 < θ < π then the positive rotation is the shorter one else the negative rotation is the shorter one, i.e., the quaternion with the positive value of the s coordinate will give the shorter path

I am abusing notation here for convenience.
Please remember we are talking about unit quaternions.

Animation

- Interpolating orientation – Unit Quaternions
  - Linear Interpolation $q = (1-k)q_1 + kq_2$
  - How to take equi-distant steps along orientation path?
  - How to pass through orientations smoothly?
  - With dual unit quaternion representations, which one to use?

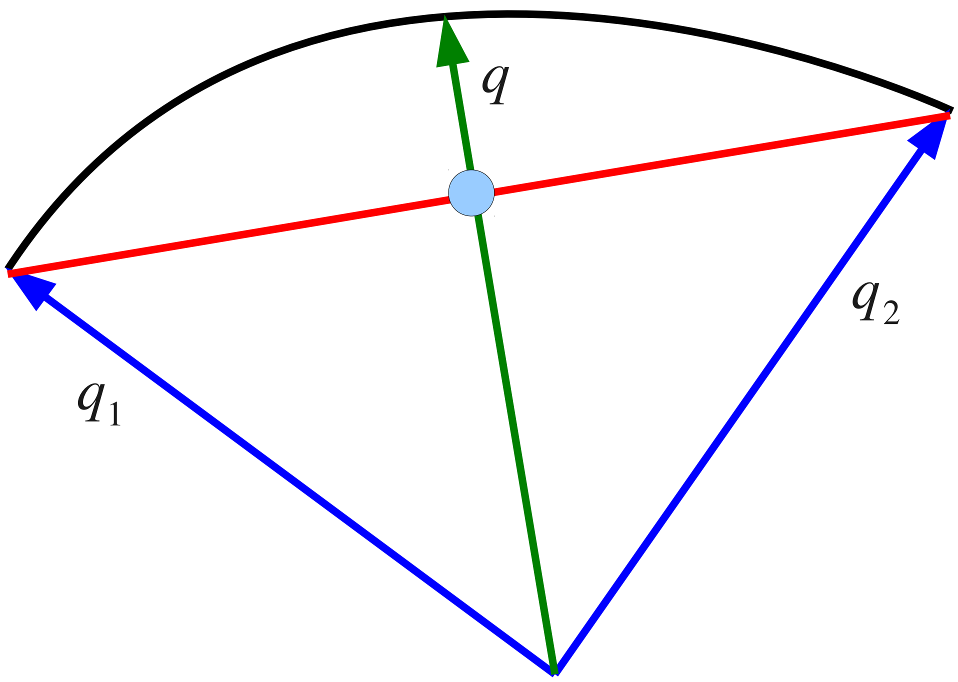
Dual representation: For Interpolation between q_1 and q_2, compute cosine between q_1 and q_2 and between q_1 and $-q_2$; choose smallest angle.

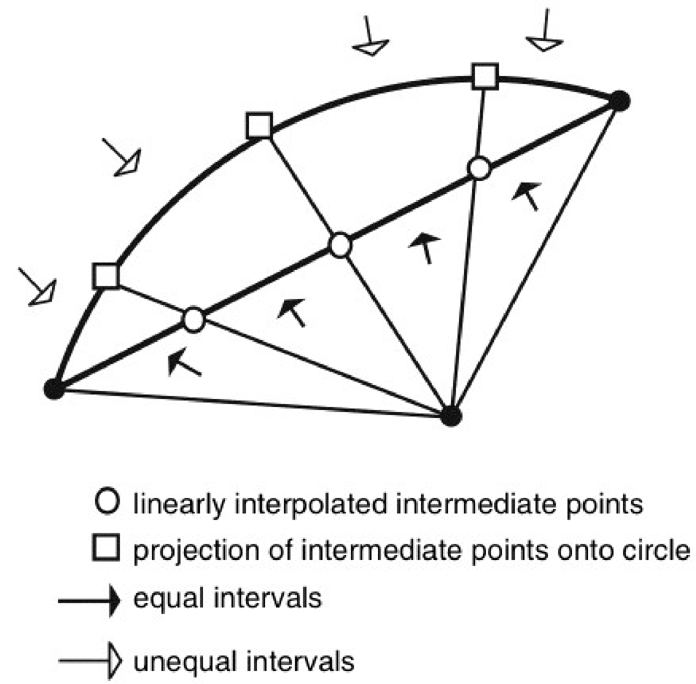


Computer Animation: Algorithms and Techniques, Rick Parent, Morgan Kaufmann, 2001

Animation

- Interpolating orientation – Unit Quaternions
  - Linear Interpolation $q = (1-k)q_1 + k q_2$
  - This is not equally spaced.





Computer Animation: Algorithms and Techniques, Rick Parent, Morgan Kaufmann, 2001

Animation

- Interpolating orientation – Unit Quaternions
  - Spherical Linear Interpolation or SLERP
  - We write, $q^{\alpha} = [\cos(\alpha\theta/2), \hat{a}\sin(\alpha\theta/2)]$
  - We want to interpolate between two rotations q_1 and q_2
  - Rotation that takes us from 1 to 2 is given by $q_2 q_1^{-1}$
  - Now we start at 1, and go to 2 in α steps as $(q_2 q_1^{-1})^{\alpha} q_1$

$$Slerp(q_{1,} q_{2,} \alpha) = (q_2 q_1^{-1})^{\alpha} q_1$$

$$Slerp(q_{1,} q_{2,} \alpha) = \frac{\sin(1-\alpha)\theta}{\sin\theta} q_1 + \frac{\sin(\alpha)\theta}{\sin\theta} q_2$$

$$\cos\theta = q_1 \cdot q_2 = s_1 s_2 + x_1 x_2 + y_1 y_2 + z_1 z_2$$