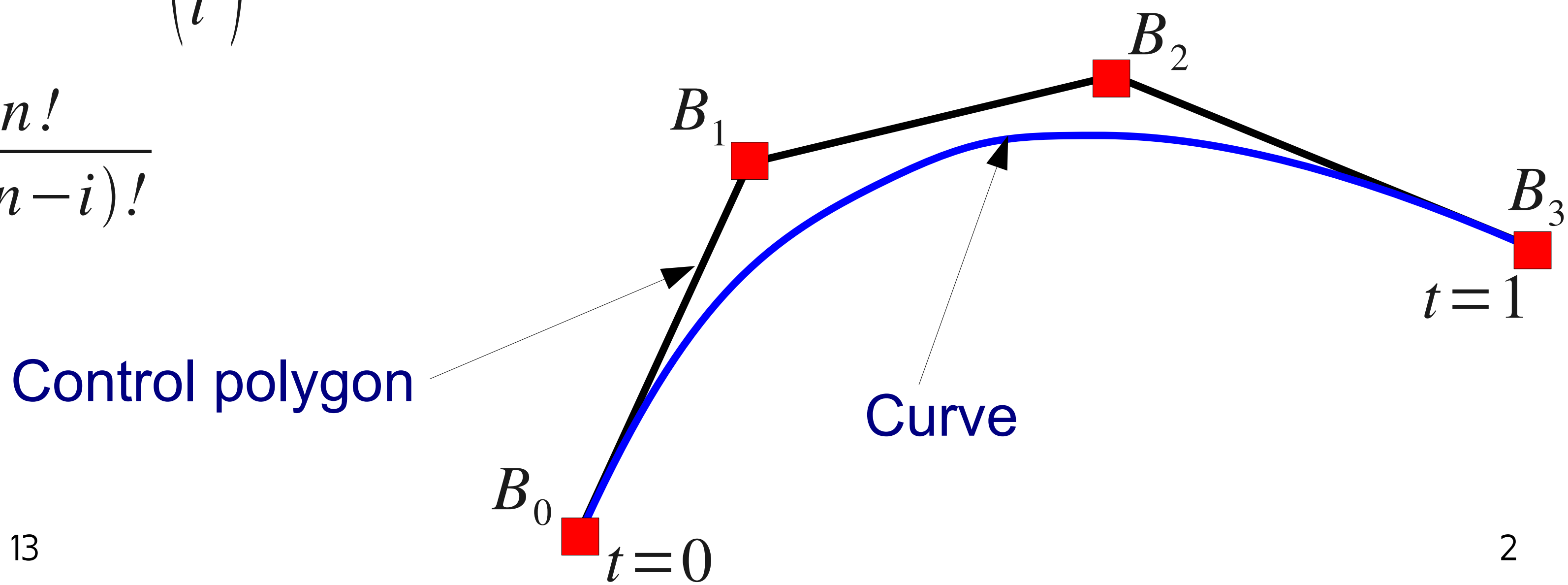
CS475/CS675
Computer Graphics

Modeling Curves: Bézier Splines

Bézier Splines

- Bézier Curves were discovered by Pierre Bézier.
- Approximating the shape of a control polygon.
- Mathematically: $P(t)=\sum_{i=0}^{n} B_i J_{n,i}(t) \quad \text{with } 0 \le t \le 1$
- Where $J_{n,i}(t)=\binom{n}{i} t^i (1-t)^{n-i}$ and is called the Bernstein basis.
- $\binom{n}{i}=\frac{n!}{i!(n-i)!}$



Bézier Splines

- $P(t)=\sum_{i=0}^{n} B_i J_{n,i}(t) \quad \text{with } 0 \le t \le 1$
- $J_{n,i}(t)=\binom{n}{i} t^i (1-t)^{n-i}$

$$\binom{n}{i}=\frac{n!}{i!(n-i)!}$$

$$\binom{n}{0}=\binom{n}{n}=1$$

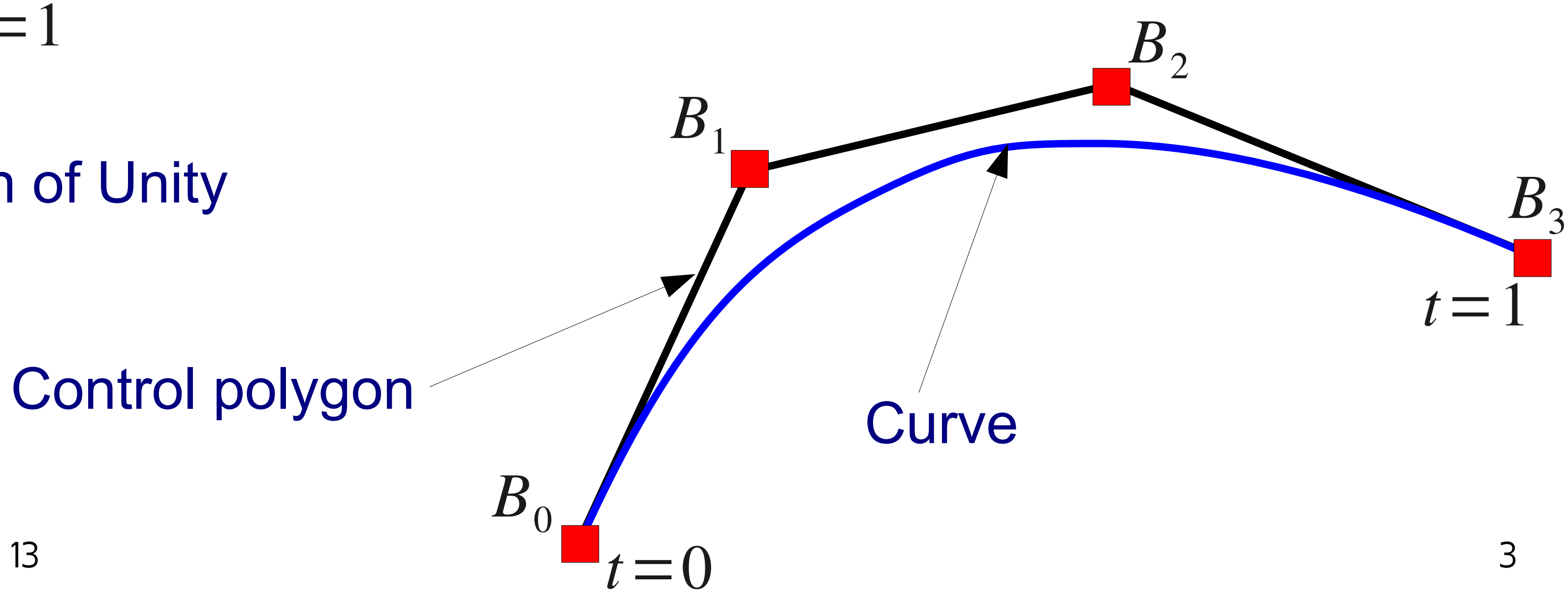
$$\binom{n}{i}=0 \text{ for } i \notin [0,n]$$

$$\binom{n}{i}>0 \text{ for } i \in [0,n]$$

Positivity

$$\sum_{i=0}^{n} J_{n,i}(t)=1$$

Partition of Unity



Bézier Splines

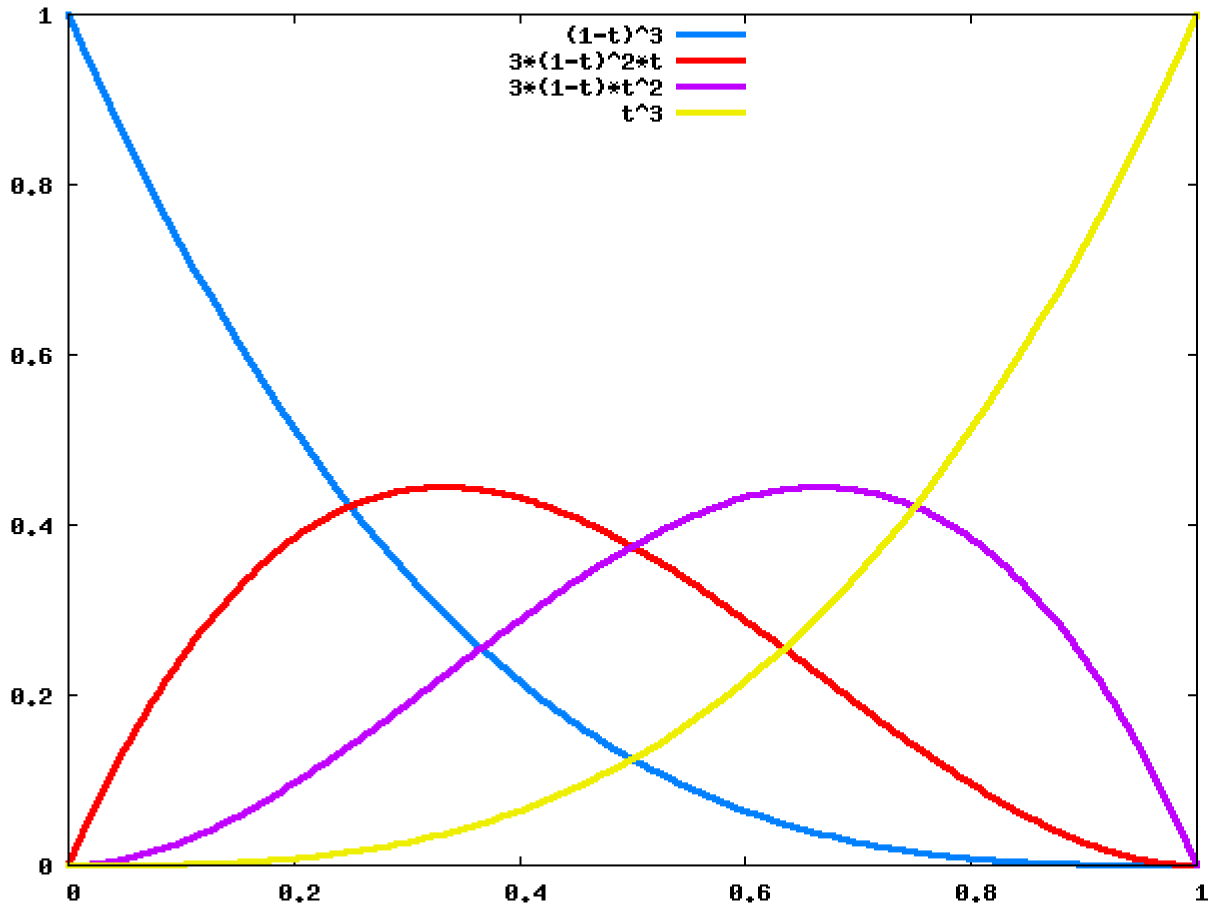
- Cubic Bézier Splines $P(t) = \sum_{i=0}^{3} B_i J_{3,i}(t) \quad \text{with } 0 \le t \le 1$

$$J_{3,0}(t) = (1-t)^3$$

$$J_{3,1}(t) = 3\mathrm{t}(1-t)^2$$

$$J_{3,2}(t) = 3\mathrm{t}^2(1-t)$$

$$J_{3,3}(t) = t^3$$



Bézier Splines

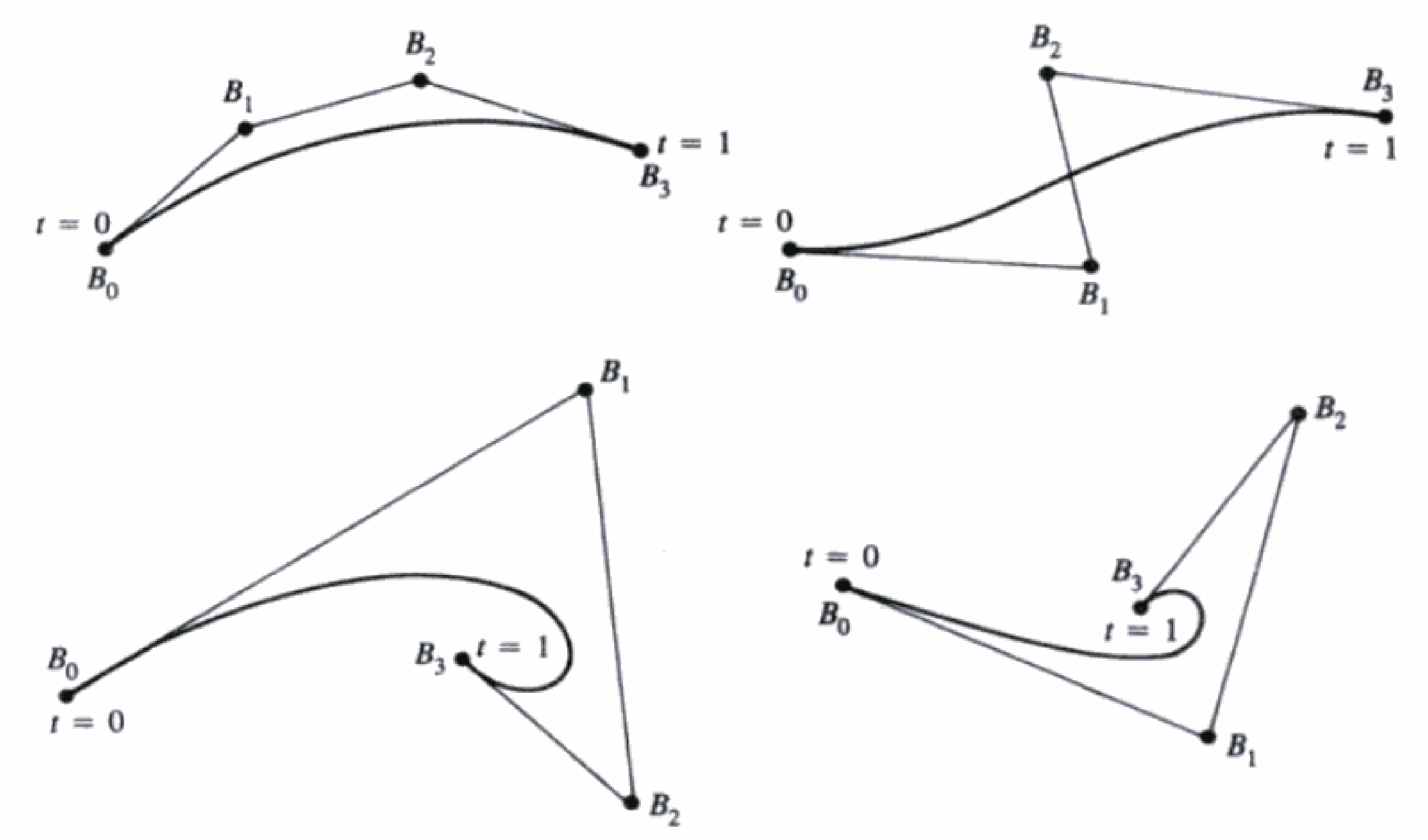
- Cubic Bézier Splines

$$P(t) = \begin{bmatrix} (1-t)^3 & 3\mathrm{t}(1-t)^2 & 3\mathrm{t}^2(1-t) & t^3 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

Bézier Splines

- Cubic Bézier Splines



Mathematical Elements for Computer Graphics, 4ed.,
D. F. Rogers and J. A. Adams

Bézier Splines

- The basis functions are real.
- Degree of the polynomial defining the Bézier curve is one less than the number of defining control polygon points.
- The curve generally follows the shape of the control polygon.
- Enpoint Interpolation: The first and last point of the curve are concident with the first and last point of the control polygon.

at $t=0$

$$i=0, J_{n,0}=\frac{n!}{n!\,0!}(0)^0(1-0)^{n-0}=1$$

$$i\neq 0, J_{n,i}=\frac{n!}{(n-i)!\,i!}(0)^i(1-0)^{n-i}=0$$

$$\Rightarrow P(0)=\sum_{i=0}^{n} B_0 J_{n,i}=B_0$$

at $t=1$

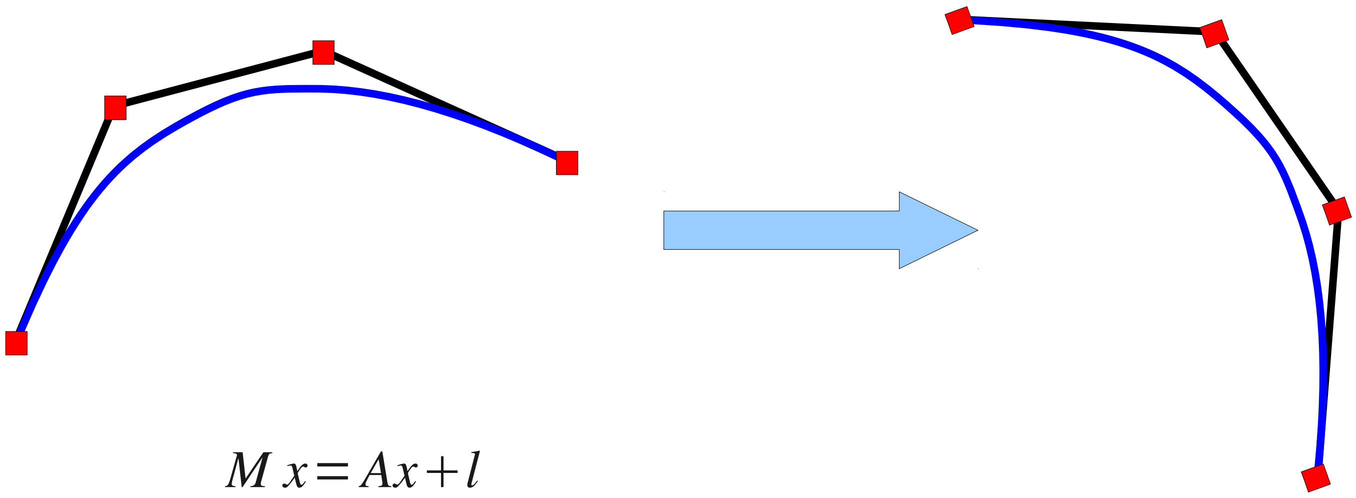
$$i=n, J_{n,n}=\frac{n!}{0!\,n!}(1)^n(1-1)^{n-n}=1$$

$$i\neq n, J_{n,i}=\frac{n!}{(n-i)!\,i!}(1)^i(1-1)^{n-i}=0$$

$$\Rightarrow P(1)=\sum_{i=0}^{n} B_n J_{n,i}=B_n$$

Bézier Splines

- Affine Invariance: Applying an affine transform to the curve is equivalent to applying the transformation to the control points.



$$
\begin{aligned}
& M\,x = Ax + l \\
& \Rightarrow M\,P(t) = A\sum_{i=0}^{n} B_i J_{n,i}(t) + l \\
& \qquad = A\sum_{i=0}^{n} B_i J_{n,i}(t) + l\sum_{i=0}^{n} J_{n,i}(t) \\
& \qquad = \sum_{i=0}^{n} (A\,B_i + l) J_{n,i}(t) = \sum_{i=0}^{n} (M\,B_i) J_{n,i}(t)
\end{aligned}
$$

Bézier Splines

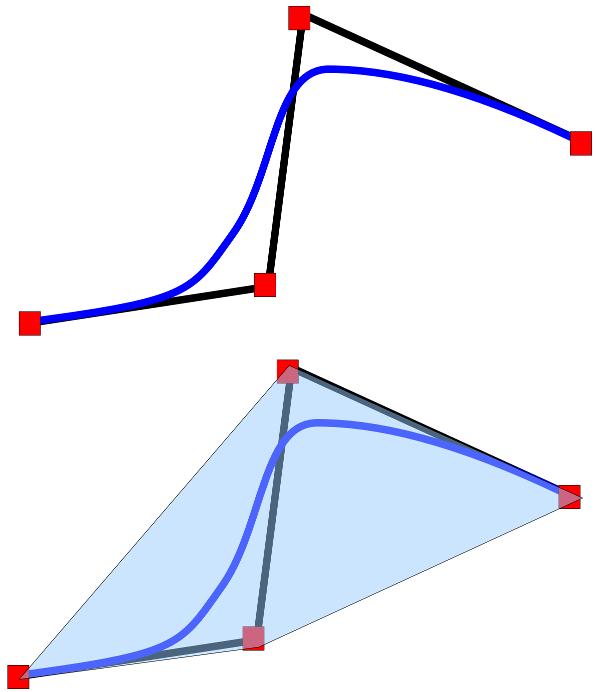
- Convex Hull: The curve lies inside the convex hull of the control points.
- Given a set of points $X=\{x_{0,} x_{1,} .... x_n\}$ the convex hull of X is given by the set of points.

$$CH(X)=\{a_0 x_0+\ldots+a_n x_n \mid \sum_{i=0}^{n} a_i=1, a_i \geq 0, a_i \in \mathbb{R}, x \in X\}$$

- Every point on the curve is of the form:

$$P(t)=\sum_{i=0}^{n} B_i J_{n,i}(t) \quad \text{with}\, 0 \leq t \leq 1$$

- and $\sum_{i=0}^{n} J_{n,i}(t)=1, J_{n,i}(t) \geq 0, J_{n,i}(t) \in \mathbb{R}$
- So every point lies in the convex hull of B_i



Bézier Splines

- Symmetry: $J_{n,i}(t) = J_{n,n-i}(1-t)$
- The curve P(t) formed by the control points $B_0,\ldots B_n$ is the same as the curve P(1-t) formed by the control points $B_n,\ldots,B_0$.

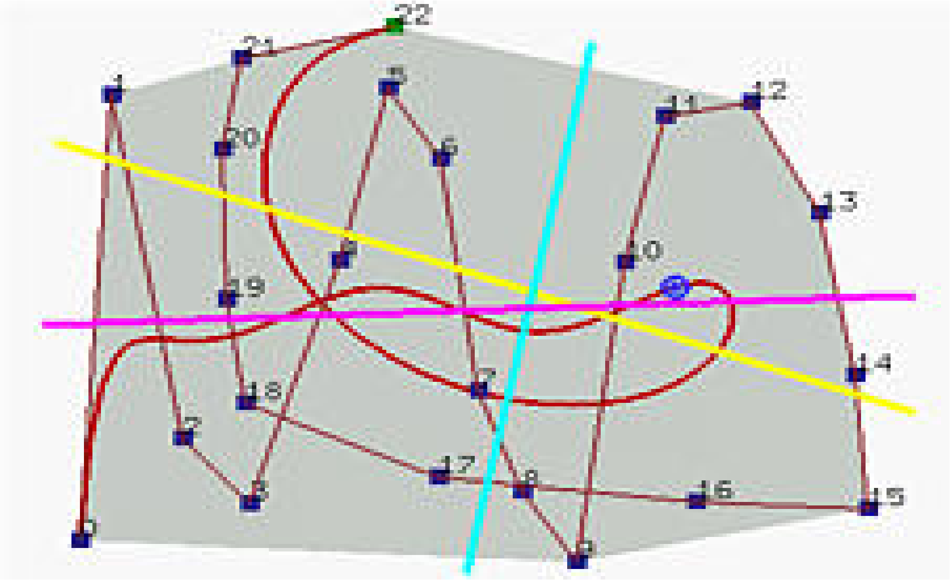
- Parameter Domain Transformation:

$$t \in [0,1], u \in [a,b]$$

$$t = \frac{u-a}{b-a} \qquad \Rightarrow P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t) = \sum_{i=0}^{n} B_i J_{n,i}\left(\frac{u-a}{b-a}\right)$$

Bézier Splines

- Variation Diminishing Property: The number of intersections of a given straight line with a planar Bézier curve is less than or equal to the number of intersections of that line with the control polygon.



http://www.cs.mtu.edu/~shene/COURSES/cs3621/NOTES/spline/Bezier/bezier-construct.html

Bézier Splines

- Tangent Vectors:

$$\frac{dP(t)}{dt}=\sum_{i=0}^{n}\frac{d}{dt}B_iJ_{n,i}(t)$$

$$\frac{dJ_{n,i}(t)}{dt}=\frac{d}{dt}\binom{n}{i}t^i(1-t)^{n-i}=\binom{n}{i}it^{i-1}(1-t)^{n-i}-\binom{n}{i}t^i(n-i)(1-t)^{n-i-1}$$

$$=n\binom{n-1}{i-1}t^{i-1}(1-t)^{n-i}-n\binom{n-1}{i}t^i(1-t)^{n-i-1}$$

$$=n(J_{n-1,i-1}(t)-J_{n-1,i}(t))$$

$$\Rightarrow\frac{dP(t)}{dt}=n\sum_{i=0}^{n}B_i(J_{n-1,i-1}(t)-J_{n-1,i}(t))=n\sum_{i=1}^{n}B_iJ_{n-1,i-1}(t)-n\sum_{i=0}^{n-1}B_iJ_{n-1,i}(t)$$

$$=n\sum_{i=0}^{n-1}B_{i+1}J_{n-1,i}(t)-n\sum_{i=0}^{n-1}B_iJ_{n-1,i}(t)=n\sum_{i=0}^{n-1}(B_{i+1}-B_i)J_{n-1,i}(t)$$

Bézier Splines

- Tangent Vectors:

$$P'(0) = n(B_1 - B_0) J_{n-1,0} = n(B_1 - B_0)$$
$$P'(1) = n(B_n - B_{n-1}) J_{n-1,n-1} = n(B_n - B_{n-1})$$

- i.e., tangent vectors at the ends of curve have the same direction as the first and last spans of the control polygon.
- Continuity: P(t) of degree n, defined by control vertices B_i

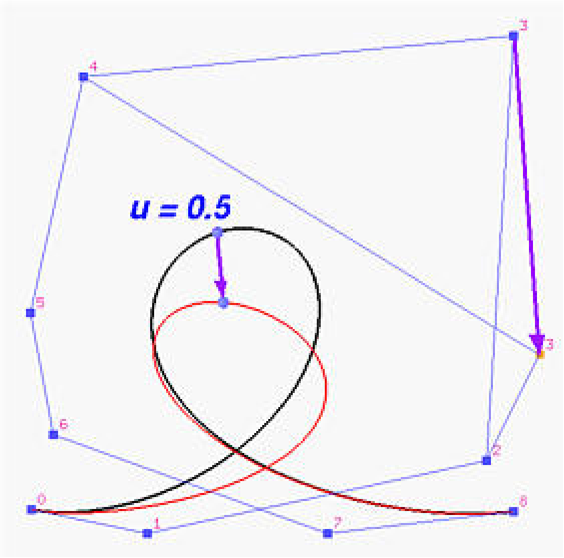
  Q(s) of degree m, defined by control vertices C_i

$$\text{for } C^1 \text{ continuity: } P'(1) = Q'(0)$$
$$\Rightarrow C_1 - C_0 = \frac{n}{m}(B_n - B_{n-1})$$
$$\Rightarrow C_1 = \frac{n}{m}(B_n - B_{n-1}) + B_n \text{ because } C_0 = B_n$$

i.e., $B_{n-1}, B_n = C_{0,} C_1$ have to be collinear for the tangents to have the same direction.

Bézier Splines

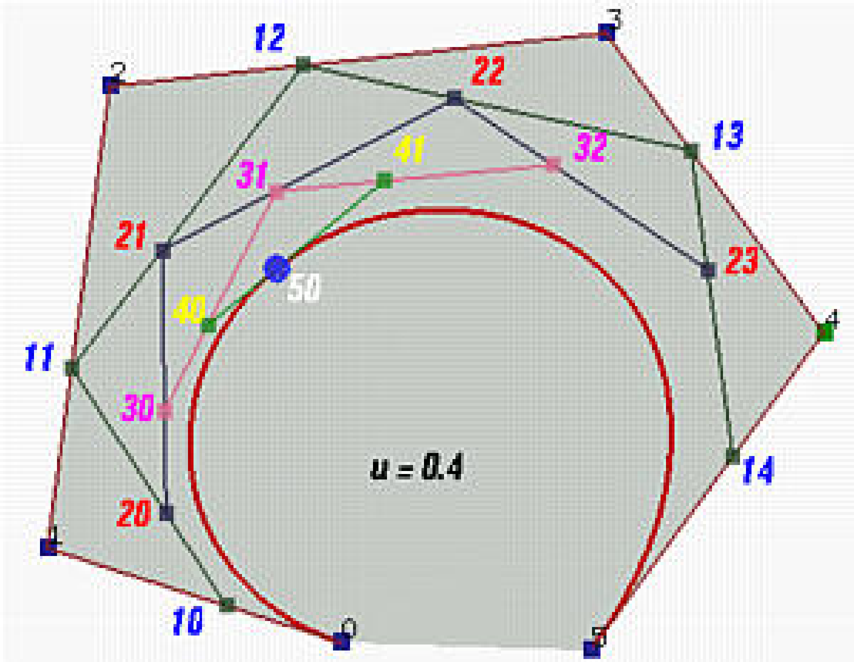
- Control:
  - How does the shape of the curve change if we move one control point?
  - Each control point is associated to a basis function.
  - The basis functions effect the shape of the curve over a range of parameter values where the basis fucntion is non-zero. In case of the Bernstien basis, this is the entire parameter range [0,1].

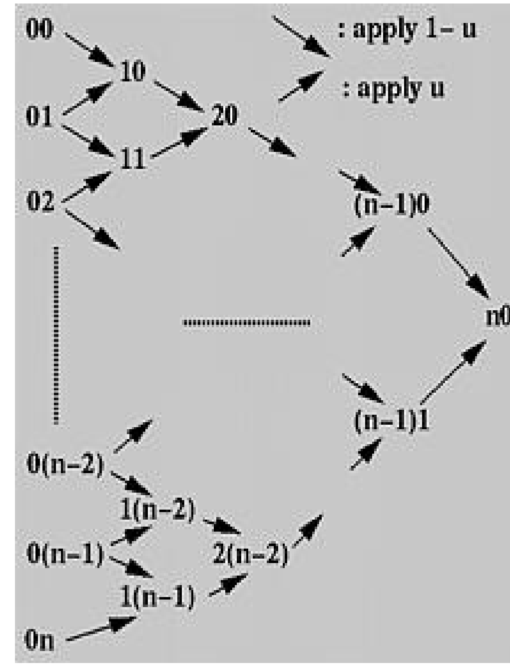


http://www.cs.mtu.edu/~shene/COURSES/cs3621/NOTES/spline/Bezier/bezier-move-ct-pt.html

Bézier Splines

- Points on the curve: De Casteljau's Algorithm





http://www.cs.mtu.edu/~shene/COURSES/cs3621/NOTES/spline/Bezier/de-casteljau.html

- The value for the entry j of column i is computed as:

$$B_{i,j} = (1-u) B_{i-1,j} + u B_{i-1,j+1} \text{ where } 1 \le i \le n,\ 0 \le j \le n-i$$

Bézier Splines

- Points on the curve: De Casteljau's Algorithm

```
deCasteljau(int i, int j)
{
  if (i == 0) then return B_{0,j}
  else
     return (1-u)* deCasteljau(i-1,j) + u* deCasteljau(i-1,j+1)
}
```

- Is this a good way to implement the algorithm?
- Why bother with the algorithm at all?
- Subdivision and Degree Elevation.