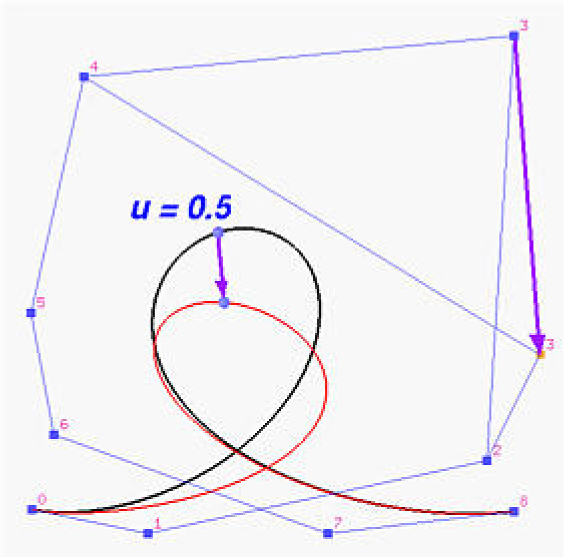
CS475/CS675
Computer Graphics

Modeling Curves: B-Splines

Bézier Splines

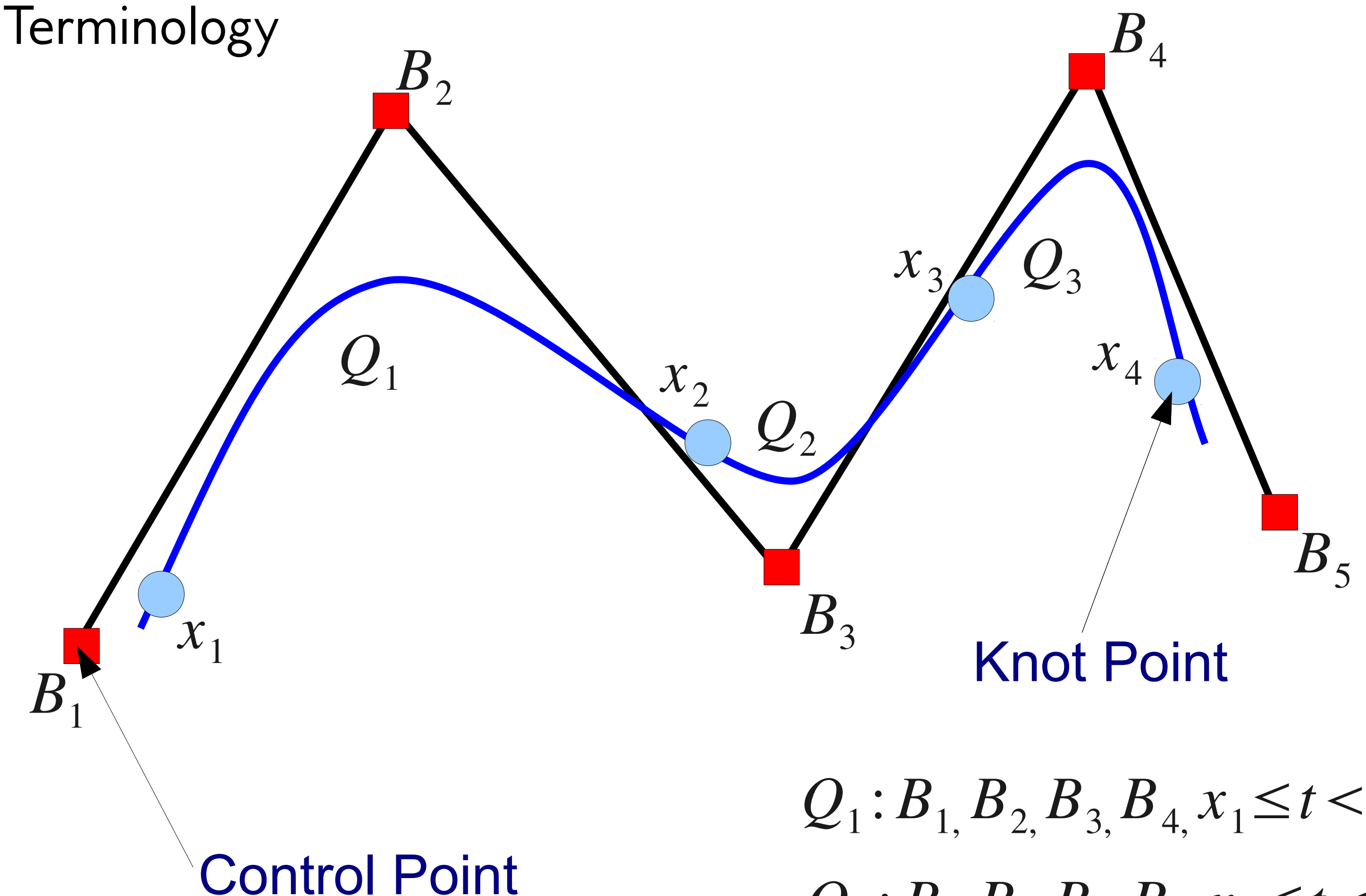
- $P(t)=\sum_{i=0}^{n} B_i J_{n,i}(t) \quad \text{with } 0 \le t \le 1$ $\qquad J_{n,i}(t)=\binom{n}{i} t^i (1-t)^{n-i}$
- No local control.
- Degree restricted by the control polygon.



http://www.cs.mtu.edu/~shene/COURSES/cs3621/NOTES/spline/Bezier/bezier-move-ct-pt.html

B-Splines

- Terminology



$$Q_1 : B_1, B_2, B_3, B_4, x_1 \leq t < x_2$$

$$Q_2 : B_2, B_3, B_4, B_5, x_2 \leq t < x_3$$

$[x_1, x_2, x_3, x_4]$ Knot Vector

B-Splines

- The B-Spline Curve of order k (degree, k-1)is given by:

$$P(t)=\sum_{i=1}^{n+1} B_i N_{i,k}(t) \quad \text{with}\; t_{min}\le t< t_{max}\; \text{and}\; 2\le k\le n+1$$

B_i - position vectors of the n+1 vertices of the control polygon

$N_{i,k}$ - normalized B-spline basis functions

$$N_{i,1}=\begin{cases} 1, \text{if } x_i \le t < x_{i+1} \\ 0, \text{otherwise} \end{cases}$$

$$N_{i,k}(t)=\frac{(t-x_i)N_{i,k-1}(t)}{x_{i+k-1}-x_i}+\frac{(x_{i+k}-t)N_{i+1,k-1}(t)}{x_{i+k}-x_{i+1}}$$

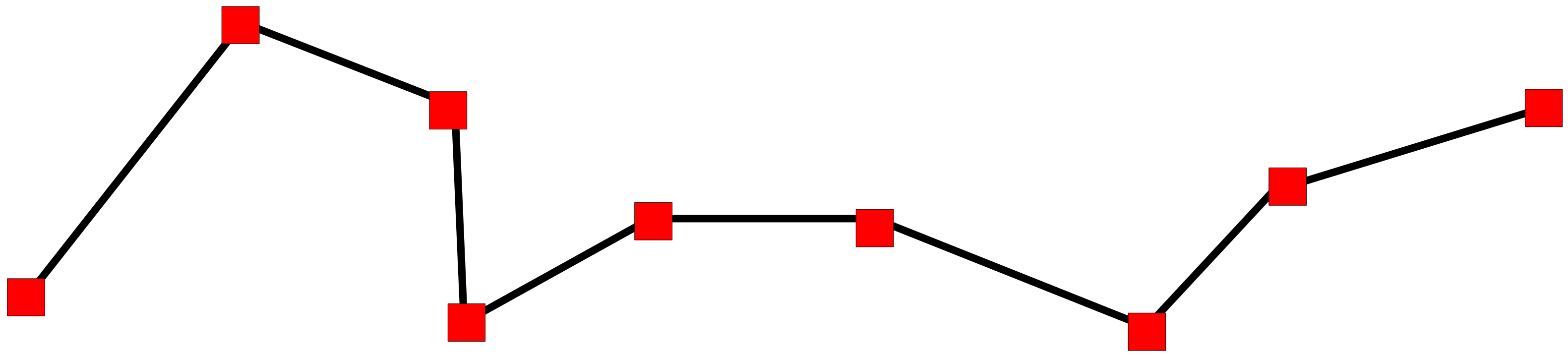
Cox-deBoor recursive formula

B-Splines

- Properties:
  - The function P(t) is a polynomial of degree k-1 on each interval $x_i \leq t < x_{i+1}$
  - i.e., A 4th-order B-spline curve is a piecewise cubic curve.
  - P(t) and its derivatives of order 1 to k-2 are all continuous over the entire curve.
  - Positivity $N_{i,k}(t) \geq 0 \; \forall \, t$
  - Partition of Unity $\sum_{i=1}^{n+1} N_{i,k}(t) = 1$ for any t and k

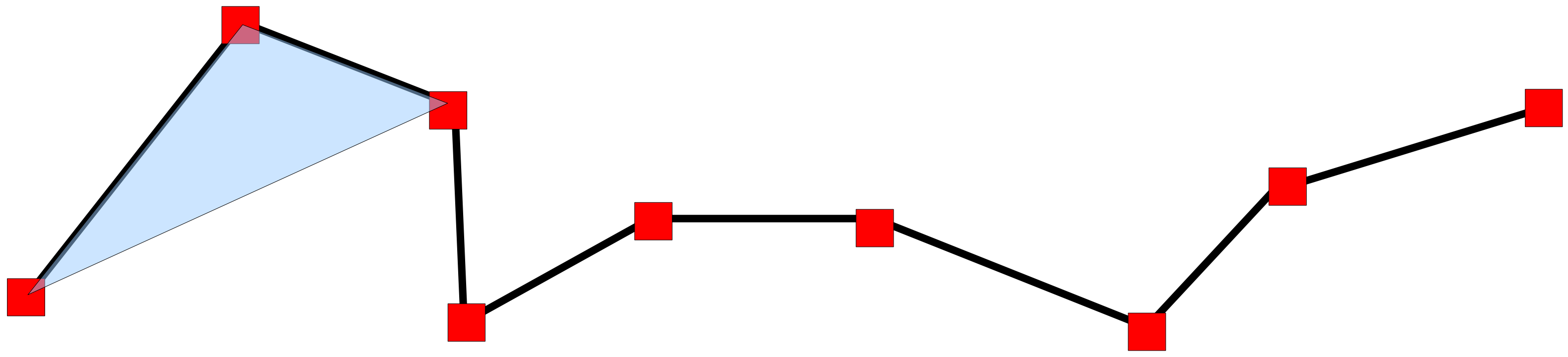
B-Splines

- Strong Convex Hull Property:
  - For a B-spline curve of order k (degree k-1), a point on the curve lies within the convex hull of k neighbouring control points.
  - For k=2



B-Splines

- Strong Convex Hull Property:
  - For a B-spline curve of order k (degree k-1), a point on the curve lies within the convex hull of k neighbouring control points.
  - For k=3



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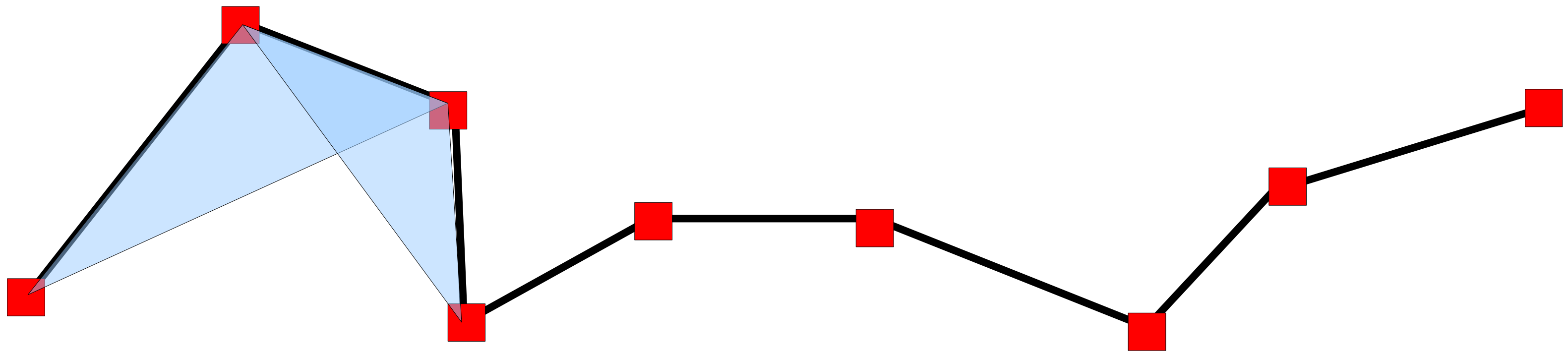

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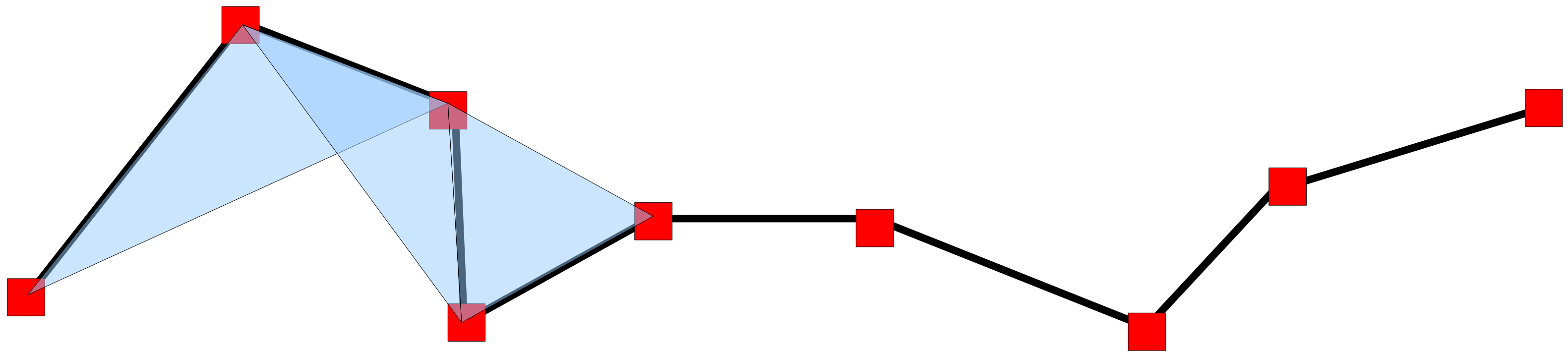

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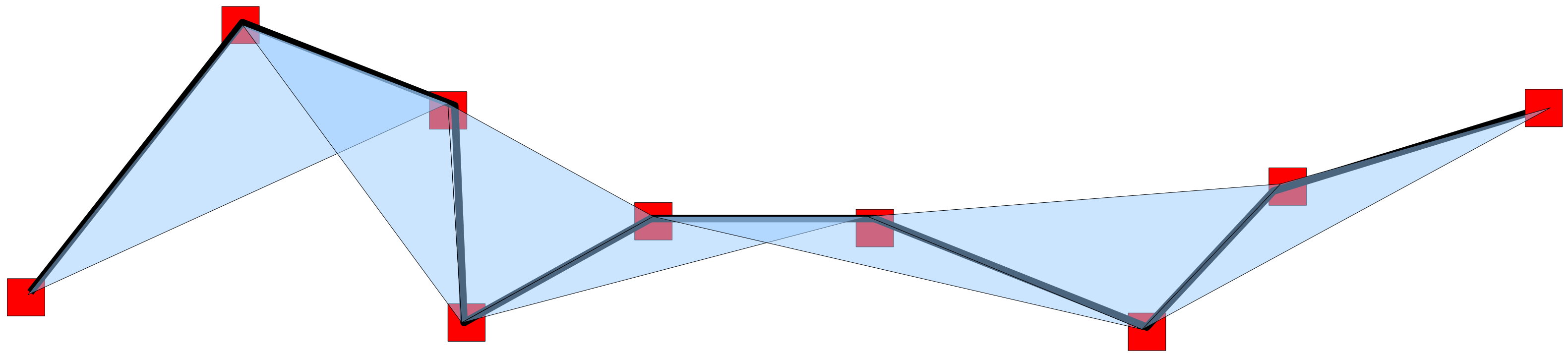
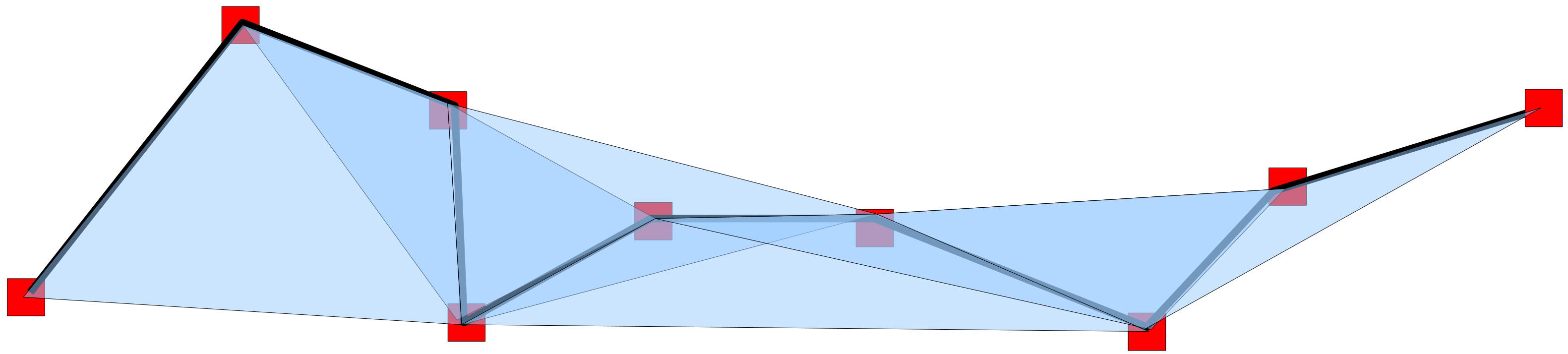


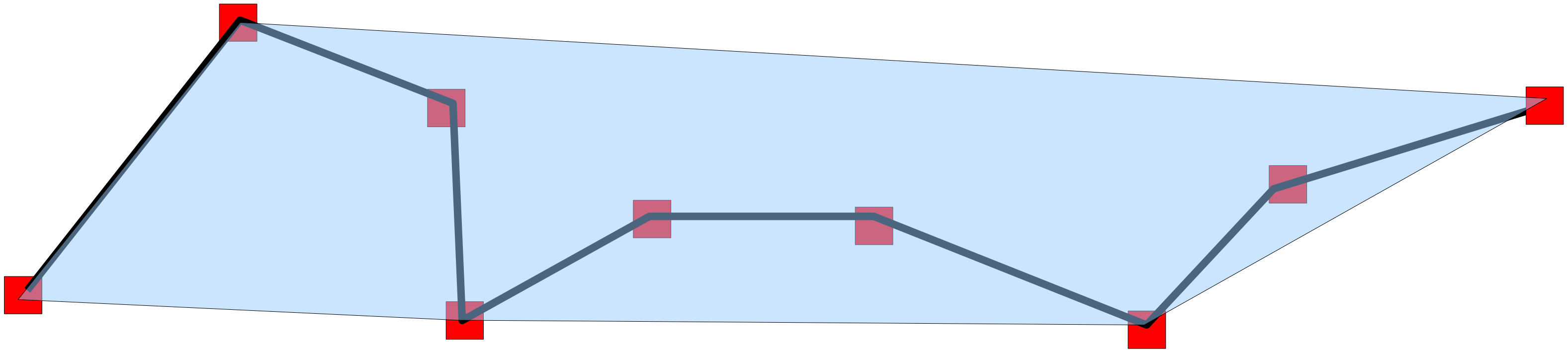
B-Splines

- Strong Convex Hull Property:
  - For a B-spline curve of order k (degree k-1), a point on the curve lies within the convex hull of k neighbouring control points.
  - For k=4



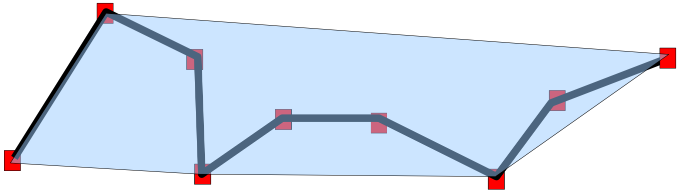
B-Splines

- Strong Convex Hull Property:
  - For a B-spline curve of order k (degree k-1), a point on the curve lies within the convex hull of k neighbouring control points.
  - For k=9



B-Splines

- Strong Convex Hull Property:
  - For a B-spline curve of order k (degree k-1), a point on the curve lies within the convex hull of k neighbouring control points.
  - If all vertices of the control polygon are collinear then the B-spline curve is a straight line.
  - If at least k-1 coincident vertices of the defining polygon occur, i.e., $B_i = B_{i+1} = \ldots = B_{i+k-2}$, then the convex hull of these vertices is the vertex itself and so the resulting curve must pass through this vertex.



B-Splines

- Knot Vectors and Basis functions

$$N_{i,1} = \begin{cases} 1, \text{if } x_i \le t < x_{i+1} \\ 0, \text{otherwise} \end{cases}$$

Cox-deBoor recursive formula

$$N_{i,k}(t) = \frac{(t - x_i) N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t) N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}$$

- The x_i's are the knot values with, $x_i \le x_{i+1}$
- The basis function $N_{i,k}(t)$ is zero outside the knot span $x_i \le x_{i+k}$
- Total number of knot values is $n+1+k$, where $n+1$ is the number of vertices of the control polygon and k is the order of the b-spline curve.
- For e.g.,

$$n+1=4, k=3$$

$$N_{1,3} \rightarrow x_1, \ldots, x_4$$

$$N_{4,3} \rightarrow x_4, \ldots, x_7$$

$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$$

B-Splines

- Types of Knot Vectors
  - *Uniform (or periodic) knot vector* : Individual knot values are evenly spaced.
  - For e.g., $[1,2,3,4]$ $[-0.2,-0.1,0.0,0.1,0.2]$
    $[0.0,0.25,0.50,0.75,1.0]$
  - For a given order k, uniform knot vectors yield periodic uniform basis functions for which

$$N_{i,k}(t)=N_{i-1,k}(t-1)=N_{i+1,k}(t+1)$$

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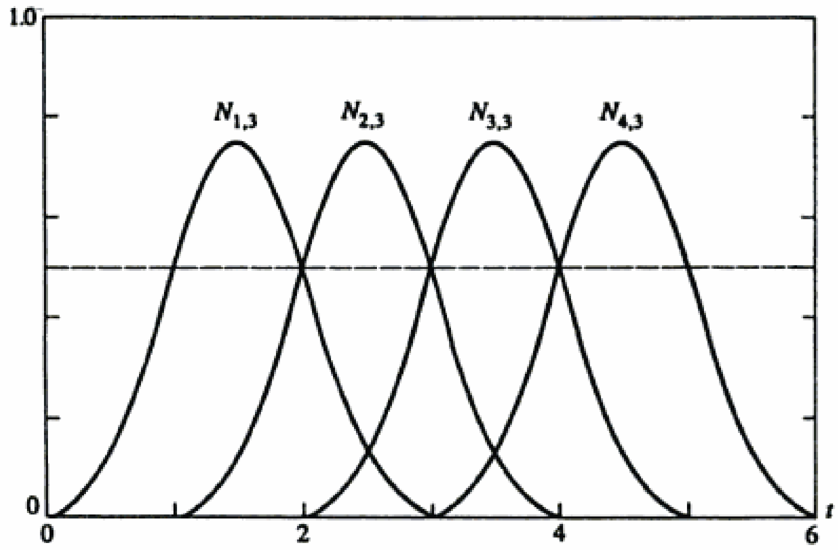

Figure 5–36 Periodic uniform B-spline basis functions, $[X] = [0\ 1\ 2\ 3\ 4\ 5\ 6]$, $n+1=4$, $k=3$.

- Thus each basis function is a translate of the other.

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B-Splines

- Types of Knot Vectors
  - *Open uniform knot vector* : this has a multiplicity of knot values at the ends of the knot vector equal to the order k of the b-spline basis function. Internal knot values are evenly spaced.
  - For e.g., $[0,0,1,2,3,4,4] \text{ for } k=2$

    $[0,0,1/4,1/2,3/4,1,1] \text{ for } k=2$
  - Formally,

$$x_i = \begin{cases} 0, & 1 \le i \le k \\ i-k, & k+1 \le i \le n+1 \\ n-k+2, & n+2 \le i \le n+k+1 \end{cases}$$

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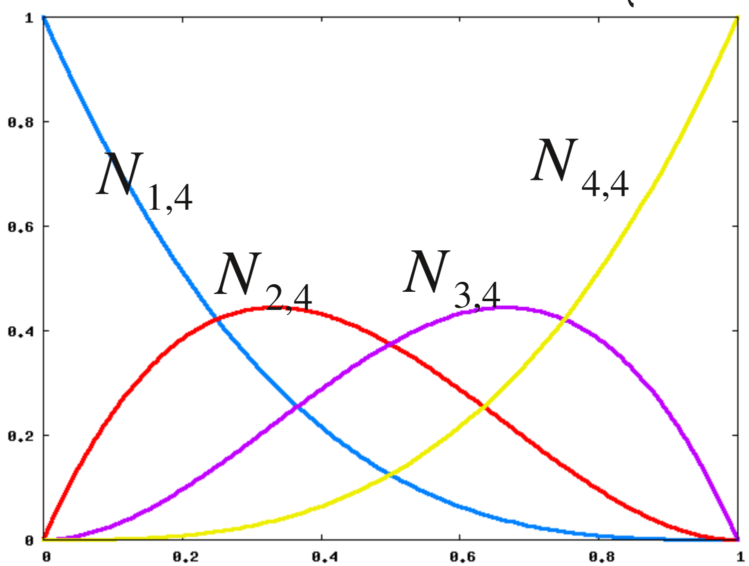

- The basis functions look as shown

$$[X] = [0.0\ 0.0\ 0.0\ 0.0\ 1.0\ 1.0\ 1.0\ 1.0]$$
$$\text{for } k=4, n+1=4$$

- Looks familiar?

B-Splines

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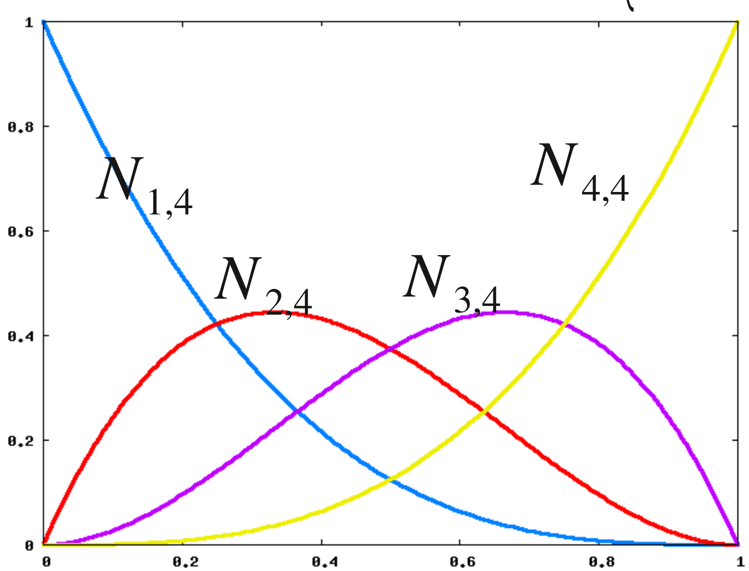


- The basis functions look as shown

  $[X] = [0.0\,0.0\,0.0\,0.0\,1.0\,1.0\,1.0\,1.0]$

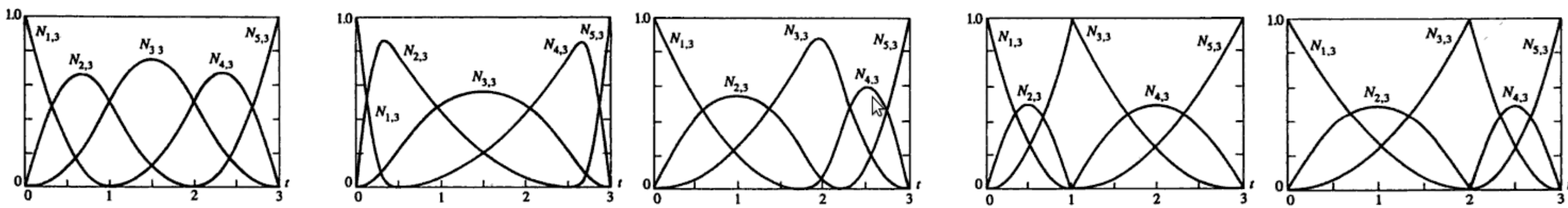
  for $k=4, n+1=4$

- The B-spline basis reduces to the Bernstien basis when $k=n+1$ and an open uniform knot vector is used.

B-Splines

- Types of Knot Vectors
  - *Non-uniform knot vector* : this has a unequally spaced and/or multiple internal knot values. They may be periodic or open.
  - For e.g.,

$$[0\,0\,0\,1\,1\,2\,2\,2], [0, 0.28, 0.5, 0.72, 1]$$



Nonuniform basis functions for $n + 1 = 5$, $k = 3$.
(a) $[X] = [0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 3\ \ 3\ \ 3]$;
(b) $[X] = [0\ \ 0\ \ 0\ \ .4\ \ 2.6\ \ 3\ \ 3\ \ 3]$;
(c) $[X] = [0\ \ 0\ \ 0\ \ 1.8\ \ 2.2\ \ 3\ \ 3\ \ 3]$;
(d) $[X] = [0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 3\ \ 3\ \ 3]$;
(e) $[X] = [0\ \ 0\ \ 0\ \ 2\ \ 2\ \ 3\ \ 3\ \ 3]$.

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B-Splines

- Buildup of b-spline basis functions

$$
\begin{array}{cccccc}
N_{i,k} & & & & & \\
N_{i,k-1} & N_{i+1,k-1} & & & & \\
N_{i,k-2} & N_{i+1,k-2} & N_{i+2,k-2} & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
N_{i,1} & N_{i+1,1} & N_{i+2,1} & N_{i+3,1} & \cdot & N_{i+k-1,1}
\end{array}
$$

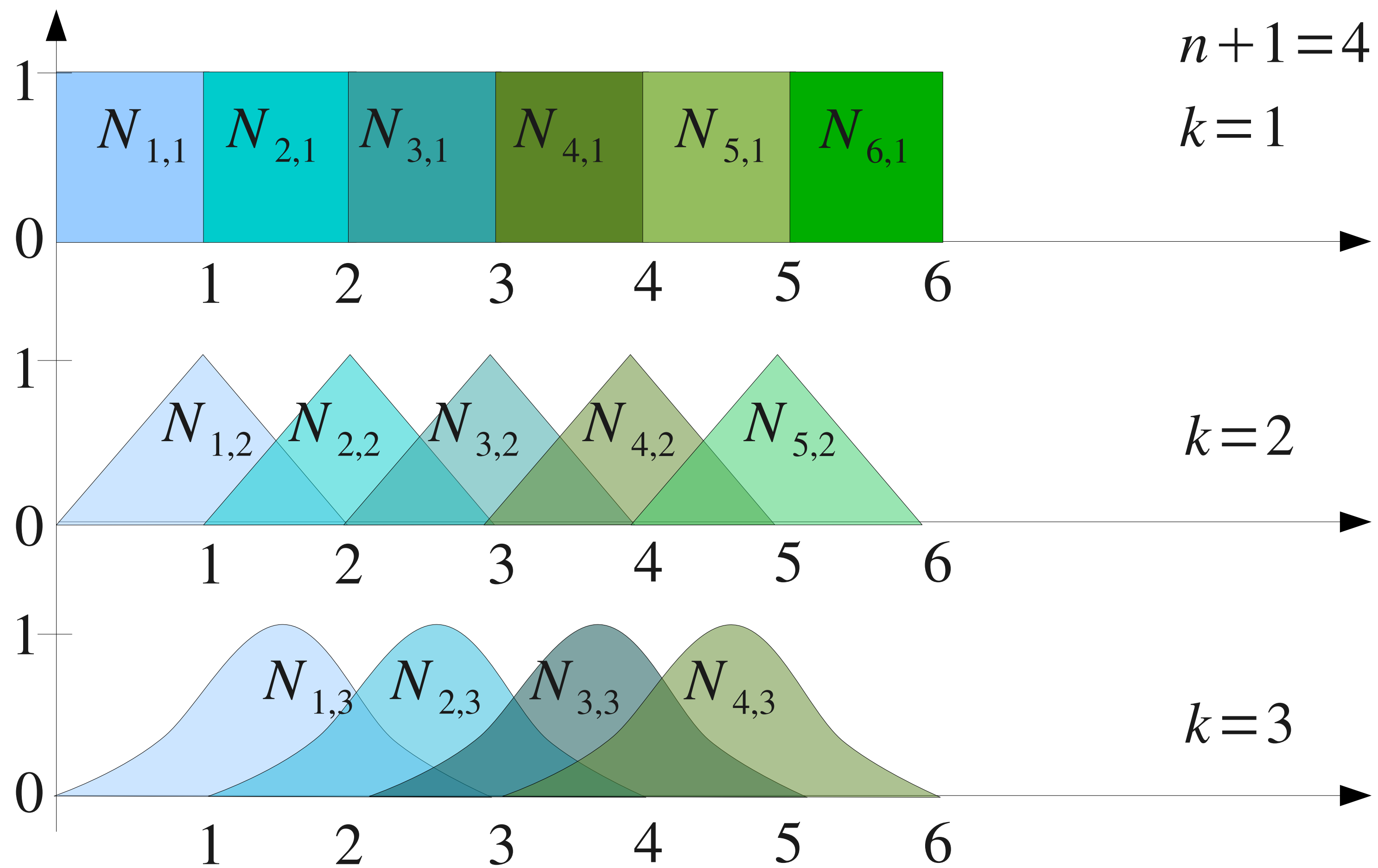
B-Splines

- Buildup of b-spline basis functions

$$
\begin{array}{ccccccccc}
N_{i-k+1,k} & \cdot & N_{i+k-1,k} & & N_{i,k} & & N_{i+1,k} & \cdot & N_{i+k-1,k} \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
 & & N_{i-1,2} & & N_{i,2} & & N_{i+1,2} & & \\
 & & & & N_{i,1} & & & &
\end{array}
$$

B-Splines

- Buildup of b-spline basis functions



B-Splines

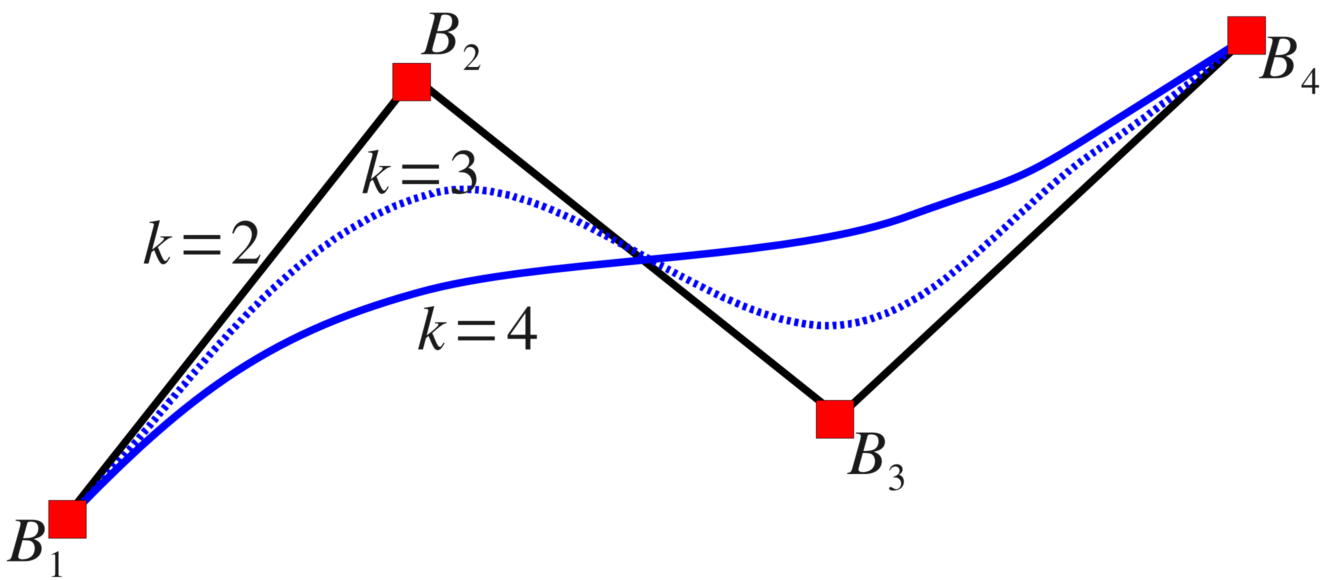
- Properties
  - Strong Convex Hull Property
  - Curve follows shape of control polygon
  - Maximum order = number of control points
  - Invariance to affine transformation
  - Variation diminishing property

B-Splines

- Controlling the B-Spline
  - Knot Vector (uniform, open, non-uniform)
  - Order of the curve
  - Number and position of control points

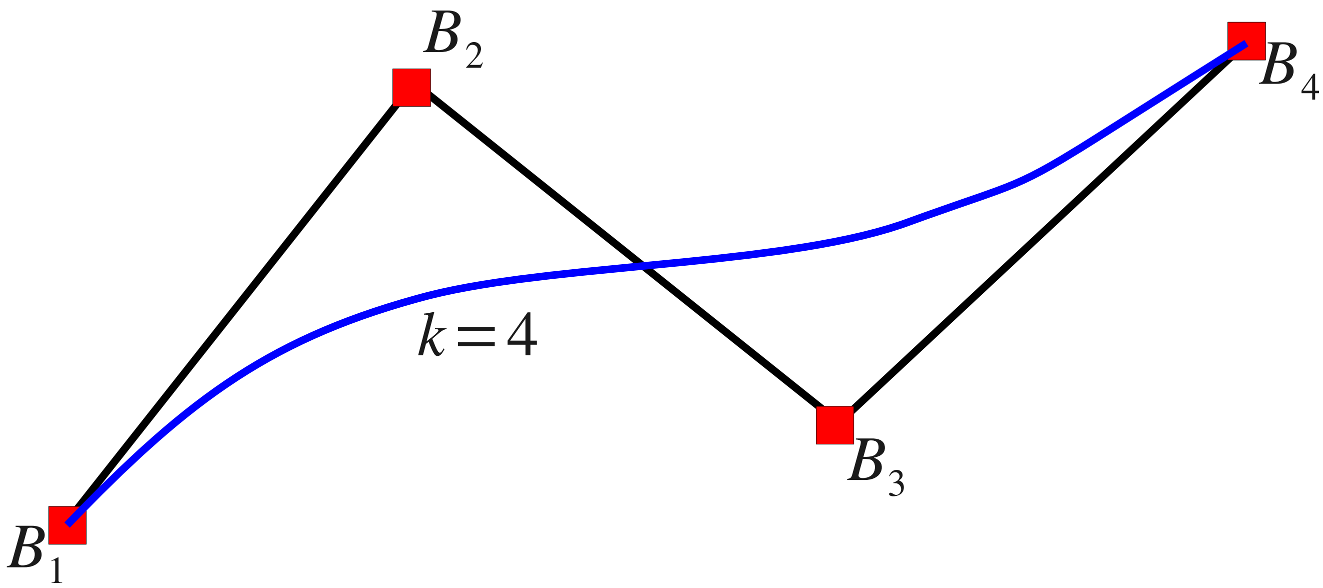
B-Splines

- Order of the b-spline curve
  - Open uniform knot vector



B-Splines

- Multiplicity of control points
  - Open uniform knot vector, k=4
  - Control Polygon B_1, B_2, B_3, B_4



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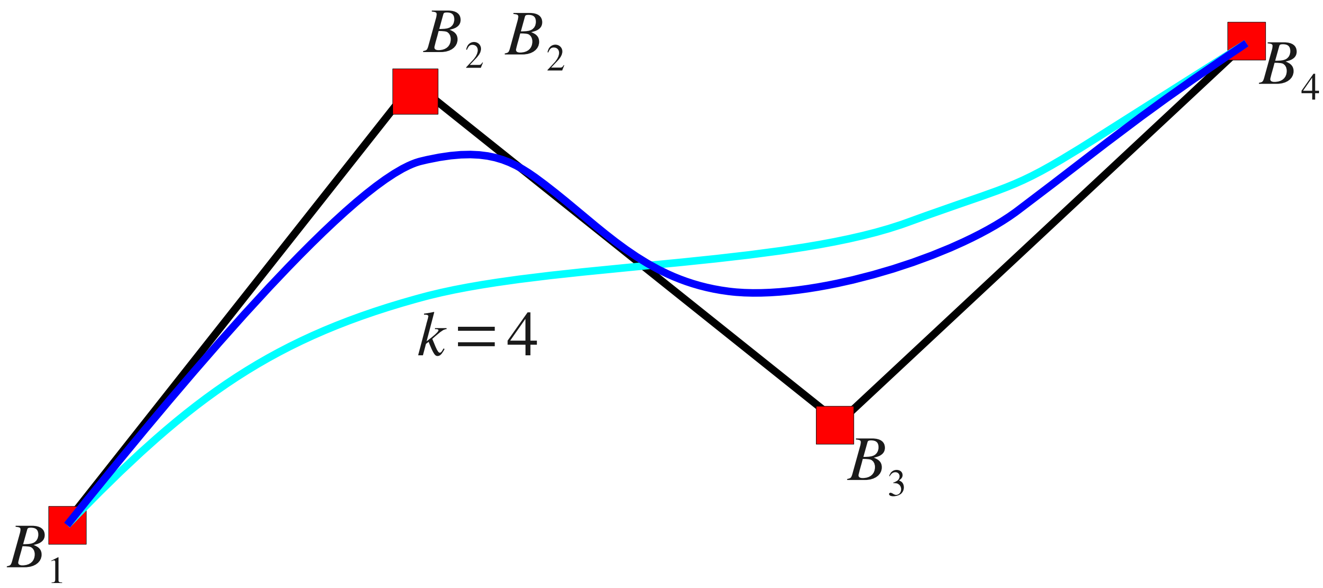


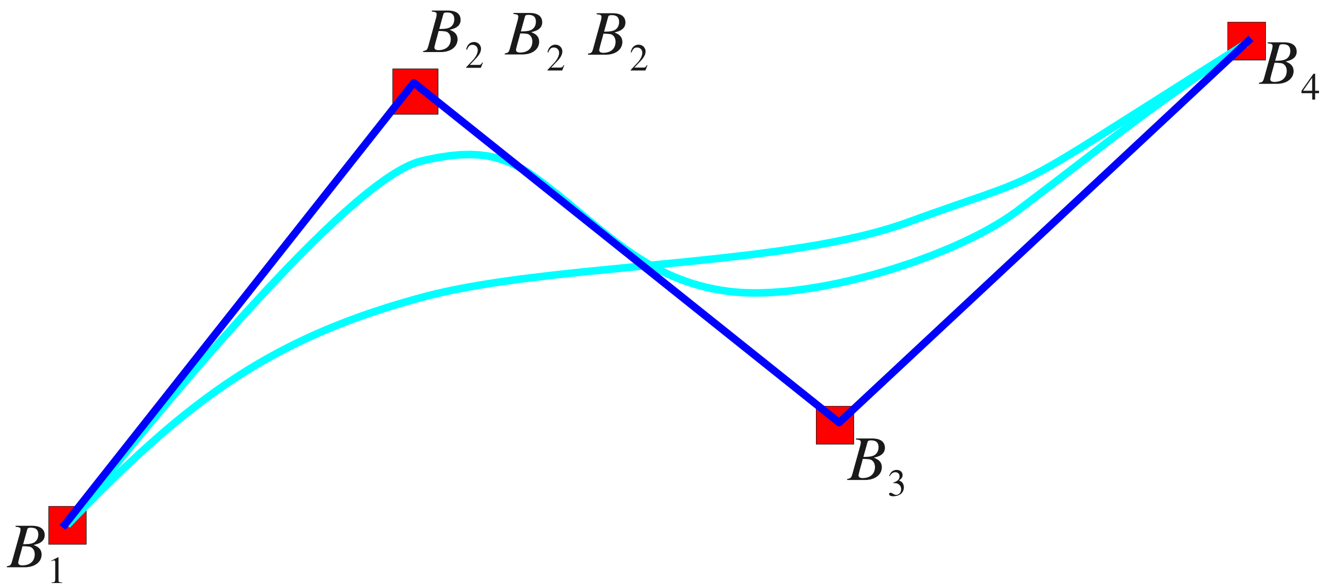
B-Splines

- Multiplicity of control points
  - Open uniform knot vector, k=4
  - Control Polygon $B_1, B_2, B_2, B_2, B_3, B_4$

$$[0\,0\,0\,0\,1\,1\,1\,1]$$

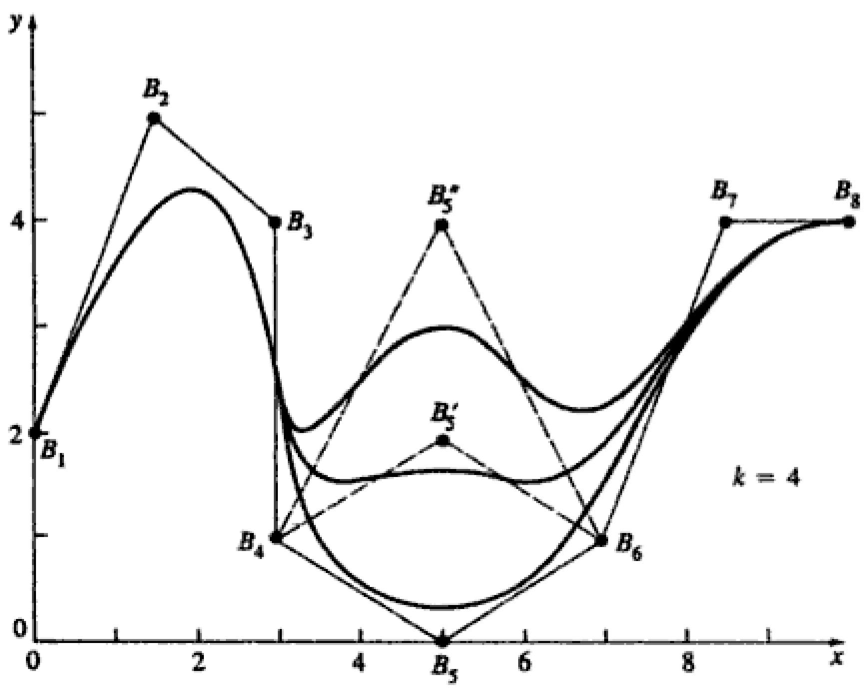
$$[0\,0\,0\,0\,1\,2\,2\,2\,2]$$

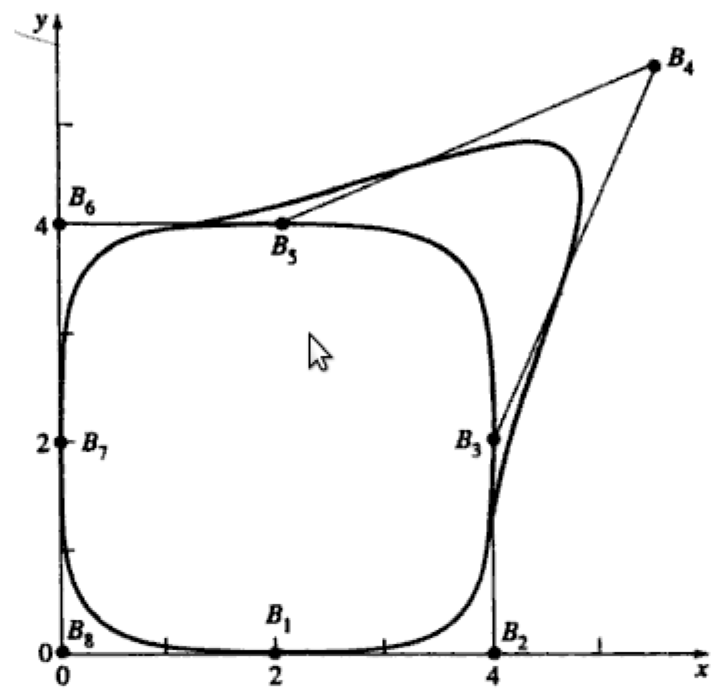
$$[0\,0\,0\,0\,1\,2\,3\,3\,3\,3]$$



B-Splines

- Moving a control point





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To get a closed curve, repeat k-2 polygon vertices at the beginning or at the end

B-Splines

- Curve Fitting or Interpolating B-splines
  - Find a B-spline passing through D_1, D_2, ...D_j

$$
\begin{aligned}
D_1(t_1) &= N_{1,k}(t_1)B_1 + N_{2,k}(t_1)B_2 + \ldots + N_{i,k}(t_1)B_i \\
D_2(t_2) &= N_{1,k}(t_2)B_1 + N_{2,k}(t_2)B_2 + \ldots + N_{i,k}(t_2)B_i \\
\vdots \quad &= \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
D_j(t_j) &= N_{1,k}(t_j)B_1 + N_{2,k}(t_j)B_2 + \ldots + N_{i,k}(t_j)B_i
\end{aligned}
$$

$$\text{where } 2 \le k \le n+1 \le j$$

B-Splines

- Curve Fitting or Interpolating B-splines
  - Find a B-spline passing through D_1, D_2, ...D_j

$$\boldsymbol{D} = \boldsymbol{N} \, . \, \boldsymbol{B}$$

$$\boldsymbol{D} = \begin{bmatrix} D_1(t_1) & D_2(t_2) & \dots & D_j(t_j) \end{bmatrix}^T$$

$$\boldsymbol{B} = \begin{bmatrix} B_1 & B_2 & \dots & B_i \end{bmatrix}^T$$

$$\boldsymbol{N} = \begin{bmatrix} N_{1,k}(t_1) & \dots & \dots & N_{i,k}(t_1) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ N_{1,k}(t_j) & \dots & \dots & N_{i,k}(t_j) \end{bmatrix}$$

B-Splines

- Curve Fitting or Interpolating B-splines
  - Find a B-spline passing through D_1, D_2, ...D_j

$$\boldsymbol{B} = \boldsymbol{N}^{-1} . \boldsymbol{D} \text{ if } 2 \leq k \leq n+1 = j$$

$$\boldsymbol{B} = [\boldsymbol{N}^T \boldsymbol{N}]^{-1} \boldsymbol{N}^T . \boldsymbol{D} \text{ if } 2 \leq k \leq n+1 < j$$

$$t_1 = 0, \qquad \frac{t_l}{t_{max}} = \frac{\sum_{s=2}^{l} \left| (D_s - D_{s-1}) \right|}{\sum_{s=2}^{j} \left| (D_s - D_{s-1}) \right|} \quad \text{for } l \geq 2$$

$$t_{max} = \text{maximum value of the knot vector}$$

B-Splines

- Rational B-Splines

$$P^h(t) = \sum_{i=1}^{n+1} B_i^h N_{i,k}(t)$$

$$P(t) = \frac{\sum_{i=1}^{n+1} B_i h_i N_{i,k}(t)}{\sum_{i=1}^{n+1} h_i N_{i,k}(t)} = \sum_{i=1}^{n+1} B_i R_{i,k}(t)$$

$$R_{i,k}(t) = \frac{h_i N_{i,k}(t)}{\sum_{i=1}^{n+1} h_i N_{i,k}(t)}$$

Question: What can rational curves do for us that polynomial curves cannot?

Non-uniform knot vector
\+ **Rational B-splines** = **NURBS**