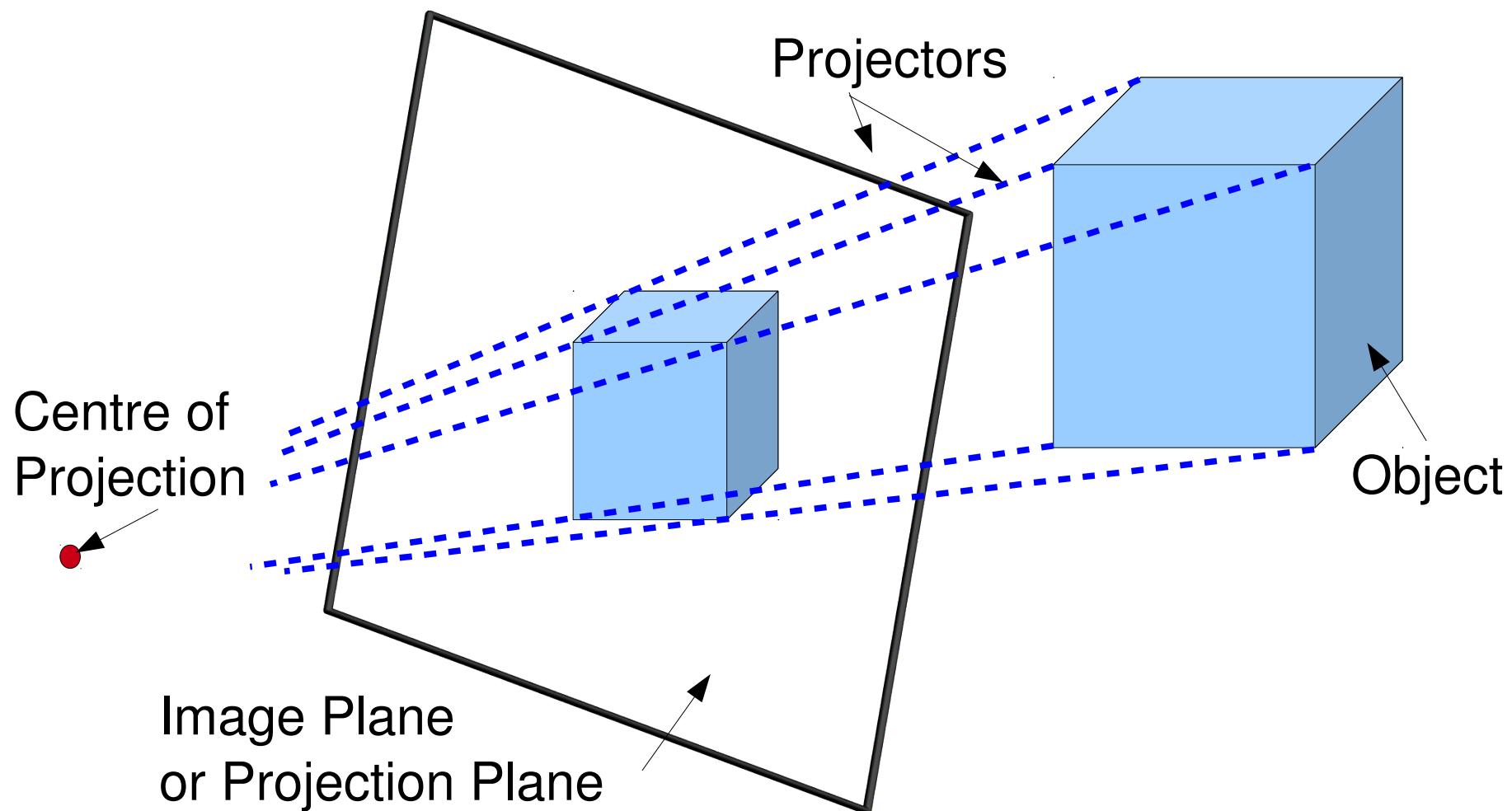




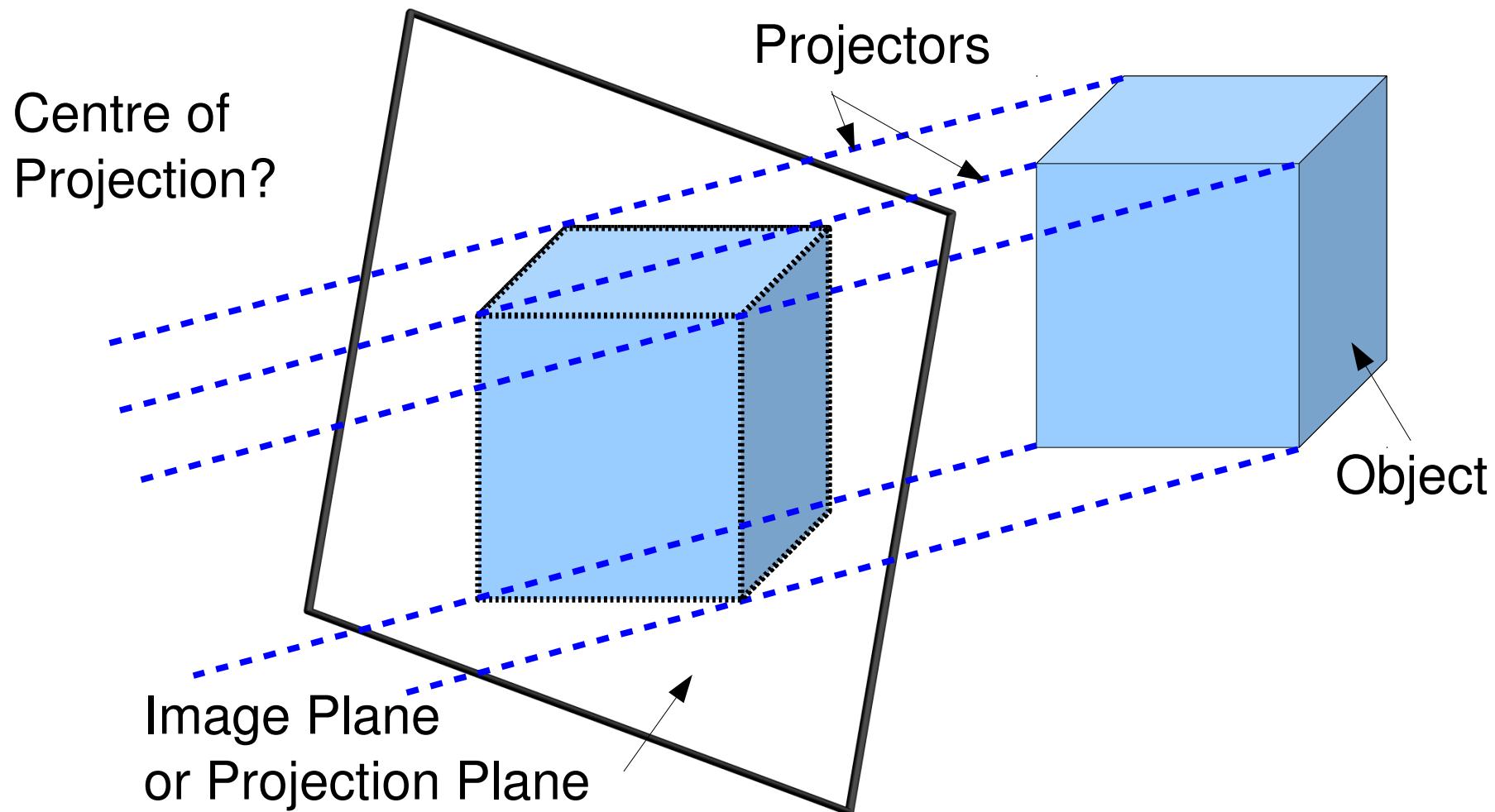
CS475/CS675 Computer Graphics

Viewing

Perspective Projection



Parallel Projection

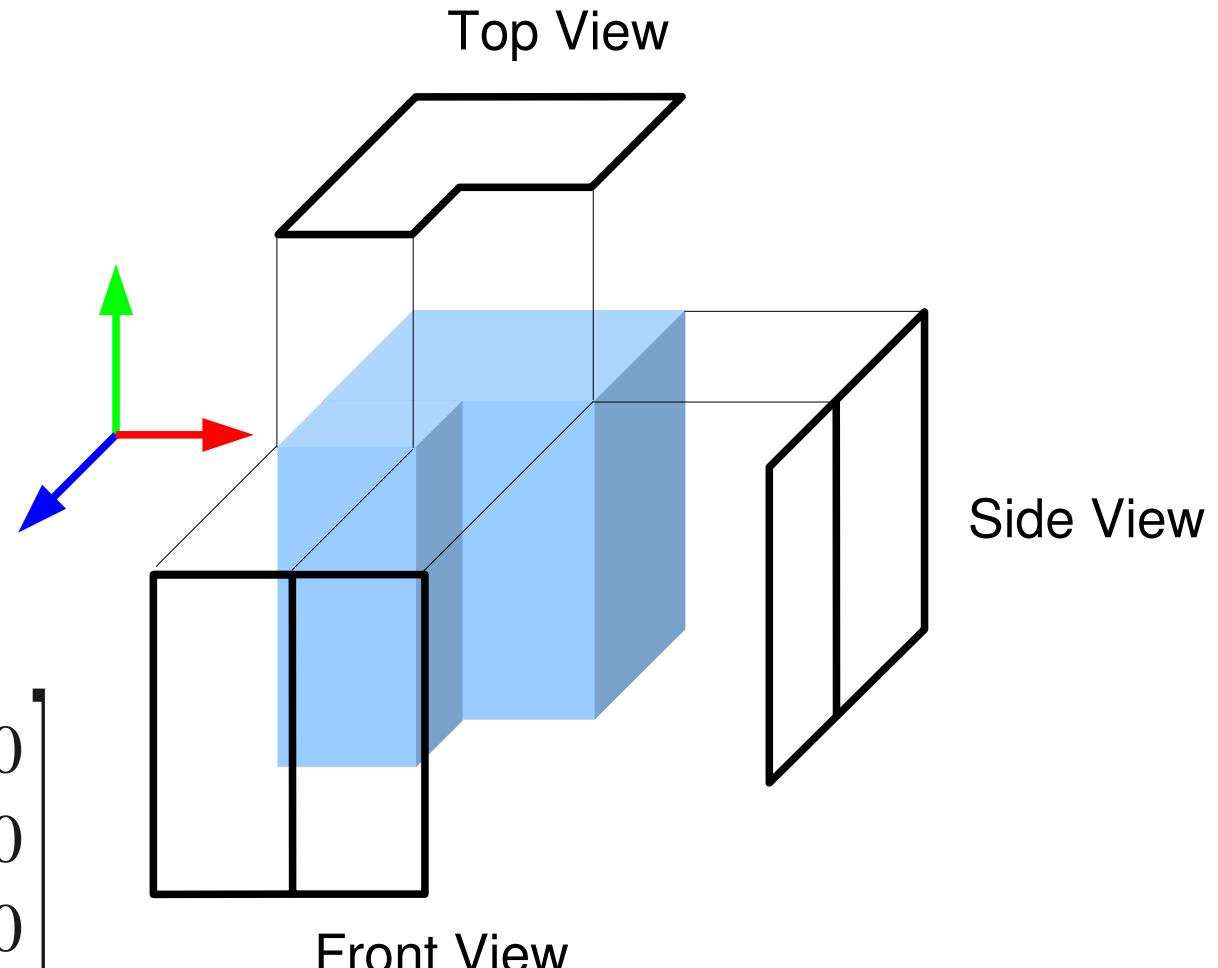


Parallel Projection

Orthographic Projection

- Multiviews ($x=0$, $y=0$ or $z=0$ or principal planes).
- True size or shape for lines.
- For projection on the $z=0$ plane we get the projection matrix as

$$P(z=0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Parallel Projection

Axonometric Projection

- Transform and then project using an orthographic projection such that at multiple adjacent faces are visible – better representation of a 3D object using 1 view. Face parallel to projection plane shows true shape and size.
- If U be the matrix formed by stacking up the unit vectors along the three axes, and T be the axonometric projection, then

$$T \cdot U = T \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} x_x' & x_y' & x_z' \\ y_x' & y_y' & y_z' \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Parallel Projection

Axonometric Projection

- If U be the matrix formed by stacking up the unit vectors along the three axes, and T be the axonometric projection, then

$$T \cdot U = T \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} x_x' & x_y' & x_z' \\ y_x' & y_y' & y_z' \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

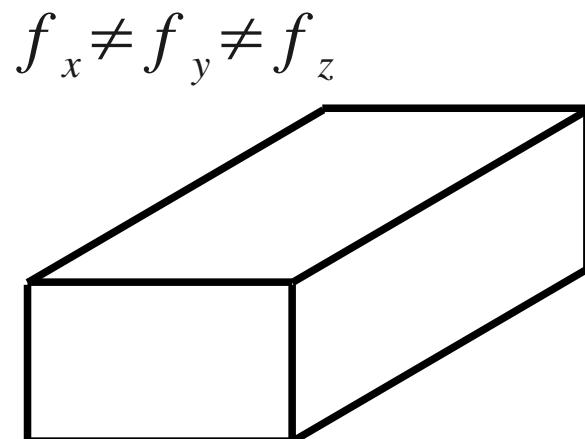
- The foreshortening ratios for each projected principal axes are then given by:

$$f_x = \sqrt{x_x'^2 + y_x'^2} \quad f_y = \sqrt{x_y'^2 + y_y'^2} \quad f_z = \sqrt{x_z'^2 + y_z'^2}$$

Parallel Projection

Axonometric Projection

- Depending on the kind of foreshortening they cause we can have three types of axonometric projections
 - Trimetric (all foreshortenings are different)

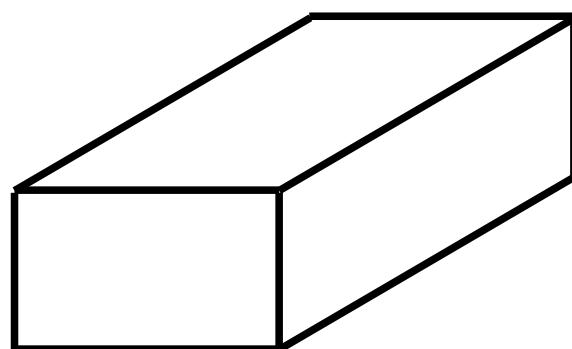


Parallel Projection

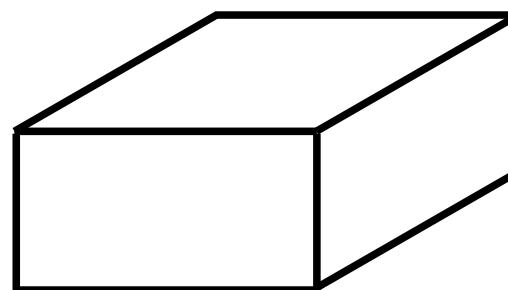
Axonometric Projection

- Depending on the kind of foreshortening they cause we can have three types of axonometric projections
 - Trimetric (all foreshortenings are different)
 - Dimetric (two foreshortenings are the same)

$$f_x \neq f_y \neq f_z$$



$$f_x = f_z$$

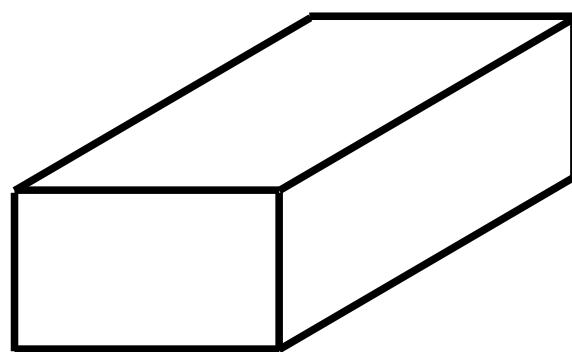


Parallel Projection

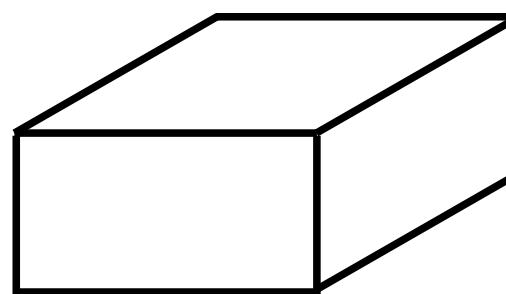
Axonometric Projection

- Depending on the kind of foreshortening they cause we can have three types of axonometric projections
 - Trimetric (all foreshortenings are different)
 - Dimetric (two foreshortenings are the same)
 - Isometric (all foreshortenings are the same)

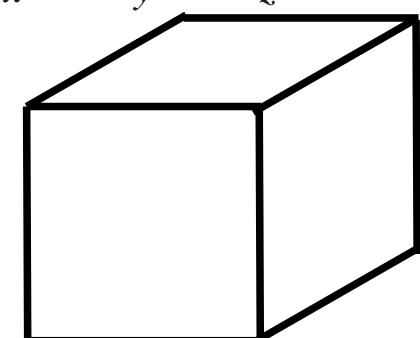
$$f_x \neq f_y \neq f_z$$



$$f_x = f_z$$



$$f_x = f_y = f_z$$



Parallel Projection

Axonometric Projection

- Assuming we rotate by $R_y(\phi)$ and $R_x(\theta)$ before we do the projection on the $z=0$ plane.

$$T = P(z=0) \cdot R_x(\theta) \cdot R_y(\phi)$$

$$\begin{aligned} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \phi & 0 & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ \sin \theta \sin \phi & \cos \theta & -\sin \theta \cos \phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Parallel Projection

Axonometric Projection

- Now we apply this axonometric projection T to U

$$\begin{aligned} T \cdot U &= \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ \sin \theta \sin \phi & \cos \theta & -\sin \theta \cos \phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos \phi & 0 & \sin \phi \\ \sin \theta \sin \phi & \cos \theta & -\sin \theta \cos \phi \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{aligned}$$

Parallel Projection

Axonometric Projection

- So the foreshortening ratios become

$$f_x^2 = \cos^2 \phi + \sin^2 \theta \sin^2 \phi$$

$$f_y^2 = \cos^2 \theta$$

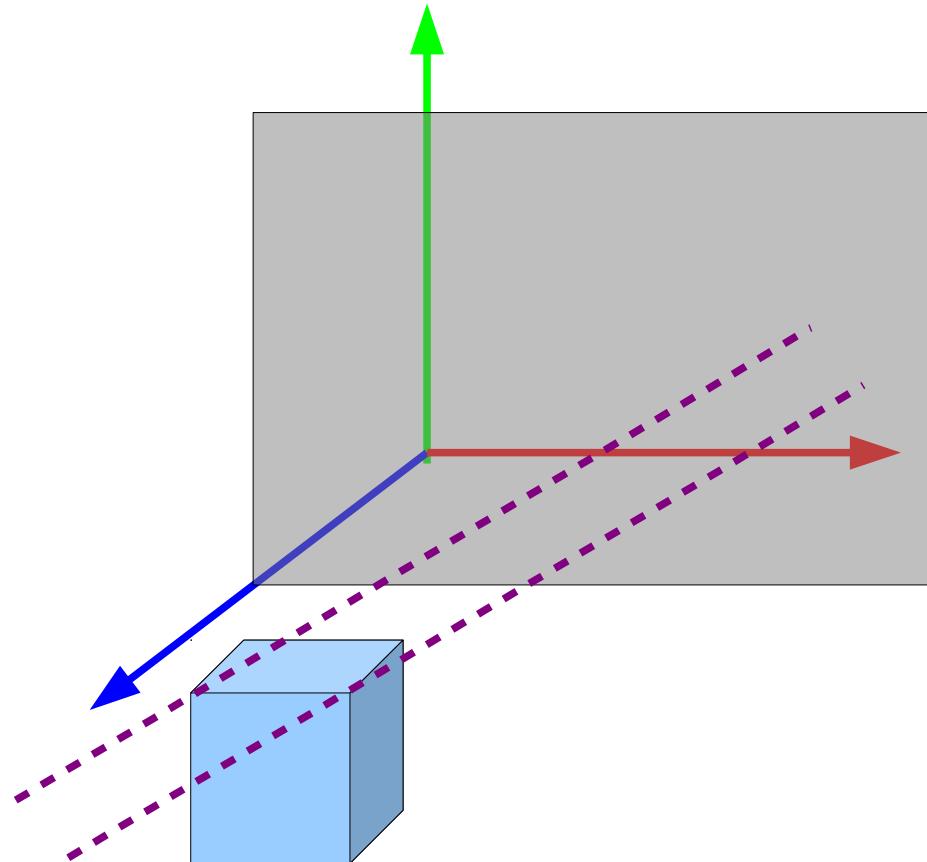
$$f_z^2 = \sin^2 \phi + \sin^2 \theta \cos^2 \phi$$

- For Isometric projections, if we solve for $f_x = f_y = f_z$ then we get $\theta = 35.26^\circ$ and $\phi = \pm 45^\circ$

Parallel Projection

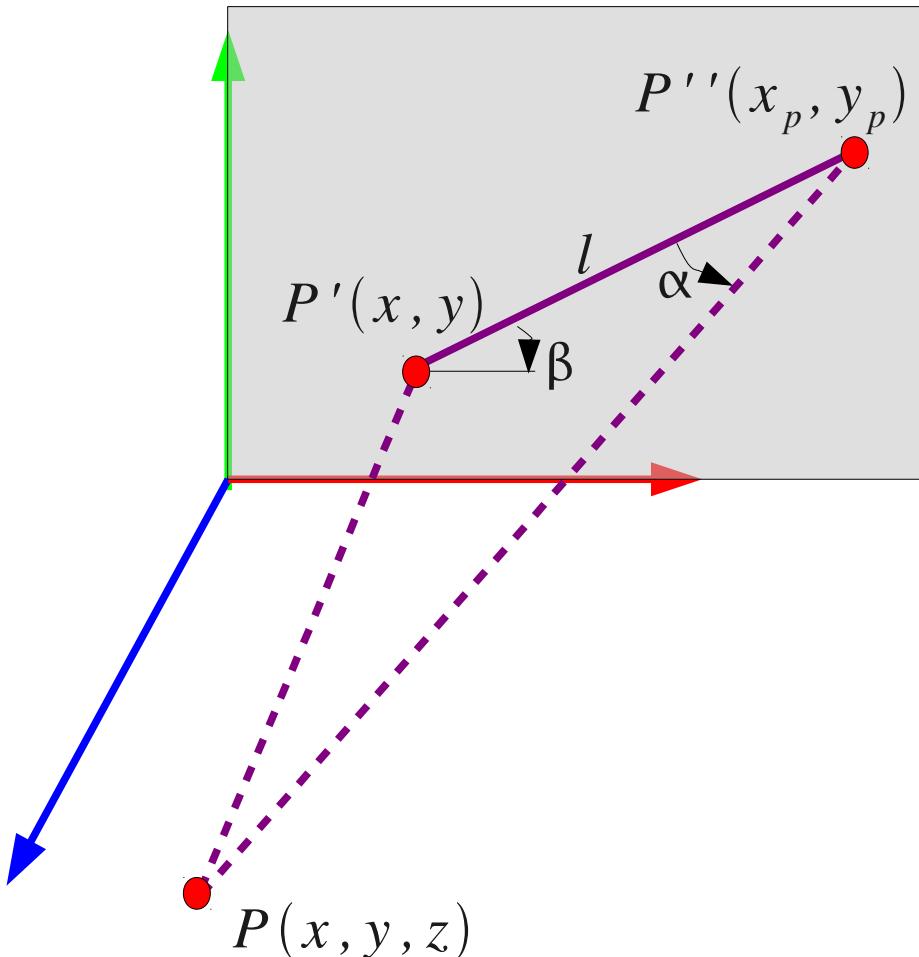
Oblique Projection

- The projectors are parallel to each other but they are not perpendicular to the plane of projection.
- Only planes parallel to plane of projection show true shape and size.



Parallel Projection

Oblique Projection



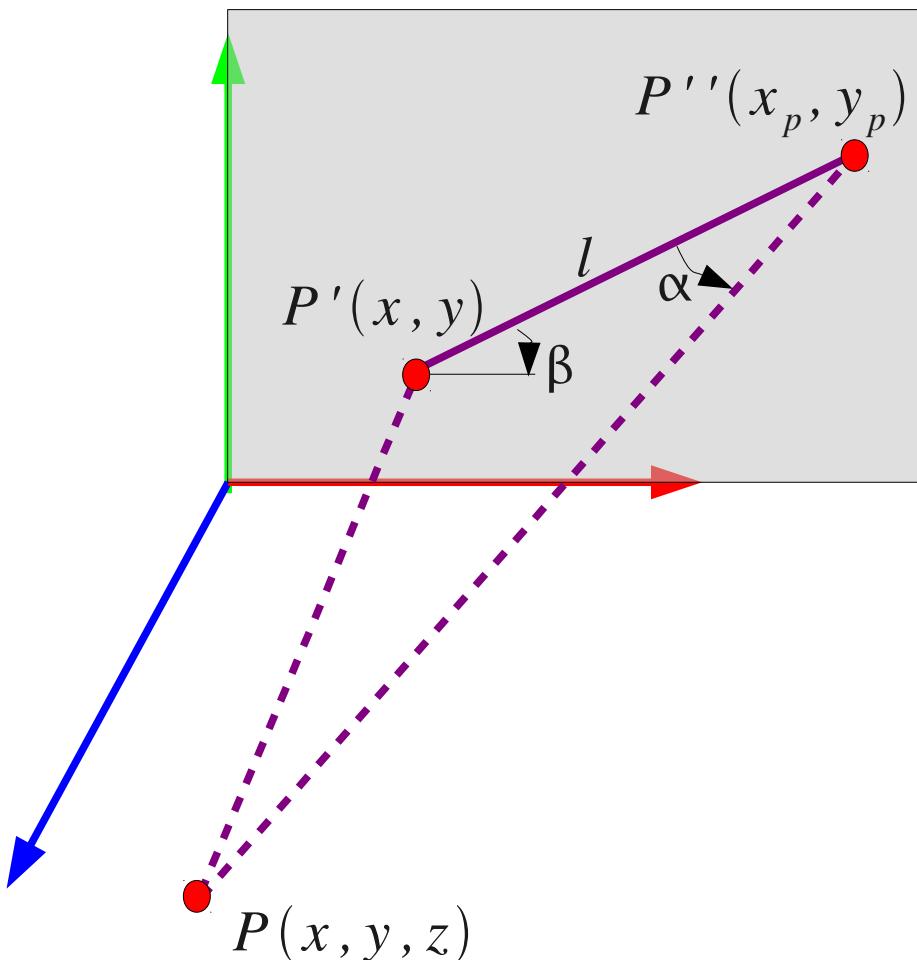
$$x_p = x + l \cos \beta$$

$$y_p = y + l \sin \beta$$

$$\tan \alpha = \frac{z}{l} \quad \text{or} \quad l = z \cot \alpha$$

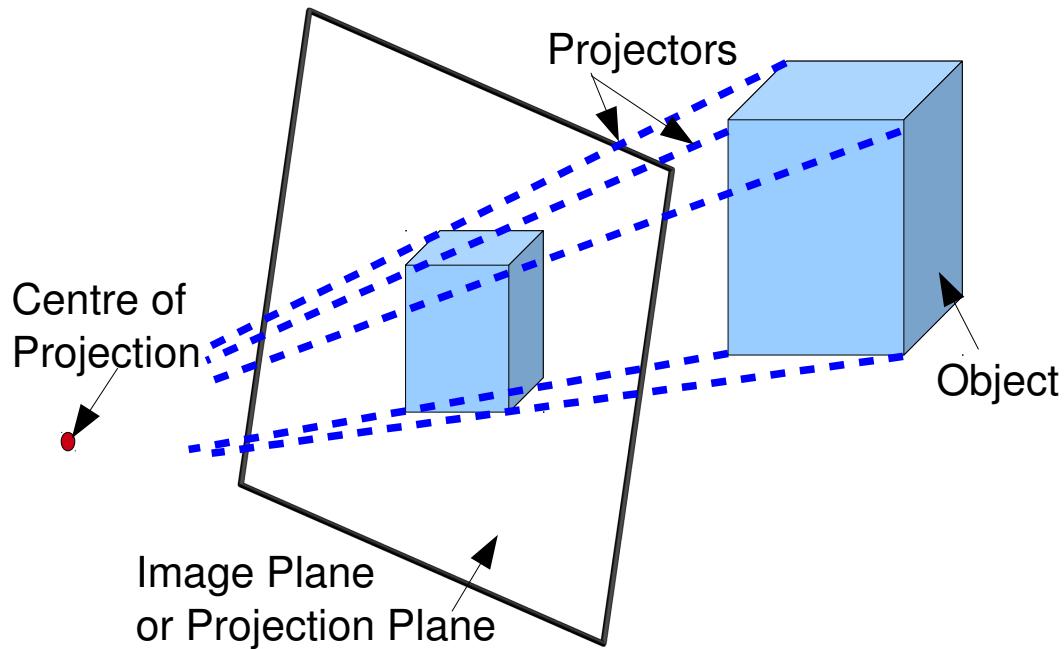
Parallel Projection

Oblique Projection



- When $\alpha=45^\circ$ we get a **Cavalier** projection. Lines perpendicular to the projection plane are not foreshortened.
- When $\cot\alpha=1/2$ we get **Cabinet** projections. Lines perpendicular to the projection plane are foreshortened by half.
- β is typically 30° or 45° .

Perspective Projection



- Projectors converge at a finite centre of projection.
- Parallel lines converge.
- We get non-uniform foreshortening.
- Shape is not preserved.
- We see in perspective – so perspective viewing seems natural and helps in depth perception.

A digression into art



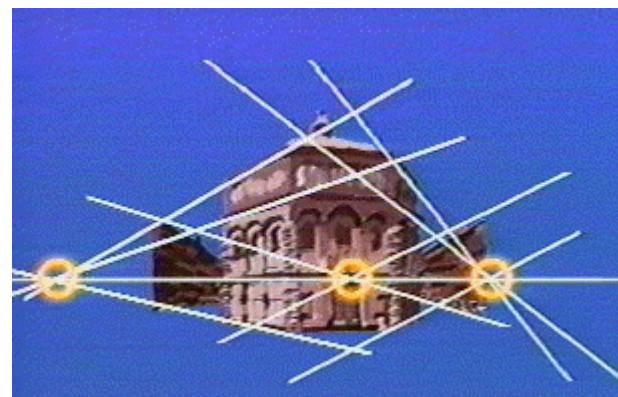
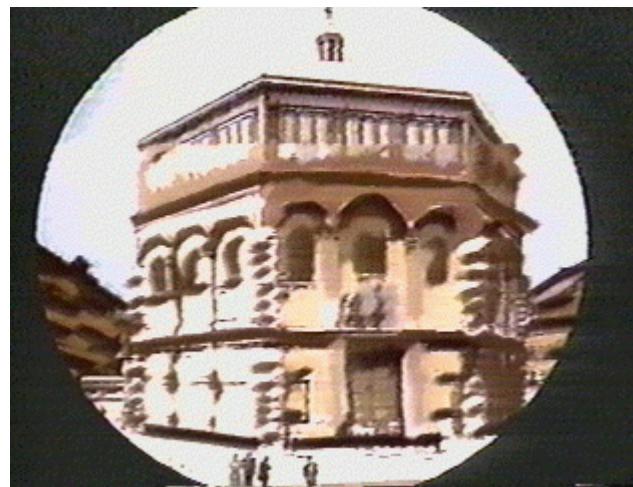
13th century , Arezzo by Giotto

A digression into art



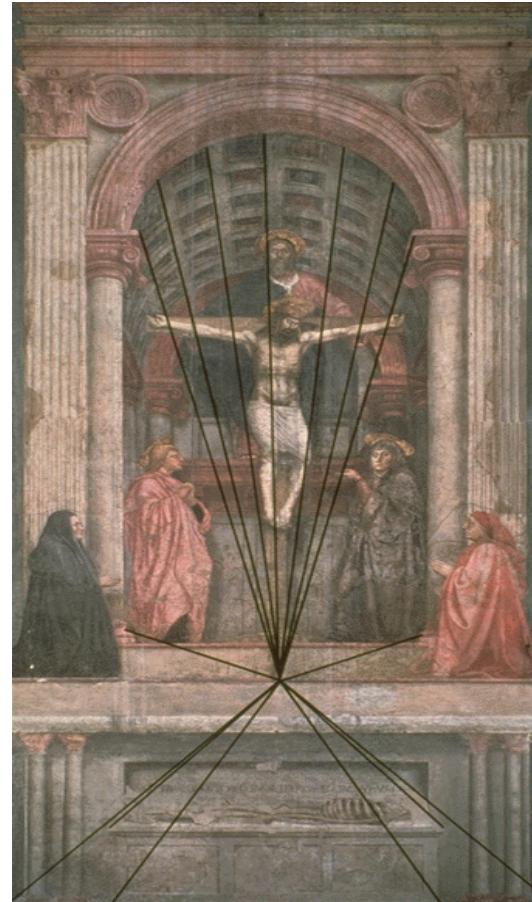
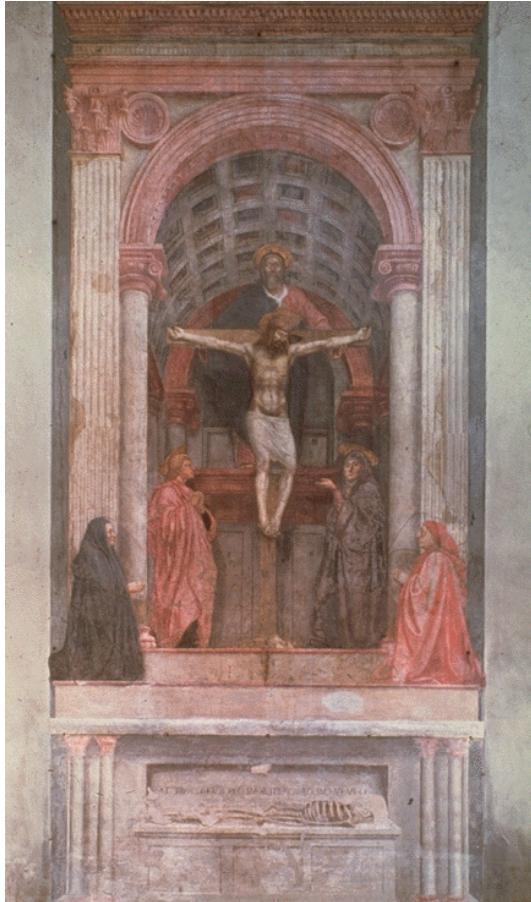
Early 15th century , The Little Garden of Paradise

A digression into art



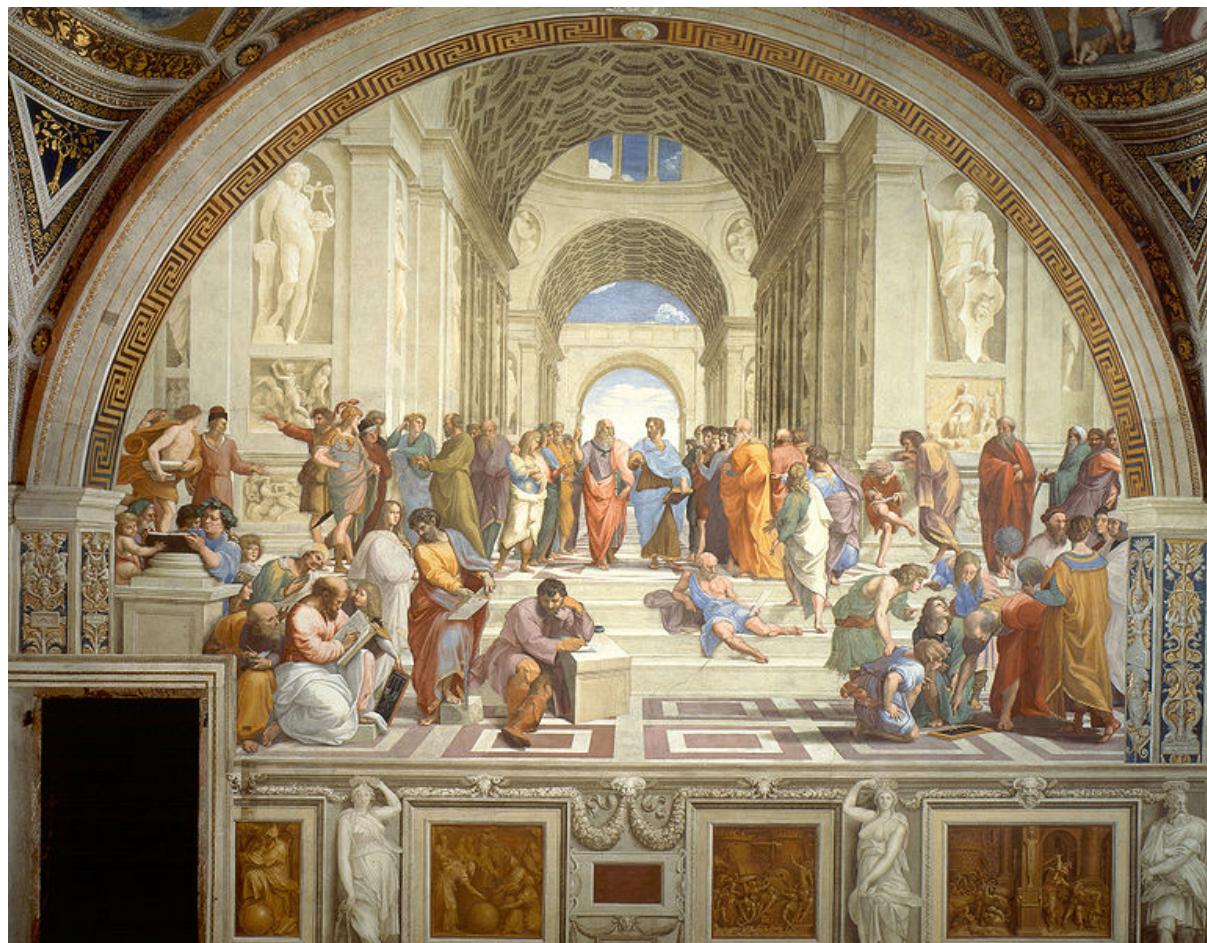
15th century , The Baptistry in Florence, Filippo Brunelleschi

A digression into art



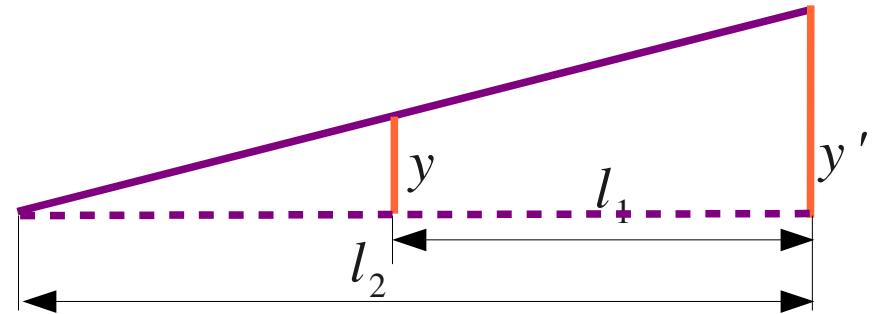
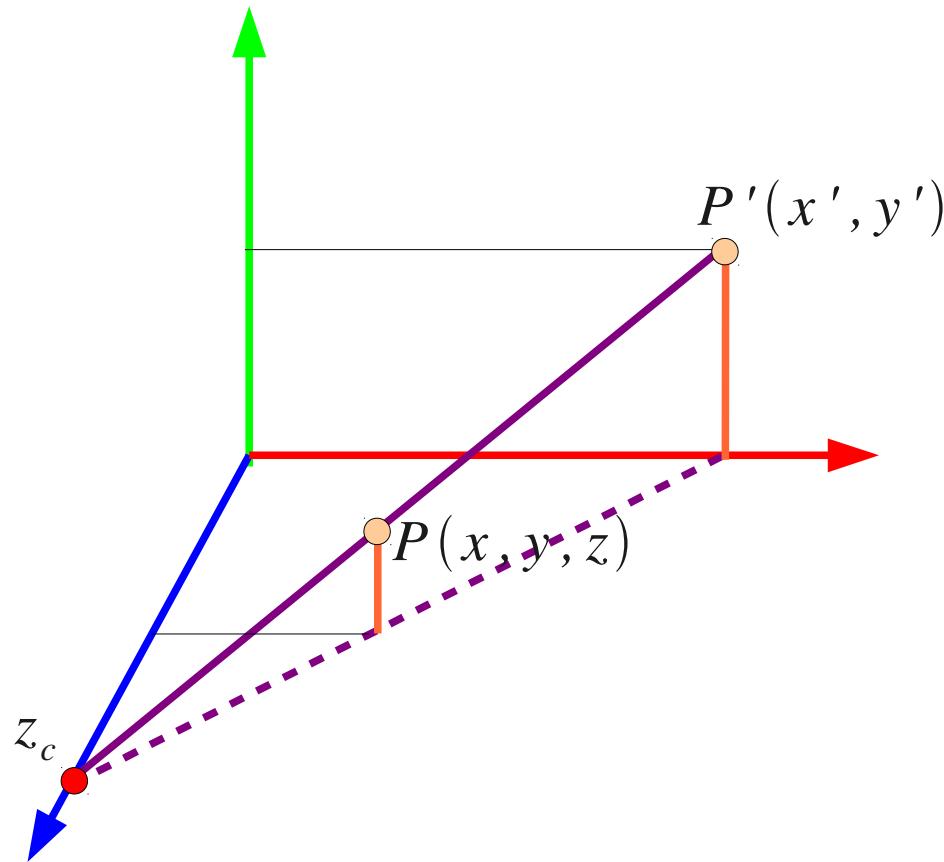
15th century , Fresco of Holy Trinity, Masaccio

A digression into art



15th century , School of Athens, Raphael

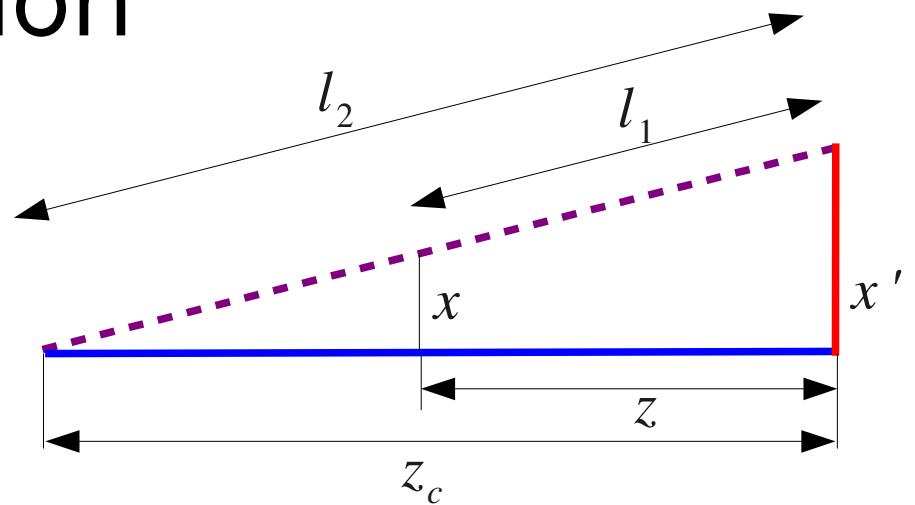
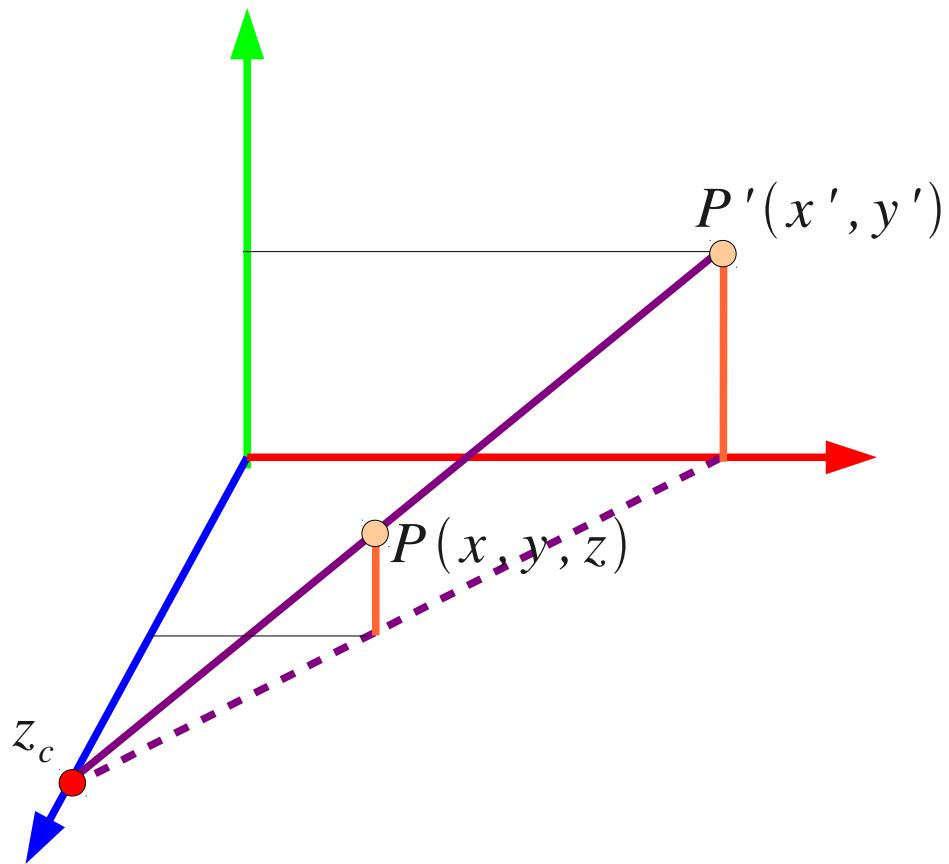
Perspective Projection



$$\frac{y'}{l_1} = \frac{y}{l_2 - l_1} \quad \frac{z_c}{l_2} = \frac{z_c - z}{l_2 - l_1}$$

$$\Rightarrow y' = \frac{y}{1 - \frac{z}{z_c}}$$

Perspective Projection



$$\frac{x'}{l_2} = \frac{x}{l_2 - l_1} \quad \frac{z_c}{l_2} = \frac{z_c - z}{l_2 - l_1}$$

$$\Rightarrow x' = \frac{x}{1 - \frac{z}{z_c}}$$

Perspective Projection

- First we apply a perspective transform to a point X that takes it to X'

$$X' = P_r \cdot X$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ rz+1 \end{bmatrix}$$

$$x' = \frac{x}{rz+1}, \quad y' = \frac{y}{rz+1}, \quad z' = \frac{z}{rz+1}$$

Perspective Projection

- First we apply a perspective transform to a point X that takes it to X'
- Now we add projection on the $z=0$ plane.

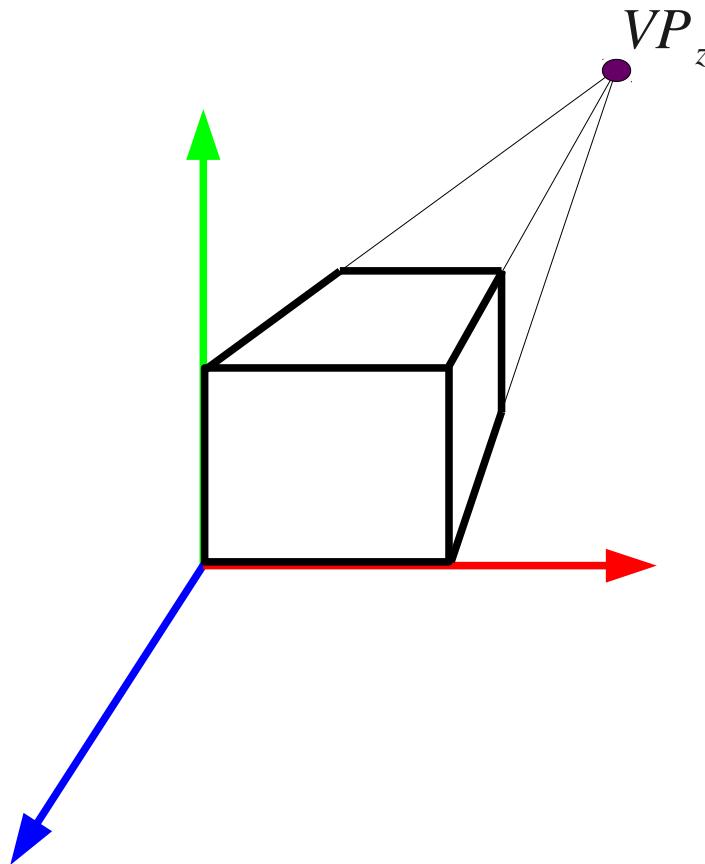
$$X' = P(z=0) \cdot P_r \cdot X$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ rz+1 \end{bmatrix}$$

$$x' = \frac{x}{rz+1}, \quad y' = \frac{y}{rz+1}, \quad z' = 0$$

- If $r = \frac{-1}{z_c}$ then we get $x' = \frac{x}{1 - \frac{z}{z_c}}, \quad y' = \frac{y}{1 - \frac{z}{z_c}}$

Perspective Projection



- Vanishing point in the z direction.
- Set of lines not parallel to the projection plane converge at a vanishing point.



Perspective Projection

- To find the vanishing point along the z direction we apply the perspective transformation to the point at infinity along the z direction.

$$X' = P_r \cdot X$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ r \end{bmatrix} \quad x' = 0, \quad y' = 0, \quad z' = \frac{1}{r}$$

- If $r = -1/z_c$ then we get $z' = -z_c$, i.e., the vanishing point lies an equal distance on the opposite side of the projection plane as the center of projection.

Perspective Projection

Single point perspective

- Centre of projection (CoP) on x axis

$$X' = P_p \cdot X$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ px+1 \end{bmatrix}$$

$$x' = \frac{x}{px+1}, \quad y' = \frac{y}{px+1}, \quad z' = \frac{z}{px+1}$$

- CoP is at $(-1/p, 0, 0, 1)$, VP is at $(1/p, 0, 0, 1)$

Perspective Projection

Single point perspective

- Centre of projection (CoP) on y axis

$$X' = P_q \cdot X$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & q & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ qy+1 \end{bmatrix}$$

$$x' = \frac{x}{qy+1}, \quad y' = \frac{y}{qy+1}, \quad z' = \frac{z}{qy+1}$$

- CoP is at $(0, -1/q, 0, 1)$, VP is at $(0, 1/q, 0, 1)$

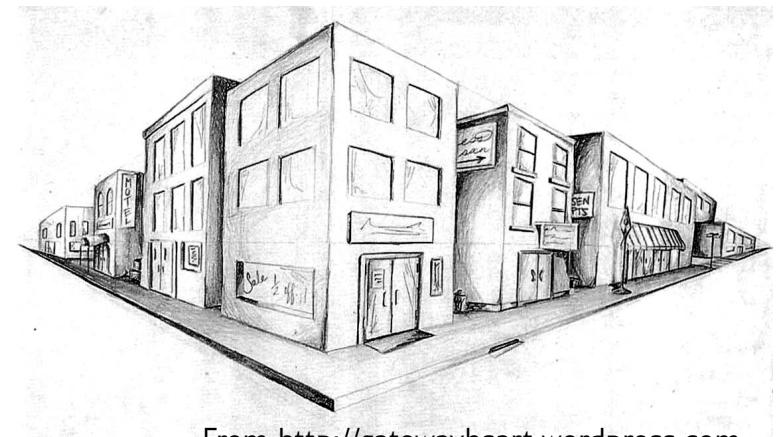
Perspective Projection

Two point perspective

$$P_{pq} = P_p \cdot P_q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & q & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & q & 0 & 1 \end{bmatrix}$$

$$x' = \frac{x}{px+qy+1}, \quad y' = \frac{y}{px+qy+1}, \quad z' = \frac{z}{px+qy+1}$$

- Two vanishing points
- Two CoPs?



From <http://gatewayhsart.wordpress.com>

Perspective Projection

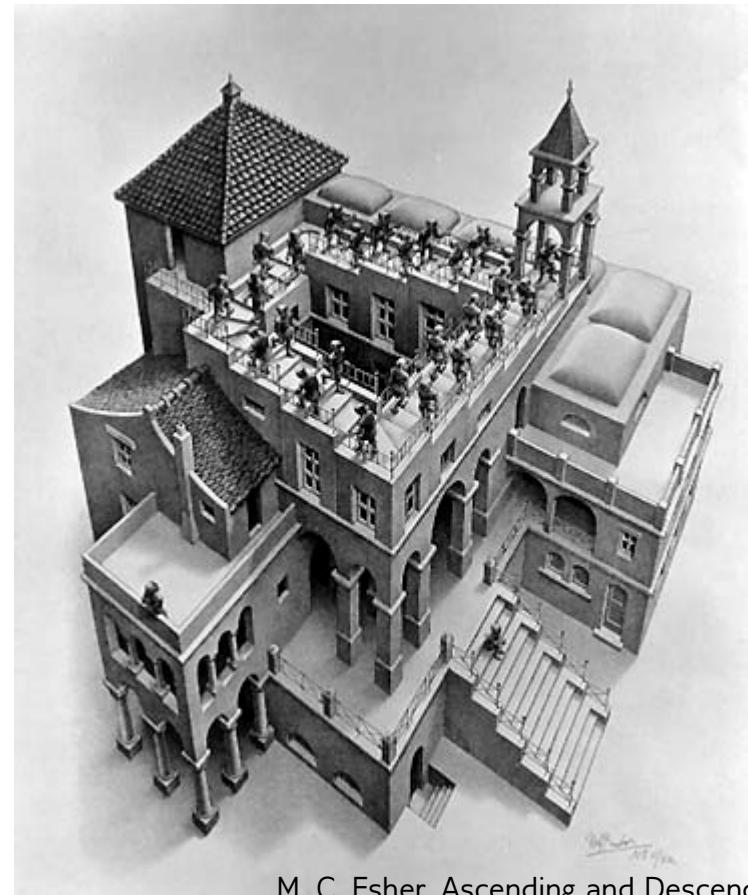
Three point perspective

$$P_{pqr} = P_p \cdot P_q \cdot P_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & q & r & 1 \end{bmatrix}$$

$$x' = \frac{x}{px + qy + rz + 1}, \quad y' = \frac{y}{px + qy + rz + 1}$$

$$z' = \frac{z}{px + qy + rz + 1}$$

- Three vanishing points
- Three CoPs?



M. C. Escher, Ascending and Descending

Perspective Projection

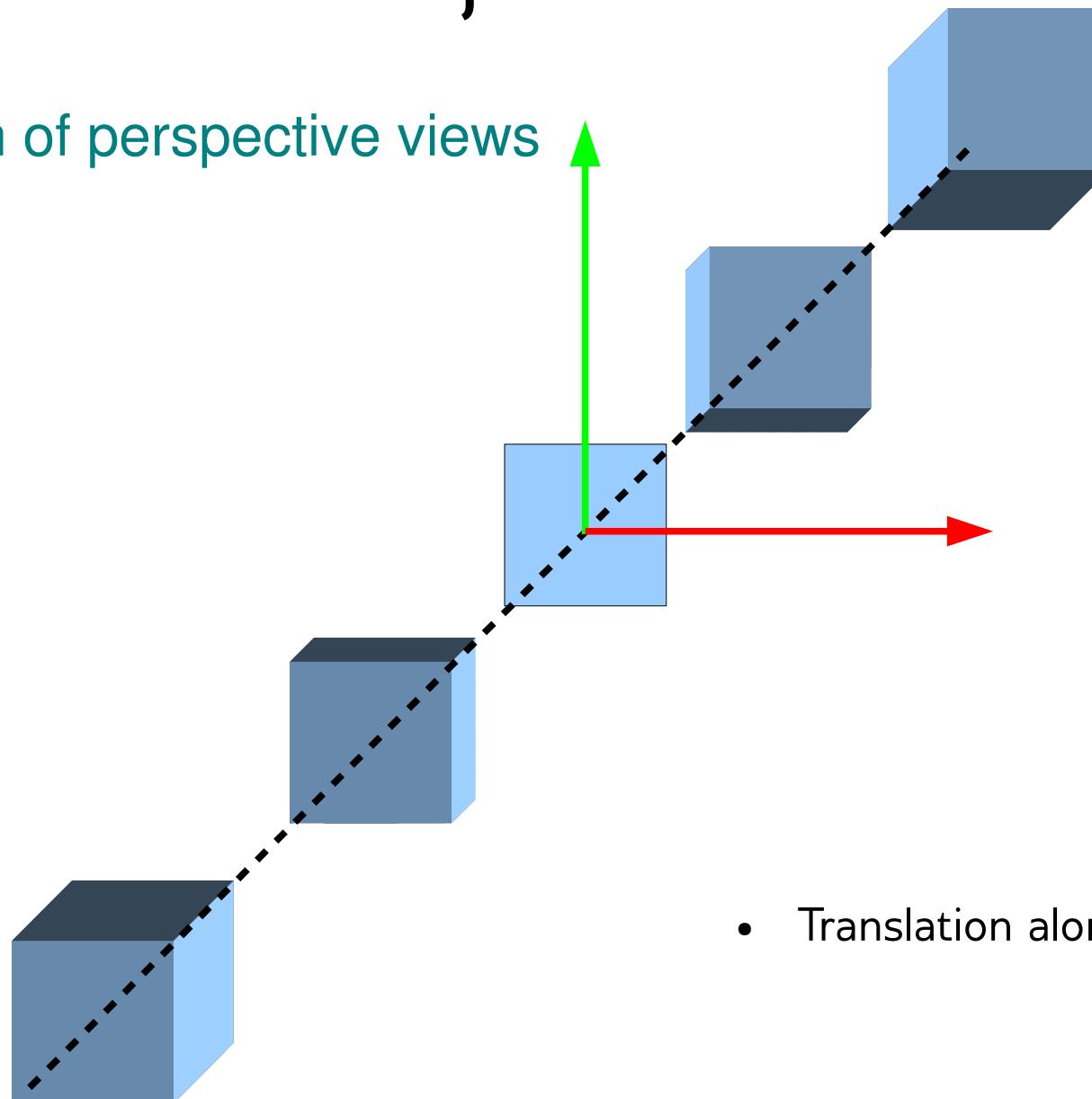
Generation of perspective views

- Transform and then apply single point perspective.
- Let us try to translate, apply a perspective and project to $z=0$

$$T = P_r(z=0) \cdot T(l, m, n) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & rn+1 \end{bmatrix}$$

Perspective Projection

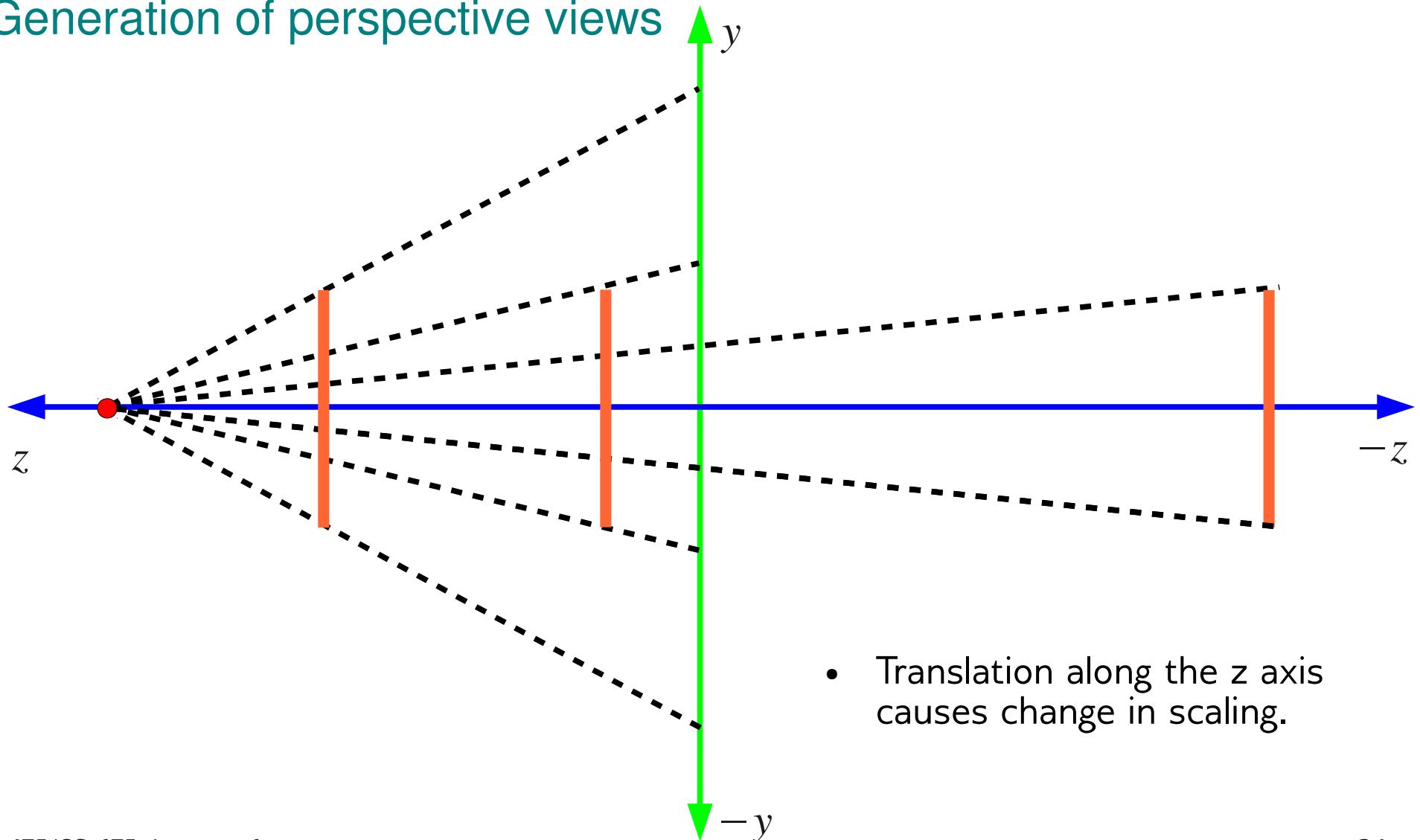
Generation of perspective views



- Translation along $x=y$ line.

Perspective Projection

Generation of perspective views



Perspective Projection

Generation of perspective views

- Rotate about y axis and then apply single point perspective projection.

$$T = P_r(z=0) \cdot R_y(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \phi & 0 & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Perspective Projection

Generation of perspective views

- Rotate about y axis and then apply single point perspective projection.

$$T = P_r(z=0) \cdot R_y(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \phi & 0 & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -r \sin \phi & 0 & r \cos \phi & 1 \end{bmatrix}$$

- We get a two point perspective.

Perspective Projection

Generation of perspective views

- Rotate about y axis, x axis and then apply single point perspective projection.

$$T = P_r(z=0) \cdot R_x(\theta) \cdot R_y(\phi)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \phi & 0 & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Perspective Projection

Generation of perspective views

- Rotate about x axis, y axis and then apply single point perspective projection.

$$T = P_r(z=0) \cdot R_y(\phi), R_x(\theta)$$

$$= \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ \sin \theta \sin \phi & \cos \theta & -\sin \theta \cos \phi & 0 \\ 0 & 0 & 0 & 0 \\ -r \cos \theta \sin \phi & r \sin \theta & r \cos \theta \cos \phi & 1 \end{bmatrix}$$

- We get a three point perspective.

Taxonomy

