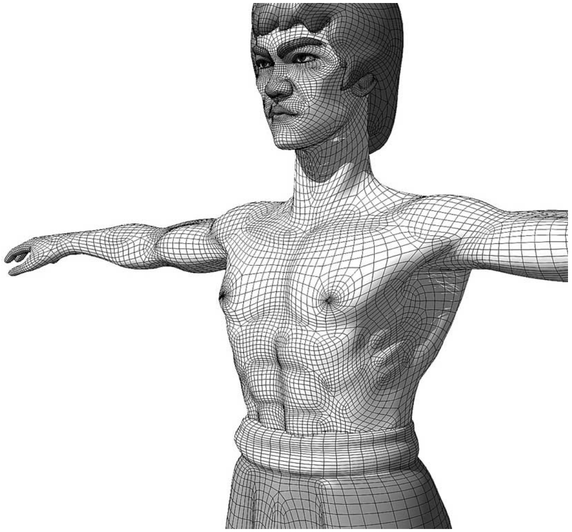
CS475/CS675
Computer Graphics

Modeling Curves: Cubic Splines

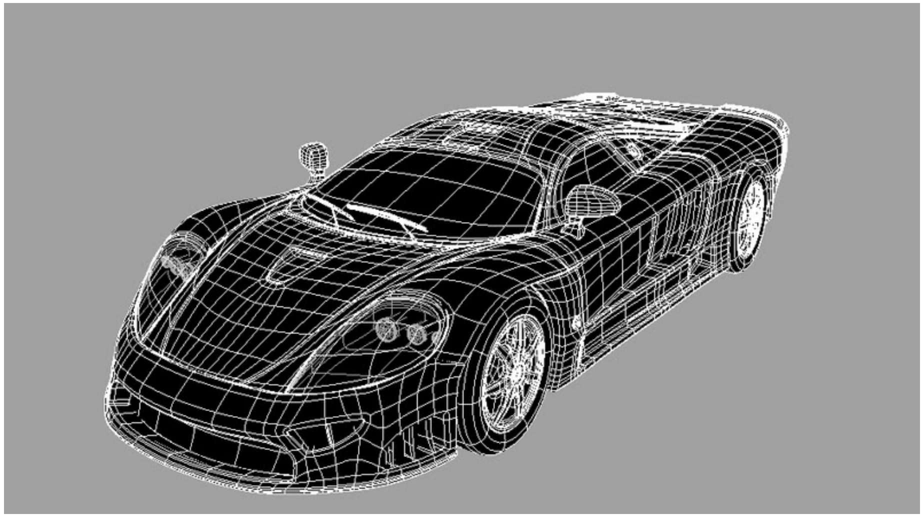
Modelling



http://www.its-ming.com/images/bruce_wireframe.jpg

- Create the virtual world
  - Create objects
  - Create animals, humans and aliens too.

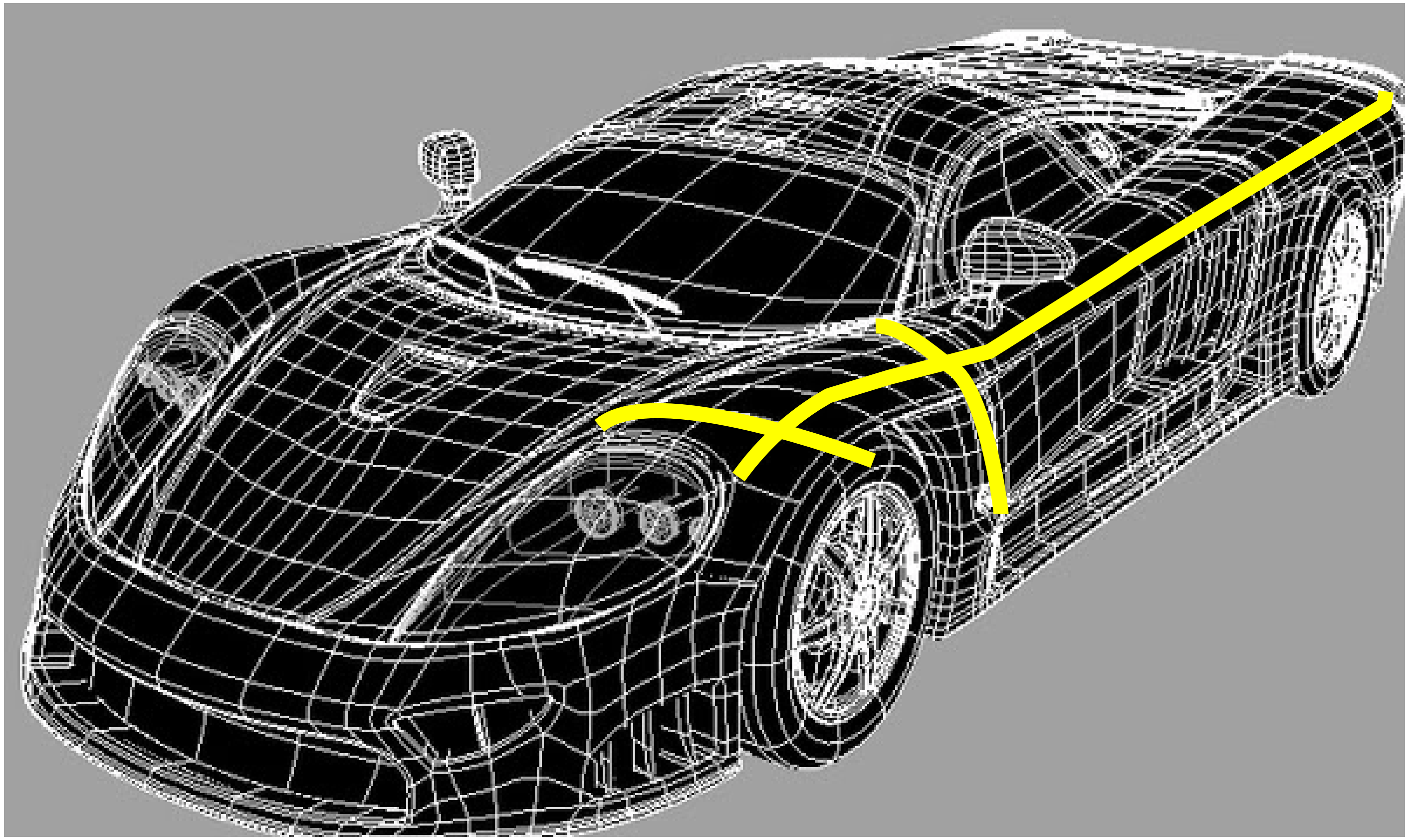
Trinity Animation Inc.



http://www.gametrailers.com/users/druie/gamepad/

Curves

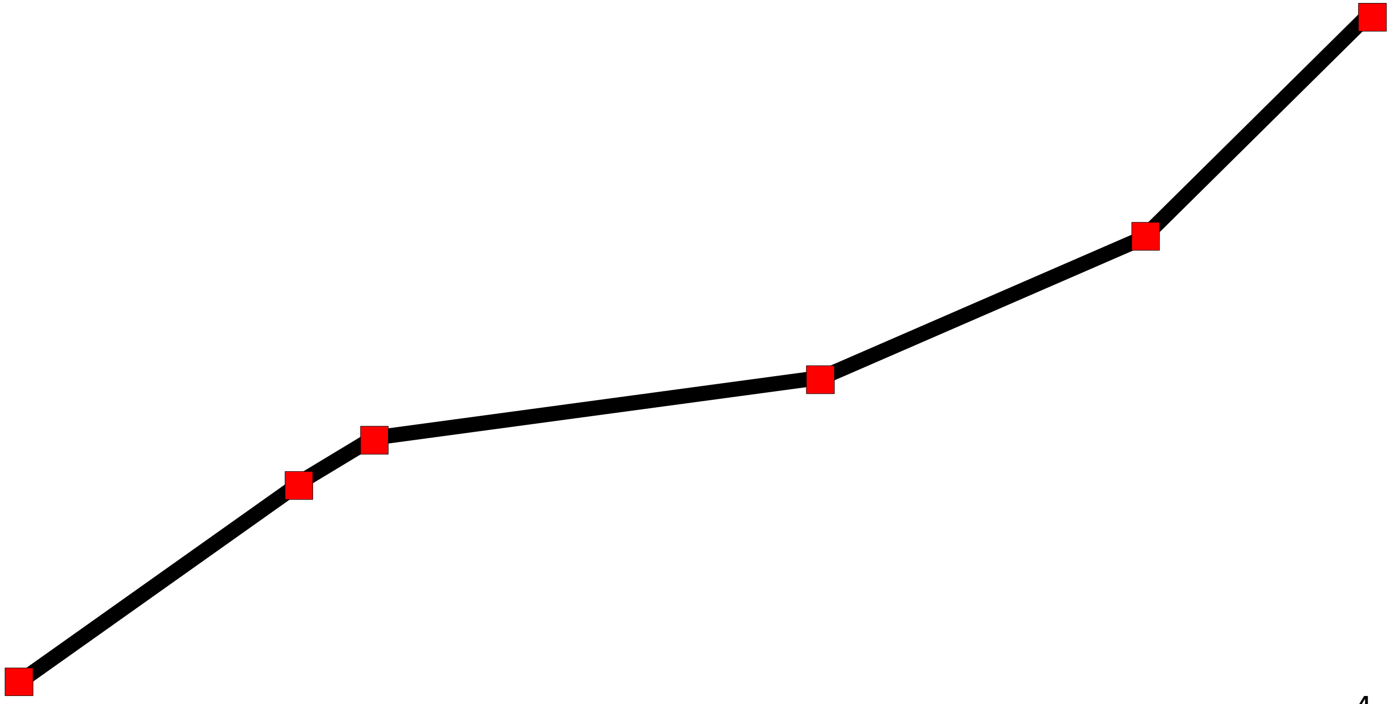
- Curves allow us to design and create complex geometry.



http://www.gametrailers.com/users/druie/gamepad/

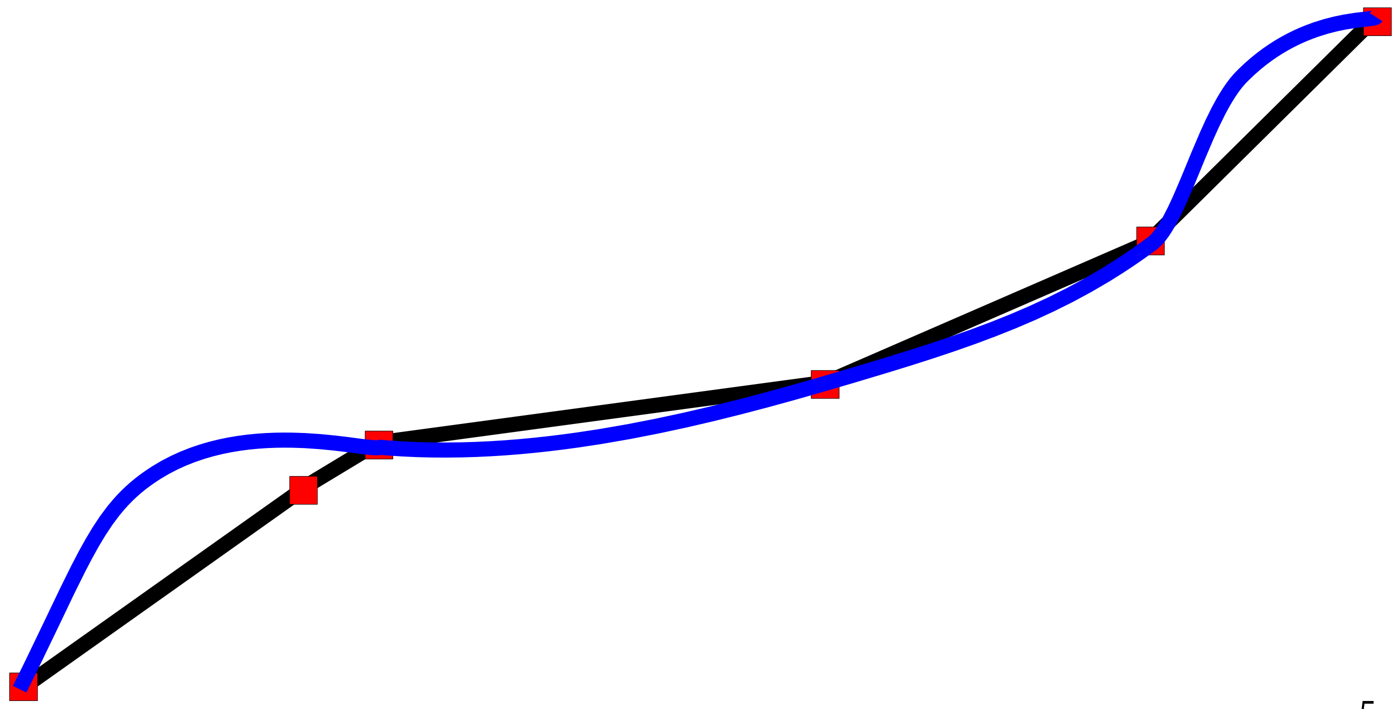
Curves

- Linear Approximation
  - Easy but not good approximation
  - Not smooth



Curves

- Higher Degree Approximations
  - Explicit $y = f(x)$
  - Implicit $f(x, y) = 0$
  - Parametric $x = f_x(t), y = f_y(t)$



Curves

- Explicit Representation
  - $y = f(x)$
  - Function plot over some interval $x \in [a, b]$
  - Simple to compute and plot
  - Simple to check whether a point lies on the curve.
  - Problem with closed and multi-valued curves

Curves

- Implicit Representation
  - $f(x, y) = 0$
  - The 'dependent' variable is not given 'explicitly' in terms of the independent variable
  - Curves defined implicitly as solution of a system of equations.
  - For e.g., $Ax + By + C = 0, x^2 + y^2 - R^2 = 0$
  - Harder to render.
  - Simple to check whether a point lies on the curve.
  - Can represent closed and multi-valued curves.

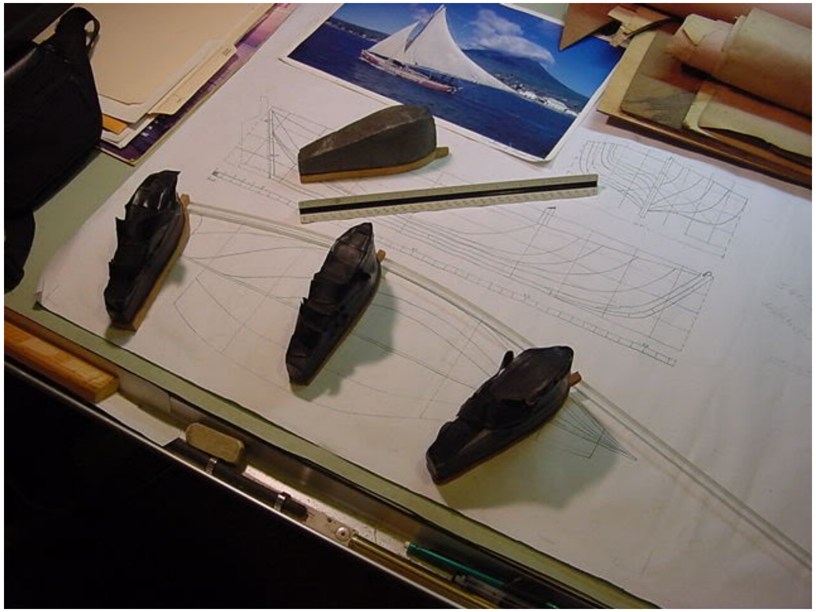
Curves

- Parametric Representation
  - $x = f_x(t), y = f_y(t)$
  - Position on the curve is given in terms of a parameter.
  - For e.g., $x(t) = A(1-t) + Bt, \quad y(t) = A(1-t) + Bt, \quad \text{with } t \in [0,1]$

    $x(t) = R\cos(t), \quad y(t) = R\sin(t), \quad \text{with } t \in [0, 2\pi]$
  - Simple to render.
  - Harder to check whether a point lies on the curve.
  - Can represent closed and multi-valued curves.

Parametric Curves

- Can represent a variety of curves
- Can be used for:
  - Interpolation
  - Approximation
- ***Splines***
  - Cubic, Hermite, Bezier, B-Splines, NURBS
  - Specification, Control, Editing



http://www.woodenboat.com

Cubic Splines

- $P(t) = B_1 + B_2 t + B_3 t^2 + B_4 t^3 = \sum_{i=1}^{4} B_i t^{i-1} \quad \text{with } t_1 \le t \le t_2$
- $x(t) = \sum_{i=1}^{4} B_{ix} t^{i-1} \quad \text{with } t_1 \le t \le t_2$
- $y(t) = \sum_{i=1}^{4} B_{iy} t^{i-1} \quad \text{with } t_1 \le t \le t_2$
- $P'(t) = B_2 + 2\mathrm{B}_3 t + 3\mathrm{B}_4 t^2 = \sum_{i=1}^{4} (i-1) B_i t^{i-2} \quad \text{with } t_1 \le t \le t_2$

Cubic Splines

$$P(t) = B_1 + B_2 t + B_3 t^2 + B_4 t^3 = \sum_{i=1}^{4} B_i t^{i-1} \quad \text{with } t_1 \leq t \leq t_2$$

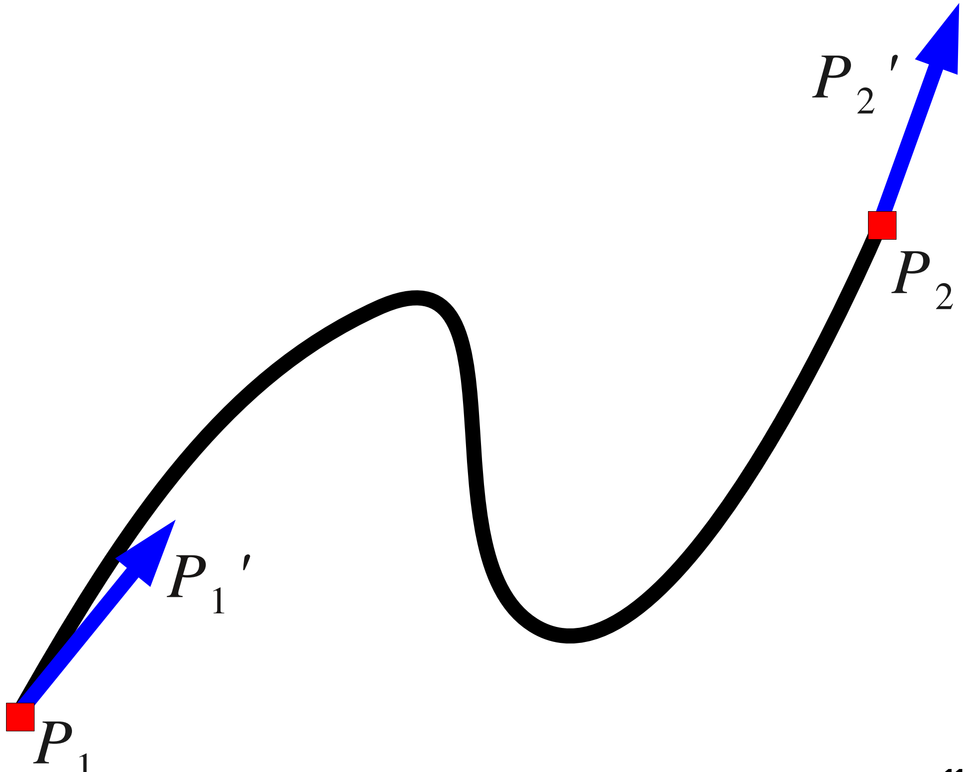
$$P'(t) = B_2 + 2\mathrm{B}_3 t + 3\mathrm{B}_4 t^2 = \sum_{i=1}^{4} (i-1) B_i t^{i-2} \quad \text{with } t_1 \leq t \leq t_2$$

- Given two points and tangent vectors at these two points find a cubic spline that satisfies these end conditions.
- Assuming

$$t_1 = 0$$

$$P(0) = P_{1,} P(t_2) = P_2$$

$$P'(0) = P_1', P'(t_2) = P_2'$$



Cubic Splines

$$P(0) = B_1 = P_1$$

$$P'(0) = B_2 = P_1'$$

$$P(t_2) = \sum_{i=1}^{4} B_i t_2^{i-1} = P_2$$

$$P'(t_2) = \sum_{i=1}^{4} (i-1) B_i t_2^{i-2} = P_2'$$

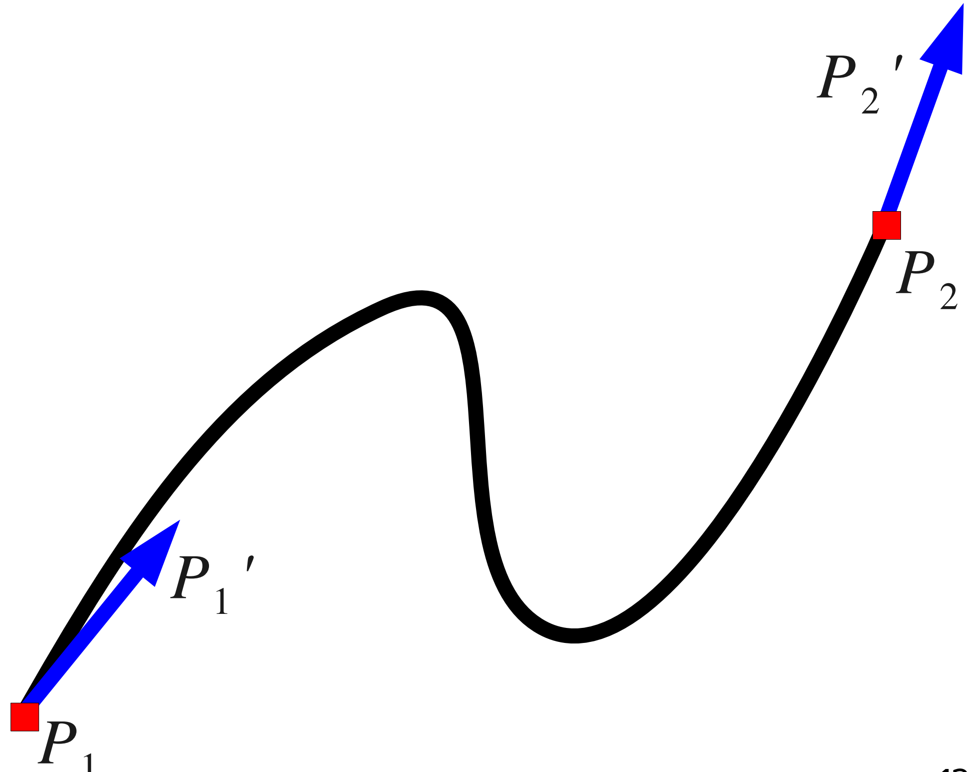
- On solving we get:

$$B_1 = P_1$$

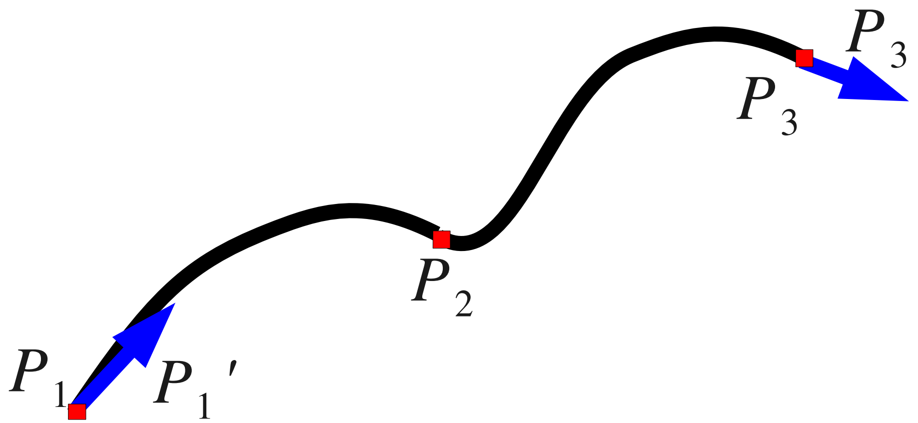
$$B_2 = P_1'$$

$$B_3 = \frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}$$

$$B_4 = \frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}$$



Cubic Splines



- Interpolate three points using cubic splines.
- We will do a piecewise polynomial interpolation with some constraints at the join to ensure "smoothness."
- But what is this notion of smoothness?

Continuity

- **Geometric**
- G^0 : Curves are joined
- G^1 : First derivatives are proportional. Tangents have same directions but not necessarily the same magnitude
- G^2 : First and second derivatives are proportional across the point of joining.

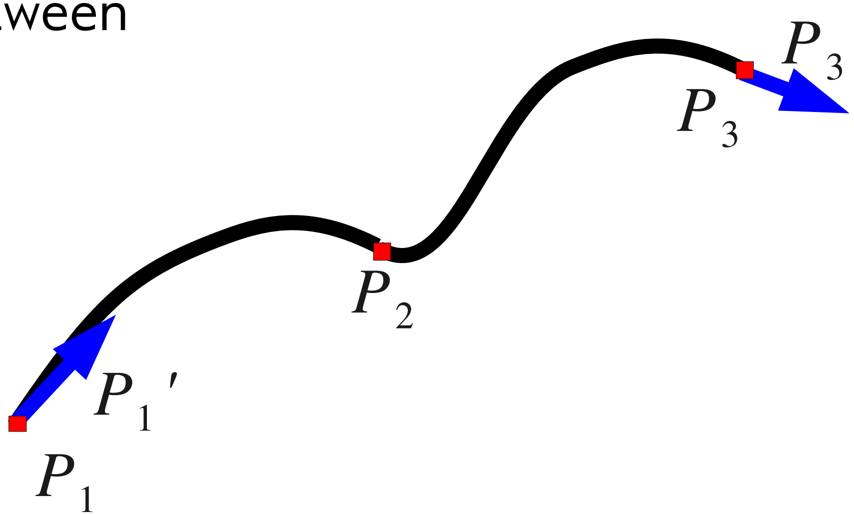
- **Parametric**
- C^0 : Curves are joined
- C^1 : First derivatives are equal. Tangents have same directions and the same magnitude
- C^2 : First and second derivatives are equal across the point of joining.

Parametric continuity of order n implies Geometric continuity, but **not** vice-versa.

Cubic Splines

- So we enforce C^2 continuity at the in-between point.

$$P''(t) = \sum_{i=1}^{4} (i-2)(i-1) B_i t^{i-3}$$



- For the first segment

$$P_1''(t_2) = 6B_4 t_2 + 2B_{3,} \text{ i.e., at } t = t_2 \text{ with } 0 \le t \le t_2$$

- For the second segment

$$P_2''(0) = 2B_{3,} \text{ i.e., at } t = 0 \text{ with } 0 \le t \le t_3$$

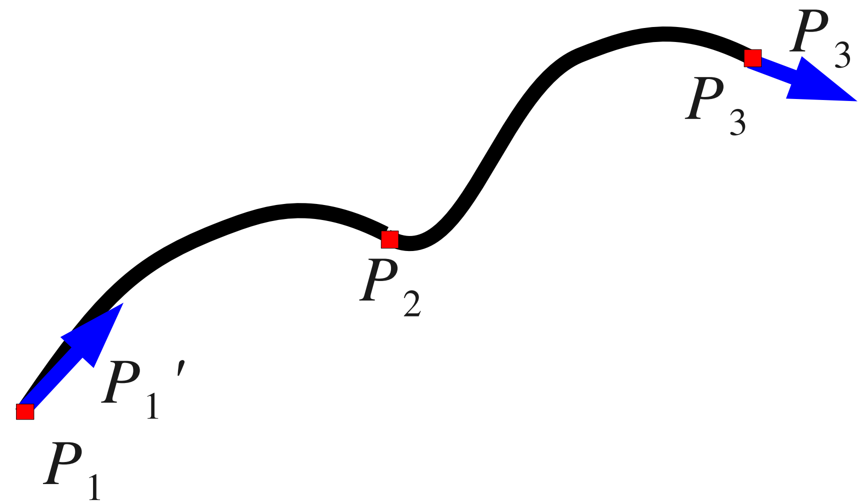
- So for C^2 continuity we have:

Please note the two B_3's in the equation are from different spans.

$$6B_4 t_2 + 2B_3 = 2B_3$$

Cubic Splines

- Substituting and solving we get:



$$t_3 P_1' + 2(t_3 + t_2) P_2' + t_2 P_3'$$

$$= \frac{3}{t_2 t_3}\left(t_2^2 (P_3 - P_2) + t_3^2 (P_2 - P_1)\right)$$

$$\Rightarrow \begin{bmatrix} t_3 & 2(t_3 + t_2) & t_2 \end{bmatrix} \begin{bmatrix} P_1' \\ P_2' \\ P_3' \end{bmatrix}$$

$$= \frac{3}{t_2 t_3}\left(t_2^2 (P_3 - P_2) + t_3^2 (P_2 - P_1)\right)$$

Cubic Splines

- Similarly for k^{th} and $k+1^{th}$ segments with $1 \le k \le n-2$

$$\begin{bmatrix} t_{k+2} & 2(t_{k+2}+t_{k+1}) & t_{k+1} \end{bmatrix} \begin{bmatrix} P_k' \\ P_{k+1}' \\ P_{k+2}' \end{bmatrix}$$

$$= \frac{3}{t_{k+1} t_{k+2}} \left(t_{k+1}^2 (P_{k+2} - P_{k+1}) + t_{k+2}^2 (P_{k+1} - P_k) \right)$$

- If we stack up the equations for all the tangent vectors we get a set of n-2 equations.

Cubic Splines

$$\begin{bmatrix} t_3 & 2(t_2+t_3) & t_2 & 0 & \cdots \\ 0 & t_4 & 2(t_3+t_4) & t_3 & \\ \vdots & & \ddots & & \vdots \\ & \cdots & t_n & 2(t_{n-1}+t_n) & t_{n-1} \end{bmatrix} \begin{bmatrix} P_1' \\ P_2' \\ \vdots \\ P_n' \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{t_2 t_3}(t_2^2(P_3 - P_2) + t_3^2(P_2 - P_1)) \\ \frac{3}{t_3 t_4}(t_3^2(P_4 - P_3) + t_4^2(P_3 - P_2)) \\ \vdots \\ \frac{3}{t_{n-1} t_n}(t_{n-1}^2(P_n - P_{n-1}) + t_n^2(P_{n-1} - P_n - 2)) \end{bmatrix}$$

Cubic Splines

$$
\begin{bmatrix}
1 & 0 & \dots & & \\
t_3 & 2(t_2+t_3) & t_2 & 0 & \dots \\
0 & t_4 & 2(t_3+t_4) & t_3 & \\
\vdots & & \ddots & & \vdots \\
 & \dots & t_n & 2(t_{n-1}+t_n) & t_{n-1} \\
 & & \dots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
P_1' \\ P_2' \\ \vdots \\ \vdots \\ P_{n-1}' \\ P_n'
\end{bmatrix}
=
\begin{bmatrix}
P_1' \\
\frac{3}{t_2 t_3}\left(t_2^2(P_3-P_2)+t_3^2(P_2-P_1)\right) \\
\frac{3}{t_3 t_4}\left(t_3^2(P_4-P_3)+t_4^2(P_3-P_2)\right) \\
\vdots \\
\frac{3}{t_{n-1} t_n}\left(t_{n-1}^2(P_n-P_{n-1})+t_n^2(P_{n-1}-P_n-2)\right) \\
P_n'
\end{bmatrix}
$$

Cubic Splines

- Solving for $B_{1,} B_{2,} B_3$ and B_4

$$B_{1k} = P_k$$

$$B_{2k} = P_k{}'$$

$$B_{3k} = \frac{3(P_{k+1} - P_k)}{t_{k+1}^2} - \frac{2P_k{}'}{t_{k+1}} - \frac{2P_{k+1}{}'}{t_{k+1}}$$

$$B_{4k} = \frac{2(P_k - P_{k+1})}{t_{k+1}^3} + \frac{P_k{}'}{t_{k+1}^2} + \frac{P_{k+1}{}'}{t_{k+1}^2}$$

Cubic Splines

or it can be rewritten as:

$$\begin{bmatrix} B_{1\mathrm{k}} \\ B_{2\mathrm{k}} \\ B_{3\mathrm{k}} \\ B_{4\mathrm{k}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/t_{k+1}^2 & -2/t_{k+1} & 3/t_{k+1}^2 & -1/t_{k+1} \\ 2/t_{k+1}^3 & 1/t_{k+1}^2 & -2/t_{k+1}^3 & 1/t_{k+1}^2 \end{bmatrix} \begin{bmatrix} P_k \\ P_k' \\ P_{k+1} \\ P_{k+1}' \end{bmatrix}$$

Cubic Splines

Now, for a curve segment of the cubic spline

$$P_k(t) = \sum_{i=1}^{4} B_{ik} t^{i-1} \quad \text{with } 0 \le t \le t_{k+1} \text{ and } 1 \le k \le n-1$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} B_{1k} \\ B_{2k} \\ B_{3k} \\ B_{4k} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/t_{k+1}^2 & -2/t_{k+1} & 3/t_{k+1}^2 & -1/t_{k+1} \\ 2/t_{k+1}^3 & 1/t_{k+1}^2 & -2/t_{k+1}^3 & 1/t_{k+1}^2 \end{bmatrix} \begin{bmatrix} P_k \\ P_k{}' \\ P_{k+1} \\ P_{k+1}{}' \end{bmatrix}$$

Cubic Splines

Substitute $\dfrac{t}{t_{k+1}}$ as τ

$$P_k(t) = \begin{bmatrix} F_1(\tau) & F_2(\tau) & F_3(\tau) & F_4(\tau) \end{bmatrix} \begin{bmatrix} P_k \\ P_{k+1} \\ P_k' \\ P_{k+1}' \end{bmatrix}$$

with $0 \le \tau \le 1$ and $1 \le k \le n-1$

where

$$F_1(\tau) = 2\tau^3 - 3\tau^2 + 1$$

$$F_2(\tau) = -2\tau^3 + 3\tau^2$$

$$F_3(\tau) = \tau(\tau^2 - 2\tau + 1)t_{k+1}$$

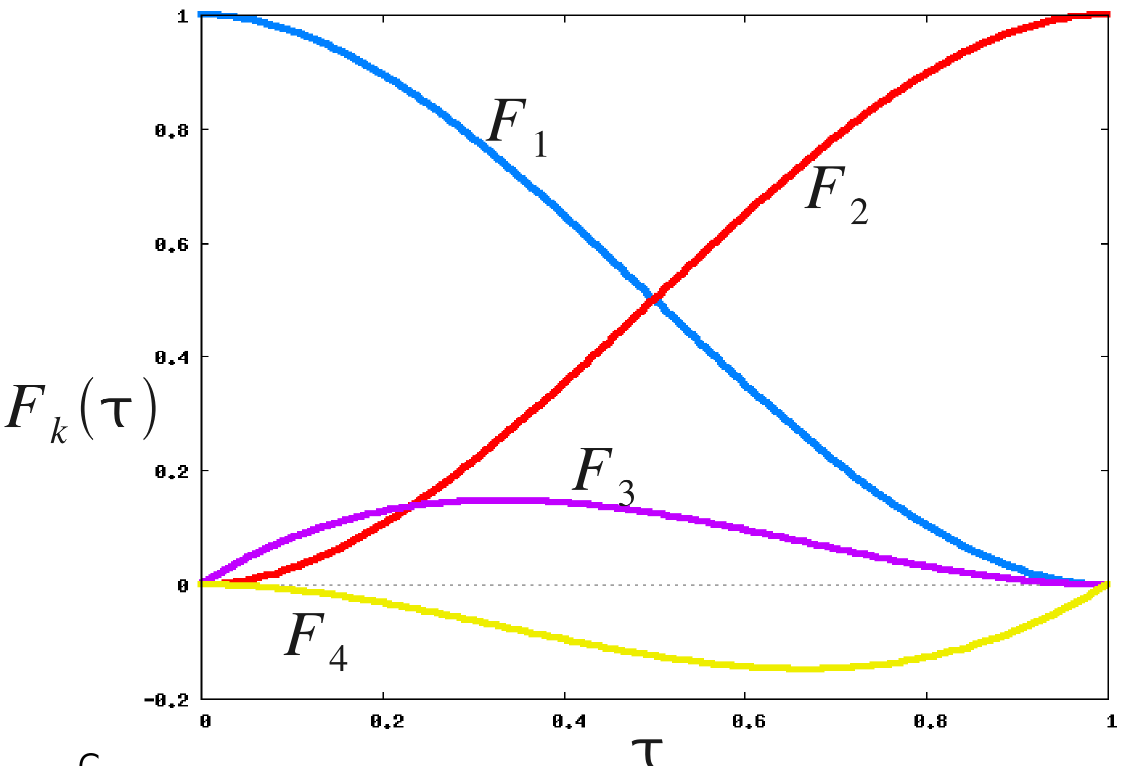
$$F_4(\tau) = \tau(\tau^2 - \tau)t_{k+1}$$

These are called the **blending** or **weighting** functions

Cubic Splines

$$P_k(\tau) = \boldsymbol{F} . \boldsymbol{B}$$

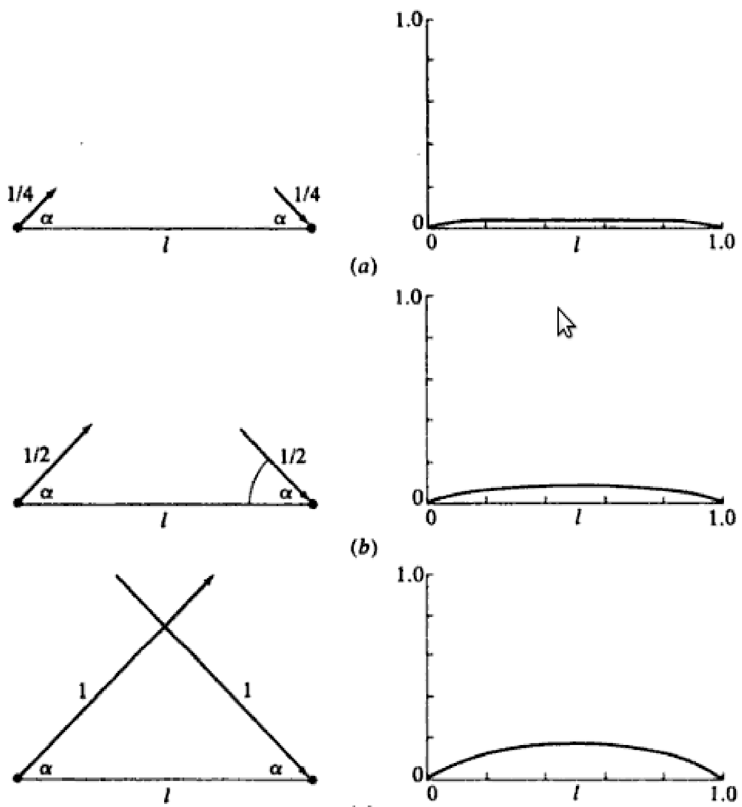
Where $\boldsymbol{F}$ is the blending function matrix and $\boldsymbol{B}$ is the geometry matrix.



- $F_1(0)=1, F_2(0)=F_3(0)=F_4(0)=0$
  Curve passes through P_1
- $F_1(1)=0, F_2(1)=1, F_3(1)=F_4(1)=0$
  Curve passes through P_2
- $F_2(\tau)=1-F_1(\tau), F_4(\tau)=-F_3(1-\tau)$
- Influence of endpoints constraints vs tangent contraints

Cubic Splines

- Piecewise cubic splines are specified using: position vectors of end points, tangent vectors of end points and parameter value t_k.
- Effect of tangent magnitude



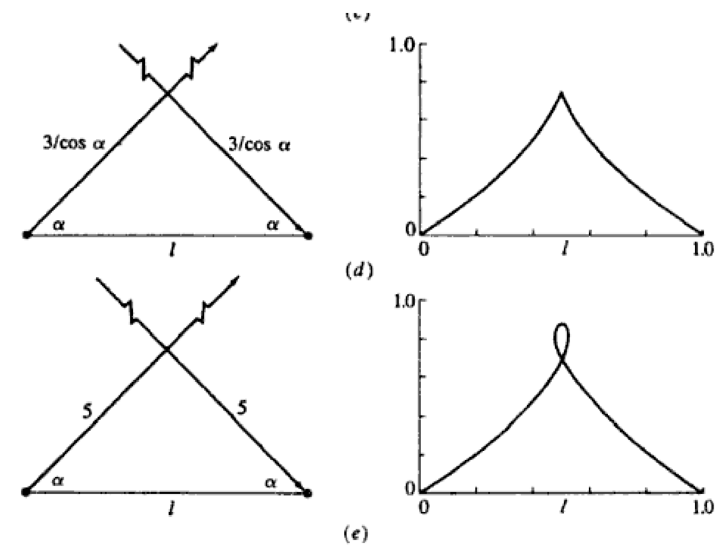


Figure 5–10 Effect of tangent vector magnitude on cubic spline segment shape, $\alpha = \pi/4$. (a) 1/4; (b) 1/2; (c) 1; (d) $3/\cos\alpha$; (e) 3/2.

Mathematical Elements for Computer Graphics, 4ed.,
D. F. Rogers and J. A. Adams

Cubic Splines

- Piecewise cubic splines are specified using: position vectors of end points, tangent vectors of end points and parameter value t_k.
- Effect of parameterization

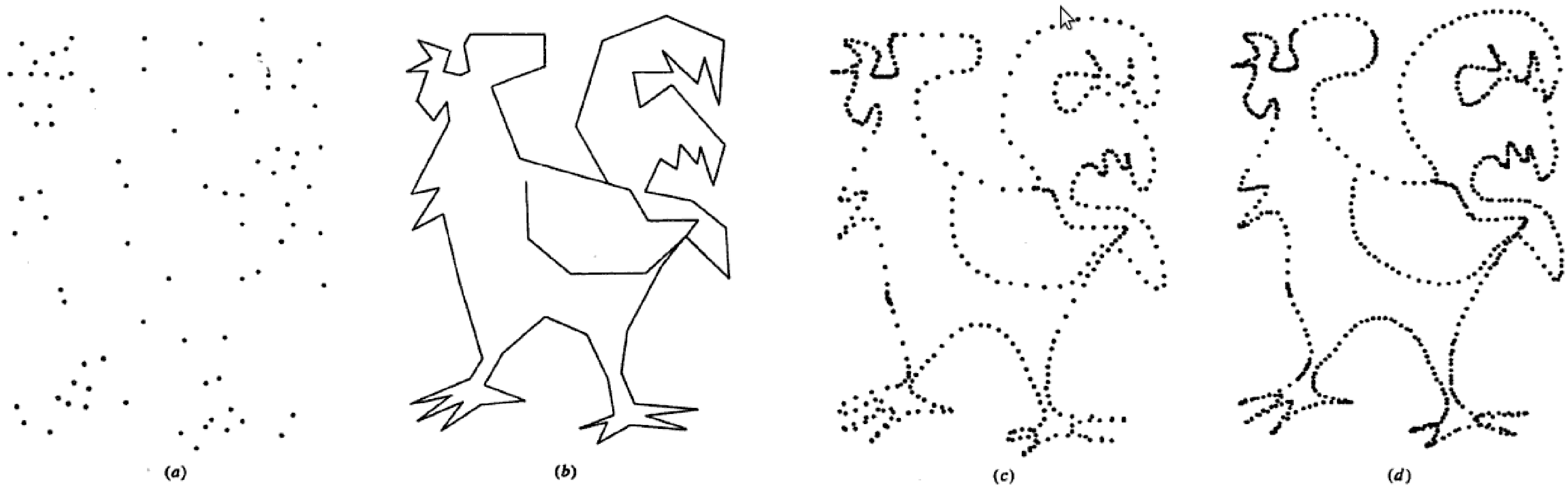


Figure 5–11 Comparison of cubic spline approximations. (a) Data; (b) connected with straight lines; (c) normalized approximation for t_k's; (d) chord length approximation for t_k's.

- Normalized parametrization all t_k =1.
- Chord length parametrization

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Cubic Splines

- Normalized Cubic Splines : Hermite Splines

$$F_1(\tau)=2\tau^3-3\tau^2+1$$
$$F_2(\tau)=-2\tau^3+3\tau^2$$
$$F_3(\tau)=\tau(\tau^2-2\tau+1)$$
$$F_4(\tau)=\tau(\tau^2-\tau)$$

or

$$\begin{bmatrix}F_1(\tau) & F_2(\tau) & F_3(\tau) & F_4(\tau)\end{bmatrix} = \begin{bmatrix}\tau^3 & \tau^2 & \tau & 1\end{bmatrix}\begin{bmatrix}2 & -2 & 1 & 1\\ -3 & 3 & -2 & -1\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}$$

- The tridiagonal system becomes:

$$\begin{bmatrix}1 & 0 & \cdots & & \\ 1 & 4 & 1 & & \cdots \\ 0 & 1 & 4 & 1 & \\ \vdots & & \ddots & & \vdots \\ & \cdots & 1 & 4 & 1 \\ & & \cdots & 0 & 1\end{bmatrix}\begin{bmatrix}P_1' \\ P_2' \\ \vdots \\ \vdots \\ P_{n-1}' \\ P_n'\end{bmatrix} = \begin{bmatrix}P_1' \\ 3((P_3-P_2)+(P_2-P_1)) \\ 3((P_4-P_3)+(P_3-P_2)) \\ \vdots \\ 3((P_n-P_{n-1})+(P_{n-1}-P_n-2)) \\ P_n'\end{bmatrix}$$

Cubic Splines

- Various end contiditions for Cubic Splines
  - Clamped: $P_1'(0)=P_1'$ and $P_n'(t_n)=P_n'$ are known
  - Relaxed/Natural: $P_1''(0) = P_n''(t_n) = 0$
  - Cyclic: $P_1'(0)=P_n'(t_n)$ and $P_1''(0)=P_n''(t_n)$
  - Anti-cyclic: $P_1'(0)=-P_n'(t_n)$ and $P_1''(0)=-P_n''(t_n)$