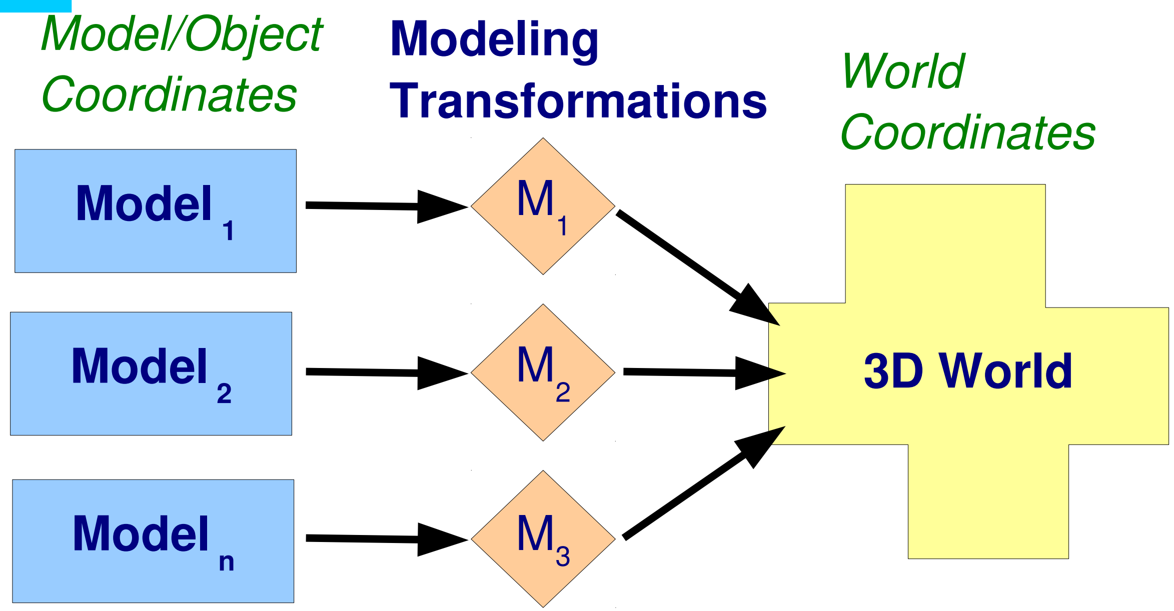
CS475/CS675
Computer Graphics

2D Transformations

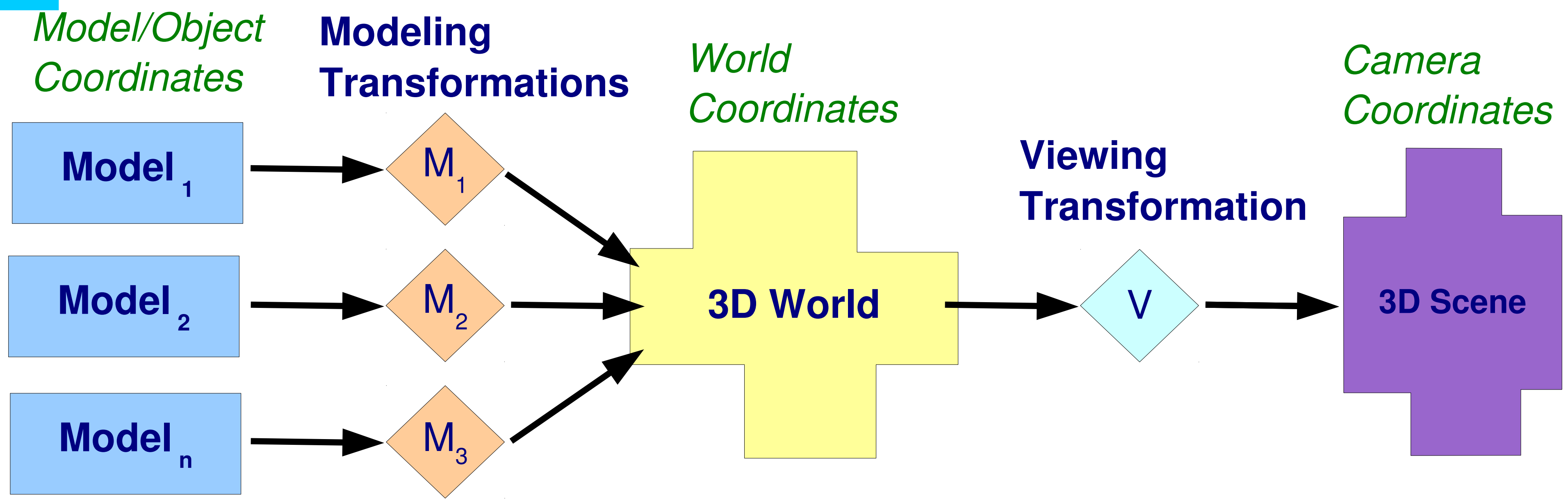
Transformations

- What is a transformation?
  - $P' = T\,P$
- Why is it useful?
  - Modelling
    - Specify object position, orientation, size in the world.
    - Create multiple instances of template shapes.
    - Specify hierarchical models.
  - Viewing
    - Makes viewing window and device independent
    - Synthetic Camera Model

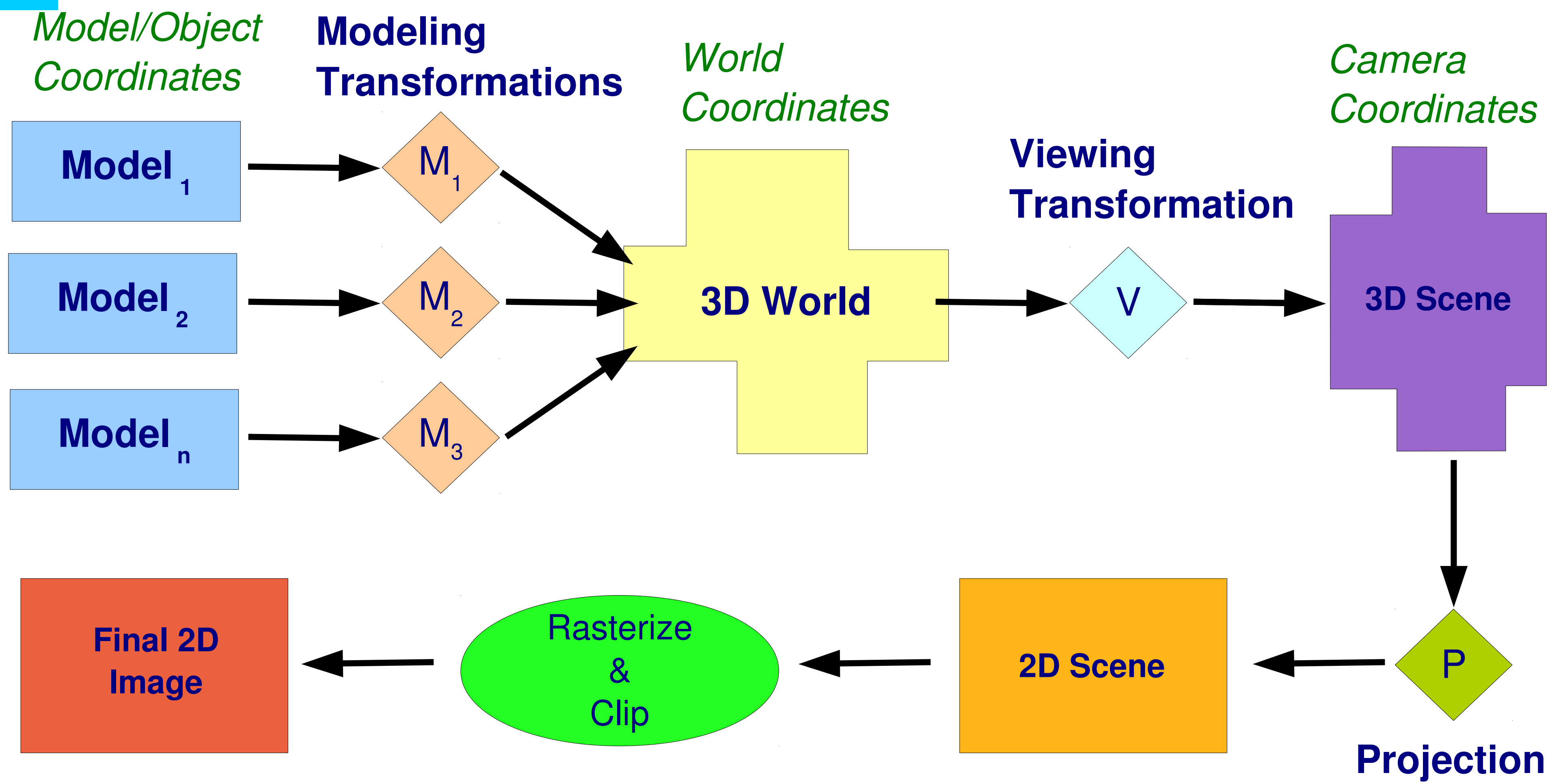
The Modeling-Viewing pipeline



The Modeling-Viewing pipeline

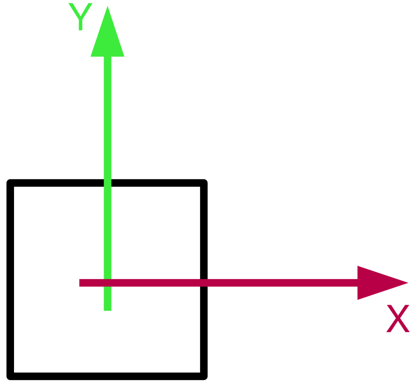


The Modeling-Viewing pipeline



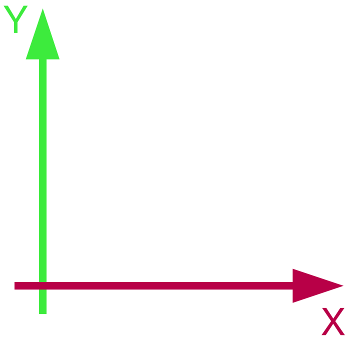
2D Transformations - Scaling

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad S = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$



$$P' = S\,P$$

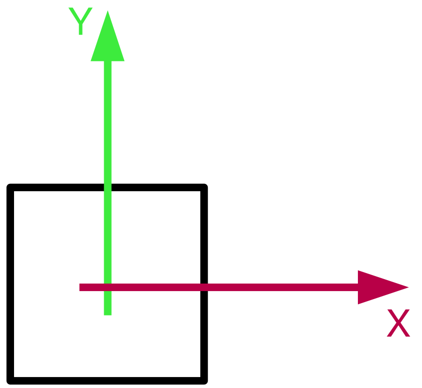
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$$x' = s_x \,.\, x$$
$$y' = s_y \,.\, y$$

2D Transformations - Scaling

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad S = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

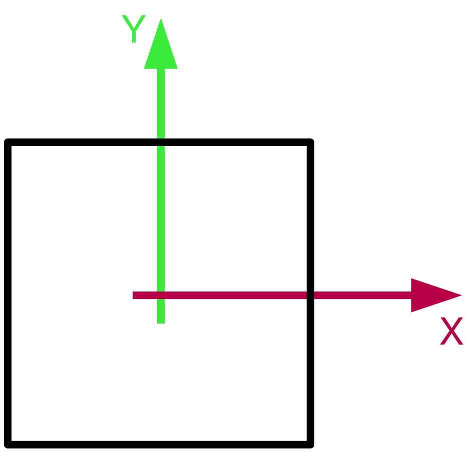


$$P' = S.P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = s_x . x$$
$$y' = s_y . y$$

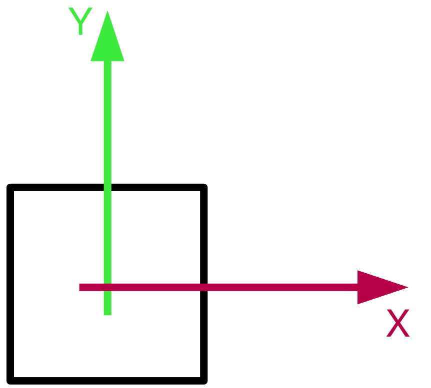
$$s_x = s_y$$



Uniform or Isotropic Scaling

2D Transformations - Scaling

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad S = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

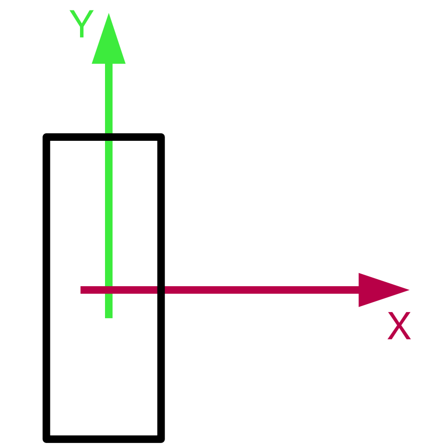


$$P' = S . P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = s_x . x$$
$$y' = s_y . y$$

$$s_x \neq s_y$$



Nonuniform or Anisotropic Scaling

2D Transformations - Rotation

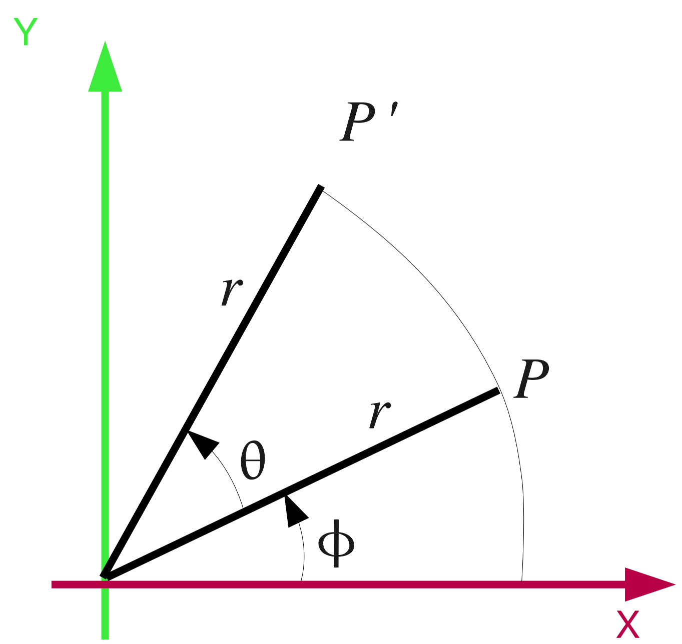
$$P=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}r\cos(\phi)\\r\sin(\phi)\end{bmatrix}$$

$$P'=\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}r\cos(\phi+\theta)\\r\sin(\phi+\theta)\end{bmatrix}=\begin{bmatrix}r\cos(\phi)\cos(\theta)-r\sin(\phi)\sin(\theta)\\r\cos(\phi)\sin(\theta)+r\sin(\phi)\cos(\theta)\end{bmatrix}$$

$$P'=R.P$$

$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}\cos(\theta)&-\sin(\theta)\\\sin(\theta)&\cos(\theta)\end{bmatrix}.\begin{bmatrix}x\\y\end{bmatrix}$$

$$R=R_\theta=\begin{bmatrix}\cos(\theta)&-\sin(\theta)\\\sin(\theta)&\cos(\theta)\end{bmatrix}$$



2D Transformations

$$P=\begin{bmatrix} x \\ y \end{bmatrix} \qquad P'=\begin{bmatrix} x' \\ y' \end{bmatrix} \qquad T=\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$P'=T\,P \quad \Rightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} a & c \\ b & d \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix}$$

$$x'=ax+cy$$
$$y'=bx+dy$$

For $a=1, d=1$ and $b=0, c=0$

$i.e.,\ T=I$ (Identity Transformation)

$$x'=x$$
$$y'=y$$

2D Transformations

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$P' = T\,P \quad \Rightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = ax + cy$$
$$y' = bx + dy$$

For $b = 0, c = 0$

$i.e.,\quad T = S$ (Scaling Transformation)

$$x' = a.x$$
$$y' = d.y$$

2D Transformations

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$P' = T\,P \quad \Rightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = ax + cy$$
$$y' = bx + dy$$

For $a = -1, d = 1$ and $b = 0, c = 0$

$i.e.,\ \ T = Rf$ (Reflection Transformation)

$$x' = -x$$
$$y' = y$$

Reflection about the line $x = 0$

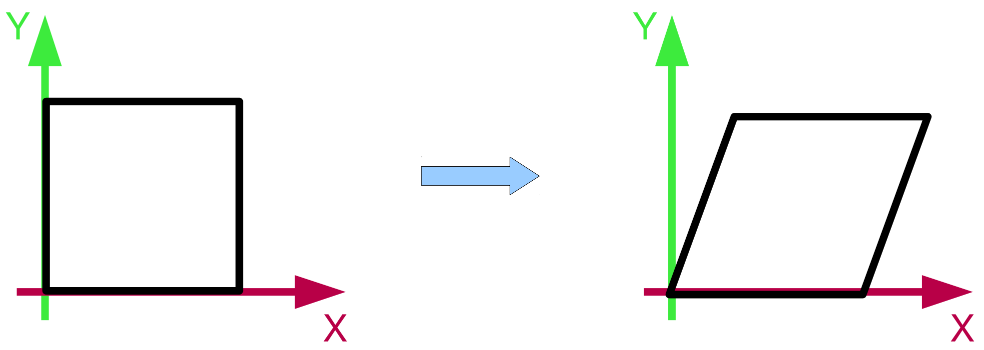
2D Transformations - Shear

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad T = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$$

$$P' = T\,P \quad \Rightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = x + cy$$
$$y' = y$$

Shearing in X



2D Transformations - Shear

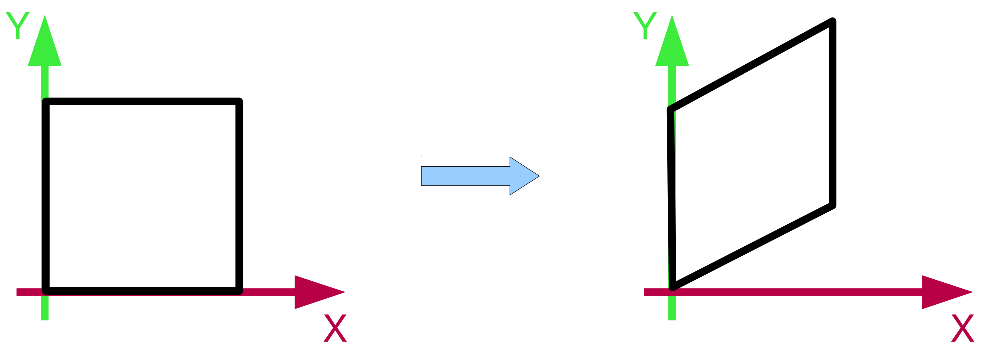
$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

$$P' = T\,P \quad \Rightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = x$$

$$y' = bx + y$$

Shearing in Y



2D Transformations - Translation

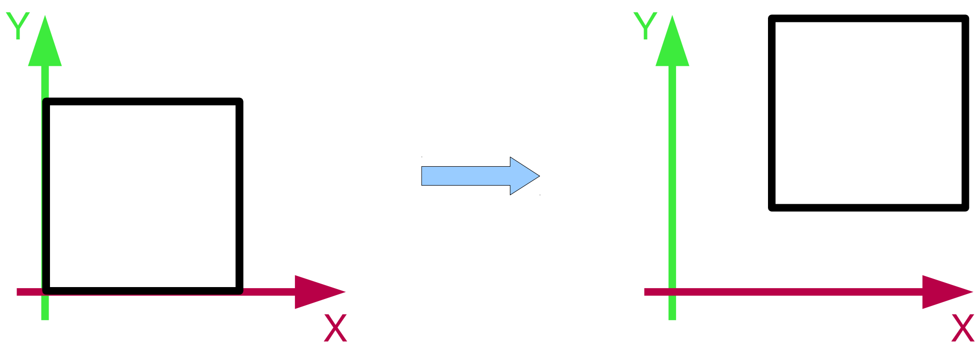
$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$P' = T + P \quad \Rightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$$

Different from other Transformations!

Cannot represent it as a matrix multiplication.

$$x' = t_x + x$$
$$y' = t_y + y$$



Homogenous Coordinates

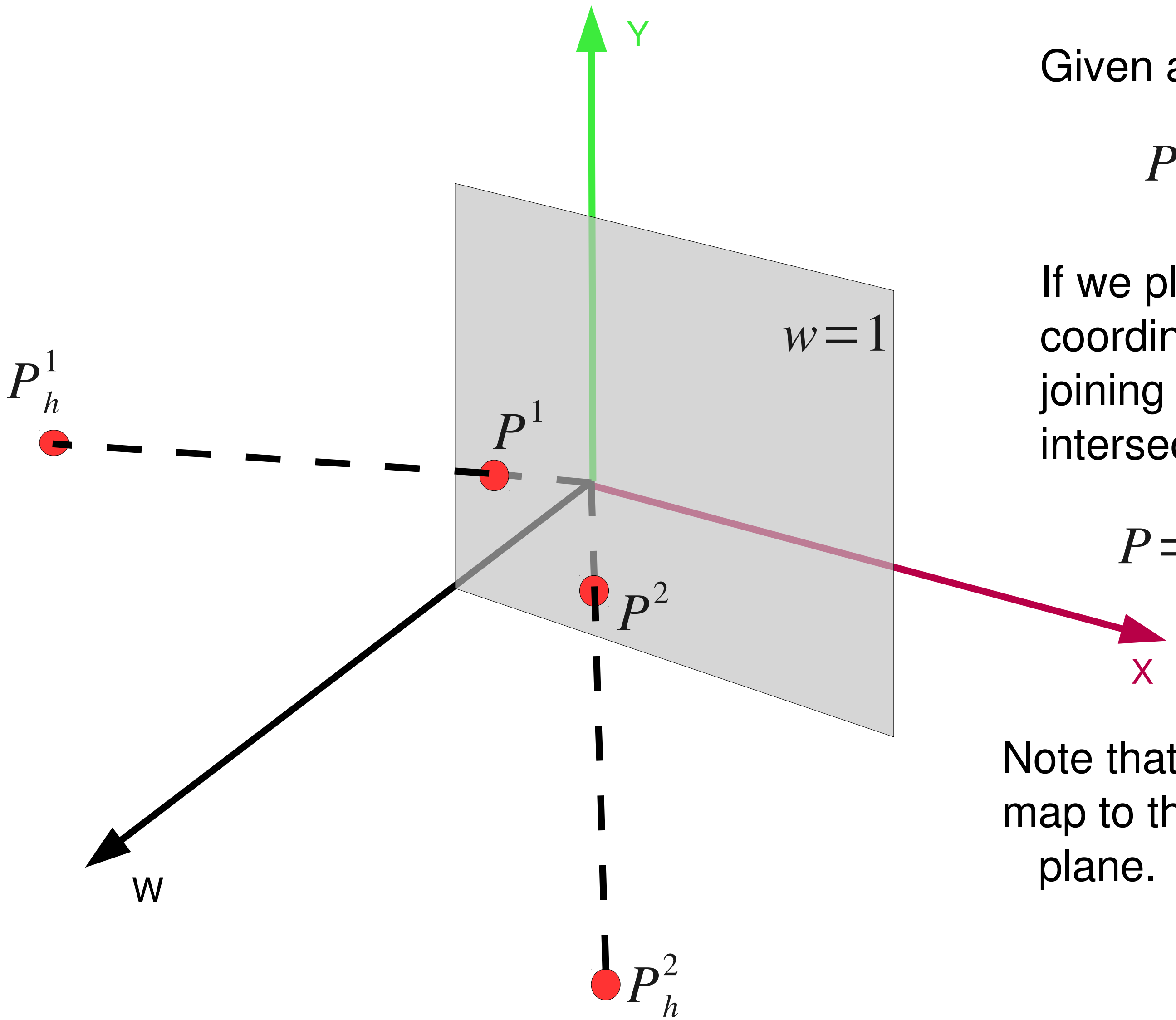
$P = \begin{bmatrix} x \\ y \end{bmatrix}$ in homogenous coordinates becomes $P_h = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

A general point in homogenous coordinates can be mapped back to usual non-homogenous coordinates as follows by dividing with the homogenous dimension.

$$P_h = \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} \sim P = \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore, the homogenous points $\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 12 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix},$ all respresent the same 2D point $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Homogenous Coordinates



Given a homogenous point:

$$P_h = \begin{bmatrix} x & y & w \end{bmatrix}^T$$

If we plot the point in the XYH coordinate system, then line joining the point with the origin intersects the $w=1$ plane at:

$$P = \begin{bmatrix} x/w & y/w & 1 \end{bmatrix}^T$$

Note that all points on the line will map to the same point on the $w=1$ plane.

2D Transformations

The general 2D Transformation matrix now becomes 3x3: $\begin{bmatrix} a & c & l \\ b & d & m \\ 0 & 0 & 1 \end{bmatrix}$

i.e.,

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & c & l \\ b & d & m \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = ax + cy + l$$

$$y' = bx + dy + m$$

$$w = 1$$

2D Transformations

So we see that for translating a point we use the matrix: $\begin{bmatrix} 1 & 0 & l \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix}$

Therefore the new transformed points become:

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & l \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = x + l$$
$$y' = y + m$$

Scaling/Rotation/Shear continue to work as before using the matrix:

$$\begin{bmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Concatenating 2D Transformations

$$T_1 = \begin{bmatrix} 1 & 0 & l_1 \\ 0 & 1 & m_1 \\ 0 & 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 1 & 0 & l_2 \\ 0 & 1 & m_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T_1 . P = \begin{bmatrix} 1 & 0 & l_1 \\ 0 & 1 & m_1 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad P'' = \begin{bmatrix} x'' \\ y'' \\ w'' \end{bmatrix} = T_2 . P' = \begin{bmatrix} 1 & 0 & l_2 \\ 0 & 1 & m_2 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$$

Successive Translations are ***additive***.

$$P'' = T_2 . T_1 . P = \begin{bmatrix} 1 & 0 & l_1 \\ 0 & 1 & m_1 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & l_2 \\ 0 & 1 & m_2 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & l_1 + l_2 \\ 0 & 1 & m_1 + m_2 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Concatenating 2D Transformations

$$S_1 = \begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad S_2 = \begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = S_1 . P = \begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad P'' = \begin{bmatrix} x'' \\ y'' \\ w'' \end{bmatrix} = S_2 . P' = \begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$$

Successive Scalings are ***multiplicative***.

$$P'' = S_2 . S_1 . P = \begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_{x1} . s_{x2} & 0 & 0 \\ 0 & s_{y1} . s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Concatenating 2D Transformations

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_\phi = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = R_\theta . P = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P'' = \begin{bmatrix} x'' \\ y'' \\ w'' \end{bmatrix} = R_\phi . P' = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$$

Concatenating 2D Transformations

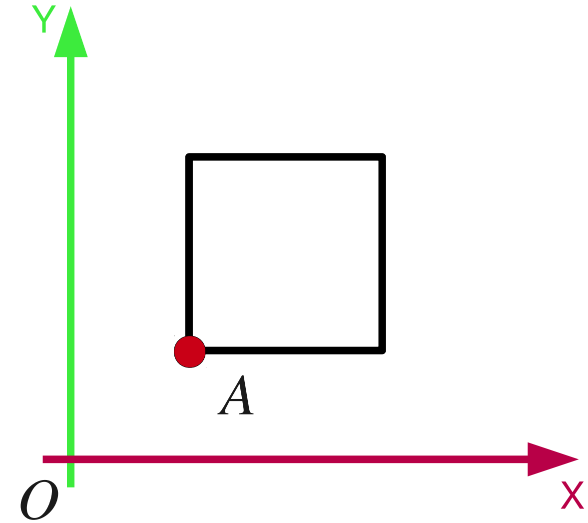
Successive Rotations are ***additive***.

$$P'' = R_{\phi} . R_{\theta} . P = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P'' = \begin{bmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) & 0 \\ \sin(\phi+\theta) & \cos(\phi+\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Concatenating 2D Transformations

Rotation about an arbitrary point A



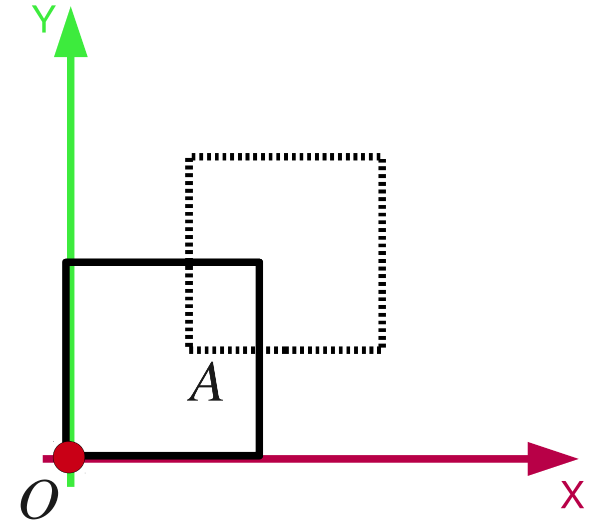
We know how to rotate about the origin O

- Translate A to O
- Rotate about O
- Translate back to A

Concatenating 2D Transformations

Rotation about an arbitrary point A

We know how to rotate about the origin O

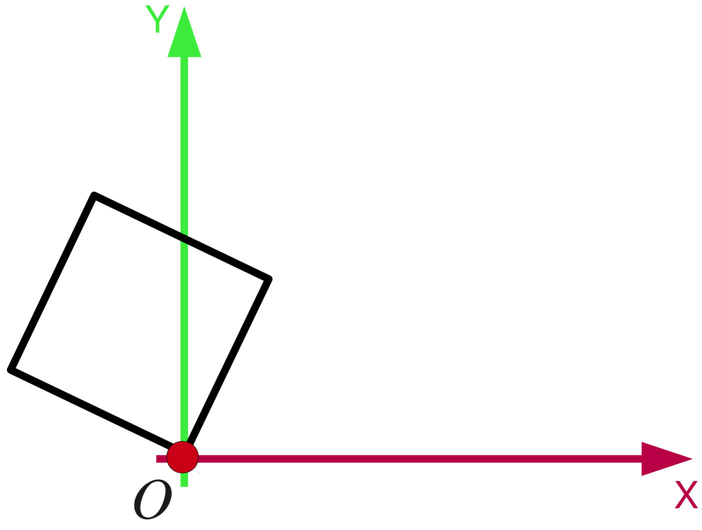


- Translate A to O

$$P' = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T_1 . P = \begin{bmatrix} 1 & 0 & l_1 \\ 0 & 1 & m_1 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Concatenating 2D Transformations

Rotation about an arbitrary point A



We know how to rotate about the origin O

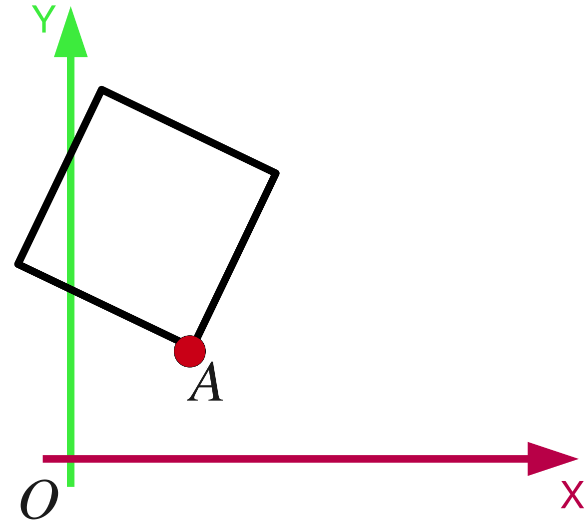
- Translate A to O
- Rotate about O

$$P'' = \begin{bmatrix} x'' \\ y'' \\ w'' \end{bmatrix} = R_\theta . P' = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$$

Concatenating 2D Transformations

Rotation about an arbitrary point A

We know how to rotate about the origin O



- Translate A to O
- Rotate about O
- Translate back to A

$$P''' = \begin{bmatrix} x''' \\ y''' \\ w''' \end{bmatrix} = T_2 . P'' = \begin{bmatrix} 1 & 0 & -l \\ 0 & 1 & -m \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x'' \\ y'' \\ w'' \end{bmatrix}$$

Concatenating 2D Transformations

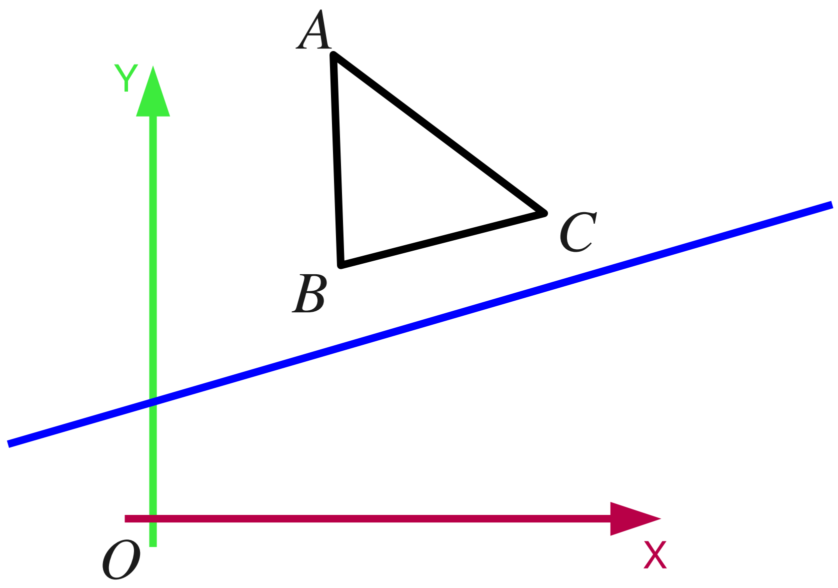
The composite transformation is then:

$$T_2 . R_\theta . T_1 = \begin{bmatrix} 1 & 0 & -l \\ 0 & 1 & -m \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & l \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & l\cos\theta - m\sin\theta - l \\ \sin\theta & \cos\theta & l\sin\theta + m\cos\theta - m \\ 0 & 0 & 1 \end{bmatrix}$$

Concatenating 2D Transformations

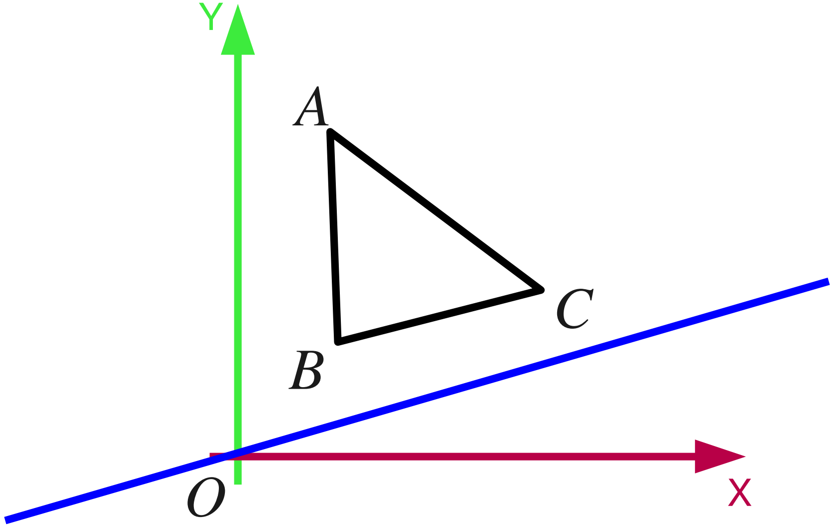
Reflection about an arbitrary line.



Concatenating 2D Transformations

Reflection about an arbitrary line.

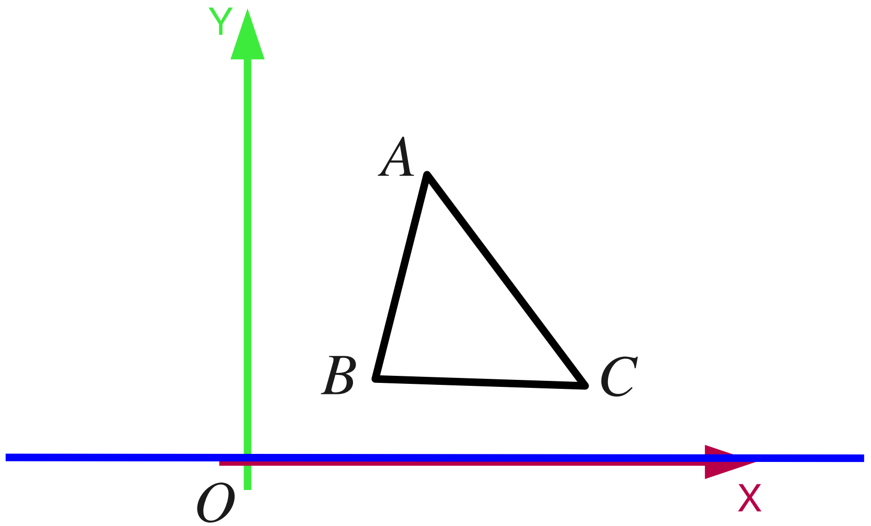
- Translation



Concatenating 2D Transformations

Reflection about an arbitrary line.

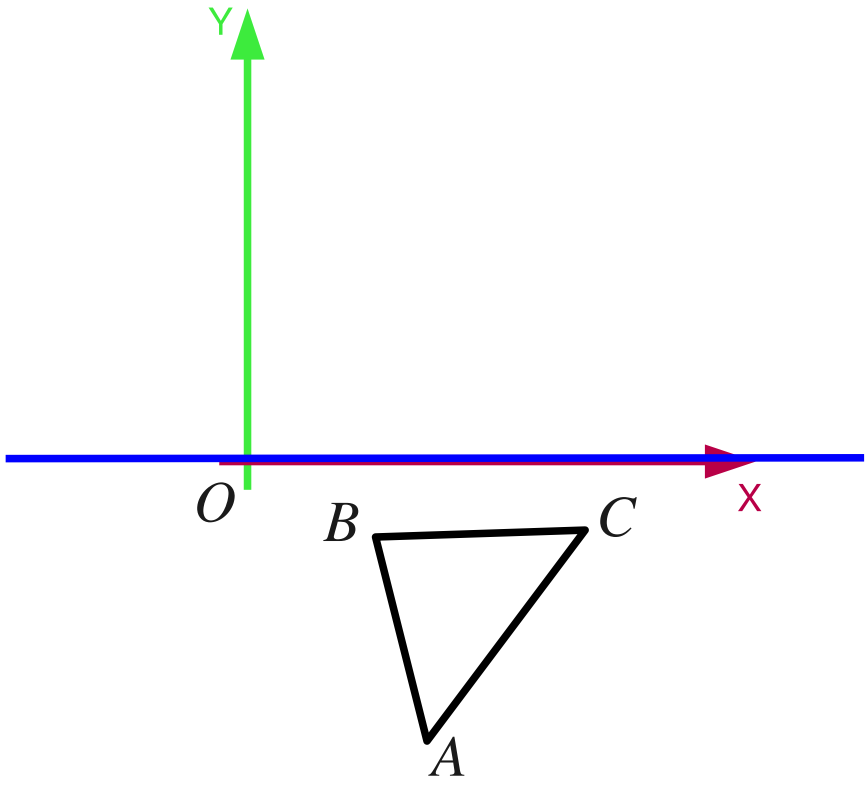
- Translation
- Rotation



Concatenating 2D Transformations

Reflection about an arbitrary line.

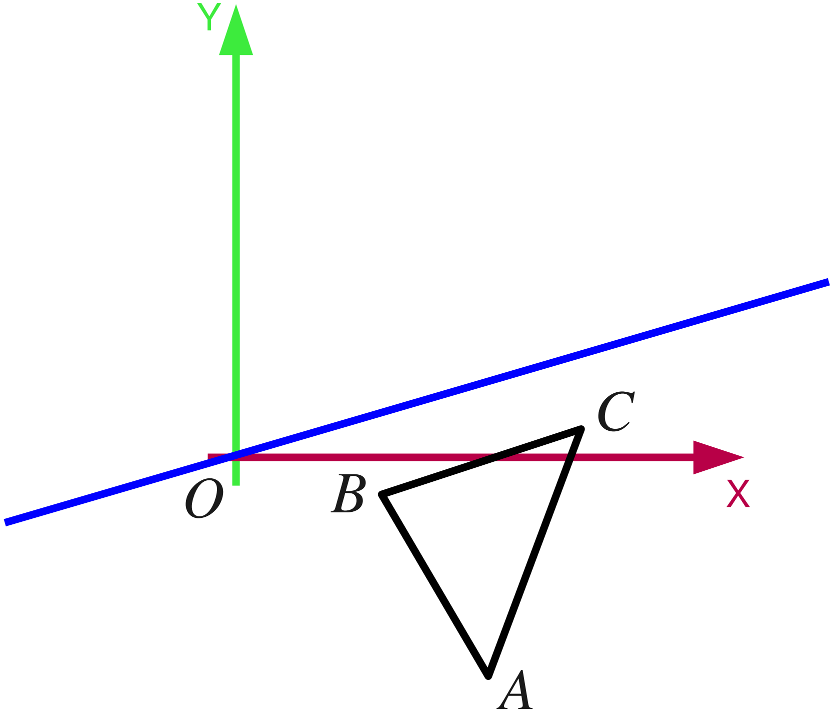
- Translation
- Rotation
- Reflection



Concatenating 2D Transformations

Reflection about an arbitrary line.

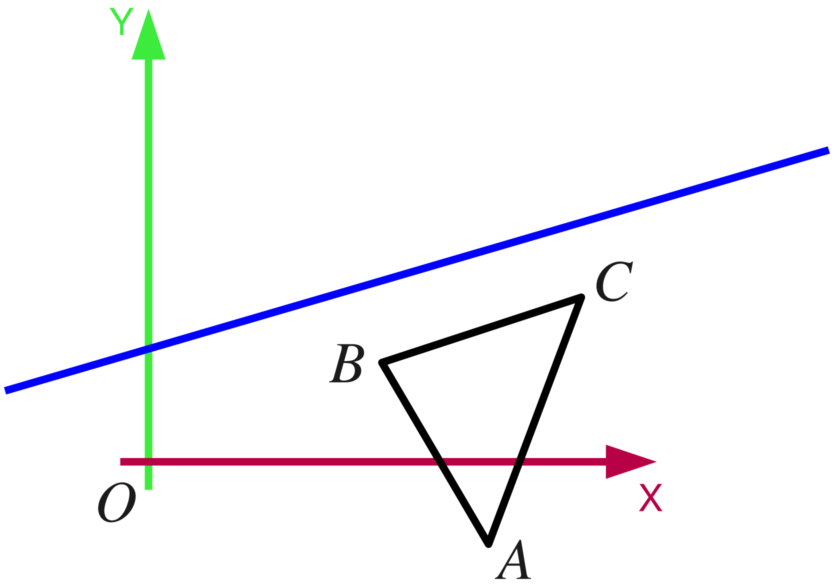
- Translation
- Rotation
- Reflection
- Rotation



Concatenating 2D Transformations

Reflection about an arbitrary line.

- Translation
- Rotation
- Reflection
- Rotation
- Translation



Concatenating 2D Transformations

Generally transformation composition is **not** commutative.

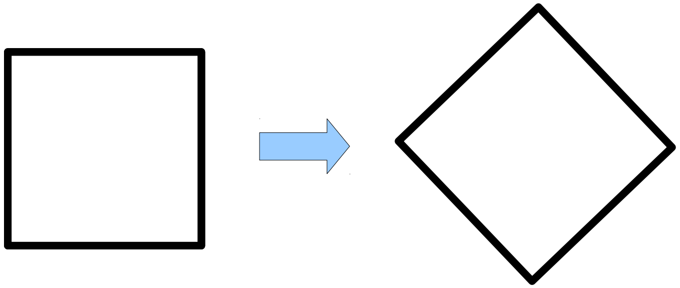
$$T_1 . T_2 \neq T_2 . T_1$$

2D Transformations

Rigid Transformations

- A square remains a square.
- Preserves lengths and angles.
- Rotations and Translations

$$T = \begin{bmatrix} r_{11} & r_{12} & l \\ r_{21} & r_{22} & m \\ 0 & 0 & 1 \end{bmatrix}$$

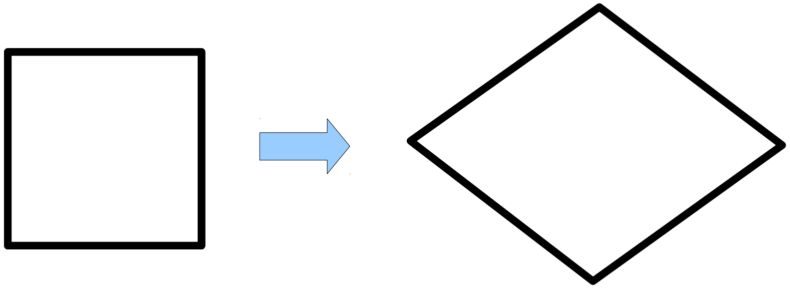


2D Transformations

Affine Transformations

- Preserves parallelism.
- Rotations, Translations, Scaling and Shears.

$$T = \begin{bmatrix} a & c & l \\ b & d & m \\ 0 & 0 & 1 \end{bmatrix}$$



2D Transformations

General 2D Transformation

$$T = \left[\begin{array}{cc|c} a & c & l \\ b & d & m \\ \hline p & q & s \end{array}\right]$$