CS 775: Advanced Computer Graphics

Lecture 4 : Monte Carlo Methods

The Rendering Equation

$$L_o(p,\omega_o) = L_e(p,\omega_o) + \int_\Omega f_r(p,\omega_o,\omega_i)\, L_i(p,\omega_i)\cos\theta_i \, d\omega_i$$

- Rendering is Integration
  - Solving the rendering equation
  - Capturing all the light paths L(D|S)*E
- Integration over
  - Lights and Objects
  - Reflected and transmitted ray directions
  - Camera apertures
  - Time

The Rendering Equation

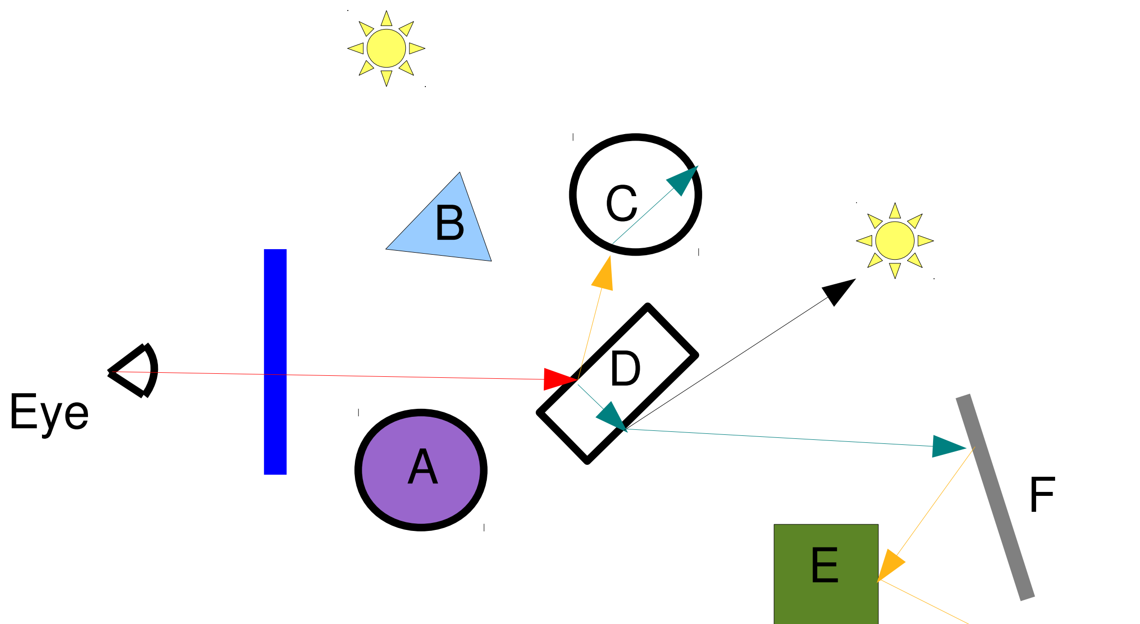
$$L_o(p,\omega_o) = L_e(p,\omega_o) + \int_{\Omega} f_r(p,\omega_o,\omega_i) L_i(p,\omega_i) \cos\theta_i \, d\omega_i$$

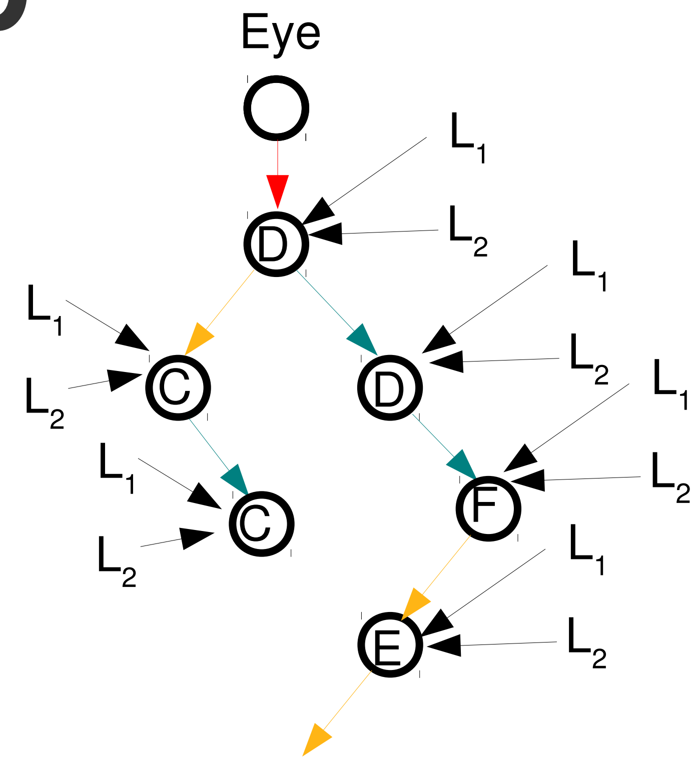
- Rendering is Integration
  - Solving the rendering equation
  - Capturing all the light paths L(D|S)*E
- Integration over
  - Lights and Objects
  - Reflected and transmitted ray directions
  - Camera apertures
  - Time

Numerical Integration is difficult and expensive

Raytracing - Recap

- Recursive Ray Tracing

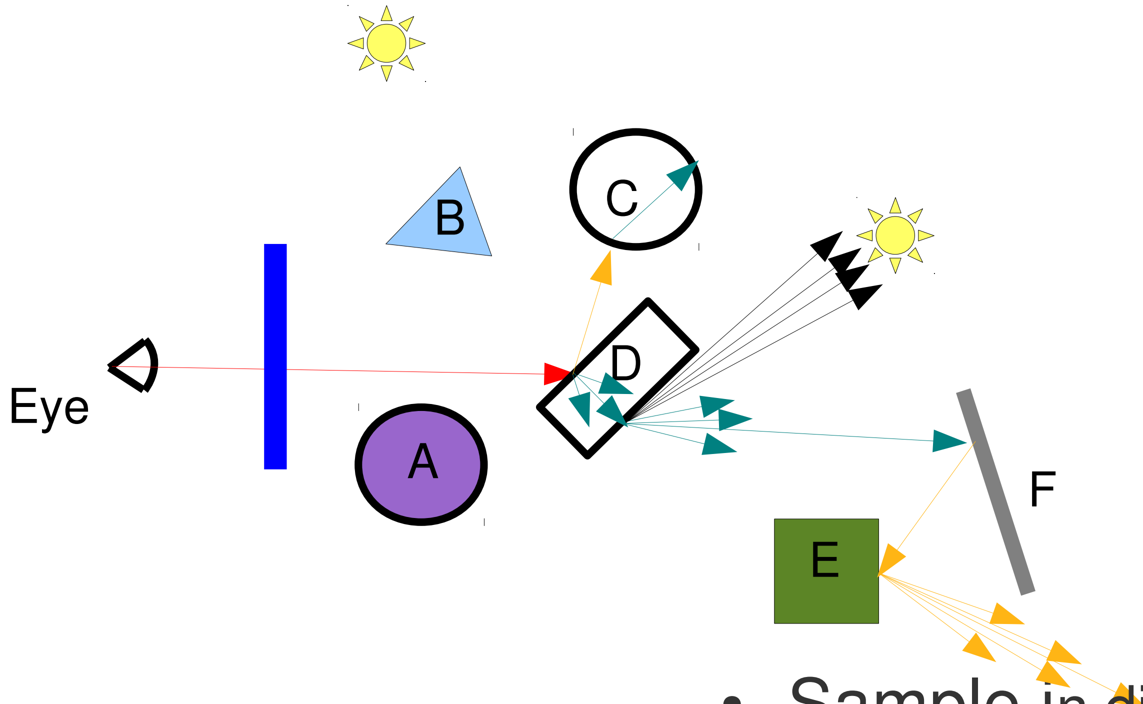


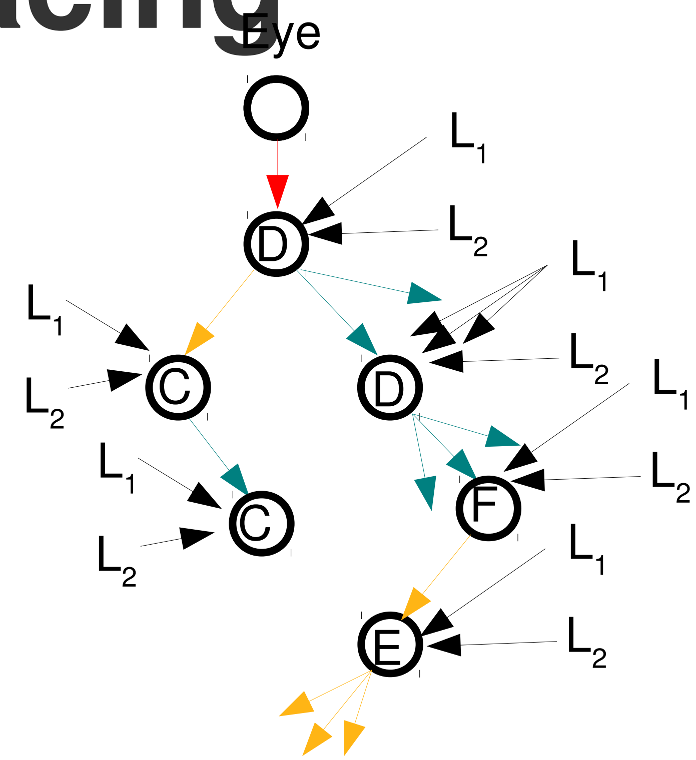


- Stop
  - When a ray leaves the scene
  - Contributed intensity is too less

Distribution Raytracing

- Stochastically sample many rays

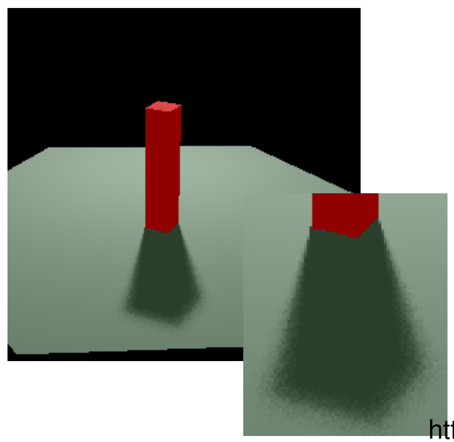
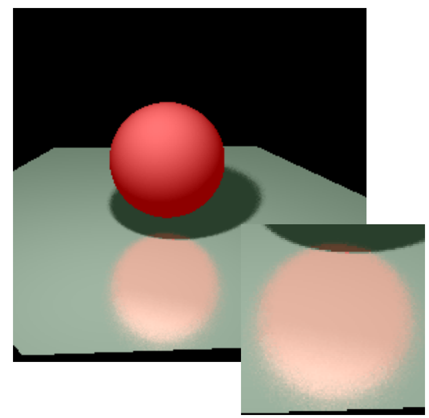
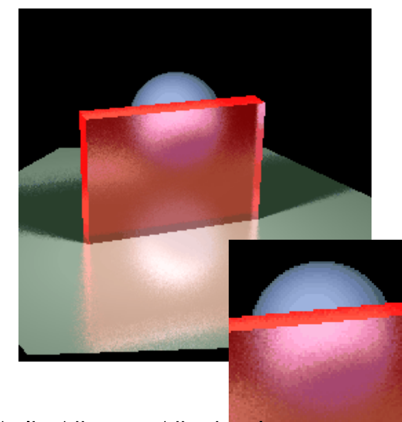




- Sample in directions preferred by the BRDF
- This is a way of better approximating the integral

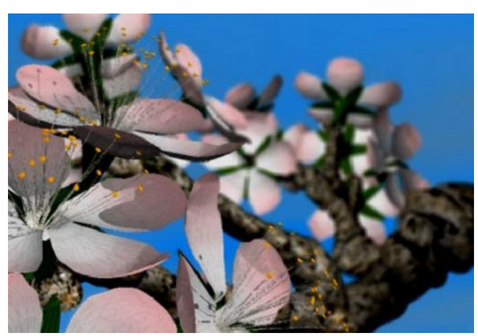
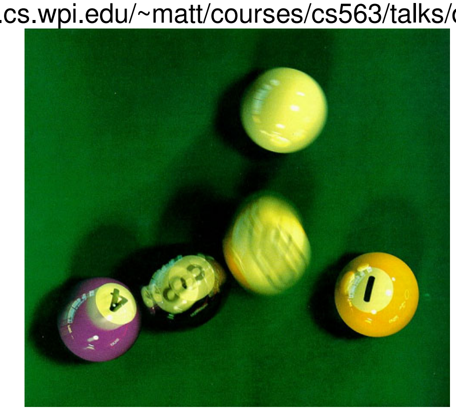
Distribution Raytracing

- Can cover all light paths
- Computationally very expensive
- Noise

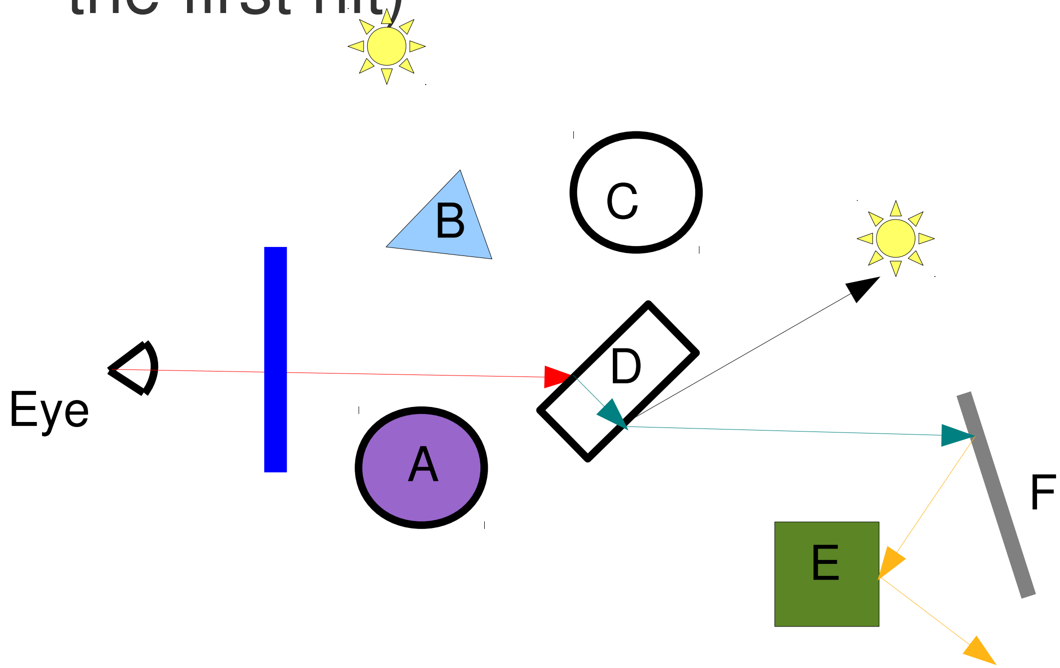
http://web.cs.wpi.edu/~matt/courses/cs563/talks/dist_ray/dist.html

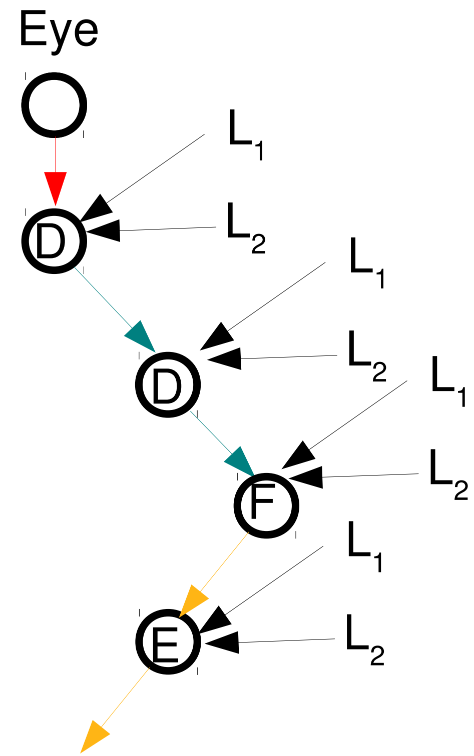
http://www.cs.utexas.edu/~fussell/

Path Tracing

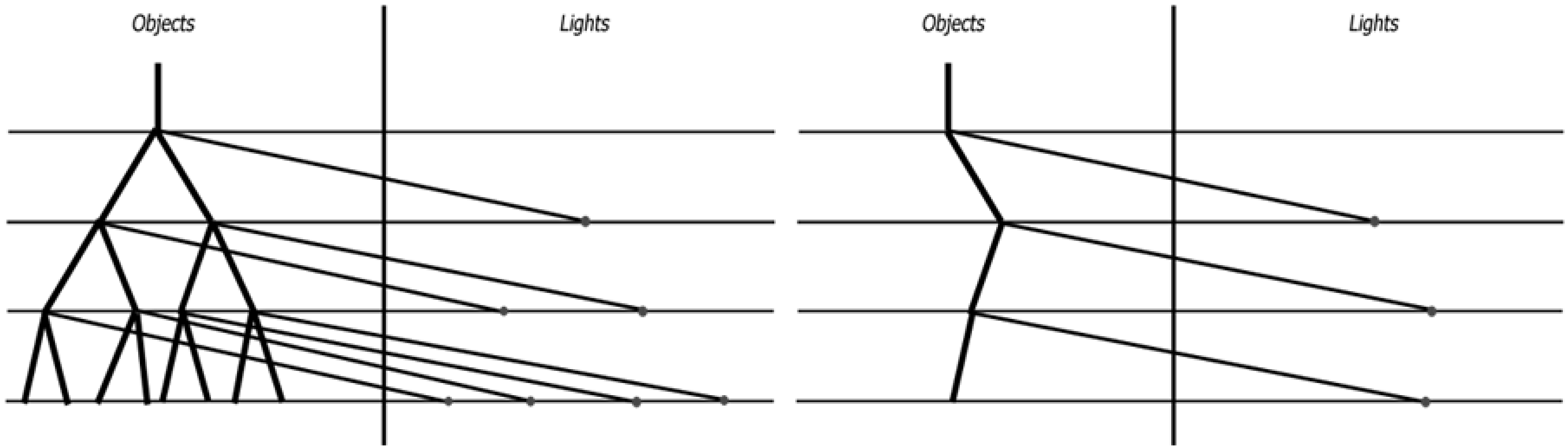
- Stochastically sample 1 ray (after the first hit)





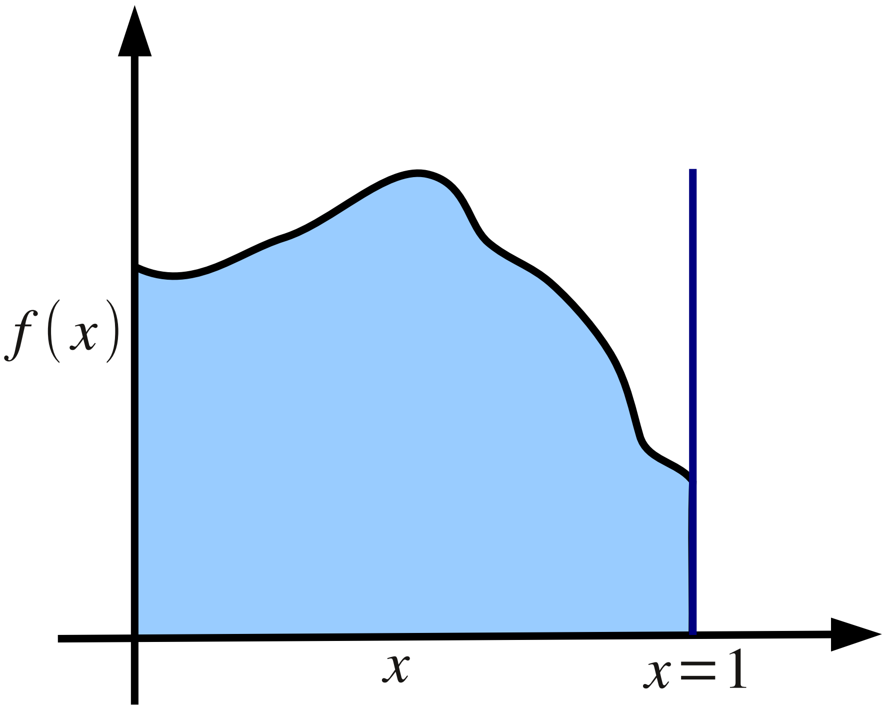
- Sample in directions preferred by the BRDF
- This is also a way of better approximating the integral

DRT vs PT



Why does random sampling work?

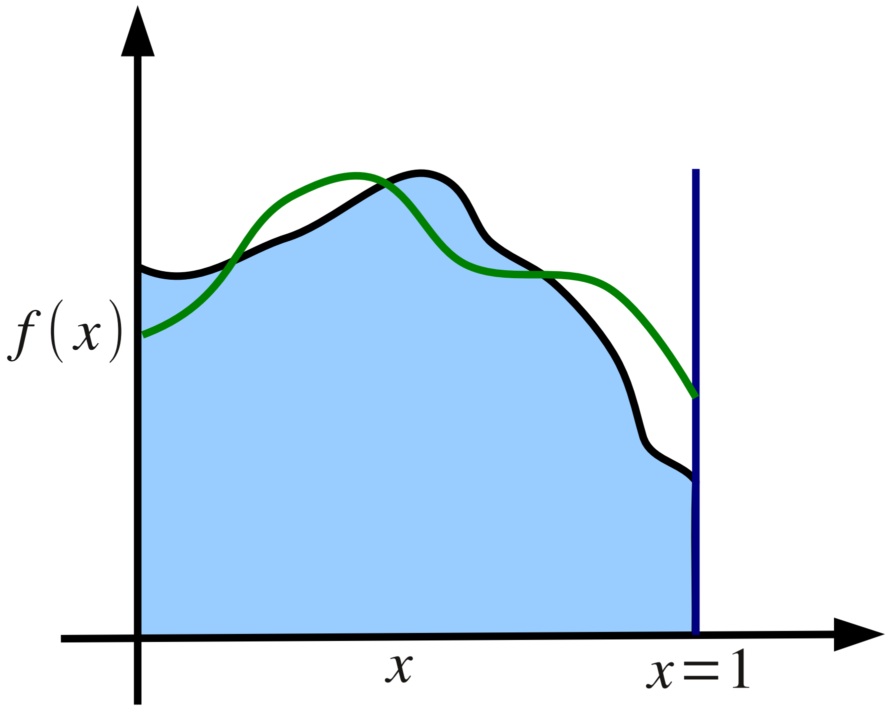
- Integration is area under the curve



$$\int_0^1 f(x)\,dx = ?$$

Why does random sampling work?

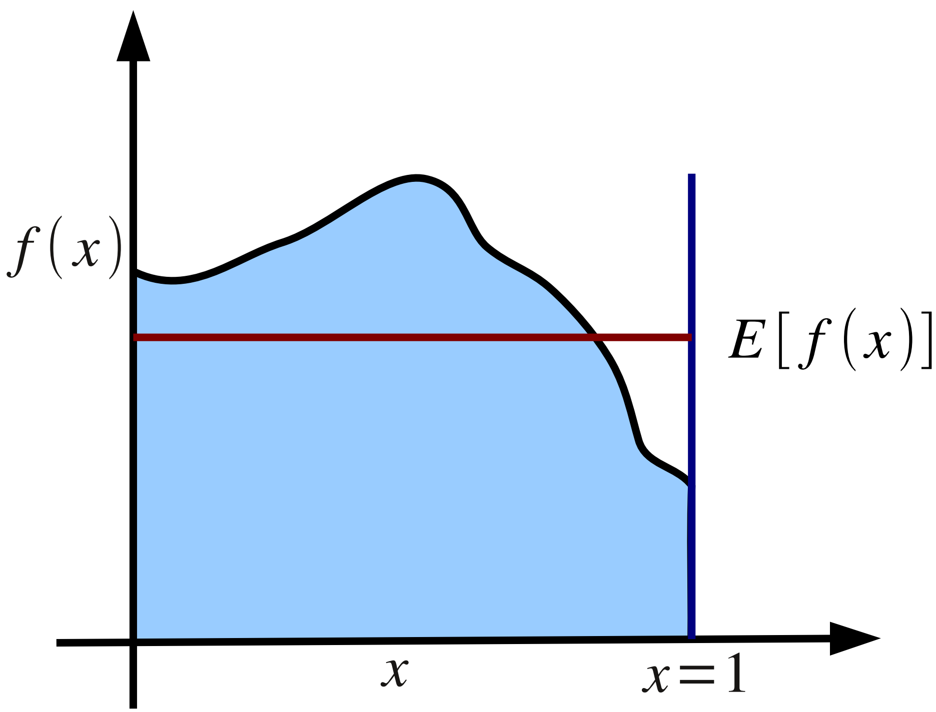
- Integration is area under the curve – integrate an approximation, that is easier to integrate



$$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx$$

Why does random sampling work?

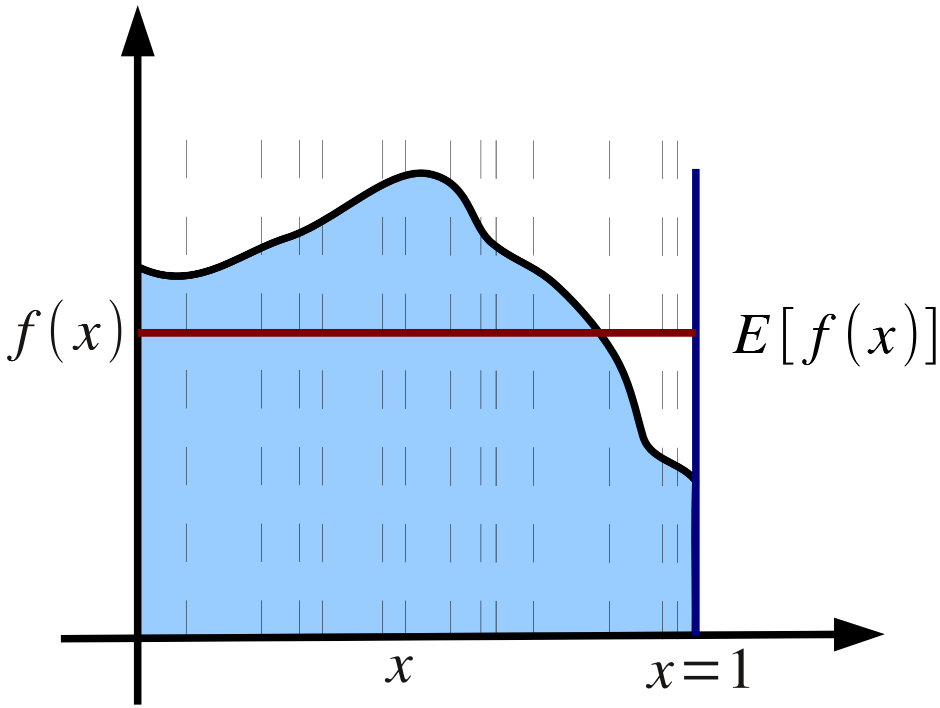
- Integration is area under the curve – or average



$$\int_0^1 f(x)dx = E[f(x)]$$

Why does random sampling work?

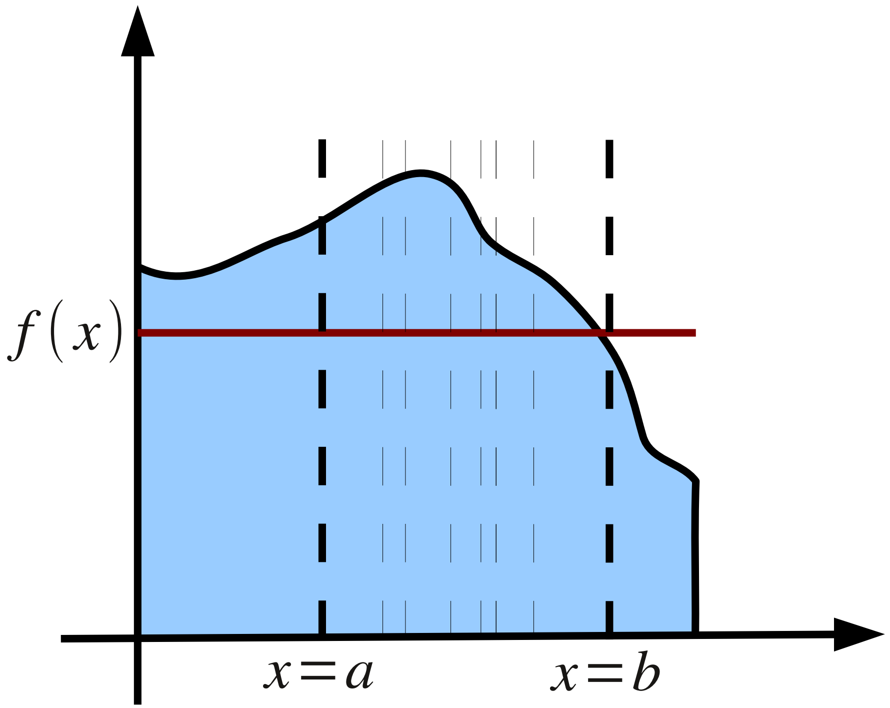
- Integration is area under the curve – estimate the average



$$\int_0^1 f(x)\,dx = E\left[\frac{1}{N}\sum_{i=1}^{N} f(x_i)\right]$$

Why does random sampling work?

- Integration is area under the curve



$$\int_a^b f(x)\,dx = E\left[\frac{(b-a)}{N}\sum_{i=1}^{N} f(x_i)\right]$$

How to choose the samples?

Probability Theory (diversion)

- Continuous random variable X
- Cumulative Distribution Function (CDF) $P(x) = Pr\{X \le x\}$
- Probability Density Function (PDF) $p(x) = \frac{dP(X)}{dx}$
- PDFs must intergrate to 1 over their domains
- For uniform random variables PDF is constant

$$\text{For } \zeta \in [0,1),\ p(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & \text{otherwise} \end{cases}$$

- Expected value of a function is defined as the average value of the function over some distribution of values over its domain. $E_p[f(x)] = \int_D f(x)\, p(x)\, dx$

Probability Theory (diversion)

- Variance $V[f(x)] = E[(f(x) - E[f(x)])^2]$
- Expected deviation of a function from its expected value
- Also, $V[f(x)] = E[(f(x))^2] - E[f(x)]^2$

The **Monte Carlo** Estimator – Given a supply of uniform random variables, $X_i \in [a, b]$, the MC estimator says that the expected value of

$$F_N = \frac{b-a}{N} \sum_{i=1}^{N} f(X_i)$$

i.e., $E[F_N]$, is the equal to the integral $\int_a^b f(x)\,dx$

Probability Theory (diversion)

- The **Monte Carlo** Estimator for random variable chosen from some arbitrary PDF is

$$F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}$$

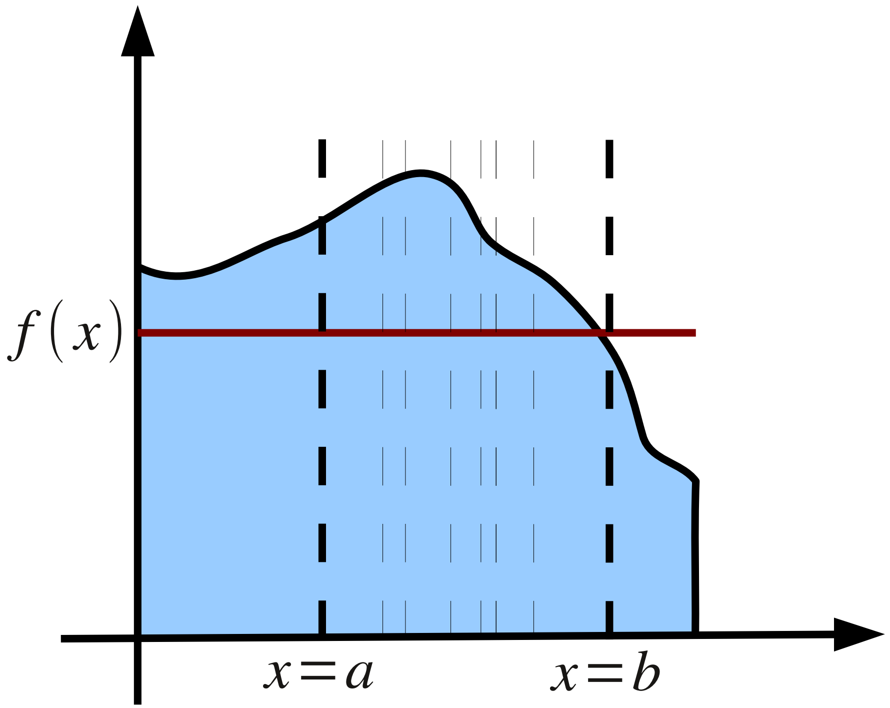
- For a multi-dimensional integral like $\int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x, y, z)\, dx\, dy\, dz$

  if samples $X_i = (x_i, y_i, z_i)$ are chosen from the box from (x_0, y_0, z_0) to (x_1, y_1, z_1), the PDF is constant and the estimator is

$$F_N = \frac{(x_1 - x_0)(y_1 - y_0)(y_1 - y_0)}{N} \sum_{i=1}^{N} f(X_i)$$

Why does random sampling work?

- Integration is area under the curve

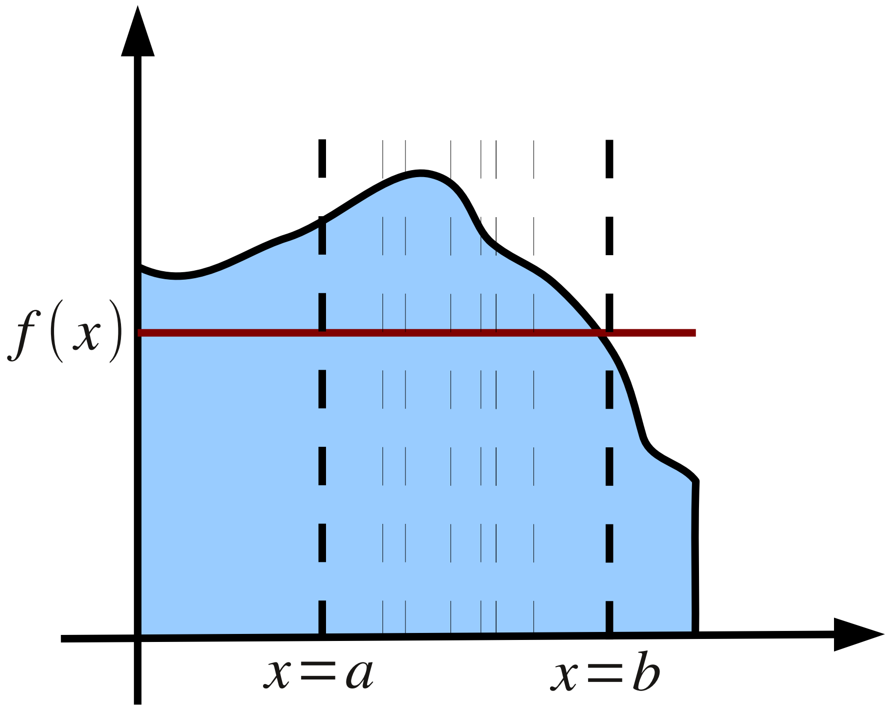


$$\int_a^b f(x)\,dx = E[F_N] = E\left[\frac{(b-a)}{N}\sum_{i=1}^{N} f(x)\right]$$

Choose uniform samples.

Why does random sampling work?

- Integration is area under the curve



$$\int_a^b f(x)\,dx = E\left[\frac{(b-a)}{N}\sum_{i=1}^{N} f(x)\right]$$

Choose uniform samples?

Let us see what happens!

Parag Chaudhuri

Monte Carlo Path Tracing

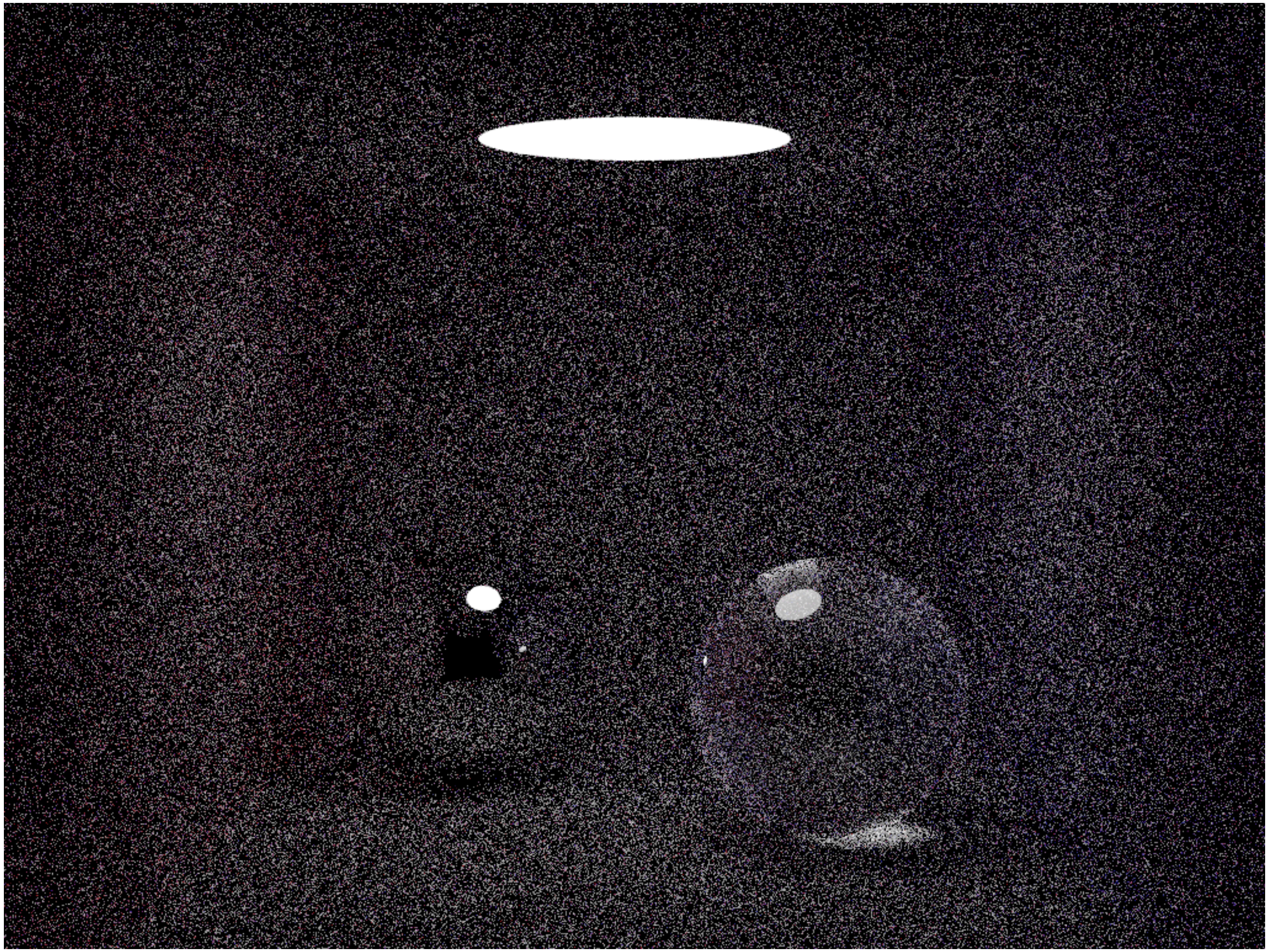
```
path_trace
 for each pixel do
 {
   color = 0
   for each sample do
   {
     pick ray through random position in pixel
     color += trace(ray)
   }
   pixel-color = color/num_samples
}
```

Monte Carlo Path Tracing

trace(ray)
find nearest intersection with objects
compute intersection point and normal
color = **shade**(point, normal)
return color

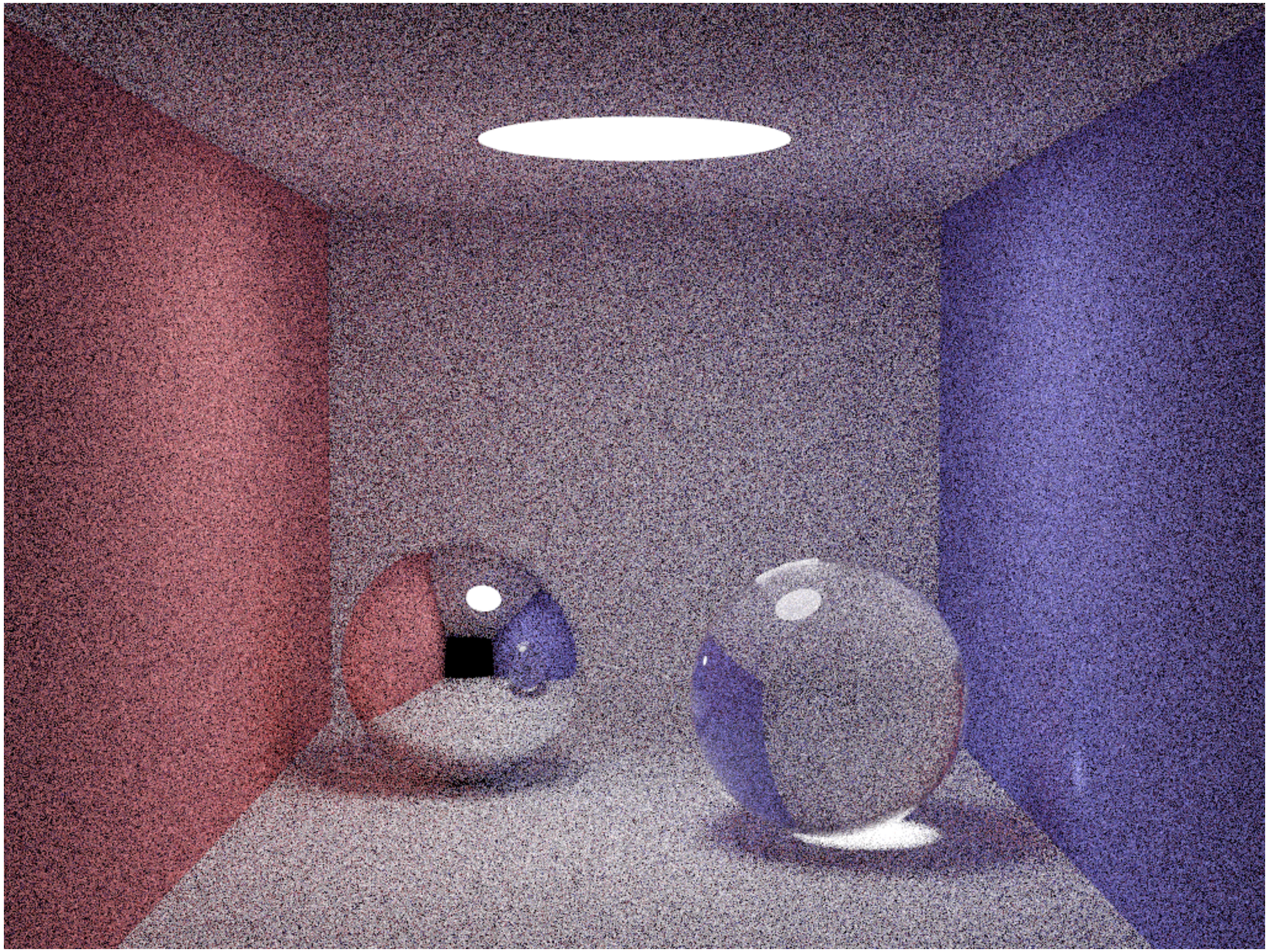
shade(point, normal)
color = 0
for each light_source **do**
{
  Test visibility of a random point on the light source
  if (visible) color += direct illumination
}
color += trace(random_ray)
return color

Monte Carlo Path Tracing



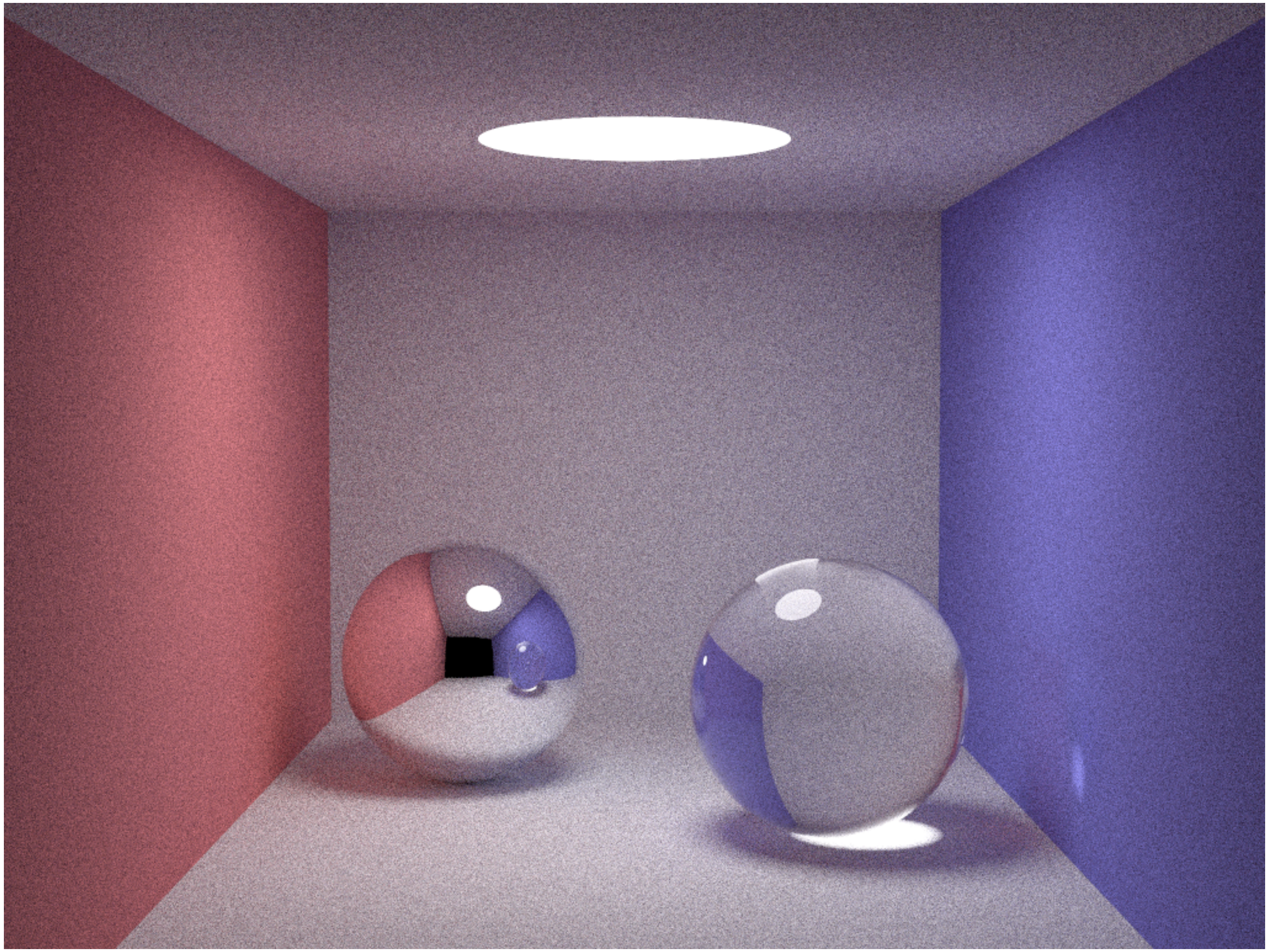
5 samples per pixel, 3s render time

Monte Carlo Path Tracing



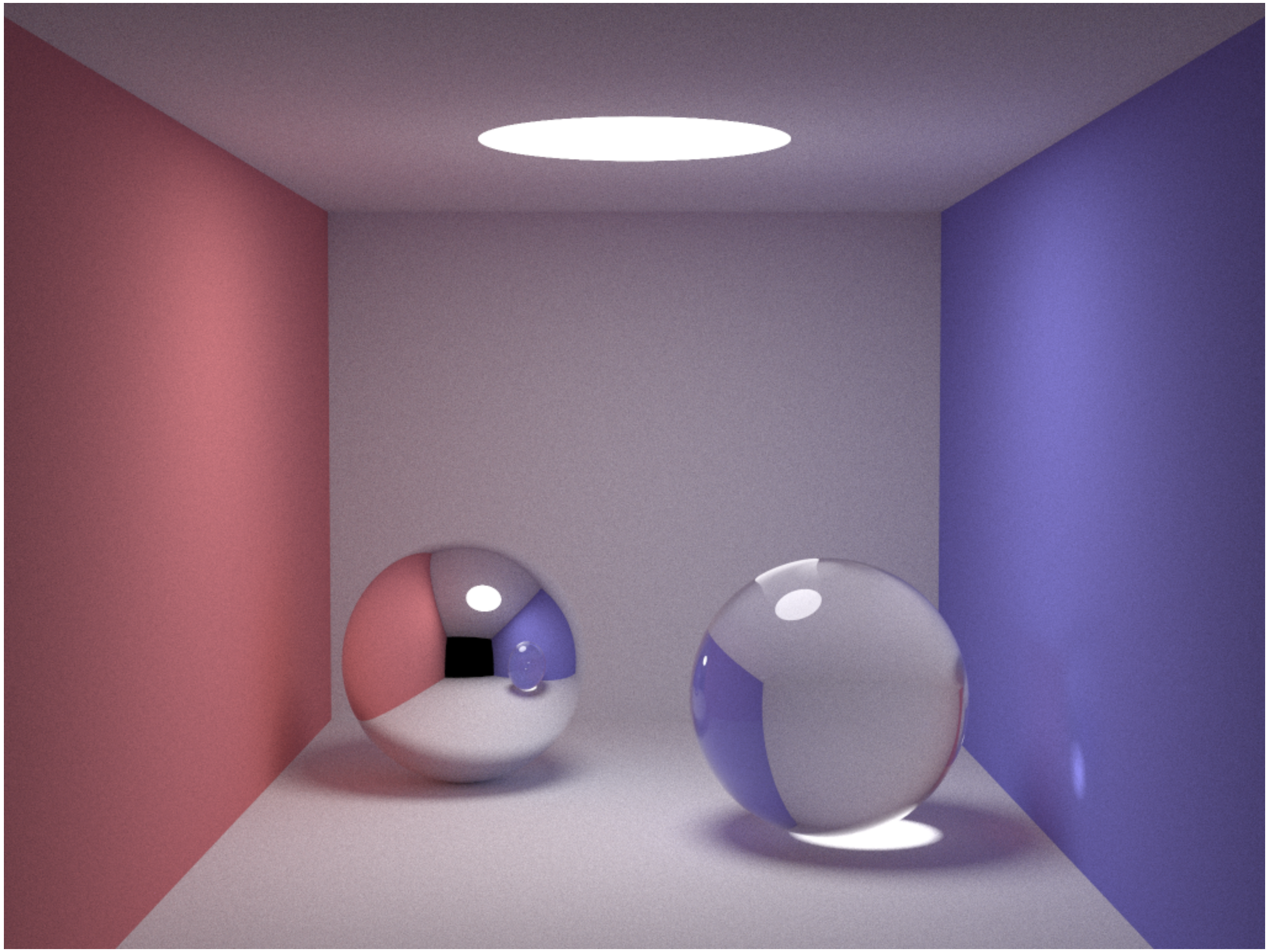
50 samples per pixel, 30s render time

Monte Carlo Path Tracing



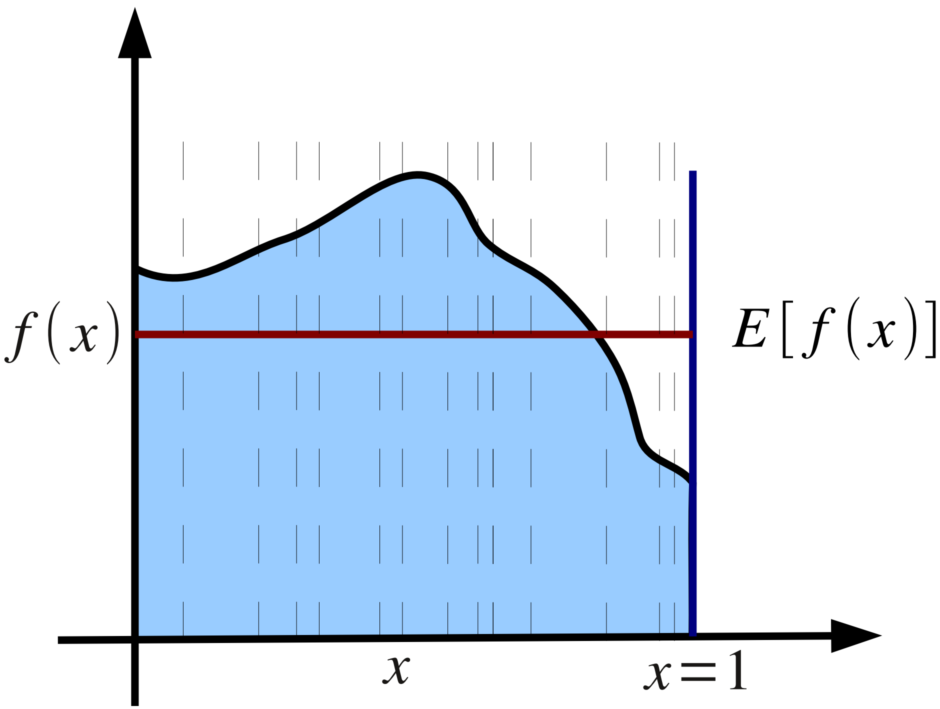
500 samples per pixel, 300s render time

Monte Carlo Path Tracing



5000 samples per pixel, ~1hr render time

Variance and Sampling



$$V[f(x)] = E[(f(x) - E[f(x)])^2]$$

$$V[f(x)] = E[(f(x))^2] - E[f(x)]^2$$

$$V[E[f(x)]] = \frac{1}{N} V[f(x)]$$

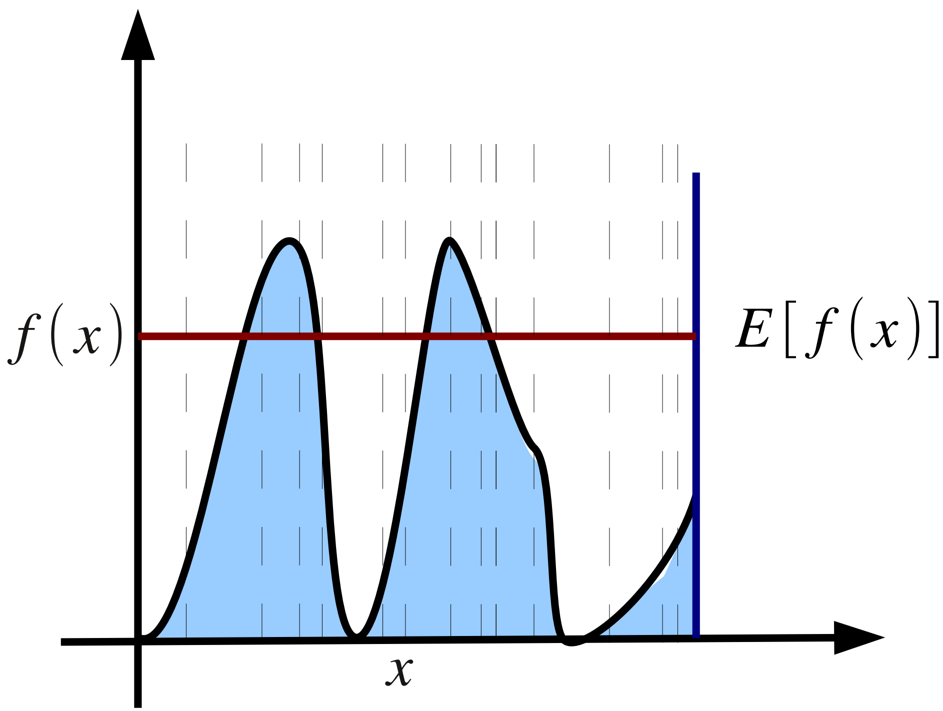
For number of samples N

Variance decreases as 1/N

Error decreases as 1/√N

Variance and Sampling

- Noise decreases slowly with number of samples.



$$\int_{\Omega} f(x)\,dx = \frac{1}{N}\sum_{i=1}^{N} Y_i$$

$$Y_i = \frac{f(x_i)}{p(x_i)}$$

$$E[Y_i] = ?$$

Importance Sampling

- MC Estimator $$F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)}$$

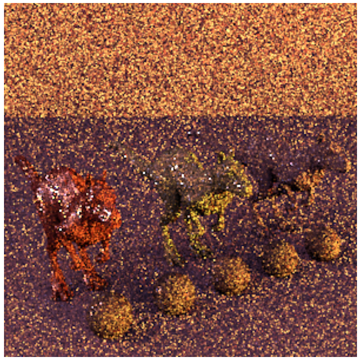
- If we know $$c = \frac{1}{\int f(x)\,dx}$$

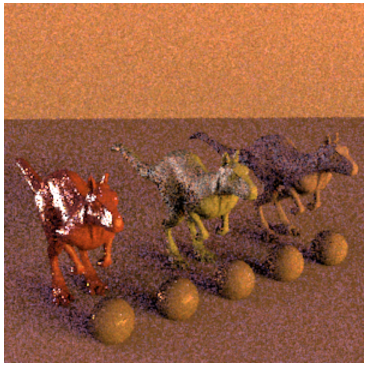
- Then $$\frac{f(X_i)}{p(X_i)} = \frac{1}{c} = \int f(x)\,d(x)$$

- What is the variance in this case?

Importance Sampling

- MC Estimator $$F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}$$
- If we know $$c = \frac{1}{\int f(x)\, dx}$$
- Then $$\frac{f(X_i)}{p(X_i)} = \frac{1}{c} = \int f(x)\, d(x)$$
- Choose $p(X_i)$ that is similar to $f(X_i)$





https://graphics.stanford.edu/wikis/cs348b-08/Assignment4

Importance Sampling

Sampling by Inversion

- The Monte Carlo estimator $F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}$
- How to sample from any PDF?
  - Compute the CDF $P(x) = \int_0^x p(x')\,dx'$
  - Compute the inverse $P^{-1}(x)$
  - Obtain a uniformly distributed random number ζ
  - Compute $X_i = P^{-1}(\zeta)$

Sampling by Inversion

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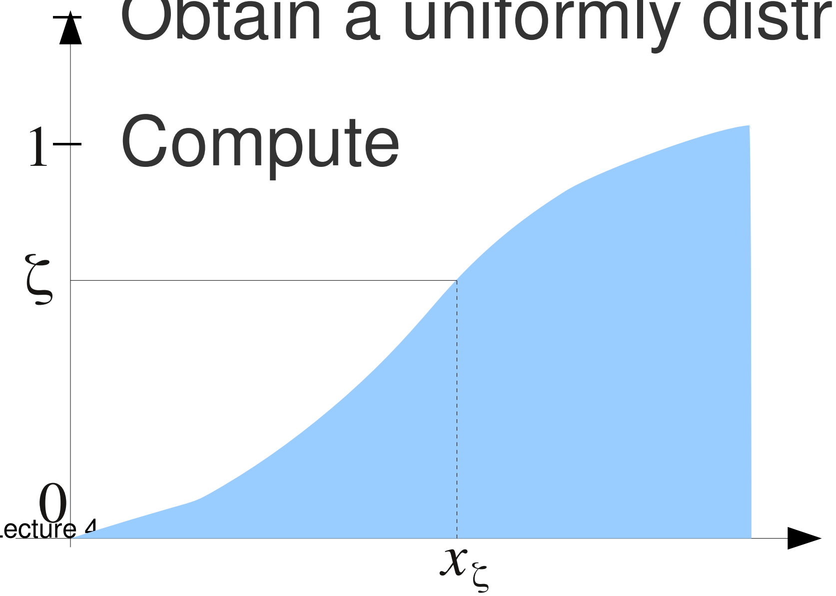
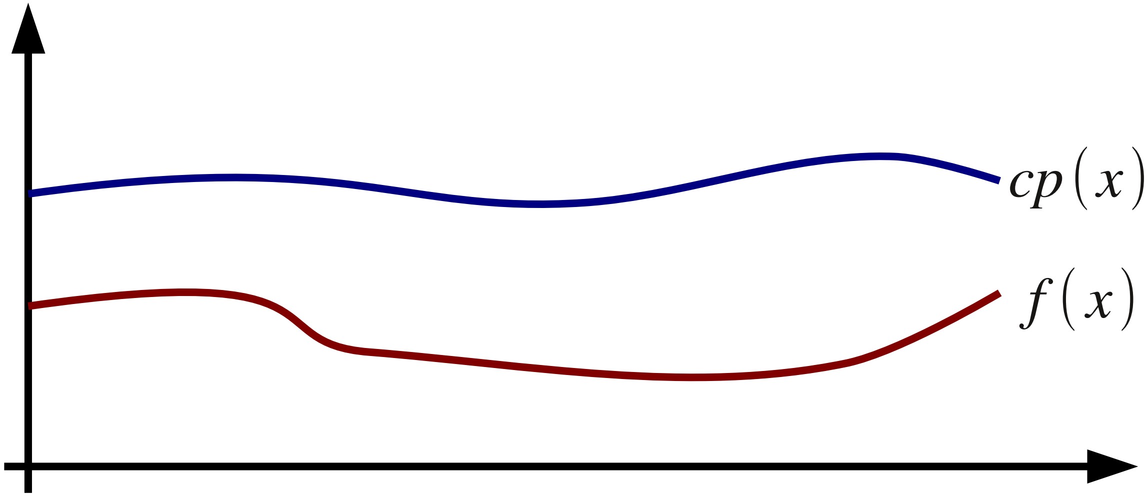


$$\zeta = P(x_\zeta) = Pr(X \le x_\zeta)$$

Method of Inversion

Sampling by Rejection

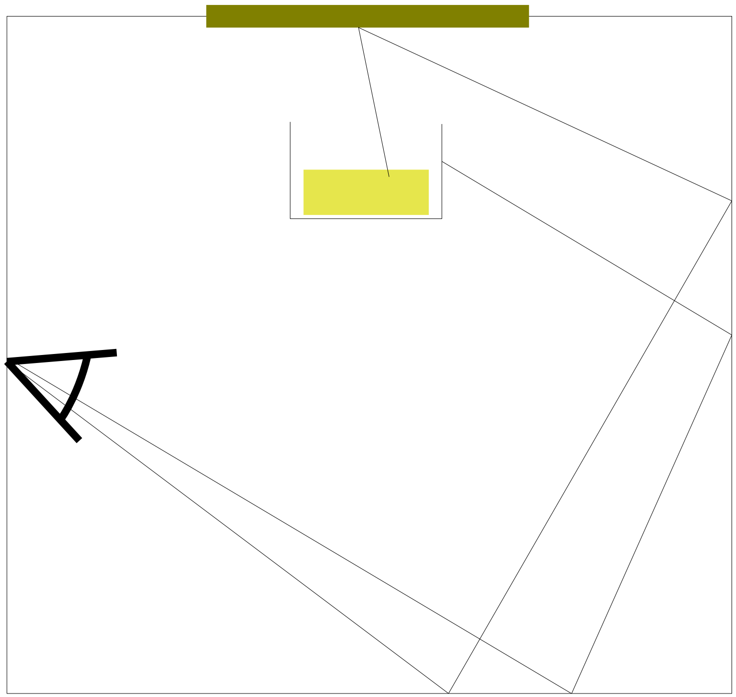
- Method of Rejection
- If we want to sample some $f(x)$
- But instead can sample from some PDF $p(x)$ $s.t.$ $f(x) < cp(x)$
- Sample X from $p(x)$
- If for some ζ , $\zeta < f(X)/(c\,p(X))$ then return X



Sampling Efficiency

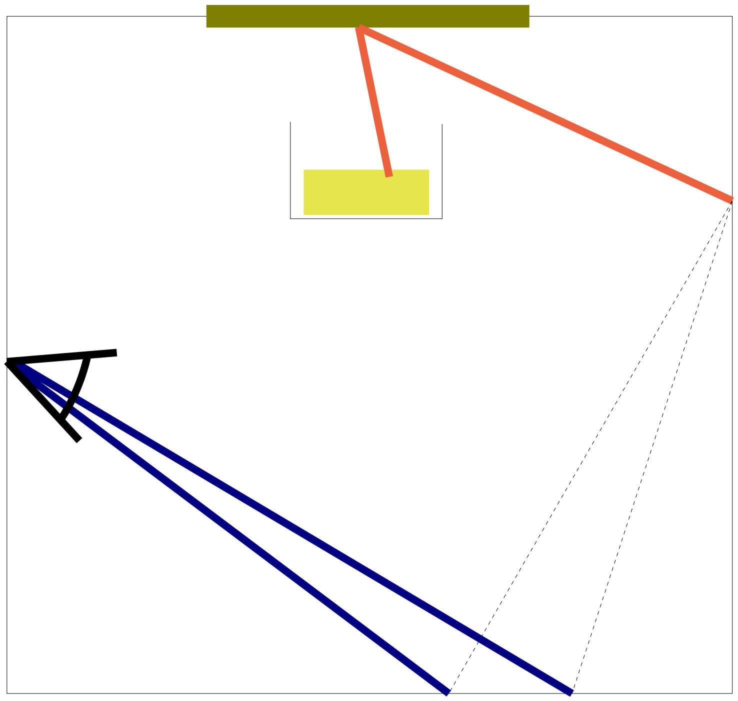
- Russian Roulette
- The integrand is not evaluated for a particular sample with some probability q and a constant value c is used.
- Integrand is evaluated with probability $1-q$ and weighted by $1/1-q$
- $$F' = \begin{cases} \dfrac{F-qc}{1-q} & \zeta > q \\ c & \text{otherwise} \end{cases}$$
- Choose the roulette weights wisely

Bidirectional Path Tracing



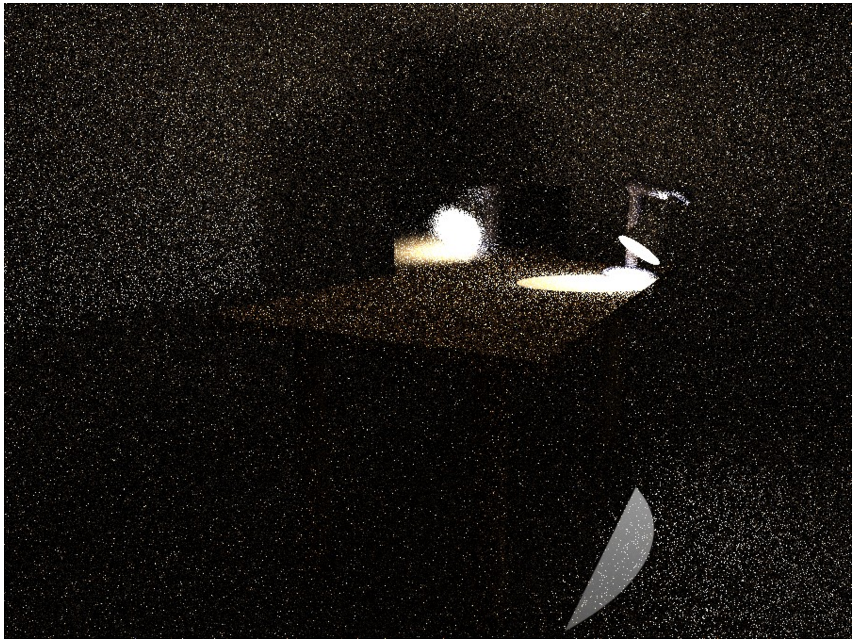
- If shot from the eye, a very small set of rays will reach the light source
- Variance will be high

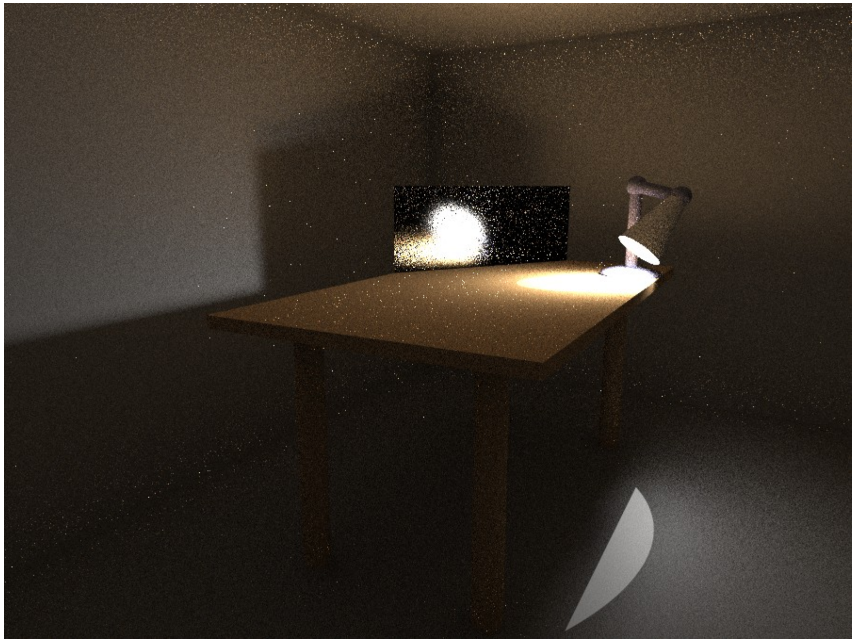
Bidirectional Path Tracing



- Shoot rays from both the light and the eye and join subpaths

Bidirectional Path Tracing





Bidirectional Estimators for Light Transport, E. Veach and L. J. Guibas, Eurographics Rendering Workshop 1994 Proceedings, pp. 147-162.