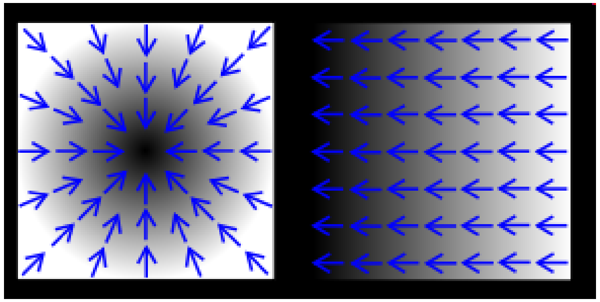
CS 775: Advanced Computer Graphics

Lecture 10: Grid Fluids

Fluid Simulation



http://en.wikipedia.org/wiki/Gradient

- Calculus Review
  - Gradient (∇): A vector pointing in the direction of the greatest rate of increment

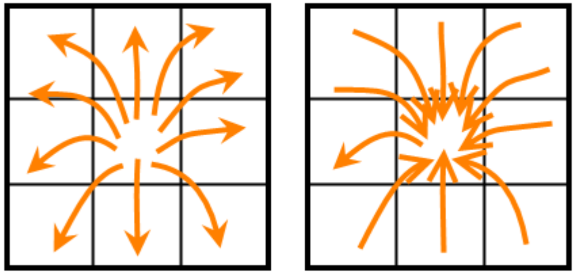
$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right)$$

u can be a scalar or a vector

  - Divergence ($\nabla\cdot$): Measure how the vectors are converging or diverging at a given location (volume density of the outward flux)

$$\nabla \cdot u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

u can only be a vector



http://www.cs.unc.edu/~lin/COMP768-S09/LEC/fluid.ppt

Fluid Simulation

- Calculus Review
  - Laplacian (∇^2): Divergence of the gradient

$$\nabla^2 u = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

u can be a scalar or a vector

  - Finite Difference: Approximating a derivative

$$\frac{\partial u}{\partial x} = \frac{u_{i+1} - u_i}{x_{i+1} - x_i}$$

over space

$$\frac{\partial u}{\partial t} = \frac{u_{i+1} - u_i}{t_{i+1} - t_i}$$

over time

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Fluid Simulation

- Calculus Review

$$\text{If } F(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$$

$$\text{then, } \nabla F = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix}$$

$$\nabla \cdot F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\text{and, } \nabla^2 F = \nabla \cdot (\nabla F) = \begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial z}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial h}{\partial z}\right) \end{bmatrix}$$

Fluid Simulation

- Navier-Stokes Equation

$$\frac{\partial \vec{u}}{\partial t} = -\vec{u} \cdot \nabla \vec{u} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} + \vec{g}$$

Momentum Equation

- Derivation Sketch

$$m \frac{D\vec{u}}{Dt} = F$$

$$m \frac{D\vec{u}}{Dt} = m\vec{g} - V \nabla p + V \mu \nabla^2 \vec{u}$$

$$\frac{D\vec{u}}{Dt} = \vec{g} - \frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \vec{u}$$

$$\frac{\partial \vec{u}}{\partial t} = -\vec{u} \cdot \nabla \vec{u} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} + \vec{g}$$

where $\frac{D\phi(x,t)}{Dt} = \frac{\partial \phi(x,t)}{\partial t} + \frac{\partial \phi(x,t)}{\partial x} \cdot \frac{\partial x}{\partial t}$
$= \frac{\partial \phi(x,t)}{\partial t} + \nabla \phi(x,t) \cdot \vec{u}$

$\vec{u}$ is fluid velocity
μ is dynamic viscosity
$\vec{g}$ is acceleration due to gravity

Fluid Simulation

- Navier-Stokes Equation

$$\nabla \cdot \vec{u} = 0$$

Incompressibility Equation

- Derivation Sketch

$$\frac{d}{dt}(Volume\,\Omega) = \int\int_S \vec{u}.\hat{n}.dS$$

$$= \int\int\int_\Omega \nabla \cdot \vec{u}.d\Omega$$

$$= 0 \qquad \text{for incompressibility}$$

$$\Rightarrow \nabla \cdot \vec{u} = 0$$

$\vec{u}$ is fluid velocity

Ω is an arbitrary chunk of fluid

$Volume\,\Omega$ is its volume

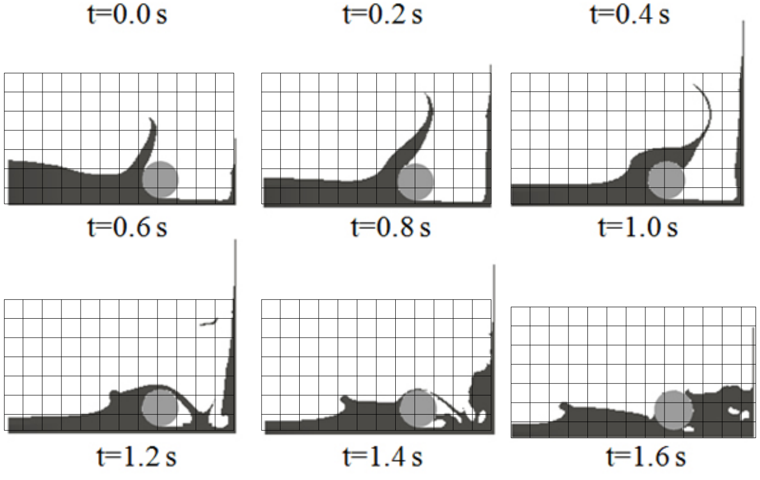
S is its surface boundary

$\vec{n}$ is normal at surface

Fluid Simulation

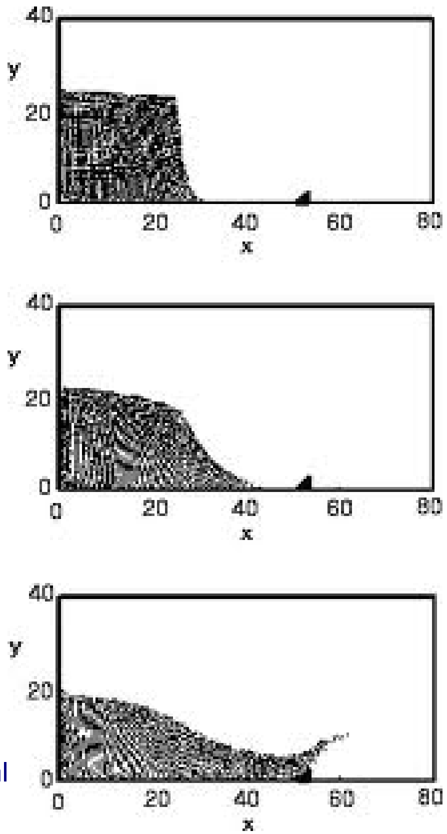
- Eulerian Viewpoint
  - Discretize the domain using *finite differences*
  - Define scalar & vector fields on the grid
  - Use the *operator splitting* technique to solve each term separately
  - Advantages
    - Derivative approximation
    - Adaptive time step/solver
  - Disadvantages
    - Memory usage & speed
    - Grid artifact/resolution limitation

http://www.eng.nus.edu.sg/EResnews/022014/images/rd02-fig1.jpg



Fluid Simulation

- Lagrangian Viewpoint
  - Discretize the domain using *particles*
  - Define interaction forces between neighbouring particles using smoothing kernels.
  - Advantages
    - Mass Conservation
    - Intuitive
  - Disadvantages
    - Surface tracking



http://mec-mail.eng.monash.edu.au/~mct/mct/docs/sph/sph.html

Fluid Simulation

- Solving the Navier-Stokes Equation
  - Eulerian Viewpoint
  - Operator Splitting

$$u^n = A + B + D + P$$

$$u^1$$

$$u^2$$

$$u^3$$

$$u^{n+1}$$

› One complicated Multi-dimensional operator solved as a series of simple, lower dimensional operators

› Each operator can have its own integration scheme

› High Modularity and Easy to debug

Fluid Simulation

- Advection

$$\frac{\partial \vec{u}}{\partial t} = -\vec{u} \cdot \nabla \vec{u}$$

- Euler

$$\frac{\vec{u}_i^{n+1} - \vec{u}_i^{n}}{\Delta t} = -\vec{u} \cdot \frac{\vec{u}_{i+1}^{n} - \vec{u}_{i-1}^{n}}{2 \Delta x}$$

Unstable!

- Semi-Lagrangian (look back in time)

$$\frac{\vec{u}_?^{n+1} - \vec{u}_i^{n}}{\Delta t} = 0$$

Unconditionally stable!
but has
Numerical Dissipation

Fluid Simulation

- Incompressibility and Pressure Solve
  - Advection may introduce compression/expansion in the field

Discrete Multiscale Vector Field Decomposition, Tong et al, 2003

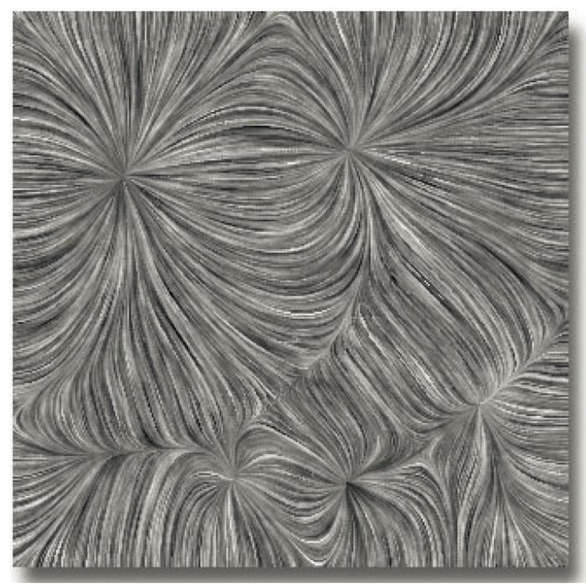
Fluid Simulation

- Hemlholtz-Hodge Decomposition

Input Velocity Field

=

Curl Free
(irrotational)



Divergence Free
(incompressible)

Discrete Multiscale Vector Field Decomposition, Tong et al, 2003

$$\vec{u} = \nabla p + \vec{u}^{\text{div\_free}}$$

Fluid Simulation

- Incompressibility and Pressure Solve

$$\vec{u}^{\text{div\_free}} = \vec{u} - \nabla p \qquad -(1)$$

$$\nabla \cdot \vec{u}^{\text{div\_free}} = 0 \qquad -(2)$$

$$\nabla \cdot \nabla p = \nabla \cdot \vec{u}$$

Solve (3), then plug into (1) to find new incompressible velocity field.

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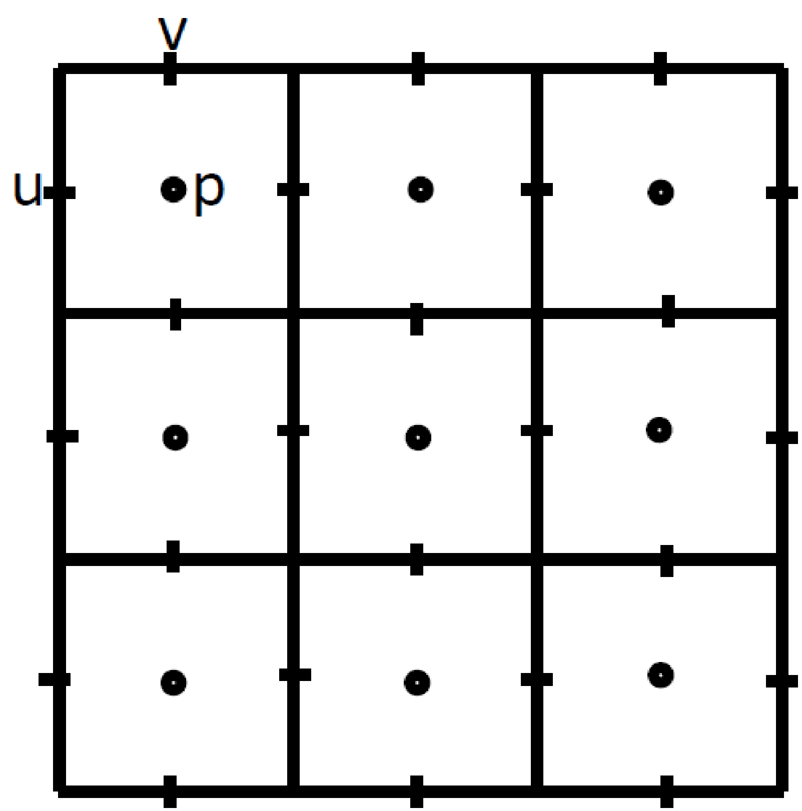
Fluid Simulation

- Incompressibility and Pressure Solve

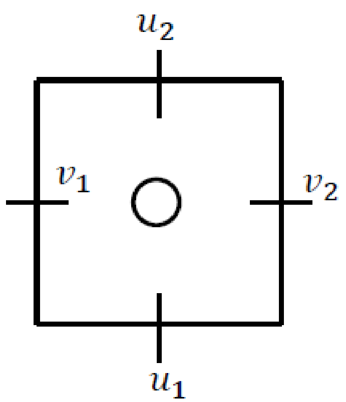


Staggered Grids

Fluid Simulation

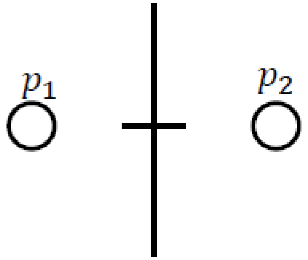
- Incompressibility and Pressure Solve
  - Divergence

$$\nabla \cdot \vec{u} = \frac{\partial \vec{u}}{\partial x} + \frac{\partial \vec{u}}{\partial y} \approx \frac{u_2 - u_1 + v_2 - v_1}{\Delta x}$$



  - Gradient

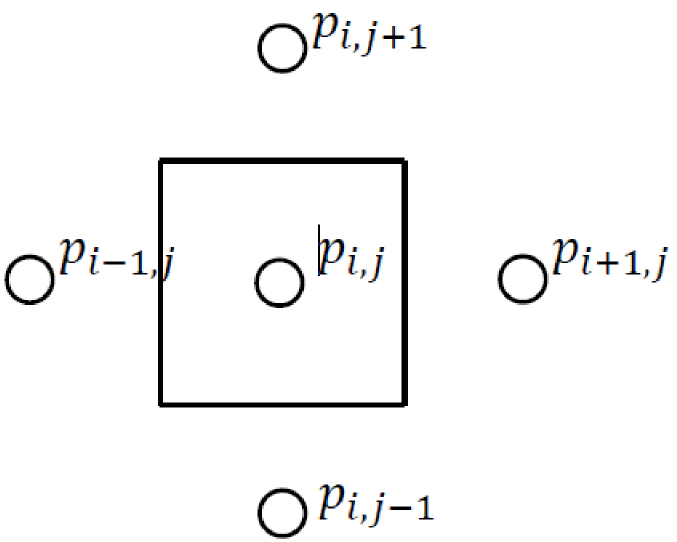
$$\nabla_x p = \frac{\partial p}{\partial x} \approx \frac{p_2 - p_1}{\Delta x}$$



Fluid Simulation

- Incompressibility and Pressure Solve
  - Laplacian (Divergence of Gradient)

$$\nabla\cdot\nabla p \approx \frac{\frac{p_{i+1,j}-p_{i,j}}{\Delta x}-\frac{p_{i,j}-p_{i-1,j}}{\Delta x}+\frac{p_{i,j+1}-p_{i,j}}{\Delta x}-\frac{p_{i,j}-p_{i,j-1}}{\Delta x}}{\Delta x}$$

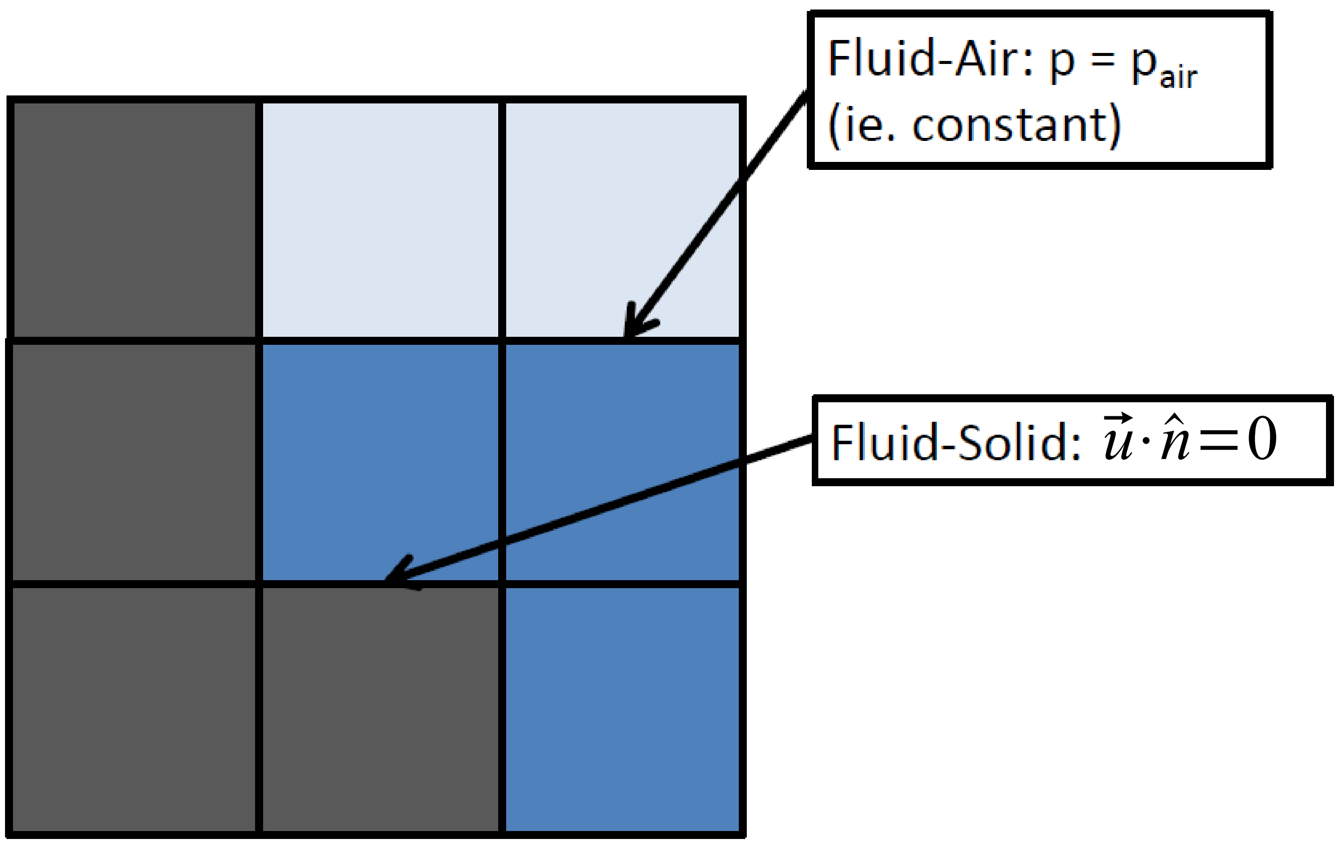


5-point stencil:

$$\begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array}$$

Fluid Simulation

- Incompressibility and Pressure Solve
  - Boundary Conditions



Fluid Simulation

- Incompressibility and Pressure Solve
  - Solving $\nabla \cdot \nabla p = \nabla \cdot \vec{u}$
    - A Poisson Equation
    - Sparse, positive definite linear system of equations
    - One equation per cell, cells globally coupled
    - Conjugate Gradients solver

Fluid Simulation

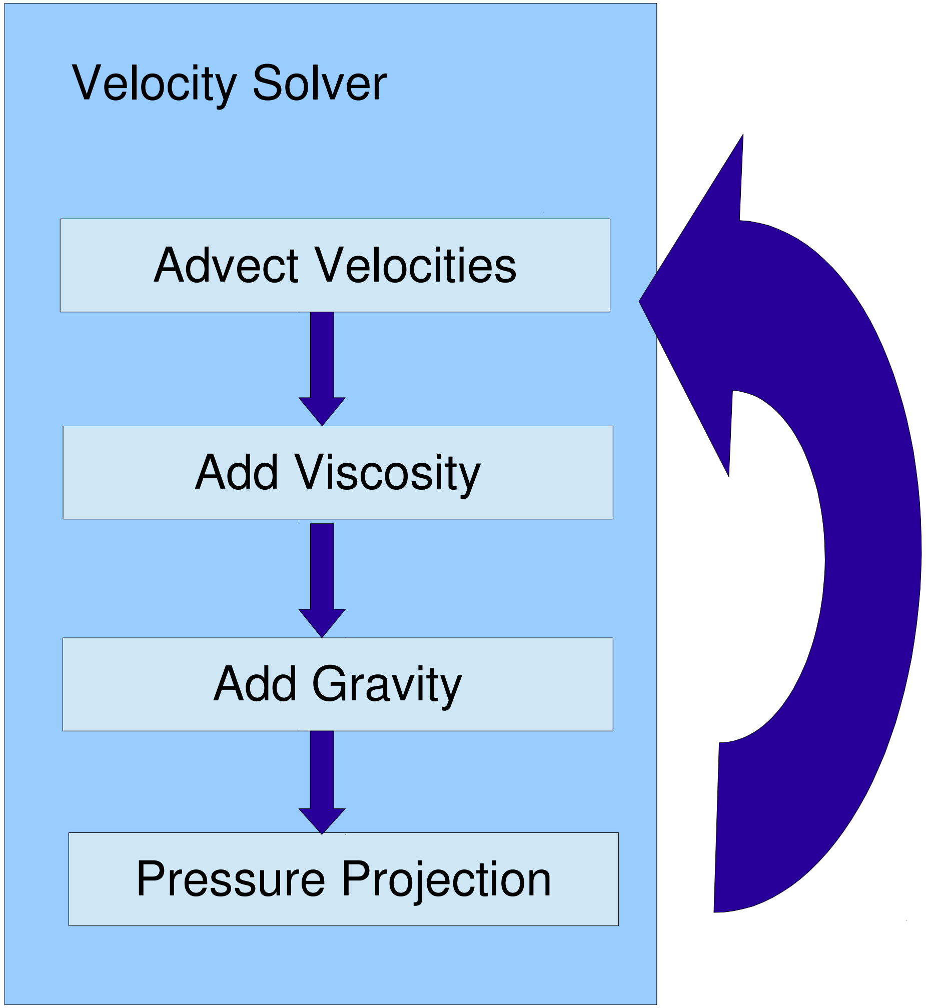
- Viscosity
  - Solving $\frac{\partial \vec{u}}{\partial t} = \nu \nabla^2 \vec{u}$
  - Dicretize and solve $\vec{u}_{new} = \vec{u}_{old} + \Delta t \, \nu \nabla^2 \vec{u}_*$
    - If $\vec{u}_*$ is $\vec{u}_{old}$, explicit integration

      No need to solve linear system
    - If $\vec{u}_*$ is $\vec{u}_{new}$, implicit integration

      Stable for high viscosities.

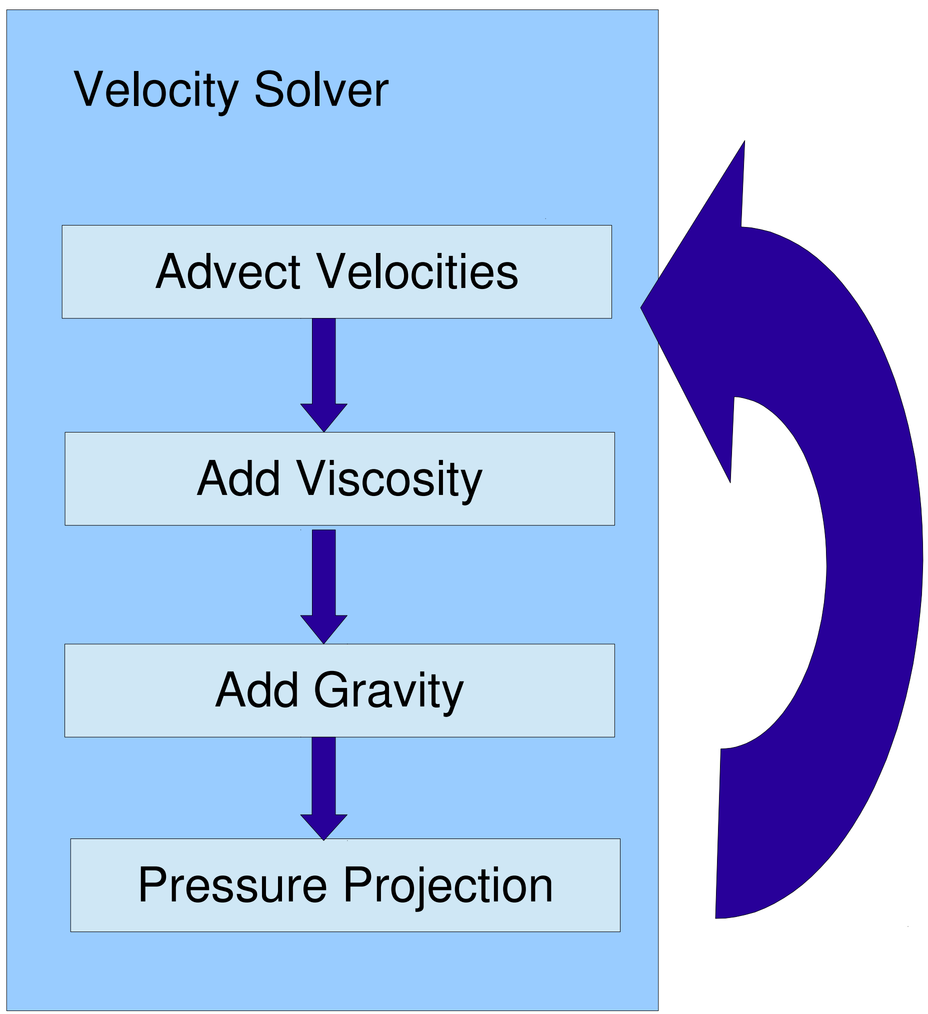
Fluid Simulation

- Complete Solver



Fluid Simulation

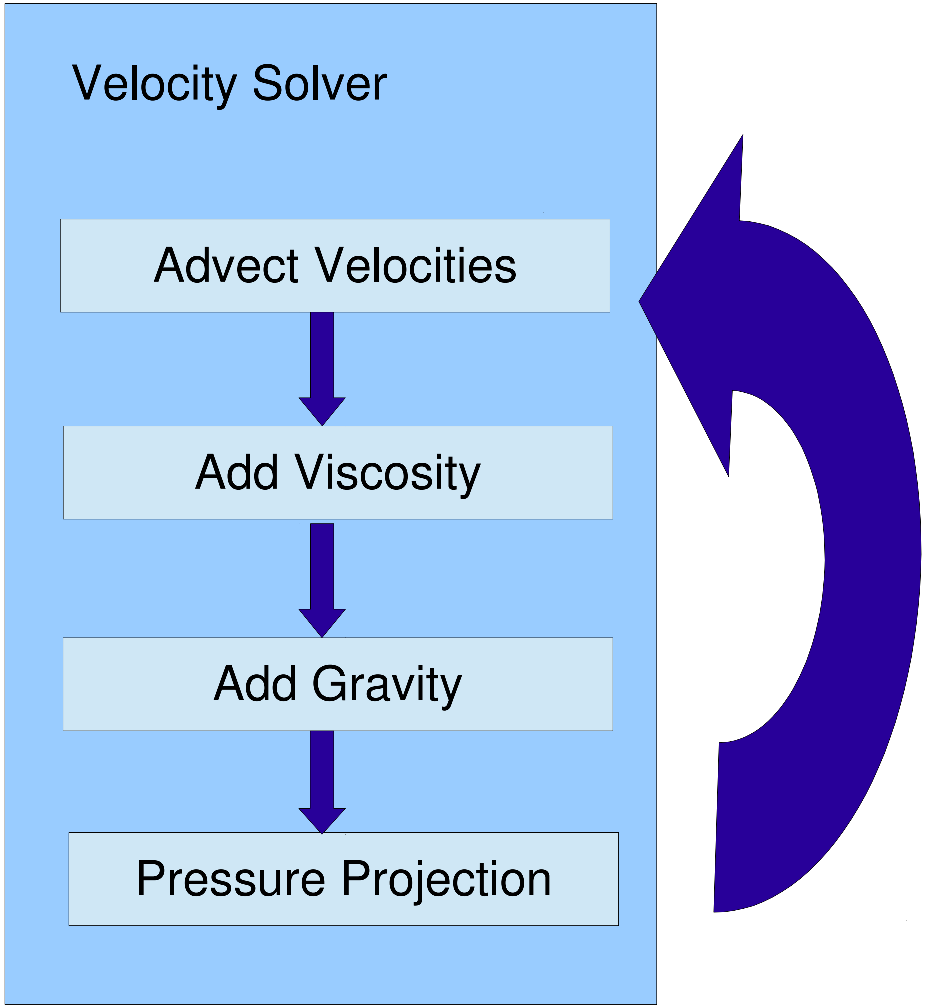
- Complete Solver



Done Finally?

Fluid Simulation

- Complete Solver



Not Quite!

Fluid Simulation

- Complete Solver