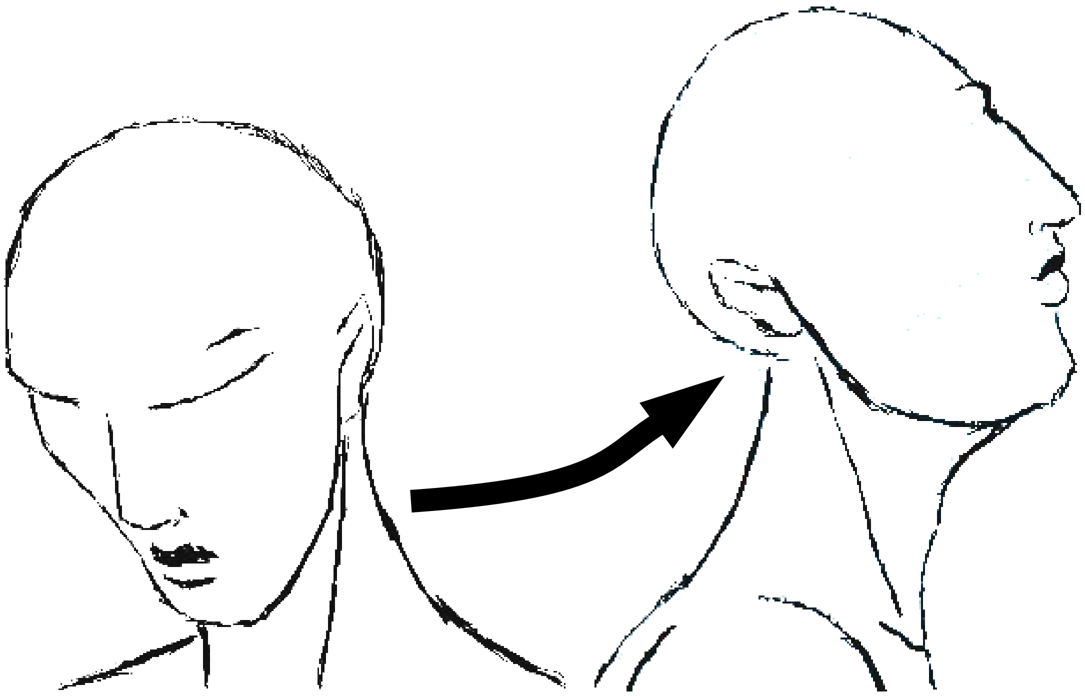
CS 775: Advanced Computer Graphics

Lecture 18 : Kinematics

Character Animation

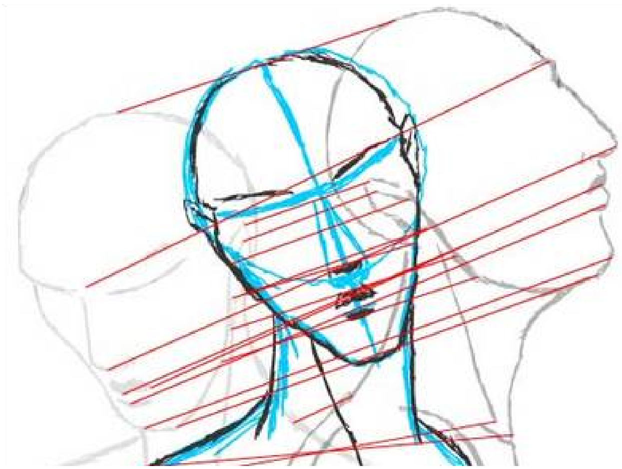
- Traditional
  - Cell Animation, hand drawn, 2D
  - Lead Animator for keyframes



http://animation.about.com/od/flashanimationtutorials/ss/flash31detanim2.htm

Character Animation

- Traditional, hand drawn animation
  - Lead Animator for keyframes and many secondary animators for the in-betweens



http://animation.about.com/od/flashanimationtutorials/ss/flash31detanim2.htm

Character Animation

- Computer assisted keyframing
  - Keyframes created/posed by hand
  - In-betweens interpolated by the semi-automatic techniques

Luxo Jr, PIXAR, 1986

How is this done?

Character Animation

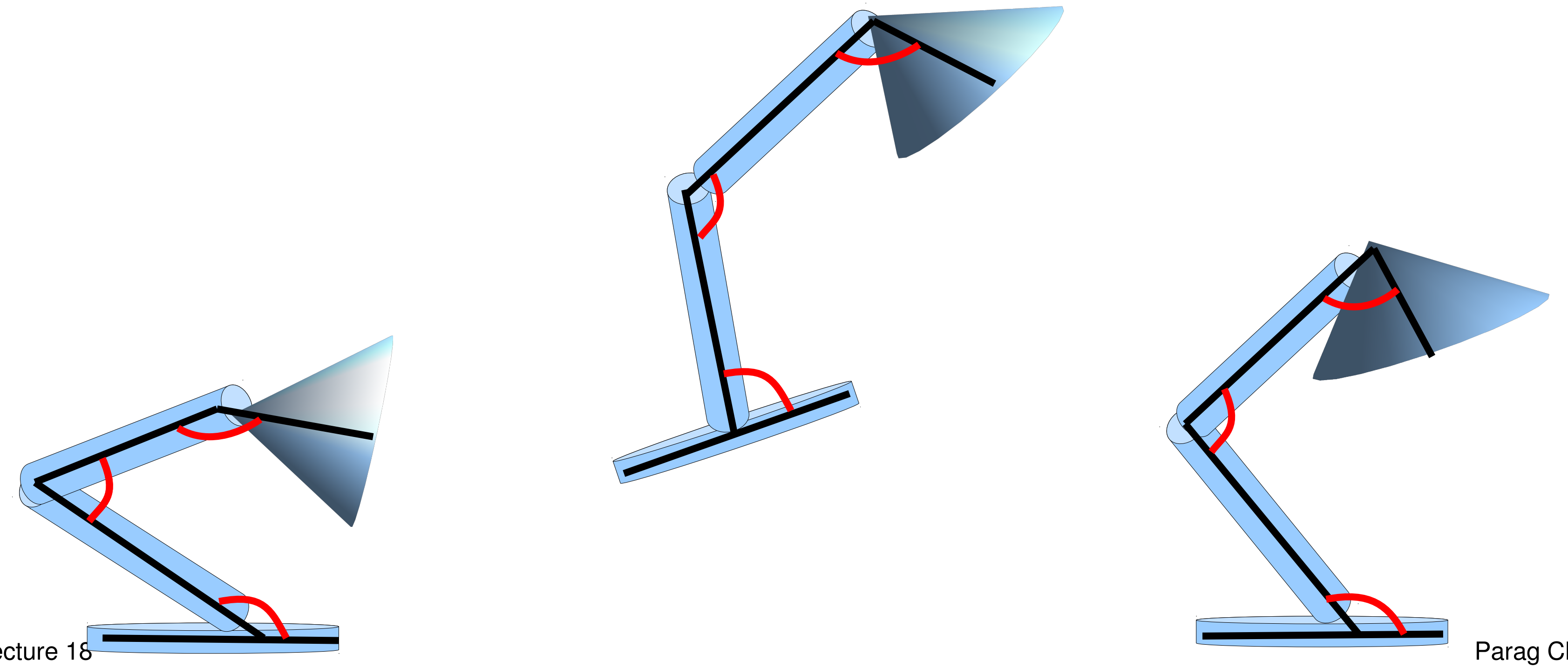
- Interpolating Position and Orientation parameters and animating (rigid) transformations.
- Linear interpolation, spline interpolation, quaternions.

Luxo Jr, PIXAR, 1986

How is this done?

Character Animation

- Creating the keyframe pose for the lamp
  - Modify the joint angles at each internal joint
  - Modify global position and orientation



Character Animation

- Kinematics
  - Study of motion of objects by studying the change in their orientation and position. The cause of the motion, i.e., the forces are *not* studied.

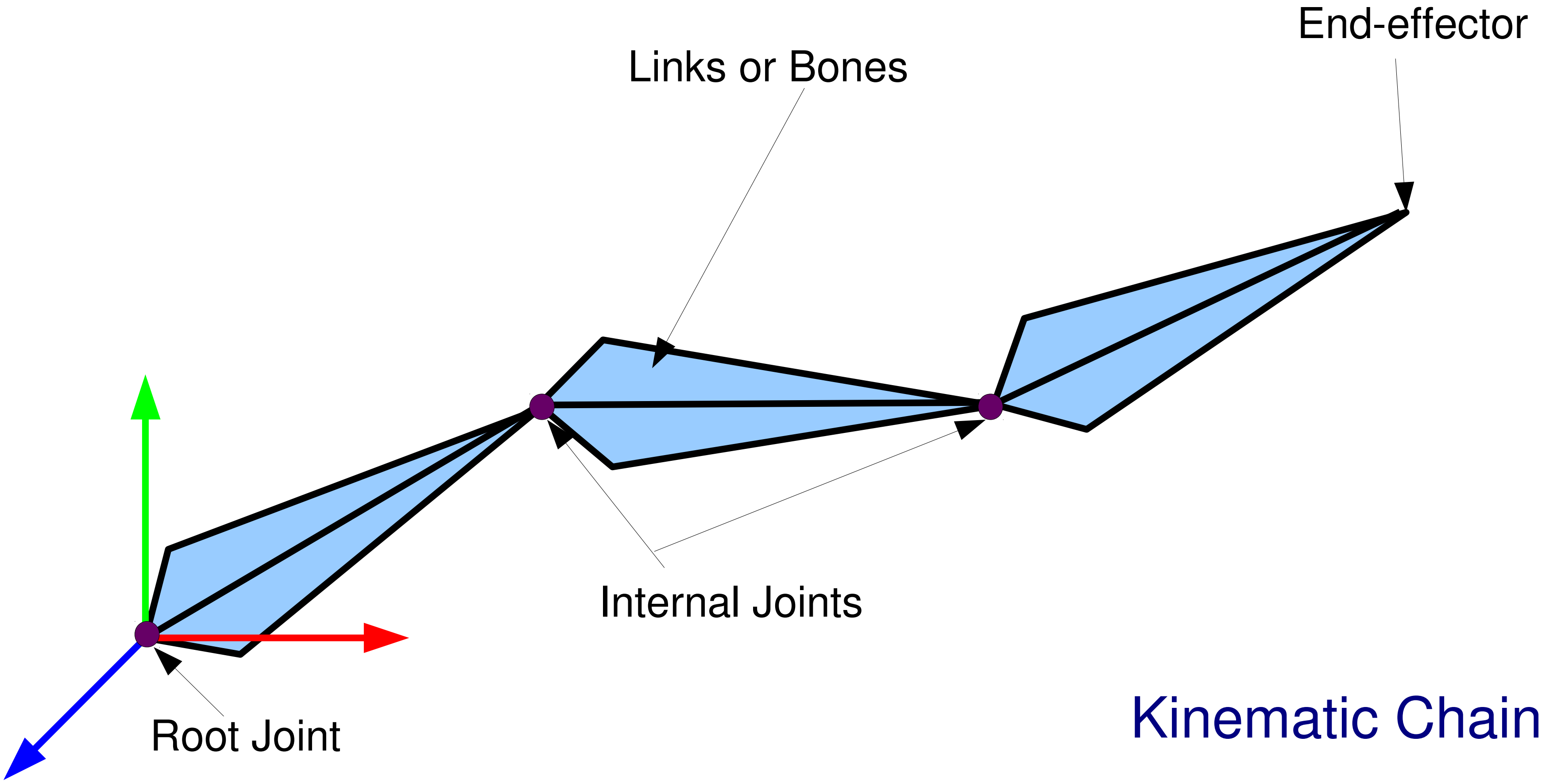
- Dynamics
  - Study of motion of objects in relation to the forces and torques that cause the motion.

Character Animation

- Kinematics
  - Forward and Inverse Kinematics – used for posing characters and interpolation in keyframed animation.
  - Faster to compute, easier to control
- Dynamics
  - Physically-based animation – used for animations involving simulations of real world physics, for e.g., collisions.
  - Harder to compute and control, more realistic (if modelled correctly),
  - Can automatically adjust to changes in the environment.

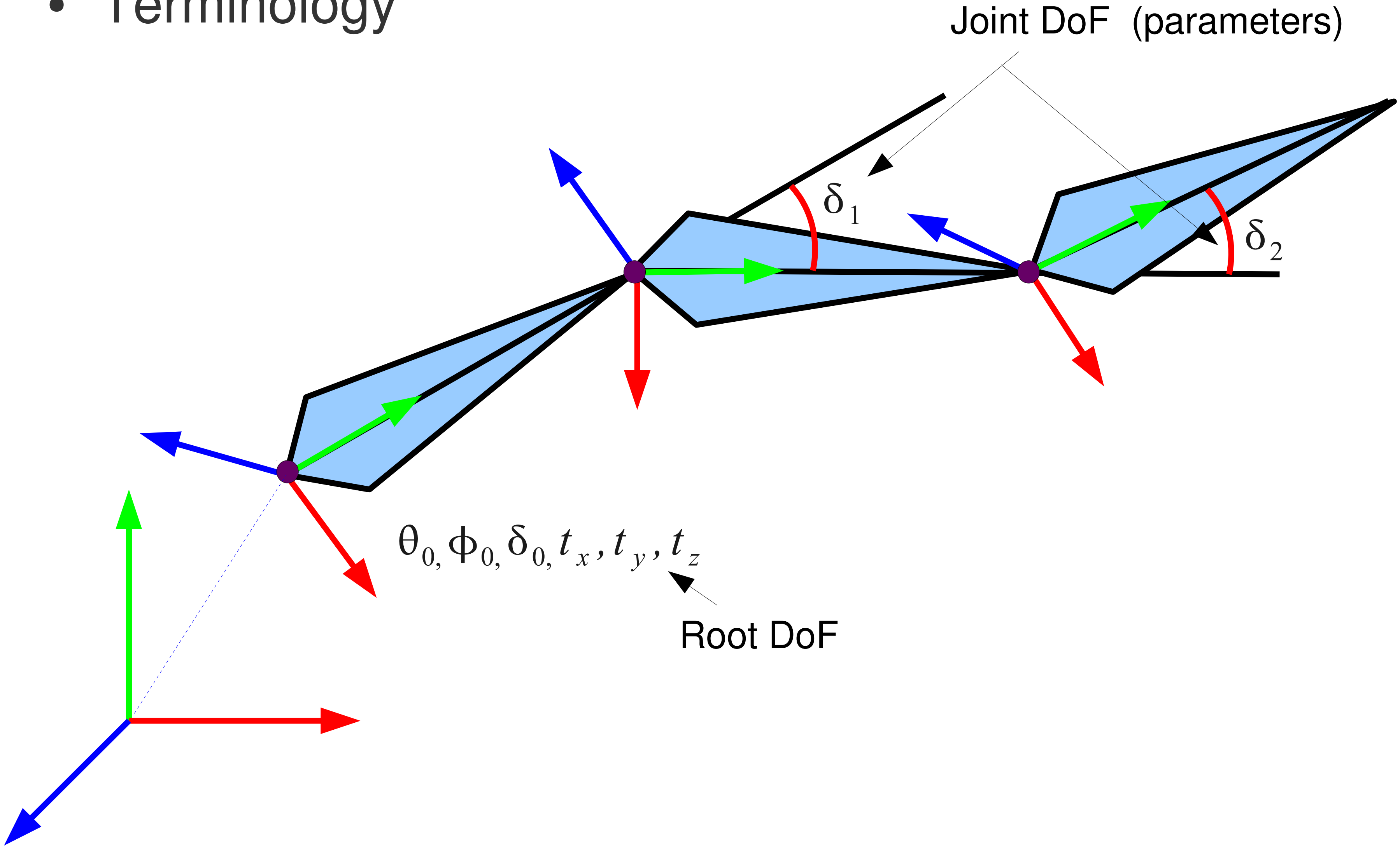
Character Animation

- Terminology



Character Animation

- Terminology



Character Animation

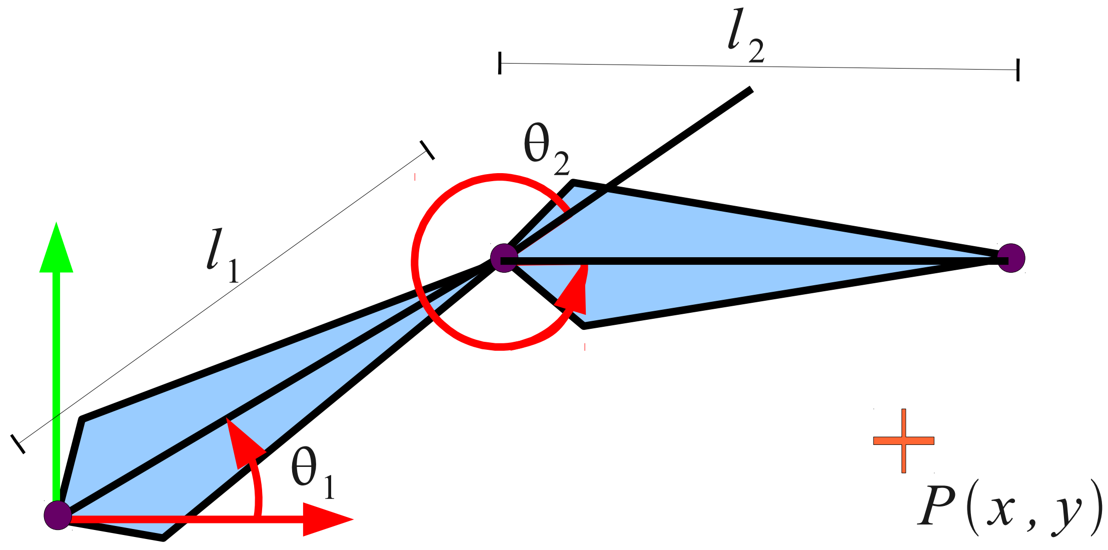
- Forward Kinematics
  - Given the joint parameters, find the position of the end effector.
  - Position of the root is given as a *global* transformation.
  - Joint parameters are given as relative rotations (*local* transformations).
  - We already know how to solve this! (from CS475/CS675)

Character Animation

- Inverse Kinematics
  - Given the desired end effector position, compute the joint parameters.
  - More interesting and easier to animate actions like – reaching, walking, grabbing.
  - Much harder to solve.
  - **Why?**

Character Animation

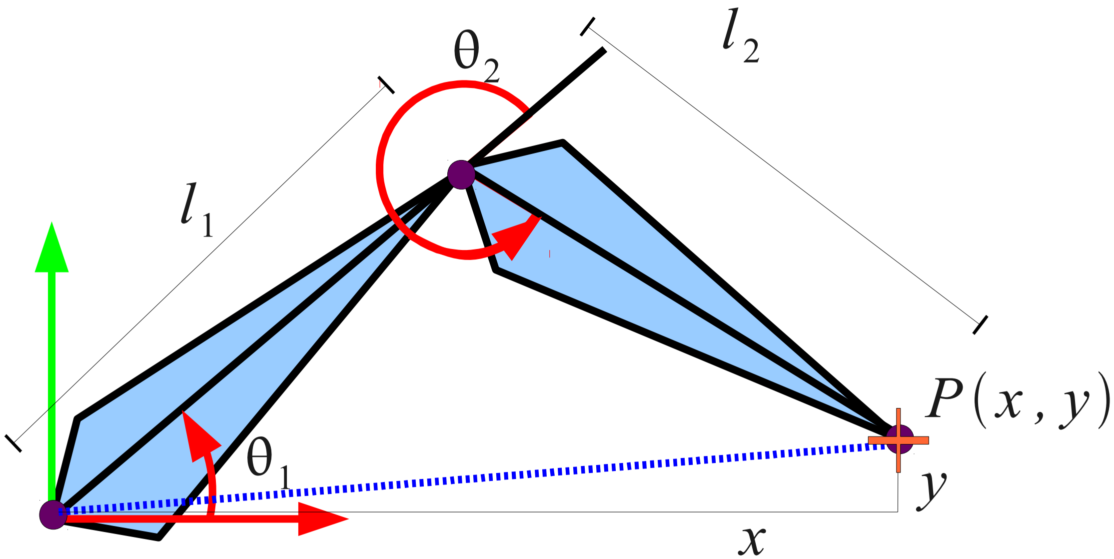
- Inverse Kinematics



Given the link lengths l_1 and l_2 the desired position of the end-effector, P find the joint parameter θ_1 and θ_2 that will make the end-effector reach the goal.

Character Animation

- Inverse Kinematics

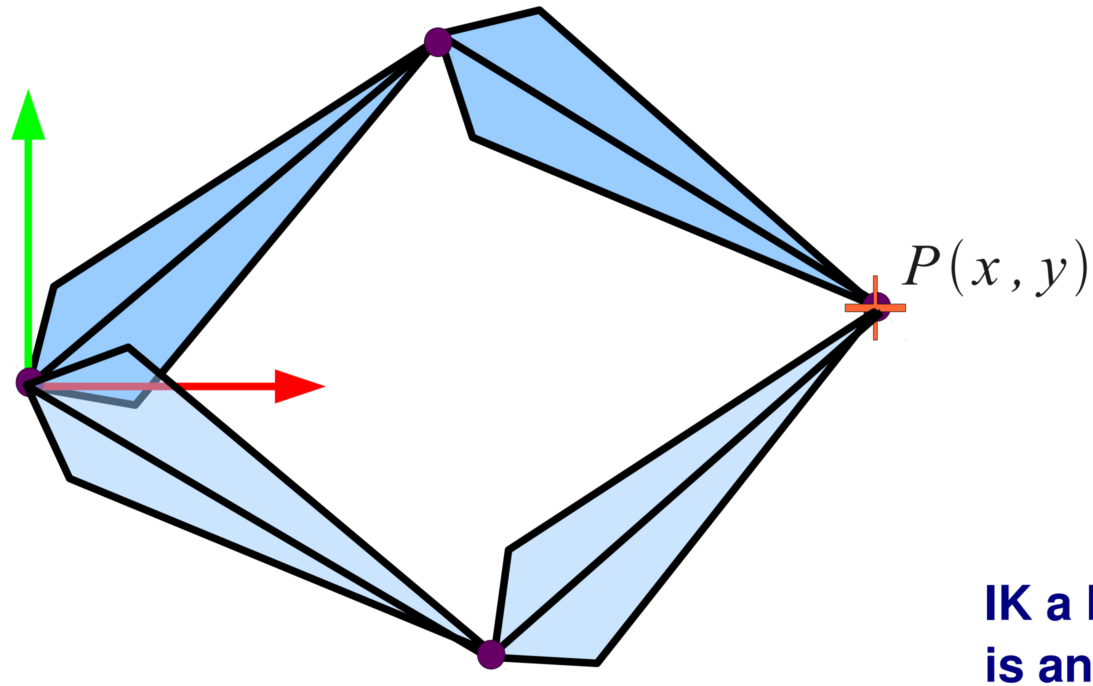


$$x = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$

$$y = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$

Character Animation

- Inverse Kinematics



IK a hard problem because it is an ill-posed problem.

Character Animation

- Posing the Kinematics problem
  - Forward

$$X = f(\Theta)$$

  - Inverse

$$\Theta = f^{-1}(X)$$

X is the vector of end-effector position and orientation. Θ is the vector of joint parameters. $f()$ is the mapping between the two.

Character Animation

- Assumptions
  - The mapping $f()$ is usually non-linear in the general case. However, we linearize the problem by localizing around the current position and use the Jacobian, i.e.,

$$dX = J(\Theta)\, d\Theta \qquad\qquad d\Theta = J^{-1}(\Theta)\, dX$$

  - Note that the Jacobian is always a function of the current set of joint parameters – so it has to recalculated every time they change!

Character Animation

- Assumptions
  - We will attempt to solve the Inverse Kinematics problems under the following assumptions:
    - All joints are hinge joints.
    - All links are rigid and do not change in length.
  - Note that these assumptions are not there because the method we will use to solve the Jacobian require these assumptions but instead to simplify our discussion a bit.

Character Animation

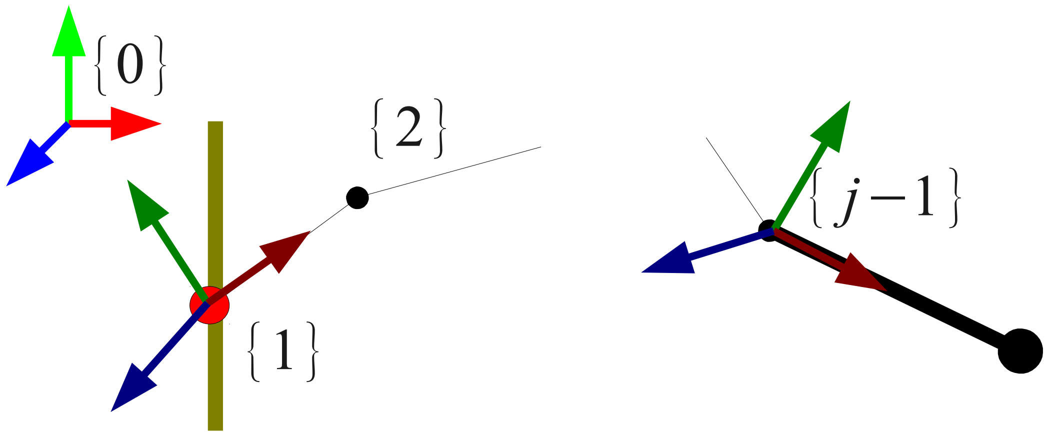
- Steps:
  - Construct the Jacobian
  - Invert the Jacobian
  - Iterate
  - Detect singularities and avoid ill-conditioning

Character Animation

- Steps:
  - **Construct the Jacobian**
  - Invert the Jacobian
  - Iterate
  - Detect singularities and avoid ill-conditioning

Character Animation

- Forming a kinematic chain (Hierarchical modelling revisit)



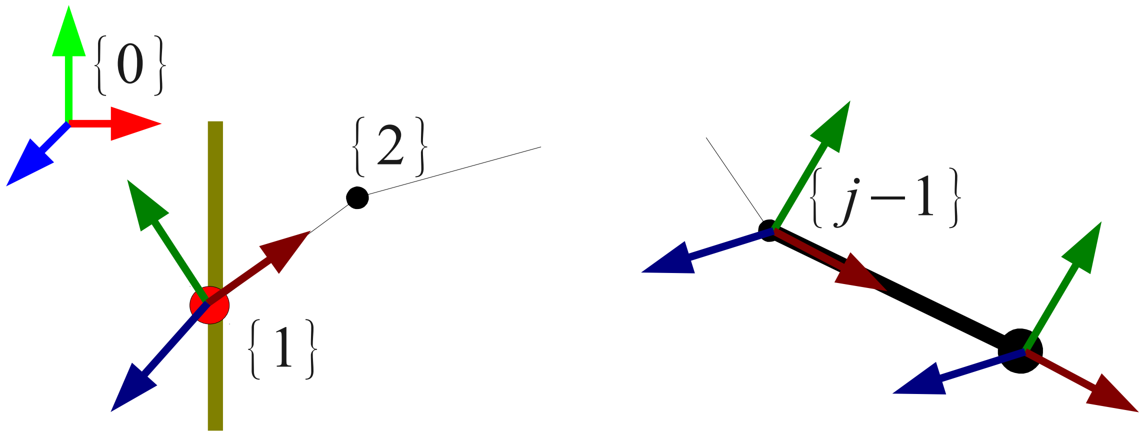
Given a kinematic chain with links (local coordinate frames) numbered from {1} to { j-1}.

- The position and orientation of the root is known in the global coordinate frame, {0}.

- Each has its local x-axis oriented along the length of the link and the local z-axis as the axis of rotation about the joint.

Character Animation

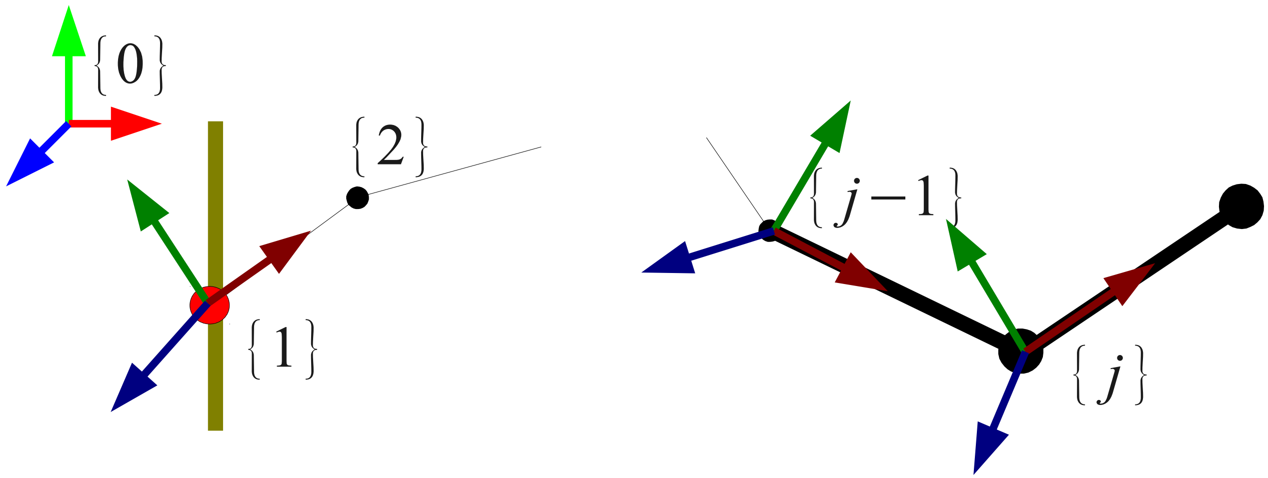
- Forming a kinematic chain



- Then the j[th] link can be added as follows:
  - Translating along the local x-axis by the length of link { j-1}, l_{j-1}

Character Animation

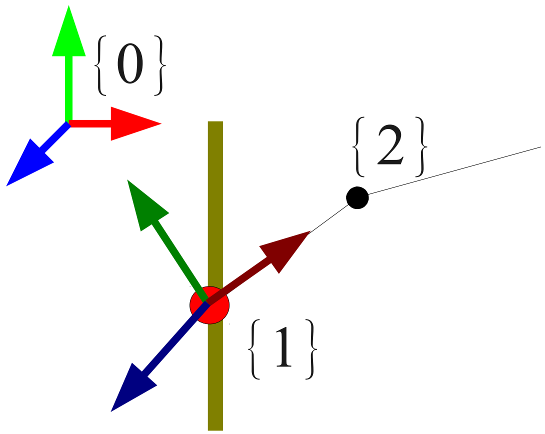
- Forming a kinematic chain

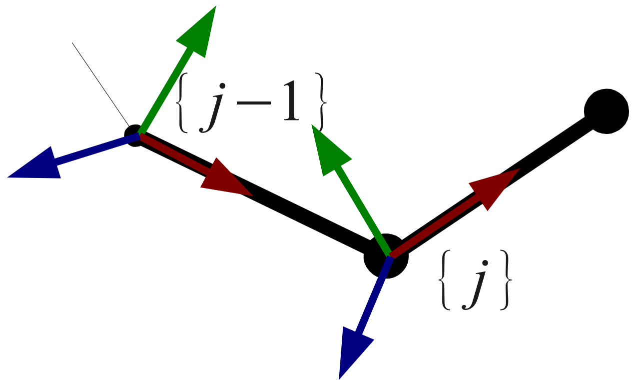


- Then the j[th] link can be added as follows:
  - Translating along the local x-axis by the length of link { j-1}, l_{j-1}
  - Rotating at the new joint origin so that the local x-axis lies along the length of the new link.

Character Animation

- Forming a kinematic chain





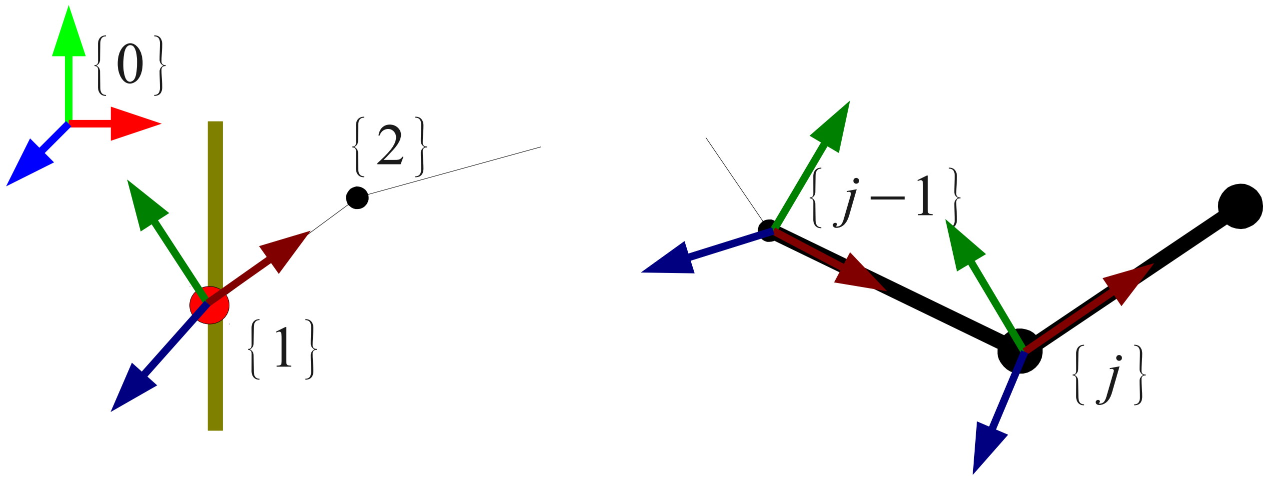
So if this the transformation between the { j} and { j-1} frame is given by ${}^{j-1}M_j$

- Any point p_j in the frame { j} can be moved to a corresponding point p_{j-1} in frame { j-1} as $p_{j-1} = {}^{j-1}M_j \, p_j$
- Therefore, the coordinates of the point in the global frame is given by

$$p_0 = M_0 . {}^{0}M_1 \ldots {}^{j-2}M_{j-1} . {}^{j-1}M_j \, p_j$$

Character Animation

- Moving to global coordinates



- The transformation from any local frame { j} to the global frame is thus, given by $T_j = {}^{0}M_1 \ldots {}^{j-2}M_{j-1} . {}^{j-1}M_j$

$$T_j = \left[\begin{array}{ccc|c} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Character Animation

- Constructing the Jacobian

$$dX = J(\Theta)\,d\Theta$$
$$\dot{X} = J(\Theta)\,\dot{\Theta}$$

- The Jacobian relates the end-effector velocities to velocities of the joint parameters.

$$\begin{bmatrix} {}^{0}V_{nx} \\ {}^{0}V_{ny} \\ {}^{0}V_{nz} \\ {}^{0}\Omega_{nx} \\ {}^{0}\Omega_{ny} \\ {}^{0}\Omega_{nz} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial f_1}{\partial \theta_{n-1}} \\ \vdots & \ddots & & & & & \vdots \\ \vdots & & & \frac{\partial f_i}{\partial \theta_j} & & & \vdots \\ \vdots & & & & & \ddots & \vdots \\ \frac{\partial f_6}{\partial \theta_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial f_6}{\partial \theta_{n-1}} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \vdots \\ \dot{\theta}_{n-2} \\ \dot{\theta}_{n-1} \end{bmatrix}$$

Character Animation

- Constructing the Jacobian

$$dX = J(\Theta)\, d\Theta$$

Known

$$\dot{X} = J(\Theta)\dot{\Theta}$$

Unknown

- The Jacobian relates the end-effector velocities to velocities of the joint parameters.

Simplifying Notation

$$\begin{bmatrix} V_{nx} \\ V_{ny} \\ V_{nz} \\ \Omega_{nx} \\ \Omega_{ny} \\ \Omega_{nz} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial f_1}{\partial \theta_{n-1}} \\ \vdots & \ddots & & & & & \vdots \\ \vdots & & & \frac{\partial f_i}{\partial \theta_j} & & & \vdots \\ \vdots & & & & & \ddots & \vdots \\ \frac{\partial f_6}{\partial \theta_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial f_6}{\partial \theta_{n-1}} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \vdots \\ \dot{\theta}_{n-2} \\ \dot{\theta}_{n-1} \end{bmatrix}$$

Character Animation

- Constructing the Jacobian

  The functions $f_{1,} f_{2,} f_3$ relate the joint parameter *angular* velocities to the linear velocity of the end-effector.

  - The functions $f_4 f_{5,} f_6$ relate the joint parameter *angular* velocities to the angular velocity of the end-effector.

  - The velocities of the end-effector are known in the **global coordinate** system. So the Jacobian has to be formed in the global coordinate system.

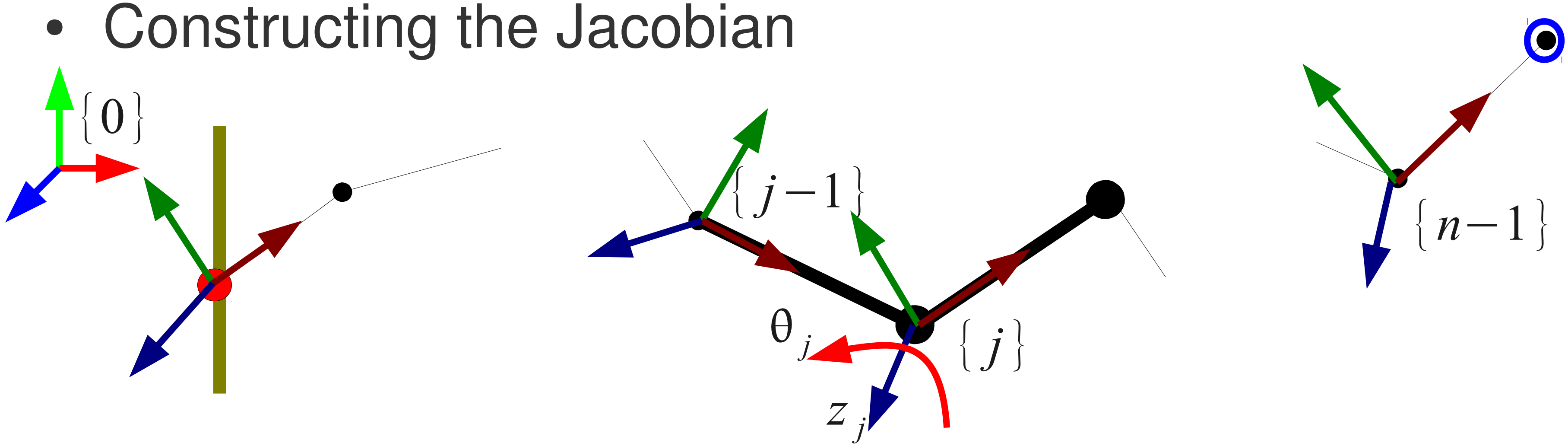
Character Animation

- Constructing the Jacobian
  - The jth column of the Jacobian relates the jth joint parameter *angular* velocity to the velocity of the end-effector.

$$J(\Theta)=\begin{bmatrix} \frac{\partial f_1}{\partial \theta_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial f_1}{\partial \theta_{n-1}} \\ \vdots & \ddots & & & & & \vdots \\ \vdots & & & \frac{\partial f_i}{\partial \theta_j} & & & \vdots \\ \vdots & & & & & \ddots & \vdots \\ \frac{\partial f_6}{\partial \theta_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial f_6}{\partial \theta_{n-1}} \end{bmatrix}$$

Character Animation

- Constructing the Jacobian



- For the angular velocity, $\Omega_j = z_j \dot{\theta}_j$
  - The contribution of this angular velocity to the angular velocity of the end-effector (in global coordinates) is given by

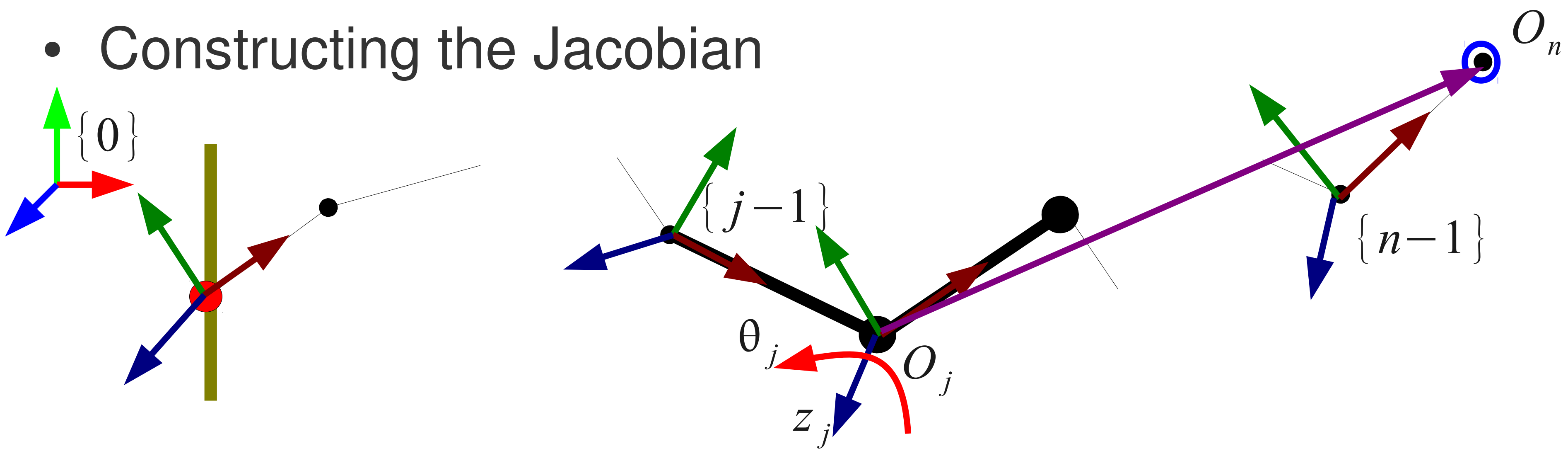
$$^0\Omega_j = T_j z_j \dot{\theta}_j = u_j \dot{\theta}_j$$

  - So

$$\frac{\partial f_4}{\partial \theta_j} = (T_j z_j)_x = u_{jx} \quad \frac{\partial f_5}{\partial \theta_j} = (T_j z_j)_y = u_{jy} \quad \frac{\partial f_6}{\partial \theta_j} = (T_j z_j)_z = u_{jz}$$

Character Animation

- Constructing the Jacobian



- For the angular velocity, $\Omega_j = z_j \dot{\theta}_j$
  - The contribution of this angular velocity to the linear velocity of the end-effector (in global coordinates) is given by

$$^0V_j = ((T_j z_j) \times (P_n - P_j))\dot{\theta}_j = v_j \dot{\theta}_j$$

  - So

where

$$P_j = T_j O_j$$

$$P_n = T_n O_n$$

$$\frac{\partial f_1}{\partial \theta_j} = v_{jx} \quad \frac{\partial f_2}{\partial \theta_j} = v_{jy} \quad \frac{\partial f_3}{\partial \theta_j} = v_{jz}$$

Character Animation

- Steps:
  - Construct the Jacobian
  - **Invert the Jacobian**
  - Iterate
  - Detect singularities and avoid ill-conditioning

Character Animation

- Inverting the Jacobian
  - To solve IK, we must invert the Jacobian

$$d\Theta = J^{-1}(\Theta)\, dX$$
$$\dot{\Theta} = J^{-1}(\Theta)\, \dot{X}$$

  - Note that the Jacobian is a *fat* matrix, i.e., it has more columns than rows.
  - So the system is underconstrained – there is more than one possible solution.
  - We choose the simplest solution, i.e., one of minimum norm:

Find the unique $\dot{\Theta}$ for which we have: $min\|\dot{\Theta}\|^2 \quad \text{s.t.} \quad \dot{X} = J(\Theta)\dot{\Theta}$

Character Animation

- Linear Least Squares (Digression)
  - For the system

$$AX = B$$

  - If A has full row rank then the unique X for which we get

$$min \|X\|^2 \quad \text{s.t.} \quad AX = B$$

  is given by

$$X = A^T (AA^T)^{-1} B = A^+ B$$

  - To see how this is so, solve the Lagrangian,

$$L = \|X\|^2 + \lambda^T (AX - B)$$

Character Animation

- Linear Least Squares (Digression)
  - For the system

$$AX = B$$

  - If A has full row rank then the unique X for which we get

$$min \|X\|^2 \quad \text{s.t.} \quad AX = B$$

    is given by $$X = A^T (AA^T)^{-1} B = A^+ B$$

    Moore-Penrose Pseudoinverse

  - To see how this is so, solve the Lagrangian,

$$L = \|X\|^2 + \lambda^T (AX - B)$$

Character Animation

- Singular Value Decomposition (Digression)
  - Any rectangular, real, *m*x*n* matrix A can be decomposed as:

$$A = U\,\Sigma\,V^T$$

$$\begin{aligned} \text{where } U &\text{ is a } m\times m \text{ orthogonal matrix} \\ \Sigma &\text{ is a } m\times n \text{ diagonal matrix} \\ V &\text{ is a } n\times n \text{ orthogonal matrix} \end{aligned}$$

  - The pseudoinverse can be computed from the SVD as

$$A^+ = V\,\Sigma^+\,U^{\mathrm{T}}$$

where Σ^+ is formed by transposing Σ
and taking a reciprocal of every non-zero singular value

Character Animation

- Inverting the Jacobian
  - So we solve

$$d\Theta = J^{-1}(\Theta)\, dX$$

  - By computing the pseudoinverse of the Jacobian using the SVD

$$J^{+} = V\Sigma^{+}U^{\mathrm{T}} = \sum_{i=1}^{r} \sigma_i^{-1} \boldsymbol{v}_i \boldsymbol{u}_i^{\boldsymbol{T}}$$

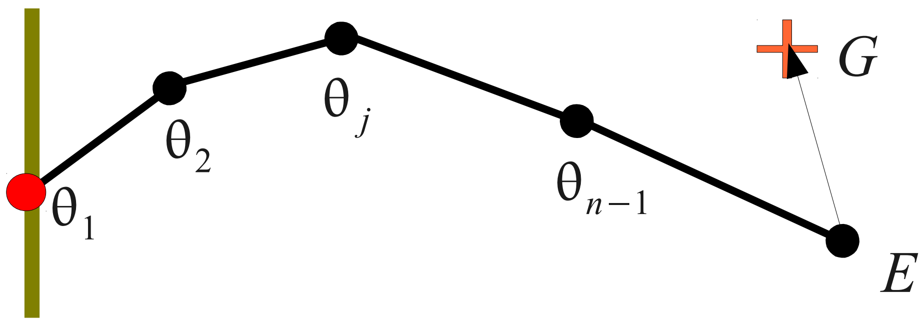
where $r = m$ if the Jacobian has full row rank

Character Animation

- Steps:
  - Construct the Jacobian
  - Invert the Jacobian
  - **Iterate**
  - Detect singularities and avoid ill-conditioning

Character Animation

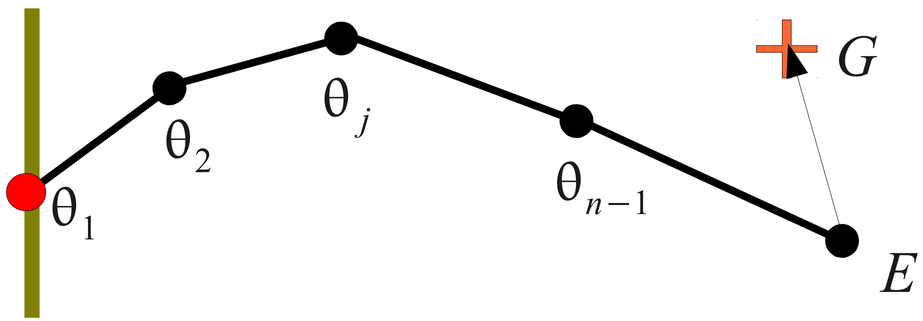
- Starting the IK solver and iterating



- Initialize the linear velocity components of the end-effector to $dX_{1..3} = G - E$, compute the Jacobian, invert, update the joints parameters and iterate.
- If dX is too large then due to local nature of the solution, tracking errors will occur, given by $\| J(\Theta) d\Theta - dX \|$
- So sometimes we go from E to G in by taking smaller steps in the $G - E$ direction.

Character Animation

- Starting the IK solver and iterating



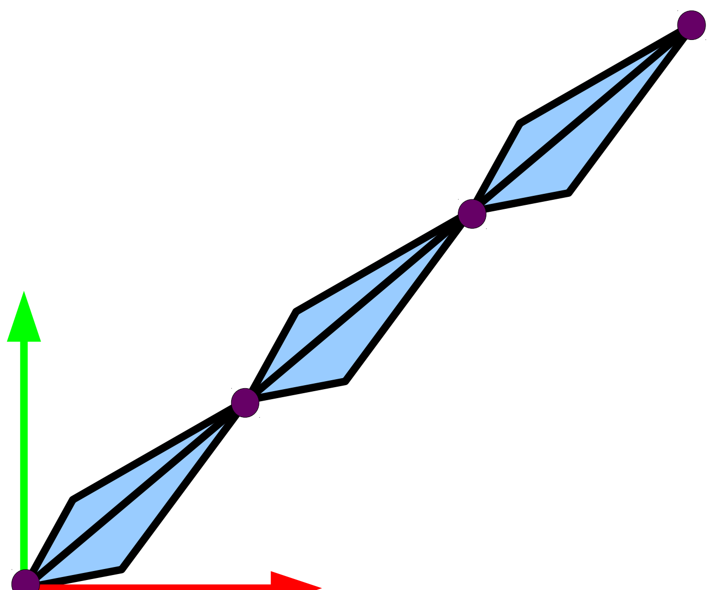
- Iterate only a fixed number of times and check whether $\|G-E\|$ has fallen below some tolerance.
- Also check after a few iterations whether the arm has become fully extended, is parallel to $G-E$ and the goal is still out of reach.
- What about the angular velocity components of the end-effector?

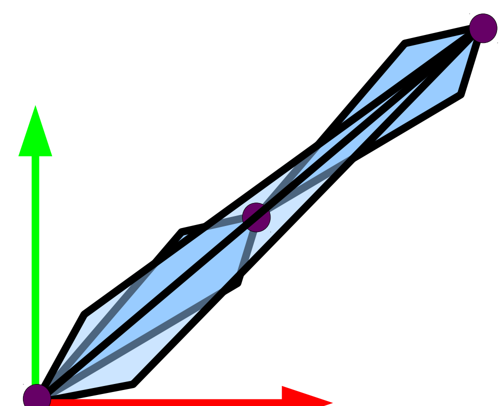
Character Animation

- Steps:
  - Construct the Jacobian
  - Invert the Jacobian
  - Iterate
  - **Detect singularities and avoid ill-conditioning**

Character Animation

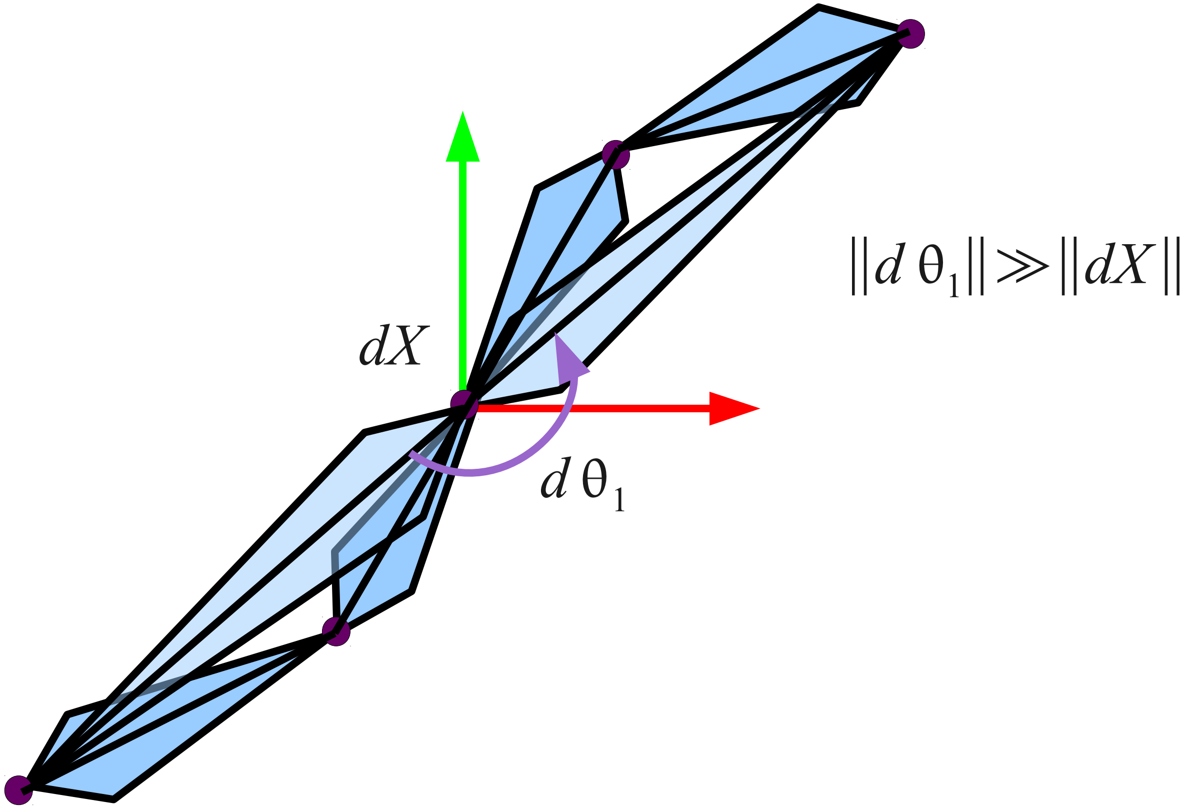
- Detect singularities and avoid ill-conditioning
  - The Jacobian becomes singular when it loses rank.
  - This is detectable using the condition number, $C = \frac{\sigma_{max}}{\sigma_{min}}$ of the Jacobian matrix.





Character Animation

- Detect singularities and avoid ill-conditioning
  - The solution also becomes ill-conditioned near a singulartiy, resulting in very high joint space velocities.



Character Animation

- Detect singularities and avoid ill-conditioning
  - To prevent ill-conditioning we damp the joint space velocities by searching for a solution that minimizes the sum: $\| J(\Theta) d\Theta - dX \|^2 + \lambda^2 \| d\Theta \|^2$
  - The solution to this is given by

$$d\Theta = J^T (J J^T + \lambda^2 I)^{-1} dX$$

$$= V \Sigma^T (\Sigma \Sigma^T + \lambda^2 I)^{-1} U^T dX$$

$$= \left(\sum_{i=1}^{r} \frac{\sigma_i}{\sigma_i^2 + \lambda^2} \boldsymbol{v_i} \boldsymbol{u_i^T} \right) dX$$

  - This is known as the *damped* least squares solution.