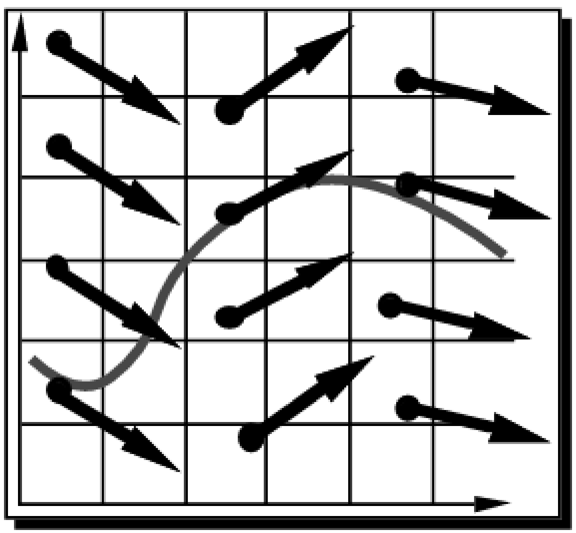
CS 775: Advanced Computer Graphics

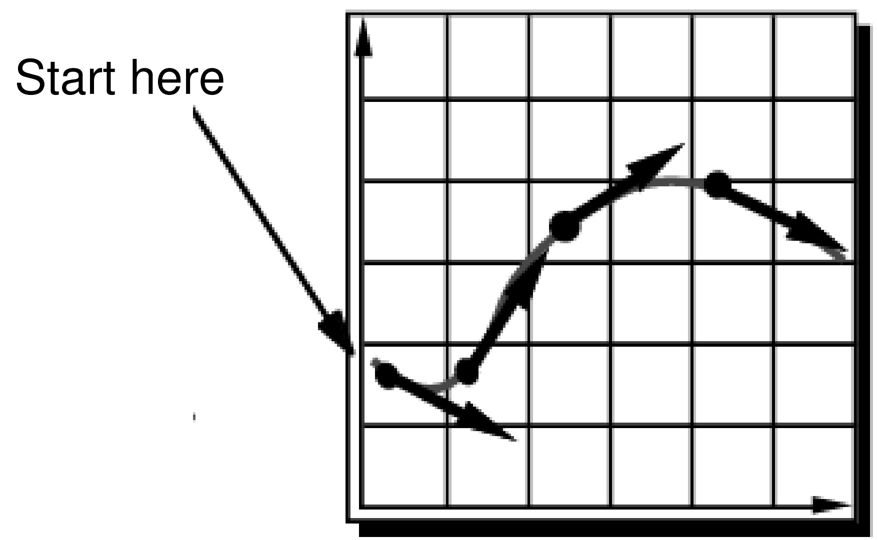
Lecture 6 : Physically-Based Animation

Physically-Based Animation

- Given the *state* of a system, $\mathbf{x}$ $\qquad \dot{\mathbf{x}} = f(\mathbf{x}, t) \qquad \dot{\mathbf{x}} \sim \frac{d\mathbf{x}}{dt}$
- Initial value problem $\quad \mathbf{x}(t_0) = \mathbf{x_0}$



The derivative function forms a vector field.



Follow the vectors.

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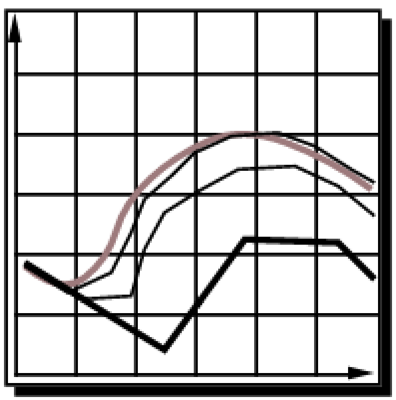
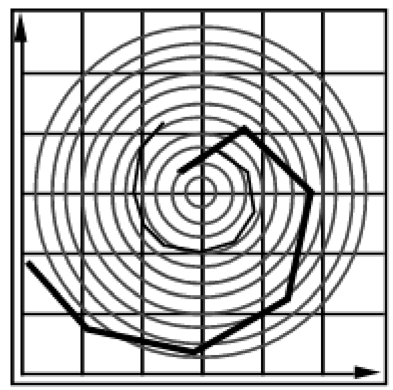
Numerical Solutions

Euler's method

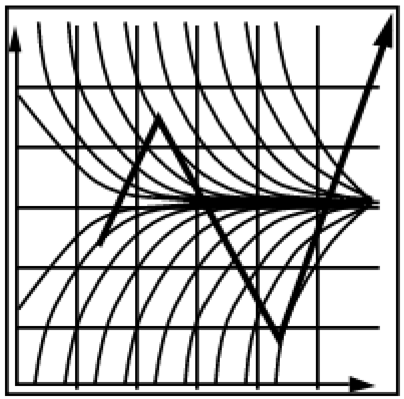
$$\mathbf{x}(t_0+h)=\mathbf{x_0}+h\,\dot{\mathbf{x}}(t_0)$$



- Simplest method
- Discrete stepsize
- Bigger steps, bigger error



- Integral curves of 2D function f are concentric circles.
- Drift caused by Euler's method moves the particle on a spiral instead



- $f=-kx$
- Oscillates $h>1/k$
- Diverges $h>2/k$

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Numerical Solutions

Euler's Method

Consider the Taylor series

$$\mathbf{x}(t_0+h)=\mathbf{x_0}+h\,\dot{\mathbf{x}}(t_0)+\frac{h^2}{2!}\ddot{\mathbf{x}}(t_0)+\ldots+\frac{h^n}{n!}\frac{\partial^n \mathbf{x}}{\partial t^n}$$

Error ϵ in Euler's method: $O(h^2)$

$$h \rightarrow \frac{h}{2} \qquad \epsilon \rightarrow \frac{\epsilon}{4} \qquad n_{steps} \rightarrow 2\,n_{steps}$$

For an interval t_0 to t_1

Numerical Solutions

Mid-point Method : Improving Euler

$$\mathbf{x}(t_0+h)=\mathbf{x_0}+h\dot{\mathbf{x}}(t_0)+\frac{h^2}{2!}\ddot{\mathbf{x}}(t_0)+O(h^3) \qquad \text{- (1)}$$

Now, $\ddot{\mathbf{x}}=\frac{\partial f}{\partial \mathbf{x}}\dot{\mathbf{x}}=f'f$ assuming $\dot{\mathbf{x}}=f(\mathbf{x}(t))$

A Taylor series of the function f gives:

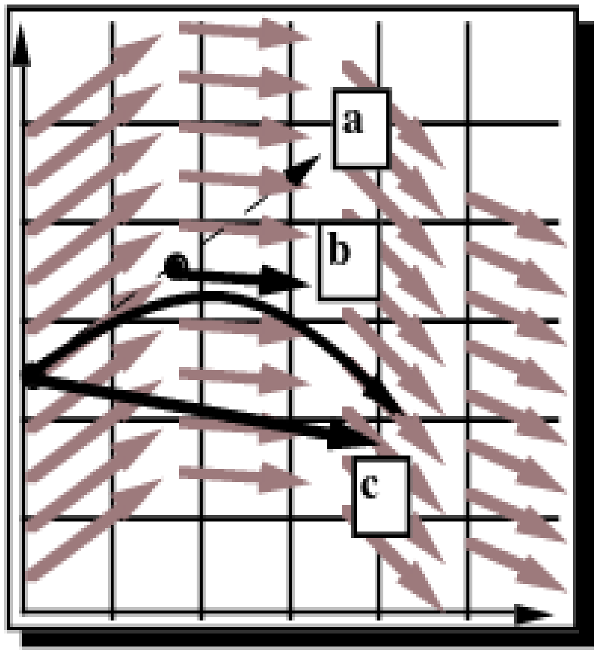
$$f(\mathbf{x}+\Delta\mathbf{x})=f(\mathbf{x_0})+\Delta\mathbf{x}f'(\mathbf{x_0})+O(\Delta\mathbf{x}^2) \qquad \text{- (2)}$$

Substitute $\Delta\mathbf{x}=\frac{h}{2}f(\mathbf{x_0})$ in the equation above, simplify, equations 1 and 2 give

$$\mathbf{x}(t_0+h)=\mathbf{x_0}+h\left(f\left(\mathbf{x_0}+\frac{h}{2}f(\mathbf{x_0})\right)\right)$$

Numerical Solutions

Mid-point Method



a. Compute an Euler step

$$\Delta \mathbf{x} = \Delta t\, \mathbf{f}(\mathbf{x}, t)$$

b. Evaluate f at the midpoint

$$\mathbf{f}_{mid} = \mathbf{f}\left(\frac{x + \Delta x}{2}, \frac{t + \Delta t}{2}\right)$$

c. Take a step using the midpoint value

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t\, \mathbf{f}_{mid}$$

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Numerical Solutions

Runge-Kutta of Order 4 (RK4)

$O(h^5)$ error

$$k_1 = hf(\mathbf{x_0}, t_0)$$

$$k_2 = hf\left(\mathbf{x_0} + \frac{k_1}{2}, t_0 + \frac{h}{2}\right)$$

$$k_3 = hf\left(\mathbf{x_0} + \frac{k_2}{2}, t_0 + \frac{h}{2}\right)$$

$$k_4 = hf(\mathbf{x_0} + k_3, t_0 + h)$$

$$\mathbf{x}(t_0 + h) = \mathbf{x_0} + \frac{1}{6}k_1 + \frac{1}{6}k_3 + \frac{1}{6}k_3 + \frac{1}{6}k_4$$

Particle Systems

- Motion of the particle is governed by Newton's laws of motion.
- Phase space equation of motion

$$[\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{v}_1, \dot{v}_2, \dot{v}_3] = [v_1, v_2, v_3, f_1/m, f_2/m, f_3/m]$$

- Every particle stores its position, velocity, mass and has a force accumulator.
- A particle system is just a collection of particles.

Particle Systems

Types of forces:

- Unary forces – gravity, drag.

$$\boldsymbol{f} = m\,\boldsymbol{g} \qquad\qquad \boldsymbol{f} = -k_d\,\boldsymbol{v}$$

- *n*-ary forces – springs connecting a set of particles.

$$\boldsymbol{f_a} = -\left[k_s(|\boldsymbol{l}| - r) + k_d \frac{\dot{\boldsymbol{l}} \cdot \boldsymbol{l}}{|\boldsymbol{l}|}\right]\frac{\boldsymbol{l}}{|\boldsymbol{l}|} \qquad\qquad \boldsymbol{f_b} = -\boldsymbol{f_a}$$

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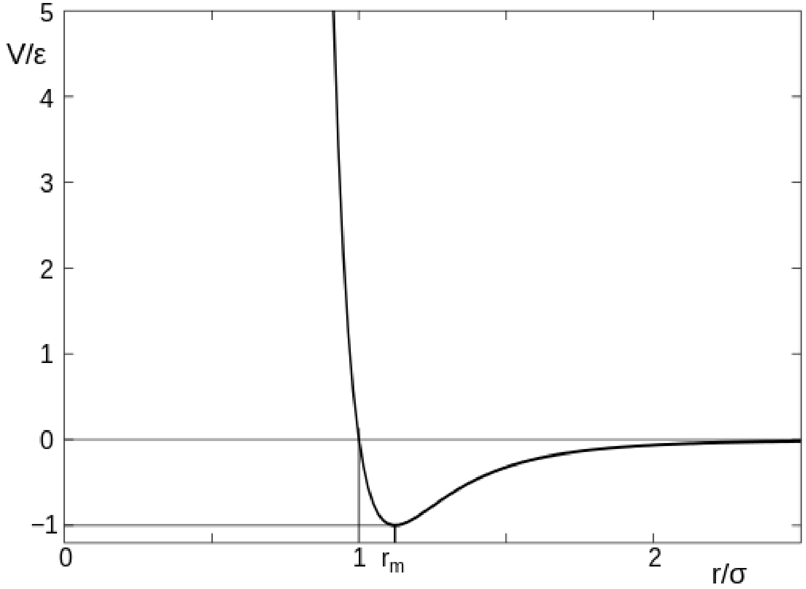
Particle Systems

Types of forces:

- Spatial interaction forces – attraction, repulsion in a neighbourhood.

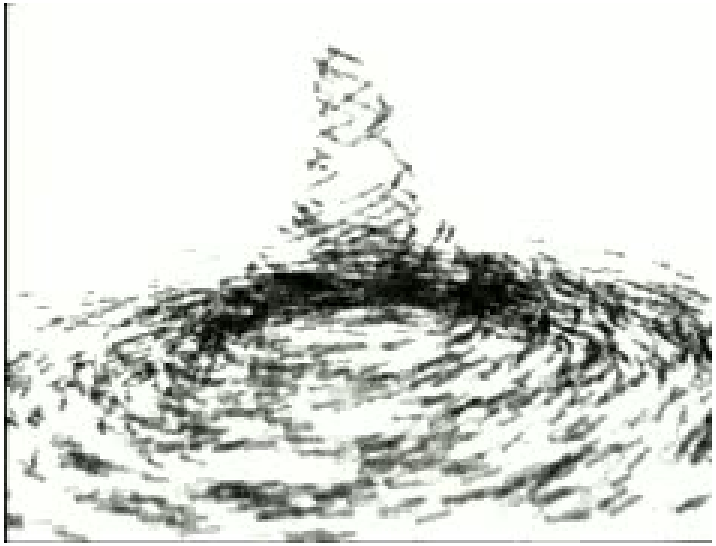


Lennard-Jones Potential

$$V_{LJ} = 4\epsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]$$

Particle Systems

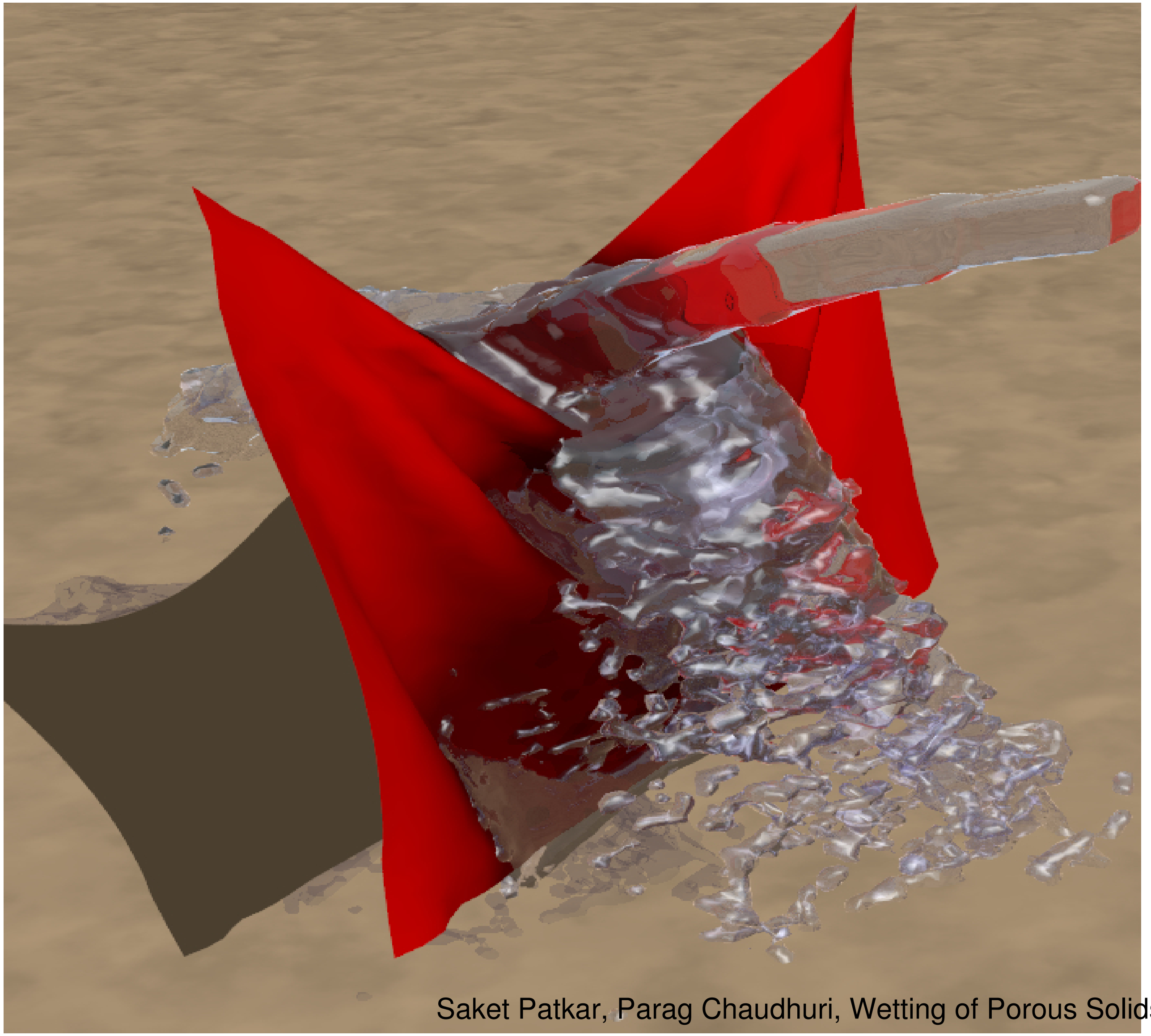
“Genesis Effect” from Start Trek 2: The Wrath of Khan



Video Clip from “Particle Dreams” by Karl Sims

- Blender Fire, Boids
- Reeves, 1983: Particle Systems – A Technique for Modeling Fuzzy Objects

Particle Fluids



Saket Patkar, Parag Chaudhuri, Wetting of Porous Solids, IEEE TVCG 2013