Computer Vision (CS 763)

Relative Orientation and the Fundamental Matrix

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Cameras to Measure Directions

An image point in a camera image defines a ray to the object point

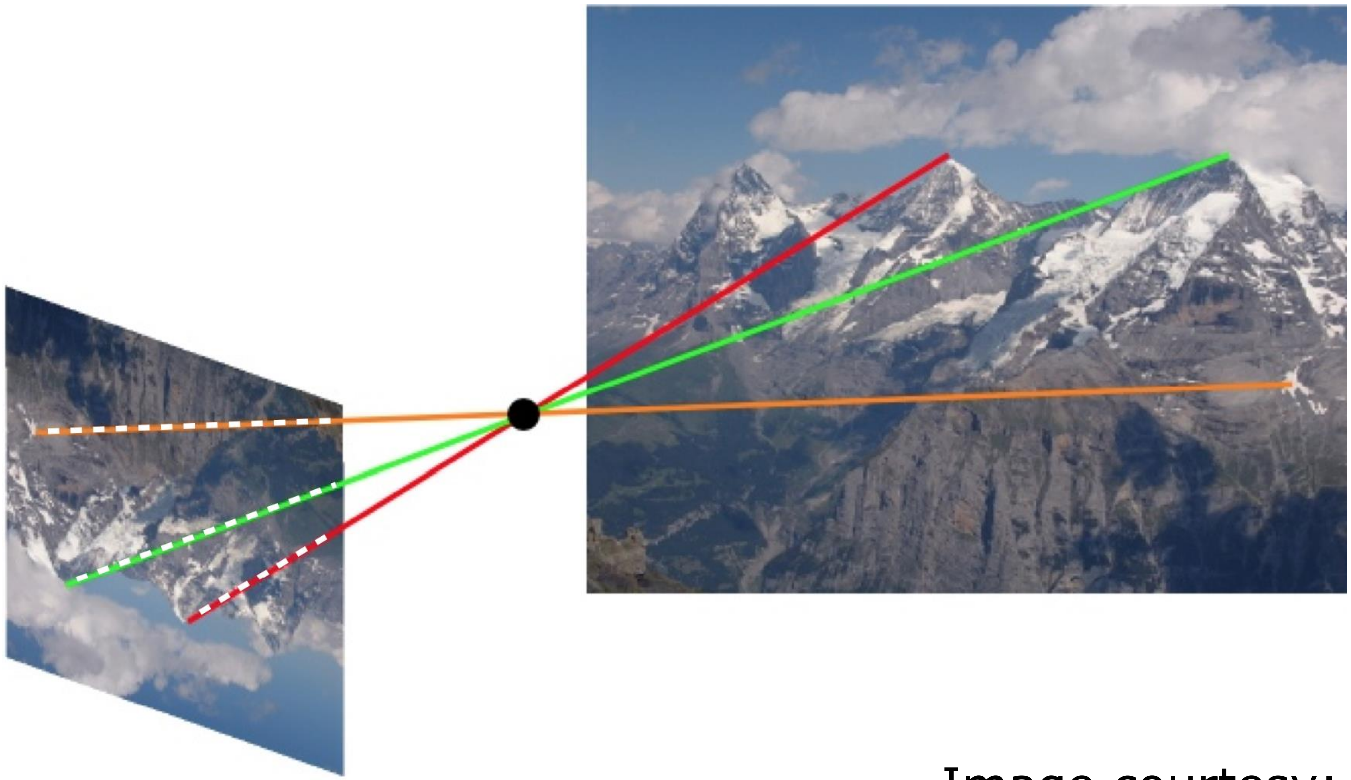


Image courtesy: Schindler

3D Perception

Multiple observations from different directions allows for estimating the 3D location of points via triangulation

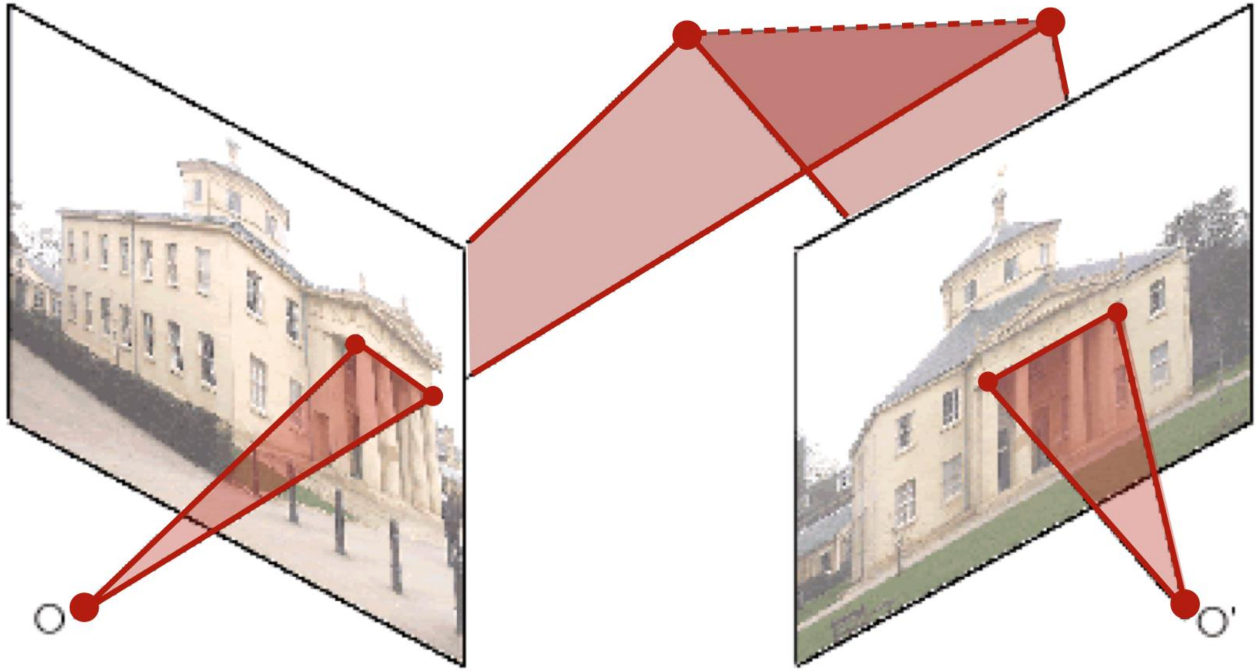
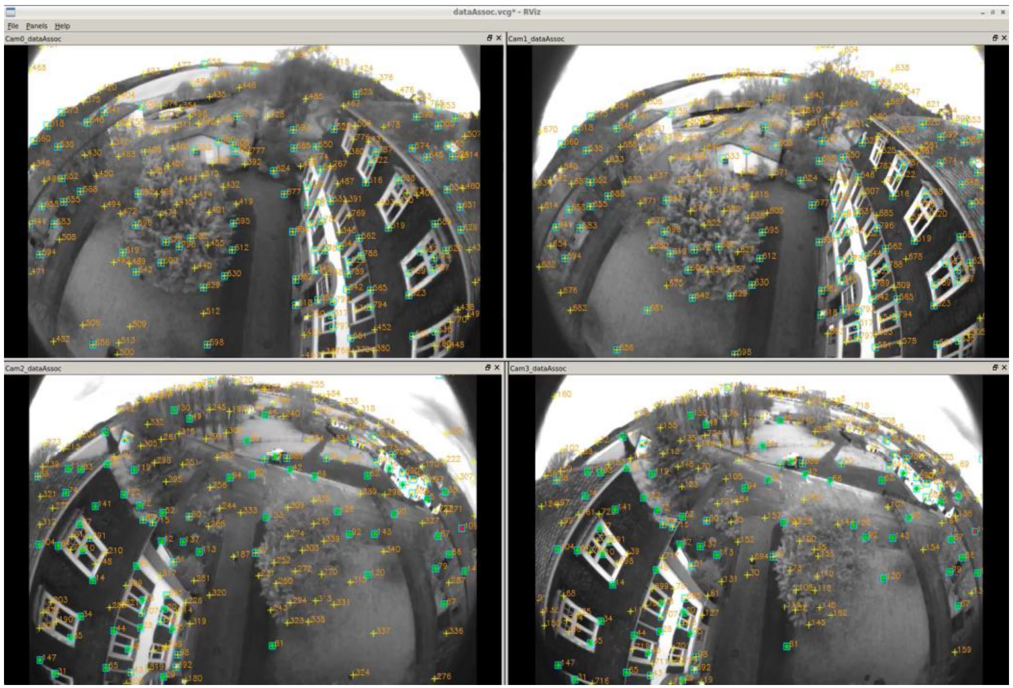
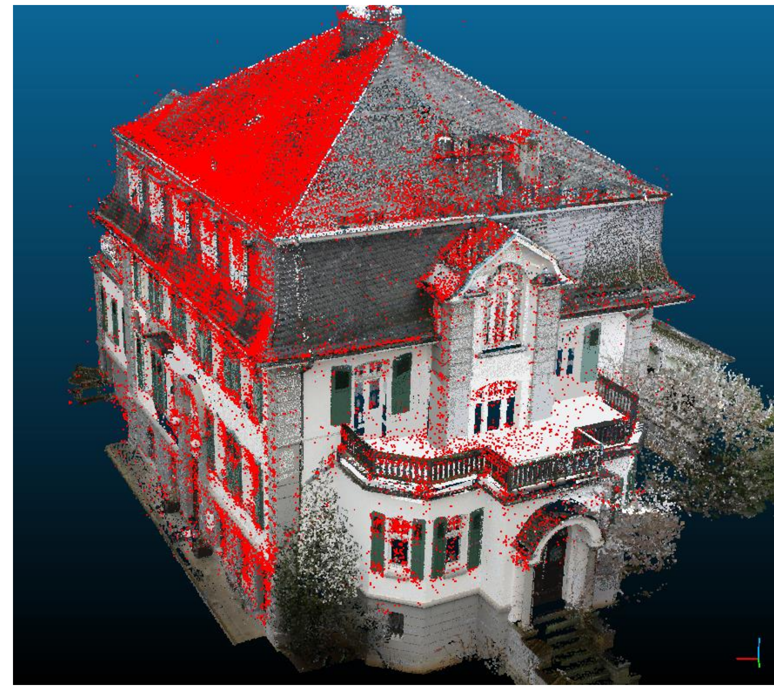


Image courtesy: Schindler

Camera Pose and Point Cloud Estimation



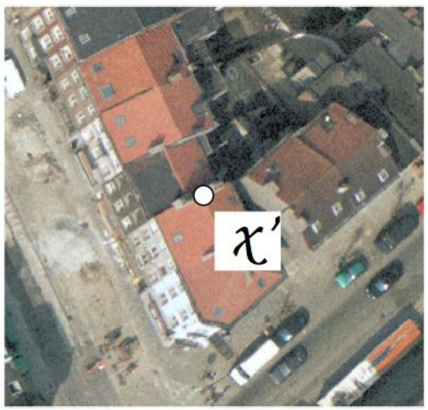


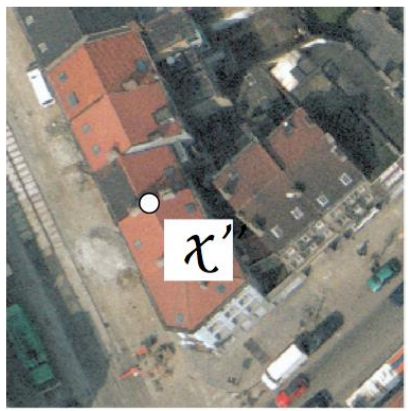
A Camera Pair

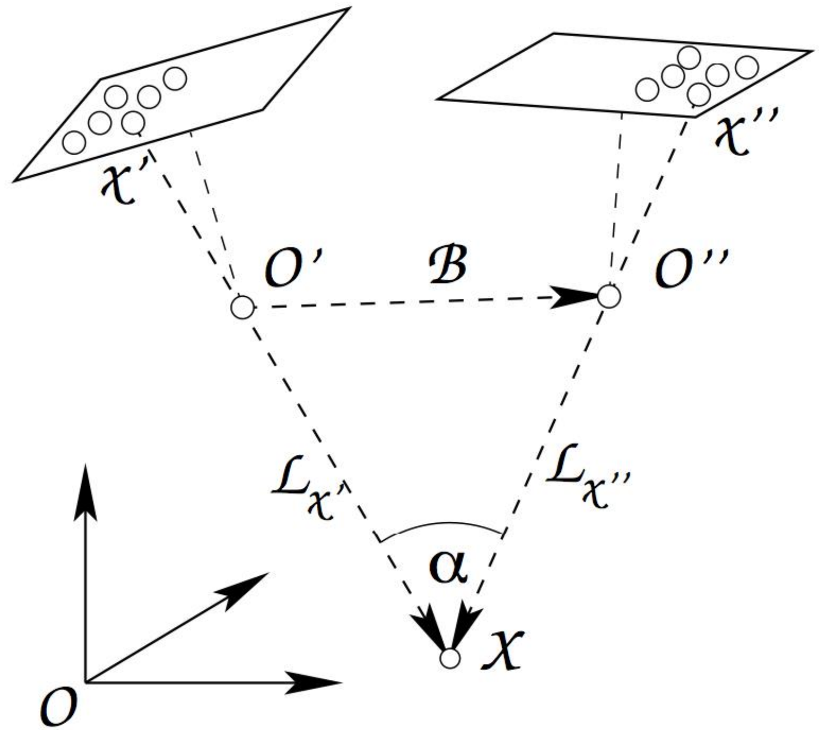
Camera Pair

- In the first part of this course, we computed the camera orientation for **single camera**
- We are now considering situation in which we have two images, potentially taken from **two cameras**

Images from different views







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Orientation Parameters for the Camera Pair and Relative Orientation

Orientation

- The orientation of the camera pair can be described using independent orientations for each camera

How many parameters are needed?

- Calibrated cameras: **?** parameters
- Uncalibrated cameras: **?** parameters

Orientation

- The orientation of the camera pair can be described using independent orientations for each camera

How many parameters are needed?

- Calibrated cameras: **12** parameters
- Uncalibrated cameras: **22** parameters

Orientation with Control Points

- The orientation of the camera pair can be described using independent orientations for each camera

- Uncalibrated cameras: 22 parameters
- Can be computed via two separate DLT steps
- Requires **?** known control points

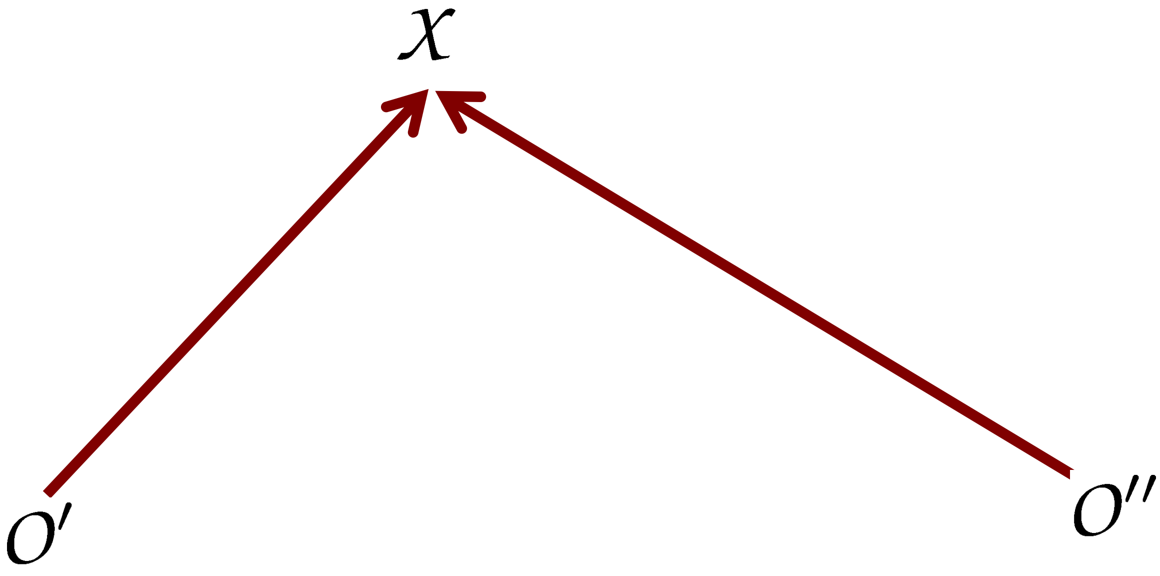
Orientation with Control Points

- The orientation of the camera pair can be described using independent orientations for each camera

- Uncalibrated cameras: 22 parameters
- Can be computed via two separate DLT steps
- Requires 6 known control points

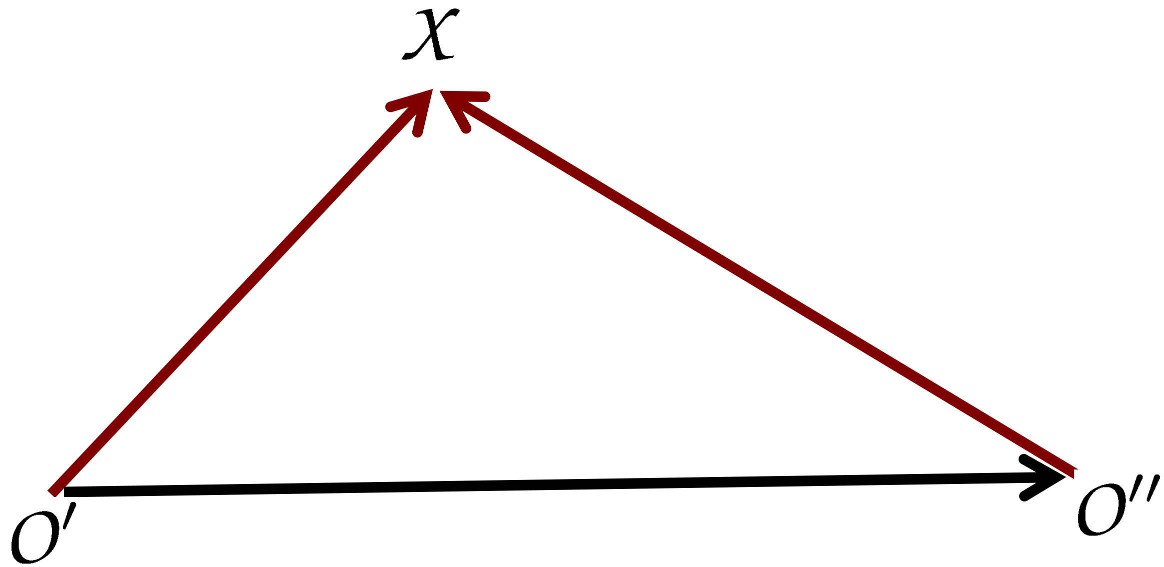
Which Parameters Can We Compute **Without Additional** Information About the Scene?

We start with a perfect orientation and the intersection of two corresponding rays



Coplanarity Constraint

- Consider perfect orientation and the intersection of two corresponding rays
- Both rays lie within one plane in 3D



For Calibrated Cameras

- We need 2x6=12 parameters for two calibrated cameras for the orientation
- Mapping of the calibrated camera is angle-preserving
- Angle-preserving model of the object
- Angle-preserving mapping is a 7 DoF similarity transformation
- Without additional information, **we cannot obtain all 12 parameters**

Which Parameters Can We Obtain?

Cameras Measure Directions

- We cannot obtain the (global) **translation** and **rotation** (if the cameras maintain their relative transformation) as well as the **scale**

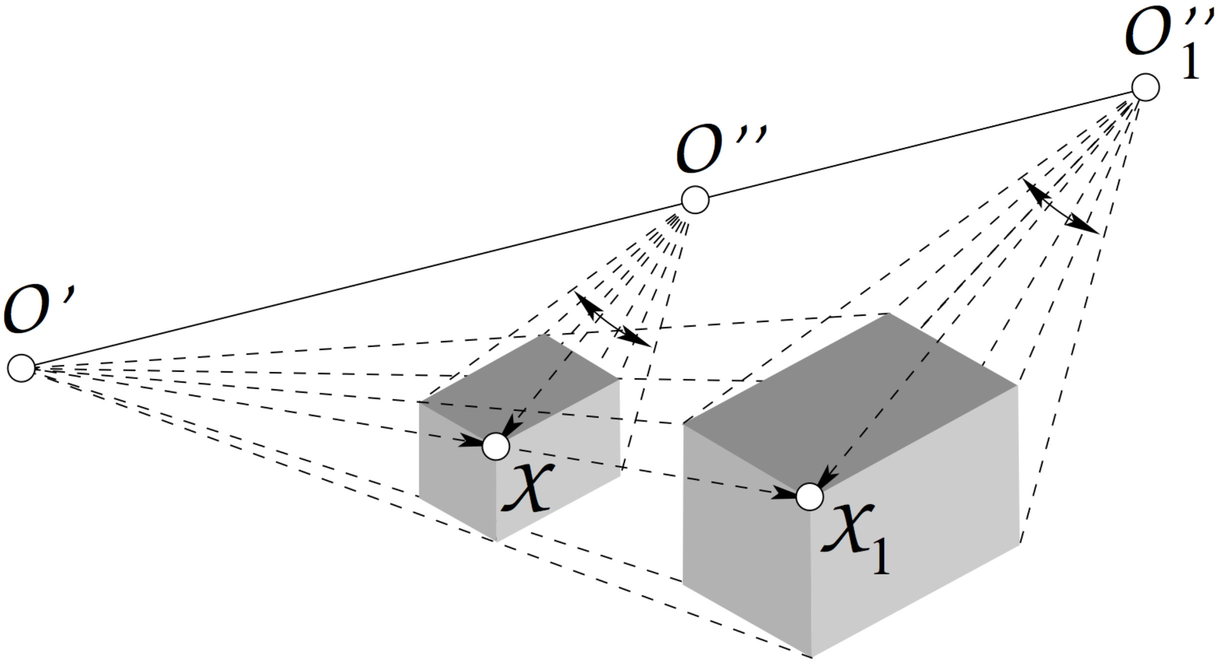


Image courtesy: Förstner & Wrobel

What We Can Compute

- The **rotation** R of the second camera w.r.t. the first one (3 parameters)
- The **direction** B of the line connecting the to centers of projection
- We do **not know** their **distance**

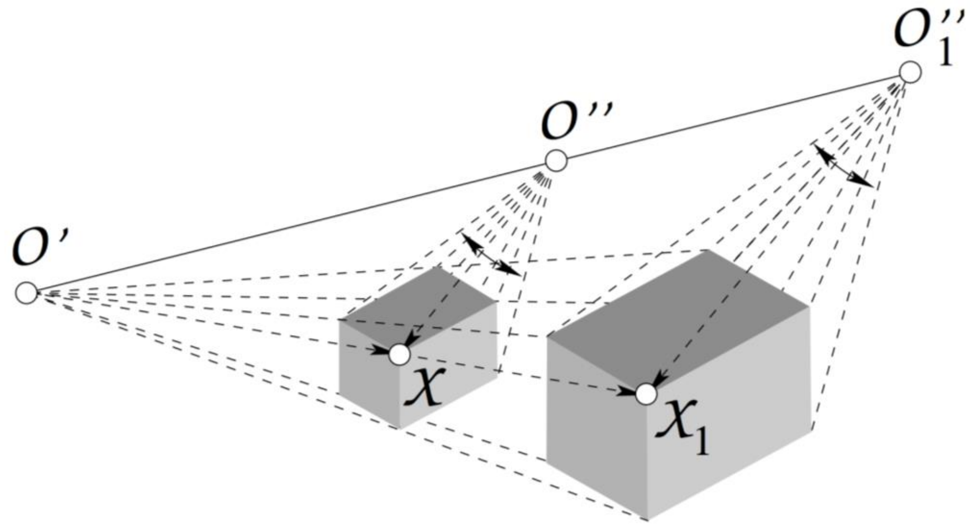


Image courtesy: Förstner & Wrobel

For Calibrated Cameras

- We need 2x6=12 parameters for two calibrated cameras for the orientation
- With a calibrated camera, we obtain an angle-preserving model of the object
- Without additional information, we can **only obtain** 12-7 = **5 parameters** (7=translation, rotation, scale)

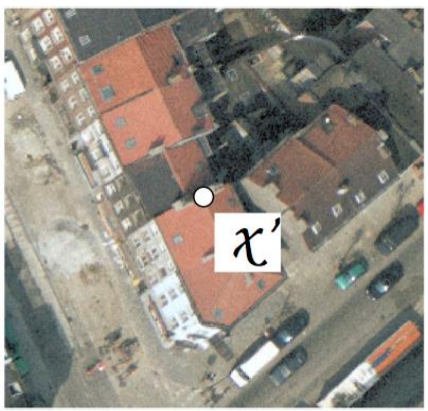
first camera

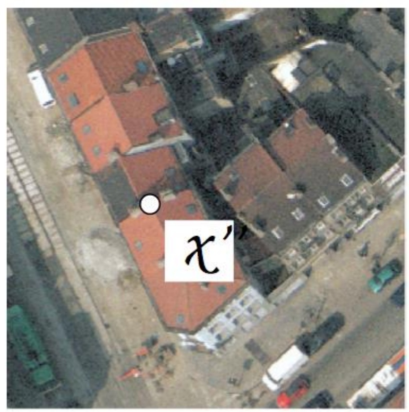
distance between cameras

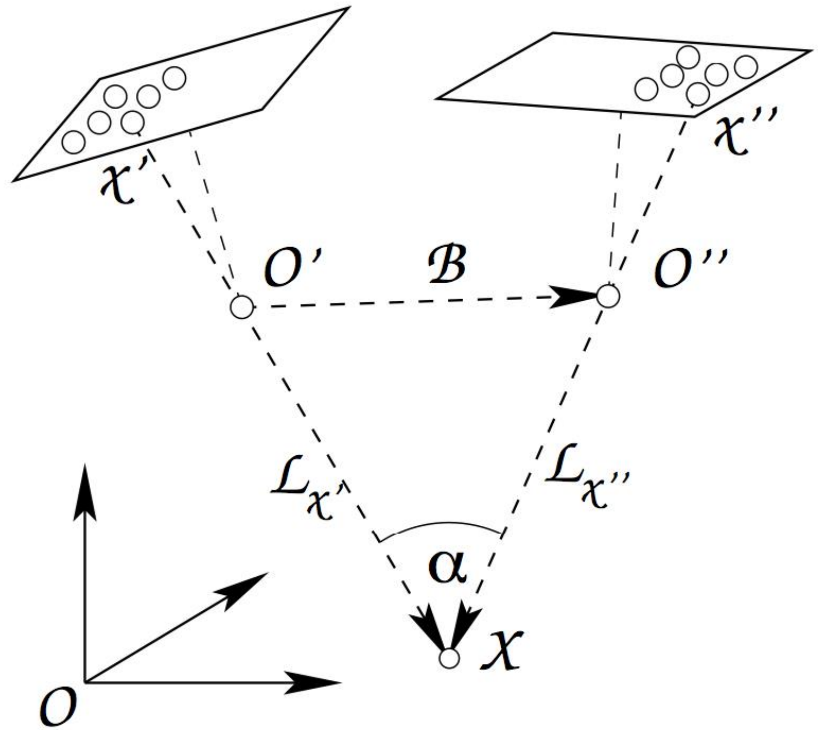
Photogrammetric Model

- Given two cameras images, we can reconstruct the object only **up to a similarity transform**
- Called a **photogrammetric model**
- The **orientation of the photogrammetric model** is called the **absolute orientation**
- For obtaining the absolute orientation, we need at least **3 points** in 3D (for 7 parameters)

What do we need for computing a **3D model** of the scene?







For Uncalibrated Cameras

- **Straight-line preserving** but **not angle preserving**
- Object can be reconstructed up to a straight-line preserving mapping
- Projective transform in 3D (**?** parameters)

For Uncalibrated Cameras

- **Straight-line preserving** but **not angle preserving**
- Object can be reconstructed up to a straight-line preserving mapping
- Projective transform (15 parameters)
- Thus, for uncalibrated cameras, we can only **obtain 22-15=7** parameters given two images
- We need at **least 5 points** in 3D (15 coordinates) for the absolute o.

Relative and Absolute Orient.

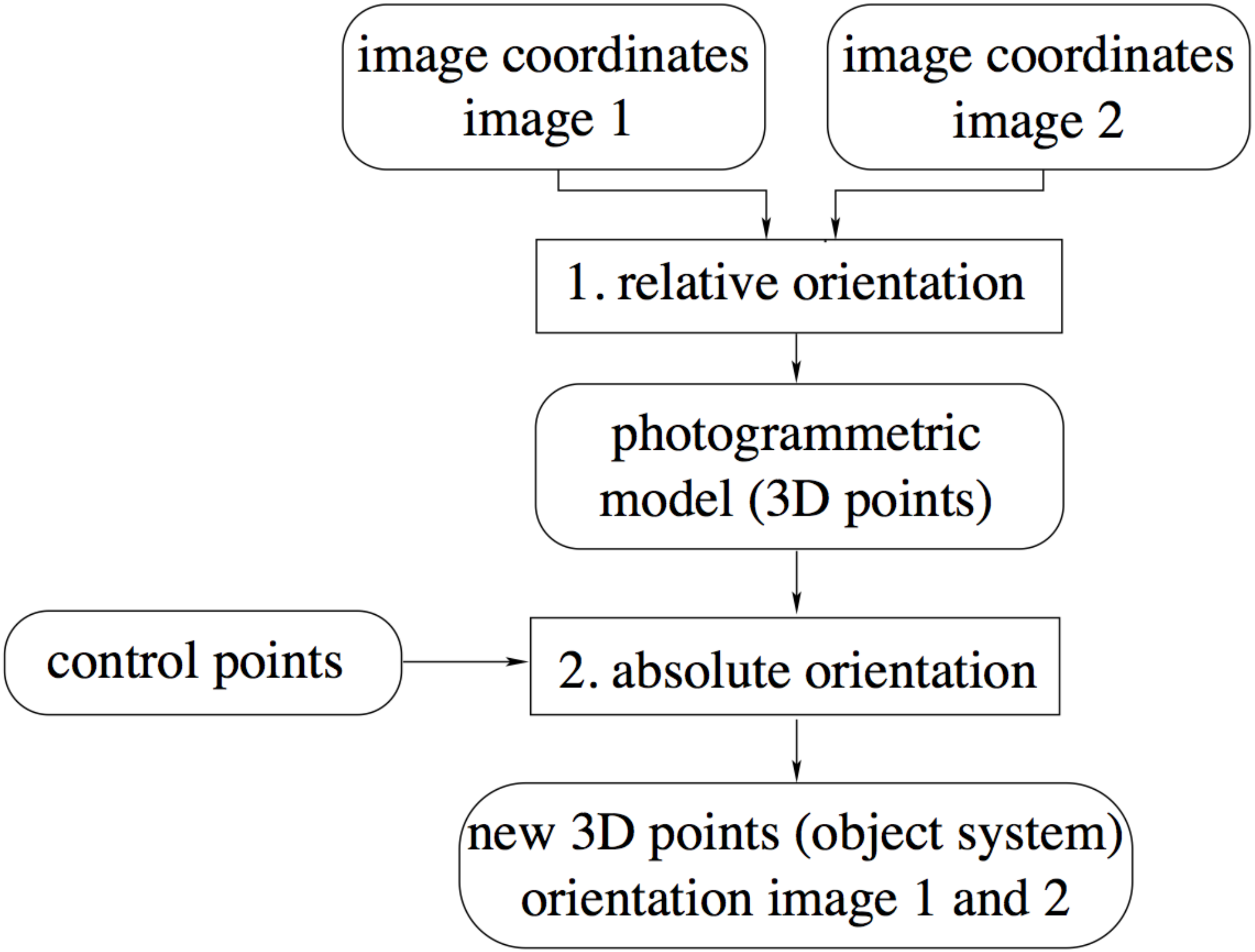


Image courtesy: Förstner & Wrobel

Summary

Cameras	#params /img	#params /img pair	#params for RO	#params for AO	min #P
calibrated	**6**	**12**	**5**	**7**	**3**
not calibrated	**11**	**22**	**7**	**15**	**5**

RO = relative orientation

AO = absolute orientation

min #P = min. number of control points

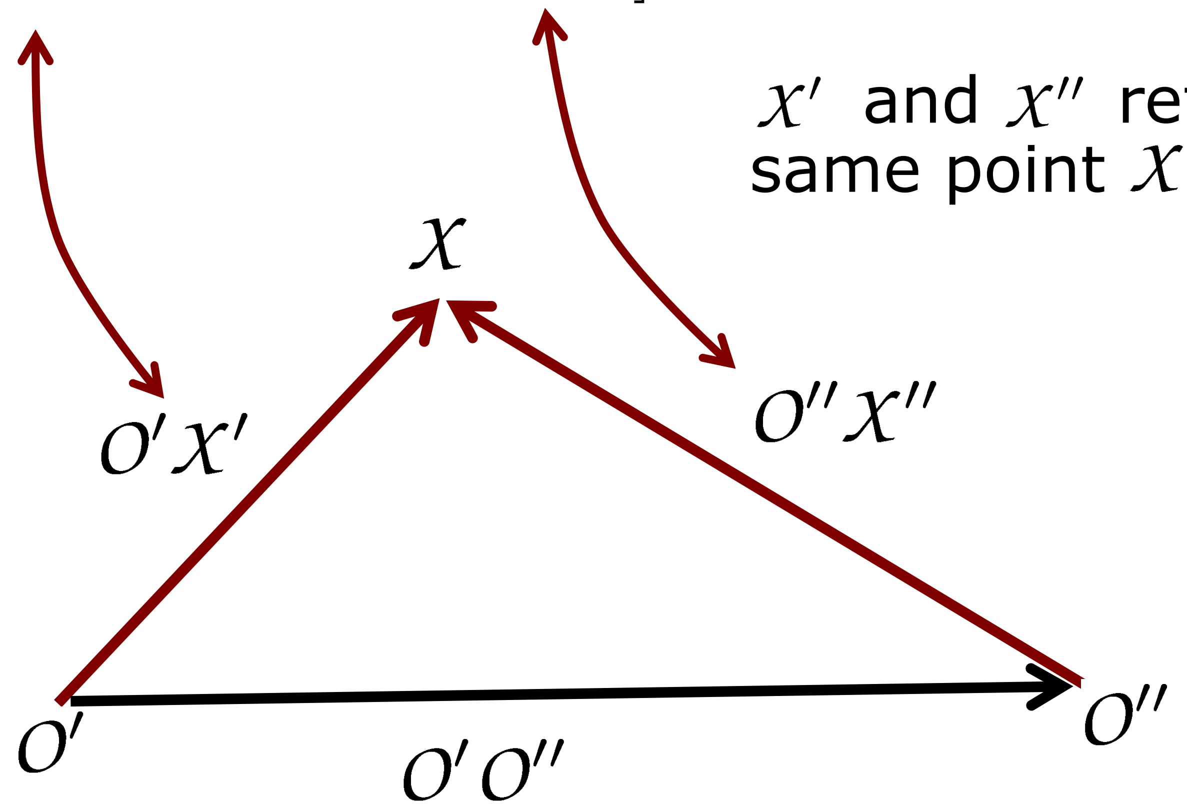
2
Coplanarity Constraint for Straight-Line Preserving (Uncalibrated) Cameras

Coplanarity Constraint for Uncalibrated Cameras

Coplanarity can be expressed by

$$[O'X' \quad O'O'' \quad O''X''] = 0$$

x' and x'' refer to the same point X in space



Scalar Triple Product

- The operator $[\cdot, \cdot, \cdot]$ is the triple product
- Dot product of one of the vectors with the cross product of the other two

$$[A, B, C] = |(A \times B) \cdot C|$$

- It is the volume of the parallelepiped of three vectors

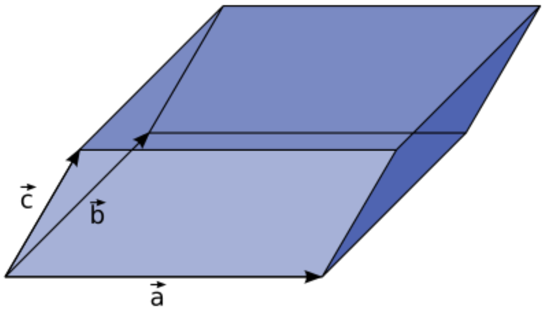


Image courtesy: Wikipedia (Niabot)

Scalar Triple Product Properties

$$[A, B, C] = (A \times B) \cdot C = A \cdot (B \times C)$$

$$[A, B, C] = (A \times B) \cdot C = -(B \times A) \cdot C = -[B, A, C]$$

$$[A, B, C] = \det \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

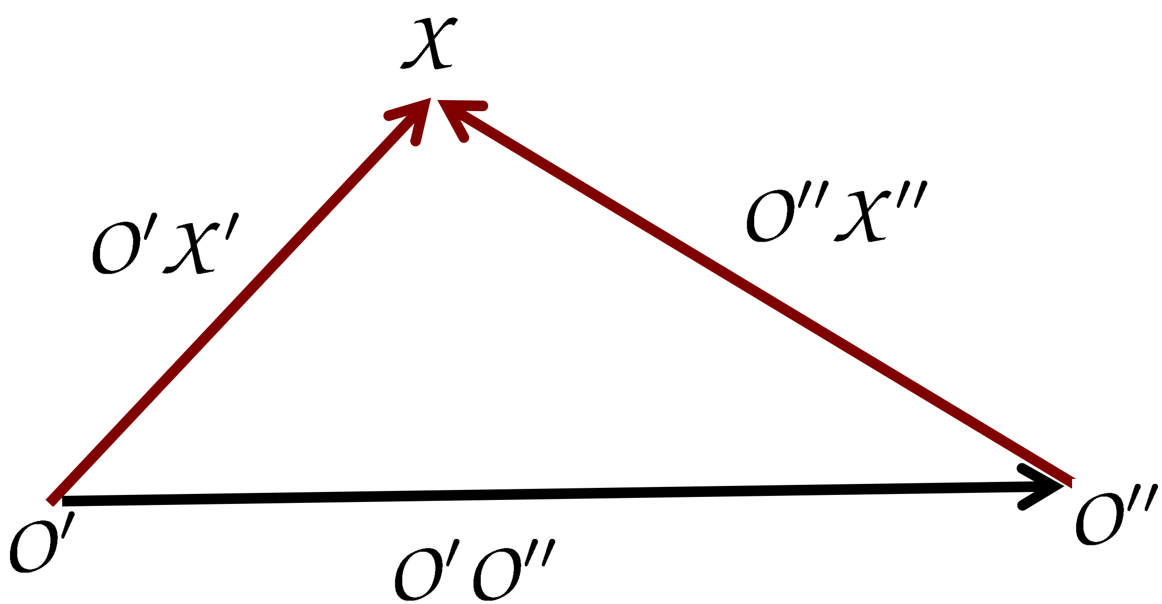
$$[A, A, B] = 0$$

$[A, B, C] = 0$ means that the three vectors lie in one plane

Coplanarity Constraint for Uncalibrated Cameras

Coplanarity can be expressed by

$$[O'X' \quad O'O'' \quad O''X''] = 0$$



Coplanarity Constraint for Uncalibrated Cameras

- The directions of the vectors $O'X'$ and $O''X''$ can be derived from the image coordinates $\mathbf{x}', \mathbf{x}''$

$$\mathbf{x}' = \mathsf{P}'\mathbf{X} \qquad\qquad \mathbf{x}'' = \mathsf{P}''\mathbf{X}$$

- with the projection matrices

$$\mathsf{P}' = \mathsf{K}'R'[\boldsymbol{I}_3| - \boldsymbol{X}_{O'}] \qquad \mathsf{P}'' = \mathsf{K}''R''[\boldsymbol{I}_3| - \boldsymbol{X}_{O''}]$$

Reminder: $[\boldsymbol{I}_3| - \boldsymbol{X}_{O''}] = \begin{bmatrix} 1 & 0 & 0 & -X_{O''} \\ 0 & 1 & 0 & -Y_{O''} \\ 0 & 0 & 1 & -Z_{O''} \end{bmatrix}$

Directions to a Point

- The directions of the vectors $O''X''$ and $O'X'$ are

$$\boxed{{}^n\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'} \leftarrow \text{image coord.}$$

- as the projection

$$ {}^n\mathbf{x}' = [I_3 | -\boldsymbol{X}_{O'}]\mathbf{X} \leftarrow \text{world coord.}$$

- provides the direction to from the center of projection to the point in 3D
- Analogous:

$$ {}^n\mathbf{x}'' = (R'')^{-1}(\mathsf{K}'')^{-1}\mathbf{x}''$$

Base Vector

- The base vector $O'O''$ directly results from the coordinates of the projection centers

$$\boxed{\mathrm{b} = \boldsymbol{B} = \boldsymbol{X}_{O''} - \boldsymbol{X}_{O'}}$$

Coplanarity Constraint

- Using the previous relations, the coplanarity constraint

$$[O'X' \quad O'O'' \quad O''X''] \quad = \quad 0$$

- can be rewritten as

$$\begin{aligned}
[\,{}^n\mathbf{x}' \quad \mathbf{b} \quad {}^n\mathbf{x}''] &= 0 \\
{}^n\mathbf{x}' \cdot (\mathbf{b} \times {}^n\mathbf{x}'') &= 0 \\
{}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' &= 0
\end{aligned}$$

skew-symmetric matrix

Derivation

- Why is this correct?

$$
\begin{aligned}
{}^{n}\mathbf{x}' \cdot (\mathbf{b} \times {}^{n}\mathbf{x}'') &= 0 \\
{}^{n}\mathbf{x}'^{\mathsf{T}} S_b \, {}^{n}\mathbf{x}'' &= 0
\end{aligned}
$$

Derivation

- Why is this correct?

$$\begin{aligned} {}^{n}\mathbf{x}' \cdot (\mathbf{b} \times {}^{n}\mathbf{x}'') &= 0 \\ {}^{n}\mathbf{x}'^{\mathsf{T}} S_b \, {}^{n}\mathbf{x}'' &= 0 \end{aligned}$$

- Results from the cross product as

$$\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}} \times \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} -b_3x_2 + b_2x_3 \\ b_3x_1 - b_1x_3 \\ -b_2x_1 + b_1x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}}_{S_b} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}}$$

- with S_b being a skew-symmetric matrix

Coplanarity Constraint

- By combining ${}^n\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'$ and ${}^n\mathbf{x}'^{\mathsf{T}} S_b\, {}^n\mathbf{x}'' \;=\; 0$
- we obtain

$$\mathbf{x}'^{\mathsf{T}}(\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}\mathsf{S}_b(R'')^{-1}(\mathsf{K}'')^{-1}\mathbf{x}'' = 0$$

Coplanarity Constraint

- By combining ${}^n\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'$ and ${}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' \; = \; 0$
- we obtain

$$\mathbf{x}'^{\mathsf{T}} \underbrace{(\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}\mathsf{S}_b(R'')^{-1}(\mathsf{K}'')^{-1}}_{\mathsf{F}} \mathbf{x}'' = 0$$

$$\begin{aligned} \mathsf{F} \; &= \; (\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}\mathsf{S}_b(R'')^{-1}(\mathsf{K}'')^{-1} \\ &= \; (\mathsf{K}')^{-\mathsf{T}}R'\mathsf{S}_bR''^{\mathsf{T}}(\mathsf{K}'')^{-1} \end{aligned}$$

Fundamental Matrix

- The matrix F is the **fundamental matrix** (for uncalibrated cameras):

$$\boxed{\mathsf{F} \quad = \quad (\mathsf{K}')^{-\mathsf{T}} R' \mathsf{S}_b R''^{\mathsf{T}} (\mathsf{K}'')^{-1}}$$

- It allow for expressing the **coplanarity constraint** by

$$\boxed{\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' \quad = \quad 0}$$

Fundamental Matrix

- The **fundamental matrix** is the matrix that fulfills the equation

$$\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' = 0$$

for corresponding points

- The fundamental matrix contains the all the available **information about the relative orientation** of **two images** from uncalibrated cameras

Alternative Definition

- In the context of many images, we will call F_{ij} that fundamental matrix which yields the constraint ${\mathbf{x}'_i}^\top \mathsf{F}_{ij} \mathbf{x}''_j = 0$
- Thus in our case, we have $\mathsf{F} = \mathsf{F}_{12}$
- Our definition of F is not the same as in the book of Hartley and Zisserman (CV)
- The definition in Hartley and Zisserman is based on ${\mathbf{x}''_i}^\top \mathsf{F}_{ij} \mathbf{x}'_j = 0$, i.e. $\mathsf{F} = \mathsf{F}_{21} = \mathsf{F}_{12}^\top$
- The transposition needs to be taken into account when comparing algebraic expressions

Fundamental Matrix From the Camera Projection Matrices

- If the projection matrices are given, we can derive the fundamental matrix
- Let the projection matrices be partitioned into a left 3×3 matrix and a 3-vector as $\mathsf{P}' = [\mathsf{A}'|\mathbf{a}']$. Then, we have

$$\mathsf{F} = (\mathsf{K}')^{-\mathsf{T}} R' \mathsf{S}_b R''^{\mathsf{T}} (\mathsf{K}'')^{-1} = \mathsf{A}'^{-\top} \mathsf{S}_{b'_{12}} \mathsf{A}''^{-1}$$

- with

$$\mathbf{b}'_{12} = \mathsf{A}''^{-1}\mathbf{a}'' - \mathsf{A}'^{-1}\mathbf{a}' \quad \text{and} \quad S_b = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$$

Fundamental Matrix From the Camera Projection Matrices

- Fundamental matrix of the form

$$\mathsf{F} = \mathsf{A}'^{-\top} \mathsf{S}_{b'_{12}} \mathsf{A}''^{-1}$$

- is a result of the projection centers

$$\boldsymbol{X}_{O'} = -\mathsf{A}'^{-1}\mathbf{a}' \qquad \boldsymbol{X}_{O''} = -\mathsf{A}''^{-1}\mathbf{a}''$$

- and $\mathsf{A}' = \mathsf{K}' R'$, $\quad \boldsymbol{X}_{O''} = -\mathsf{A}''^{-1}\mathbf{a}''$
- This yields $\mathbf{b}'_{12} = \mathsf{A}''^{-1}\mathbf{a}'' - \mathsf{A}'^{-1}\mathbf{a}'$

Fundamental matrix depends on the projection matrices and not the corresponding points, same for all correspondences

Next Week:
Computing the Fundamental Matrix from Corresponding Points

Fundamental Matrix from Corresponding Points

- The coplanarity constraint is bilinear in the homogenous image coordinates $\mathbf{x}'$ and $\mathbf{x}''$ and linear in the elements of the fundamental matrix F
- This is the basis for a simple determination of the fundamental matrix from corresponding points

Degrees of Freedom

- The fundamental F matrix has seven degrees of freedom. This is because F is homogeneous and singular, as the skew symmetric matrix S_b is singular with rank two.
- Any matrix of the form

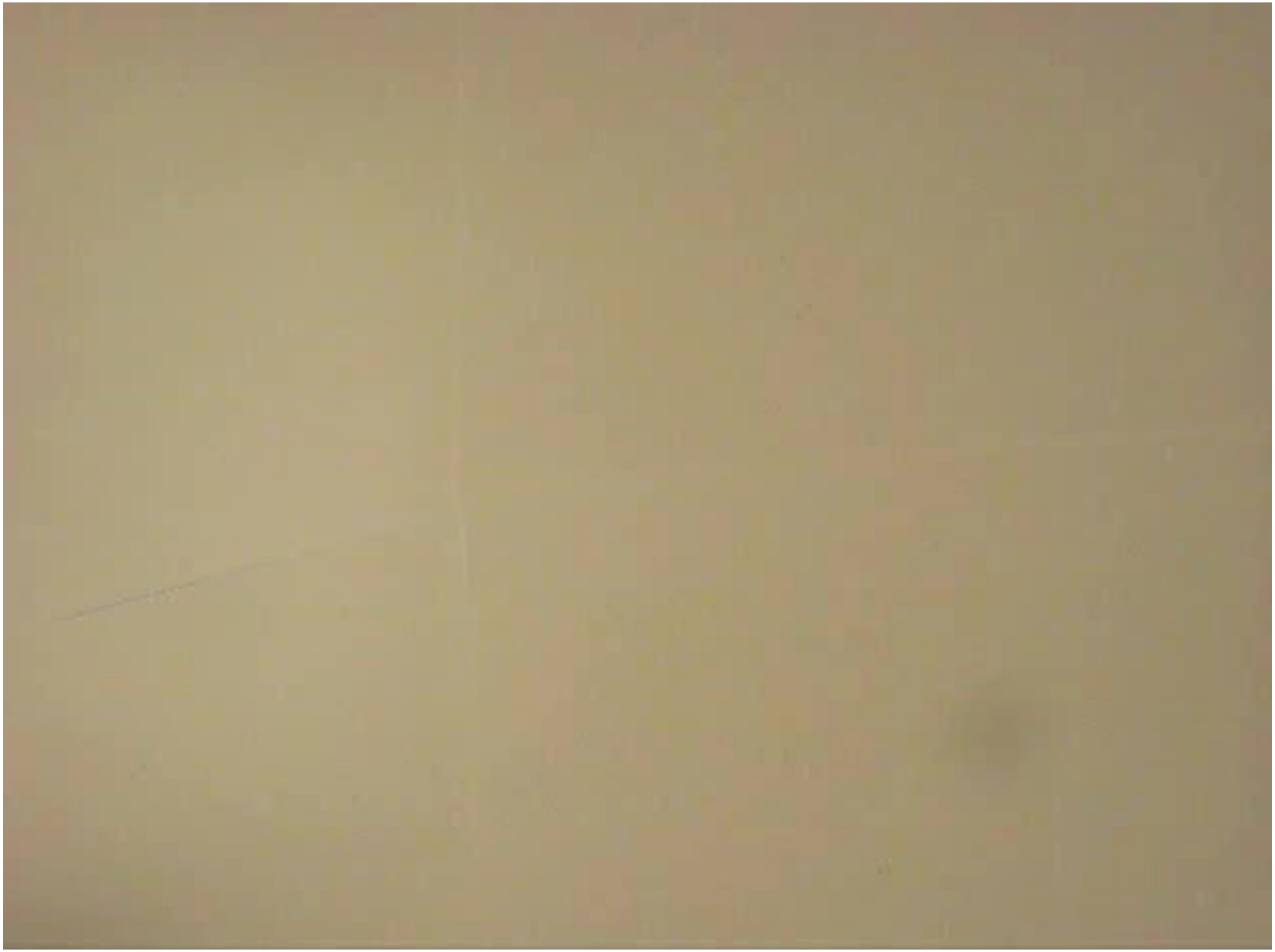
$$\mathsf{F} = U\mathrm{Diag}(s_1, s_2, 0)V^{\mathsf{T}} \quad \text{with } s_i > 0$$

- with orthogonal matrices U and V is a fundamental matrix

Corresponding Points

- We need **7 corresponding points** to compute the fundamental matrix
- We will study a **direct method that needs 8 points** (next week)

The Fundamental Matrix Song



Video courtesy: Daniel Wedge
http://danielwedge.com/fmatrix/

Summary

- Geometry of image pairs
- Relative orientation
- Absolute orientation
- Corresponding points
- Fundamental matrix
- Correspondence test

Literature

- Förstner, Wrobel: Photogrammetric Computer Vision, Ch. 12.2.1 – 12.2.2