Computer Vision

Epipolar Geometry and the Essential Matrix

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Administrative Matters

- Final project demo dates:
  - 5th May (Sat) vs. 6th May (Sun)
  - Outstanding cases?
  - Cribs on same days
- Assignment 5 feedback?
- Assignment 6:
  - Stereo Vision

Recap:
Coplanarity Constraint for Straight-Line Preserving (Uncalibrated) Cameras

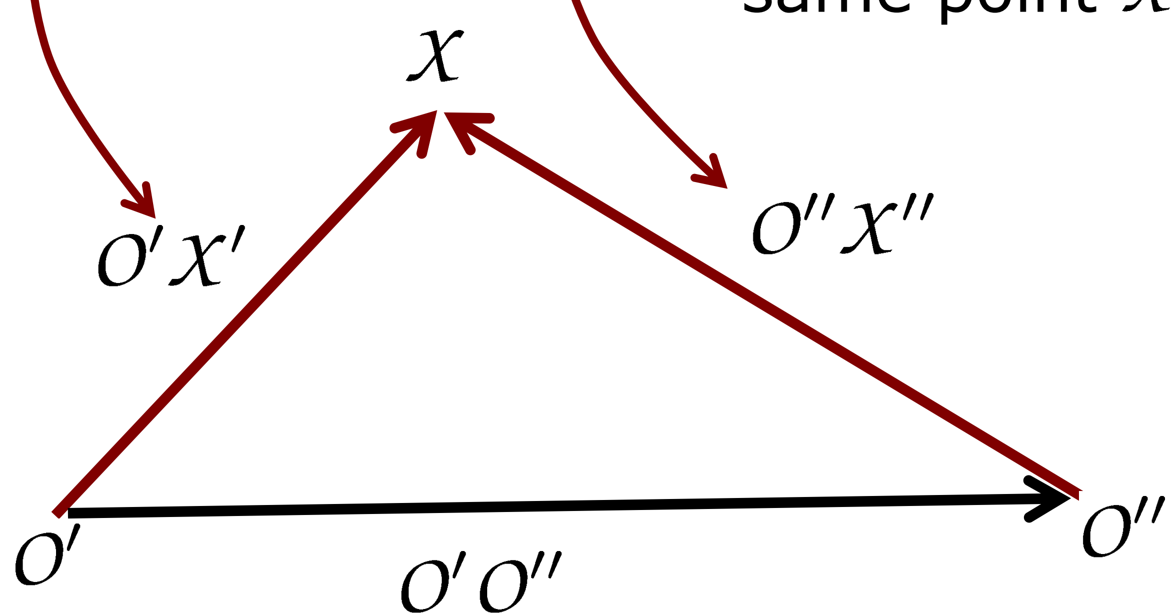
Coplanarity Constraint for Uncalibrated Cameras

Coplanarity can be expressed by

$$[O'X' \quad O'O'' \quad O''X''] = 0$$

Scalar Triple Product

x' and x'' refer to the same point X in space



Refresher from Camera Geometry

- Recall that:

3D Homogeneous

$$\begin{aligned}\lambda \mathbf{x} &= \mathsf{P}\mathbf{X} = \mathsf{K}R[I_3| - \boldsymbol{X}_O]\mathbf{X} \\ &= [\mathsf{K}R| - \mathsf{K}R\boldsymbol{X}_O]\begin{bmatrix}\boldsymbol{X} \\ 1\end{bmatrix} \\ &= \mathsf{K}R\boldsymbol{X} - \mathsf{K}R\boldsymbol{X}_O\end{aligned}$$

3D Euclidean

factor resulting from the H.C.

Refresher from Camera Geometry

- Starting from $\lambda \mathbf{x} = \mathsf{K}R\boldsymbol{X} - \mathsf{K}R\boldsymbol{X}_O$
- we obtain

$$\begin{aligned} \boldsymbol{X} &= (\mathsf{K}R)^{-1}\mathsf{K}R\boldsymbol{X}_O + \lambda(\mathsf{K}R)^{-1}\mathbf{x} \\ &= \boldsymbol{X}_O + \lambda\underbrace{(\mathsf{K}R)^{-1}\mathbf{x}}_{\text{3x1 Euclidean}} \end{aligned}$$

- The term $(\mathsf{K}R)^{-1}\mathbf{x}$ describes the direction of the ray from the camera origin $\boldsymbol{X}_O$ to the 3D point $\boldsymbol{X}$

Coplanarity Constraint for Uncalibrated Cameras

- The directions of the vectors $\mathcal{O}'\mathcal{X}'$ and $\mathcal{O}''\mathcal{X}''$ can be derived from the image coordinates $\mathbf{x}', \mathbf{x}''$

Homogenous coord.

Pixel space

$$\mathbf{x}' = \mathsf{P}'\mathbf{X} \qquad \mathbf{x}'' = \mathsf{P}''\mathbf{X}$$

World space

- with the projection matrices

$$\mathsf{P}' = \mathsf{K}'R'[I_3| - \boldsymbol{X}_{O'}] \qquad \mathsf{P}'' = \mathsf{K}''R''[I_3| - \boldsymbol{X}_{O''}]$$

Euclidean coord.

Reminder: $[I_3| - \boldsymbol{X}_{O''}] = \begin{bmatrix} 1 & 0 & 0 & -X_{O''} \\ 0 & 1 & 0 & -Y_{O''} \\ 0 & 0 & 1 & -Z_{O''} \end{bmatrix}$

Directions to a Point

- The directions of the vectors $O''X''$ and $O'X'$ are

3x1 Euclidean

$$\boxed{{}^{n}\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'}$$

pixel space,
Homogenous coord

- provides the direction to from the center of projection to the point in 3D

- Analogous:

$${}^{n}\mathbf{x}'' = (R'')^{-1}(\mathsf{K}'')^{-1}\mathbf{x}''$$

Base Vector

- The base vector $O'O''$ directly results from the coordinates of the projection centers

$$\boxed{\mathrm{b} = \boldsymbol{B} = \boldsymbol{X}_{O''} - \boldsymbol{X}_{O'}}$$

Coplanarity Constraint

- Using the previous relations, the coplanarity constraint

$$[O'X' \quad O'O'' \quad O''X''] \quad = \quad 0$$

- can be rewritten as

$$\begin{aligned} [\,{}^{n}\mathbf{x}' \quad \mathbf{b} \quad {}^{n}\mathbf{x}''] &= 0 \\ {}^{n}\mathbf{x}' \cdot (\mathbf{b} \times {}^{n}\mathbf{x}'') &= 0 \\ {}^{n}\mathbf{x}'^{\mathsf{T}} S_b \, {}^{n}\mathbf{x}'' &= 0 \end{aligned}$$

skew-symmetric matrix

Derivation

- Why is this correct?

$$
\begin{aligned}
{}^{n}\mathbf{x}' \cdot (\mathbf{b} \times {}^{n}\mathbf{x}'') &= 0 \\
{}^{n}\mathbf{x}'^{\mathsf{T}} S_b \, {}^{n}\mathbf{x}'' &= 0
\end{aligned}
$$

Derivation

- Why is this correct?

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\begin{aligned}
{}^n\mathbf{x}' \cdot (\mathbf{b} \times {}^n\mathbf{x}'') &= 0 \\
{}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' &= 0
\end{aligned}
$$

- Results from the cross product as

$$
\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}} \times \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} -b_3 x_2 + b_2 x_3 \\ b_3 x_1 - b_1 x_3 \\ -b_2 x_1 + b_1 x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}}_{S_b} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}}
$$

- with S_b being a skew-symmetric matrix

Coplanarity Constraint

- By combining ${}^{n}\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'$ and ${}^{n}\mathbf{x}'^{\mathsf{T}} S_b \, {}^{n}\mathbf{x}'' \;=\; 0$
- we obtain

$$\mathbf{x}'^{\mathsf{T}}(\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}\mathsf{S}_b(R'')^{-1}(\mathsf{K}'')^{-1}\mathbf{x}'' = 0$$

Coplanarity Constraint

- By combining ${}^{n}\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'$ and ${}^{n}\mathbf{x}'^{\mathsf{T}} S_b \, {}^{n}\mathbf{x}'' \;=\; 0$
- we obtain

$$\mathbf{x}'^{\mathsf{T}} \underbrace{(\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}\mathsf{S}_b(R'')^{-1}(\mathsf{K}'')^{-1}}_{\mathsf{F}} \mathbf{x}'' = 0$$

$$\begin{aligned} \mathsf{F} &= (\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}\mathsf{S}_b(R'')^{-1}(\mathsf{K}'')^{-1} \\ &= (\mathsf{K}')^{-\mathsf{T}} R' \mathsf{S}_b R''^{\mathsf{T}} (\mathsf{K}'')^{-1} \end{aligned}$$

Fundamental Matrix

- The matrix F is the **fundamental matrix** (for uncalibrated cameras):

$$\boxed{\mathsf{F} \;=\; (\mathsf{K}')^{-\mathsf{T}} R' \mathsf{S}_b R''^{\mathsf{T}} (\mathsf{K}'')^{-1}}$$

- It allow for expressing the **coplanarity constraint** by

$$\boxed{\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' \;=\; 0}$$

Summary: Fundamental Matrix

- F encodes the R.O. of two images from uncalibrated cameras
- F has 7 DoF (5 pose, 2 intrinics)
- F can be computed easily from $\mathsf{P}', \mathsf{P}''$
- Fundamental matrix is defined as

$$\mathsf{F} = (\mathsf{K}')^{-\mathsf{T}} R' \mathsf{S}_b {R''}^{\mathsf{T}} (\mathsf{K}'')^{-1}$$

- Coplanarity constraint

$$\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' = 0$$

The Fundamental Matrix

Video courtesy: Daniel Wedge
http://danielwedge.com/fmatrix/

Table of Contents

1
Epipolar Geometry

Epipolar Geometry – Motivation

- **Given** a point $\boxed{x'}$ in the plane of the **first** image
- **Find** the corresponding point $\boxed{x''}$ in the **second** image plane

Epipolar Geometry

- Epipolar geometry is used to describe geometric relations in image pairs
- Enables efficient search for and prediction of corresponding points
- Given a straight-line preserving mapping, the search space reduces from 2D (whole image) to 1D (line)

Epipolar Geometry

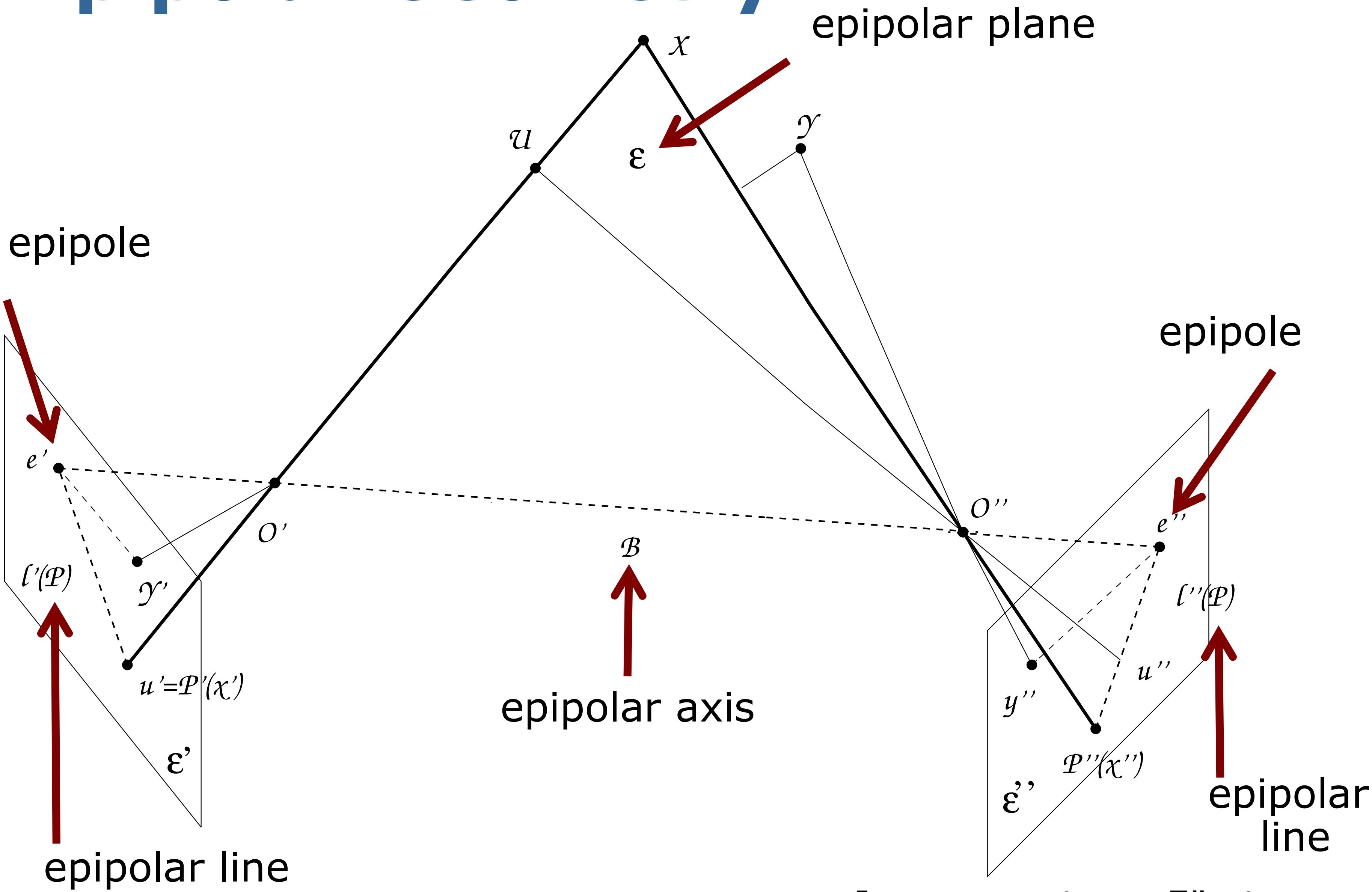


Image courtesy: Förstner

Important Elements (1)

- **Epipolar axis** $\mathcal{B} = (O'O'')$ is the line through the two projection centers
- **Epipolar plane** $\varepsilon = (O'O''X)$ depends on the projection centers and the point

Multiple Epipolar Planes, One Axis

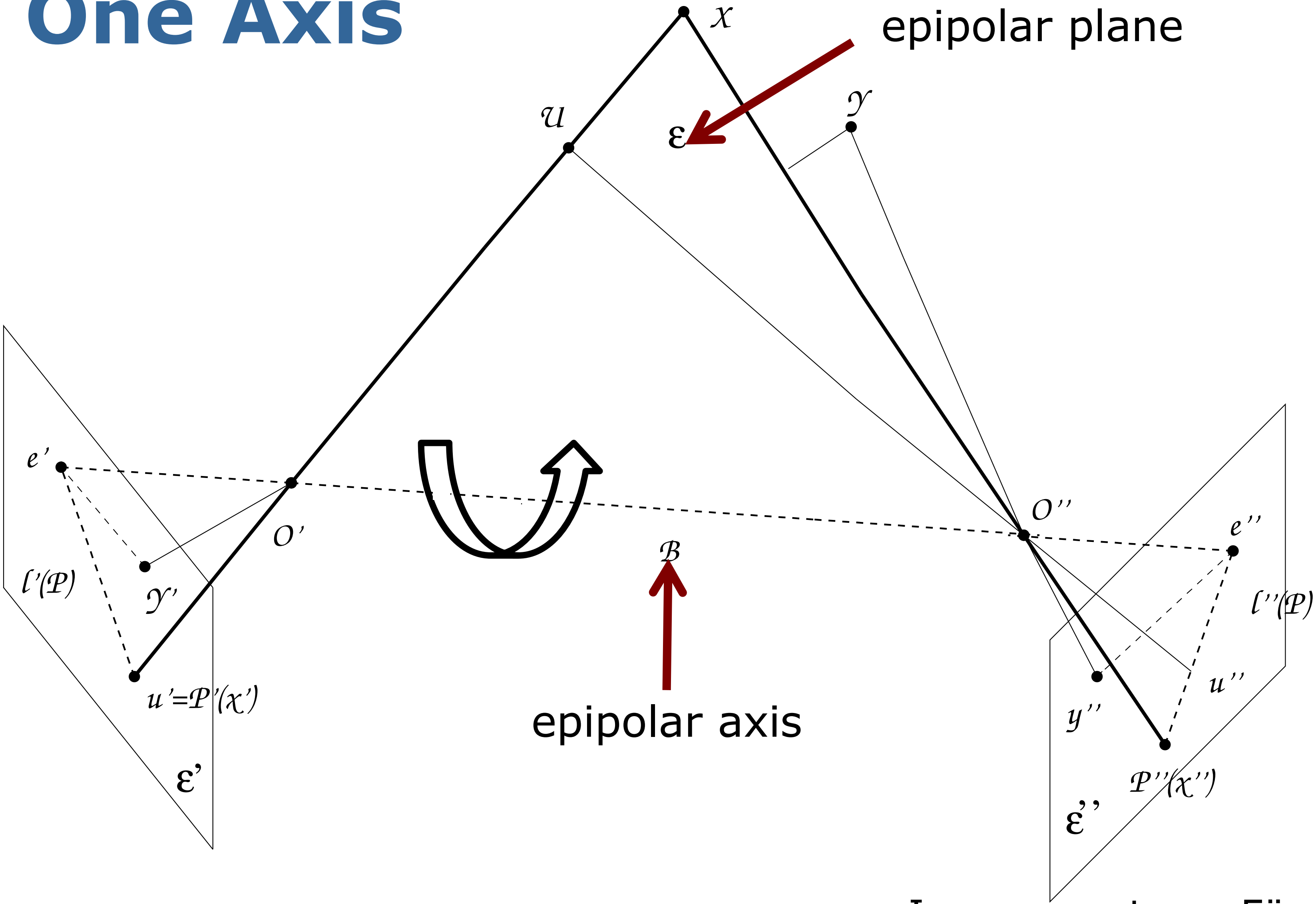


Image courtesy: Förstner

Important Elements (2)

- **Epipolar axis** $\mathcal{B} = (O'O'')$ is the line through the two projection centers
- **Epipolar plane** $\varepsilon = (O'O''X)$ depends on the projection centers and the point
- **Epipoles** $e' = (O'')'$, $e'' = (O')''$ are the images of the projection centers
- **Epipolar lines** $\mathcal{l}'(X) = (O''X)'$, $\mathcal{l}''(X) = (O'X)''$ are are the images of the rays $(O''X)$ and $(O'X)$ in the other image respectively

Epipoles and Epipolar Lines

- **Epipoles** $e' = (O'')', \, e'' = (O')''$ are the images of the projection centers
- They can also be written as

$$e' = (O'O'') \cap \varepsilon' \qquad e'' = (O'O'') \cap \varepsilon''$$

- **Epipolar lines** $\ell'(X) = (O''X)', \, \ell''(X) = (O'X)''$ are are the images of the rays $(O''X)$ and $(O'X)$ in the other image respectively and can be written as

$$\ell'(X) = \varepsilon \cap \varepsilon' \qquad \ell''(X) = \varepsilon \cap \varepsilon''$$

Epipolar Geometry

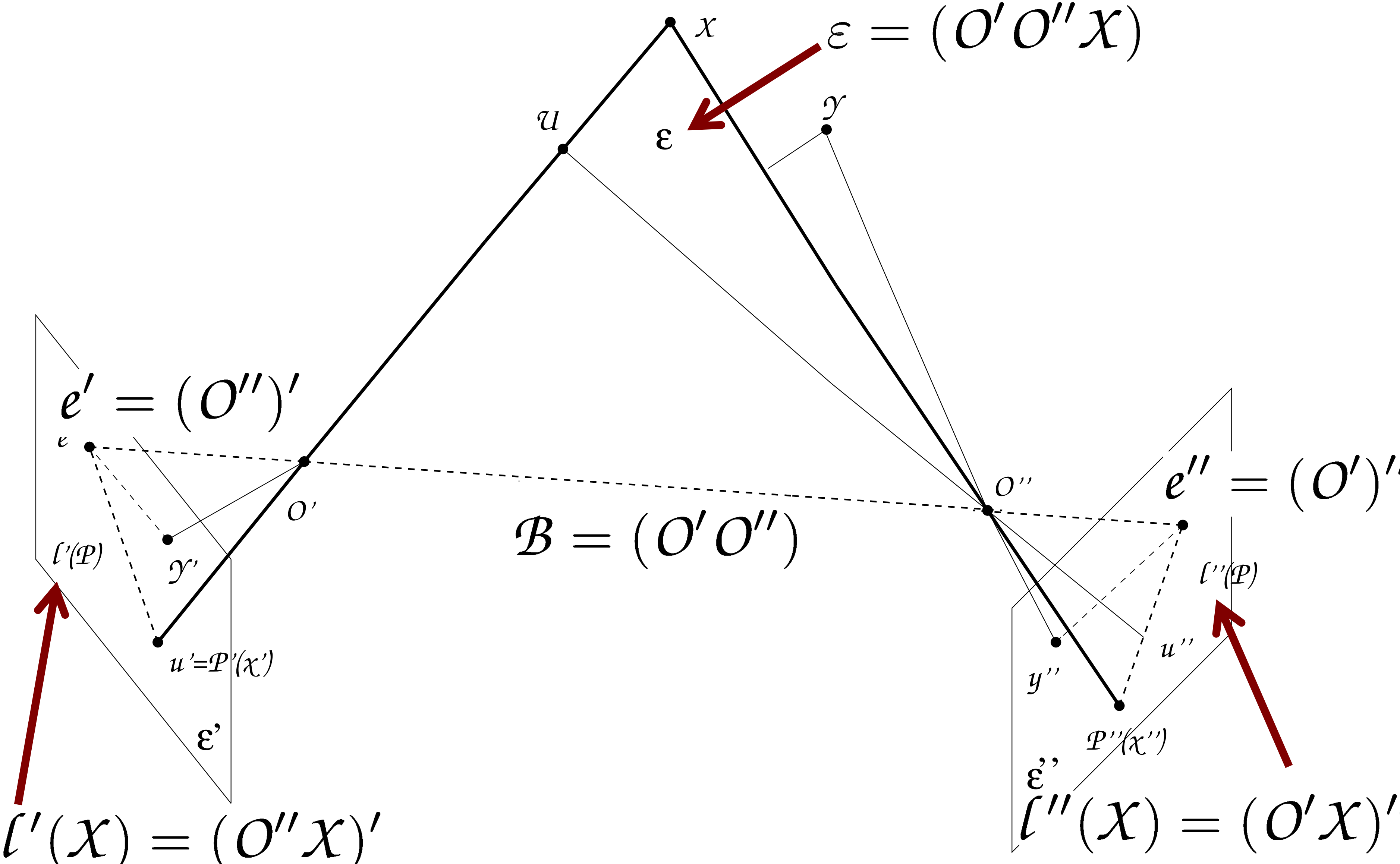


Image courtesy: Förstner 28

In the Epipolar Plane...

Using a distortion-free lens,

- the projection centers O', O''
- the point X
- the epipolar lines $l'(X), l''(X)$
- the image points x', x''

all lie in the epipolar plane ε

This simplifies the task of predicting the location of corresponding a point in the other image

Predicting the Location of Corresponding Points

- Task: Predict the location x'' given x'
- The epipolar plane through $\varepsilon = (O' O'' X')$
- The intersection of the epipolar plane and the second image plane ε'' yields the epipolar line $l''(x)$
- The corresponding point x'' lies on the epipolar line $l''(x)$

Reduces the search space from 2D to 1D

Computing the Key Elements of the Epipolar Geometry

Computing the Key Elements of the Epipolar Geometry

- We described the important quantities geometrically, but do not know yet how to obtain them.

- We will show how to compute them based on the **projection matrices** and the **fundamental matrix**

Computing Epipolar Lines

- The image points lie on the epipolar lines, i.e., $x' \in l'$, $x'' \in l''$
- For x':

2D Homogenous coord.

$$\mathbf{x}'^{\mathsf{T}} \mathbf{l}' = 0$$

- We can exploit the coplanarity constraint for both points x', x''

$$\mathbf{x}'^{\mathsf{T}} \underbrace{\mathsf{F}\mathbf{x}''}_{\mathbf{l}'} = 0$$

$$\boxed{\mathbf{l}' = \mathsf{F}\mathbf{x}''}$$

Computing Epipolar Lines

- The same for the point x''

$$\mathbf{l}''^{\mathsf{T}} \mathbf{x}'' = 0$$

- We can again exploit the coplanarity constraint $\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' = 0$ and obtain

$$\mathbf{l}''^{\mathsf{T}} = \mathbf{x}'^{\mathsf{T}} \mathsf{F}$$

$$\Rightarrow \boxed{\mathbf{l}'' = \mathsf{F}^{\mathsf{T}} \mathbf{x}'}$$

this is our prediction (1D space)

Computing the Epipoles

- The epipoles are the projections of the projection centers in the other image
- Both can be computed easily using the projection matrices

$$\mathbf{e}' = \qquad\qquad \mathbf{e}'' =$$

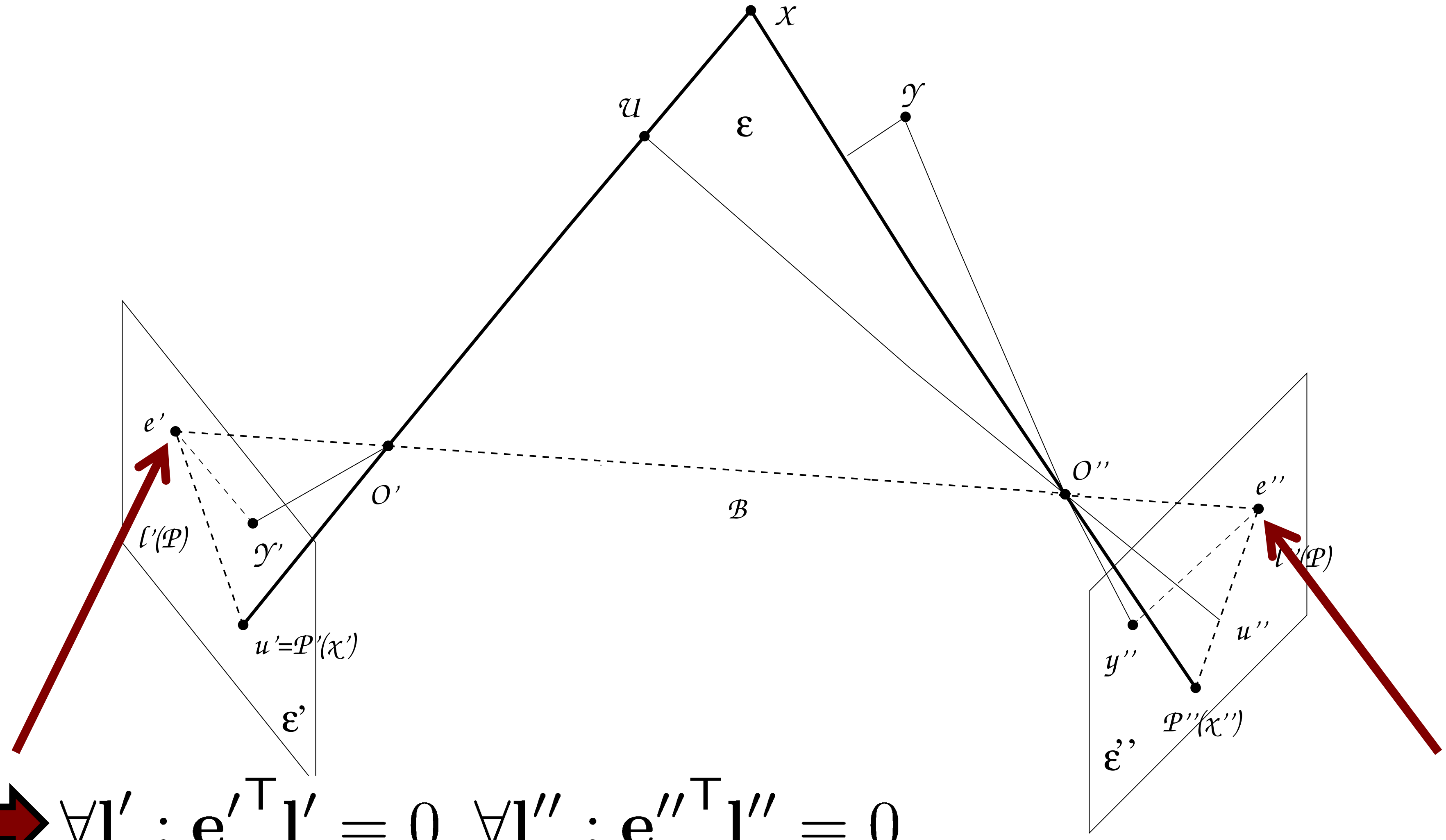
Computing the Epipoles

- The epipoles are the projections of the projection centers in the other image
- Both can be computed easily using the projection matrices

$$\boxed{\mathbf{e}' = \mathsf{P}'\mathbf{X}_{O''} \qquad \mathbf{e}'' = \mathsf{P}''\mathbf{X}_{O'}}$$

3D Homogenous coord.

Epipole is the Intersection of All Epipolar Lines in an Image



$$\forall \mathbf{l}' : \mathbf{e}'^{\mathsf{T}} \mathbf{l}' = 0 \quad \forall \mathbf{l}'' : \mathbf{e}''^{\mathsf{T}} \mathbf{l}'' = 0$$

Image courtesy: Förstner

Epipoles and Fundamental Mat.

- The epipoles are the intersection of all epipolar lines, i.e.,

$$\forall \mathbf{l}' : \mathbf{e}'^{\mathsf{T}} \mathbf{l}' = 0 \qquad \forall \mathbf{l}'' : \mathbf{e}''^{\mathsf{T}} \mathbf{l}'' = 0$$

- Combining this with the epipolar lines defined through F, i.e., $\mathbf{l}' = \mathsf{F}\mathbf{x}''$ yields for all points $\mathbf{x}''$

$$\boxed{\mathbf{e}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' = 0}$$

- The epipole is the **null space** of F^{T}

Epipoles and Fundamental Mat.

- Analogous, we obtain for the second epipole $\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{e}'' = 0$
- Thus,

$$\mathrm{null}(\mathsf{F}^{\mathsf{T}}) = \mathbf{e}' \qquad \mathrm{null}(\mathsf{F}) = \mathbf{e}''$$

Eigenvectors of F

The epipoles are the left and right eigenvectors of the fundamental matrix

$$\mathbf{e}'^{\mathsf{T}}\mathsf{F} = \mathbf{0} \qquad \mathsf{F}\mathbf{e}'' = \mathbf{0}$$

$$\mathrm{null}(\mathsf{F}^{\mathsf{T}}) = \mathbf{e}' \qquad \mathrm{null}(\mathsf{F}) = \mathbf{e}''$$

(they correspond to an eigenvalue of zero)

Epipolar Geometry Summary

- We only assumed a straight-line preserving (uncalibrated) camera
- We discussed the epipolar geometry and key elements such as epipolar lines, axis, plane, and epipoles
- **Insight**: epipolar geometry reduces the search for correspondences in image pairs from a 2D space to 1D space

2
Essential Matrix
(for Calibrated Cameras)

Using Calibrated Cameras

- Most photogrammetric systems rely on calibrated cameras
- Calibrated cameras simplify the orientation problem
- Often, we assume that both cameras have the same calibration matrix
- Assumption here: no distortions or other imaging errors

Refresher from Camera Geometry

- Starting from $\lambda \mathbf{x} = \mathsf{K}R\boldsymbol{X} - \mathsf{K}R\boldsymbol{X}_O$
- we obtain

$$\begin{aligned}\boldsymbol{X} &= (\mathsf{K}R)^{-1}\mathsf{K}R\boldsymbol{X}_O + \lambda(\mathsf{K}R)^{-1}\mathbf{x} \\ &= \boldsymbol{X}_O + \lambda(\mathsf{K}R)^{-1}\mathbf{x}\end{aligned}$$

3x1 Euclidean

- The term $(\mathsf{K}R)^{-1}\mathbf{x}$ describes the direction of the ray from the camera origin $\boldsymbol{X}_O$ to the 3D point $\boldsymbol{X}$

Coplanarity Constraint

- For calibrated cameras the coplanarity constraint can be simplified
- Based on the calibration matrices, we obtain the **directions** as

$$\underbrace{{}^{k}\mathbf{x}'}_{\text{direction in camera frame}} = \mathsf{K}'^{-1} \underbrace{\mathbf{x}'}_{\text{coordinates in the image}} \qquad {}^{k}\mathbf{x}'' = \mathsf{K}''^{-1} \mathbf{x}''$$

Coplanarity Constraint

- Exploiting the fundamental matrix

$$
\begin{aligned}
\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' &= 0 \\
\mathbf{x}'^{\mathsf{T}} \underbrace{(\mathsf{K}')^{-\mathsf{T}} (R')^{-\mathsf{T}} \mathsf{S}_b (R'')^{-1} (\mathsf{K}'')^{-1}}_{\mathsf{F}} \mathbf{x}'' &= 0
\end{aligned}
$$

Coplanarity Constraint

- Exploiting the fundamental matrix

$$
\begin{aligned}
\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' &= 0 \\
\mathbf{x}'^{\mathsf{T}} \underbrace{(\mathsf{K}')^{-\mathsf{T}} (R')^{-\mathsf{T}} \mathsf{S}_b (R'')^{-1} (\mathsf{K}'')^{-1}}_{\mathsf{F}} \mathbf{x}'' &= 0 \\
\underbrace{\mathbf{x}'^{\mathsf{T}} (\mathsf{K}')^{-\mathsf{T}}}_{{}^{k}\mathbf{x}'^{\mathsf{T}}} R' \mathsf{S}_b R''^{\mathsf{T}} \underbrace{(\mathsf{K}'')^{-1} \mathbf{x}''}_{{}^{k}\mathbf{x}''} &= 0
\end{aligned}
$$

Coplanarity Constraint

- Exploiting the fundamental matrix

$$
\begin{aligned}
\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' &= 0 \\
\mathbf{x}'^{\mathsf{T}} \underbrace{(\mathsf{K}')^{-\mathsf{T}} (R')^{-\mathsf{T}} \mathsf{S}_b (R'')^{-1} (\mathsf{K}'')^{-1}}_{\mathsf{F}} \mathbf{x}'' &= 0 \\
\underbrace{\mathbf{x}'^{\mathsf{T}} (\mathsf{K}')^{-\mathsf{T}}}_{{}^k\mathbf{x}'^{\mathsf{T}}} R' \mathsf{S}_b R''^{\mathsf{T}} \underbrace{(\mathsf{K}'')^{-1} \mathbf{x}''}_{{}^k\mathbf{x}''} &= 0 \\
{}^k\mathbf{x}'^{\mathsf{T}} R' \mathsf{S}_b R''^{\mathsf{T}}\ {}^k\mathbf{x}'' &= 0
\end{aligned}
$$

same form as the fundamental matrix but for calibrated cameras

Essential Matrix

- From F to the essential matrix E

$$
\begin{aligned}
\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' &= 0 \\
\mathbf{x}'^{\mathsf{T}} \underbrace{(\mathsf{K}')^{-\mathsf{T}} (R')^{-\mathsf{T}} \mathsf{S}_b (R'')^{-1} (\mathsf{K}'')^{-1}}_{\mathsf{F}} \mathbf{x}'' &= 0 \\
\underbrace{\mathbf{x}'^{\mathsf{T}} (\mathsf{K}')^{-\mathsf{T}}}_{{}^k\mathbf{x}'^{\mathsf{T}}} R' \mathsf{S}_b R''^{\mathsf{T}} \underbrace{(\mathsf{K}'')^{-1} \mathbf{x}''}_{{}^k\mathbf{x}''} &= 0 \\
{}^k\mathbf{x}'^{\mathsf{T}} \underbrace{R' \mathsf{S}_b R''^{\mathsf{T}}}_{\mathsf{E}} \, {}^k\mathbf{x}'' &= 0 \\
{}^k\mathbf{x}'^{\mathsf{T}} \mathsf{E} \, {}^k\mathbf{x}'' &= 0
\end{aligned}
$$

essential matrix

Essential Matrix

- We derived a **specialization of the fundamental matrix**
- For the calibrated cameras, it is called the **essential matrix**

$$\boxed{\mathsf{E} = R' \mathsf{S}_b R''^{\mathsf{T}}}$$

- We can write the coplanarity constraint for calibrated cameras as

$$\boxed{{}^k\mathbf{x}'^{\mathsf{T}} \mathsf{E}\, {}^k\mathbf{x}'' = 0}$$

Essential Matrix

- The essential matrix as **five** degrees of freedom
- There are **five** parameters that determine the relative orientation of the image pair for calibrated cameras
- There are 4=9-5 constraints to its 9 elements (3 by 3 matrix)
- The essential matrix is homogenous and singular

$$ {}^{k}\mathbf{x}'^{\mathsf{T}} \mathsf{E} \, {}^{k}\mathbf{x}'' = 0 $$

4
Popular Parameterizations for the Relative Orientation

Five Parameters – How?

- Five parameters that determine the relative orientation of the image pair

How to parameterize the essential matrix?

The Popular Parameterizations

- Five parameters that determine the relative orientation of the image pair
- Three popular parameterizations

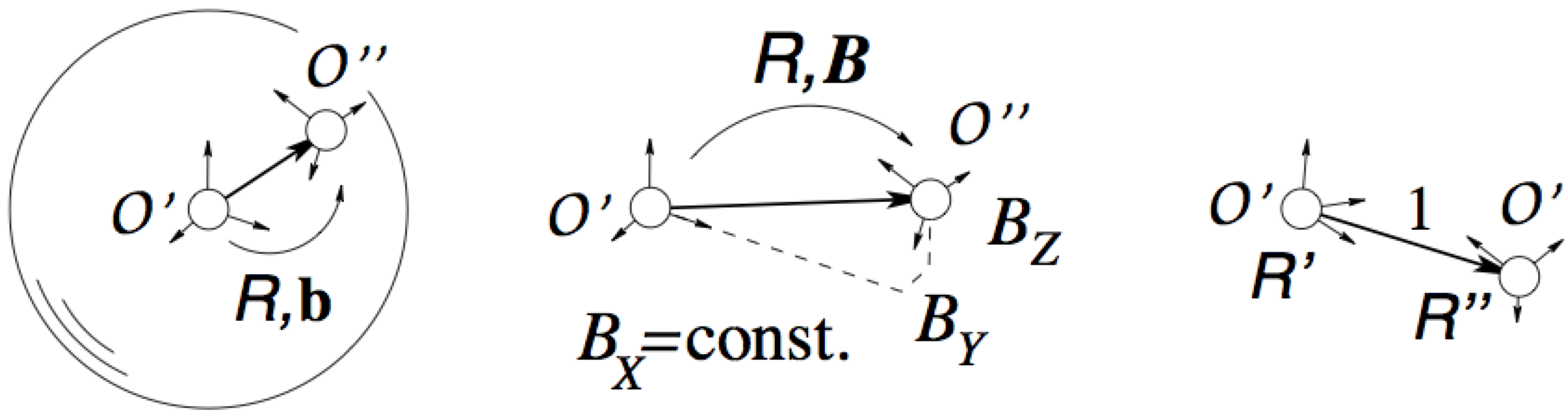


Image courtesy: Förstner

The Popular Parameterizations

1. General parameterization of dependent images
2. Photogrammetric parameterization of dependent images
3. Parameterization with independent images

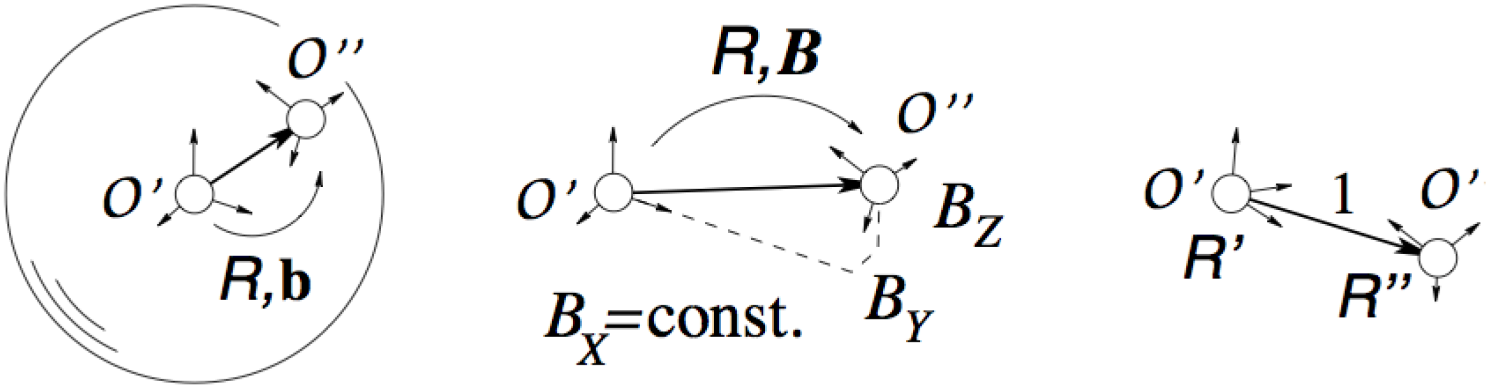
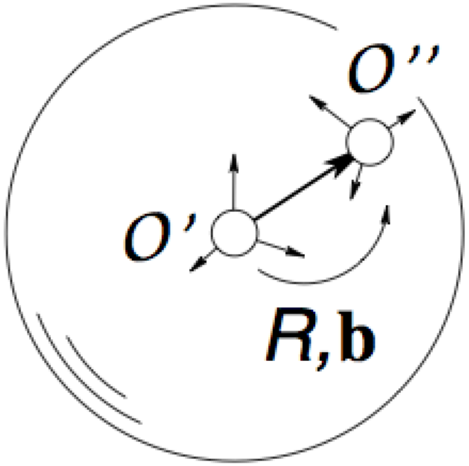


Image courtesy: Förstner

General Parameterization of Dependent Images

The general parameterization of dependent images uses a

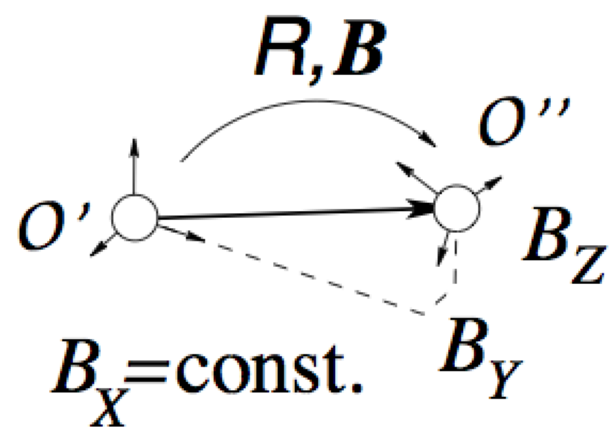
- **normalized direction vector** b
- **rotation matrix** R



Photogrammetric parametrizat. of Dependent Images

Photogrammetric parameterization of dependent images uses

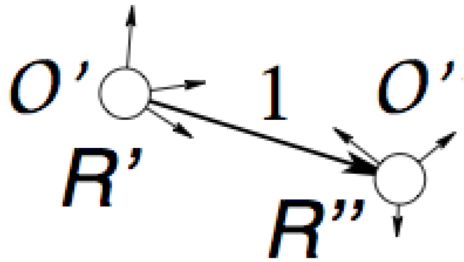
- **two components B_X and B_Z of the base direction**
- a **rotation matrix** R



Parameterization with Independent Images

The parameterization with independent images uses

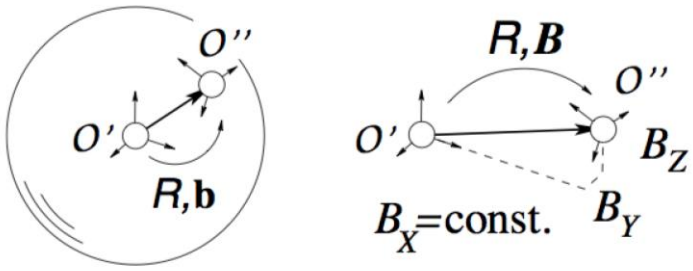
- a **rotation matrix** $R'(\omega', \phi', \kappa')$
- a **rotation matrix** $R''(\omega'', \phi'', \kappa'')$
- a fixed basis of constant length



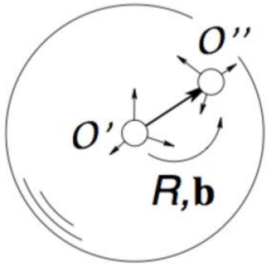
Parameterization of Dependent Images

For the Parameterizations of Dependent Images

- The reference frame is the frame of the first camera
- Describe the second camera relative to the first one
- Rotation mat. of the first cam is $R' = I_3$
- The rotation of the R.O. is then $R = R''$



General Parameterization of Dependent Images

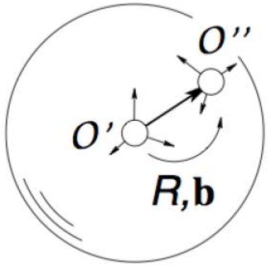


- The orientation of the second camera is $R = R''$ and we obtain from the coplanarity constraint

$$ {}^{k}\mathbf{x}'^{\mathsf{T}} S_b R^{\mathsf{T}}\, {}^{k}\mathbf{x}'' = 0 \qquad \text{with} \qquad |\mathbf{b}| = 1 $$

- 6 parameters + 1 constraint $|\mathbf{b}| = 1$

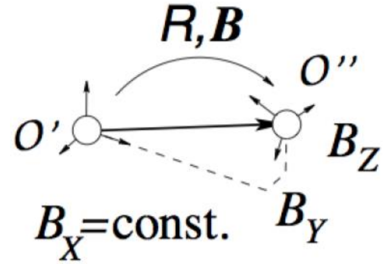
General Parameterization of Dependent Images



- The resulting 5 degree of freedom are

$$(\underbrace{B_X, B_Y, B_Z}_{\mathbf{b}}, \underbrace{\omega, \phi, \kappa}_{R}) \quad \text{with} \quad B_X^2 + B_Y^2 + B_Z^2 = 1$$

Photogrammetric Parametrizat. of Dependent Images

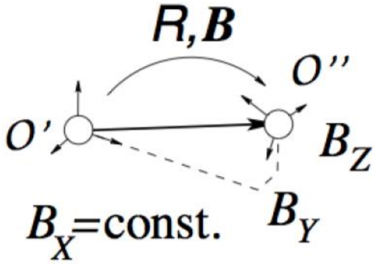


- As before, the first camera defines the reference frame
- $R = R''$
- The basis directs towards the x axis
- The component B_X is constant

$$ {}^k\mathbf{x}'^{\mathsf{T}} \mathsf{S}_b R^{\mathsf{T}}\, {}^k\mathbf{x}'' = 0 \quad \text{with} \quad B_X = const. $$

- B_Y, B_Z are parameters of the R.O.

Photogrammetric Parametrizat. of Dependent Images

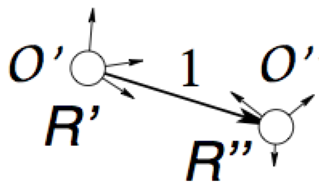


- The resulting 5 parameters are

$$\underbrace{(B_Y, B_Z}_{\boldsymbol{B}}, \underbrace{\omega, \phi, \kappa)}_{R}$$

Parameterization with Independent Images

Parameterization with Independent Images

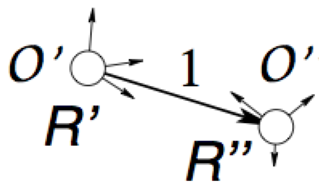


- The center of the reference frame is the projection center O' of the 1st cam
- The x-axis $e_1^{[3]}$ of the object c.s. is the basis

$$\boldsymbol{B}_r = \begin{bmatrix} B_{X_r} \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{X}_{O''_r} - \boldsymbol{X}_{O'_r}$$

- with $\mathbf{b} = \boldsymbol{B}_r = (B_{X_r}, 0, 0)^\mathsf{T}$, $B_{X_r} = const.$

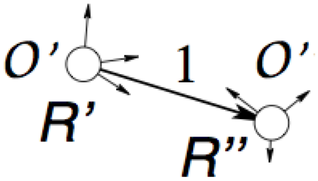
Parameterization with Independent Images



- We have 6 rotation parameters but one rotation around the basis cannot be obtained
- It would result in a change in the exterior orientation of the camera pair
- Thus, one omits the rotation ω' or uses the difference $\Delta\omega = \omega' - \omega''$

$$ {}^k\mathbf{x}'^{\mathsf{T}} R' S R''^{\mathsf{T}} \, {}^k\mathbf{x}'' = 0 \quad \text{with} \quad \omega', S = const. $$

Parameterization with Independent Images

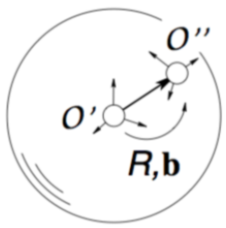


- The resulting 5 parameters are

$$(\Delta\omega, \phi', \kappa', \phi'', \kappa'')$$

Parameterizations Summary

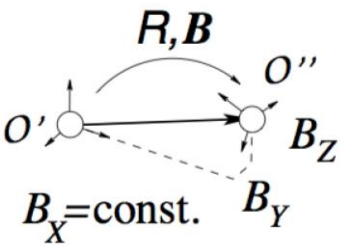
1. General parameterization of dependent images



$$(B_X, B_Y, B_Z, \omega, \phi, \kappa) \quad \text{with} \quad B_X^2 + B_Y^2 + B_Z^2 = 1$$

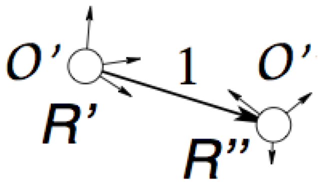
2. Photogrammetric parameterization of dependent images

$$(B_Y, B_Z, \omega, \phi, \kappa)$$



3. Parameterization with independent images

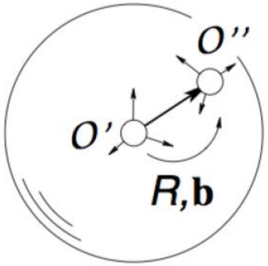
$$(\Delta\omega, \phi', \kappa', \phi'', \kappa'')$$



Remark

- Two parameterizations are general and can represent **all geometric configurations**
- The classical photogrammetric parameterization **has a singularity**
- Singularity: If the base vector is directed orthogonal to the X axis, the base components B_Y and B_Z will be infinitely large in general
- This parameterization therefore leads to instabilities

General Parameterization of Dependent Images



- This **general parameterization** is the **most frequently used one**
- The resulting parameters are

$$(\underbrace{B_X, B_Y, B_Z}_{\mathbf{b}}, \underbrace{\omega, \phi, \kappa}_{R}) \quad \text{with} \quad B_X^2 + B_Y^2 + B_Z^2 = 1$$

$$(|\mathbf{b}| = 1)$$

Essential Matrix Summary

- The essential matrix $\mathsf{E} = R' \mathsf{S}_b R''^{\mathsf{T}}$ encodes the coplanarity const. for calibrated cams, i.e., ${}^k\mathbf{x}'^{\mathsf{T}} \mathsf{E}\, {}^k\mathbf{x}'' = 0$
- It encode the relative orientation
- It has 5 degree of freedom
- Given the general parameterization of dependent images, this yields

$$E(B_X, B_Y, B_Z, \omega, \phi, \kappa) = \mathsf{S}_{(B_X, B_Y, B_Z)} R(\omega, \phi, \kappa)$$

$$\text{with} \quad B_X^2 + B_Y^2 + B_Z^2 = 1$$

4
Normalized Stereo Pairs

Stereo Normal Case

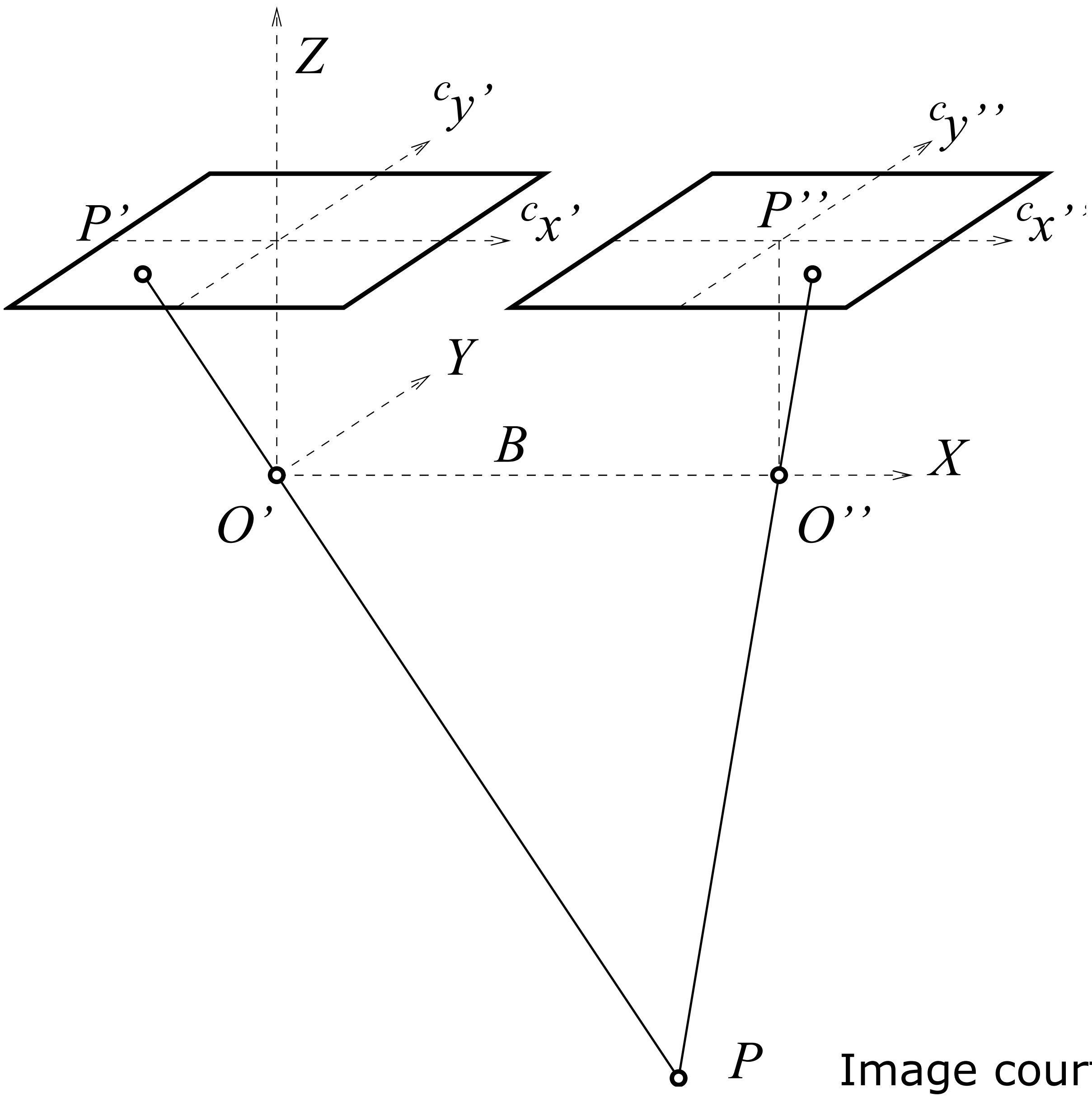


Image courtesy: Förstner

Stereo Normal Case

- Images planes are parallel and not rotated w.r.t. each other
- Offset **only** in x direction
- Y-parallaxes are zero for all points

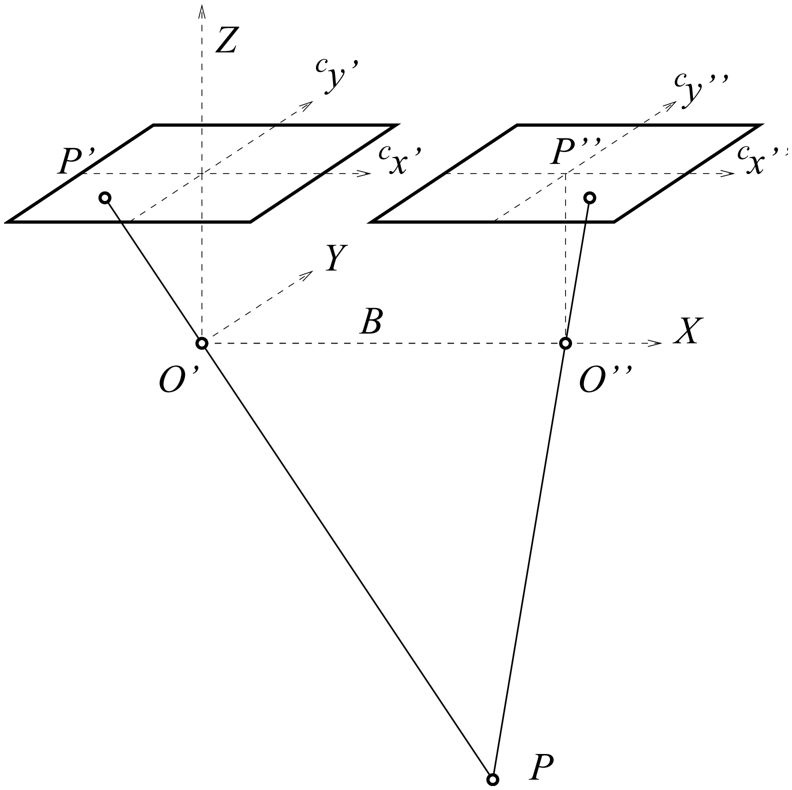
$$p_y = {}^{c}y'' - {}^{c}y' = 0$$

Stereo Normal Case

$$R' = R'' = I_3 \qquad \mathbf{b} = \begin{bmatrix} B_X \\ 0 \\ 0 \end{bmatrix}$$

$$\mathsf{K} \doteq \mathsf{K}' = \mathsf{K}'' = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathsf{E} = \mathsf{S}_b R^{\mathsf{T}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -B_X \\ 0 & B_X & 0 \end{bmatrix}$$



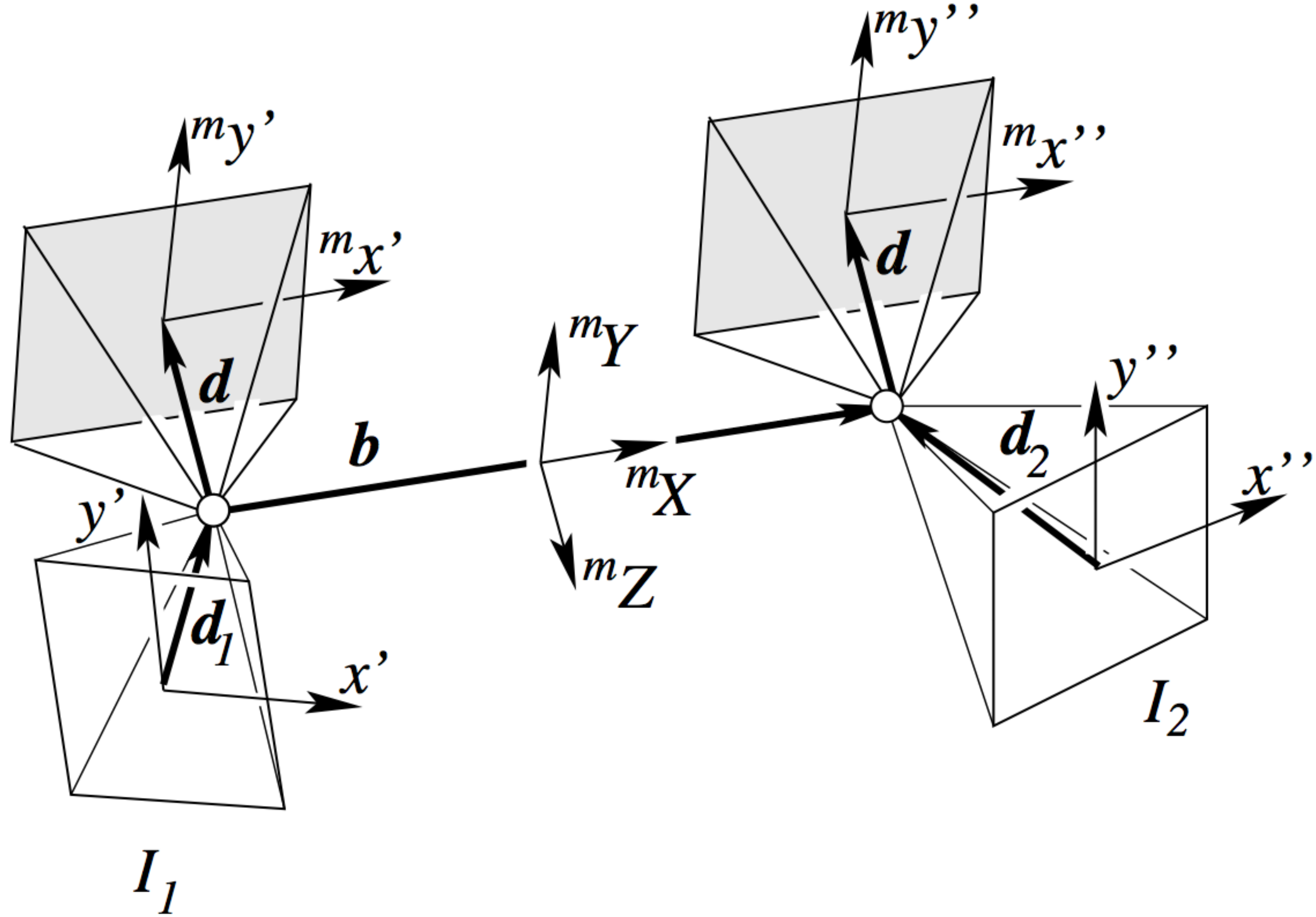
$$\Rightarrow \left[\ {}^c x' \quad {}^c y' \quad c \right] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -B_X \\ 0 & B_X & 0 \end{bmatrix} \begin{bmatrix} {}^c x'' \\ {}^c y'' \\ c \end{bmatrix} = cB_X \left({}^c y'' - {}^c y' \right) = 0$$

Stereo Normal Case

- Image planes are parallel and not rotated w.r.t. each other
- Offset **only** in x direction
- Y-parallaxes are zero for all points
- Approximately for aerial images
- **Not the case** for most image pairs

Can we generate stereo normal pairs?

Generating Norm. Stereo Pairs

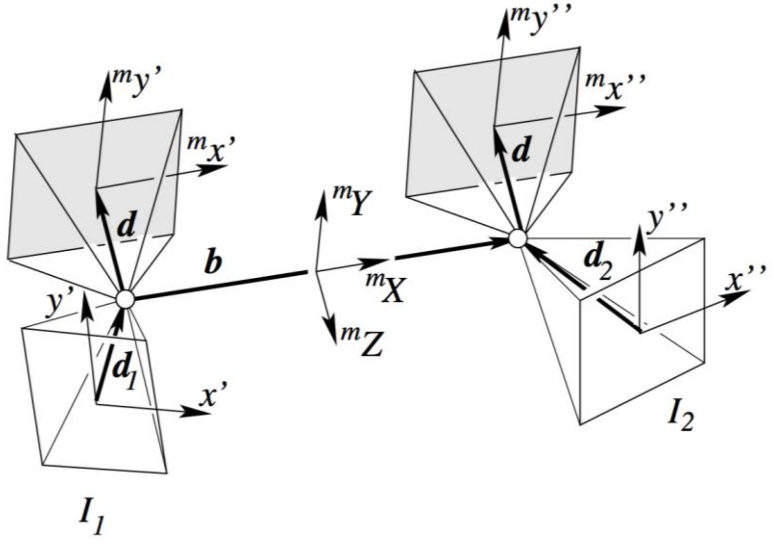


The normalized images have a common calibration and rotation matrix, the common viewing direction d being the average of the two viewing directions d_1 and d_2.

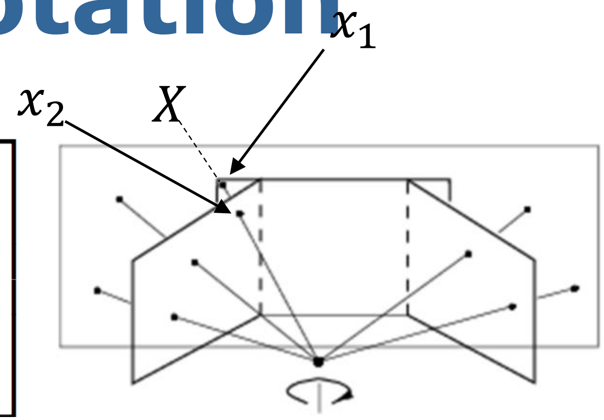
Generating Norm. Stereo Pairs

- Such that, the bundle of rays stays unchanged
- But now the image planes can change.
- Possibly have different zoom/focal length.

What type of mapping is needed?



Homography: Pure Rotation



$$\lambda_1 \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \overbrace{\begin{bmatrix} f_1 & 0 & 0 \\ 0 & f_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{K_1} \overbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & \cancel{t_x} \\ r_{21} & r_{22} & r_{23} & \cancel{t_y} \\ r_{31} & r_{32} & r_{33} & \cancel{t_z} \end{bmatrix}}^{R_1} \begin{bmatrix} X \\ Y \\ Z \\ \cancel{1} \end{bmatrix}$$

$$\lambda_2 \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \overbrace{\begin{bmatrix} f_2 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{K_2} \overbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & \cancel{t_x} \\ r_{21} & r_{22} & r_{23} & \cancel{t_y} \\ r_{31} & r_{32} & r_{33} & \cancel{t_z} \end{bmatrix}}^{R_2} \begin{bmatrix} X \\ Y \\ Z \\ \cancel{1} \end{bmatrix}$$

$$\lambda \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \underbrace{K_2 R_2 {R_1}^{-1} {K_1}^{-1}}_{H} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

Convert between 2D locations on 2 images with pure camera rotation with a 3X3 matrix H

Generating Norm. Stereo Pairs

- Can be expressed by a **homography (projective transformation)**

$$\mathbf{x}' = \mathsf{H}'_m \ {}^m\mathbf{x}' \qquad \mathbf{x}'' = \mathsf{H}''_m \ {}^m\mathbf{x}''$$

homographies

$${}^m\mathbf{x}' = {}^m\mathsf{H}'\mathbf{x}' \qquad {}^m\mathbf{x}'' = {}^m\mathsf{H}''\mathbf{x}''$$

NB: ${}^mH' = (H'_m)^{-1}$ and ${}^mH'' = (H''_m)^{-1}$

Computing the Homographies

- Given $\mathsf{P}'(\mathsf{K}', R', \mathbf{X}_{O'}), \mathsf{P}''(\mathsf{K}'', R'', \mathbf{X}_{O''})$
- Compute $\mathsf{H}'_m, \mathsf{H}''_m$ or ${}^m\mathsf{H}', {}^m\mathsf{H}''$
- We have

$$
\begin{aligned}
\mathbf{x}' &= \mathsf{P}'\mathbf{X} = \mathsf{K}'R'[I_3 | -\boldsymbol{X}_{O'}]\mathbf{X} \\
\mathbf{x}'' &= \mathsf{P}''\mathbf{X} = \mathsf{K}''R''[I_3 | -\boldsymbol{X}_{O''}]\mathbf{X}
\end{aligned}
$$

- Define the system for the stereo normal case (keeping the projection centers)

Computing the Homographies

- Given $\mathsf{P}'(\mathsf{K}', R', \mathbf{X}_{O'}), \mathsf{P}''(\mathsf{K}'', R'', \mathbf{X}_{O''})$
- Compute $\mathsf{H}'_m, \mathsf{H}''_m$ or ${}^m\mathsf{H}', {}^m\mathsf{H}''$
- We have

$$\begin{aligned} \mathbf{x}' &= \mathsf{P}'\mathbf{X} = \mathsf{K}'R'[I_3 | -\boldsymbol{X}_{O'}]\mathbf{X} \\ \mathbf{x}'' &= \mathsf{P}''\mathbf{X} = \mathsf{K}''R''[I_3 | -\boldsymbol{X}_{O''}]\mathbf{X} \end{aligned}$$

- Define K, R so that

$$\begin{aligned} {}^m\mathbf{x}' &= {}^m\mathsf{P}'\mathbf{X} = \mathsf{K}R[I_3 | -\boldsymbol{X}_{O'}]\mathbf{X} \\ {}^m\mathbf{x}'' &= {}^m\mathsf{P}''\mathbf{X} = \mathsf{K}R[I_3 | -\boldsymbol{X}_{O''}]\mathbf{X} \end{aligned}$$

- Typically: $\mathsf{K} = \mathrm{Diag}([c, c, 1])$

Computing the Homographies

$$\mathbf{x}' = \mathsf{P}'\mathbf{X} = \mathsf{K}'R'\underbrace{\left[I_3 | -\boldsymbol{X}_{O'}\right]}_{\mathsf{A}'}\mathbf{X} = \mathsf{K}'R'\mathsf{A}'\mathbf{X}$$

$$^{m}\mathbf{x}' = {}^{m}\mathsf{P}'\mathbf{X} = \mathsf{K}R\underbrace{\left[I_3 | -\boldsymbol{X}_{O'}\right]}_{\mathsf{A}'}\mathbf{X} = \mathsf{K}R\mathsf{A}'\mathbf{X}$$

Computing the Homographies

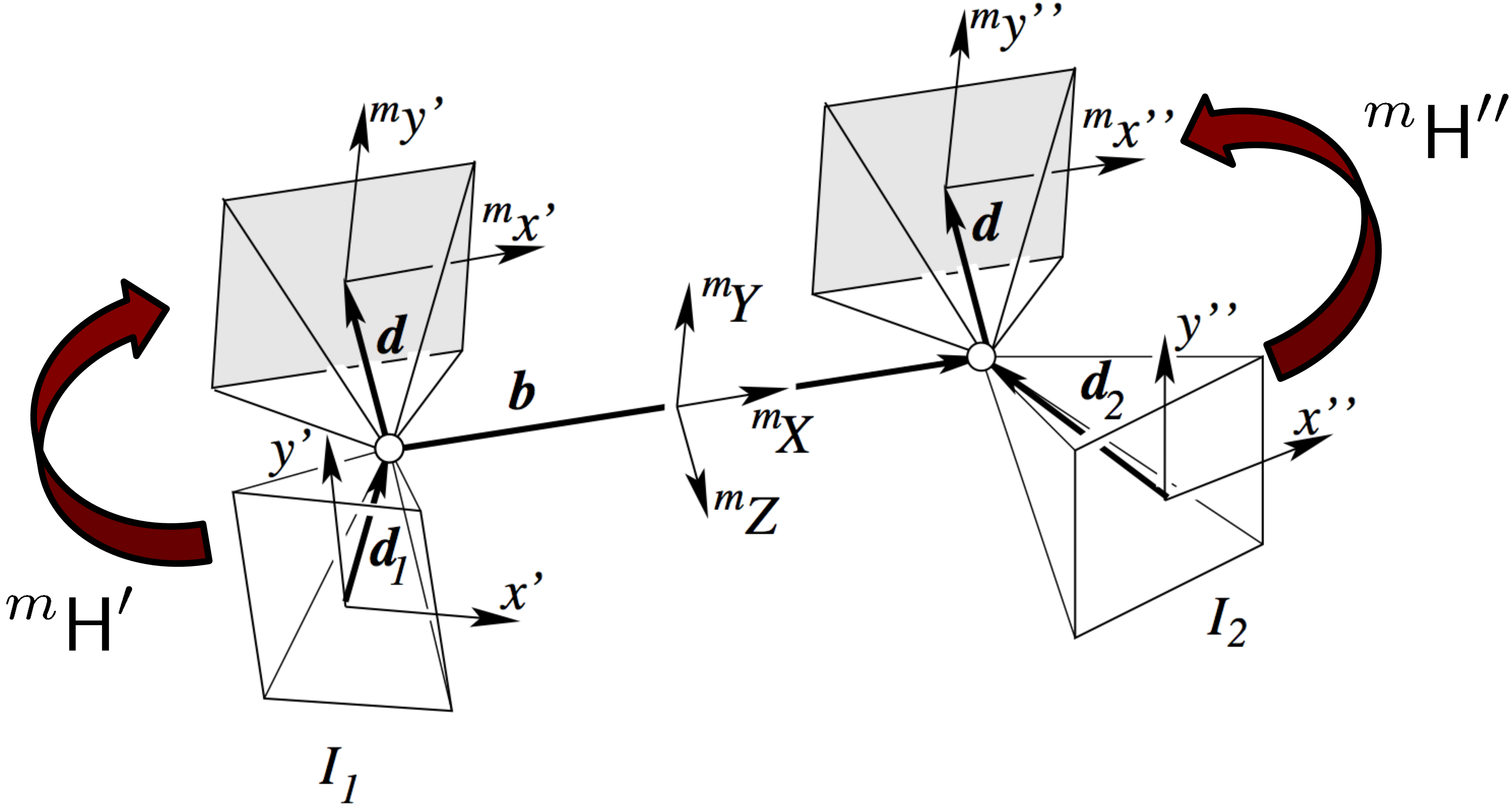
$$\mathbf{x}' = \mathsf{P}'\mathbf{X} = \mathsf{K}'R'\underbrace{[I_3|-\boldsymbol{X}_{O'}]}_{\mathsf{A}'}\mathbf{X} = \boxed{\mathsf{K}'R'}\mathsf{A}'\mathbf{X}$$

$$^m\mathbf{x}' = {}^m\mathsf{P}'\mathbf{X} = \mathsf{K}R\underbrace{[I_3|-\boldsymbol{X}_{O'}]}_{\mathsf{A}'}\mathbf{X} = \boxed{\mathsf{K}R}\mathsf{A}'\mathbf{X}$$

$$\Rightarrow \quad {}^m\mathsf{H}' = \mathsf{K}RR'^{\mathsf{T}}\mathsf{K}'^{-1}$$

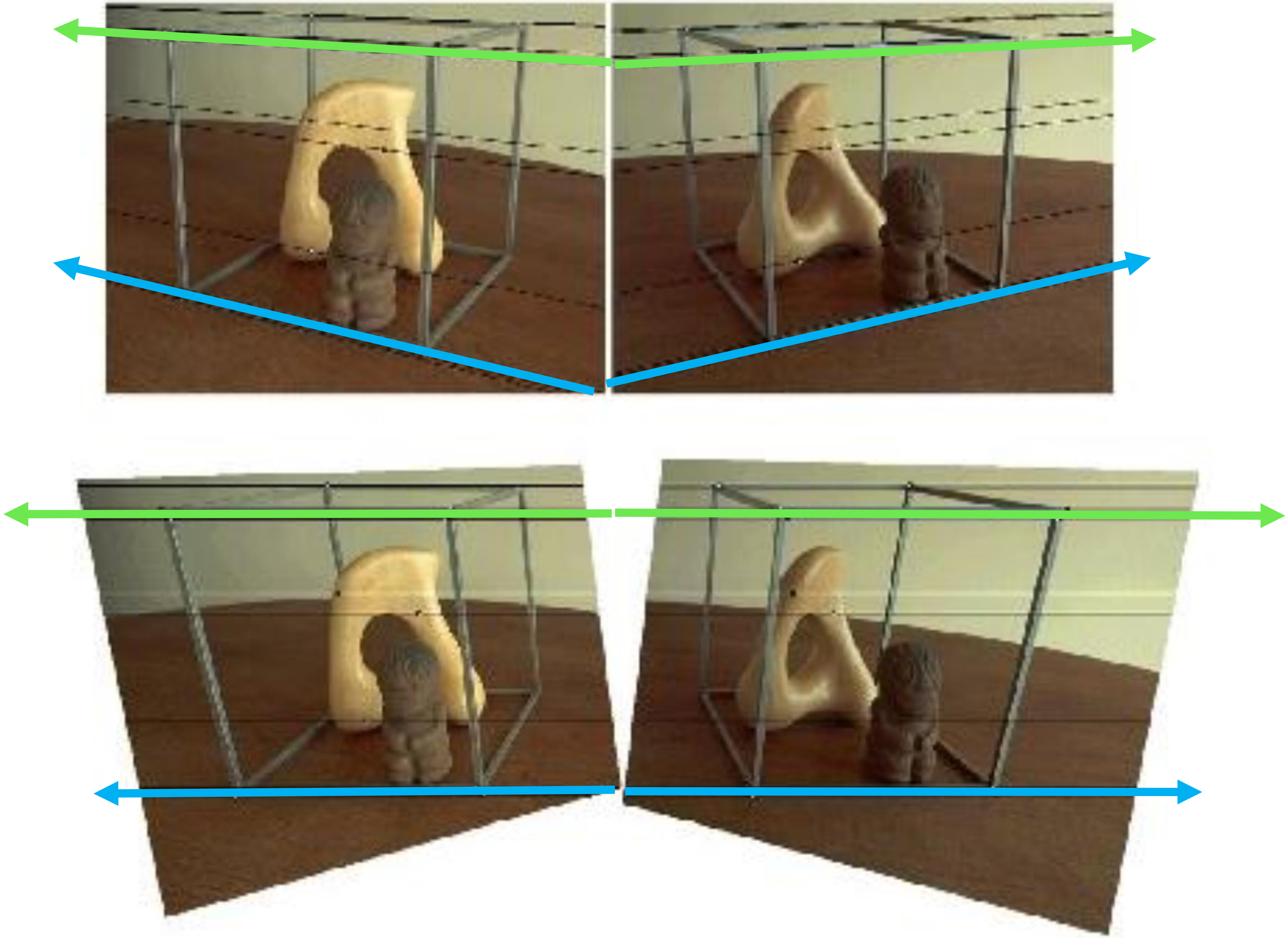
$$\Rightarrow \quad {}^m\mathbf{x}' = {}^m\mathsf{H}'\mathbf{x}'$$

Generating Norm. Stereo Pairs



$${}^m\mathbf{x}' = {}^m\mathsf{H}'\mathbf{x}' \qquad {}^m\mathbf{x}'' = {}^m\mathsf{H}''\mathbf{x}''$$

Example



Summary

- Epipolar geometry (epipoles, epipolar lines, planes, axes)
- Essential matrix
- Three key parameterization for the relative orientation
- Generating normalized stereo pairs

Literature

- Förstner, Wrobel: Photogrammetric Computer Vision, Ch. 12.2.3-12.2.7

Slide Information

- The slides have been created by Cyrill Stachniss as part of the photogrammetry and robotics courses.
- I tried to acknowledge all people from whom I used images or videos. In case I made a mistake or missed someone, please let me know.
- The photogrammetry material heavily relies on the very well written lecture notes by Wolfgang Förstner and the Photogrammetric Computer Vision book by Förstner & Wrobel.
- Parts of the robotics material stems from the great Probabilistic Robotics book by Thrun, Burgard and Fox.

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