Computer Vision (CS763)

Teaching cameras to “see”

Homogeneous Coordinates and Important Transformations

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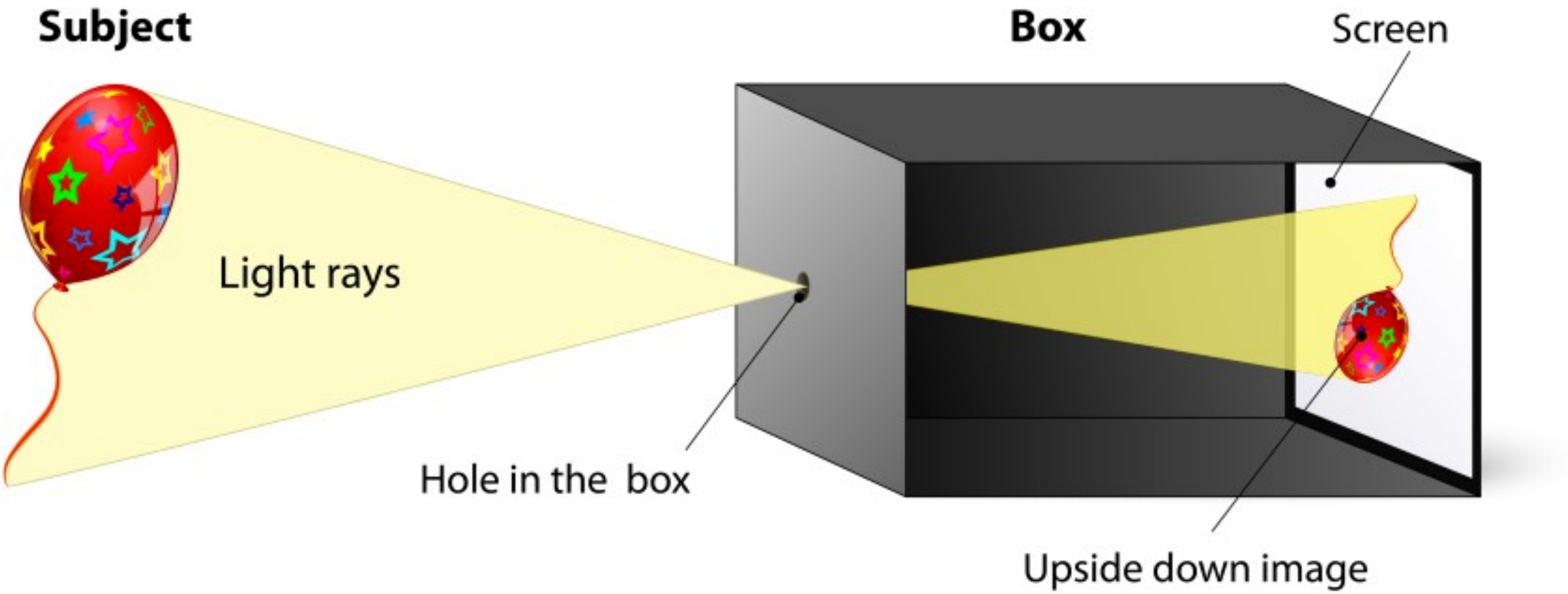
Pin Hole Camera Model

1. A box with an infinitesimal hole
2. Camera center is the intersection point of all incoming rays
3. Back wall is the image plane
4. Distance between the hole and the back wall is called **camera constant**

Pin Hole Camera Model



Pin Hole Camera Model

Geometry and Images

What can we say about the geometry?

Image courtesy: Förstner

Pinhole Camera Properties

- **Line-preserving:** straight lines are mapped to straight lines
- **Not length-preserving:** size of objects is inverse proportional to the distance
- **Not angle-preserving:** Angles between lines change

Perspective Projection

- Straight lines stay straight
- Parallel lines may intersect

Image courtesy: Förstner

Rectified Images

Image courtesy: Förstner

Central Questions in Photogrammetry

- Relationship between the object in the scene and the object in the image
- Relationship between points in the image and the rays to the object
- Orientation of the camera in the scene
- Inferring the geometry of an object or a scene given an image

Information Loss Caused by the Projection of the Camera

- A camera **projects from** the **3D** world **to** a **2D** image
- This causes a **loss of information**
- 3D information can only be recovered if additional information is available

Information Loss Caused by the Projection of the Camera

- A camera **projects from** the **3D** world **to** a **2D** image
- This causes a **loss of information**
- 3D information can only be recovered if additional information is available
  - Multiple images
  - Details about the camera
  - Known size of objects
  - ...

Cameras to Measure Directions

An image point in a camera image defines a ray to the object point

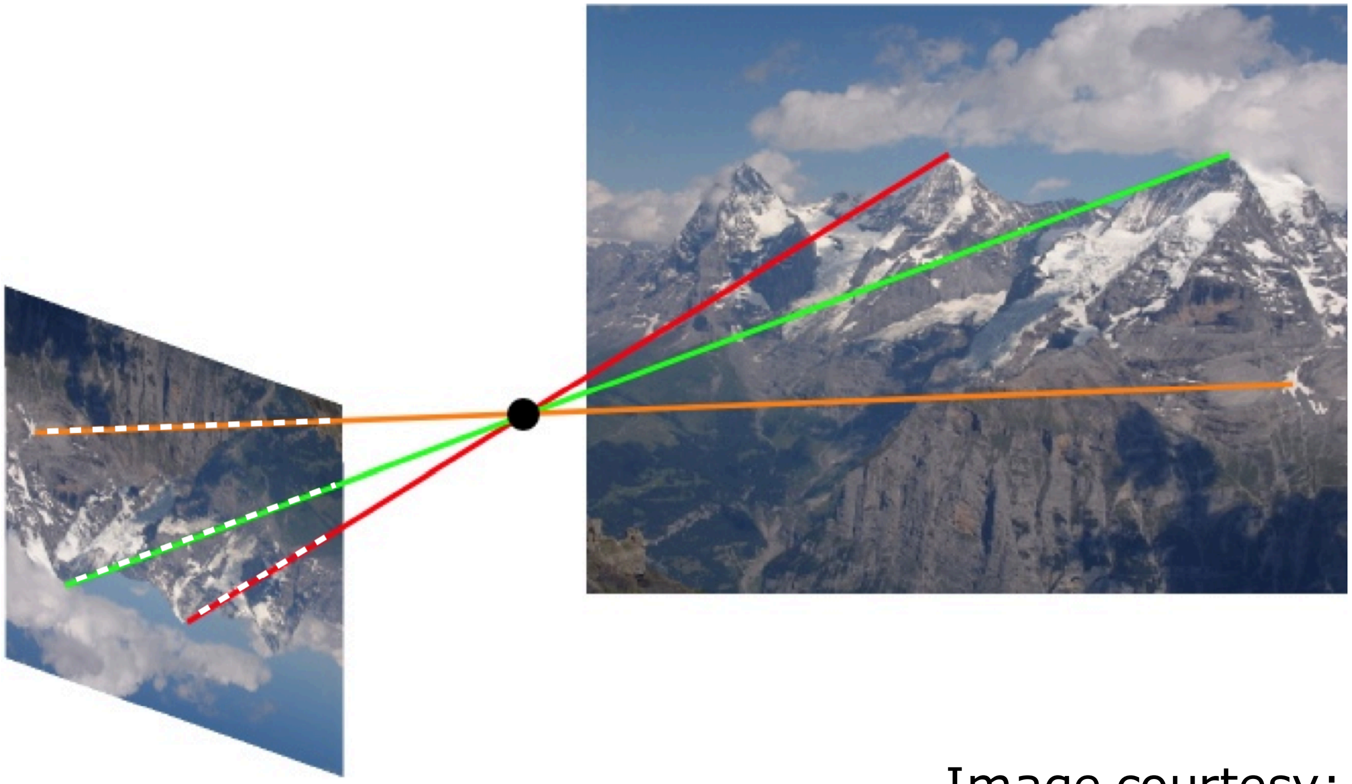


Image courtesy: Schindler

3D Perception

Multiple observations from different directions allow for estimating the 3D location of points via triangulation

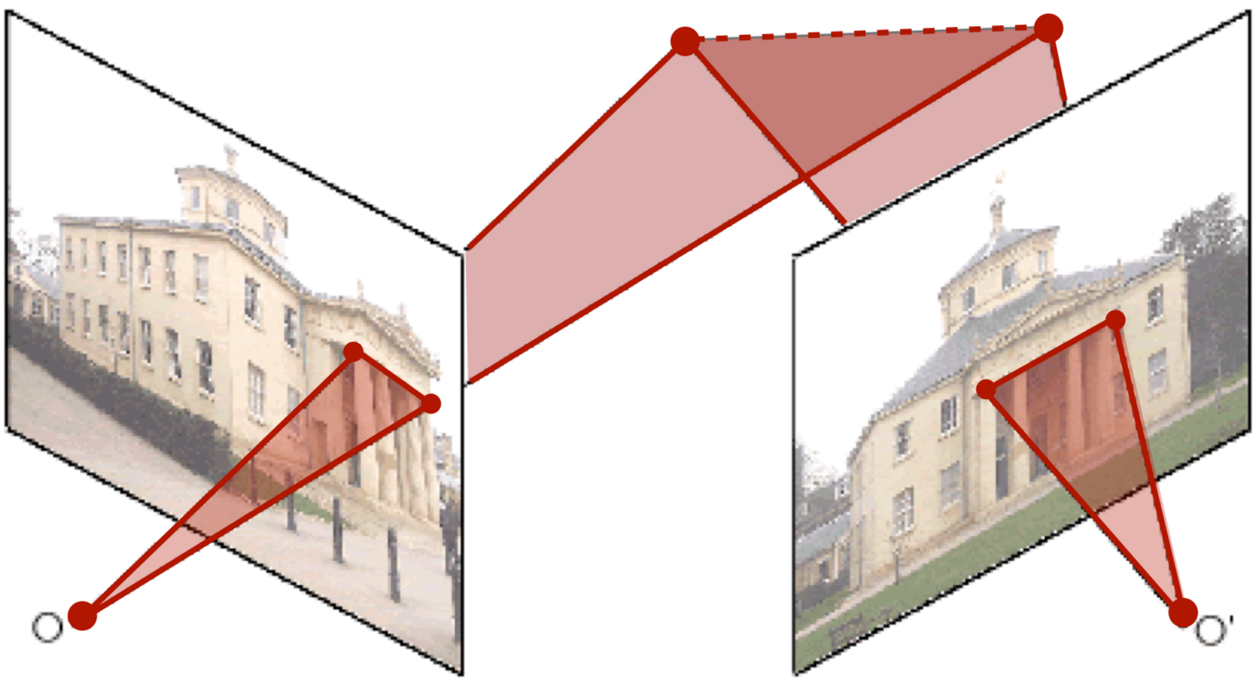


Image courtesy: Schindler

Consider the Previous Example

- Straight lines stay straight
- Parallel lines may not remain parallel

Image courtesy: Förstner

Vanishing Point

Image Courtesy: J. Jannene

Vanishing Points

- Parallel lines are not parallel anymore
- All mapped parallel lines intersect in a vanishing point
- The vanishing point is the “point at infinity” for the parallel lines
- Every direction has exactly one vanishing point

Vanishing Points

- Parallel lines are not parallel anymore
- All mapped parallel lines intersect in a vanishing point
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- Every direction has exactly one vanishing point

How to describe "points at infinity"?

Projective Geometry Motivation

- Cameras generate a projected image of the world
- **Euclidian geometry is suboptimal to describe the central projection**
- In Euclidian geometry, the math can get difficult
- Projective geometry is an alternative algebraic representation of geometric objects and transformations

Advantages of the Projective Geometry

- Math becomes simpler
- Projective geometry does not change the geometric relations
- Computations can also be done in Euclidian geometry (but more difficult)

But at what cost?

Homogeneous Coordinates

- H.C. are a system of coordinates used in projective geometry
- Formulas involving H.C. are often simpler than in the Cartesian world
- Points at infinity can be represented using finite coordinates
- A single matrix can represent transformations and projections

Homogeneous Coordinates

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- **A single matrix can represent transformations and projections**

Homogeneous Coordinates

Definition

The representation $\mathbf{x}$ of a geometric object is **homogeneous** if $\mathbf{x}$ and $\lambda\mathbf{x}$ represent the same object for $\lambda \neq 0$

Example

$$\mathbf{x} = \lambda\,\mathbf{x}$$

homogeneous

$$x \neq \lambda\,x$$

Euclidian

Homogeneous Coordinates

- H.C. use a n+1 dimensional vector to represent the same (n-dim.) point
- Example for $\mathbb{R}^2/\mathbb{P}^2$

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = w \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

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Example

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} wx \\ wy \\ w1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

homogeneous

Euclidian

Definition

- Homogeneous Coordinates of a point χ in the plane $\mathbb{R}^2$ is a 3-dim. vector

$$\chi : \quad \mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \text{ with } |\mathbf{x}|^2 = u^2 + v^2 + w^2 \neq 0$$

- it corresponds to Euclidian coordinates

$$\chi : \quad \boldsymbol{x} = \begin{bmatrix} u/w \\ v/w \end{bmatrix} \text{ with } w \neq 0$$

Example: Projective Plane

The projective plane $\mathbb{P}^2(\mathbb{R})$ or $\mathbb{P}^2$ contains

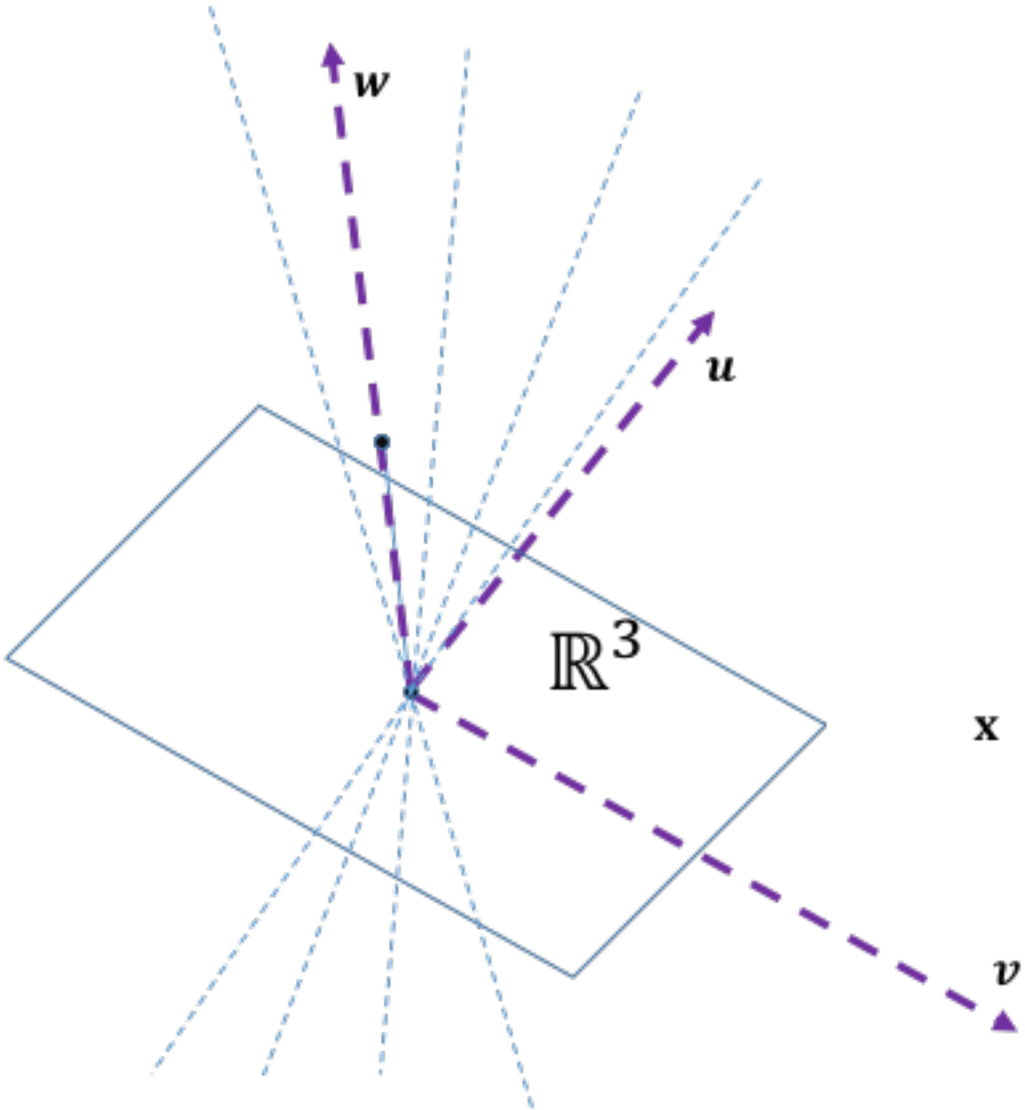
- All points $\mathcal{X}$ of the Euclidian plane $\mathbb{R}^2$ with $\boldsymbol{x} = [x, y]^\top$ expressed through the 3-valued vector (e.g., $\mathbf{x} = [x, y, 1]^\top$)
- and all points at infinity, i.e.,

  $$\mathbf{x} = [x, y, 0]^\top$$

- except $[0, 0, 0]^\top$

2D Projective Plane

Think of $\mathbb{P}^2$ to be made of all lines going through the origin in $\mathbb{R}^3$



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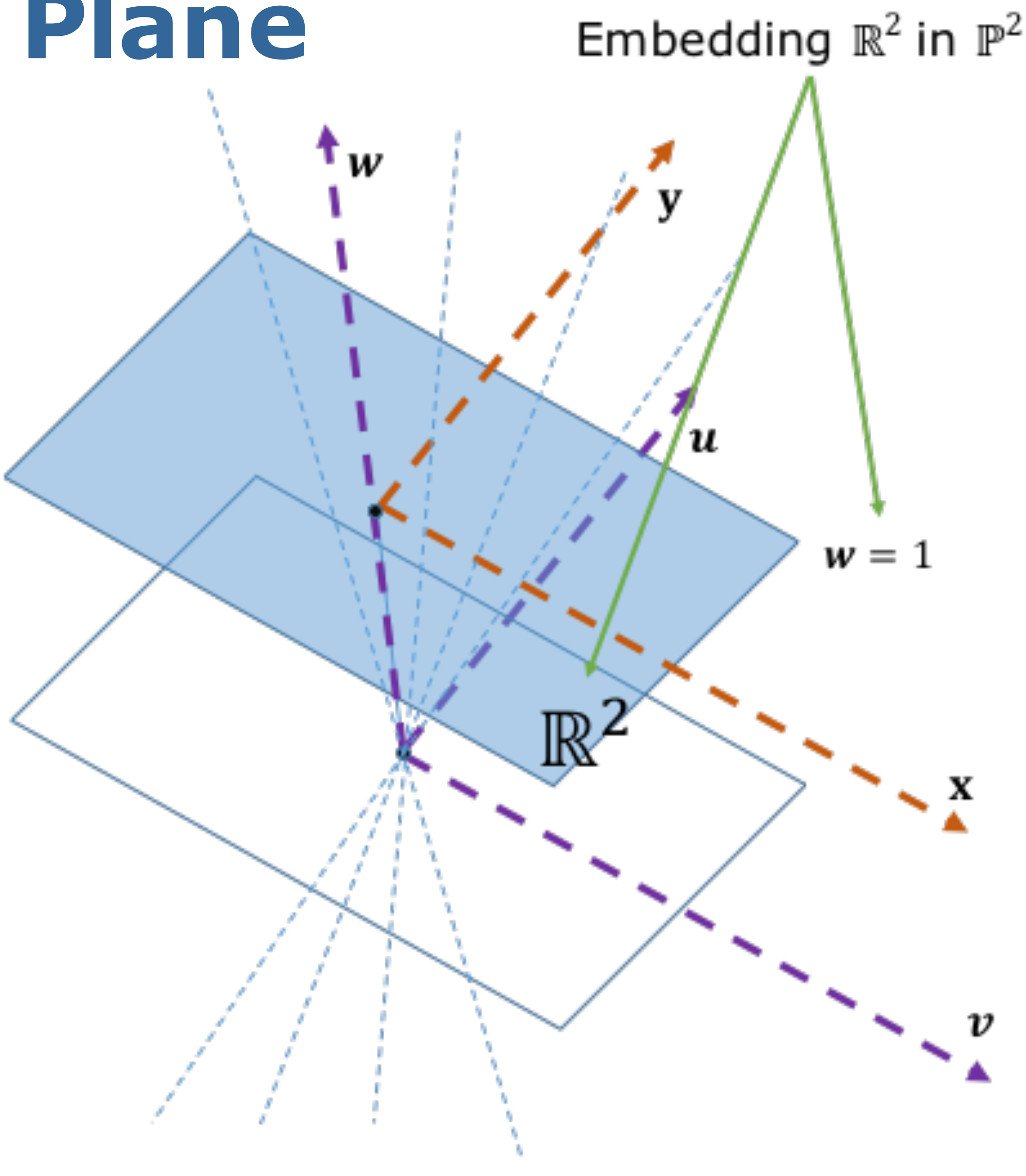
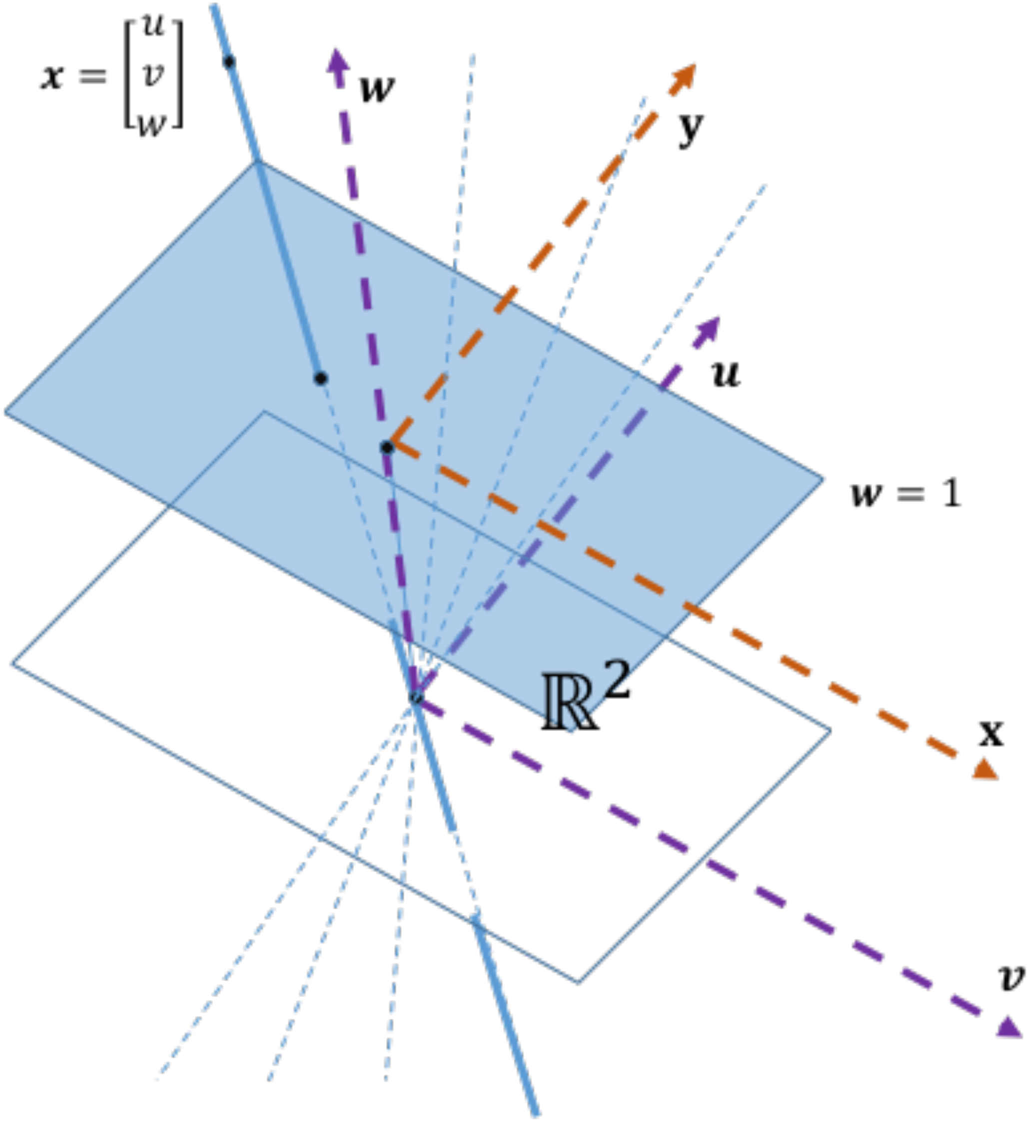


2D Projective Plane

Think of $\mathbb{P}^2$ to be made of all lines going through the origin in $\mathbb{R}^3$

Associate a point $\mathbf{x} = [x, y]^T$ of $\mathbb{R}^2$ with a projective point $\varphi(\mathbf{x}) = [u, v, 1]^T$ in $\mathbb{P}^2$

$\mathbf{x} \to \varphi(\mathbf{x})$ is clearly one-to-one as distinct points of $\boldsymbol{w} = \mathbf{1}$ give rise to distinct radial lines through them.



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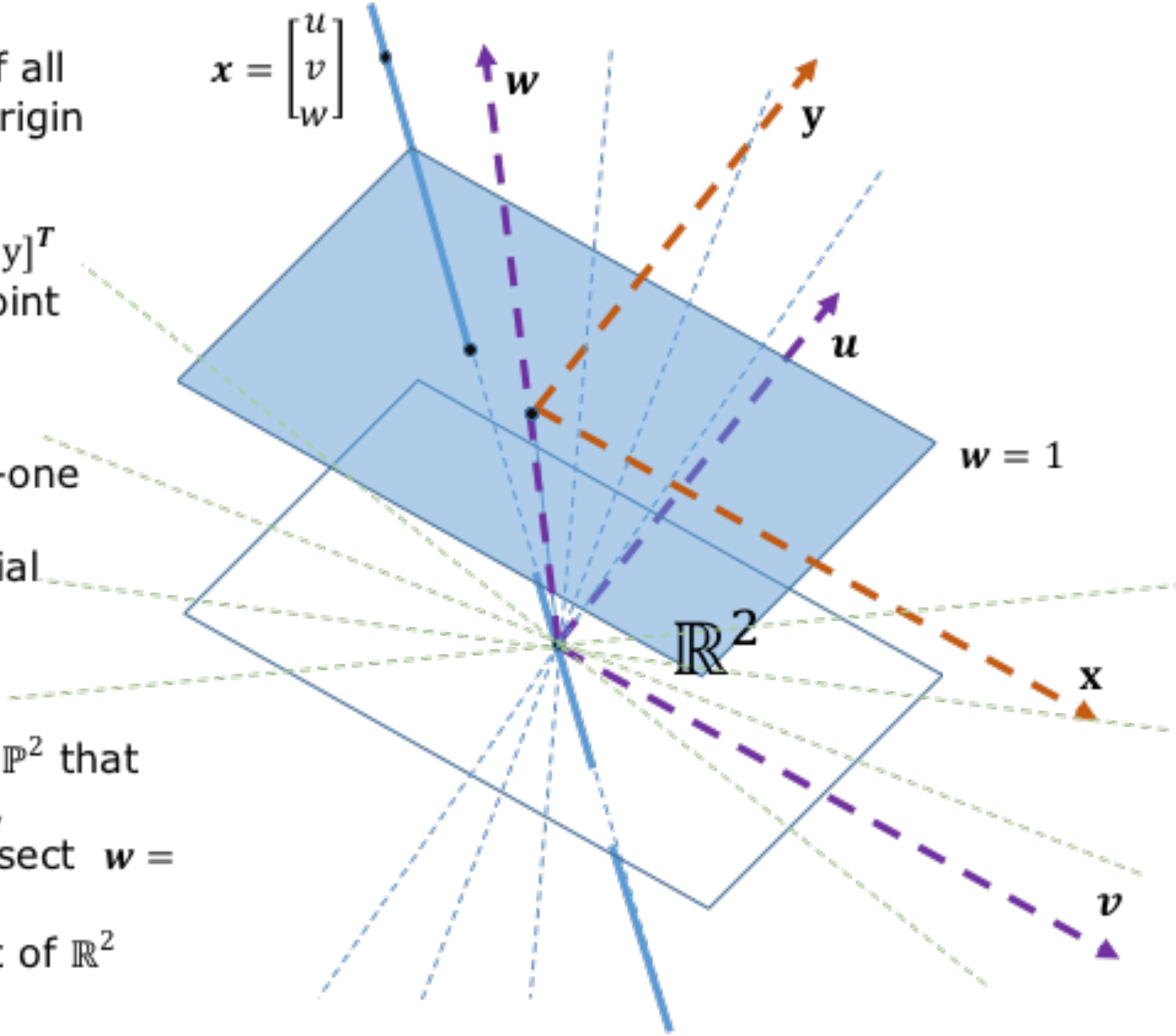
It's not onto as points of $\mathbb{P}^2$ that lie on the plane $\boldsymbol{w} = \mathbf{0}$ or, equivalently, do not intersect $\boldsymbol{w} = \mathbf{1}$ and, therefore, are not associated with any point of $\mathbb{R}^2$



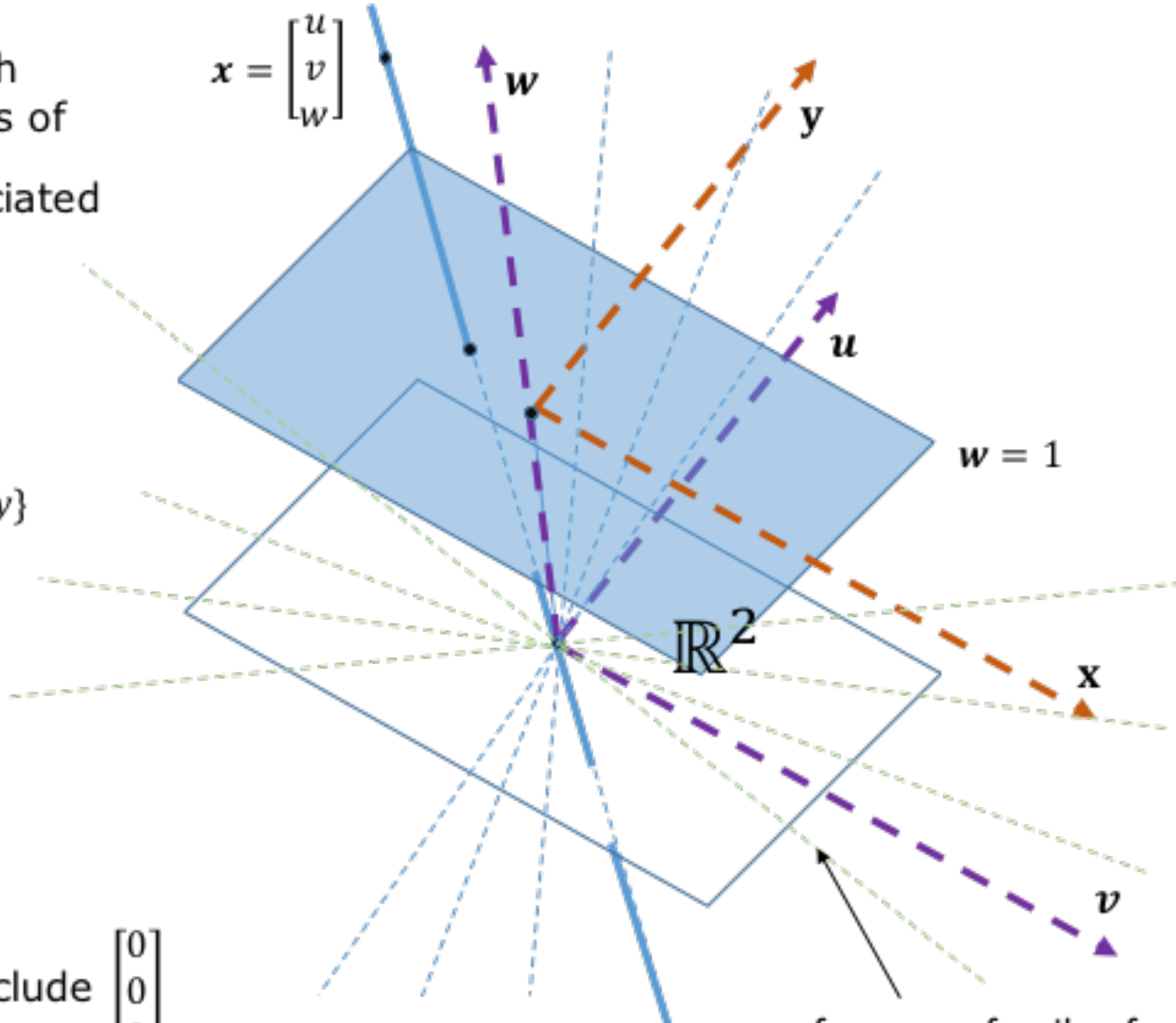
2D Projective Plane

Precisely, points of $\mathbb{P}^2$ with homogeneous coordinates of the form $\begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$ are not associated with any point of $\mathbb{R}^2$

$$\mathbb{P}^2 = \mathbb{R}^2 \cup \{points\ at\ infinity\}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u/w \\ v/w \end{bmatrix} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix}$$

Reminder: $\mathbb{P}^2$ does not include $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$



one for every family of parallel lines in $\mathbb{R}^2$

2D Projective Plane

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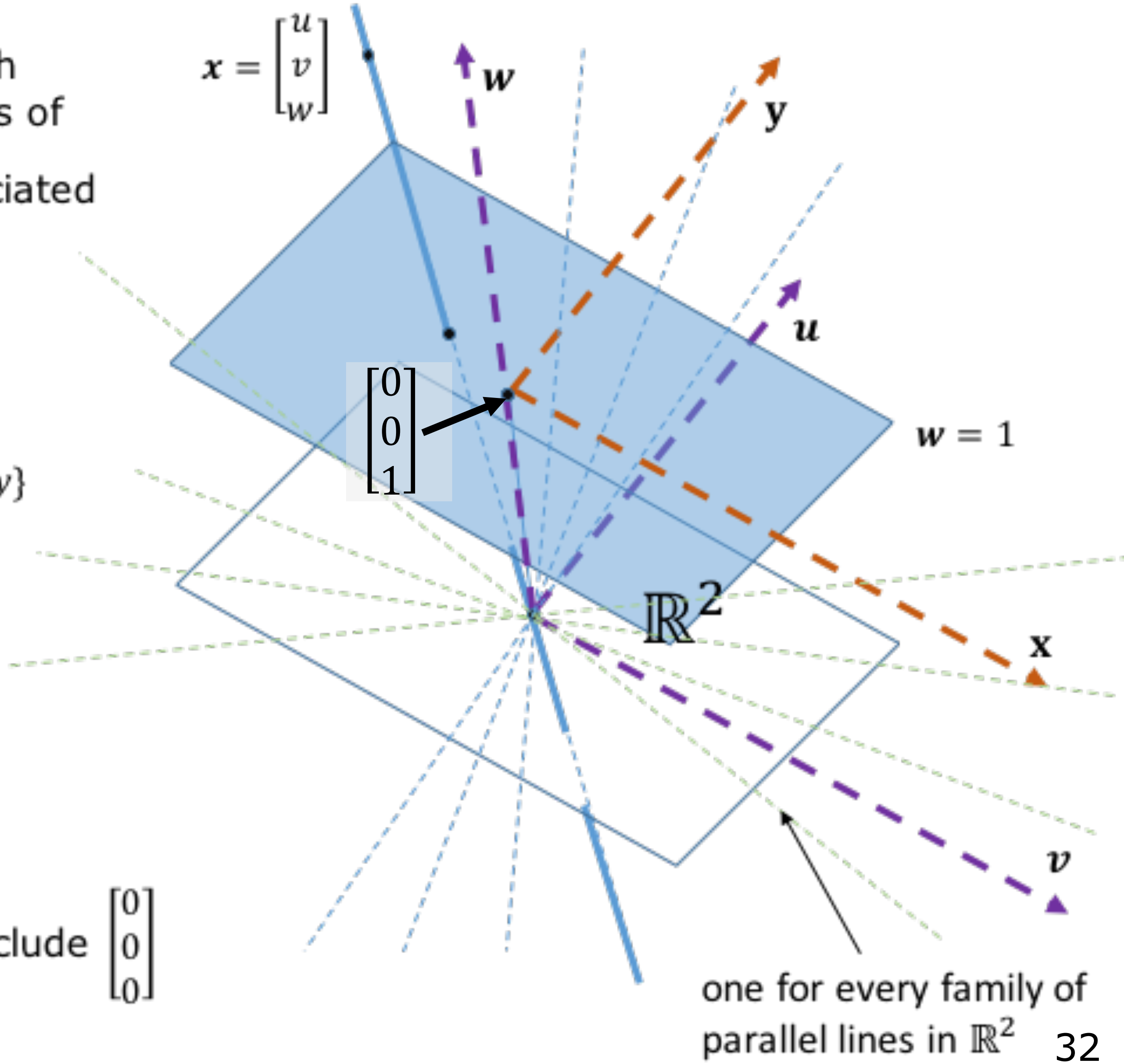


3D Points

Analogous for points in 3D Euclidian space $\mathbb{R}^3$

homogeneous Euclidian

$$\mathbf{X} = \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix} = \begin{bmatrix} U/T \\ V/T \\ W/T \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} U/T \\ V/T \\ W/T \end{bmatrix}$$

Lines

Points and Lines in $\mathbb{P}^2$

- We know that a line in $\mathbb{R}^2$ can be represented by the equation $l_1 x + l_2 y + l_3 = 0$ where not all l_1, l_2, l_3 are zeros

- If we think of $\boldsymbol{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ as a point in $\mathbb{P}^2$ then $\boldsymbol{x}$ belongs to the line $\boldsymbol{l} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$ if $l_1 x + l_2 y + l_3 = 0$

- Note that $l_1 x + l_2 y + l_3 = \boldsymbol{l}^T \boldsymbol{x} = \boldsymbol{x}^T \boldsymbol{l}$

- If we use $\boldsymbol{x} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix}$ to represent $\boldsymbol{x}$ instead, we get the scalar product $\lambda l_1 x + \lambda l_2 y + \lambda l_3 = \lambda(l_1 x + l_2 y + l_3) = 0$

Thus, it does not matter what representation we use for $\boldsymbol{x}$

Points and Lines in $\mathbb{P}^2$

- Again, lets start with a line in $\mathbb{R}^2$ that can be represented by the equation $l_1 x + l_2 y + l_3 = 0$ where not all l_1, l_2, l_3 are zeros

- Lets go from $\mathbb{R}^2$ to $\mathbb{P}^2$ by using $\begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix} = \begin{bmatrix} u/w \\ \mathrm{v}/w \end{bmatrix}$ and replace the same in the above equation of the line:

$$l_1 \frac{u}{w} + l_2 \frac{v}{w} + l_3 = 0$$
$$\Rightarrow l_1 u + l_2 v + l_3 w = 0$$

- Thus, if we represent a line of $\mathbb{P}^2$ by the 3-vector $\boldsymbol{l} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$, all not zero

Points and Lines in $\mathbb{P}^2$

- A point $\boldsymbol{m}$ is located on a line $\boldsymbol{l}$ iff

  $\boldsymbol{l}^{\boldsymbol{T}}\boldsymbol{x} = \boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{l} = \mathbf{0}$ $(ul_1 + vl_2 + wl_3 = 0)$

- A line $\boldsymbol{l}$ passes through a point $\boldsymbol{x}$ iff

  $\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{l} = \boldsymbol{l}^{\boldsymbol{T}}\boldsymbol{x} = \mathbf{0}$ $(ul_1 + vl_2 + wl_3 = 0)$

Points and Lines in $\mathbb{P}^2$– Duality

- A point of $\mathbb{P}^2$ $\boldsymbol{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$, all not zero
- A line of $\mathbb{P}^2$ $\boldsymbol{l} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$, all not zero
- A point $\boldsymbol{m}$ is located on a line $\boldsymbol{l}$ iff
  $\boldsymbol{l}^{\boldsymbol{T}}\boldsymbol{x} = \boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{l} = \boldsymbol{0}$ $(ul_1 + vl_2 + wl_3 = 0)$
- A line $\boldsymbol{l}$ passes through a point $\boldsymbol{x}$ iff
  $\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{l} = \boldsymbol{l}^{\boldsymbol{T}}\boldsymbol{x} = \boldsymbol{0}$ $(ul_1 + vl_2 + wl_3 = 0)$
- This **symmetry** in the equation shows that there is no formal difference between points and lines in the projective plane.

This is known as the principle of *duality*.

Points and Lines in $\mathbb{P}^2$ – Duality

- In $\mathbb{P}^3$, this duality is between points and planes.

Test If a Point Lies on a Line

- A point

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- lies on a line

$$\mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$$

- if $\mathbf{x} \cdot \mathbf{l} = 0$

Intersecting Lines

- Given two lines l, m expressed in H.C., we look for the intersection $x = l \cap m$

How to find the intersection of two lines?

Intersecting Lines

- Given two lines ℓ, m expressed in H.C., we look for the intersection $\chi = \ell \cap m$
- Find the point $\boldsymbol{x} = [x, y]^T$ through the following system linear equations

$$\begin{bmatrix} \mathbf{l} \cdot \mathbf{x} \\ \mathbf{m} \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -l_3 \\ -m_3 \end{bmatrix}.$$

Reminder: Cramer's rule

- A system of linear equations can be solved via Cramer's rule

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \qquad\qquad x_i = \frac{\det(A_i)}{\det(A)}$$

- with A_i being the matrix in which the i^{th} column is replaced by $\boldsymbol{b}$
- Easily applicable for 2 by 2 systems

Intersecting Lines

- Solution of

$$\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -l_3 \\ -m_3 \end{bmatrix}$$

- through Cramer's rule

$$x = \frac{D_1}{D_3} \qquad y = \frac{D_2}{D_3}$$

$$D_1 = l_2 m_3 - l_3 m_2$$
$$D_2 = l_3 m_1 - l_1 m_3$$
$$D_3 = l_1 m_2 - l_2 m_1$$

- which can be expressed in vector form as

$$\mathbf{l} \times \mathbf{m} = \boldsymbol{D} = \frac{1}{D_3} \boldsymbol{D} = \mathbf{x}$$

Intersecting Lines

- The intersection of two lines in H.C. is

$$\boxed{\mathcal{x} = \ell \cap m : \quad \mathbf{x} = \mathbf{l} \times \mathbf{m}}$$

- **Simple way for computing the intersection of two lines using H.C.**

Line Between Two Points

- H.C. also offer a simple way for computing a line through two points
- Given two points $\chi = [x_i], y = [y_i]$, find the line $\ell = [l_i]$ connecting both points

How to find a line that connects two given points?

Line Between Two Points

- H.C. also offer a simple way for computing a line through two points
- Given two points $\chi = [x_i], \mathcal{y} = [y_i]$, find the line $\ell = [l_i]$ connecting both points
- We write that as $\ell = \chi \wedge \mathcal{y}$ ("wedge")
- Solution via a system of linear eqns.

$$\begin{bmatrix} \mathbf{x} \cdot \mathbf{l} \\ \mathbf{y} \cdot \mathbf{l} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} -x_3 \, l_3 \\ -y_3 \, l_3 \end{bmatrix}$$

Line Between Two Points

- Cramer's rule again solves

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} -x_3\, l_3 \\ -y_3\, l_3 \end{bmatrix}$$

- by

$$l_1 = \frac{D_1}{D_3} \qquad l_2 = \frac{D_2}{D_3}$$

- with

$$\begin{aligned} D_1 &= l_3(x_2 y_3 - y_2 x_3) \\ D_2 &= l_3(x_3 y_1 - y_3 x_1) \\ D_3 &= x_1 y_2 - x_2 y_1 \end{aligned}$$

Line Between Two Points

- Cramer's leads to

$$l_1 = \frac{D_1}{D_3} \qquad l_2 = \frac{D_2}{D_3}$$

$$\begin{aligned} D_1 &= l_3(x_2 y_3 - y_2 x_3) \\ D_2 &= l_3(x_3 y_1 - y_3 x_1) \\ D_3 &= x_1 y_2 - x_2 y_1 \end{aligned}$$

- and we use

$$l_3 = l_3 \frac{D_3}{D_3}$$

- which results in

$$\mathbf{l} = \left[\frac{D_1}{D_3}, \frac{D_2}{D_3}, l_3 \frac{D_3}{D_3} \right]^\top \longrightarrow \mathbf{l} = \frac{l_3}{D_3} \begin{bmatrix} x_2 y_3 - y_2 x_3 \\ x_3 y_2 - y_2 x_1 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$

Line Between Two Points

- We again exploit the cross product

$$\boldsymbol{n} = \mathbf{x} \times \mathbf{y}$$

- and obtain

$$\mathbf{l} = \frac{l_3}{D_3} \begin{bmatrix} x_2 y_3 - y_2 x_3 \\ x_3 y_2 - y_2 x_1 \\ x_1 y_2 - x_2 y_1 \end{bmatrix} = \frac{l_3}{D_3} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \mathbf{x} \times \mathbf{y}$$

- exploiting the homogeneous property
- Thus,

$$\boxed{\mathcal{l} = \mathcal{x} \wedge \mathcal{y} : \quad \mathbf{l} = \mathbf{x} \times \mathbf{y}}$$

Summary

- A point lies on a line if

$$\mathbf{x} \cdot \mathbf{l} = 0$$

- Intersection of two lines

$$\mathcal{x} = \ell \cap m : \quad \mathbf{x} = \mathbf{l} \times \mathbf{m}$$

- A line through two given points

$$\ell = \mathcal{x} \wedge y : \quad \mathbf{l} = \mathbf{x} \times \mathbf{y}$$

Points and Lines at Infinity

Points at Infinity

- It is possible to **explicitly** model infinitively distant points **with finite coordinates**

$$\mathcal{X}_\infty : \quad \mathbf{x}_\infty = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

- We can **maintain the direction** to that infinitively distant point
- Great tool when working with cameras as they are bearing-only sensors

Intersection at Infinity

- All lines ℓ with $\ell \cdot x_\infty = 0$ pass through x_∞
- This means $[u, v] \cdot [\cos\phi, \sin\phi] = 0$
- This hold for any line $\mathbf{m} = [\cos\phi, \sin\phi, *]^T$ i.e. for any line that is parallel to ℓ

All parallel lines meet at one point at infinity!

Intersection at Infinity

- All lines ℓ with $\ell \cdot \chi_\infty = 0$ pass through χ_∞
- This means $[u, v] \cdot [\cos\phi, \sin\phi] = 0$
- This hold for any line $\mathbf{m} = [\cos\phi, \sin\phi, *]^T$ i.e. for any line that is parallel to ℓ
- This can also be seen by

$$\mathbf{l} \times \mathbf{m} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} a \\ b \\ d \end{bmatrix} = \begin{bmatrix} bd - bc \\ ac - ad \\ ab - ab \end{bmatrix} = \begin{bmatrix} bd - bc \\ ac - ad \\ 0 \end{bmatrix}$$

All parallel lines meet at one point at infinity!

Parallel Lines Meet at Infinity

Image Courtesy: J. Jannene

Infinitely Distant Objects

- Infinitely distant point

$$\mathcal{X}_\infty : \quad \mathbf{x}_\infty = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

- The Infinitely distant line is the **ideal line**

$$\mathcal{l}_\infty : \quad \mathbf{l}_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- $\mathcal{l}_\infty$ can be interpreted as the horizon

Infinitely Distant Objects

- All points at infinity lie on the line at infinity called the **ideal line** given by

$$\mathbf{x}_{\infty} \cdot \mathbf{l}_{\infty} = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

- The ideal line can be seen as the horizon

2D Projective Plane

- A homogeneous point $x = [u\ v\ w]^T$ maps to the point of intersection of the corresponding line $t(u, v, w)$ and the unit sphere centered at the origin

$$u^2 + v^2 + w^2 = 0$$

- In other words

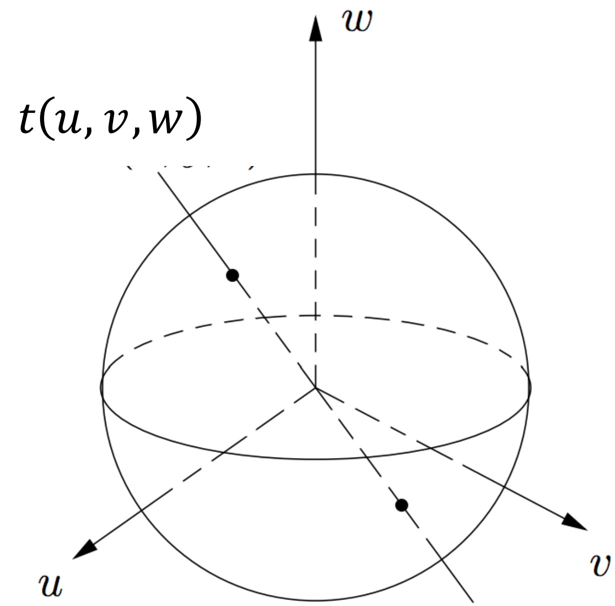
$$(u, v, w) \mapsto \pm \frac{(u, v, w)}{\sqrt{u^2 + v^2 + w^2}}$$

Also known as Spherical Normalization

2D Projective Plane

- Antipodal points represent the same homogeneous point

- Thus, it suffices to consider the upper half sphere together with (half) the equator



2D Projective Plane

- Every 2D point with H.C. coordinates $x = [u\ v\ 0]^T$ is a point at infinity

- Where do these points lie in our unit sphere model?

- A line in 2D corresponds to a great circle (intersection of the sphere with the plane containing the origin and that 2D line elevated to the z = 1 plane)

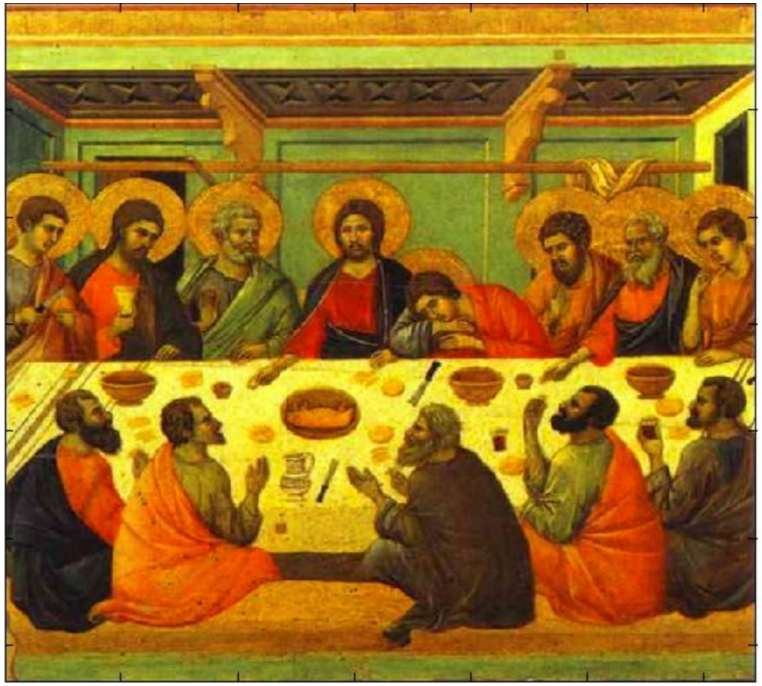
2D Projective Plane

- If the intersection of two lines corresponding to the intersection points of the two great circles on the sphere lies on the equator, its last coordinate is zero, indicating the point is at infinity; thus, the two lines are parallel.

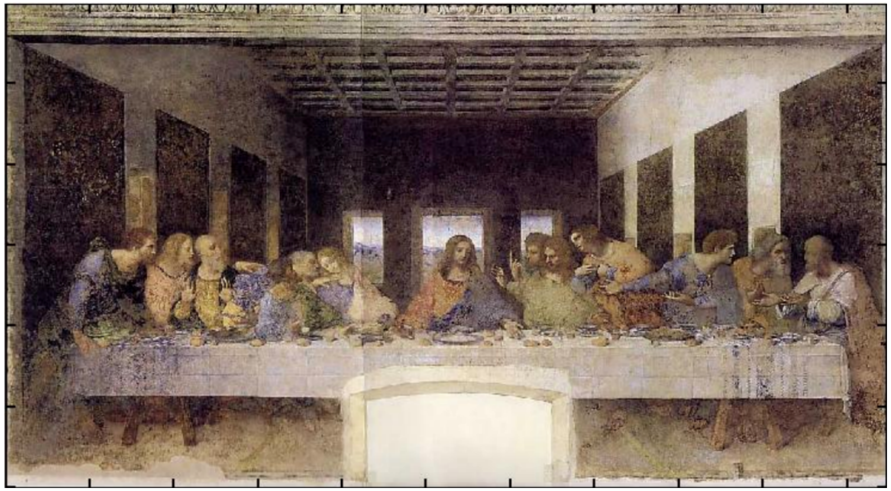
- What about the Ideal Line?

The Last Supper

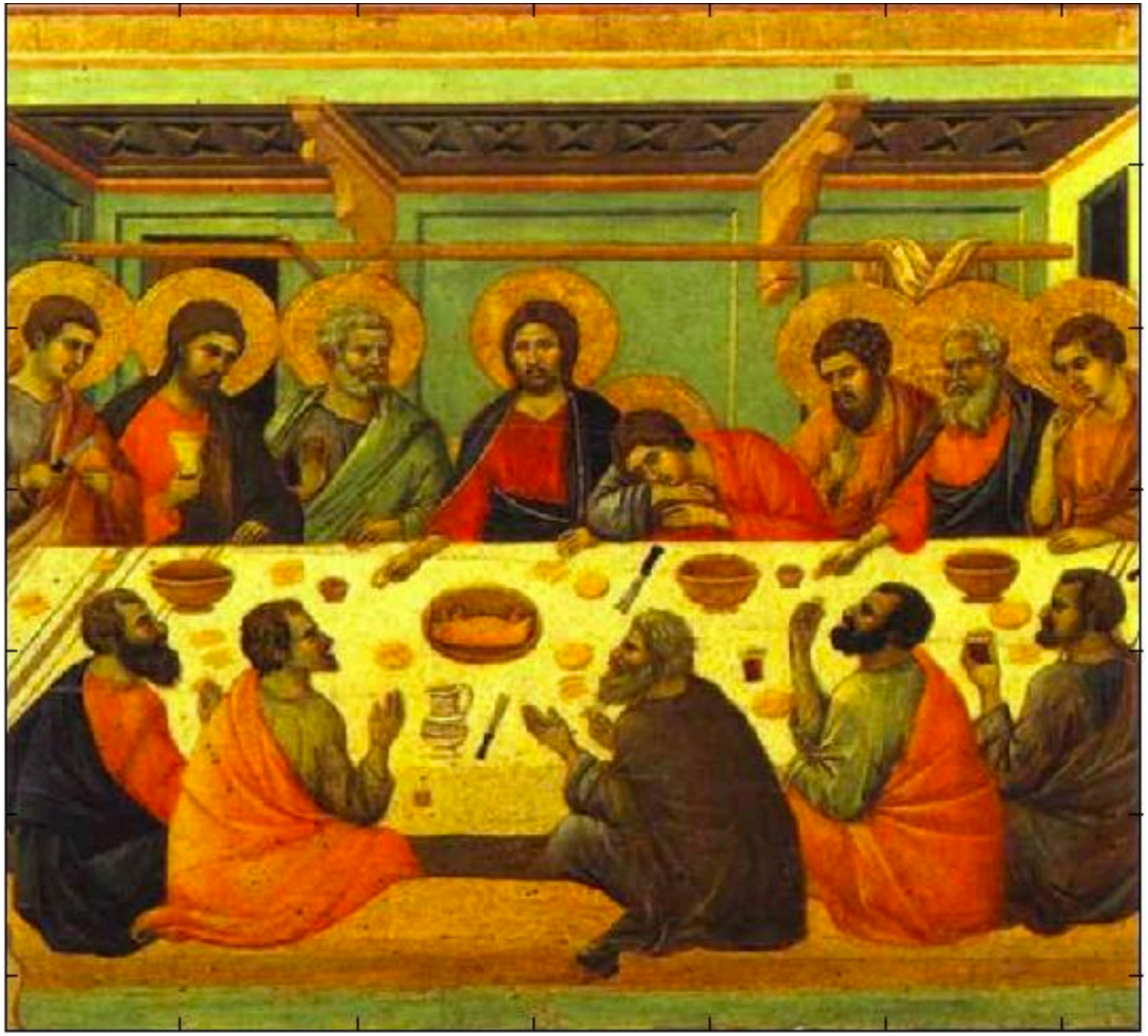
by Duccio (around 1310)



Da Vinci (1499)



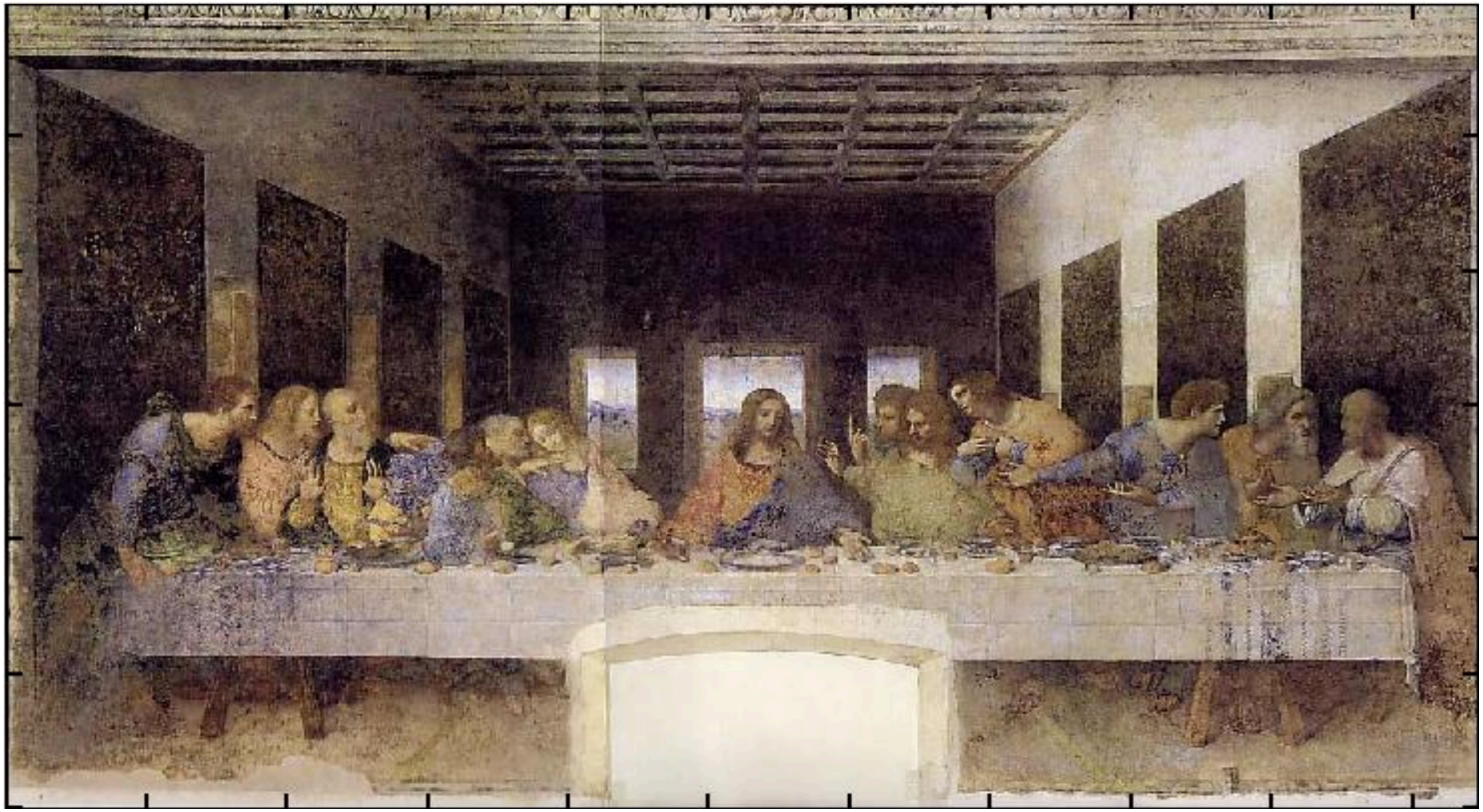
The Last Supper by Duccio (around 1310)



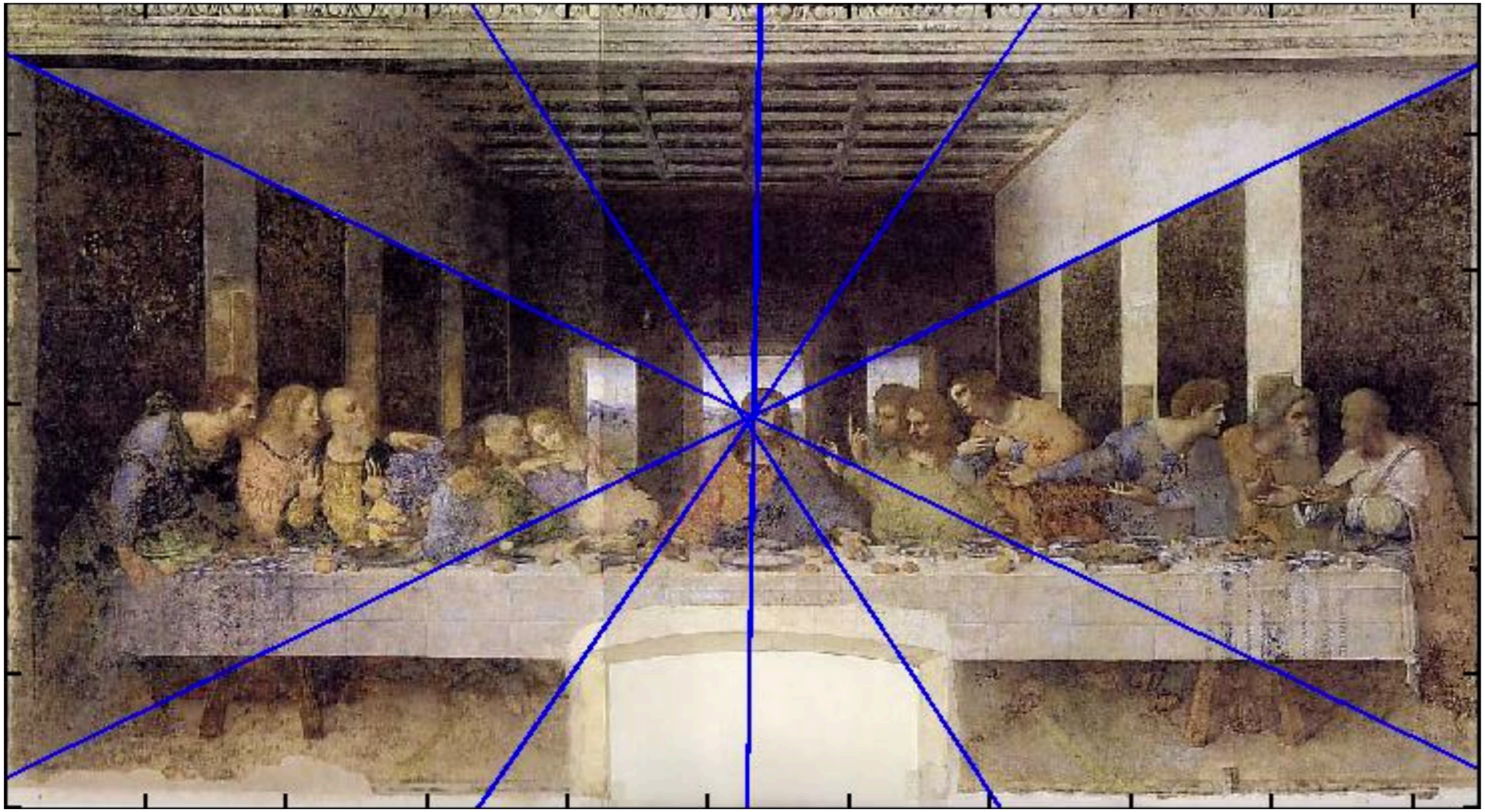
The Last Supper by Duccio (around 1310)

Parallel lines do not
meet at single vanishing point

The Last Supper by Da Vinci (1499)



The Last Supper by Da Vinci (1499)



Parallel lines meet at single vanishing point!

Analogous for 3D Objects

- 3D point

$$\mathbf{X} = \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix} = \begin{bmatrix} U/T \\ V/T \\ W/T \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} U/T \\ V/T \\ W/T \end{bmatrix}$$

- Plane

$$\mathbf{A} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

Point on a Plane

- Via the scalar product, we can again test if a point lies on a plane

$$\mathbf{A} \cdot \mathbf{X} = \mathbf{A}^{\mathsf{T}}\mathbf{X} = \mathbf{X}^{\mathsf{T}}\mathbf{A} = 0$$

- which is based on

$$AX + BY + CZ + D = 0 \quad \text{or} \quad N \cdot X - S = 0$$

3D Objects at Infinity

- 3D point

$$\mathbf{P}_\infty = \begin{bmatrix} U \\ V \\ W \\ 0 \end{bmatrix}$$

- Plane

$$\mathbf{A}_\infty = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Points at Infinity

- Note that points at infinity in 3D can 'project' to non-infinity points in 2D (remember to think about this when we learn about 3D to 2D projection)

$$\mathbf{P}_\infty = \begin{bmatrix} U \\ V \\ W \\ 0 \end{bmatrix}$$

Transformations

Transformations

- A projective transformation is an invertible linear mapping

$$\mathbf{X}' = \mathrm{H}\mathbf{X}$$

Fundamental Theorem of Projective Geometry

- Every one-to-one, straight-line preserving mapping of a projective space $\mathbb{P}^n$ onto itself is a homography (projectivity) for $2 \leq n < \infty$

- Implies that all one-to-one, straight-line preserving transformations are linear if we use projective(homogeneous) coordinates

Important 3D Transformations

- General projective mapping

$$\mathbf{X}' = \mathsf{H}\mathbf{X}$$

- Translation: 3 parameters
  (3 translations)

$$\mathsf{H} = \lambda \begin{bmatrix} \boldsymbol{I} & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

homogeneous property

Important 3D Transformations

- Rotation: 3 parameters
  (3 rotation)

$$\mathrm{H} = \lambda \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

rotation matrix

Recap – Rotation Matrices

- 2D:

$$R^{2D}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

- 3D:

$$R_x^{3D}(\omega) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega) & -\sin(\omega) \\ 0 & \sin(\omega) & \cos(\omega) \end{bmatrix} \quad R_y^{3D}(\phi) = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

$$R_z^{3D}(\kappa) = \begin{bmatrix} \cos(\kappa) & -\sin(\kappa) & 0 \\ \sin(\kappa) & \cos(\kappa) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^{3D}(\omega, \phi, \kappa) = R_z^{3D}(\kappa) R_y^{3D}(\phi) R_x^{3D}(\omega)$$

Important 3D Transformations

- Rotation: 3 parameters
  (3 rotation)

$$\mathsf{H} = \lambda \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

- Rigid body transformation: 6 params
  (3 translation + 3 rotation)

$$\mathsf{H} = \lambda \begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

Important 3D Transformations

- Similarity transformation: 7 params
  (3 trans + 3 rot + 1 scale)

$$\mathsf{H} = \lambda \begin{bmatrix} mR & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

(angle-preserving)

- Affine transformation: 12 parameters
  (3 trans + 3 rot + 3 scale + 3 sheer)

$$\mathsf{H} = \lambda \begin{bmatrix} A & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

(not angle-preserving but parallel lines remain parallel)

Important 3D Transformations

- Projective transformation: 15 params.

$$\mathrm{H} = \lambda \begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{a}^{\mathsf{T}} & 1 \end{bmatrix}$$

affine transformation + 3 parameters

- These 3 parameters are the projective part and they are the reason that **parallel lines may not stay parallel**

Important 3D Transformations

- Projective transformation: 15 params.

$$\mathrm{H} = \lambda \begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{a}^{\mathsf{T}} & 1 \end{bmatrix}$$

affine transformation + 3 parameters

- These 3 parameters are the projective part and they are the reason that **parallel lines may not stay parallel**

Transformations for 2D

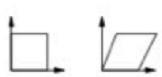
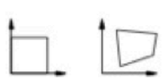
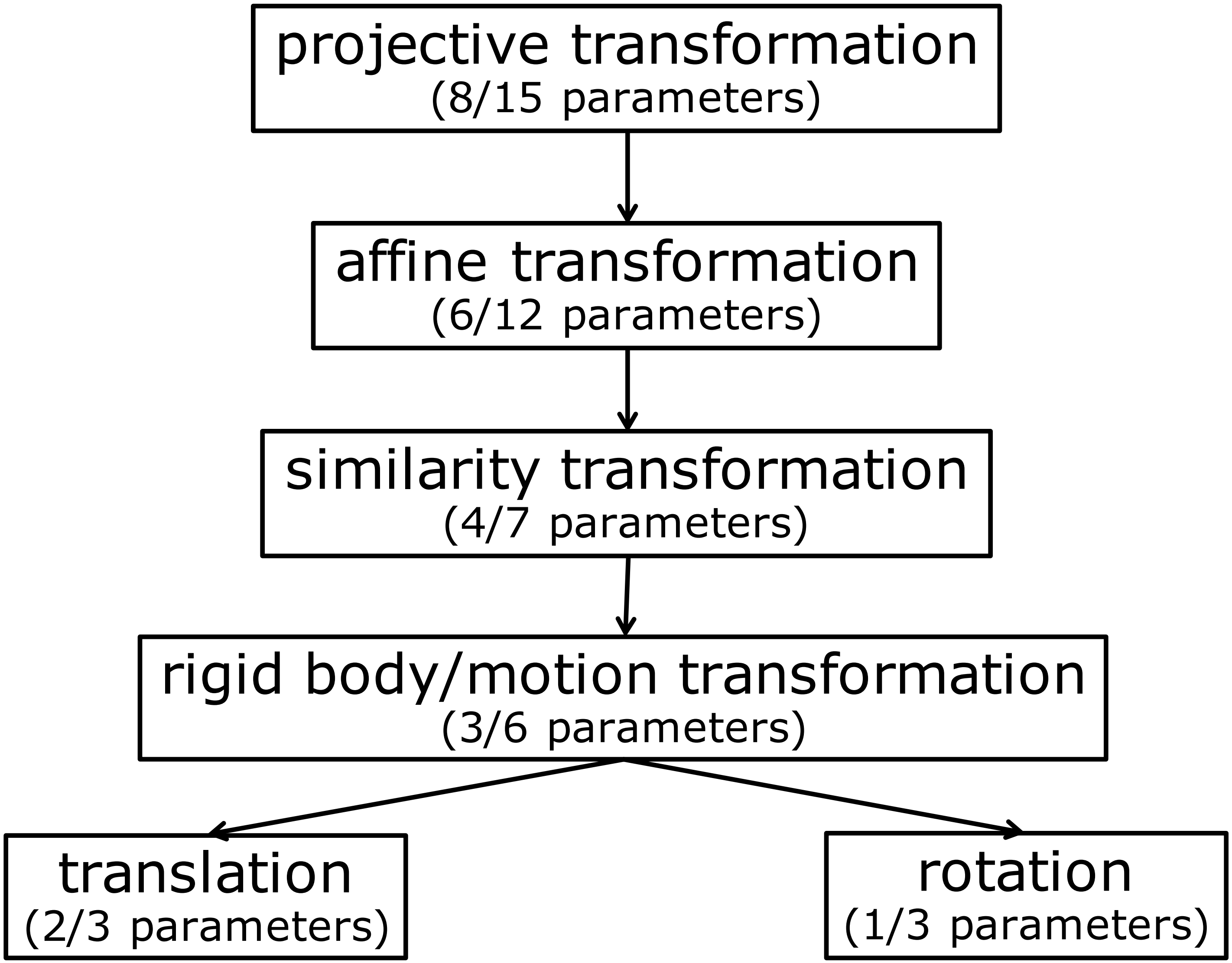
2D Transformation	Figure	d. o. f.	H	H
Translation		2	$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} I & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Mirroring at y-axis		1	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} Z & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Rotation		1	$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Motion		3	$\begin{bmatrix} \cos\varphi & -\sin\varphi & t_x \\ \sin\varphi & \cos\varphi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Similarity		4	$\begin{bmatrix} a & -b & t_x \\ b & a & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \lambda R & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Scale difference		1	$\begin{bmatrix} 1+m/2 & 0 & 0 \\ 0 & 1-m/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} D & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Shear		1	$\begin{bmatrix} 1 & s/2 & 0 \\ s/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} S & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Asym. shear		1	$\begin{bmatrix} 1 & s' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} S' & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Affinity		6	$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Projectivity		8	$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$	$\begin{bmatrix} A & \mathbf{t} \\ \mathbf{p}^\top & 1/\lambda \end{bmatrix}$

Image courtesy: Schindler

Transformations Hierarchy



Inverting and Chaining

- Inverting a transformation

$$
\begin{aligned}
\mathbf{X}' &= \mathsf{H}\mathbf{X} \\
\mathbf{X} &= \mathsf{H}^{-1}\mathbf{X}'
\end{aligned}
$$

- Chaining transformations via matrix products (not commutative)

$$
\begin{aligned}
\mathbf{X}' &= \mathsf{H}_1\mathsf{H}_2\mathbf{X} \\
&\neq H_2\mathsf{H}_1\mathbf{X}
\end{aligned}
$$

What about 3D to 2D Transformations?

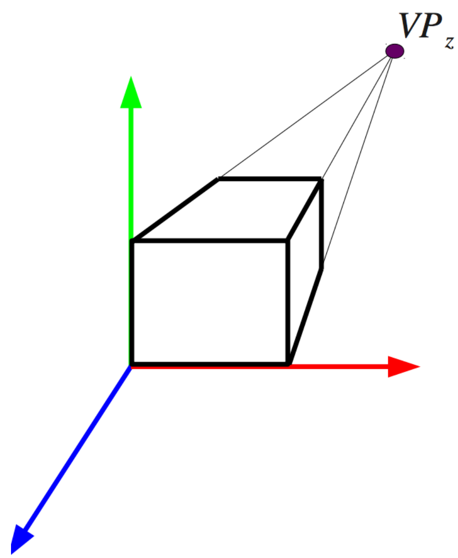
- Projections: coming up next lecture

What about 3D to 2D Transformations?

- Single point perspective

$$\begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{a}^{\mathsf{T}} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & 0 & 0 & 1 \end{bmatrix}.$$



- Vanishing point in the z direction.
- Set of lines not parallel to the projection plane converge at a vanishing point.

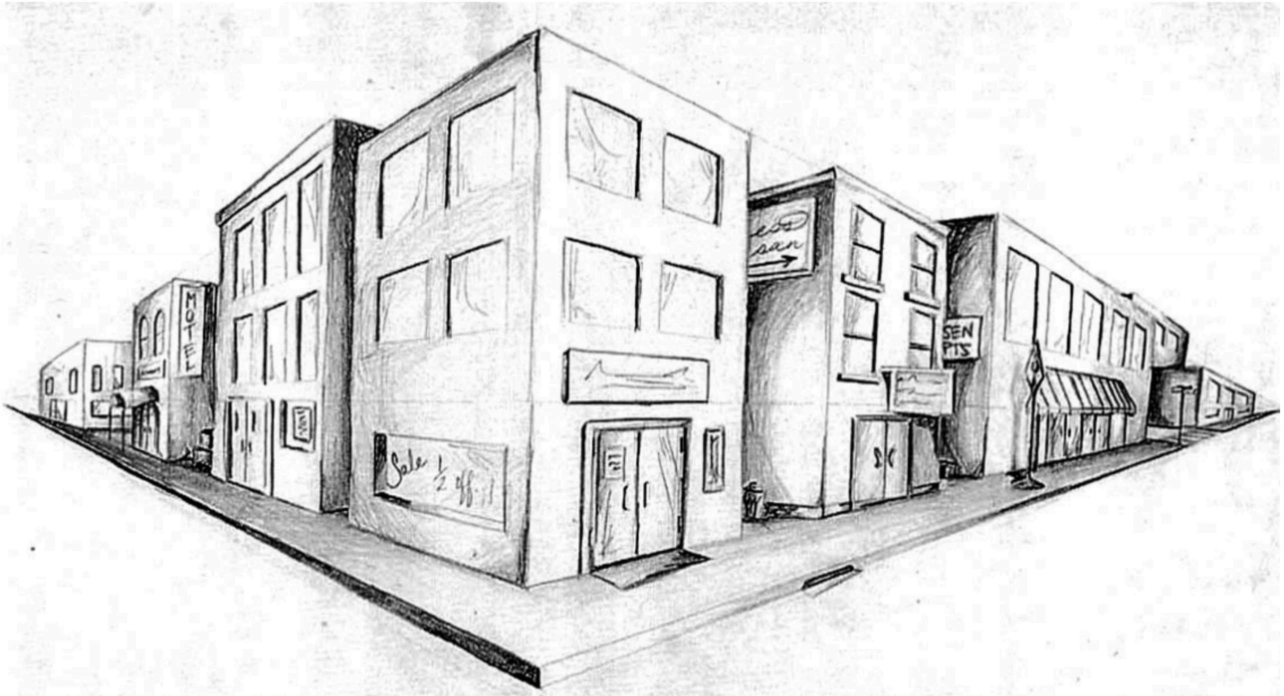
https://www.cse.iitb.ac.in/~paragc/teaching/2017/cs475/lectures/06_viewing_1.pdf

What about 3D to 2D Transformations?

- Two point perspective

$$\begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{a}^\top & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & q & 0 & 1 \end{bmatrix}.$$



From http://gatewayhsart.wordpress.com

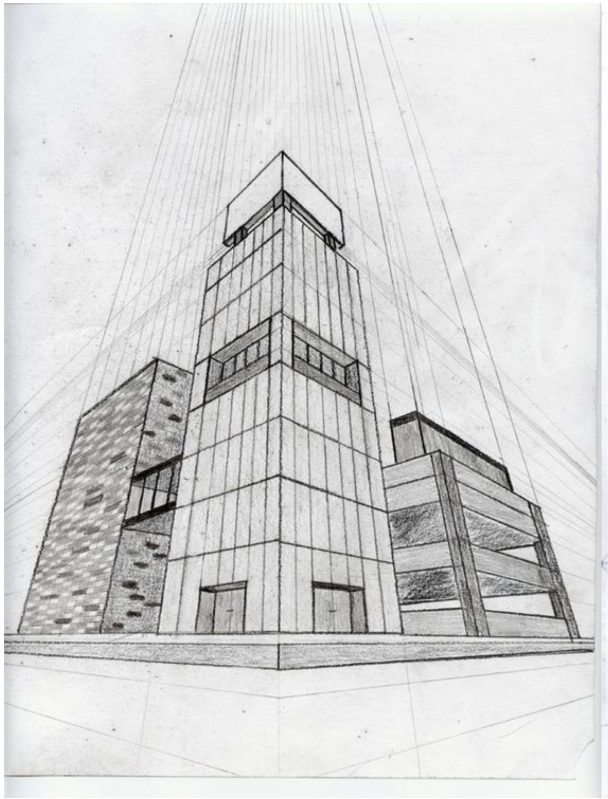
https://www.cse.iitb.ac.in/~paragc/teaching/2017/cs475/lectures/06_viewing_1.pdf

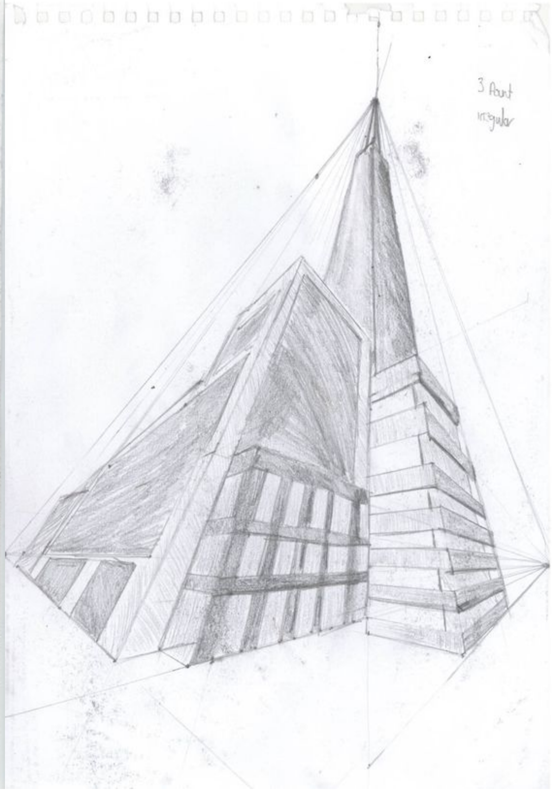
What about 3D to 2D Transformations?

- Three point perspective

$$\begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{a}^{\mathsf{T}} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & q & r & 1 \end{bmatrix}$$





https://www.cse.iitb.ac.in/~paragc/teaching/2017/cs475/lectures/06_viewing_1.pdf

Conclusion

- Homogeneous coordinates are an alternative representation for geometric objects
- They can simplify mathematical expressions
- They can model points at infinity
- Easy chaining and inversion of transformations
- Modeled through an extra dimension
- Equivalence up to scale

Slide Information

- The slides have been created by Cyrill Stachniss (cyrill.stachniss@igg.uni-bonn.de) as part of the photogrammetry and robotics courses.
- A lot of material from Ajit Rajwade's CS763 course
- Thanks to Parag for slides 86-89
- **I tried to acknowledge all people from whom I used images or videos. In case I made a mistake or missed someone, please let me know.**
- The photogrammetry material heavily relies on the very well written lecture notes by Wolfgang Förstner and the Photogrammetric Computer Vision book by Förstner & Wrobel.
- Parts of the robotics material stems from the great Probabilistic Robotics book by Thrun, Burgard and Fox.
- If you are a university lecturer, feel free to use the course material. If you adapt the course material, please make sure that you keep the acknowledgements to others and please acknowledge me as well. To satisfy my own curiosity, please send me email notice if you use my slides.

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