Beyond the limits of the Curry-Howard isomorphism

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The well-known Curry-Howard isomorphism relates functions with proofs and can be considered as one of the conceptional bases of Martin-Löf's type theory (MLTT).

For our considerations, the crucial correspondence is the one between (intuitionistic) proofs of an implication $A \rightarrow B$ and functions of the type $A \rightarrow B$. To make sense out of this correspondence, the functions need to be, of course, *total*, i.e., for every element of A the function needs to associate an element of B.

Although totality of functions is a desirable property, it does not match with computational reality. It is not only the case that non-terminating computations appear natural. Even more importantly, there is no Turing complete computable model of computation in which all functions are total, therefore there is no Turing complete programming language based on total functions.

Partial functions are easily integrated in *type-free* contexts, and we may address shortly some interesting historical considerations regarding *functional self-application*, cf. [Kah07, Appendix]. The main aim of this talk is, however, to illustrate the role of partial functions in the formalization of the *extended predicative Mahlo universe* (EPM), [KS10].

A weakly Mahlo cardinal is a regular cardinal κ such that for every function $f : \kappa \rightarrow \kappa$ there exists a regular cardinal $\pi < \kappa$ s.t. $f : \pi \rightarrow \pi$ [Rat90]. This definition has been translated into MLTT [Set00] and Feferman's theory of Explicit Mathematics (EM) [JS01]. In EM, a Mahlo universe is a universe M such that for every $a \in \mathsf{M}$ and $f : \mathsf{M} \rightarrow \mathsf{M}$ there exists a subuniverse $\mathsf{m}(a, f)$ of M, which is an element of M, contains a, and is closed under f.

This axiomatization, which we call axiomatic Mahlo (AxM), is clearly highly impredicative, since, if viewed as an introduction rule, the definition of M requires to add $\mathsf{m}(a, f)$ for all total $f : \mathsf{M} \rightarrow \mathsf{M}$, where the set of total f is only known after M is complete.

In the EPM, formulated in EM, the totality of $f : \mathsf{M} \rightarrow \mathsf{M}$ is no longer required. Instead one requires f only to be total on the subset $\mathsf{m}(a, f)$ of M, and not on elements added after the addition of $\mathsf{m}(a, f)$ to M – the reason for adding $\mathsf{m}(a, f)$ is not destroyed by the addition of $\mathsf{m}(a, f)$ or any element added after $\mathsf{m}(a, f)$. More precisely one tries to build subsets $\mathsf{m}(a, f)$ closed under f and a. If that succeeds $\mathsf{m}(a, f)$ is added to M. M is constructed from below, because the reason for adding $\mathsf{m}(a, f)$ depends only on $\mathsf{m}(a, f)$ and not on all of M.

It turns out that the permission of partial functions is crucial for the axiomatization of the EPM, and because of this, we have not been able to port it to MLTT yet. Due to the use of partial functions one can define in EPM an elimination rule for M, which makes M a *least Mahlo universe*. This implies that M contains only elements introduced by its introduction rules. In AxM such an elimination rule results in a contradiction (see [Pal98] for a proof in the context of MLTT).

The reason for calling the resulting theory *extended* predicative is that it goes beyond the proof theoretic definition of predicativity (i.e., theories with the proof-theoretic strength of Γ_0). It goes as well beyond its use in MLTT, where it seems to be limited to types defined by function types and inductive and inductive-recursive definitions [Dyb00, DS03]. In this sense the EPM goes beyond what is undoubtedly considered as predicative in MLTT.[1]

[1] One should note that many type theoretists consider the Mahlo universe in MLTT, which goes beyond inductive and inductive-recursive definitions, as predicative as well.

This result suggests, on the proof-theoretic side, that the formulation of a least Mahlo universe, and as well of a Mahlo universe which is without any doubts accepted as predicative in nature, may exceed the limit of MLTT (or at least require a paradigm shift in MLTT). On the more conceptional side, it challenges the restriction to total functions in the Curry-Howard correspondence, although the question, what the meaning of a "partial implication" could be, remains a desideratum (see also [Kah1x]).

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