Reflecting Universes in Type Theory

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1 Structure, common to Π_n-refl, Π_α-refl, Π^1_1-refl

1.1 Dealing with booleans

- $\mathbb{B}$: Set.
  - $\underline{\mathrm{tt}} : \mathbb{B}$.
  - $\underline{\mathrm{ff}} : \mathbb{B}$.
- $\mathrm{atom} : \mathbb{B} \to \mathrm{Set}$.
  - $\mathrm{atom}(\mathrm{tt}) = \mathrm{N}_0$. ($\mathrm{N}_k$ defined below)
  - $\mathrm{atom}(\mathrm{ff}) = \mathrm{N}_1$.

1.2 Dealing with N

- $\mathrm{N}^{\mathrm{fin}} : \mathrm{N} \to \mathrm{Set}$.
- (We write N_k for $\mathrm{N}^{\mathrm{fin}}(k)$).
- $\underline{\max} : (n : \mathrm{N}) \to \mathrm{N}_{\mathrm{S}(n)}$.
- $\underline{\mathrm{emb}} : (n : \mathrm{N}, k : \mathrm{N}_n) \to \mathrm{N}_{\mathrm{S}(n)}$.
- $0_1 := \max_1 : \mathrm{N}_1$.
- $* := \mathrm{inl}(0_1)$ (: $\mathrm{N}_1 + A$ for any A).
- $\mathrm{embfin} : (n : \mathrm{N}, \mathrm{N}_n) \to \mathrm{N}$.
  - $\mathrm{embfin}(\mathrm{S}(n), \max_n) = n$.

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- $\mathrm{embfin}(\mathrm{S}(n), \mathrm{emb}_n(k)) = \mathrm{embfin}(n,k)$.

- $\mathrm{tr} : (n : \mathrm{N}, k : \mathrm{N}_n, l : \mathrm{N}_{\mathrm{S}(\mathrm{embfin}(n,k))}) \to \mathrm{N}_n$.
  - $\mathrm{tr}(\mathrm{S}(n), \mathrm{max}_n, l) = l$.
  - $\mathrm{tr}(\mathrm{S}(n), \mathrm{emb}_n(k), l) = \mathrm{emb}_n(\mathrm{tr}(n,k,l))$

- $?_< : (n : \mathrm{N}, l : \mathrm{N}_n, k : \mathrm{N}_n) \to \mathrm{N}_{\mathrm{embfin}_n(l)} + \mathrm{N}_1$.
  ($?_<(n,l,k) = \mathrm{inl}(k')$, if k represents a number $k' < l$, $\mathrm{inr}(0_1)$ otherwise.
  - $?_<(\mathrm{S}(n), l, \mathrm{max}_n) = \mathrm{inr}(0_1)$.
  - $?_<(\mathrm{S}(n), \mathrm{max}_n, \mathrm{emb}_n(k')) = \mathrm{inl}(k')$.
  - $?_<(\mathrm{S}(n), \mathrm{emb}_n(l'), \mathrm{emb}_n(k')) = ?_<(n, l', k')$.

- $?_= : (n : \mathrm{N}, l : \mathrm{N}_n, k : \mathrm{N}_n) \to \mathbb{B}$.
  ($?_=(n,l,k) = \mathrm{tt}$, if $k = l$, ff otherwise).
  - $?_=(\mathrm{S}(n), \mathrm{max}_n, \mathrm{max}_n) = \mathrm{tt}$.
  - $?_=(\mathrm{S}(n), \mathrm{max}_n, \mathrm{emb}_n(k')) = \mathrm{ff}$.
  - $?_=(\mathrm{S}(n), \mathrm{emb}_n(l'), \mathrm{max}_n) = \mathrm{ff}$.
  - $?_=(\mathrm{S}(n), \mathrm{emb}_n(l'), \mathrm{emb}_n(k')) = ?_=(n, l', k')$

1.3 Notations

- We write $\langle a, b, c, d\rangle$ instead of $\langle a, \langle b, \langle c, d\rangle\rangle\rangle$.
- $f \,\tilde{\circ}\, g := (a,b)f(g(a), b)$.

1.4 The main universe structure

$\mathbb{U} : \mathrm{Set}$ $\qquad$ $\mathbb{T} : \mathbb{U} \to \mathrm{Set}$

$\mathrm{Univ} : \mathrm{Set}$ $\qquad$ $\mathrm{U} : \mathrm{Univ} \to \mathrm{Set}$

$\widehat{\mathrm{T}} : (u : \mathrm{Univ}, \mathrm{U}_u) \to \mathbb{U}$

$\mathrm{T}_u(a) := \mathbb{U}(\widehat{\mathrm{T}}_u(a))(: \mathrm{Set})$

$\mathrm{fpar} : (u : \mathrm{Univ}, a : \mathrm{U}_u, \mathrm{T}_u(b)) \to \mathbb{U}$

$\mathrm{gpar} : (u : \mathrm{Univ}, a : \mathrm{U}_u, \mathrm{T}_u(b), \mathbb{T}(\mathrm{fpar}_u(a,b))) \to \mathbb{U}$

$\underline{\widehat{\mathrm{f}}} : (u : \mathrm{Univ}, a : \mathrm{U}_u, b : \mathrm{T}_u(a) \to \mathrm{U}_u) \to \mathrm{U}_u$

$\widehat{\mathrm{T}}_u(\widehat{\mathrm{f}}_u(a,b)) = \mathrm{fpar}_u(a,b)$

$\underline{\widehat{\mathrm{g}}} : (u : \mathrm{Univ}, a : \mathrm{U}_u, b : \mathrm{T}_u(a) \to \mathrm{U}_u, c : \mathbb{T}(f(a,b))) \to \mathrm{U}_u$

$\widehat{\mathrm{T}}_u(\widehat{\mathrm{g}}_u(a,b)) = \mathrm{gpar}_u(a,b)$

$\underline{\mathrm{univ}} : \mathrm{Univ} \to \mathbb{U}$ $\qquad$ $\mathbb{T}(\mathrm{univ}(u)) = \mathrm{U}_u$

$\mathrm{Univ}^+ : \mathrm{Set}$ $\qquad$ $\mathrm{Univ}^+ = \mathrm{N}_1 + \mathrm{Univ}$

$\mathrm{U}^+ : \mathrm{Univ}^+ \to \mathrm{Set}$ $\qquad$ $\mathrm{U}^+_* = \mathbb{U}$

$\mathrm{U}^+_{\mathrm{inr}(u)} = \mathrm{U}_u$

$\mathrm{T}^+ : (u : \mathrm{Univ}^+, \mathrm{U}^+_u) \to \mathrm{Set}$ $\qquad$ $\mathrm{T}^+_*(a) = \mathbb{T}(a)$

$\mathrm{T}^+_{\mathrm{inl}(u)}(a) = \mathrm{T}_u(a)$

1.5 Degrees

$\mathrm{P} : \mathrm{Univ}^+ \to \mathrm{Set}$

$$\widehat{\mathrm{Q}} : (u : \mathrm{Univ}^+, \mathrm{P}(u), a : \mathrm{U}^+_u, b : \mathrm{T}^+_u(a) \to \mathrm{U}^+_u) \to \mathrm{U}^+_u$$

$$\mathrm{Q}(u, a, b, p) = \mathrm{T}^+_u(\widehat{\mathrm{Q}}(u, a, b, p))$$

$$\mathrm{R} : (u : \mathrm{Univ}^+, p : \mathrm{P}(u), a : \mathrm{U}^+_u, b : \mathrm{T}^+_u(a) \to \mathrm{U}^+_u, q : \mathrm{Q}(u, p, a, b)) \to \mathrm{P}(u)$$

2 Π_{n+2}-reflection

Assume $n \in \mathrm{N}$.

2.1 Degrees of elements of Univ

$$\mathrm{deg} : (v : \mathrm{Univ}, \mathrm{N}_n) \to \\ (w : \mathrm{Univ}^+) \times \mathrm{P}(w) \times (f : (\mathrm{U}_v \to \mathrm{U}^+_w)) \times ((a : \mathrm{U}_v.\mathrm{T}^+_w(f(v)) \to \mathrm{T}_v(a)))$$

$\mathrm{w}(u,l) := \mathrm{deg}(u,l)0(: \mathrm{Univ}^+)$

$\mathrm{Pdeg}(u,l)_{\mathrm{p}} := \mathrm{deg}(u,l)10(: \mathrm{P}(\mathrm{w}(u,l)))$

$\mathrm{lift}(u,l,a) := \mathrm{deg}(u,l)110(a)(: \mathrm{U}_v \to \mathrm{U}^+_{\mathrm{w}(u,l)})$

$\mathrm{emblift}(u,l,a,b) := \mathrm{deg}(u,l)111(a,b)$
$(: (a : \mathrm{U}_v, \mathrm{T}^+_{\mathrm{w}(u,l)}(\mathrm{lift}(u,l,a))) \to \mathrm{T}_v(a))$

2.2 First subuniverse

Assume

- $f : (a : \mathbb{U}, b : \mathbb{T}(a) \to \mathbb{U}) \to \mathbb{U}$.
- $g : (a : \mathbb{U}, b : \mathbb{T}(a) \to \mathbb{U}, \mathbb{T}(f(a,b))) \to \mathbb{U}$.
- $p : \mathrm{N}_n \to \mathrm{P}(*)$

Then

- $\mathrm{u}_0(f,g,p) : \mathrm{Univ}$

Let $\mathrm{v} := \mathrm{u}_0(f,g,p)$. Then

- $\mathrm{deg}(\mathrm{v},l) = \langle *, p(l), \widehat{\mathrm{T}}_u, (a,b)b \rangle$
- $\mathrm{fpar}_{\mathrm{v}}(a,b) = f(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b)$
- $\mathrm{gpar}_{\mathrm{v}}(a,b,c) = g(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b, c)$

2.3 Second subuniverse

Assume

- $u : \mathrm{Univ}.$
- $f : (a : \mathrm{U}_u, b : \mathrm{T}_u(a) \to \mathrm{U}_u) \to \mathrm{U}_u.$
- $g : (a : \mathrm{U}_u, b : \mathrm{T}_u(a) \to \mathrm{U}_u, \mathrm{T}_u(f(a,b))) \to \mathrm{U}_u.$
- $l : \mathrm{N}_n,$
- $a : \mathrm{U}_u$
- $b : \mathrm{T}_u(a) \to \mathrm{U}_u$
- $p : \mathrm{Q}(\mathrm{w}(u,l), \mathrm{Pdeg}(u,l), \mathrm{lift}(u,l,a), (x)\mathrm{lift}(u,l,b(\mathrm{emblift}(u,l,a,x)))).$
- $q : \mathrm{N}_{\mathrm{embfin}_n(l)} \to \mathrm{P}(\mathrm{inr}(u)).$

Let $\vec{fg} := u, f, g, l, a, b, p, q$. Then

- $\mathrm{u}_1(\vec{fg}) : \mathrm{Univ},$
- $\underline{\widehat{\mathrm{u}}}_1(\vec{fg}) : \mathrm{U}_u.$
- $\underline{\widehat{\mathrm{t}}}_1(\vec{fg}) : \mathrm{U}_{\mathrm{u}_1(\vec{fg})} \to \mathrm{U}_u.$

Let $\mathrm{v} := \mathrm{u}_1(\vec{fg})$. Then

- $\widehat{\mathrm{T}}_u(\widehat{\mathrm{u}}_1(\vec{fg})) = \mathrm{univ}(\mathrm{v}).$
- $\widehat{\mathrm{T}}_u(\widehat{\mathrm{t}}_1(\vec{fg}, c)) = \widehat{\mathrm{T}}_{\mathrm{v}}(c).$
- $$\begin{array}{l}\mathrm{deg}(\mathrm{v}, l') = \text{case } (?_<(n,l,l')) \text{ of}\\ \quad \mathrm{inl}(l'') \longrightarrow \langle \mathrm{inr}(u), q(l''), \widehat{\mathrm{t}}_1(\vec{fg}), (a,b)b\rangle\\ \quad \mathrm{inr}(0_1) \longrightarrow (\text{case } ?_=(n,l,l') \text{ of}\\ \qquad\qquad \mathrm{tt} \longrightarrow \langle \mathrm{w}(u,l),\\ \qquad\qquad\qquad \mathrm{R}(\mathrm{w}(u,l), \mathrm{Pdeg}(u,l), \mathrm{lift}(u,l,a),\\ \qquad\qquad\qquad\quad (x)\mathrm{lift}(u,l,b(\mathrm{emblift}(u,l,a,x))), p),\\ \qquad\qquad\qquad \mathrm{lift}(u,l) \circ \widehat{\mathrm{t}}_1(\vec{fg}),\\ \qquad\qquad\qquad \mathrm{emblift}(u,l) \,\tilde{\circ}\, \widehat{\mathrm{t}}_1(\vec{fg})\rangle\\ \qquad\qquad \mathrm{ff} \longrightarrow \langle \mathrm{w}(u,l'),\\ \qquad\qquad\qquad \mathrm{Pdeg}(u,l'),\\ \qquad\qquad\qquad \mathrm{lift}(u,l') \circ \widehat{\mathrm{t}}_1(\vec{fg}),\\ \qquad\qquad\qquad \mathrm{emblift}(u,l') \,\tilde{\circ}\, \widehat{\mathrm{t}}_1(\vec{fg})\rangle).\end{array}$$
- $\mathrm{fpar}_{\mathrm{v}}(a,b) = \widehat{\mathrm{T}}_u(f(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b))$
- $\mathrm{gpar}_{\mathrm{v}}(a,b,c) = \widehat{\mathrm{T}}_u(g(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b, c))$

3 Π_α-reflection

3.1 Ordinals up to ordinals

- Let Ord be a set of ordinal notations for ordinals $< \alpha$.
- Let $\prec_{\mathbb{B}}$: Ord $\to$ Ord $\to \mathbb{B}$ be the ordering on Ord.
- $\beta \prec \gamma := \mathrm{atom}(\beta \prec_{\mathbb{B}} \gamma)$.

3.2 Degrees of elements of Univ

$\mathrm{length} : \mathrm{Univ} \to \mathrm{N}$

$\mathrm{deg} : (u : \mathrm{Univ}, l : \mathrm{N}_{\mathrm{S}(\mathrm{length}(u))}) \to$
$\quad \mathrm{Ord} \times (w : \mathrm{Univ}^+) \times \mathrm{P}(w) \times (l : (\mathrm{U}_u \to \mathrm{U}^+_w)) \times ((a : \mathrm{U}_u, \mathrm{T}^+_w(l(a))) \to \mathrm{T}_u(a))$

$\mathrm{ord}(u, l) := \mathrm{deg}(u, l)0(: \mathrm{Ord})$

$\mathrm{w}(u, l) := \mathrm{deg}(u, l)10(: \mathrm{Univ}^+)$

$\mathrm{Pdeg}(u, l) := \mathrm{deg}(u, l)110(: \mathrm{P}_{\mathrm{w}(u,l)})$

$\mathrm{lift}(u, l) := \mathrm{deg}(u, l)1110(: \mathrm{U}_u \to \mathrm{U}^+_{\mathrm{w}(u,l)})$

$\mathrm{emblift}(u, l) := \mathrm{deg}(u, l)1111(: (a : \mathrm{U}_u, \mathrm{T}^+_{\mathrm{w}(u,l)}(\mathrm{lift}(u, l))) \to \mathrm{T}_u(a))$

3.3 First subuniverse

Assume

- $f : (a : \mathbb{U}, b : \mathbb{T}(a) \to \mathbb{U}) \to \mathbb{U}$.
- $g : (a : \mathbb{U}, b : \mathbb{T}(a) \to \mathbb{U}, \mathbb{T}(f(a, b))) \to \mathbb{U}$.
- β : Ord.
- $p : \mathrm{P}(*)$.

Then

- $\mathrm{u}_0(f, g, \beta, p)$: Univ

Let $\mathrm{v} := \mathrm{u}_0(f, g, \beta, p)$. Then

- $\mathrm{length}(\mathrm{v}) = 0$
- $\mathrm{deg}(\mathrm{v}, l) = \langle \beta, *, p, \widehat{\mathrm{T}}_u, (a, b)b \rangle$.
- $\mathrm{fpar}_{\mathrm{v}}(a, b) = f(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b)$
- $\mathrm{gpar}_{\mathrm{v}}(a, b, c) = g(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b, c)$

3.4 Second subuniverse

Assume

- $u : \mathrm{Univ}$.
- $f : (a : \mathrm{U}_u, b : \mathrm{T}_u(a) \to \mathrm{U}_u) \to \mathrm{U}_u$.
- $g : (a : \mathrm{U}_u, b : \mathrm{T}_u(a) \to \mathrm{U}_u, \mathrm{T}_u(f(a,b))) \to \mathrm{U}_u$.
- $l : \mathrm{N}_{\mathrm{S}(\mathrm{length}(u))}$,
- $a : \mathrm{U}_u$
- $b : \mathrm{T}_u(a) \to \mathrm{U}_u$
- $p : \mathrm{Q}(\mathrm{w}(u,l), \mathrm{Pdeg}(u,l), \mathrm{lift}(u,l,a), (x)\mathrm{lift}(u,l,b(\mathrm{emblift}(u,l,a,x))))$.

Let $\vec{fg} := u, f, g, l, a, b, p$. Then

- $\mathrm{u}_1(\vec{fg}) : \mathrm{Univ}$,
- $\underline{\widehat{\mathrm{u}}}_1(\vec{fg}) : \mathrm{U}_u$.
- $\underline{\widehat{\mathrm{t}}}_1(\vec{fg}) : \mathrm{U}_{\mathrm{u}_1(\vec{fg})} \to \mathrm{U}_u$.

Let $\mathrm{v} := \mathrm{u}_1(\vec{fg})$. Then

- $\widehat{\mathrm{T}}_u(\widehat{\mathrm{u}}_1(\vec{fg})) = \mathrm{univ}(\mathrm{v})$.
- $\widehat{\mathrm{T}}_u(\widehat{\mathrm{t}}_1(\vec{fg}, c)) = \widehat{\mathrm{T}}_{\mathrm{v}}(c)$.
- $\mathrm{length}(\mathrm{v}) = \mathrm{embfin}_{\mathrm{S}(\mathrm{length}(u))}(l)$.
- $\mathrm{deg}(\mathrm{v}, \mathrm{max}_l) =$
$$\begin{aligned}&\langle \mathrm{ord}(u,l),\\ &\ \mathrm{w}(u,l),\\ &\ \mathrm{Q}(\mathrm{w}(u,l), \mathrm{Pdeg}(u,l), \mathrm{lift}(u,l,a),\\ &\qquad\qquad (x)\mathrm{lift}(u,l,b(\mathrm{emblift}(u,l,a,x))), p),\\ &\ \mathrm{lift}(u,l) \circ \widehat{\mathrm{t}}_1(\vec{fg}),\\ &\ \mathrm{emblift}(u,l) \,\tilde{\circ}\, \widehat{\mathrm{t}}_1(\vec{fg})\rangle\end{aligned}$$
- $\mathrm{deg}(\mathrm{v}, \mathrm{emb}_l(l')) =$
$$\begin{aligned}&\mathrm{let}\ \{l'' = \mathrm{tr}(\mathrm{length}(u), l, \mathrm{emb}_l(l')) : \mathrm{N}_{\mathrm{S}(\mathrm{length}(u))}\}\\ &\mathrm{in}\ \langle \mathrm{ord}(u,l''),\\ &\qquad \mathrm{w}(u,l''),\\ &\qquad \mathrm{Pdeg}(u,l''),\\ &\qquad \mathrm{lift}(u,l'') \circ \widehat{\mathrm{t}}_1(\vec{fg}),\\ &\qquad \mathrm{emblift}(u,l'') \,\tilde{\circ}\, \widehat{\mathrm{t}}_1(\vec{fg})\rangle.\end{aligned}$$

- $\mathrm{fpar}_{\mathrm{v}}(a, b) = \widehat{\mathrm{T}}_u(f(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b))$
- $\mathrm{gpar}_{\mathrm{v}}(a, b, c) = \widehat{\mathrm{T}}_u(g(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b, c))$

3.5 Third subuniverse

Assume

- $u : \mathrm{Univ}.$
- $f : (a : \mathrm{U}_u, b : \mathrm{T}_u(a) \to \mathrm{U}_u) \to \mathrm{U}_u.$
- $g : (a : \mathrm{U}_u, b : \mathrm{T}_u(a) \to \mathrm{U}_u, \mathrm{T}_u(f(a, b))) \to \mathrm{U}_u.$
- $n := \mathrm{length}(u).$
- $a : \mathrm{U}_u$
- $b : \mathrm{T}_u(a) \to \mathrm{U}_u$
- $\beta : \mathrm{Ord},$
- $q : \beta \prec \mathrm{ord}(u, n).$
- $p : \mathrm{P}(\mathrm{inr}(u)).$

Let $\vec{fg} := u, f, g, a, b, \beta, q, p$. Then

- $\mathrm{u}_2(\vec{fg}) : \mathrm{Univ},$
- $\underline{\widehat{\mathrm{u}}}_2(\vec{fg}) : \mathrm{U}_u.$
- $\underline{\widehat{\mathrm{t}}}_2(\vec{fg}) : \mathrm{U}_{\mathrm{u}_2(\vec{fg})} \to \mathrm{U}_u.$

Let $\mathrm{v} := \mathrm{u}_2(\vec{fg})$. Then

- $\widehat{\mathrm{T}}_u(\widehat{\mathrm{u}}_2(\vec{fg})) = \mathrm{univ}(\mathrm{v}).$
- $\widehat{\mathrm{T}}_u(\widehat{\mathrm{t}}_2(\vec{fg}, c)) = \widehat{\mathrm{T}}_{\mathrm{v}}(c).$
- $\mathrm{length}(\mathrm{v}) = \mathrm{S}(n).$
- $\mathrm{deg}(\mathrm{v}, \mathrm{max}_{\mathrm{S}(n)}) =$
  $\langle \beta, u, p, \widehat{\mathrm{t}}_2(\vec{fg}), (a', b')b' \rangle$
- $\mathrm{deg}(\mathrm{v}, \mathrm{emb}_{\mathrm{S}(n)}(l)) =$
  $\langle \mathrm{ord}(u, l), \mathrm{w}(u, l), \mathrm{Pdeg}(u, l), \mathrm{lift}(u, l) \circ \widehat{\mathrm{t}}_2(\vec{fg}), \mathrm{emblift}(u, l)\, \tilde{\circ}\, \widehat{\mathrm{t}}_2(\vec{fg}) \rangle.$
- $\mathrm{fpar}_{\mathrm{v}}(a, b) = \widehat{\mathrm{T}}_u(f(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b))$
- $\mathrm{gpar}_{\mathrm{v}}(a, b, c) = \widehat{\mathrm{T}}_u(g(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b, c))$

4 Π^1_1-reflection

4.1 Degrees of elements of Univ

$\mathrm{length} : \mathrm{Univ} \to \mathrm{N}$

$\deg : (u : \mathrm{Univ}, l : \mathrm{N}_{\mathrm{S}(\mathrm{length}(u))}) \to$
$\quad \mathrm{P}(*) \times (w : \mathrm{Univ}^+) \times \mathrm{P}(w) \times (l : (\mathrm{U}_u \to \mathrm{U}^+_w)) \times ((a : \mathrm{U}_u, \mathrm{T}^+_w(l(a))) \to \mathrm{T}_u(a))$

$\mathrm{Ddeg}(u,l) := \deg(u,l)0(: \mathrm{P}(*))$

$\mathrm{w}(u,l) := \deg(u,l)10(: \mathrm{Univ}^+)$

$\mathrm{Pdeg}(u,l) := \deg(u,l)110(: \mathrm{P}_{\mathrm{w}(u,l)})$

$\mathrm{lift}(u,l) := \deg(u,l)1110(: \mathrm{U}_u \to \mathrm{U}^+_{\mathrm{w}(u,l)})$

$\mathrm{emblift}(u,l) := \deg(u,l)1111(: (a : \mathrm{U}_u, \mathrm{T}^+_{\mathrm{w}(u,l)}(\mathrm{lift}(u,l))) \to \mathrm{T}_u(a))$

4.2 First subuniverse

Assume

- $f : (a : \mathbb{U}, b : \mathbb{T}(a) \to \mathbb{U}) \to \mathbb{U}$.
- $g : (a : \mathbb{U}, b : \mathbb{T}(a) \to \mathbb{U}, \mathbb{T}(f(a,b))) \to \mathbb{U}$.
- $d, p : \mathrm{P}(*)$.

Then

- $\mathrm{u}_0(f,g,d,p) : \mathrm{Univ}$

Let $\mathrm{v} := \mathrm{u}_0(f,g,d,p)$. Then

- $\mathrm{length}(\mathrm{v}) = 0$
- $\deg(\mathrm{v},l) = \langle d, *, p, \widehat{\mathrm{T}}_u, (a,b)b \rangle$.
- $\mathrm{fpar}_{\mathrm{v}}(a,b) = f(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b)$
- $\mathrm{gpar}_{\mathrm{v}}(a,b,c) = g(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b, c)$

4.3 Second subuniverse

Assume

- $u : \mathrm{Univ}$.
- $f : (a : \mathrm{U}_u, b : \mathrm{T}_u(a) \to \mathrm{U}_u) \to \mathrm{U}_u$.
- $g : (a : \mathrm{U}_u, b : \mathrm{T}_u(a) \to \mathrm{U}_u, \mathrm{T}_u(f(a,b))) \to \mathrm{U}_u$.
- $l : \mathrm{N}_{\mathrm{S}(\mathrm{length}(u))}$,
- $a : \mathrm{U}_u$
- $b : \mathrm{T}_u(a) \to \mathrm{U}_u$
- $p : \mathrm{Q}(\mathrm{w}(u,l), \mathrm{Pdeg}(u,l), \mathrm{lift}(u,l,a), (x)\mathrm{lift}(u,l,b(\mathrm{emblift}(u,l,a,x))))$.

Let $\vec{fg} := u, f, g, l, a, b, p$. Then

- $\mathrm{u}_1(\vec{fg}) : \mathrm{Univ}$,
- $\underline{\widehat{\mathrm{u}}}_1(\vec{fg}) : \mathrm{U}_u$.
- $\underline{\widehat{\mathrm{t}}}_1(\vec{fg}) : \mathrm{U}_{\mathrm{u}_1(\vec{fg})} \to \mathrm{U}_u$.

Let $\mathrm{v} := \mathrm{u}_1(\vec{fg})$. Then

- $\widehat{\mathrm{T}}_u(\widehat{\mathrm{u}}_1(\vec{fg})) = \mathrm{univ}(\mathrm{v})$.
- $\widehat{\mathrm{T}}_u(\widehat{\mathrm{t}}_1(\vec{fg}, c)) = \widehat{\mathrm{T}}_{\mathrm{v}}(c)$.
- $\mathrm{length}(\mathrm{v}) = \mathrm{embfin}_{\mathrm{S}(\mathrm{length}(u))}(l)$.
- $\mathrm{deg}(\mathrm{v}, \mathrm{max}_l) =$
  $\langle \mathrm{Ddeg}(u,l),$
  $\mathrm{w}(u,l),$
  $\mathrm{Q}(\mathrm{w}(u,l), \mathrm{Pdeg}(u,l), \mathrm{lift}(u,l,a),$
  $(x)\mathrm{lift}(u,l,b(\mathrm{emblift}(u,l,a,x))), p),$
  $\mathrm{lift}(u,l) \circ \widehat{\mathrm{t}}_1(\vec{fg}),$
  $\mathrm{emblift}(u,l) \mathbin{\tilde{\circ}} \widehat{\mathrm{t}}_1(\vec{fg})\rangle$
- $\mathrm{deg}(\mathrm{v}, \mathrm{emb}_l(l')) =$
  let $\{l'' = \mathrm{tr}(\mathrm{length}(u), l, \mathrm{emb}_l(l')) : \mathrm{N}_{\mathrm{S}(\mathrm{length}(u))}\}$
  in $\langle \mathrm{Ddeg}(u,l''), \mathrm{w}(u,l''), \mathrm{Pdeg}(u,l''), \mathrm{lift}(u,l'') \circ \widehat{\mathrm{t}}_1(\vec{fg}),$
  $\mathrm{emblift}(u,l'') \mathbin{\tilde{\circ}} \widehat{\mathrm{t}}_1(\vec{fg})\rangle$.
- $\mathrm{fpar}_{\mathrm{v}}(a,b) = \widehat{\mathrm{T}}_u(f(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b))$
- $\mathrm{gpar}_{\mathrm{v}}(a,b,c) = \widehat{\mathrm{T}}_u(g(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b, c))$

4.4 Third subuniverse

Assume

- $u : \mathrm{Univ}$.
- $f : (a : \mathrm{U}_u, b : \mathrm{T}_u(a) \to \mathrm{U}_u) \to \mathrm{U}_u$.
- $g : (a : \mathrm{U}_u, b : \mathrm{T}_u(a) \to \mathrm{U}_u, \mathrm{T}_u(f(a,b))) \to \mathrm{U}_u$.
- $n := \mathrm{length}(u)$,
- $a : \mathrm{U}_u$
- $b : \mathrm{T}_u(a) \to \mathrm{U}_u$
- $d : \mathrm{Q}(*, \mathrm{Ddeg}(u, \max_n), \widehat{\mathrm{T}}_u(a), \widehat{\mathrm{T}}_u \circ b)$,
- $p : \mathrm{P}(\mathrm{inr}(u))$.

Let $\vec{fg} := u, f, g, a, b, d, p$. Then

- $\mathrm{u}_2(\vec{fg}) : \mathrm{Univ}$,
- $\underline{\widehat{\mathrm{u}}}_2(\vec{fg}) : \mathrm{U}_u$.
- $\underline{\widehat{\mathrm{t}}}_2(\vec{fg}) : \mathrm{U}_{\mathrm{u}_2(\vec{fg})} \to \mathrm{U}_u$.

Let $\mathrm{v} := \mathrm{u}_2(\vec{fg})$. Then

- $\widehat{\mathrm{T}}_u(\widehat{\mathrm{u}}_2(\vec{fg})) = \mathrm{univ}(\mathrm{v})$.
- $\widehat{\mathrm{T}}_u(\widehat{\mathrm{t}}_2(\vec{fg}, c)) = \widehat{\mathrm{T}}_{\mathrm{v}}(c)$.
- $\mathrm{length}(\mathrm{v}) = \mathrm{S}(n)$.
- $\mathrm{deg}(\mathrm{v}, \max_{\mathrm{S}(n)}) =$
  $\langle \mathrm{R}(*, \mathrm{Ddeg}(u, \max_n), \widehat{\mathrm{T}}_u(a), \widehat{\mathrm{T}}_u \circ b, d),$
  $u, p, \widehat{\mathrm{t}}_2(\vec{fg}), (a', b')b' \rangle$
- $\mathrm{deg}(\mathrm{v}, \mathrm{emb}_{\mathrm{S}(n)}(l)) =$
  $\langle \mathrm{Ddeg}(u, l), \mathrm{w}(u, l), \mathrm{Pdeg}(u, l), \mathrm{lift}(u, l) \circ \widehat{\mathrm{t}}_2(\vec{fg}), \mathrm{emblift}(u, l) \,\tilde{\circ}\, \widehat{\mathrm{t}}_2(\vec{fg}) \rangle$.
- $\mathrm{fpar}_{\mathrm{v}}(a, b) = \widehat{\mathrm{T}}_u(f(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b))$
- $\mathrm{gpar}_{\mathrm{v}}(a, b, c) = \widehat{\mathrm{T}}_u(g(\widehat{\mathrm{T}}_{\mathrm{v}}(a), \widehat{\mathrm{T}}_{\mathrm{v}} \circ b, c))$