Type universes and ramifications

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Russell's Theory of Types (p2)

Bertrand Russell (1908): *Mathematical Logic as Based on the Theory of Types.*

Russell's presentation was partly informal (e.g. regarding substitution). A modern reconstruction using lambda calculus notation is

F. Kamareddine, T. Laan, R. Nederpelt. *Types in Logic and Mathematics Before 1940.* Bulletin of Symbolic Logic 2002.

Simple type theory (p3)

The simple types are defined inductively

- Basic types of individuals are $\mathbf{N}$ $(=\{0, s(0), s(s(0)), \ldots\})$ and $\mathbf{1}$ $(=\{\star\})$
- For types A and B the product $A \times B$ is a type.
- For a type A the propositional functions on A constitute a type $\mathbf{P}(A)$

If $\varphi(x)$ is a proposition then $(\lambda x : A)\varphi(x) : \mathbf{P}(A)$ is a propositional function.

(p4)

The propositional functions can equivalently be regarded as subsets: Write

- $\{x : A \mid \varphi(x)\} =_{\mathrm{def}} (\lambda x : A)\varphi(x)$
- $a \in F =_{\mathrm{def}} F(a)$ where $F : \mathbf{P}(A)$ and $a : A$

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There is no restriction on the proposition $\varphi(x)$. It may very well contain a quantifier $(\forall F : \mathbf{P}(A))$. Thus

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is constructed using quantification over the totality to which it self belongs.

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is constructed using quantification over the totality to which it self belongs.
This is an impredicative construction and instance of the *vicious circle principle.*

Ramified types (p5)

To avoid possible paradoxes and vicious circles Russell introduced a *stratification* of the propositions and hence propositional functions. Thus for every type S we have a stratified sequence of power sets of S

$$\mathrm{P}_0(S) \subseteq \mathrm{P}_1(S) \subseteq \mathrm{P}_2(S) \subseteq \cdots$$

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it is required that $\varphi(x)$ contains only quantifiers over individuals or over $\mathrm{P}_n(S)$ where $n < k$. **It quantifies only over objects already constructed or given.**

Ramified type symbols (p6)

Define the ramified type symbols $\mathcal{R} = \cup_{n\geq 0}\mathcal{R}_n$. inductively:

- $\mathcal{R}_0$ contains $\mathbf{1}$, $\mathbf{N}$ and is closed under $\times$
- $\mathcal{R}_{n+1}$ contains $\mathcal{R}_n$ and is closed under $\times$ and $\mathbf{P}_k(\cdot)$ for $k \leq n$.

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We have e.g.

$$A = \mathbf{P}_3(\mathbf{N} \times \mathbf{P}_2(\mathbf{P}_1(\mathbf{1}))) \in \mathcal{R}_4$$

$$B = \mathbf{P}_1(\mathbf{P}_2(\mathbf{1})) \in \mathcal{R}_3$$

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(According to Russell's typing (K-L-N 2002) only A would be ramified. B is however rarely useful in his system.)

Ramified formulas (p7)

The set of formulas of level n, $\mathcal{F}_n$, include

- $\bot$, $s =_A t$ for $A \in \mathcal{R}_n$,
- $X(t)$ for $X : \mathbf{P}_k(A)$ and $k \leq n$

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Restricted (predicative) comprehension principle: for $\varphi(x) \in \mathcal{F}_n$

$$\{x : A \mid \varphi(x)\} : \mathbf{P}_n(A).$$

Reducibility axiom (p8)

Russell's *axiom of reducibility* can be phrased

$$(\forall X : \mathbf{P}_n(A))(\exists Y : \mathbf{P}_0(A)(X = Y).$$

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In an intuitionistic setting, some special instance the axiom of reducibility are indeed predicatively valid.

Type universes (p9)

Standard formulations of Martin-Löf type theory (1984) include a cumulative hierarchy of type universes $U_0, U_1, U_2, \ldots$.

$$\frac{A : U_n}{A \text{ type}} \qquad \frac{A : U_n}{A : U_{n+1}} \qquad U_n : U_{n+1}$$

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U_0 contains basic types: $N_0, N_1, N : U_0$

Each U_n is closed under type operations $\Sigma, \Pi, +, \mathrm{Id}(\cdot,\cdot,\cdot)$. E.g.

$$\frac{A : U_n \qquad B(x) : U_n \ (x : A)}{(\Sigma x : A)B(x) : U_n}.$$

Setoids $\sim$ Errett Bishop's notion of set (p10)

- A *setoid* $A = (|A|, =_A)$ is a type $|A|$ together with an equivalence relation $=_A$.
- An *(extensional) function* $f : A \longrightarrow B$ between setoids is a function (operation) $|A| \longrightarrow |B|$ together with a proof that the operation respects the equalities $=_A$ and $=_B$.

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When based on Martin-Löf type theory this forms a good category of sets for constructive mathematics, supporting several choice principles: *Axiom of Unique Choice, Dependent Choice* and *Aczel's Presentation Axiom*.

Stratified setoids (p11)

A setoid A is an (m, n)-setoid if

$$|A| : U_m \qquad\qquad =_A : |A| \to |A| \to U_n .$$

- m-setoid $=_{\mathrm{def}}$ (m, m)-setoid
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- m-setoid $=_{\mathrm{def}}$ (m,m)-setoid
- m-classoid $=_{\mathrm{def}}$ $(m+1,m)$-setoid
- ("Replacement") $f : A \to B$, A m-setoid, B m-classoid $\Longrightarrow$ $\mathrm{Im}(f)$ m-setoid.

Examples of stratified setoids (p12)

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- $\mathbb{N} = (N, \mathrm{Id}(N, \cdot, \cdot)$ is a 0-setoid.
- $\Omega_n = (U_n, \leftrightarrow)$ propositions of level n with logical equivalence constitute an n-classoid.
- For an n-setoid A, the setoid of extensional propositional functions of level n

$$P_n(A) = [A \to \Omega_n]$$

  is an n-classoid.

Exponent setoids (p13)

For setoids A and B the *exponent setoid* $[A \to B] = B^A$ is given by

$$|B^A| =_{\mathrm{def}} (\Sigma f : |A| \to |B|)(\forall x, y : |A|)(x =_A y \Rightarrow f(x) =_B f(y))$$

and

$$(f, p) =_{B^A} (g, q) \Longleftrightarrow_{\mathrm{def}} (\forall x : |A|)(f(x) =_B g(x))$$

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If A is an (m, n)-setoid and B is an (m', n')-setoid, then $[A \to B]$ is an $(\max(m, m', n, n'), \max(m, n'))$-setoid.

In particular, the category of n-setoids are closed under exponentiation, but the n-classoids are not.

A natural model of ramified types in MLTT (p14)

For each type symbol $S \in \mathcal{R}$ define a setoid S^* in Martin-Löf type theory, by recursion:

- $\mathbf{N}^* = (N, \mathrm{Id}(N, \cdot, \cdot)$
- $\mathbf{1}^* = (N_1, \mathrm{Id}(N_1, \cdot, \cdot)$
- $(A \times B)^* = A^* \times B^*$ (cartesian product)
- $\mathbf{P}_n(A)^* = [A^* \to \Omega_n]$ (exponent construction)

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This gives a hierarchy of type which satisfies the extensionality axioms. We have

$$A \in \mathcal{R}_n \Longrightarrow A \text{ } n\text{-setoid}$$

Local sets (p15)

Following practice in topos theory (Bell 1988) define a *local set* to be a type together with a subset.
A *local set* of grade (m, n) is a pair $A = (\tau_A, \rho_A)$ where

$$\tau_A : \mathcal{R}_m \qquad\qquad \rho : \mathbf{P}_n(\tau_A)^*.$$

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$$\tau_A : \mathcal{R}_m \qquad\qquad \rho : \mathbf{P}_n(\tau_A)^*.$$

Let A and B be local sets. A *map* $F : A \to B$ is a relation $F : \mathbf{P}_\ell(\tau_A \times \tau_B)^*$, for some ℓ, such that

$$(\forall x : {\tau_A}^*)(\rho_A(x) \Rightarrow (\exists! y : {\tau_B}^*)(\rho_B(y) \wedge F(x, y)))$$

and

$$(\forall x : {\tau_A}^*)(\forall y : {\tau_B}^*)(F(x, y) \Rightarrow \rho_A(x) \wedge \rho_B(y))$$

(p16)

We remark that the types used in local sets are almost as simple as those used in impredicative simple type theory or topos logic (higher order logic).

In particular, dependent types are not used (other than as propositions).

(p17)

For a each local (m,n)-set $A = (\tau_A, \rho_A)$ we associate a corresponding setoid

$$\check{A} = ((\Sigma x : {\tau_A}^*)\rho_A(x), =')$$

where

$$(x,p) =' (y,q) \Longleftrightarrow_{\mathrm{def}} x =_{{\tau_A}^*} y.$$

This setoid has index $(\max(m,n), m)$.

(p18)

For local sets A and B, we have a setoid $[\check{A} \to \check{B}]$ which can represent all maps $A \to B$. For $f : [\check{A} \to \check{B}]$ define

$$G_f(x, y) = (\exists p : \rho_A(x))(\exists q : \rho_B(y))(f(x, p) =_{\check{B}} (y, q)).$$

Then for any map $F : A \to B$ there is by Axiom of Unique Choice some unique $f : [\check{A} \to \check{B}]$ with

$$F = G_f.$$

A reducibility principle for functions (p19)

Reducibility for functions:

Thus for local sets A and B there some level ℓ so that for a map $F : A \to B$ (of arbitrary level) there is a map $G : A \to B$ with $G : \mathbf{P}_\ell(\tau_A \times \tau_B)^*$ such that

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Note that this would not work predicatively in a classical setting, as it implies a full comprehension principle.

Quotients sets (p20)

Quotients sets are constructed as sets of equivalence classes.
Let $A = (\tau_A, \rho_A)$ be an (m, n)-subset and suppose $E : \mathbf{P}_k(\tau_A \times \tau_A)^*$ is an equivalence relation on A, i.e. it satisfies

- $\rho_A(x) \Leftrightarrow E(x, x)$
- $E(x, y) \Rightarrow E(y, x)$
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- $E(x, y) \Rightarrow E(y, x)$
- $E(x, y) \wedge E(y, z) \Rightarrow E(x, z)$

Define the quotient subset $A/E = (\tau_B, \rho_B)$ by $\tau_B = \mathbf{P}_\ell(\tau_A)$ where $\ell = \max(m, n, k)$, and

$$\rho_B(S) = (\exists x : {\tau_A}^*)(\rho_A(x) \wedge (\forall y : {\tau_A}^*)(S(y) \Leftrightarrow E(x, y))).$$

Category of local sets (p21)

The category of local sets admits constructions of function sets, quotients, pullbacks etc.

Though not yet fully verified, we consider it very likely that it satisfies some standard axioms for a "predicative topos", E.g. locally cartesian closed pretopos.

Constructive varieties of categories of sets (p22)

1. Setoids
2. PERs - types with partial equivalence relations and functional relations as maps
3. TPERs - types with partial equivalence relations and as maps functions total on the type
4. Local sets for a ramified or simple type theory
5. Sets within CZF

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Which is most suitable depends on whether you want a standard category or a E-category, and whether you need chosen pullbacks (cf. Makkai).

References (p23)

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