Making constructive set theory explicit

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Formal systems for constructive mathematics Bishop's style (Bishop 1967)

1. **Martin–Löf type theory** (Martin–Löf 1975)

2. **Constructive set theory** (Myhill 1975)

Constructive Zermelo–Fraenkel (CZF) (Aczel 1978)

3. **Explicit mathematics (EM)** (Feferman 1975)

Aim: build a bridge between 2 and 3

Constructive Operational Set Theory (COST)

Operational set theory (OST / IZFR):

(Classical) operational set theory, Feferman 2001; 2006

Intuitionistic set theory with rules, Beeson 1988

Jaeger 2006, 2008 on classical operational set theory

Constructive Zermelo–Fraenkel (CZF)

a *generalised predicative* version of **ZF** based on intuitionistic logic

Intuitionistic logic: Foundation is stated in a positive, constructive way: **set–induction**

No full axiom of **choice**

Predicativity: we implement restrictions on those **ZF** -axioms which can give rise to impredicativity:

- **Δ_0–separation**

- Powerset is replaced by a "predicative" version of it
  subset collection

Note

- we only talk about **sets** (no urelements)

- the theory is fully **extensional**

Explicit mathematics

a theory of ***operations*** (or rules) and ***classes***

Characteristics

- classes are thought of as successively generated from preceding ones
- operations and classes are ***intensional***
- operations and classes are not interreducible
- operations may be applied to classes and to operations
- ***self–application*** is allowed
- in general operations are ***partial***

Constructive Operational Set Theory (COST)

Characteristics

- an ***intensional*** **notion of operation** along with an ***extensional*** **notion of set**
- **urelements** for natural numbers and elements of a combinatory algebra
- **uniform operations** on sets
- there is a limited form of **self–application**

Motivation

- Have an **extensional context** for developing **mathematics** and an **intensional** one for studying the **computational side**.
- **Natural numbers** and **recursive functions** are taken as **primitive**
- **Uniformity of (some) operations on sets**

The theory **COST** (sketch)

Language: applicative extension of first order language of **ZF**:

- the combinators K and S;
- constants 0, SUC, PR, D;
- predicates: App (application), $\mathcal{S}$ (sets), $\mathcal{N}$ (natural numbers) and $\mathcal{U}$ (elements of combinatory algebra)

Constants:

- el (operation representing membership);
- pair, un, im, exp, sep (set operations);
- $\varnothing$, Nat and Ur (set constant)

A formula is *App-bounded*, or Δ_0^{App} iff it is bounded (or Δ_0) and it does **not** contain formulas of the form $App(x, y, z)$

COST

- First order intuitionistic logic with equality
- Ontological axioms and extensionality for sets
- Applicative axioms
- Membership
- Set theoretic axioms (uniform)
- Induction and collection principles

- **Ontological axioms and extensionality**

  (a) $\neg(\mathcal{U}(x) \wedge \mathcal{S}(x))$

  (b) $\mathcal{U}(x) \vee \mathcal{S}(x)$

  (c) $\mathcal{N}(x) \rightarrow \mathcal{U}(x)$

  (d) $x \in y \rightarrow \mathcal{S}(y)$

  (e) $\forall x\,(x \in a \leftrightarrow x \in b) \rightarrow a = b$

Convention on variables

$u, v, x, y, z, \ldots$: generic variables

$a, b, \ldots$: sets, but $F, G, \ldots$: sets which are functions

$f, g, \ldots$: urelements as well as sets, when used as operations

$p, q, \ldots$: urelements

$k, m, n, \ldots$: natural numbers

- **General applicative axioms and $\mathcal{N}$-closure**

  (a) $App(x,y,z) \wedge App(x,y,w) \rightarrow z = w$

  (b) $Kxy = x \wedge Sxy\downarrow \wedge Sxyz \simeq xz(yz)$

  (c) $\mathcal{N}(0) \wedge \forall n\,(\mathcal{N}(SUCn) \wedge SUC\,n \neq 0)$

  (d) $PR\,0 = 0 \wedge \forall n\,(\mathcal{N}(PR\,n) \wedge PR\,(SUC\,n) = n)$

  (e) $Dxynn = x \wedge (n \neq m \rightarrow Dxynm = y)$

  (f) $\exists r\; App(p,q,r)$

  (g) $\forall r\,(pr \simeq qr) \rightarrow p = q$

  (h) $\mathcal{U}(K) \wedge \mathcal{U}(S) \wedge \mathcal{U}(SUC) \wedge \mathcal{U}(PR) \wedge \mathcal{U}(D)$

- **Membership operation**

  (a) $\mathsf{el} : \mathbf{V}^2 \rightarrow \Omega$ and $\mathsf{el}\,xy \simeq \top \leftrightarrow x \in y$

- **Set constructors**

  (a) $\mathcal{S}(\varnothing) \wedge \forall x\, (x \notin \varnothing)$

  (b) $\mathcal{S}(Ur) \wedge \forall x\, (x \in Ur \leftrightarrow \mathcal{U}(x))$

  (c) $\mathcal{S}(Nat) \wedge \forall x\, (x \in Nat \leftrightarrow \mathcal{N}(x))$

  (d) $\mathcal{S}(\mathsf{pair}\, xy) \wedge \forall z\, (z \in \mathsf{pair}\, xy \leftrightarrow z = x \vee z = y)$

  (e) $\mathcal{S}(\mathsf{un}\, a) \wedge \forall z\, (z \in \mathsf{un}\, a \leftrightarrow \exists y \in a(z \in y))$

  (f) $(f : a \rightarrow \Omega) \rightarrow \mathcal{S}(\mathsf{sep}\, fa) \wedge \forall x\, (x \in \mathsf{sep}\, fa \leftrightarrow x \in a \wedge fx \simeq \top)$

  (g) $(f : a \rightarrow V) \rightarrow \mathcal{S}(\mathsf{im}\, fa) \wedge \forall x\, (x \in \mathsf{im}\, fa \leftrightarrow \exists y \in a(x \simeq fy))$

  (h) $\mathcal{S}(\mathsf{exp}\, ab) \wedge \forall x(x \in \mathsf{exp}\, ab \leftrightarrow (Fun(x) \wedge Dom(x) = a \wedge Ran(x) \subseteq b)$

- **Induction**

  Due to **separation** between **natural numbers** and **sets**, we can define 2 principles of induction: one for sets and one for numbers:

  Induction on the natural numbers

  Set- induction

- **Collection Principles**

  (a) Subset Collection: a predicative variant of powerset

  (b) Strong Collection scheme: a strengthening of replacement

$\mathbf{COST}_b$ is the system obtained from **COST** by restricting induction on the natural numbers (induction axiom) but leaving full set-induction

Main results

- **Intensionality of operations** is essential
- (proof theory) $\mathbf{COST}_b$ has the same **proof theoretic strength** as **PA**.
- This theory is quite expressive, for example it recasts Aczel's *class inductive definitions*
- **Choice** is still **problematic** also for operations

Lemma 2 There are application terms eq, and, all, exists, imp, or, ur, nat, set, representing in a natural way the corresponding notions

Lemma 3 Uniform comprehension for Δ_0^{App} formulas

Corollary 4 Heyting Arithmetic **HA** is interpretable in $\mathbf{COST}_b$

Lemma 5 Let $\varphi(x, y)$ be Δ_0^{App} (with the free variables shown). Then there exists an operation D_φ such that $D_\varphi abu \downarrow$ and

$$D_\varphi abuv = \begin{cases} a, & \text{if } \varphi(u, v); \\ b, & \text{else} \end{cases}$$

Proof: There exists a total operation D_φ such that

$$D_\varphi = \lambda a \lambda b \lambda u \lambda v.\{x \in a : \varphi(u, v)\} \cup \{x \in b : \neg\varphi(u, v)\}.$$

Refuting extensionality and totality of operations:

Proposition 6: $\mathbf{COST}_b$ refutes extensionality for operations

Proposition 7: $\mathbf{COST}_b$ refutes totality of application for operations

Proposition 8: $\mathbf{COST}_b$ with uniform separation for conditions containing $\simeq$ proves $\perp$

[*Extensionality for operations*

$$\forall x\,(fx \simeq gx) \to f = g]$$

Operations vs. set theoretic functions

In **COST** we have set theoretic functions and operations

What is the relationship between them?

Beeson's axiom **FO**:

$$
\textbf{(FO)} \qquad \forall z\,(Fun(z) \wedge Dom(z) = a \wedge Ran(z) \subseteq b \\
\rightarrow \forall x \in a\, \exists y \in b\; zx \simeq y)
$$

i.e. *"every set theoretic function is an operation"*

FO can be consistently added to **COST**

FO implies that every element of the set $\mathsf{exp}\, ab$ is an operation from a to b

Is it consistent to assume the existence of the **set**

$$\mathsf{op}ab := \{\, f : \forall x \in a\, \exists y \in b\, (fx \simeq y)\}$$

of *all operations* from a to b?

Lemma 9 (Pierluigi Minari): $\mathbf{COST}_b + \forall a \forall b\, \exists c(\mathsf{op}\, ab = c)$ is inconsistent

The axiom of choice:

In *extensional* set theories like **CZF** the full axiom of choice, **AC**, is problematic since it implies the law of excluded middle by a well known argument

When translated in type theoretic contexts (e.g. Martin–Löf type theory) **AC** is valid due to the *intensionality* of type theory (or Curry–Howard isomorphism)

Question: What is the status of the axiom of choice in **COST***?*

AC in its usual form ***fails*** in **COST** by the same argument as for **CZF** due to extensionality of sets

What about an *axiom of choice for operations*?

We formulate two variants of **AC** for operations:

OAC

$$\forall x \in a\, \exists y\, \varphi(x, y) \to \exists f\, \forall x \in a\, \varphi(x, fx)$$

and its generalized form **GAC**

$$\forall x\, (\varphi(x) \to \exists y\, \psi(x, y)) \to \exists f\, \forall x\, (\varphi(x) \to \psi(x, fx))$$

GAC! denotes **GAC** with the uniqueness restriction on the quantifier $\exists y$ in the antecedent of **GAC**

Lemma 12:

- $\mathbf{COST}_b + \mathbf{OAC}$ proves $\varphi \vee \neg\varphi$ for arbitrary bounded formulas
- Moreover, $\mathbf{COST}_b + \mathbf{GAC}$ and $\mathbf{COST}^- + \mathbf{GAC}!$ are inconsistent

Proof theoretic strength of the theory $\mathbf{COST}_b$

(*Assigning a combinatory structure to the universe of constructive sets*)

(1) We define an auxiliary theory $\mathbf{CZF}_b^{op}$

Here urelements represent natural numbers and application terms, but ***application for sets is not allowed***

(2) We interpret the theory $\mathbf{COST}_b$ in $\mathbf{CZF}_b^{op}$

We recast application on sets by a ***class-inductive-definition***

(this makes essential use full set induction)

(3) We introduce a classical theory, $\mathbf{T_c}$, of partial (non–extensional) classes in the style of explicit mathematics (see Cantini 1996)

This is a theory with a truth predicate

(4) We translate $\mathbf{CZF}_b^{op}$ in $\mathbf{T_c}$ by use of an appropriate notion of realizability

(5) We show that the proof theoretic strength of $\mathbf{T_c}$ is the same as **PA**'s

Note: the proof theoretic weakness is due to the restriction on the Nat-induction

Thank you!