The extended predicative Mahlo Universe in Explicit Mathematics – Model Construction

Swansea University
http://www.swansea.ac.uk/compsci/
Joint Work with Reinhard Kahle, Lisbon

Seminar Logic and Theory Group, Bern, Switzerland
http://www.ltg.unibe.ch/home
http://www.ltg.unibe.ch/node/1872

5 October 2017

Goal of the Talk

- Introduce an alternative formalisation of the Mahlo universe in the context of explicit Mathematics.
- Give a model showing consistency.
- Not yet but almost – determine upper bound for the proof theoretic strength of extended predicative Mahlo universe.

Universes in Type Theory

- In Martin-Löf type theory a universe is a type the elements of which represent (via a decoding function) types.
- Usually a universe should be closed under basic constructs for forming types.
- Allows to form internal models of type theory.
- Corresponds to
  - large cardinals in set theory or
  - admissibles in Kripke Platek Set Theory.
- Universes allow to reach higher proof theoretic strength.

Hierarchy of Universes

- Martin-Löf introduced a hierarchy of universe

$$\mathrm{U}_0 \overset{\in}{\subseteq} \mathrm{U}_1 \overset{\in}{\subseteq} \cdots$$

- Palmgren introduced the super universe operator, which defines for a every family of types a universe containing those types.
- He added a universe closed under the super universe operator.

Illustration of the Super Universe

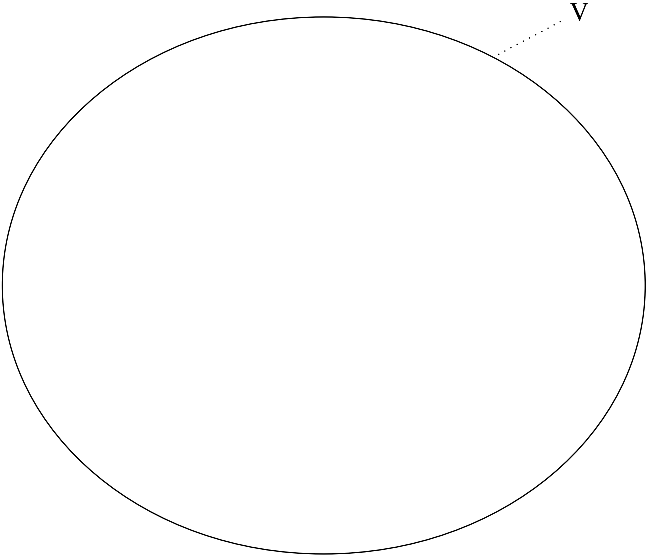


Illustration of the Super Universe

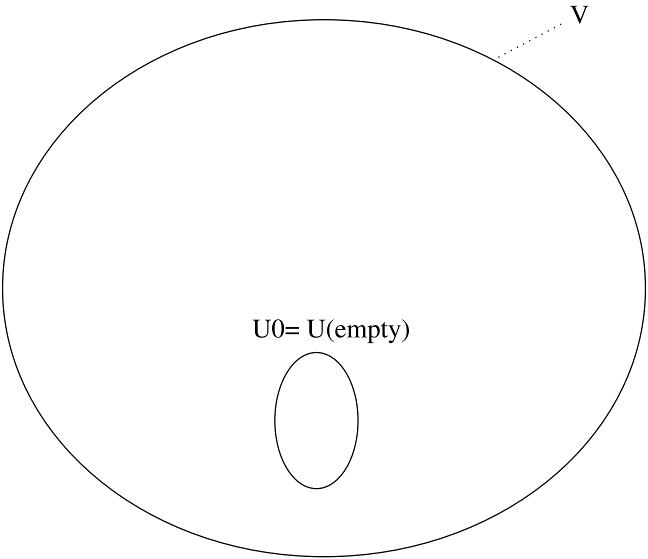


Illustration of the Super Universe

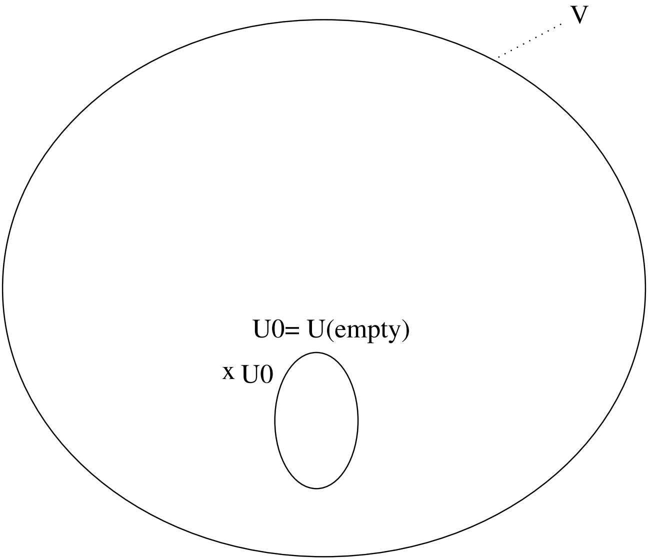


Illustration of the Super Universe

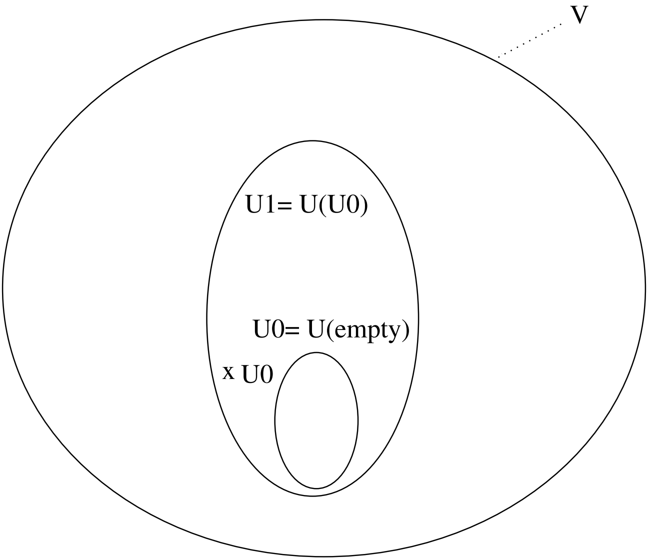


Illustration of the Super Universe

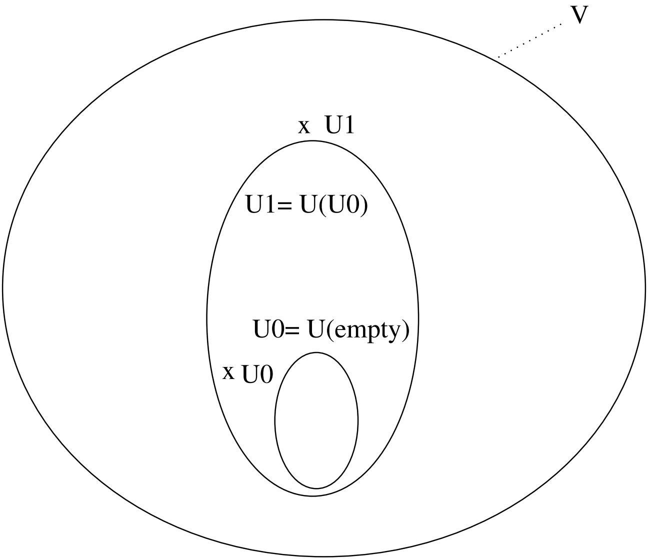


Illustration of the Super Universe

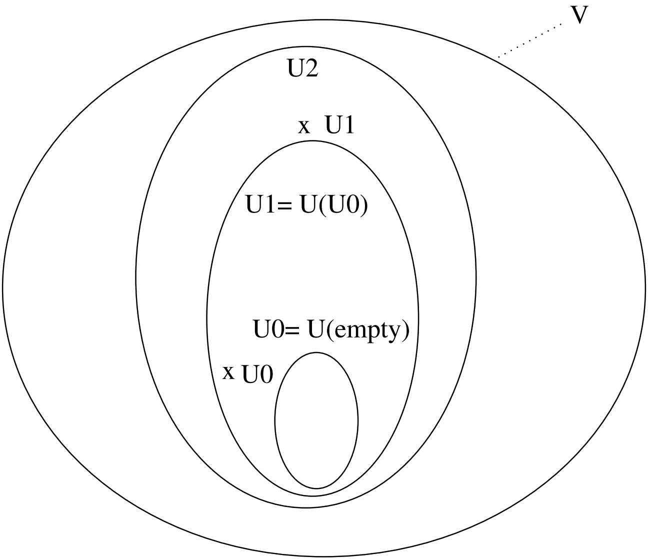


Illustration of the Super Universe

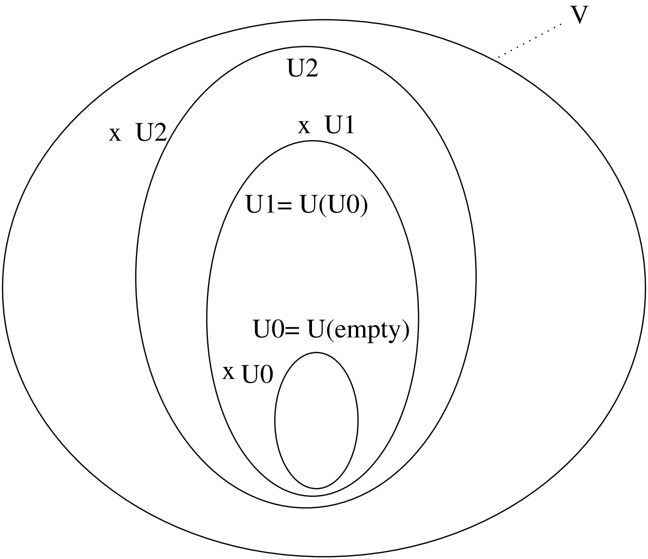
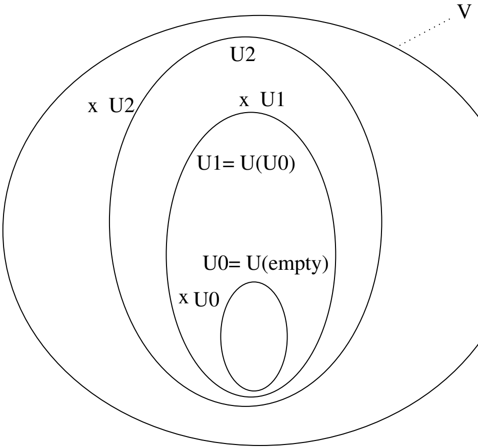


Illustration of the Super Universe



One can form as well universes above families of universes.

Super^n-Universes

- The above can be continued:
  We can form a
  - super^2-universe V,
  - closed under a super-universe operator, forming a super universe above a family of sets in V.
- And we can iterate the above n-many times, and even go beyond.
- Up to now everything was an instance of inductive-recursion.

Mahlo Universe

- The Mahlo universe in type theory is
  - a universe $\underset{\sim}{\mathrm{V}}$,
  - which has not only subuniverses corresponding to some operators, but subuniverses corresponding to all operators it is closed under:
  - for every universe operator on V,
    - i.e. every $f : \mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})$,
  - there exists a universe $\underset{\sim}{\mathrm{U}_f}$ closed under f.
- There exist as well a formalisation in Explicit Mathematics, which will be discussed below.

Illustration of the Mahlo Universe

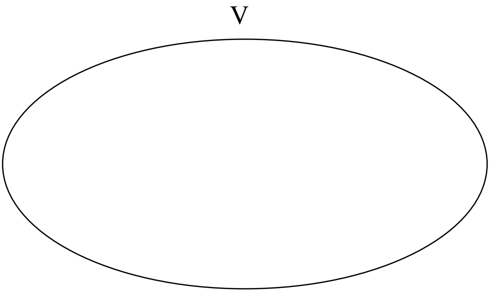


Illustration of the Mahlo Universe

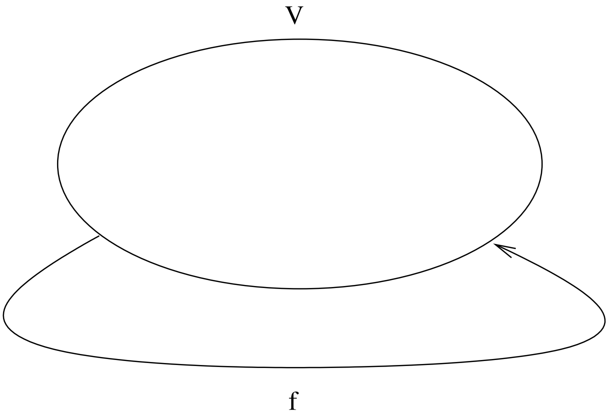


Illustration of the Mahlo Universe

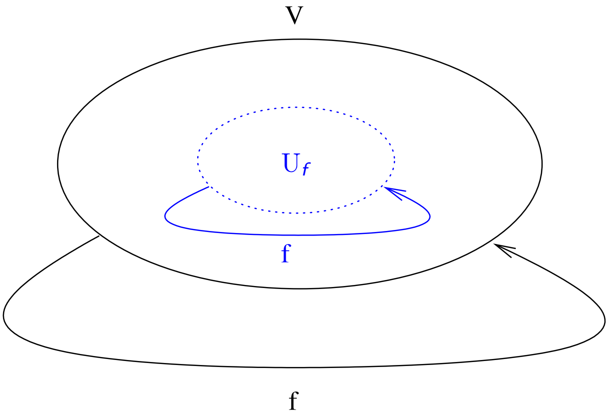
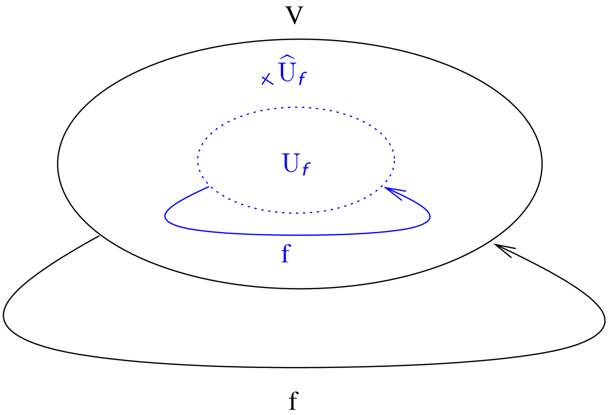


Illustration of the Mahlo Universe



Formulation of Mahlo Universe

$$
\begin{array}{l}
\mathrm{mutual} \\
\quad \mathrm{data}\ \mathrm{V} : \mathrm{Set}\ \mathrm{where} \\
\quad\quad \begin{array}{lll}
\widehat{\Pi} & : & (x : \mathrm{V}) \to (\mathrm{T_V}\ x \to \mathrm{V}) \to \mathrm{V} \\
\ldots & & \\
\widehat{\mathrm{U}} & : & (f : (x : \mathrm{V}) \to (\mathrm{T_V}\ x \to \mathrm{V}) \to \mathrm{V}) \\
& & \to (g : (x : \mathrm{V}) \to (\mathrm{T_V}\ x \to \mathrm{V}) \to (\mathrm{T_V}\ (f\ x\ y) \to \mathrm{V} \to \mathrm{V}) \\
& & \to \mathrm{V}
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\mathrm{T_V} : \mathrm{V} \to \mathrm{Set} \\
\begin{array}{lll}
\mathrm{T_V}\ (\widehat{\Pi}\ a\ b) & = & (x : \mathrm{T_V}\ a) \to \mathrm{T_V}\ (b\ x) \\
\ldots & & \\
\mathrm{T_V}\ (\widehat{\mathrm{U}}\ f\ g) & = & \mathrm{U}\ f\ g
\end{array}
\end{array}
$$

Mahlo Universe in Agda

$$
\begin{array}{l}
\mathrm{data}\ \mathrm{U}\ (f : (x : \mathrm{V}) \to (\mathrm{T_V}\ x \to \mathrm{V}) \to \mathrm{V}) \\
\qquad\qquad (g : (x : \mathrm{V}) \to (\mathrm{T_V}\ x \to \mathrm{V}) \to (\mathrm{T_V}\ (f\ x\ y) \to \mathrm{V})\mathrm{V}) \\
\qquad\qquad : \mathrm{Set\ where}
\end{array}
$$

$$
\begin{array}{lcl}
\widehat{\Pi} & : & (x : \mathrm{U}_{f,g}) \to (\mathrm{T}_{f,g}\ x \to \mathrm{U}_{f,g}) \to \mathrm{U}_{f,g} \\
\ldots & & \\
\widehat{\mathrm{f}} & : & (x : \mathrm{U}_{f,g}) \to (\mathrm{T}_{f,g}\ x \to \mathrm{U}_{f,g}) \to \mathrm{U}_{f,g} \\
\widehat{\mathrm{g}} & : & (x : \mathrm{U}_{f,g}) \\
& & \to (y : \mathrm{T}_{f,g}\ x \to \mathrm{U}_{f,g}) \\
& & \to \mathrm{T_V}\ (f\ (\widehat{\mathrm{T}}_{f,g}\ x)\ (\widehat{\mathrm{T}}_{f,g} \circ y)) \\
& & \to \mathrm{U}_{f,g}
\end{array}
$$

Mahlo Universe in Agda

$$
\begin{array}{lcl}
\multicolumn{3}{l}{\widehat{\mathrm{T}}\ (f : (x : \mathrm{V}) \to (\mathrm{T}_{\mathrm{V}}\ x \to \mathrm{V}) \to \mathrm{V})} \\
\multicolumn{3}{l}{\quad (g : (x : \mathrm{V}) \to (\mathrm{T}_{\mathrm{V}}\ x \to \mathrm{V}) \to (\mathrm{T}_{\mathrm{V}}\ (f\ x\ y) \to \mathrm{V})\mathrm{V})} \\
\multicolumn{3}{l}{\quad : \mathrm{U}_{f,g} \to \mathrm{V}} \\
\widehat{\mathrm{T}}_{f,g}\ (\widehat{\Pi}\ a\ b) & = & \widehat{\Pi}\ (\widehat{\mathrm{T}}_{f,g}\ a)\ (\widehat{\mathrm{T}}_{f,g} \circ b) \\
\cdots & & \\
\widehat{\mathrm{T}}_{f,g}\ (\widehat{\mathrm{f}}\ a\ b) & = & f\ (\widehat{\mathrm{T}}_{f,g}\ a)\ (\widehat{\mathrm{T}}_{f,g} \circ b) \\
\widehat{\mathrm{T}}_{f,g}\ (\widehat{\mathrm{g}}\ a\ b\ c) & = & g\ (\widehat{\mathrm{T}}_{f,g}\ a)\ (\widehat{\mathrm{T}}_{f,g} \circ b)\ c
\end{array}
$$

Problem of Mahlo Universe

- Elements of V are constructed, depending on total functions

$$f : \mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})$$

- So we introduce elements of V by referring to the totality of V.
- Justification is possible.
- However, many type theoretist doubt the validity of the Mahlo universe as a foundation of mathematics.

Problem of Mahlo Universe

- This reference to the totality of the functions, and therefore impredicativity, can be avoided by observing that for defining U_f, only the restriction of f to $\mathrm{Fam}(\mathrm{U}_f)$ is needed to be total.
- However, in order to make sense of U_f which are not total, we need to refer to arbitrary **partial** functions f.
  - For partial functions f we construct U_f.
  - If we can complete it, i.e. f is total, then we add $\widehat{\mathrm{U}}_f$ to V.
- Problem: In Martin-Löf Type Theory all functions are total.
- We use therefore Feferman's explicit mathematics where we have access to the collection of untyped programs.

External vs Internal Mahlo Universe

- The External Mahlo Universe is a slightly weaker variant of the Mahlo universe (also called Internal Mahlo universe).
- Instead of formalisation that there exist a set V with the Mahlo property, one formalises that the collection of sets Set has this property.
- Usually considered as unproblematic since Set is considered as open ended.
- Often one talks about
  - the external Mahlo universe as the **green Mahlo universe**
  - the internal Mahlo universe as the red Mahlo universe

Simplification of the Mahlo Universe

- One can consider a reformulation of the Mahlo universe, where the subuniverses U_f are not closed under universe constructions.
  - Except for the identity type.
- However, one needs to add an extra parameter a, a family of sets, which are represented in $\mathrm{U}_{a,f}$.
  - Otherwise U_f would be empty.
- For every endofunction f on families of sets you can find an a', and f' such that $\mathrm{U}_{a',f'}$ is closed under f and universe constructions.

Explicit Mathematics

- Framework introduced by Solomon Feferman in order to formalise constructive mathematics.
  - Later classical logic was added.
- It is an untyped alternative to type theory.
- In its intuitionistic form could be deveoped into an alternative to Martin-Löf type theory as a foundation of mathematics.
- It is presented as a second order language, where first order objects (individuals) can be considered as programs or terms.
- Second order quantifiers range over sets which have names, where names are specific individuals.
- Use of second order quantifiers could be avoided.

Explicit Mathematics

- Types can be named by individuals, and we have a naming relation

$$\Re(x, U)$$

and have that every type

$$\forall U.\exists x.\Re(x, U)$$

- We define now

$$\begin{aligned} \Re(s) &:= \exists X.\Re(s, X), \\ s \mathrel{\dot{\in}} t &:= \exists X.\Re(t, X) \land s \in X, \end{aligned}$$

Explicit Mathematics

$$
\begin{array}{lcl}
\Re(s) & := & \exists X.\Re(s, X), \\
s \mathrel{\dot{\in}} t & := & \exists X.\Re(t, X) \wedge s \in X, \\
\exists x \mathrel{\dot{\in}} s.\phi(x) & := & \exists x.x \mathrel{\dot{\in}} s \wedge \phi(x), \\
\forall x \mathrel{\dot{\in}} s.\phi(x) & := & \forall x.x \mathrel{\dot{\in}} s \rightarrow \phi(x), \\
s \mathrel{\dot{\subset}} t & := & \forall x \mathrel{\dot{\in}} s.x \mathrel{\dot{\in}} t, \\
s \doteq t & := & s \mathrel{\dot{\subset}} t \wedge t \mathrel{\dot{\subset}} s, \\
\Re_{\Re}(s) & := & \Re(s) \wedge \forall x \mathrel{\dot{\in}} s.\Re_{\Re}(x), \\
f \in (\Re \rightarrow \Re) & := & \forall x.\Re(x) \rightarrow \Re(f\,x), \\
f \in (s \rightarrow s) & := & \forall x.x \mathrel{\dot{\in}} s \rightarrow f\,x \mathrel{\dot{\in}} s, \\
f \in (s^2 \rightarrow s) & := & \forall x, y.x \mathrel{\dot{\in}} s \wedge y \mathrel{\dot{\in}} s \rightarrow f\,(x, y) \in .
\end{array}
$$

Explicit Mathematics

- We can avoid the from a foundational point of view problematic 2nd order language by rewriting it in a first order setting by having a relation

$$\Re(s)$$

selecting individuals which are names for sets and a relation

$$s \mathrel{\dot{\in}} t$$

together with

$$s \mathrel{\dot{\in}} t \rightarrow \Re(t)$$

Inductive Generation

- Inductive generation in Explicit Mathematics defines the accessible part of a relation.
- It plays the rôle of the W-type in type theory.
- It is axiomatised as follows:
  Define

$$\begin{array}{lcl} x \prec_b y & := & (x, y) \mathrel{\dot\in} b \\ \mathcal{C}l_{\mathsf{i}}(a, b, c) & := & \forall x \mathrel{\dot\in} a.(\forall y \mathrel{\dot\in} a. y \prec_b x \to y \mathrel{\dot\in} c) \to x \mathrel{\dot\in} c \end{array}$$

Assume $\Re(a) \ \wedge \ \Re(b)$ Then

$\Re(\mathsf{i}\,(a, b))$

$\mathcal{C}l_{\mathsf{i}}(a, b, \mathsf{i}\,(a, b))$

$\mathcal{C}l_{\mathsf{i}}(a, b, \varphi) \to \forall x \mathrel{\dot\in} \mathsf{i}\,(a, b).\varphi(x)$

Universes

- Universes are names, the elements of which are names, and which are closed under the standard constructs of explicit mathematics for forming sets:

$$
\begin{array}{lll}
a \in \Gamma_{\mathrm{univ}}(u) := & & \\
\quad a = \mathsf{nat} & & \text{(natural numbers)} \\
\quad \vee & a = \mathrm{id}, & \text{(identity)} \\
\quad \vee & (\exists x \mathrel{\dot{\in}} u.a = \mathsf{co}\, x) & \text{(complement)} \\
\quad \vee & (\exists x, y \mathrel{\dot{\in}} u.a = \mathsf{int}\,(x, y)) & \text{(intersection)} \\
\quad \vee & (\exists x \mathrel{\dot{\in}} u.a = \mathsf{dom}\, x) & \text{(domain)} \\
\quad \vee & (\exists f.\exists x \mathrel{\dot{\in}} u.a = \mathsf{inv}\,(f, x)) & (f^{-1}(x)) \\
\quad \vee & (\exists x \mathrel{\dot{\in}} u.\exists f \mathrel{\dot{\in}} (x \to u).a = \mathsf{j}\,(x, f)) & (\Sigma\text{-type})
\end{array}
$$

- Now define with

$$\Re_{\Re}(u) := \Re(u) \ \wedge \ (\forall x \mathrel{\dot{\in}} u.\Re(x))$$

$$\mathcal{U}(u) := \Re_{\Re}(u) \ \wedge \ \forall x \in \Gamma_{\mathrm{univ}}(u) \to x \mathrel{\dot{\in}} u$$

Pre-Universes

- We want to formalise the idea of
  - Whenever we can form a subuniverse U_f closed under f of the Mahlo universe, then we have an element $\widehat{\mathrm{U}}_f$ of the Mahlo universe representing U_f.
- In explicit mathematics traditionally the subuniverses of the Mahlo universe have an additional parameter a, determining an element contained in it.
- In order to formalise this idea we first formulate the notion of a pre universe $\mathrm{pu}\,(a, f, v)$.
- $\mathrm{pu}\,(a, f, v)$ is a universe containing a and closed under f, provided the elements created are in v.
- So it is a subuniverse of v.

Pre-Universes

- We first define the set of potential elements of the pre universe:

$$x \in \Gamma^{\mathrm{pot}}_{\mathrm{pu}}(a, f, u) \quad := \quad x \in \Gamma_{\mathrm{univ}}(u) \ \vee \ x = a \ \vee \ \exists y \mathrel{\dot{\in}} u.x = f\ y$$

- The closure property of a pre universe is now:

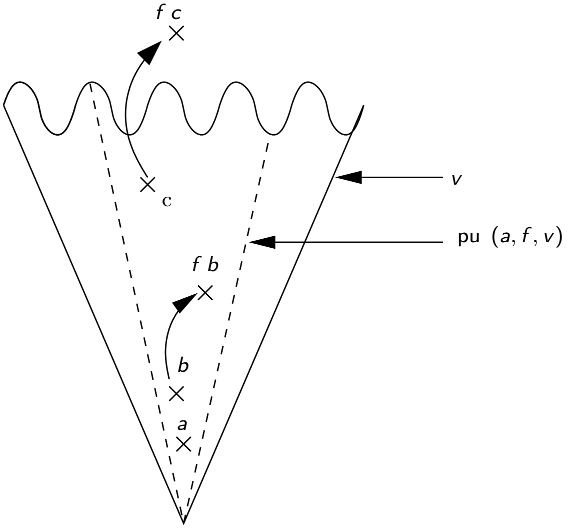
$$\mathcal{C}l_{\mathrm{pu}}\,(a, f, u, v) \quad := \quad \forall x \in \Gamma^{\mathrm{pot}}_{\mathrm{pu}}(a, f, u) \ \wedge \ x \mathrel{\dot{\in}} v \to x \mathrel{\dot{\in}} u$$

- Now we formalise that $\mathrm{pu}\,(a, f, v)$ is the least pre-universe closed under a, f relative to v:
  Assume $\Re_{\Re}(v)$:

$$\begin{array}{l} \mathcal{C}l_{\mathrm{pu}}\,(a, f, \mathrm{pu}\,(a, f, v), v) \\ \mathcal{C}l_{\mathrm{pu}}\,(a, f, \varphi, v) \to \forall x \mathrel{\dot{\in}} \mathrm{pu}\,(a, f, v).\varphi(x) \end{array}$$

  We will only need the pre universe for v being the Mahlo universe.

pu (a,f,v)



Indep(a,f,u,v)

- That we have a "subuniverse closed under a and f" corresponds now to the fact that the pre universe is independent of v:
  - The condition that the elements we add to the pre-universe need to be in v is always fulfilled, i.e. the pre universe is independent of v:

$$\mathrm{Indep}(a, f, u, v) \quad := \quad \forall x \in \Gamma^{\mathsf{pot}}_{\mathsf{pu}}(a, f, u).x \mathrel{\dot{\in}} v$$

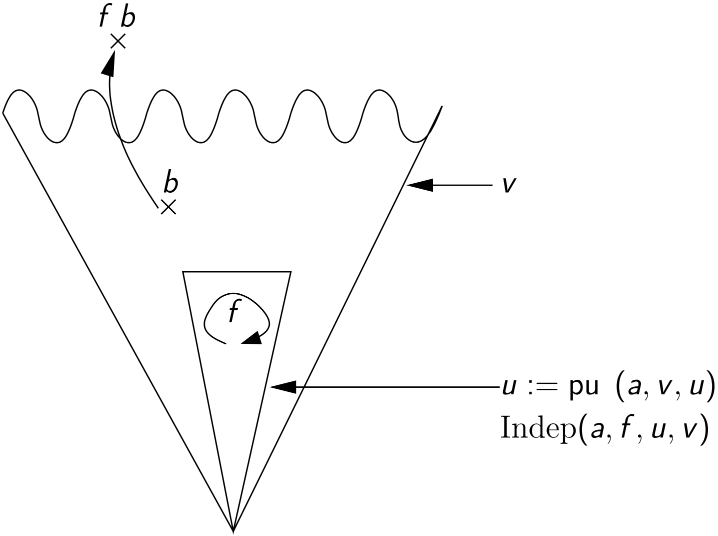
  - Once we have

$$\mathrm{Indep}(a, f, \mathsf{pu}\ (a, f, \mathrm{M}), \mathrm{M})$$

    we know that $\mathsf{pu}\ (a, f, \mathrm{M})$ does not depend on future elements of M introduced.
  - We get immediately

$$\mathrm{Indep}(a, f, \mathsf{pu}\ (a, f, v), v) \rightarrow \forall x \in \Gamma^{\mathsf{pot}}_{\mathsf{pu}}(a, f, u, x).x \mathrel{\dot{\in}} \mathsf{pu}\ (a, f, v)$$

Indep(a,f,u,v)

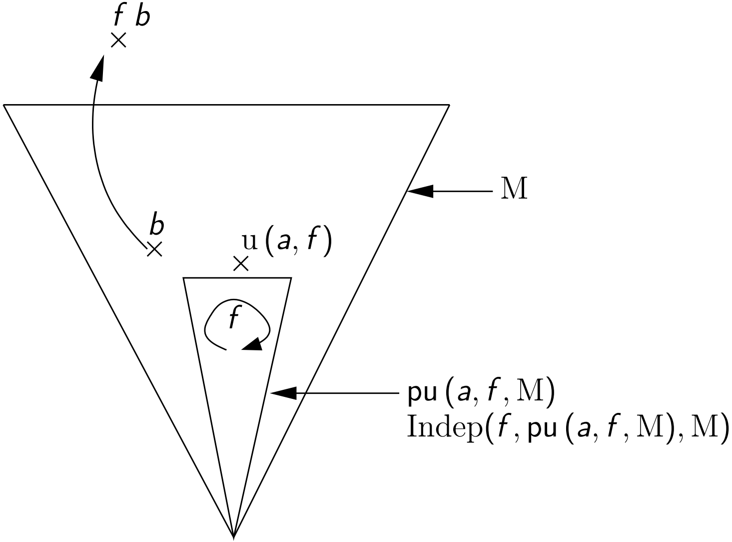


The Extended Predicative Mahlo Universe

- The extended predicative Mahlo universe is a universe, closed under inductive generation, such that for a, f we have
  - If $\mathsf{pu}\,(a, f, \mathrm{M})$ is independent of M, then we have an element $\mathrm{u}\,(a, f)$ of M representing $\mathsf{pu}\,(a, f, \mathrm{M})$.

$$
\begin{array}{l}
\mathcal{U}(\mathrm{M}) \\
\forall a, b \,\dot{\in}\, \mathrm{M}.\mathrm{i}\ \ (a, b) \in \mathrm{M} \\
\forall a, f.\mathrm{Indep}(a, f, \mathsf{pu}\,(a, f, \mathrm{M}), \mathrm{M}) \to \Re(\mathrm{u}\,(a, f)) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \wedge\, \mathrm{u}\,(a, f) \,\dot{\in}\, \mathrm{M} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \wedge\, \mathrm{u}\,(a, f) \doteq \mathsf{pu}\,(a, f, \mathrm{M}) \ \ .
\end{array}
$$

The Extended Predicative Mahlo Universe



The Least Mahlo Universe

We can formulate the notion of a least Mahlo universe as follows:

$$
\begin{array}{l}
\mathcal{U}(u) \wedge \\
(\forall a, b \,\dot{\in}\, u.\mathrm{i}\,(a, b) \,\dot{\in}\, u) \wedge \\
(\forall f, a.\mathrm{Indep}(a, f, \mathrm{pu}\,(a, f, u), u) \rightarrow \mathrm{u}\,(a, f) \,\dot{\in}\, u) \\
\rightarrow \mathrm{M} \,\dot{\subset}\, u \quad .
\end{array}
$$

Note that there are no consistent elimination rules for the standard Mahlo universe.

Axiomatic Mahlo in Explicit Mathematics

- Normally by the Mahlo universe in explicit mathematics one means the external Mahlo universe where $\Re$ has the property of Mahloness.
- One can formalise an extended predicative external Mahlo universe as well.
- Tupailo formalised the internal axiomatic Mahlo universe in explicit mathematics as follows:

$$
\begin{array}{l}
\mathcal{U}(\mathrm{M}) \\
\forall a, b \mathrel{\dot\in} \mathrm{M}.\mathsf{i}\,(a, b) \mathrel{\dot\in} \mathrm{M} \\
a \mathrel{\dot\in} \mathrm{M} \ \wedge\ f \in (\mathrm{M} \to \mathrm{M}) \to \mathcal{U}(\mathrm{u}\,(a, f)) \\
\qquad\qquad\qquad\qquad\qquad \wedge\ \mathrm{u}\,(a, f) \mathrel{\dot\subset} \mathrm{M}, \\
\qquad\qquad\qquad\qquad\qquad \wedge\ a \mathrel{\dot\in} \mathrm{u}\,(a, f) \\
\qquad\qquad\qquad\qquad\qquad \wedge\ f \in (\mathrm{u}\,(a, f) \to \mathrm{u}\,(a, f)) \\
\qquad\qquad\qquad\qquad\qquad \wedge\ \mathrm{u}\,(a, f) \mathrel{\dot\in} \mathrm{M}
\end{array}
$$

Extended Predicative Mahlo Universe is a Mahlo Universe

Theorem

The extended predicative Mahlo universe fulfils the axioms of the axiomatic Mahlo universe.

Basic Setup for the Model

- We define the model as subset of $\mathbb{N}^3$.
  If $P \subseteq \mathbb{N}^3$, we interpret $\Re$, $\dot{\in}$, $\dot{\notin}$ as follows:

$$\begin{array}{ll} \Re_P(a) & := P(a,0,0), \\ b \in_P a & := P(a,b,1), \\ b \notin_P a & := P(a,b,2). \end{array}$$

- On the other hand from relations R, $\in'$ $\notin'$ we can define a subset of $\mathbb{N}^3$:

$$\begin{array}{ll} \mathrm{Pred}_{\Re,\in',\notin'}(a,b,c) & := (R(a) \ \wedge \ b = 0 \ \wedge \ c = 0) \vee \\ & \quad\ (a \in' b \ \wedge \ c = 1) \vee \\ & \quad\ (a \notin' b \ \wedge \ c = 2). \\ \mathrm{Pred}(R, \in', \notin') & := \{(a,b,c) \mid \mathrm{Pred}_{R,\in',\notin'}(a,b,c)\} \end{array}$$

- The model will be defined by iterating an operator up to a large ordinal.

Operator for Defining the Model - j

$$
\begin{array}{lcl}
\Re^{\mathsf{j}}_{P}(a,u,f) & := & a = \widehat{\mathsf{j}}(u,f) \ \wedge\ \Re_{P}(u) \wedge \\
 & & (\forall^{\mathsf{pos}} x \in_{P} u.\exists z.\{f\}(x) \simeq z \ \wedge\ \Re_{P}(z)) \\
\Re^{\mathsf{j},+}_{P}(a) & := & \exists u,f.\Re^{\mathsf{j}}_{P}(a,u,f) \\
b \in^{\mathsf{j}}_{P} a & := & \exists u,f.\Re^{\mathsf{j}}_{P}(a,u,f) \ \wedge\ \exists y,z,z'.b = \langle y,z\rangle \ \wedge\ y \in_{P} u \\
 & & \qquad\qquad\qquad\qquad\qquad\qquad \wedge\ \{f\}(y) \simeq z' \ \wedge\ z \in_{P} z' \\
b \notin^{\mathsf{j}}_{P} a & := & \exists u,f.\Re^{\mathsf{j}}_{P}(a,u,f) \\
 & & \quad \wedge\ \forall y,z,z'.b = \langle y,z\rangle \\
 & & \qquad\qquad \rightarrow (y \notin_{P} u \vee \{f\}(y) \not\simeq z' \vee z \notin_{P} z') \\
\Gamma^{\mathsf{j}}(P) & := & \mathrm{Pred}(\Re^{\mathsf{j},+}_{P}, \in^{\mathsf{j}}_{P}, \notin^{\mathsf{j}}_{P})
\end{array}
$$

Operator for Defining the Model - i

$$
\begin{array}{lcl}
\Re_P^{\mathrm{pre-i}}(a,u,v) & := & a = \widehat{\mathrm{i}}(u,v) \ \wedge \ \Re_P(u) \ \wedge \ \Re_P(v) \\
b \in_P^{\mathrm{i}} a & := & \exists u,v.\Re_P^{\mathrm{pre-i}}(a,u,v) \ \wedge \ b \in_P u \\
 & & \qquad \wedge \ \forall^{\mathrm{pos}} x \in_P u.\langle x,b\rangle \in_P v \rightarrow^{\mathrm{pos}} x \in_P a \\
\mathcal{C}l_P^{\mathrm{i}}(u,v) & := & \forall x \in_P^{\mathrm{i}} \widehat{\mathrm{i}}(u,v).x \in_P \widehat{\mathrm{i}}(u,v) \\
\Re_P^{\mathrm{i}}(a,u,v) & := & \Re_P^{\mathrm{pre-i}}(a,u,v) \ \wedge \ \mathcal{C}l_P^{\mathrm{i}}(u,v) \\
\Re_P^{\mathrm{i},+}(a) & := & \exists u,v.\Re_P^{\mathrm{i}}(a,u,v) \\
b \notin_P^{\mathrm{i}} a & := & \exists u,v.\Re_P^{\mathrm{i}}(a,u,v) \ \wedge \ b \notin_P a \\
\Gamma^{\mathrm{i}}(P) & := & \mathrm{Pred}(\Re_P^{\mathrm{i},+}, \in_P^{\mathrm{i}}, \notin_P^{\mathrm{i}})
\end{array}
$$

Operator for Defining the Model - pu

$$
\begin{array}{lcl}
\Re_P^{\mathrm{pre-pu}}(a', a, f, v) & := & a' = \widehat{\mathrm{pu}}(a, f, v) \\
b \in_P^{\mathrm{pre,pot}} a' & := & \exists a, f, v.\Re_P^{\mathrm{pre-pu}}(a', a, f, v) \\
 & & \qquad \wedge (b = a \vee (\exists x \in_P a'.b \simeq \{f\}(x)) \\
 & & \qquad\quad \vee \Gamma_P^{\mathrm{univ}}(\widehat{\mathrm{pu}}(a, f, v), b)) \\
b \in_P^{\mathrm{pu}} a' & := & \exists a, f, v.b \in_P^{\mathrm{pu,pot}} a' \wedge \Re_P^{\mathrm{pre-pu}}(a', a, f, v) \wedge b \in_P v \\
\mathcal{C}l_P^{\mathrm{pu}}(a, f, v) & := & \forall b \in_P^{\mathrm{pu}} \widehat{\mathrm{pu}}(a, f, v).b \in_P \widehat{\mathrm{pu}}(a, f, v) \\
\mathrm{Indep}_P^{\mathrm{pu}}(a', v) & := & \exists a, f.\Re_P^{\mathrm{pu}}(a', a, f, v) \wedge \forall b \in_P^{\mathrm{pu,pot}} a'.b \in_P v \\
\Re_P^{\mathrm{pu,pot}}(a', a, f, v) & := & \Re_P^{\mathrm{pre-pu}}(a', a, f, v) \\
 & & \wedge (\Re_P(v) \vee \mathrm{Indep}_P^{\mathrm{pu}}(a', v)) \\
\Re_P^{\mathrm{pu}}(a', a, f, v) & := & \Re_P^{\mathrm{pu,pot}}(a', a, f, v) \wedge \mathcal{C}l_P^{\mathrm{pu}}(a, f, v) \\
\Re_P^{\mathrm{pu},+}(a') & := & \exists a, f, v.\Re_P^{\mathrm{pu}}(a', a, f, v) \\
b' \notin_P^{\mathrm{pu}} a' & := & \exists a, f, v.\Re_P^{\mathrm{pu}}(a' a, f, v) \wedge \neg(b' \in_P a') \\
\Gamma^{\mathrm{pu}}(P) & := & \mathrm{Pred}(\Re_P^{\mathrm{pu},+}, \in_P^{\mathrm{pu}}, \notin_P^{\mathrm{pu}})
\end{array}
$$

Operator for Defining the Model - u

$$
\begin{array}{lcl}
\Re_P^{\mathsf{u,pre}}(a', a, f) & := & a' = \widehat{\mathsf{u}}(a, f) \\
\Re_P^{\mathsf{u,pot}}(a', a, f) & := & \Re_P^{\mathsf{u,pre}}(a', a, f) \ \wedge \ \mathrm{Indep}_P^{\mathsf{pre}}(\widehat{\mathsf{pu}}(a, f, \mathrm{M}), \mathrm{M}) \\
\Re_P^{\mathsf{u,next}}(a', a, f) & := & \Re_P^{\mathsf{u,pot}}(a', a, f) \ \wedge \ \mathcal{C}l^{\mathsf{pu}}(\widehat{\mathsf{pu}}(a, f, \mathrm{M}), \mathrm{M}) \\
\Re_P^{\mathsf{u}}(a', a, f) & := & \Re_P^{\mathsf{u,next}}(a', a, f) \ \wedge \ \forall x \in_P \widehat{\mathsf{pu}}(a, f, \mathrm{M}). x \in_P \widehat{\mathsf{u}}(a, f) \\
\Re_P^{\mathsf{u},+}(a') & := & \exists a, f. \Re_P^{\mathsf{u}}(a', a, f) \\
b \in_P^{\mathsf{u}} a' & := & \exists a, f. \Re_P^{\mathsf{u,next}}(a', a, f) \\
 & & \qquad \wedge \ b \in_P \widehat{\mathsf{pu}}(a, f, \mathrm{M}) \\
b \notin_P^{\mathsf{u}} a' & := & \exists a, f. \Re_P^{\mathsf{u}}(a', a, f) \\
 & & \qquad \wedge \ b \notin_P \widehat{\mathsf{pu}}(a, f, \mathrm{M}) \\
\Gamma^{\mathsf{u}}(P) & := & \mathsf{Pred}(\Re_P^{\mathsf{u},+}, \in_P^{\mathsf{u}}, \notin_P^{\mathsf{u}})
\end{array}
$$

Operator for Defining the Model - M

$$
\begin{array}{lcl}
\Re^{\mathsf{pre-M}}_{P}(a) & := & a = \mathrm{M} \\
b \in^{\mathrm{M}}_{P} a & := & \Re^{\mathsf{pre-M}}_{P}(a) \\
 & & \wedge\ (\Gamma^{\mathsf{univ}}_{P}(\mathrm{M}, b) \vee \Gamma^{\mathsf{i}}_{P}(\mathrm{M}, b) \vee \exists a, f.\Re^{\mathsf{u}}_{P}(b, a, f)) \\
\mathcal{C}l^{\mathrm{M}}_{P} & := & \forall b \in^{\mathrm{M},\mathsf{pot}}_{P} \mathrm{M}.b \in_{P} \mathrm{M} \\
\Re^{\mathrm{M}}_{P}(a) & := & \Re^{\mathsf{pre-M}}_{P}(a)\ \wedge\ \mathcal{C}l^{\mathrm{M}}_{P} \\
b \notin^{\mathrm{M}}_{P} a & := & \Re^{\mathrm{M}}_{P}(a)\ \wedge\ \neg(b \in_{P} \mathrm{M}) \\
\Gamma^{\mathrm{M}}(P) & := & \mathsf{Pred}(\Re^{\mathrm{M}}_{P}, \in^{\mathrm{M}}_{P}, \notin^{\mathrm{M}}_{P})
\end{array}
$$

Monotonicity of Operators

Definition

1. For P, Q ternary predicates we define

$$\begin{array}{rcl} P \preceq Q & :\Leftrightarrow & \Re_P \subseteq \Re_Q \\ & & \wedge \forall a, b.(a \in_P b \rightarrow a \in_Q b) \\ & & \qquad \wedge (a \notin_P b \rightarrow a \notin_Q b) \\ & & \qquad \wedge (\Re_P(b) \rightarrow (a \in_P b \leftrightarrow a \in_Q b) \wedge \\ & & \qquad\qquad (a \notin_P b \leftrightarrow a \notin_Q b)) \end{array}$$

2. Let $\mathcal{B} : \mathcal{P}(\mathbb{N}^3) \rightarrow \mathcal{P}(\mathbb{N}^3)$.
   $\mathcal{B}$ is $\preceq$-monotone iff

$$\forall P, Q.P \preceq Q \rightarrow \mathcal{B}(P) \preceq \mathcal{B}(Q)$$

Monotonicity of Operators

Definition

1. Let $\mathcal{B} : \mathcal{P}(\mathbb{N}^3) \to \mathcal{P}(\mathbb{N}^3)$.
   $\mathcal{B}$ is weakly $\preceq$-monotone iff

$$\forall P, Q.(P \preceq Q \wedge \mathsf{El}_P \restriction \Re_{\mathcal{B}(P)} = \mathsf{El}_Q \restriction R_{\mathcal{B}(P)}) \to \mathcal{B}(P) \preceq \mathcal{B}(Q)$$

2. $\mathcal{B} : \mathcal{P}(\mathbb{N}^3) \to \mathcal{P}(\mathbb{N}^3)$ is a closure operator if $\mathcal{B}$ is weakly monotone and

$$\mathsf{El}_{\mathcal{B}(P)} \restriction (\Re_{\mathcal{B}(P)} \setminus \Re_P) \subseteq \mathsf{El}_P \restriction (\Re_{\mathcal{B}(P)} \setminus \Re_P)$$

3. $\mathcal{B} : \mathcal{P}(\mathbb{N}^3) \to \mathcal{P}(\mathbb{N}^3)$ is inflationary, iff

$$(P \preceq \mathcal{B}(P)) \to (\mathcal{B}(P) \preceq \mathcal{B}^2(P))$$

Lemma

Lemma

1. Γ^{c} *is monotone for each basic set constructor* c.
2. Γ^{i} *is a closure operator.*
3. Γ^{pu} *is a closure operator.*
4. $\Gamma^{\mathsf{u}} \cup \Gamma^{\mathsf{pu}}$ *is a closure operator.*
5. $\Gamma^{\mathsf{i}} \cup \Gamma^{\mathsf{u}} \cup \Gamma^{\mathsf{pu}} \cup \Gamma^{\mathrm{M}}$ *is a closure operator.*

Lemma

The following lemma still needs to be scrutinised:

Lemma

Let Γ *be the overall operator.*

1. *If* κ *is an admissible,* $\Re_{\Gamma^{<\kappa}}(a), \Re_{\Gamma^{<\kappa}}(b)$*. Then* $\Re_{\Gamma^{\kappa}}(\widehat{\mathsf{i}}(a,b))$*.*
2. *If* κ *is an admissible,* $\mathrm{Indep}^{\mathsf{pre}}_{\Gamma^{<\kappa}}(\widehat{\mathsf{pu}}(a,f,u),u)$*, then* $\Re_{\Gamma^{\kappa}}(\widehat{\mathsf{pu}}(a,f,u))$*.*
3. *Let* Mahlo *be a recursively Mahlo ordinal*
   If $\mathrm{Indep}^{\mathsf{pre}}_{\Gamma^{<\mathsf{Mahlo}}}(\widehat{\mathsf{pu}}(a,f,\mathrm{M}),\mathrm{M})$*, then* $\Re_{\Gamma^{<\mathsf{Mahlo}}}(\widehat{\mathsf{pu}}(a,f,\mathrm{M}))$*,*
   $\Re_{\Gamma^{<\mathsf{Mahlo}}}(\widehat{\mathsf{u}}(a,f))$*, and* $\widehat{\mathsf{u}}(a,f) \in_{\Gamma^{<\mathsf{Mahlo}}} \mathrm{M}$*.*
4. $\mathrm{M} \in \Gamma^{\mathsf{Mahlo}}$*.*
5. *Let* κ *be a recursively inaccessible ordinal above* Mahlo*.*
   Then Γ^{κ} *is a model of the extended predicative Mahlo universe.*

Future Work

- Carry out the model in an extension of KPM, should give an upper bound of the proof theoretic strength of the extended predicative Mahlo universe.
  - Upper bound should be KPI + (M).
- Restriction of the model to external extended predicative Mahlo universe.
  - Expected strength should be that of KPM.
- Adapt the well-ordering proof for Type theory + Mahlo universe to obtain a lower bound for the proof theoretic strength.

Reformulation of the Extended Predicative Mahlo Universe

- Reformulate explicit mathematics and the extended predicative Mahlo universe using introduction and elimination rules.
- This would as well a step towards proof assistants based on Explicit Mathematics.
- First step: Formalisation in Agda.

Example Formalisation in Agda

```
{-# OPTIONS --no-positivity-check #-}

-- This is only a partial formalisation of
-- the extended predicative Mahlo universe
-- especially closure under univeres operations
-- and equality of terms
-- is missing


data Term : Set where
 nat zero :  Term
 m     : Term
 suc   : Term → Term
 ap : (s t   : Term) → Term
 pu : (a  f   : Term) → Term
```

Example Formalisation in Agda

```
data Nat : Term → Set where
  zerop : Nat zero
  sucp  : (t : Term) → Nat t → Nat (suc t)

data M  : Term → Set where
  u  :  (a f  : Term) →
        (ma : M a) →
        (indep : (x : Term) (xpu : PU a f x) → M (ap f x))
        → M (pu a f)
```

Example Formalisation in Agda

```
data R : Term → Set where
  natp : R nat
  up   : (a f  : Term) →
         (ma : M a) →
        (indep : (x : Term) (xpu : PU a f x) → M (ap f x))
        → R (pu a f)
  Mp : R m

MR  : (t : Term) → M t → R t
MR .(pu a f) (u a f m′ indep) = up a f m′ indep
```

Example Formalisation in Agda

```
data PU  (a f : Term) : (x : Term) →  Set  where
  aproof : (m : M a) →  PU a f a
  fproof : (x : Term) → PU a f x → M (ap f x) → PU a f (ap f x)
```

```
data  U  (a f : Term)
         (ma  : M a)
         (indep : (x : Term) (xpu : PU a f x) → M (ap f x))
         : (x : Term) → Set  where
  aproof :  U a f ma indep a
  fproof  :  (x : Term) → U a f ma indep x
             → U a f ma indep (ap f x)
```

Example Formalisation in Agda

```
U2PU  :  (a f : Term)
         (ma  : M a)
         (indep : (x : Term) (xpu : PU a f x) → M (ap f x))
         (x : Term)
         (up : U a f ma indep x)
         → PU a f x
U2PU a f ma indep .a aproof = aproof ma
U2PU a  f ma indep .(ap f x) (fproof x up') =
         fproof  x (U2PU a f ma indep x up')
                   (indep x (U2PU a f ma indep x up'))
```

Example Formalisation in Agda

$$\mathsf{PUM} : (a\ \ f\ x : \mathsf{Term}) \to (ispu : \mathsf{PU}\ a\ \ f\ x) \to \mathsf{M}\ x$$
$$\mathsf{PUM}\ a\ \ f\ .a\ (\mathsf{aproof}\ m') = m'$$
$$\mathsf{PUM}\ a\ \ f\ .(\mathsf{ap}\ \ f\ x_1)\ (\mathsf{fproof}\ x_1\ \ ispu\ x) = x$$

$$\begin{aligned}
\mathsf{U2M}\ \ :\ \ & (a\ \ f : \mathsf{Term}) \\
& (ma\ \ : \mathsf{M}\ \ a) \\
& (indep : (x : \mathsf{Term})\ (xpu : \mathsf{PU}\ a\ \ f\ x) \to \mathsf{M}\ (\mathsf{ap}\ \ f\ x)) \\
& (x : \mathsf{Term}) \\
& (up : \mathsf{U}\ \ a\ \ f\ \ ma\ indep\ x) \\
& \to \mathsf{M}\ x
\end{aligned}$$
$$\mathsf{U2M}\ a\ \ f\ \ ma\ indep\ x\ up' = \mathsf{PUM}\ \ a\ \ f\ \ x\ (\mathsf{U2PU}\ \ a\ \ f\ \ ma\ indep\ x\ up')$$

Example Formalisation in Agda

$$\mathsf{PUM} : (a\ \ f\ x : \ \mathsf{Term}) \to (ispu : \mathsf{PU}\ a\ \ f\ \ x) \to \mathsf{M}\ x$$
$$\mathsf{PUM}\ a\ \ f\ .a\ (\mathsf{aproof}\ m') = m'$$
$$\mathsf{PUM}\ a\ \ f\ .(\mathsf{ap}\ \ f\ x_1)\ (\mathsf{fproof}\ x_1\ \ ispu\ x) = x$$

$$\begin{aligned}\mathsf{U2M}\ \ :\ \ & (a\ \ f\ :\ \mathsf{Term}) \\ & (ma\ \ :\ \mathsf{M}\ \ a) \\ & (indep : (x : \ \mathsf{Term})\ (xpu : \mathsf{PU}\ a\ \ f\ \ x) \to \mathsf{M}\ (\mathsf{ap}\ \ f\ \ x)) \\ & (x : \ \mathsf{Term}) \\ & (up : \mathsf{U}\ \ a\ \ f\ \ ma\ indep\ x) \\ & \to \mathsf{M}\ \ x\end{aligned}$$
$$\mathsf{U2M}\ a\ \ f\ \ ma\ indep\ x\ up' = \mathsf{PUM}\ a\ \ f\ \ x\ (\mathsf{U2PU}\ a\ \ f\ \ ma\ indep\ x\ up')$$