The limits of the Curry-Howard isomorphism

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The well-known Curry-Howard isomorphism relates functions with proofs and can be considered as one of the conceptional bases of Martin-Löf's type theory.

For our considerations, the crucial correspondence is the one between (intuitionistic) proofs of an implication $A \to B$ and functions of the type $A \to B$. To make sense out of this correspondence, the functions need to be, of course, *total*, i.e., for every element of A the function need to associate an element of B.

Nontheless that totality of functions is a desirable property, it does not match with computational reality. It is not only the case that non-terminating computations appear natural; even more importantly, modern recursion theory gains its power from the admission of *partial* functions.

.... partial recursive functions etc pp - auch Zitat von Kleene bzgl. Diagonalisierung von "Church's These" ...

Partial functions are easily integrated in *type-free* contexts, and we may address shortly some intresting historical considerations regarding *functional self-application*, cf. [**?**]. The main of this talk is, however, to illustrate the role of partial functions in the formalization of Anton Setzer's *extended predicative Mahlo universe*, [KS10].

A Mahlo universe is ....

Axiomatically, such a Mahlo universe can be introduced roughly as follows: ... Strahm...

This axiomatizations is clearly highly impredicative, as (the type of) the function used in the premise refers to the Mahlo universe which is some "to be constructed".

A corresponding axiomatization in Martin-Löf type theory was given by Setzer [Set00], whose formalization is, however, more involved due to the specific technicalities of this type theory, but which suffers from the same impredicativity.

Setzer proposed an alternative construction of a Mahlo universe "from below", which has a predicative flavour. Since the resulting theory will

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proof-theoretically go beyond what is called the limit of predicativity (i.e., theories with the proof-theoretic strength of Γ_0), this approach is dubbed *extended predicative.*

The rough idea is to ....

In doing so, we will construct "sub-universes" of M which are closed under a function f which *need not to be total on M*; thus, f is a partial function (seen from M).

The sub-universes, together with their corresponding functions under which they are closed, are now essential to formulate a sound induction rule for such a Mahlo universe (corresponding to an elimination rule in Martin-Löf type theory) or, what's the same, defining a *least Mahlo universe.*

It turns out that the permission of partial functions is substantial for the axiomatization of a least Mahlo universe, which, hence, cannot be formulated in Martin-Löf type theory.

This result suggests, on the proof-theoretic side, that the least Mahlo universe already exceeds the "limit of Martin-Löf type theory"; on the more conceptional side, it challenges the restriction to total functions in the Curry-Howard correspondence, although the question what the meaning of a "partial implication" could be remains a desideratum.

References

[KS10] Reinhard Kahle and Anton Setzer. An extended predicative definition of the Mahlo universe. In Ralf Schindler, editor, *Ways of Proof Theory*, Ontos Series in Mathematical Logic, pages 309 – 334. Ontos Verlag, 2010.

[Set00] Anton Setzer. Extending Martin-Löf type theory by one Mahlo-universe. *Arch. Math. Log.*, 39:155 – 181, 2000.