Strong Normalization for Intuitionistic Simple Type Theory

Wolfram Pohlers

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The system ITT

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The non logical basic symbols of ITT comprise

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- Symbols for n–ary functions mapping ground object to ground objects

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- If $A \vdash 1$ and $B \vdash 1$ then $A \longrightarrow B \vdash 1$
- If $A \vdash 1$ and τ is a basic type, x basic variable then $(\forall x^\tau)A \vdash 1$.
- If $A \vdash 1$, $\tau_1, \ldots, \tau_n$ are basic types and x is a basic variable then $\{(x^{\tau_1} \ldots x^{\tau_n}) |\ A\} \vdash (\tau_1, \ldots, \tau_n)$

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- Assume $A \vdash 1$, τ is a basic type, x a basic variable and $a \vdash A$ such that x^τ does not occur in any type σ of a variable y^σ occurring freely in a. Then $\lambda x^\tau.a \vdash (\forall x^\tau)A$

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- Let $A \vdash 1$, $\tau_1, \ldots, \tau_n$ be basic types, $t_i \vdash \tau_i$ for $i = 1, \ldots, n$. Then $a \vdash A \binom{t_1, \ldots, t_n}{x^{\tau_1}, \ldots, x^{\tau_n}}$ iff $a \vdash (\{(x^{\tau_1} \ldots x^{\tau_n}) |\ A\} t_1 \ldots t_n)$.

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- If $a \vdash A \rightarrow B$ and $b \vdash A$ then $(ab) \vdash B$
- If τ is a basic type, $a \vdash (\forall x^\tau)A$ and $t \vdash \tau$ then $(at) \vdash A\binom{t}{x^\tau}$

The type of an ITT–term is either a basic type (denoted by $\tau, \tau_1, \ldots$) or a formula denoted by $A, B, \ldots F, G \ldots$
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By $\overline{FV(a)}$ we denote the set of all free variables x^σ occurring in a such that $\sigma \vdash 1$.

A formula A is derivable in simple type theory if there is an $a \vdash A$ such that $\overline{FV(a)} = \emptyset$.

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- $A \wedge B$ is defined by $(\forall x^1)[(A \to (B \to x^1)) \to x^1]$
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- $A \vee B$ is defined by $(\forall x^1)[(A \to x^1) \to ((B \to x^1) \to x^1)]$
- $(\exists x^\tau)F(x^\tau)$ is defined by $(\forall x^1)[(\forall x^\tau)(F(x^\tau) \to x^1) \to x^1]$.

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We define the simple reduction step $a \succ_1 b$ by the following clauses

(R_0) $a \succ_0 b \;\Rightarrow\; a \succ_1 b$

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A term a is normal iff there is no term b such that $a \succ_1 b$.

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- An ITT–term a is well–founded, if every reduction chain for a is finite.

Back

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Theorem

Every closed ITT*–term is well-founded.*

We assume familiarity with the type free lambda calculus λK. As a pecularity we require that there is a constant u in the λ–calculus. Conversion $\succ_0$, simple reduction $\succ_1$ and reduction chains and the reduction relation $a \succ b$ are defined as for ITT.

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- We call a reduction chain strongly normal if it is normal and its last term is strongly normal.

Lemma

If $\mathfrak{R} = \langle c_i |\ i \in I \rangle$ *is a reduction chain for an* λ*–term* (ab) *and* i *is its least index such that* $c_i \succ_1 c_{i+1}$ *is a reduction according to* R_0, *i.e., a conversion, then for all* $j \leq i$ *the term* c_j *has the form* $(a'_j b'_j)$ *such that* $a \succ a'_j$ *and* $b \succ b'_j$

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Lemma (Lemma I)

If a and b are well–founded λ–terms then every reduction chain $\mathfrak{R}$ for (ab) which contains no conversion according to R_0 is strongly normal.

Back to Main Lemma (3.1) Back to Main Lemma (3.2)

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(F_{12}) $a \in \mathfrak{F}^1$ and $a \succ b \;\Rightarrow\; b \in \mathfrak{F}^1$

(F_2) A foundation–predicate of type $(\tau_1, \ldots, \tau_n)$ is a mapping which assigns a foundation–predicate of type 1 to any tuple $\mathfrak{F}_1^{\tau_1}, \ldots, \mathfrak{F}_n^{\tau_n}$ of foundation–predicates of type $\tau_i (i = 1, \ldots, n)$.

The theory IFT

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- If $\mathfrak{F}^\tau$ is a name for a foundation–predicate of type τ, then $\mathfrak{F}^\tau \vdash \tau$.
- Let $\tau_1, \ldots, \tau_n$ be a tuple of basic types, $t \vdash \tau_i$ for $i = 1, \ldots, n$ and $\mathfrak{F}^\tau$ a foundation–predicate of type $\tau = (\tau_1, \ldots, \tau_n)$. Then $(\mathfrak{F}^\tau t_1 \ldots t_n) \vdash 1$.

Evaluating IFT–terms

Defining $|a|$

Definition

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(B_1) For a closed λ–term t we put $t \in |\mathfrak{F}^1|$ if every reduction chain $\mathfrak{R}$ of t which is not strongly normal contains an element $t' \in \mathfrak{F}^1$ which is not preceeded by an L–term in $\mathfrak{R}$.

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(B_2) If $\tau = (\tau_1, \ldots, \tau_n)$ is a basic type, a a closed IFT–term such that $a \vdash \tau$ and $t_1, \ldots, t_n$ are closed IFT-terms with $t_i \vdash \tau_i$ for $i = 1, \ldots, n$, then $t \in |at_1 \ldots t_n|$ holds true iff t is a closed λ–term such that every reduction chain $\mathfrak{R}$ of t which is not strongly normal contains an element $s \in |a|(|t_1|, \ldots, |t_n|)$ which is not preceeded by an L–term in $\mathfrak{R}$.

Definition

(continued)

(B_3) $t \in |A \rightarrow B|$ holds true iff t is a closed λ–term such that every reduction chain $\mathfrak{R}$ of t, which is not strongly normal, contains a first L–term $\lambda x.\, t'$ such that $t' \binom{b}{x} \in |B|$ for all $b \in |A|$

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- (B_4) $t \in |(\forall x^\tau A)|$ holds true iff t is a closed λ–term such that every reduction chain $\mathfrak{R}$ of t, which is not strongly normal, contains a first L–term $\lambda x.\, t'$ such that $t' \binom{u}{x} \in |A \binom{\mathfrak{F}^\tau}{x^\tau}|$ for all foundation–predicates $\mathfrak{F}^\tau$ of type τ.

Definition

(continued)

(B_5) If $\tau := (t_1, \ldots, t_n)$ is a basis type and $\mathfrak{F}^\tau$ is a name for a foundation–predicate of type τ, then then $|\mathfrak{F}^\tau|$ is the mapping defined by $|\mathfrak{F}^\tau|(\mathfrak{F}_1, \ldots, \mathfrak{F}_n) := |\mathfrak{F}^\tau(\mathfrak{F}_1, \ldots, \mathfrak{F}_n)|$ for any tuple $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ of foundation–predicates of types $\tau_1, \ldots, \tau_n$.

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(continued)

(B_5) If $\tau := (t_1, \ldots, t_n)$ is a basis type and $\mathfrak{F}^\tau$ is a name for a foundation–predicate of type τ, then then $|\mathfrak{F}^\tau|$ is the mapping defined by $|\mathfrak{F}^\tau|(\mathfrak{F}_1, \ldots, \mathfrak{F}_n) := |\mathfrak{F}^\tau(\mathfrak{F}_1, \ldots, \mathfrak{F}_n)|$ for any tuple $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ of foundation–predicates of types $\tau_1, \ldots, \tau_n$.

(B_6) If $\tau = (\tau_1, \ldots, \tau_n)$ is a basic type then $|\{(x^{\tau_1} \ldots x^{\tau_n})|\ A\}|(\mathfrak{F}_1, \ldots, \mathfrak{F}_n) := |A \begin{pmatrix} \mathfrak{F}_1, \ldots, \mathfrak{F}_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}|$ for any tuple $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ of foundation–predicates of types $\tau_1, \ldots, \tau_n$.

Lemma

If τ is a basic type and $a \vdash \tau$ is a closed IFT*–term then $|a|$ is a foundation–predicate of type τ.*

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Proof (B_1) Proof (B_2) Proof (B_3) Proof (B_4) Proof (B_5) Proof (B_6)

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Lemma

Let A be an IFT*–sentence and t a λ–functional. If every reduction chain $\mathfrak{R}$ of t, which is not strongly normal, contains an element $t' \in |A|$ which is not preceeded by L–terms in $\mathfrak{R}$ then $t \in |A|$.*

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▸ Proof (atomic case) ▸ Proof ($\longrightarrow$ case) ▸ Proof ($\forall$ case)

Lemma

Let τ *be a basic type and* $a \vdash \tau$ *for a closed* IFT*–term* a*. Then* $|a| = \|a\|$.

Proof

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Let τ be a basic type and $a \vdash \tau$ for a closed IFT*–term a. Then $|a| = \|a\|$.*

▸ Proof

Corollary

Let $FV(a) = \{x^\tau\}$. For any closed IFT*–term $b \vdash \tau$ we obtain $|a\begin{pmatrix} b \\ x^\tau \end{pmatrix}| = |a\begin{pmatrix} |b| \\ x^\tau \end{pmatrix}|$.*

Embedding IFT into λK

Defining the embedding

Definition

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- $\overline{\lambda x^\sigma . a} = \lambda x . \overline{a}$.
- $\overline{(ab)} := (\overline{a}\ \overline{b})$.

Lemma

- $\overline{a\binom{b}{x^{\sigma}}} = \overline{a}\binom{\overline{b}}{x}$.

Lemma

- $\overline{a\begin{pmatrix} b \\ x^{\sigma} \end{pmatrix}} = \overline{a}\begin{pmatrix} \overline{b} \\ x \end{pmatrix}$.
- *$a \succ_0 b$ implies $\overline{a} \succ_0 \overline{b}$.*

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- $\overline{a\begin{pmatrix} b \\ x^\sigma \end{pmatrix}} = \overline{a}\begin{pmatrix} \overline{b} \\ x \end{pmatrix}$.
- $a \succ_0 b$ *implies* $\overline{a} \succ_0 \overline{b}$.
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- *$a \succ_1 b$ implies $\overline{a} \succ_1 \overline{b}$.*
- *If $\mathfrak{R} = \langle c_i \mid i \in I \rangle$ is a reduction chain for a then $\overline{\mathfrak{R}} := \langle \overline{c_i} \mid i \in I \rangle$ is a reduction chain for $\overline{a}$.*

Lemma

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Theorem

Well–foundedness of $\overline{a}$ entails the well–foundedness of a.

Lemma

Let $A \vdash 1$ *and* $a \vdash A$ *be* IFT*–terms. Let* $FV(a) \setminus \overline{FV(a)} = \{x^{\tau_1}, \ldots, x^{\tau_n}\}$ *and* c_i *closed terms such that* $c_i \vdash \tau_i$ *for* $i = 1, \ldots, n$*. Then* $FV(a\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}) = \{y^{B_1}, \ldots, y^{B_m}\}$ *and* $b_i \in |B_i|$ *imply*

$$\overline{a\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}}\begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} \in \left|A\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}\right|.$$

Proof (case 1) Proof (case 2.1) Proof (case 2.2) Proof (case 3.1) Proof (case 3.2)

Results

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Let A be a sentence and $a \vdash A$ for a closed IFT*–term a. Then $\bar{a} \in |A|$.*

Results

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Let A be a sentence and $a \vdash A$ for a closed IFT*–term a. Then $\overline{a} \in |A|$.*

Corollary

Every closed ITT*–term is strongly normalizable.*

Thank you for your attention

Proof of the Evaluation Lemma

Case (B_1)

We prove the lemma by induction on the definition of $|a|$.

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Proof of the Evaluation Lemma

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We prove the lemma by induction on the definition of $|a|$.
The claim holds trivially for (B_0).
To handle the case of (B_1) let $t \in |\mathfrak{F}^1|$. Let $\mathfrak{R}$ be a reduction chain for t which is not strongly normal.

Proof of the Evaluation Lemma

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To handle the case of (B_1) let $t \in |\mathfrak{F}^1|$. Let $\mathfrak{R}$ be a reduction chain for t which is not strongly normal. Then $\mathfrak{R}$ contains an element $t' \in \mathfrak{F}^1$.

Proof of the Evaluation Lemma

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To handle the case of (B_1) let $t \in |\mathfrak{F}^1|$. Let $\mathfrak{R}$ be a reduction chain for t which is not strongly normal. Then $\mathfrak{R}$ contains an element $t' \in \mathfrak{F}^1$. Since $\mathfrak{F}^1$ is a foundation–predicate, every reduction chain $\mathfrak{R}'$ for t' is finite.

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Now assume $t \succ s$ and let $\mathfrak{R}$ be a reduction chain for s. Then $\mathfrak{R}$ is a subchain of a reduction chain $\mathfrak{R}'$ for t. If $\mathfrak{R}$ is strongly normal then so is $\mathfrak{R}'$. Assume that $\mathfrak{R}$ is not strongly normal. Then $\mathfrak{R}'$ is also not strongly normal and therefore contains a λ–functional $s' \in \mathfrak{F}^1$ which is not preceeded by L–terms in $\mathfrak{R}'$.

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If $s' \notin \mathfrak{R}$ we get $s' \succ s$ and thus also $s \in \mathfrak{F}^1$. Hence $s \in |\mathfrak{F}^1|$ by (B_1).

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If $s' \notin \mathfrak{R}$ we get $s' \succ s$ and thus also $s \in \mathfrak{F}^1$. Hence $s \in |\mathfrak{F}^1|$ by (B_1).
If $s' \in \mathfrak{R}$ then there are no L–terms in $\mathfrak{R}$ which preceed s'. Hence $s \in |\mathfrak{F}^1|$ by (B_1).

Back

Proof of the Evaluation Lemma

To handle the case of (B_2) let $\tau = (\tau_1, \ldots, \tau_n)$ be a basic type, a a closed IFT–term such that $a \vdash \tau$ and $t_1, \ldots, t_n$ be closed IFT-terms such that $t_i \vdash \tau_i$ for $i = 1, \ldots, n$ and $t \in |at_1 \ldots t_n|$.

Proof of the Evaluation Lemma

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Proof of the Evaluation Lemma

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Proof of the Evaluation Lemma

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To handle the case of (B_2) let $\tau = (\tau_1, \ldots, \tau_n)$ be a basic type, a a closed IFT–term such that $a \vdash \tau$ and $t_1, \ldots, t_n$ be closed IFT-terms such that $t_i \vdash \tau_i$ for $i = 1, \ldots, n$ and $t \in |at_1 \ldots t_n|$. Let $\mathfrak{R}$ be a reduction chain for t which is not strongly normal. Then $\mathfrak{R}$ contains an element $t' \in |a|(|t_1|, \ldots, |t_n|)$. Since by induction hypothesis $|a|$ is a foundation–predicate of type τ and $|t_i|$ a foundation–predicate of type τ_i we obtain $|a|(|t_1|, \ldots, |t_n|)$ as a foundation–predicate of type 1.Thus every reduction chain $\mathfrak{R}'$ for t' is finite. But then $\mathfrak{R}$ is finite, too. Therfore t is well–founded.

Proof of the Evaluation Lemma

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Proof of the Evaluation Lemma

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To handle the case of (B_2) let $\tau = (\tau_1, \ldots, \tau_n)$ be a basic type, a a closed IFT–term such that $a \vdash \tau$ and $t_1, \ldots, t_n$ be closed IFT-terms such that $t_i \vdash \tau_i$ for $i = 1, \ldots, n$ and $t \in |at_1 \ldots t_n|$. Let $\mathfrak{R}$ be a reduction chain for t which is not strongly normal. Then $\mathfrak{R}$ contains an element $t' \in |a|(|t_1|, \ldots, |t_n|)$. Since by induction hypothesis $|a|$ is a foundation–predicate of type τ and $|t_i|$ a foundation–predicate of type τ_i we obtain $|a|(|t_1|, \ldots, |t_n|)$ as a foundation–predicate of type 1.Thus every reduction chain $\mathfrak{R}'$ for t' is finite. But then $\mathfrak{R}$ is finite, too. Therfore t is well–founded.

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If $s' \notin \mathfrak{R}$ we get $s' \succ s$ and thus also $s \in |a|(|t_1|, \ldots, |t_n|)$.

Proof of the Evaluation Lemma

Case (B_2)

To handle the case of (B_2) let $\tau = (\tau_1, \dots, \tau_n)$ be a basic type, a a closed IFT–term such that $a \vdash \tau$ and $t_1, \dots, t_n$ be closed IFT-terms such that $t_i \vdash \tau_i$ for $i = 1, \dots, n$ and $t \in |at_1 \dots t_n|$. Let $\mathfrak{R}$ be a reduction chain for t which is not strongly normal. Then $\mathfrak{R}$ contains an element $t' \in |a|(|t_1|, \dots, |t_n|)$. Since by induction hypothesis $|a|$ is a foundation–predicate of type τ and $|t_i|$ a foundation–predicate of type τ_i we obtain $|a|(|t_1|, \dots, |t_n|)$ as a foundation–predicate of type 1.Thus every reduction chain $\mathfrak{R}'$ for t' is finite. But then $\mathfrak{R}$ is finite, too. Therfore t is well–founded.

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If $s' \notin \mathfrak{R}$ we get $s' \succ s$ and thus also $s \in |a|(|t_1|, \dots, |t_n|)$. Since s is not preceeded by any term in $\mathfrak{R}$ we get $s \in |at_1 \dots t_n|$ by (B_3).

Proof of the Evaluation Lemma

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To handle the case of (B_2) let $\tau = (\tau_1, \dots, \tau_n)$ be a basic type, a a closed IFT–term such that $a \vdash \tau$ and $t_1, \dots, t_n$ be closed IFT-terms such that $t_i \vdash \tau_i$ for $i = 1, \dots, n$ and $t \in |at_1 \dots t_n|$. Let $\mathfrak{R}$ be a reduction chain for t which is not strongly normal. Then $\mathfrak{R}$ contains an element $t' \in |a|(|t_1|, \dots, |t_n|)$. Since by induction hypothesis $|a|$ is a foundation–predicate of type τ and $|t_i|$ a foundation–predicate of type τ_i we obtain $|a|(|t_1|, \dots, |t_n|)$ as a foundation–predicate of type 1.Thus every reduction chain $\mathfrak{R}'$ for t' is finite. But then $\mathfrak{R}$ is finite, too. Therfore t is well–founded.

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If $s' \notin \mathfrak{R}$ we get $s' \succ s$ and thus also $s \in |a|(|t_1|, \dots, |t_n|)$. Since s is not preceeded by any term in $\mathfrak{R}$ we get $s \in |at_1 \dots t_n|$ by (B_3).

If $s' \in \mathfrak{R}$ then there are no L–terms in $\mathfrak{R}$ which preceed s'. Hence $s \in |at_1 \dots t_n|$ by (B_3). Back

Proof of the Evaluation Lemma

Case (B_3)

To handle the case of (B_3) let $t \in |A \to B|$.

Proof of the Evaluation Lemma

Case (B_3)

To handle the case of (B_3) let $t \in |A \to B|$. Let $\mathfrak{R}$ be a reduction chain for t which is not strongly normal.

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Proof of the Evaluation Lemma

Case (B_3)

To handle the case of (B_3) let $t \in |A \to B|$. Let $\mathfrak{R}$ be a reduction chain for t which is not strongly normal. Then $\mathfrak{R}$ contains an element $\lambda x.\, t'$ such that $t' \binom{s}{x} \in |B|$ for all $s \in |A|$. By induction hypothesis $|B|$ is a foundation–predicate of type 1.

Proof of the Evaluation Lemma

Case (B_3)

To handle the case of (B_3) let $t \in |A \to B|$. Let $\mathfrak{R}$ be a reduction chain for t which is not strongly normal. Then $\mathfrak{R}$ contains an element $\lambda x.\, t'$ such that $t' \binom{s}{x} \in |B|$ for all $s \in |A|$. By induction hypothesis $|B|$ is a foundation–predicate of type 1.Therefore $t' \binom{s}{x}$ is well–founded which implies that $\lambda x.\, t'$ and thus also t is well–founded.

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If $\lambda x.\, t' \notin \mathfrak{R}$ we get $\lambda x.\, t' \succ s$ which implies that s is a functional $\lambda x.\, s'$ such that $t' \succ s'$. This entails $t' \binom{r}{x} \succ s' \binom{r}{x}$ for all $r \in |A|$. Hence $s' \binom{r}{x} \in |B|$ which implies $\lambda x.\, s' \in |A \to B|$.
If $\lambda x.\, t' \in \mathfrak{R}$ then there are no L–terms in $\mathfrak{R}$ which preceed $\lambda x.\, t'$. Hence $s \in |A \to B|$.

Back

To handle the case of (B_4) let $t \in |(\forall x^{\tau})B|$.

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Proof of the Evaluation Lemma

Case (B_4)

To handle the case of (B_4) let $t \in |(\forall x^\tau)B|$. Let $\mathfrak{R}$ be a reduction chain for t which is not strongly normal. Then $\mathfrak{R}$ contains an element $\lambda x.\, t'$ such that $t' \binom{u}{x} \in |B \binom{\mathfrak{F}}{x^\tau}|$ for all foundation–predicates $\mathfrak{F}$ of type τ.

Proof of the Evaluation Lemma

Case (B_4)

To handle the case of (B_4) let $t \in |(\forall x^\tau)B|$. Let $\mathfrak{R}$ be a reduction chain for t which is not strongly normal. Then $\mathfrak{R}$ contains an element $\lambda x.\, t'$ such that $t' \binom{u}{x} \in |B \binom{\mathfrak{F}}{x^\tau}|$ for all foundation–predicates $\mathfrak{F}$ of type τ. By induction hypothesis $|B \binom{\mathfrak{F}}{x^\tau}|$ is a foundation–predicate of type

1.

Proof of the Evaluation Lemma

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Now assume $t \succ s$ and let $\mathfrak{R}$ be a reduction chain for s. Then $\mathfrak{R}$ is a subchain of a reduction chain $\mathfrak{R}'$ for t. If $\mathfrak{R}$ is not strongly normal then so is $\mathfrak{R}'$ and thus contains a λ–functional $\lambda x.\, t'$ which is not preceeded by L–terms in $\mathfrak{R}'$ such that $t' \binom{u}{x} \in |B \binom{\mathfrak{F}}{x^\tau}|$ for all foundation–predicates $\mathfrak{F}$ of type τ.

Proof of the Evaluation Lemma

Case (B_4) (continued)

If $\lambda x.\, t' \notin \mathfrak{R}$ we get $\lambda x.\, t' \succ s$ which implies that s is a functional $\lambda x.\, s'$ such that $t' \succ s'$.

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Proof of the Evaluation Lemma

Case (B_4) (continued)

If $\lambda x.\, t' \notin \mathfrak{R}$ we get $\lambda x.\, t' \succ s$ which implies that s is a functional $\lambda x.\, s'$ such that $t' \succ s'$. This entails $t' \binom{u}{x} \succ s' \binom{u}{x}$. Hence $s' \binom{u}{x} \in |B \binom{\mathfrak{F}}{x^\tau}|$ which implies $\lambda x.\, s' \in |(\forall x^\tau) B|$.
If $\lambda x.\, t' \in \mathfrak{R}$ then there are no L–terms in $\mathfrak{R}$ which preceed $\lambda x.\, t'$. Hence $s \in |(\forall x^\tau) B|$.

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In the case of (B_5) we get $|\mathfrak{F}^{\tau}\mathfrak{F}_1 \ldots \mathfrak{F}_n|$ as a foundation–predicate of type 1. Therefore $|\mathfrak{F}^{\tau}|$ maps tuples of foundation–predicates of adequate types to foundation–predicates of type 1. So $|\mathfrak{F}^{\tau}|$ is a foundation–predicates of type τ.

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Proof of the Evaluation Lemma

Case (B_6)

In the case of (B_6) we get
$|\{(x^{\tau_1} \dots x^{\tau_n})| \; A\}|(\mathfrak{F}_1, \dots, \mathfrak{F}_n) := |A \begin{pmatrix} \mathfrak{F}_1, \dots, \mathfrak{F}_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}|$ which is a foundation–predicate of type 1 by induction hypothesis.

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$|\{(x^{\tau_1}\dots x^{\tau_n})|\ A\}|(\mathfrak{F}_1,\dots,\mathfrak{F}_n) := |A\begin{pmatrix}\mathfrak{F}_1,\dots,\mathfrak{F}_n\\ x^{\tau_1},\dots,x^{\tau_n}\end{pmatrix}|$ which is a foundation–predicate of type 1 by induction hypothesis. Therefore $|\{(x^{\tau_1}\dots x^{\tau_n})|\ A\}|$ maps tuples of foundation–predicates of adequate types to foundation–predicates of type 1. So $|\{(x^{\tau_1}\dots x^{\tau_n})|\ A\}|$ is a foundation–predicates of type τ.

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Proof of the Lemma

We distinguish the following cases:
(Atomic case) Let A be the formula $(at_1 \ldots t_n)$ for $n \geq 0$ (If $n = 0$ we assume that a is of type 1).

Proof of the Lemma

We distinguish the following cases:
(Atomic case) Let A be the formula $(at_1 \dots t_n)$ for $n \geq 0$ (If $n = 0$ we assume that a is of type 1). Let t by a λ–functional and $\mathfrak{R}$ be a reduction chain for t which is not strongly normal but contains an element $t' \in |a|(|t_1|, \dots, |t_n|)$. Let $\mathfrak{R}'$ be the subchain which starts with t'. Then $\mathfrak{R}'$ is not strongly normal. Therefore it contains an element $t'' \in \mathfrak{R}'$ which is not preceeded by an L–functional in $\mathfrak{R}'$. But then t'' is also not preceeded by an L–functional in $\mathfrak{R}$. Hence $t \in |at_1 \dots t_n|$.

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Proof of the Lemma
(continued)

($\rightarrow$ case) Now assume that A is a formula $B \rightarrow C$ and let $\mathfrak{R}$ be a reduction chain of a λ–functional t which is not strongly normal.

Proof of the Lemma
(continued)

($\rightarrow$ case) Now assume that A is a formula $B \rightarrow C$ and let $\mathfrak{R}$ be a reduction chain of a λ–functional t which is not strongly normal. Then $\mathfrak{R}$ contains an element $t' \in |B \rightarrow C|$ which is not preceeded by an L–functional in $\mathfrak{R}$.

Proof of the Lemma

(continued)

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Proof of the Lemma

(continued)

($\to$ case) Now assume that A is a formula $B \to C$ and let $\mathfrak{R}$ be a reduction chain of a λ–functional t which is not strongly normal. Then $\mathfrak{R}$ contains an element $t' \in |B \to C|$ which is not preceeded by an L–functional in $\mathfrak{R}$. Let $\mathfrak{R}'$ be the subchain of $\mathfrak{R}$ which starts with t'. Then $\mathfrak{R}'$ contains a first functional $\lambda x.\, t''$ such that $t'' \binom{r}{x} \in |C|$ for all $r \in |B|$.

Proof of the Lemma

(continued)

($\to$ case) Now assume that A is a formula $B \to C$ and let $\mathfrak{R}$ be a reduction chain of a λ–functional t which is not strongly normal. Then $\mathfrak{R}$ contains an element $t' \in |B \to C|$ which is not preceeded by an L–functional in $\mathfrak{R}$. Let $\mathfrak{R}'$ be the subchain of $\mathfrak{R}$ which starts with t'. Then $\mathfrak{R}'$ contains a first functional $\lambda x.\, t''$ such that $t'' \binom{r}{x} \in |C|$ for all $r \in |B|$. But then $\lambda x.\, t''$ is also the first functional with this property in $\mathfrak{R}$. Hence $t \in |B \to C|$.

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Proof of the Lemma

Continued

($\forall$)–case In the last case assume that A is a sentence $(\forall x^{\tau})B$ and let $\mathfrak{R}$ be a reduction chain of t which is not strongly normal.

($\forall$)–case In the last case assume that A is a sentence $(\forall x^{\tau})B$ and let $\mathfrak{R}$ be a reduction chain of t which is not strongly normal. Then $\mathfrak{R}$ contains an element $t' \in |(\forall x^{\tau})B|$.

Proof of the Lemma

Continued

$(\forall)$–case In the last case assume that A is a sentence $(\forall x^{\tau})B$ and let $\mathfrak{R}$ be a reduction chain of t which is not strongly normal. Then $\mathfrak{R}$ contains an element $t' \in |(\forall x^{\tau})B|$. Let $\mathfrak{R}'$ be the subchain of $\mathfrak{R}$ starting with t'. Then $\mathfrak{R}'$ is not strongly normal and therefore contains a functional $\lambda x.\, t''$ which is not preceeded in $\mathfrak{R}'$ by L–functional such that $t'' \binom{u}{x} \in |B \binom{\mathfrak{F}}{x^{\tau}}|$ for all foundation–predicates $\mathfrak{F}$ of type τ.

Proof of the Lemma

Continued

$(\forall)$–case In the last case assume that A is a sentence $(\forall x^\tau)B$ and let $\mathfrak{R}$ be a reduction chain of t which is not strongly normal. Then $\mathfrak{R}$ contains an element $t' \in |(\forall x^\tau)B|$. Let $\mathfrak{R}'$ be the subchain of $\mathfrak{R}$ starting with t'. Then $\mathfrak{R}'$ is not strongly normal and therefore contains a functional $\lambda x.\, t''$ which is not preceeded in $\mathfrak{R}'$ by L–functional such that $t'' \binom{u}{x} \in |B \binom{\mathfrak{F}}{x^\tau}|$ for all foundation–predicates $\mathfrak{F}$ of type τ. But then $\lambda x.\, t''$ is also the first functional with this property in $\mathfrak{R}$ which implies $t \in |(\forall x^\tau)B|$. $\square$

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Proof of the Lemma

Let $a \vdash \tau$. Firstly we observe that $|a|$ as well as $\|a\|$ are foundation–predicates predicates of type τ.

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$\tau = 0$. Then $|a| = a$ which trivially entails $\|a\| = |a| = a$.

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$\tau = 0$. Then $|a| = a$ which trivially entails $\|a\| = |a| = a$.

$\tau = 1$. If $t \in |a|$ then every reduction chain of t contains an element in $|a|$. Hence $t \in \|a\|$.

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If vice versa $t \in \|a\|$ then every reduction chain $\mathfrak{R}$ of t which is not strongly normal contains an element t' which is not preceeded in $\mathfrak{R}$ by an L–term. By the previous lemma we then get $t \in |a|$.

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If vice versa $t \in \|a\|$ then every reduction chain $\mathfrak{R}$ of t which is not strongly normal contains an element t' which is not preceeded in $\mathfrak{R}$ by an L–term. By the previous lemma we then get $t \in |a|$.

$\tau = (\tau_1, \ldots, \tau_n)$. Then a is a name $\mathfrak{F}^\tau$ for a foundation–predicate or a is a term $\{((x^{\tau_1} \ldots x^{\tau_n}))|\ A\}$. In both cases $|a|$ is a foundation–predicate of type τ. In the first case we get for any tuple $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ of foundation–predicates of adequate types
$\|\mathfrak{F}^\tau\|(\mathfrak{F}_1, \ldots, \mathfrak{F}_n) = ||\mathfrak{F}^\tau|(\mathfrak{F}_1, \ldots, \mathfrak{F}_n)| = \|\mathfrak{F}^\tau(\mathfrak{F}_1, \ldots, \mathfrak{F}_n)\| =$
$|\mathfrak{F}^\tau(\mathfrak{F}_1, \ldots, \mathfrak{F}_n)| = |\mathfrak{F}^\tau|(\mathfrak{F}_1, \ldots, \mathfrak{F}_n)$. $\square$

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Proof of the Lemma
(continued)

In the second case we get $\|\{(x^{\tau_1}\dots x^{\tau_n})|\ A\}\|(\mathfrak{F}_1,\dots,\mathfrak{F}_n) =$
$||\{(x^{\tau_1}\dots x^{\tau_n})|\ A\}|(\mathfrak{F}_1,\dots,\mathfrak{F}_n)| = \|A\begin{pmatrix}\mathfrak{F}_1,\dots,\mathfrak{F}_n\\ x^{\tau_1},\dots,x^{\tau_n}\end{pmatrix}\| =$
$|A\begin{pmatrix}\mathfrak{F}_1,\dots,\mathfrak{F}_n\\ x^{\tau_1},\dots,x^{\tau_n}\end{pmatrix}| = |\{(x^{\tau_1}\dots x^{\tau_n})|\ A\}|(\mathfrak{F}_1,\dots,\mathfrak{F}_n)$.

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Proof of the Main Lemma

We prove the lemma by induction on the length of a.

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Firstly assume $a = y^A$. Then $a\binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}} = y^{A\binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}}} = y^{B_i}$ for some $i \in \{1, \ldots, m\}$.

Proof of the Main Lemma

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Firstly assume $a = y^A$. Then $a\binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}} = y^{A\binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}}} = y^{B_i}$ for some $i \in \{1, \ldots, m\}$. Hence $\overline{a\binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}}} = y$, which for $b \in |B_i|$ implies $\overline{a\binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}}}\binom{b_1, \ldots, b_m}{y_1, \ldots, y_m} = b \in |B_i| = |A\binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}}|$.

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Proof of the Main Lemma

continued

Assume that $a = \lambda x^{\sigma}.d$.

Proof of the Main Lemma

continued

Assume that $a = \lambda x^\sigma . d$.
If σ is a basic type τ then A is a formula $(\forall x^\tau)C$ and $d \vdash C$ with $FV(d) \setminus \overline{FV(d)} = \{x^{\tau_1}, \dots, x^{\tau_n}, x^\tau\}$.

Proof of the Main Lemma

continued

Assume that $a = \lambda x^\sigma . d$.

If σ is a basic type τ then A is a formula $(\forall x^\tau)C$ and $d \vdash C$ with $FV(d) \setminus \overline{FV(d)} = \{x^{\tau_1}, \ldots, x^{\tau_n}, x^\tau\}$. We have to show that $\overline{d \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} \in |C \begin{pmatrix} c_1, \ldots, c_m \\ x^{\tau_1}, \ldots, x^{\tau_m} \end{pmatrix} \begin{pmatrix} \mathfrak{F} \\ x^\tau \end{pmatrix}|$ holds true for every foundation–predicate $\mathfrak{F}$ of type τ.

Proof of the Main Lemma

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Assume that $a = \lambda x^{\sigma}. d$.
If σ is a basic type τ then A is a formula $(\forall x^{\tau})C$ and $d \vdash C$ with $FV(d) \setminus \overline{FV(d)} = \{x^{\tau_1}, \ldots, x^{\tau_n}, x^{\tau}\}$. We have to show that
$\overline{d \binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}}} \binom{b_1, \ldots, b_m}{y_1, \ldots, y_m} \binom{u}{x} \in |C \binom{c_1, \ldots, c_m}{x^{\tau_1}, \ldots, x^{\tau_m}} \binom{\mathfrak{F}}{x^{\tau}}|$ holds true
for every foundation–predicate $\mathfrak{F}$ of type τ.
By the induction hypothesis we have

$$\overline{d \binom{c_1, \ldots, c_m}{x^{\tau_1}, \ldots, x^{\tau_m}} \binom{\mathfrak{F}}{x^{\tau}}} \binom{b_1, \ldots, b_m}{y_1, \ldots, y_m} \in |C \binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}} \binom{\mathfrak{F}}{x^{\tau}}|$$

Proof of the Main Lemma

continued

Assume that $a = \lambda x^\sigma . d$.
If σ is a basic type τ then A is a formula $(\forall x^\tau)C$ and $d \vdash C$ with $FV(d) \setminus \overline{FV(d)} = \{x^{\tau_1}, \ldots, x^{\tau_n}, x^\tau\}$. We have to show that
$\overline{d\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} \in |C \begin{pmatrix} c_1, \ldots, c_m \\ x^{\tau_1}, \ldots, x^{\tau_m} \end{pmatrix} \begin{pmatrix} \mathfrak{F} \\ x^\tau \end{pmatrix}|$ holds true for every foundation–predicate $\mathfrak{F}$ of type τ.
By the induction hypothesis we have
$\overline{d\begin{pmatrix} c_1, \ldots, c_m \\ x^{\tau_1}, \ldots, x^{\tau_m} \end{pmatrix} \begin{pmatrix} \mathfrak{F} \\ x^\tau \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} \in |C \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix} \begin{pmatrix} \mathfrak{F} \\ x^\tau \end{pmatrix}|$
and compute easily $\overline{d\begin{pmatrix} c_1, \ldots, c_m \\ x^{\tau_1}, \ldots, x^{\tau_m} \end{pmatrix} \begin{pmatrix} \mathfrak{F} \\ x^\tau \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} =$
$\overline{d\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix}$.

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Proof of the Main Lemma

continued

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Proof of the Main Lemma

continued

Assume that $a = \lambda x^{\sigma}.d$.
If σ is a formula B then A is a formula $B \to C$ and $d \vdash C$. Then $FV(d \binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}}) = \{y^{B_1}, \ldots, y^{B_n}, x^B\}$.

Proof of the Main Lemma

continued

Assume that $a = \lambda x^\sigma . d$.
If σ is a formula B then A is a formula $B \rightarrow C$ and $d \vdash C$. Then $FV(d \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}) = \{y^{B_1}, \ldots, y^{B_n}, x^B\}$. We have to show that $\overline{d \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} \begin{pmatrix} b \\ x \end{pmatrix} \in |C \begin{pmatrix} c_1, \ldots, c_m \\ x^{\tau_1}, \ldots, x^{\tau_m} \end{pmatrix}|$ holds true for all $b \in |B|$.

Proof of the Main Lemma

continued

Assume that $a = \lambda x^\sigma . d$.

If σ is a formula B then A is a formula $B \to C$ and $d \vdash C$. Then $FV(d \binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}}) = \{y^{B_1}, \ldots, y^{B_n}, x^B\}$. We have to show that $\overline{d \binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}}} \binom{b_1, \ldots, b_m}{y_1, \ldots, y_m} \binom{b}{x} \in |C \binom{c_1, \ldots, c_m}{x^{\tau_1}, \ldots, x^{\tau_m}}|$ holds true for all $b \in |B|$.

But this is in fact the induction hypothesis. Back

Proof of the Main Lemma

continued

Assume that $a = (de)$ such that $d \vdash (\forall x^\tau)D$ and $e \vdash \tau$ for a basic type τ.

Assume that $a = (de)$ such that $d \vdash (\forall x^\tau)D$ and $e \vdash \tau$ for a basic type τ. Then A is the formula $D\binom{e}{x^\tau}$, $\overline{e} = u$ and

$$\overline{a\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} = \left(\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}u\right)\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} =$$
$$\left(\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m}\right)u.$$

Proof of the Main Lemma
continued

Assume that $a = (de)$ such that $d \vdash (\forall x^\tau)D$ and $e \vdash \tau$ for a basic type τ. Then A is the formula $D\binom{e}{x^\tau}$, $\overline{e} = u$ and $\overline{a\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} = \left(\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}u\right)\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} = \left(\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m}\right)u$. By induction hypothesis we have $\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} \in \left|(\forall x^\tau)D\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}\right|$.

Proof of the Main Lemma

continued

Assume that $a = (de)$ such that $d \vdash (\forall x^\tau)D$ and $e \vdash \tau$ for a basic type τ. Then A is the formula $D\binom{e}{x^\tau}$, $\overline{e} = u$ and $\overline{a\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} = \left(\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}u\right)\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} = \left(\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m}\right)u$. By induction hypothesis we have $\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} \in |(\forall x^\tau)D\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}|$. Hence $\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m}$ is well–founded.

Proof of the Main Lemma

continued

Assume that $a = (de)$ such that $d \vdash (\forall x^\tau)D$ and $e \vdash \tau$ for a basic type τ. Then A is the formula $D\binom{e}{x^\tau}$, $\overline{e} = u$ and $\overline{a\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} = \left(\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}u\right)\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} = \left(\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m}\right)u$. By induction hypothesis we have $\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} \in |(\forall x^\tau)D\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}|$. Hence $\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m}$ is well–founded. Let $\mathfrak{R} = \langle r_i |\ i \in I\rangle$ be a reduction chain of $\overline{a\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m}$.

Proof of the Main Lemma

continued

Assume that $a = (de)$ such that $d \vdash (\forall x^\tau)D$ and $e \vdash \tau$ for a basic type τ. Then A is the formula $D\binom{e}{x^\tau}$, $\overline{e} = u$ and $\overline{a\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} = \left(\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}u\right)\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} = \left(\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m}\right)u$. By induction hypothesis we have $\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} \in |(\forall x^\tau)D\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}|$. Hence $\overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m}$ is well–founded. Let $\mathfrak{R} = \langle r_i|\ i \in I\rangle$ be a reduction chain of $\overline{a\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m}$. If $\mathfrak{R}$ does not contain a conversion according to R_0 then $\mathfrak{R}$ is strongly normal by Lemma I.

Proof of the Main Lemma

continued

Otherwise let k be the least index such that $r_k \succ r_{k+1}$ is a reduction according to R_0. Then $r_k = (\lambda x.\, r')u$ and
$\overline{d\begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \dots, b_m \\ y_1, \dots, y_m \end{pmatrix} \succ \lambda x.\, r'$.

Proof of the Main Lemma
continued

Otherwise let k be the least index such that $r_k \succ r_{k+1}$ is a reduction according to R_0. Then $r_k = (\lambda x.\, r')u$ and
$\overline{d \binom{c_1, \dots, c_n}{x^{\tau_1}, \dots, x^{\tau_n}}} \binom{b_1, \dots, b_m}{y_1, \dots, y_m} \succ \lambda x.\, r'$. Hence
$\lambda x.\, r' \in |(\forall x^\tau) D \binom{c_1, \dots, c_n}{x^{\tau_1}, \dots, x^{\tau_n}}|$

Proof of the Main Lemma

continued

Otherwise let k be the least index such that $r_k \succ r_{k+1}$ is a reduction according to R_0. Then $r_k = (\lambda x.\, r')u$ and $\overline{d\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}}\begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} \succ \lambda x.\, r'$. Hence $\lambda x.\, r' \in |(\forall x^{\tau})D\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}|$ which implies that $r'\begin{pmatrix} u \\ x \end{pmatrix} \in |D\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}\begin{pmatrix} \mathfrak{F} \\ x^{\tau} \end{pmatrix}|$ for all foundation–predicates $\mathfrak{F}$ of type τ.

Proof of the Main Lemma

continued

Otherwise let k be the least index such that $r_k \succ r_{k+1}$ is a reduction according to R_0. Then $r_k = (\lambda x.\, r')u$ and $\overline{d\begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \dots, b_m \\ y_1, \dots, y_m \end{pmatrix} \succ \lambda x.\, r'$. Hence $\lambda x.\, r' \in |(\forall x^\tau) D \begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}|$ which implies that $r' \begin{pmatrix} u \\ x \end{pmatrix} \in |D \begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix} \begin{pmatrix} \mathfrak{F} \\ x^\tau \end{pmatrix}|$ for all foundation–predicates $\mathfrak{F}$ of type τ. But $|e|$ is a foundation–predicate of type τ.

Proof of the Main Lemma

continued

Otherwise let k be the least index such that $r_k \succ r_{k+1}$ is a reduction according to R_0. Then $r_k = (\lambda x.\, r')u$ and $\overline{d \binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}}} \binom{b_1, \ldots, b_m}{y_1, \ldots, y_m} \succ \lambda x.\, r'$. Hence $\lambda x.\, r' \in |(\forall x^{\tau}) D \binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}}|$ which implies that $r' \binom{u}{x} \in |D \binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}} \binom{\mathfrak{F}}{x^{\tau}}|$ for all foundation–predicates $\mathfrak{F}$ of type τ. But $|e|$ is a foundation–predicate of type τ. Therefore we get $r_{k+1} = r' \binom{u}{x} \in |D \binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}} \binom{|e|}{x^{\tau}}| = |D \binom{c_1, \ldots, c_n}{x^{\tau_1}, \ldots, x^{\tau_n}} \binom{e}{x^{\tau}}|$.

Proof of the Main Lemma

continued

Otherwise let k be the least index such that $r_k \succ r_{k+1}$ is a reduction according to R_0. Then $r_k = (\lambda x.\, r')u$ and $\overline{d\begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}}\begin{pmatrix} b_1, \dots, b_m \\ y_1, \dots, y_m \end{pmatrix} \succ \lambda x.\, r'$. Hence $\lambda x.\, r' \in |(\forall x^\tau) D\begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}|$ which implies that $r'\begin{pmatrix} u \\ x \end{pmatrix} \in |D\begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}\begin{pmatrix} \mathfrak{F} \\ x^\tau \end{pmatrix}|$ for all foundation–predicates $\mathfrak{F}$ of type τ. But $|e|$ is a foundation–predicate of type τ. Therefore we get $r_{k+1} = r'\begin{pmatrix} u \\ x \end{pmatrix} \in |D\begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}\begin{pmatrix} |e| \\ x^\tau \end{pmatrix}| = |D\begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}\begin{pmatrix} e \\ x^\tau \end{pmatrix}|$. Since r_{k+1} is not preceeded by L–functionals in $\mathfrak{R}$ we obtain $\overline{a\begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}}\begin{pmatrix} b_1, \dots, b_m \\ y_1, \dots, y_m \end{pmatrix} \in |D\begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}\begin{pmatrix} e \\ x^\tau \end{pmatrix}|$.

Back

Proof of the Main Lemma

continued

Assume that $a = (de)$ such that $d \vdash B \to A$ and $e \vdash B$ for a formula B.

Proof of the Main Lemma

continued

Assume that $a = (de)$ such that $d \vdash B \to A$ and $e \vdash B$ for a formula B. By the induction hypothesis we get

$$d' := \overline{d\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} \in |(B \to A)\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}| \text{ and}$$

$$e' := \overline{e\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_n \\ y_1, \ldots, y_n \end{pmatrix} \in |B\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}|$$

Proof of the Main Lemma

continued

Assume that $a = (de)$ such that $d \vdash B \to A$ and $e \vdash B$ for a formula B. By the induction hypothesis we get

$$d' := \overline{d\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} \in |(B \to A)\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}| \text{ and}$$

$$e' := \overline{e\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_n \\ y_1, \ldots, y_n \end{pmatrix} \in |B\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}| \text{ and}$$

$$a' := \overline{a\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} = (d'e').$$

Proof of the Main Lemma

continued

Assume that $a = (de)$ such that $d \vdash B \to A$ and $e \vdash B$ for a formula B. By the induction hypothesis we get

$$d' := \overline{d\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} \in |(B \to A)\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}| \text{ and}$$

$$e' := \overline{e\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_n \\ y_1, \ldots, y_n \end{pmatrix} \in |B\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}| \text{ and}$$

$$a' := \overline{a\begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} = (d'e').$$

Let $\mathfrak{R}$ be a reduction chain of a'.

Proof of the Main Lemma

continued

Assume that $a = (de)$ such that $d \vdash B \to A$ and $e \vdash B$ for a formula B. By the induction hypothesis we get

$$d' := \overline{d \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} \in |(B \to A) \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}| \text{ and}$$

$$e' := \overline{e \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_n \\ y_1, \ldots, y_n \end{pmatrix} \in |B \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}| \text{ and}$$

$$a' := \overline{a \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} = (d'e').$$

Let $\mathfrak{R}$ be a reduction chain of a'. If $\mathfrak{R}$ does not contain conversions according to R_0 then $\mathfrak{R}$ is strongly normal by Lemma I since d' and e' are well–founded.

Proof of the Main Lemma

continued

Assume that $a = (de)$ such that $d \vdash B \to A$ and $e \vdash B$ for a formula B. By the induction hypothesis we get

$$d' := \overline{d \begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \dots, b_m \\ y_1, \dots, y_m \end{pmatrix} \in |(B \to A) \begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}| \text{ and}$$

$$e' := \overline{e \begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \dots, b_n \\ y_1, \dots, y_n \end{pmatrix} \in |B \begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}| \text{ and}$$

$$a' := \overline{a \begin{pmatrix} c_1, \dots, c_n \\ x^{\tau_1}, \dots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \dots, b_m \\ y_1, \dots, y_m \end{pmatrix} = (d'e').$$

Let $\mathfrak{R}$ be a reduction chain of a'. If $\mathfrak{R}$ does not contain conversions according to R_0 then $\mathfrak{R}$ is strongly normal by ▸ Lemma I since d' and e' are well–founded. If $\mathfrak{R}$ is not strongly normal then it contains an element with minimal index k such that $r_k \succ_1 r_{k+1}$ is a conversion according to R_0. Then r_k is of the form $(\lambda x.\, r')s$ such that $d' \succ \lambda x.\, r'$ and $e' \succ s$.

Proof of the Main Lemma

continued

Assume that $a = (de)$ such that $d \vdash B \to A$ and $e \vdash B$ for a formula B. By the induction hypothesis we get

$d' := \overline{d \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} \in |(B \to A) \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}|$ and

$e' := \overline{e \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_n \\ y_1, \ldots, y_n \end{pmatrix} \in |B \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}|$ and

$a' := \overline{a \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}} \begin{pmatrix} b_1, \ldots, b_m \\ y_1, \ldots, y_m \end{pmatrix} = (d'e')$.

Let $\mathfrak{R}$ be a reduction chain of a'. If $\mathfrak{R}$ does not contain conversions according to R_0 then $\mathfrak{R}$ is strongly normal by ‣ Lemma I since d' and e' are well–founded. If $\mathfrak{R}$ is not strongly normal then it contains an element with minimal index k such that $r_k \succ_1 r_{k+1}$ is a conversion according to R_0. Then r_k is of the form $(\lambda x.\, r')s$ such that $d' \succ \lambda x.\, r'$ and $e' \succ s$. Hence $\lambda x.\, r' \in |(B \to A) \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}|$ and $s \in |B \begin{pmatrix} c_1, \ldots, c_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}|$

Proof of the Main Lemma

continued

Assume that $a = (de)$ such that $d \vdash B \to A$ and $e \vdash B$ for a formula B. By the induction hypothesis we get

$$d' := \overline{d\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} \in |(B \to A)\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}| \text{ and}$$

$$e' := \overline{e\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_n}{y_1,\ldots,y_n} \in |B\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}| \text{ and}$$

$$a' := \overline{a\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}}\binom{b_1,\ldots,b_m}{y_1,\ldots,y_m} = (d'e').$$

Let $\mathfrak{R}$ be a reduction chain of a'. If $\mathfrak{R}$ does not contain conversions according to R_0 then $\mathfrak{R}$ is strongly normal by Lemma I since d' and e' are well–founded. If $\mathfrak{R}$ is not strongly normal then it contains an element with minimal index k such that $r_k \succ_1 r_{k+1}$ is a conversion according to R_0. Then r_k is of the form $(\lambda x.\, r')s$ such that $d' \succ \lambda x.\, r'$ and $e' \succ s$. Hence $\lambda x.\, r' \in |(B \to A)\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}|$ and $s \in |B\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}|$ and therefore $r'\binom{s}{x} \in |A\binom{c_1,\ldots,c_n}{x^{\tau_1},\ldots,x^{\tau_n}}|$.

Proof of the Main Lemma

continued

But r_{k+1} is the term $r'\binom{s}{x}$. Since r_{k+1} is not preceeded by L–functionals in $\mathfrak{R}$ we have $a' \in |A\begin{pmatrix} b_1, \ldots, b_n \\ x^{\tau_1}, \ldots, x^{\tau_n} \end{pmatrix}|$. $\square$

Back

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