Church encodings of ordinals, and simulation of ordinal functions

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Ordinals as iterators

"A number is the exponent of an operation." *(T, 6.021)*

$$\left.\begin{array}{l} X : \mathsf{Set} \\ z : X \\ s : X \to X \\ l : X^N \to X \end{array}\right\} \quad \overline{x} : (1 + X + X^N) \to X$$

$$\begin{array}{l} \mathsf{Br} : \mathsf{Set} \to \mathsf{Set} \\ \mathsf{Br}\, X = 1 + X + X^N \\ \Omega = \mu\, \mathsf{Br} \end{array}$$

$$\begin{tikzcd} \mathsf{Br}\, X \arrow[r, "\overline{x}"] & X \\ \mathsf{Br}\, \Omega \arrow[u, "\mathsf{Br}\, (\![.]\!)"] \arrow[r, "\langle 0,(+1),\mathsf{sup}\rangle"'] & \Omega \arrow[u, "(\![.]\!)"'] \end{tikzcd}$$

AMEN

$$
\begin{aligned}
(\![\alpha + \beta]\!)\, X\, z\, s\, l &= (\![\beta]\!)\, X\, ((\![\alpha]\!)\, X\, z\, s\, l)\, s\, l \\
(\![\alpha \times \beta]\!)\, X\, z\, s\, l &= (\![\beta]\!)\, X\, z, (x \mapsto (\![\alpha\, X\, x\, s\, l]\!))\, s\, l \\
(\![\alpha \uparrow \beta]\!)\, X\, z\, s\, l &= (\![\beta]\!)\, (X \to X)\, s \\
&\qquad\qquad\qquad (f, x \mapsto (\![\alpha]\!)\, X\, x\, f\, l) \\
&\qquad\qquad\qquad (g, x \mapsto l(n \mapsto g\, n\, x)) \\
&\qquad\qquad\qquad z \\
(\![0]\!)\, X\, z\, s\, l &= z \\
(\![\omega]\!)\, X\, z\, s\, l &= l(n \mapsto s^n\, z)
\end{aligned}
$$

Algebra

Modulo $\beta\eta$,

- $(0, +)$ a monoid.
- $(1, \times)$ a monoid.
- $\alpha \times 0 = 0, \quad \alpha \times (\beta + \gamma) = \alpha \times \beta + \alpha \times \gamma$
- $\begin{aligned} \alpha \uparrow 0 &= 1, & \alpha \uparrow (\beta + \gamma) &= \alpha \uparrow \beta \times \alpha \uparrow \gamma \\ \alpha \uparrow 1 &= \alpha, & \alpha \uparrow (\beta \times \gamma) &= (\alpha \uparrow \beta) \uparrow \gamma \end{aligned}$

In particular,

$$\begin{aligned} \alpha + 0 &= \alpha & \alpha + (\beta + 1) &= (\alpha + \beta) + 1 \\ \alpha \times 0 &= 0 & \alpha \times (\beta + 1) &= (\alpha \times \beta) + \alpha \\ \alpha \uparrow 0 &= 1 & \alpha \uparrow (\beta + 1) &= (\alpha \uparrow \beta) \times \alpha \end{aligned}$$

So, our definitions are correct.

$$(\!\![\phi\,\alpha]\!\!)\,X\,\underbrace{z\,s\,l}_{\overline{x}} \quad = D\,\overline{x}((\!\![\alpha]\!\!)(F\,X)(U\,\overline{x}))$$

where

- F : Set $\rightarrow$ Set
- $U : (\mathrm{Br}\,X \rightarrow X) \rightarrow (\mathrm{Br}(F\,X) \rightarrow F\,X)$: 'uplifts' a Br-algebra on carrier X to another on $F\,X$.
- $D : (\mathrm{Br}\,X \rightarrow X) \rightarrow F\,X \rightarrow X$: 'drops' from $F\,X$ to X.

Example (ω^{α}):

- $F\,X = X \rightarrow X$,
- $U\,\overline{x} = s, (f, x \mapsto l(n \mapsto f^n\,x)), (g, x \mapsto l(n \mapsto g\,n\,x))$,
- $D\,\overline{x} = (f \mapsto f\,z)$.

Some nice closure properties

- Closed under composition. $\phi \cdot \psi$ simulated by

$$
\begin{array}{l}
F_\psi \cdot F_\phi \\
\overline{x} \mapsto U_\psi(U_\phi \overline{x}) \\
\overline{x} \mapsto (D_\phi\, \overline{x}) \cdot D_\psi\,(U_\phi\, \overline{x})
\end{array}
$$

- How about 'countable composition' $\sup_n(\phi_n \cdot \phi_{n-1} \cdot \dots \cdot \phi_0)$?

Well, yes, it works. It is the basis for simulating the Veblen hierarchy χ^α_β. But it is a little heavy with subscripts, so let's just look at $\phi^\omega = \sup_n(\underbrace{\phi \cdot \phi \cdot \dots \cdot \phi}_{n})$.

Sup of a sequence

- Given $F : \mathrm{Set} \to \mathrm{Set}$, form $F' X = (\prod n : N)\, F^n X$.
- Given $U : (\mathrm{Br}\, X \to X) \to (\mathrm{Br}(F\, X) \to F\, X)$,
  form $U_n : (Br\, X \to X) \to (\mathrm{Br}(F^n\, X) \to F^n\, X)$.
- Now, eliding some of the more bureaucratic arguments, we have an inverse chain:

$$X \xleftarrow{D\ldots} F\, X \xleftarrow{D\ldots} F^2\, X \xleftarrow{D\ldots} \cdots$$

- Given $\xi : F' X = (\prod n : N)\, F^n X$, form the 'sup' of $\xi_0 = \xi$, $\xi_1 = (n \mapsto D(\ldots)\xi_0(n+1))$, $\xi_2 = \ldots$ using the sup at each level.
- (Rough) claim: if (F, U, D) simulates ϕ, which is normal, then the operation $\xi \mapsto \xi_\omega$ maps $F' X$ onto the inverse limit of the above chain, and simulates ϕ^ω. Call this op C.
- Define $U' \ldots$ (giving a Br-algebra on $F' X$) by applying/postcomposing C to (U_n) above.

THE (F, U, D)'S FORM A LARGE BR-ALGEBRA

- The zero: take $F\,X = X \to X, \ldots$, that simulates ω^α.
- The successor: the operation that takes (F, U, D) to $X \mapsto (\prod n : N)\,F^n\,X, \ldots$ as on the previous slide. (More or less, takes us from a normal function ϕ to its Veblen derivative.
- The limit: we have an ω-sequence of (F_n, U_n, D_n). The idea is quite similar to what we do in the successor case, except the steps in the chain are heterogeneous.

With no universes, we can define approximants up to ε_0. Then with one universe by iterating the large Br-algebra through these approximants, we can define approximants up to $\phi_{\varepsilon_0}0$. And so on ... with a tower of universes, up to Γ_0.

Rash claim: I expect that the same techniques (with essentially no new ideas), can be used to obtain similar (lower bounds) results for a superuniverse, a super^2 universe, and so on.