Rules for the Π_3-reflecting Universe

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Abstract

We introduce a universe, which yields the strength of KP + Π_3-reflection + ω admissibles above it.

1 Rules for the Π_3-reflecting unverse

The basic structure

$\mathbb{U}$ is a universe, with decoding function $\mathrm{S}(x)$.
Univ is a subset of $\mathbb{U}$, each element u of it is a subuniverses of it $\mathbb{U}$ with decoding function $\widehat{\mathrm{T}}_u$.
M is the set of Mahlo-degrees, and $\mathrm{m}(u)$ is the Mahlo-degree of a universe u. If u is a universe then it will be Mahlo with respect to every $\mathrm{m}(u)^{a,b}[i]$ for $i : \tau^{a,b}(\mathrm{m}(u))$ for every family of sets a, b in u. So the degree of Mahloness depends on $\mathrm{m}(u)$ and the universe u.

In order to distinguish recursive and inductive definitions we underline all constructors when they are introduced the first time (afterwards we don't underline them, to reduce the amount of syntax). So, whenever we have a rule which yields a set or an element of a set and have an expression which does not start with an underlined constructor, this is a recursive definition, and whenever we introduce a new element of the set by recursion on which it is defined (and introducing means, that the element starts with a constructor), we have to tell, how to evaluate it. For instance, $\mathrm{S}(a)$ is defined recursively by recursion on $a : \mathbb{U}$, and it is defined by $\mathrm{S}(\mathrm{univ}(u)) = \mathrm{U}_u$, $\mathrm{S}(\widehat{\Sigma}(a, b)) = \Sigma(\mathrm{S}(a), \mathrm{S} \circ b)$, where univ and $\widehat{\Sigma}$ are the constructors of $\mathbb{U}$.

$\underline{\mathbb{U}} : \mathrm{Set}$

$\underline{\mathrm{Univ}} : \mathrm{Set}$

$$\frac{u : \mathrm{Univ}}{\underline{\mathrm{univ}}(u) : \mathbb{U}}$$

$$\frac{a : \mathbb{U}}{\mathrm{S}(a) : \mathrm{Set}}$$

$$\frac{u : \mathrm{Univ}}{\underline{\mathrm{U}}_u : \mathrm{Set}}$$

$$\frac{u : \mathrm{Univ}}{\mathrm{S}(\mathrm{univ}(u)) = \mathrm{U}_u : \mathrm{Set}}$$

$$\frac{u : \mathrm{Univ} \qquad a : \mathrm{U}_u}{\widehat{\mathrm{T}}_u(a) : \mathbb{U}}$$

If $u : \mathrm{Univ}$, $a : \mathrm{U}_u$ then $\mathrm{T}_u(a) := \mathrm{S}(\widehat{\mathrm{T}}_u(a)) : \mathrm{Set}$.

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$$\underline{\mathrm{M}} : \mathrm{Set} \qquad \frac{u : \mathrm{Univ}}{\mathrm{m}(u) : \mathrm{M}}$$

$$\frac{a : \mathbb{U} \qquad b : \mathrm{S}(a) \to \mathbb{U} \qquad m : \mathrm{M}}{\widehat{\tau}^{a,b}(m) : \mathbb{U}} \qquad \tau^{a,b}(m) := \mathrm{S}(\widehat{\tau}^{a,b}(m))$$

$$\frac{m : \mathrm{M} \qquad a : \mathbb{U} \qquad b : \mathrm{S}(a) \to \mathbb{U} \qquad i : \tau^{a,b}(m)}{m^{a,b}[i] : \mathrm{M}}$$

$\mathbb{U}$ is a universe

$$\frac{a : \mathbb{U} \qquad b : \mathrm{S}(a) \to \mathbb{U}}{\underline{\widetilde{\Sigma}}(a,b) : \mathbb{U}} \qquad \frac{a : \mathbb{U} \qquad b : \mathrm{S}(a) \to \mathbb{U}}{\mathrm{S}(\widetilde{\Sigma}(a,b)) = \Sigma(\mathrm{S}(a), \mathrm{S} \circ b)}$$

Similarly closure under N_0, N_1, N_2, N, $+$, Π, W, I.

The elements of Univ are universes

$$\frac{u : \mathrm{Univ} \qquad a : \mathrm{U}_u \qquad b : \mathrm{T}_u(a) \to \mathrm{U}_u}{\underline{\widehat{\Sigma}}(a,b) : \mathrm{U}_u}$$

$$\frac{u : \mathrm{Univ} \qquad a : \mathrm{U}_u \qquad b : \mathrm{T}_u(a) \to \mathrm{U}_u}{\widehat{\mathrm{T}}_u(\widehat{\Sigma}(a,b)) = \widetilde{\Sigma}(\widehat{\mathrm{T}}_u(a), \widehat{\mathrm{T}}_u \circ b) : \mathbb{U}}$$

Similarly closure under $\widetilde{\mathrm{N}}_0$, $\widetilde{\mathrm{N}}_1$, $\widetilde{\mathrm{N}}_2$, $\widetilde{\mathrm{N}}$, $\widetilde{+}$, $\widetilde{\Pi}$, $\widetilde{\mathrm{W}}$, $\widetilde{\mathrm{I}}$.

Universes are Mahlo with respect to $\cdot^{\cdot}[\cdot]$

$$\frac{\begin{array}{cc} u : \mathrm{Univ} & r : \mathrm{U}_u \\ s : \mathrm{T}_u(r) \to \mathrm{U}_u & i : \tau^{\widehat{\mathrm{T}}_u(r), \widehat{\mathrm{T}}_u \circ s}(\mathrm{m}(u)) \\ f : (x : \mathrm{U}_u, y : \mathrm{T}_u(x) \to \mathrm{U}_u) \to \mathrm{U}_u & g : (x : \mathrm{U}_u, y : \mathrm{T}_u(x) \to \mathrm{U}_u, \mathrm{T}_u(f(x,y))) \to \mathrm{U}_u \end{array}}{\underline{\mathrm{su}}_{u,r,s,i,f,g} : \mathrm{Univ}}$$

Assume now in the following the assumptions of the last rule.
We write $\vec{\mathrm{fg}}$ for u, r, s, i, f, g.

$$\mathrm{m}(\mathrm{su}_{\vec{\mathrm{fg}}}) = (\mathrm{m}(u))^{\widehat{\mathrm{T}}_u(r), \widehat{\mathrm{T}}_u \circ s}[i] : \mathrm{M} \qquad \mathrm{SU}_{\vec{\mathrm{fg}}} := \mathrm{U}_{\mathrm{su}_{\vec{\mathrm{fg}}}} : \mathrm{Set}$$

$\mathrm{SU}_{\vec{\mathrm{fg}}}$ is a subuniverse of u:

$$\frac{a : \mathrm{SU}_{\vec{\mathrm{fg}}}}{\underline{\mathrm{s}}_{\vec{\mathrm{fg}}}(a) : \mathrm{U}_u} \qquad \frac{a : \mathrm{SU}_{\vec{\mathrm{fg}}}}{\widehat{\mathrm{T}}_u(\mathrm{s}_{\vec{\mathrm{fg}}}(a)) = \widehat{\mathrm{T}}_{\mathrm{su}_{\vec{\mathrm{fg}}}}(a) : \mathbb{U}}$$

$\mathrm{SU}_{\vec{\mathrm{fg}}}$ is closed under f, g:

$$\frac{a : \mathrm{SU}_{\vec{\mathrm{fg}}} \qquad b : \mathrm{T}_{\mathrm{su}_{\vec{\mathrm{fg}}}}(a) \to \mathrm{SU}_{\vec{\mathrm{fg}}}}{\underline{\mathrm{appf}}_{\vec{\mathrm{fg}}}(a,b) : \mathrm{SU}_{\vec{\mathrm{fg}}}}$$

$$\frac{a : \mathrm{SU}_{\vec{\mathrm{fg}}} \qquad b : \mathrm{T}_{\mathrm{su}_{\vec{\mathrm{fg}}}}(a) \to \mathrm{SU}_{\vec{\mathrm{fg}}}}{\widehat{\mathrm{T}}_{\mathrm{su}_{\vec{\mathrm{fg}}}}(\mathrm{appf}_{\vec{\mathrm{fg}}}(a,b)) = \widehat{\mathrm{T}}_u(f(\mathrm{s}_{\vec{\mathrm{fg}}}(a), \mathrm{s}_{\vec{\mathrm{fg}}} \circ b)) : \mathbb{U}}$$

$$\frac{\begin{array}{c} a : \mathrm{SU}_{\vec{\mathrm{fg}}} \\ b : \mathrm{T}_{\mathrm{su}_{\vec{\mathrm{fg}}}}(a) \to \mathrm{SU}_{\vec{\mathrm{fg}}} \quad c : \mathrm{T}_u(f(\mathrm{s}_{\vec{\mathrm{fg}}}(a), \mathrm{s}_{\vec{\mathrm{fg}}} \circ b)) \end{array}}{\underline{\mathrm{appg}}_{\vec{\mathrm{fg}}}(a, b, c) : \mathrm{SU}_{\vec{\mathrm{fg}}}}$$

$$\frac{\begin{array}{c} a : \mathrm{SU}_{\vec{\mathrm{fg}}} \\ b : \mathrm{T}_{\mathrm{su}_{\vec{\mathrm{fg}}}}(a) \to \mathrm{SU}_{\vec{\mathrm{fg}}} \qquad c : \mathrm{T}_u(f(\mathrm{s}_{\vec{\mathrm{fg}}}(a), \mathrm{s}_{\vec{\mathrm{fg}}} \circ b)) \end{array}}{\widehat{\mathrm{T}}_{\mathrm{su}_{\vec{\mathrm{fg}}}}(\mathrm{appg}_{\vec{\mathrm{fg}}}(a, b, c)) = \widehat{\mathrm{T}}_u(g(\mathrm{s}_{\vec{\mathrm{fg}}}(a), \mathrm{s}_{\vec{\mathrm{fg}}} \circ b, c)) : \mathbb{U}}$$

Introduction rules for M

$$\frac{F : (a : \mathbb{U}, b : \mathrm{S}(a) \to \mathbb{U}) \to \mathbb{U} \quad G : (a : \mathbb{U}, b : \mathrm{S}(a) \to \mathbb{U}, \mathrm{U}_{F(a,b)}) \to \mathrm{M}}{\underline{\mathrm{mahlo}}(F, G) : \mathrm{M}}$$

And under the assumptions of the last rule we have:

$\widehat{\tau}^{a,b}(\mathrm{mahlo}(F, G)) = F(a, b) : \mathbb{U}$ $\qquad$ $\mathrm{mahlo}(F, G)^{a,b}[c] = G(a, b, c) : \mathrm{M}$

Existence of Mahlo universe

For every Mahlo-degree we have universe of its degree:

$$\frac{m : \mathrm{M}, \qquad f : (x : \mathbb{U}, y : \mathrm{S}(x) \to \mathbb{U}) \to \mathbb{U} \qquad g : (x : \mathbb{U}, y : \mathrm{S}(x) \to \mathbb{U}, z : \mathrm{S}(f(x, y))) \to \mathbb{U}}{\underline{\mathrm{v}}_{m,f,g} : \mathrm{Univ}}$$

Assume now in the following the assumptions of the last rule.

$\mathrm{m}(\mathrm{v}_{m,f,g}) = m : \mathrm{M}$

$\mathrm{V}_{m,f,g} := \mathrm{U}_{\mathrm{v}_{m,f,g}} : \mathrm{Set}$

$\mathrm{V}_{m,f,g}$ is closed under f, g:

$$\frac{a : \mathrm{V}_{m,f,g} \quad b : \mathrm{T}_{\mathrm{v}_{m,f,g}}(a) \to \mathrm{V}_{m,f,g}}{\underline{\mathrm{Appf}}_{m,f,g}(a, b) : \mathrm{V}_{m,f,g}}$$

$$\frac{a : \mathrm{V}_{m,f,g} \qquad\qquad b : \mathrm{T}_{\mathrm{v}_{m,f,g}}(a) \to \mathrm{V}_{m,f,g}}{\widehat{\mathrm{T}}_{\mathrm{v}_{m,f,g}}(\mathrm{Appf}_{m,f,g}(a, b)) = f(\widehat{\mathrm{T}}_{\mathrm{v}_{m,f,g}}(a), \widehat{\mathrm{T}}_{\mathrm{v}_{m,f,g}} \circ b) : \mathbb{U}}$$

$$\frac{\begin{array}{c} a : \mathrm{V}_{m,f,g} \\ b : \mathrm{T}_{\mathrm{v}_{m,f,g}}(a) \to \mathrm{V}_{m,f,g} \quad c : \mathrm{T}_u(f(\widehat{\mathrm{T}}_{\mathrm{v}_{m,f,g}}(a), \widehat{\mathrm{T}}_{\mathrm{v}_{m,f,g}} \circ b)) \end{array}}{\underline{\mathrm{Appg}}_{m,f,g}(a, b, c) : \mathrm{V}_{m,f,g}}$$

$$\frac{\begin{array}{c} a : \mathrm{V}_{m,f,g} \\ b : \mathrm{T}_{\mathrm{v}_{m,f,g}}(a) \to \mathrm{V}_{m,f,g} \qquad c : \mathrm{T}_u(f(\widehat{\mathrm{T}}_{\mathrm{v}_{m,f,g}}(a), \widehat{\mathrm{T}}_{\mathrm{v}_{m,f,g}} \circ b)) \end{array}}{\widehat{\mathrm{T}}_{\mathrm{v}_{m,f,g}}(\mathrm{Appg}_{m,f,g}(a, b, c)) = g(\widehat{\mathrm{T}}_{\mathrm{v}_{m,f,g}}(a), \widehat{\mathrm{T}}_{\mathrm{v}_{m,f,g}} \circ b), c) : \mathbb{U}}$$

Elimination rules for the universe

non applicable

2 The Π_n-reflecting universe

The following rules are tentative, whether we get the strength is not known yet.

Let $n \geq 3$ be fixed. $\mathrm{J} := \{2, \ldots, n\}$, $\mathrm{I} := \{(i_1, \ldots, i_l) \mid n \geq i_1 > \cdots > i_l \geq 2, l \geq 0\}$. If $\vec{i} = (i_1, \ldots, i_l)$, $\vec{i}, j := (i_1, \ldots, i_l, j)$. In the following $\vec{i}$ ranges over I, j over J, and whenever we write $\vec{i}, j$ we assume $\vec{i}, j \in \mathrm{I}$.

$$\underline{\mathbb{U}} : \mathrm{Set} \qquad\qquad \underline{\mathrm{Univ}}^{\vec{i}} : \mathrm{Set}$$

$$\frac{u : \mathrm{Univ}^{\vec{i}}}{\underline{\mathrm{univ}}^{\vec{i}}(U) : \mathbb{U}} \qquad\qquad \frac{a : \mathbb{U}}{\mathrm{S}(a) : \mathrm{Set}}$$

$$\frac{u : \mathrm{Univ}^{\vec{i}}}{\underline{\mathrm{U}}^{\vec{i}}_u : \mathrm{Set}} \qquad\qquad \frac{u : \mathrm{Univ}^{\vec{i}}}{\mathrm{S}(\mathrm{univ}^{\vec{i}}(u)) = \mathrm{U}^{\vec{i}}_u : \mathrm{Set}}$$

$$\frac{u : \mathrm{Univ}^{\vec{i}} \qquad a : \mathrm{U}^{\vec{i}}_u}{\widehat{\mathrm{T}}^{\vec{i}}_u(a) : \mathbb{U}}$$

If $u : \mathrm{Univ}^{\vec{i}}$, $a : \mathrm{U}^{\vec{i}}_u$ then $\mathrm{T}^{\vec{i}}_u(a) := \mathrm{S}(\widehat{\mathrm{T}}^{\vec{i}}_u(a)) : \mathrm{Set}$.

$$\frac{u : \mathrm{Univ}^{\vec{i}}}{\underline{\mathrm{P}}^{\vec{i},j}(u) : \mathrm{Set}} \qquad\qquad \frac{u : \mathrm{Univ}^{\vec{i},j,\vec{k}}}{\mathrm{lift}^{\vec{i};j;\vec{k}}(u) : \mathrm{Univ}^{\vec{i}}}$$

$$\frac{u : \mathrm{Univ}^{\vec{i},j,\vec{k}} \qquad a : \mathrm{U}^{\vec{i},j,\vec{k}}_u}{\underline{\mathrm{emb}}^{\vec{i};j;\vec{k}}_u(a) : \mathrm{U}^{\vec{i}}_{\mathrm{lift}_{\vec{i},j,\vec{k}}(u)}} \qquad\qquad \frac{u : \mathrm{Univ}^{\vec{i},j,\vec{k}}}{\mathrm{p}^{\vec{i};j;\vec{k}}(u) : \mathrm{P}^{\vec{i},j}(\mathrm{lift}^{\vec{i};j;\vec{k}}(u))}$$

$$\frac{u : \mathrm{Univ}^{\vec{i}} \qquad a : \mathrm{U}^{\vec{i}}_u \qquad b : \mathrm{T}^{\vec{i}}_u(a) \to \mathrm{U}^{\vec{i}}_u \qquad p : \mathrm{P}^{\vec{i},j}(u)}{\widehat{\tau}^{\vec{i},u,a,b}(p) : \mathrm{U}^{\vec{i}}_u}$$

$$\tau^{\vec{i},u,a,b}(p) := \mathrm{T}^{\vec{i}}_u(\widehat{\tau}^{\vec{i},u,a,b}(p))$$

$$\frac{u : \mathrm{Univ}^{\vec{i}} \qquad p : \mathrm{P}^{\vec{i},j}(u) \qquad a : \mathrm{U}^{\vec{i}}_u \qquad b : \mathrm{T}^{\vec{i}}_u(a) \to \mathrm{U}^{\vec{i}}_u \qquad i : \tau^{\vec{i},u,a,b}(m)}{p^{\vec{i},u,a,b}[i] : \mathrm{P}^{\vec{i},j}(u)}$$

$\mathbb{U}$ is a universe

$$\frac{a : \mathbb{U} \qquad b : \mathrm{S}(a) \to \mathbb{U}}{\underline{\widetilde{\Sigma}}(a,b) : \mathbb{U}} \qquad\qquad \frac{a : \mathbb{U} \qquad b : \mathrm{S}(a) \to \mathbb{U}}{\mathrm{S}(\widetilde{\Sigma}(a,b)) = \Sigma(\mathrm{S}(a), \mathrm{S} \circ b)}$$

Similarly closure under N_0, N_1, N_2, N, +, Π, W, I.

The elements of $\mathrm{Univ}^{\vec{i}}$ are universes

$$\frac{u : \mathrm{Univ}^{\vec{i}} \qquad a : \mathrm{U}^{\vec{i}}_u \qquad b : \mathrm{T}^{\vec{i}}_u(a) \to \mathrm{U}^{\vec{i}}_u}{\underline{\widehat{\Sigma}}(a,b) : \mathrm{U}^{\vec{i}}_u}$$

$$\frac{u : \mathrm{Univ}^{\vec{i}} \qquad a : \mathrm{U}^{\vec{i}}_u \qquad b : \mathrm{T}^{\vec{i}}_u(a) \to \mathrm{U}^{\vec{i}}_u}{\widehat{\mathrm{T}}^{\vec{i}}_u(\widehat{\Sigma}(a,b)) = \widetilde{\Sigma}(\widehat{\mathrm{T}}^{\vec{i}}_u(a), \widehat{\mathrm{T}}^{\vec{i}}_u \circ b) : \mathbb{U}}$$

Similarly closure under $\widetilde{\mathrm{N}}_0$, $\widetilde{\mathrm{N}}_1$, $\widetilde{\mathrm{N}}_2$, $\widetilde{\mathrm{N}}$, $\widetilde{+}$, $\widetilde{\Pi}$, $\widetilde{\mathrm{W}}$, $\widetilde{\mathrm{I}}$.

$\mathrm{Univ}^{\vec{i},j}$ is Mahlo and Π^k reflecting for k such that $\vec{i}, k \in \mathrm{I}$

Assume the following assumptions:

$$
\begin{array}{lcl}
u & : & \mathrm{Univ}^{\vec{i},j,\vec{k}} \\
r & : & \mathrm{U}_u^{\vec{i},j,\vec{k}} \\
s & : & \mathrm{T}_u^{\vec{i},j}(r) \to \mathrm{U}_u^{\vec{i},j} \\
\nu & : & \tau^{\vec{i},\mathrm{lift}^{\vec{i};j;\vec{k}}(u),\mathrm{emb}_u^{\vec{i};j;\vec{k}}(r),\mathrm{emb}_u^{\vec{i};j;\vec{k}} \circ s}(\mathrm{p}^{\vec{i};j;\vec{k}}(u)) \\
f & : & (x : \mathrm{U}_u^{\vec{i},j,\vec{k}}, y : \mathrm{T}_u^{\vec{i},j,\vec{k}}(x) \to \mathrm{U}_u^{\vec{i},j,\vec{k}}) \to \mathrm{U}_u^{\vec{i},j,\vec{k}} \\
g & : & (x : \mathrm{U}_u^{\vec{i},j,\vec{k}}, y : \mathrm{T}_u^{\vec{i},j,\vec{k}}(x) \to \mathrm{U}_u^{\vec{i},j,\vec{k}}, \mathrm{T}_u^{\vec{i},j,\vec{k}}(f(x,y))) \to \mathrm{U}_u^{\vec{i},j,\vec{k}} \\
f_l & : & (x : \mathrm{U}_u^{\vec{i},j,\vec{k}}, y : \mathrm{T}_u^{\vec{i},j,\vec{k}}(x) \to \mathrm{U}_u^{\vec{i},j,\vec{k}}) \to \mathrm{U}_u^{\vec{i},j,\vec{k}} \\
g_l & : & (x : \mathrm{U}_u^{\vec{i},j,\vec{k}}, y : \mathrm{T}_u^{\vec{i},j,\vec{k}}(x) \to \mathrm{U}_u^{\vec{i},j,\vec{k}}, \mathrm{T}_u^{\vec{i},j,\vec{k}}(f_i(x,y))) \to \mathrm{P}^{\vec{i},j,\vec{k},l}(u)
\end{array}
$$

********* Bis hier geaendert, ab jetzt noch einiges umzubauen ********

$$
\frac{\begin{array}{cc}
u : \mathrm{Univ}^{\vec{i},j} & r : \mathrm{U}_u^{\vec{i},j} \\
s : \mathrm{T}_u^{\vec{i},j}(r) \to \mathrm{U}_u^{\vec{i},j} & \nu : \tau^{\vec{i},\mathrm{lift}^{\vec{i},j}(u),\widehat{\mathrm{T}}_u^{\vec{i},j}(r),\widehat{\mathrm{T}}_u^{\vec{i},j} \circ s}(\mathrm{p}^{\vec{i},j}(u)) \\
f : (x : \mathrm{U}_u^{\vec{i},j}, y : \mathrm{T}_u^{\vec{i},j}(x) \to \mathrm{U}_u^{\vec{i},j}) \to \mathrm{U}_u^{\vec{i},j} & g : (x : \mathrm{U}_u^{\vec{i},j}, y : \mathrm{T}_u^{\vec{i},j}(x) \to \mathrm{U}_u^{\vec{i},j}, \mathrm{T}_u^{\vec{i},j}(f(x,y))) \to \mathrm{U}_u^{\vec{i},j} \\
s_k : \mathrm{U}_u^{\vec{i},j} \ (2 \le k < j) & p_k : \mathrm{T}_u^{\vec{i},j}(s_k) \to \mathrm{P}^{\vec{i},j,k}(u) \ (2 \le k < j)
\end{array}}{\underline{\mathrm{su}}_{\vec{i},j,u,r,s,\nu,f,g,s_{j-1},p_{j-1},\ldots,s_2,p_2} : \mathrm{Univ}^{\vec{i},j}}
$$

Assume now in the following the assumptions of the last rule.
We write $\vec{\mathrm{fg}}$ for $\vec{i}, j, u, r, s, \nu, f, g, s_{j-1}, p_{j-1}, \ldots, s_2, p_2$.

$\mathrm{lift}^{\vec{i},j}(\mathrm{su}_{\vec{\mathrm{fg}}}) = \mathrm{lift}^{\vec{i},j}(u) : \mathrm{Univ}_{\vec{i}}$

$\mathrm{p}^{\vec{i},j}(\mathrm{su}_{\vec{\mathrm{fg}}}) = (\mathrm{p}(u))^{\vec{i},j,\mathrm{lift}^{\vec{i},j}(u),\widehat{\mathrm{T}}_u^{\vec{i},j}(r),\widehat{\mathrm{T}}_u^{\vec{i},j} \circ s}[\nu] : \mathrm{P}^{\vec{i},j}(\mathrm{lift}^{\vec{i},j}(u))$

$\mathrm{SU}_{\vec{\mathrm{fg}}} := \mathrm{U}_{\mathrm{su}_{\vec{\mathrm{fg}}}}^{\vec{i},j} : \mathrm{Set}$

$\mathrm{SU}_{\vec{\mathrm{fg}}}$ is a subuniverse of u:

$$
\frac{a : \mathrm{SU}_{\vec{\mathrm{fg}}}}{\underline{\mathrm{s}}_{\vec{\mathrm{fg}}}(a) : \mathrm{U}_u^{\vec{i},j}}
\qquad\qquad
\frac{a : \mathrm{SU}_{\vec{\mathrm{fg}}}}{\widehat{\mathrm{T}}_u^{\vec{i},j}(\mathrm{s}_{\vec{\mathrm{fg}}}(a)) = \widehat{\mathrm{T}}_{\mathrm{su}_{\vec{\mathrm{fg}}}}^{\vec{i},j}(a) : \mathbb{U}}
$$

$\mathrm{SU}_{\vec{\mathrm{fg}}}$ is closed under f, g:

$$
\frac{a : \mathrm{SU}_{\vec{\mathrm{fg}}} \quad b : \mathrm{T}_{\mathrm{su}_{\vec{\mathrm{fg}}}}^{\vec{i},j}(a) \to \mathrm{SU}_{\vec{\mathrm{fg}}}}{\underline{\mathrm{appf}}_{\vec{\mathrm{fg}}}(a,b) : \mathrm{SU}_{\vec{\mathrm{fg}}}}
$$

$$
\frac{a : \mathrm{SU}_{\vec{\mathrm{fg}}} \qquad\qquad b : \mathrm{T}_{\mathrm{su}_{\vec{\mathrm{fg}}}}^{\vec{i},j}(a) \to \mathrm{SU}_{\vec{\mathrm{fg}}}}{\widehat{\mathrm{T}}_{\mathrm{su}_{\vec{\mathrm{fg}}}}^{\vec{i},j}(\mathrm{appf}_{\vec{\mathrm{fg}}}(a,b)) = \widehat{\mathrm{T}}_u^{\vec{i},j}(f(\mathrm{s}_{\vec{\mathrm{fg}}}(a), \mathrm{s}_{\vec{\mathrm{fg}}} \circ b)) : \mathbb{U}}
$$

$$
\frac{\begin{array}{l}
a : \mathrm{SU}_{\vec{\mathrm{fg}}} \\
b : \mathrm{T}_{\mathrm{su}_{\vec{\mathrm{fg}}}}^{\vec{i},j}(a) \to \mathrm{SU}_{\vec{\mathrm{fg}}} \quad c : \mathrm{T}_u^{\vec{i},j}(f(\mathrm{s}_{\vec{\mathrm{fg}}}(a), \mathrm{s}_{\vec{\mathrm{fg}}} \circ b))
\end{array}}{\underline{\mathrm{appg}}_{\vec{\mathrm{fg}}}(a,b,c) : \mathrm{SU}_{\vec{\mathrm{fg}}}}
$$

$$\frac{a : \mathrm{SU}_{\vec{\mathrm{fg}}} \qquad b : \mathrm{T}^{\vec{i},j}_{\mathrm{su}_{\vec{\mathrm{fg}}}}(a) \to \mathrm{SU}_{\vec{\mathrm{fg}}} \qquad c : \mathrm{T}^{\vec{i},j}_{u}(f(\mathrm{s}_{\vec{\mathrm{fg}}}(a), \mathrm{s}_{\vec{\mathrm{fg}}} \circ b))}{\widehat{\mathrm{T}}^{\vec{i},j}_{\mathrm{su}_{\vec{\mathrm{fg}}}}(\mathrm{appg}_{\vec{\mathrm{fg}}}(a,b,c)) = \widehat{\mathrm{T}}^{\vec{i},j}_{u}(g(\mathrm{s}_{\vec{\mathrm{fg}}}(a), \mathrm{s}_{\vec{\mathrm{fg}}} \circ b, c)) : \mathbb{U}}$$

$\mathrm{SU}_{\vec{\mathrm{fg}}}$ has Π_k-degrees p_k $(2 \leq k < j)$

$$\underline{\mathrm{P}}^{+}_{\vec{\mathrm{fg}},k} : \mathrm{Set} \qquad \frac{a : \mathrm{P}^{+}_{\vec{\mathrm{fg}},k}}{\mathrm{pemb}_{+,\vec{\mathrm{fg}},k}(a) : \mathrm{P}^{\vec{i},j,k}(u)}$$

$$\frac{a : \mathrm{T}^{\vec{i},j}_{u}(s_k)}{\underline{\widehat{\mathrm{p}}}_{+,\vec{\mathrm{fg}},k}(a) : \mathrm{P}^{+}_{\vec{\mathrm{fg}},k}} \qquad \frac{a : \mathrm{T}^{\vec{i},j}_{u}(s_k)}{\mathrm{pemb}_{+,\vec{\mathrm{fg}},k}(\widehat{\mathrm{p}}_{+,\vec{\mathrm{fg}},k}(a)) = p_k(a) : \mathrm{P}^{\vec{i},j,k}(u)}$$

$$\frac{a : \mathrm{P}^{+}_{\vec{\mathrm{fg}},k} \qquad b : \mathrm{SU}_{\vec{\mathrm{fg}}} \qquad c : \mathrm{T}^{\vec{i},j}_{\mathrm{su}_{\vec{\mathrm{fg}}}}(b) \to \mathrm{SU}_{\vec{\mathrm{fg}}} \qquad \mu : \tau^{\vec{i},j,k,u,\mathrm{s}_{\vec{\mathrm{fg}}}(b),\mathrm{s}_{\vec{\mathrm{fg}}} \circ c}(\mathrm{inp}_{+,\vec{\mathrm{fg}},k}(a))}{a^{+,\vec{i},j,k,\vec{\mathrm{fg}},b,c}[\mu] : \mathrm{P}^{+}_{\vec{\mathrm{fg}},k}}$$

$$\frac{a : \mathrm{P}^{+}_{\vec{\mathrm{fg}},k} \qquad b : \mathrm{SU}_{\vec{\mathrm{fg}}} \qquad c : \mathrm{T}^{\vec{i},j}_{\mathrm{su}_{\vec{\mathrm{fg}}}}(b) \to \mathrm{SU}_{\vec{\mathrm{fg}}} \qquad \mu : \tau^{\vec{i},j,k,\mathrm{su}_{\vec{\mathrm{fg}}},\mathrm{s}_{\vec{\mathrm{fg}}}(b),\mathrm{s}_{\vec{\mathrm{fg}}} \circ c}(\mathrm{inp}_{+,\vec{\mathrm{fg}},k}(a))}{\mathrm{pemb}_{+,\vec{\mathrm{fg}},k}(a^{+,\vec{i},j,k,\vec{\mathrm{fg}},b,c}[\mu]) = (\mathrm{pemb}_{+,\vec{\mathrm{fg}},k}(a))^{\vec{i},j,k,u,\mathrm{s}_{\vec{\mathrm{fg}}}(b),\mathrm{s}_{\vec{\mathrm{fg}}} \circ c}[\mu] : \mathrm{P}^{\vec{i},j,k}(u)}$$

$$\frac{a : \mathrm{P}^{+}_{\vec{\mathrm{fg}},k}}{\underline{\mathrm{inp}}_{+,\vec{\mathrm{fg}},k}(a) : \mathrm{P}^{\vec{i},j,k}(\mathrm{su}_{\vec{\mathrm{fg}}})}$$

$$\frac{a : \mathrm{P}^{+}_{\vec{\mathrm{fg}},k} \qquad b : \mathrm{SU}_{\vec{\mathrm{fg}}} \qquad c : \mathrm{T}^{\vec{i},j}_{\mathrm{su}_{\vec{\mathrm{fg}}}}(b) \to \mathrm{SU}_{\vec{\mathrm{fg}}}}{\widehat{\tau}^{\vec{i},j,k,\mathrm{su}_{\vec{\mathrm{fg}}},b,c}(\mathrm{inp}_{+,\vec{\mathrm{fg}},k}(a)) = \widehat{\tau}^{\vec{i},j,k,u,\mathrm{s}_{\vec{\mathrm{fg}}}(b),\mathrm{s}_{\vec{\mathrm{fg}}} \circ c}(\mathrm{pemb}_{+,\vec{\mathrm{fg}},k}(a))}$$

$$\frac{a : \mathrm{P}^{+}_{\vec{\mathrm{fg}},k} \qquad b : \mathrm{SU}_{\vec{\mathrm{fg}}} \qquad c : \mathrm{T}^{\vec{i},j}_{\mathrm{su}_{\vec{\mathrm{fg}}}}(b) \to \mathrm{SU}_{\vec{\mathrm{fg}}} \qquad \mu : \tau^{\vec{i},j,k,\mathrm{su}_{\vec{\mathrm{fg}}},b,c}(\mathrm{inp}_{+,\vec{\mathrm{fg}},k}(a))}{(\mathrm{inp}_{+,\vec{\mathrm{fg}},k}(a))^{\vec{i},j,k,\mathrm{su}_{\vec{\mathrm{fg}}},b,c}[\mu] = \mathrm{inp}_{+,\vec{\mathrm{fg}},k}(a^{+,\vec{i},j,k,\vec{\mathrm{fg}},b,c}[\mu])}$$

Introduction rules for $\mathrm{P}^{\vec{i},j}(u)$

$$\frac{u : \mathrm{Univ}^{\vec{i}} \qquad F : (a : \mathrm{U}^{\vec{i}}_{u}, b : \mathrm{T}^{\vec{i}}_{u}(a) \to \mathrm{U}^{\vec{i}}_{u}) \to \mathrm{U}^{\vec{i}}_{u} \quad G : (a : \mathrm{U}^{\vec{i}}_{u}, b : \mathrm{T}^{\vec{i}}_{u}(a) \to \mathrm{U}^{\vec{i}}_{u}, \mathrm{U}^{\vec{i}}_{F(a,b)}) \to \mathrm{P}^{\vec{i},j}(u)}{\underline{\mathrm{mahlo}}_{\vec{i},j,u}(F,G) : \mathrm{P}^{\vec{i},j}(u)}$$

And under the assumptions of the last rule we have:

$\widehat{\tau}^{a,b}(\mathrm{mahlo}_{\vec{i},j,u}(F,G)) = F(a,b) : \mathrm{U}^{\vec{i}}_{u}$

$\mathrm{mahlo}_{\vec{i},j,u}(F,G)^{a,b}[c] = G(a,b,c) : \mathrm{P}^{\vec{i},j}(u)$

A universe is Π^k reflecting with all its degrees

$$\frac{\begin{array}{c} u : \mathrm{Univ}^{\vec{i}} \\ p : \mathrm{P}^{\vec{i},j}(u) \\ f : (x : \mathrm{U}^{\vec{i}}_u, y : \mathrm{T}^{\vec{i}}_u(x) \to \mathrm{U}^{\vec{i}}_u) \to \mathrm{U}^{\vec{i}}_u \\ g : (x : \mathrm{U}^{\vec{i}}_u, y : \mathrm{T}^{\vec{i}}_u(x) \to \mathrm{U}^{\vec{i}}_u, z : \mathrm{T}^{\vec{i}}_u(f(x,y))) \to \mathrm{U}^{\vec{i}}_u \\ s_k : \mathrm{U}^{\vec{i}}_u \ (2 \le k < j) \\ p_k : \mathrm{T}^{\vec{i}}_u(s_k) \to \mathrm{P}^{\vec{i},k}(u) \ (2 \le k < j) \end{array}}{\underline{\mathrm{v}}_{u,p,f,g,s_2,p_2,\ldots,s_{j-1},p_{j-1}} : \mathrm{Univ}^{\vec{i},j}}$$

Assume now in the following the assumptions of the last rule.

$\vec{\mathrm{fg}} := u, p, f, g, s_2, p_2, \ldots, s_{j-1}, p_{j-1}$.

$\mathrm{lift}^{\vec{i},j}(\mathrm{v}_{\vec{\mathrm{fg}}}) = u : \mathrm{Univ}^{\vec{i}}$ $\qquad$ $\mathrm{p}^{\vec{i},j}(\mathrm{v}_{\vec{\mathrm{fg}}}) = p : \mathrm{P}^{\vec{i},j}(u)$

$\mathrm{V}_{\vec{\mathrm{fg}}} := \mathrm{U}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}} : \mathrm{Set}$

$\mathrm{V}_{\vec{\mathrm{fg}}}$ is a subuniverse of u:

$$\frac{a : \mathrm{V}_{\vec{\mathrm{fg}}}}{\underline{\mathrm{s}}_{-,\vec{\mathrm{fg}}}(a) : \mathrm{U}^{\vec{i}}_u} \qquad \frac{a : \mathrm{V}_{\vec{\mathrm{fg}}}}{\widehat{\mathrm{T}}_u(\mathrm{s}_{-,\vec{\mathrm{fg}}}(a)) = \widehat{\mathrm{T}}_{\mathrm{v}_{\vec{\mathrm{fg}}}}(a) : \mathbb{U}}$$

$\mathrm{V}_{\vec{\mathrm{fg}}}$ is closed under f, g:

$$\frac{a : \mathrm{V}_{\vec{\mathrm{fg}}} \quad b : \mathrm{T}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}}(a) \to \mathrm{V}_{\vec{\mathrm{fg}}}}{\underline{\mathrm{Appf}}_{\vec{\mathrm{fg}}}(a,b) : \mathrm{V}_{\vec{\mathrm{fg}}}}$$

$$\frac{a : \mathrm{V}_{\vec{\mathrm{fg}}} \qquad b : \mathrm{T}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}}(a) \to \mathrm{V}_{\vec{\mathrm{fg}}}}{\widehat{\mathrm{T}}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}}(\mathrm{Appf}_{\vec{\mathrm{fg}}}(a,b)) = f(\widehat{\mathrm{T}}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}}(a), \widehat{\mathrm{T}}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}} \circ b) : \mathrm{Univ}^{\vec{i},j}}$$

$$\frac{\begin{array}{l} a : \mathrm{V}_{\vec{\mathrm{fg}}} \\ b : \mathrm{T}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}}(a) \to \mathrm{V}_{\vec{\mathrm{fg}}} \quad c : \mathrm{T}^{\vec{i},j}_u(f(\widehat{\mathrm{T}}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}}(a), \widehat{\mathrm{T}}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}} \circ b)) \end{array}}{\underline{\mathrm{Appg}}_{\vec{\mathrm{fg}}}(a,b,c) : \mathrm{V}_{\vec{\mathrm{fg}}}}$$

$$\frac{\begin{array}{l} a : \mathrm{V}_{\vec{\mathrm{fg}}} \\ b : \mathrm{T}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}}(a) \to \mathrm{V}_{\vec{\mathrm{fg}}} \qquad c : \mathrm{T}^{\vec{i},j}_u(f(\widehat{\mathrm{T}}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}}(a), \widehat{\mathrm{T}}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}} \circ b)) \end{array}}{\widehat{\mathrm{T}}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}}(\mathrm{Appg}_{\vec{\mathrm{fg}}}(a,b,c)) = g(\widehat{\mathrm{T}}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}}(a), \widehat{\mathrm{T}}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}} \circ b), c) : \mathrm{Univ}^{\vec{i},j}}$$

$\mathrm{V}_{\vec{\mathrm{fg}}}$ has Π_k-degrees p_k $(2 \le k < j)$

$\underline{\mathrm{P}}^{-}_{\vec{\mathrm{fg}},k} : \mathrm{Set}$

$$\frac{a : \mathrm{P}^{-}_{\vec{\mathrm{fg}},k}}{\mathrm{pemb}_{-,\vec{\mathrm{fg}},k}(a) : \mathrm{P}^{\vec{i},j,k}(u)}$$

$$\frac{a : \mathrm{T}^{\vec{i}}_u(s_k)}{\underline{\widehat{\mathrm{p}}}_{-,\vec{\mathrm{fg}},k}(a) : \mathrm{P}^{-}_{\vec{\mathrm{fg}},k}} \qquad \frac{a : \mathrm{T}^{\vec{i}}_u(s_k)}{\mathrm{pemb}_{-,\vec{\mathrm{fg}},k}(\widehat{\mathrm{p}}_{-,\vec{\mathrm{fg}},k}(a)) = p_k(a) : \mathrm{P}^{\vec{i},k}(u)}$$

$$\frac{a : \mathrm{P}^{-}_{\vec{\mathrm{fg}},k} \qquad b : \mathrm{V}_{\vec{\mathrm{fg}}} \qquad c : \mathrm{T}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}}(b) \to \mathrm{V}_{\vec{\mathrm{fg}}} \qquad \mu : \tau^{\vec{i},k,u,\mathrm{v}_{\vec{\mathrm{fg}}}(b),\mathrm{v}_{\vec{\mathrm{fg}}}\circ c}(\mathrm{inp}_{-,\vec{\mathrm{fg}},k}(a))}{a^{-,\vec{i},j,k,\vec{\mathrm{fg}},b,c}[\mu] : \mathrm{P}^{-}_{\vec{\mathrm{fg}},k}}$$

$$\frac{a : \mathrm{P}^{-}_{\vec{\mathrm{fg}},k} \qquad b : \mathrm{V}_{\vec{\mathrm{fg}}} \qquad c : \mathrm{T}^{\vec{i},j}_{\mathrm{V}_{\vec{\mathrm{fg}}}}(b) \to \mathrm{V}_{\vec{\mathrm{fg}}} \qquad \mu : \tau^{\vec{i},k,u,\mathrm{s}_{-\vec{\mathrm{fg}}}(b),\mathrm{s}_{-,\vec{\mathrm{fg}}}\circ c}(\mathrm{inp}_{-,\vec{\mathrm{fg}},k}(a))}{\mathrm{pemb}_{-,\vec{\mathrm{fg}},k}(a^{-,\vec{i},j,k,\vec{\mathrm{fg}},b,c}[\mu]) = (\mathrm{pemb}_{-,\vec{\mathrm{fg}},k}(a))^{\vec{i},k,u,\mathrm{s}_{-,\vec{\mathrm{fg}}}(b),\mathrm{s}_{\vec{\mathrm{fg}}}\circ c}[\mu] : \mathrm{P}^{\vec{i},k}(u)}$$

$$\frac{a : \mathrm{P}^{-}_{\vec{\mathrm{fg}},k}}{\underline{\mathrm{inp}}_{-,\vec{\mathrm{fg}},k}(a) : \mathrm{P}^{\vec{i},j,k}(\mathrm{v}_{\vec{\mathrm{fg}}})}$$

$$\frac{a : \mathrm{P}^{-}_{\vec{\mathrm{fg}},k} \qquad b : \mathrm{V}_{\vec{\mathrm{fg}}} \qquad c : \mathrm{T}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}}(b) \to \mathrm{V}_{\vec{\mathrm{fg}}}}{\widehat{\tau}^{\vec{i},j,k,\mathrm{v}_{\vec{\mathrm{fg}}},b,c}(\mathrm{inp}_{-,\vec{\mathrm{fg}},k}(a)) = \widehat{\tau}^{\vec{i},k,u,\mathrm{s}_{\vec{\mathrm{fg}}}(b),\mathrm{s}_{\vec{\mathrm{fg}}}\circ c}(\mathrm{pemb}_{-,\vec{\mathrm{fg}},k}(a))}$$

$$\frac{a : \mathrm{P}^{-}_{\vec{\mathrm{fg}},k} \qquad b : \mathrm{V}_{\vec{\mathrm{fg}}} \qquad c : \mathrm{T}^{\vec{i},j}_{\mathrm{v}_{\vec{\mathrm{fg}}}}(b) \to \mathrm{V}_{\vec{\mathrm{fg}}} \qquad \mu : \tau^{\vec{i},j,k,\mathrm{v}_{\vec{\mathrm{fg}}},b,c}(\mathrm{inp}_{-,\vec{\mathrm{fg}},k}(a))}{\mathrm{inp}_{-,\vec{\mathrm{fg}},k}(a)^{\vec{i},j,k,\mathrm{v}_{\vec{\mathrm{fg}}},b,c}[\mu] = \mathrm{inp}_{-,\vec{\mathrm{fg}},k}(a^{-,\vec{i},j,k,\vec{\mathrm{fg}},b,c}[\mu])}$$

Existence of Universes

$\underline{*} : \mathrm{Univ}^{()}$

Elimination rules for the universe

non applicable