Girard's cut elimination for iterated inductive definitions revisited

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1 $ID(\underline{Q})$

General Assumption 1.1 *(a) In the following, we will often introduce in the language of PA, extended by some predicates, new predicates. We will write $\lambda x.\phi(x)$, if ϕ is a formula, for the predicate P s.t. $P(x) :\Leftrightarrow \phi(x)$, similarly for more variables, eg $\lambda x, y, z.\phi(x, y, z)$ introduces a predicate with 3 arguments.*
If we have a formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_l)$, then $\phi(r_1, \ldots, r_n)$ stands for the l-ary predicate $\lambda y_1, \ldots, y_l.\phi(r_1, \ldots, r_n, y_1, \ldots, y_l)$.

(b) We will use capital letters X, Y, Z for indicate predicates. So $\phi(x, y, X)$ will be a formula, having (possibly) free variables x and y and one predicate P. Then $\phi(s, t, P)$ is the result of substituting x by s, y by t and (if X is eg a binary predicate) $X(r, r')$ by $P(r, r')$ (which means, that, if $P \equiv \lambda x, y.\psi(x, y)$, we have to replace $X(r, r')$ by $\psi(r, r')$).

(c) We define, if P is a n-ary predicate,$(r_1, \ldots, r_n) \in P :\equiv P(r_1, \ldots, r_n)$, esp. if $n = 1$ $r \in P :\equiv P(r)$.

Definition 1.2 *(a) For a binary predicate P, $P_\nu(s) := P(\nu, s)$, $P_{\prec\nu}(r, s) := r \prec \nu \wedge P(r, s)$, similar for a 4-ary predicate Φ, $\Phi_\nu(X, Y, x) := \Phi(\nu, X, Y, x)$. Therefore P_ν is a unary predicate, similarily for $P_{\prec\nu}$, Φ_ν.*
If A, B are unary (possibly $A \equiv P_\nu$), $A \subset B :\equiv \forall x.A(x) \to B(x)$, $A = B :\equiv A \subset B \wedge B \subset A$. We will talk in the same way of A as if it were a set, referring always to the formulas mentioned. Therefore P_ν denotes the set, s.t. $x \in P_\nu \Leftrightarrow P_\nu(x)$.

(b) The language of ID_R (we will omit in the following the mentioning of R) is the language of arithmetic, extended by a binary predicate I.

(c) We assume, that R is a primitive recursive linear ordering (usually a well-ordering). Therefore $R(x, y) \equiv f(x, y) = 0$ for some primitive recursive function f. $\underline{R}(x) :\equiv R(x, x)$, $\nu \prec \mu :\equiv R(\nu, \mu) \wedge \nu \neq \mu$, $\nu \preceq \mu :\equiv R(\nu, \mu)$, $\forall \nu \prec \rho.\phi :\equiv \forall \nu.\nu \prec \rho \to \phi$ etc.

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(d) *A formula $\Phi(\nu, X, Y, x)$ is fair, if X is a only positively occurring unary, Y a (arbitrarily occurring) binary predicate, ν, x are variables (esp. I does not occurr in I).*
We assume that we have some fair Φ fixed, and $\Phi_\nu(X, Y, x)$ will denote the set s.t. $x \in \Phi_\nu(X, Y) \leftrightarrow \Phi_\nu(X, Y, x)$ in the convention as in (a).
$\Phi_\nu(X, Y)$.

(e) *The axioms of ID_R or ID are axioms of PA (in the extended language) plus the axioms:*

(ID1)
$$\forall \nu \in \underline{R}.\Phi_\nu(P_\nu, P_{\prec\nu}) \subset P_\nu$$

(ID2)
$$\forall \nu \in \underline{R}.(\Phi_\nu(B, P_{\prec\nu}) \subset B) \to P_\nu \subset B$$

where in the last statement B is an arbitrary unary predicate (e.g. definable by a formula).

Definition 1.3 *If α, β are ordinals, $f : \alpha \to \beta$, we define $f(\alpha) := \beta$. Note, that this definition depends on the choice of β, so whenever we define a function we will explicitly declare domain and codomain of it.*

Definition 1.4 *(a) Let $\mathcal{O} := \{(\alpha, \beta, \Omega) | \alpha < \beta < \Omega\}$. We will assume that if $\underline{O} \in \mathcal{O}$, $\underline{O} = (\alpha, \beta, \Omega)$, and if $\underline{O}' \in \mathcal{O}$, $\underline{O}' = (\alpha', \beta', \Omega')$. Unless stated differently, $\underline{O}, \underline{O}'$ are always assumed to be elements of $\mathcal{O}$.*

(b) *If $\underline{O}, \underline{O}' \in \mathcal{O}$, then $f : \underline{O} \to \underline{O}'$ means, that $f : \Omega \to \Omega'$ and $f(\alpha) = \alpha'$, $f(\beta) = f(\beta')$.*

(c) *If $\underline{O}, \underline{O}' \in \mathcal{O}$, then we define $\underline{O} < \underline{O}'$ iff $\alpha = \alpha'$, $\beta = \beta'$, $\Omega < \Omega'$.*
We write $\iota_{\underline{O}\underline{O}'}$ for the inclusion: $\underline{O} \to \underline{O}'$, $\iota_{\underline{O}\underline{O}'}(z) := z$ (which implies $\iota_{\underline{O},\underline{O}'}(\Omega) = \Omega'$.

(d) *If $\Omega \le \Omega'$ (not necessarily $\underline{O} \le \underline{O}'$), then $\tilde{\iota}_{\underline{O}\underline{O}'}$ is the inclusion, $\Omega \to \Omega'$, $\tilde{\iota}_{\underline{O}\underline{O}'}(z) := z$.*

Definition 1.5 *(a) If γ is an ordinal, $\gamma^{\le} := (\gamma + 1)$ (considered as a set), $M_{-1} := M \cup \{-1\}$, (i.e. $\gamma^{\le}_{-1} = (\gamma + 1) \cup \{-1\}$).*
$\omega_{-1} = \omega \cup \{-1\}$).

(b) *If $\underline{O} \in \mathcal{O}$, then $Deg(\underline{O}) := \alpha^{\le}_{-1} \times \Omega^{\le}_{-1} \times \omega_{-1}$.*

Definition 1.6 *Assume $\underline{O} \in \mathcal{O}$.*

(a) *For $u', u \in \Omega^{\le}_{-1}$ we define $u' < u :\Leftrightarrow u' <_{\underline{O}} u :\Leftrightarrow (u \neq -1 \wedge (u' = -1 \vee u' < u)$.*
$u' \ll u :\Leftrightarrow u' \ll_{\underline{O}} u :\Leftrightarrow u' < u < \Omega \vee (u' < \beta \wedge u = \Omega)$.
The definition for the value -1 is only an auxiliary one, needed for $Deg(\underline{O})$ in (c). In the following by "for all $u' < s$" we mean "for all $u' < s$ s.t. $u' \neq -1$", similar for $\ll$.

(b) *For $(t, u), (t', u') \in \Omega^{\le}_{-1} \times \alpha^{\le}_{-1}$ we define $(t', u') \ll (t, u) \Leftrightarrow (t', u') \ll_{\underline{O}} (t, u) \Leftrightarrow (t' < t \wedge u' < \beta) \vee (t' = t \wedge u' \ll_{\underline{O}} u)$.*
$(t', u') < (t, u) :\Leftrightarrow t' < t \vee (t' = t \wedge u' < u)$, $(t', u') = (t, u) :\Leftrightarrow t' = t \wedge u' = u$.
For the values -1, the same applies as in (a), so by "for all $(t', u') < (t, u)$" we mean "for all $(t', u') < (t, u)$ s.t. $t' \neq -1$ and $u' \neq -1$".

(c) *For $(t, u, n), (t', u', n') \in Deg(\underline{O})$ we define $(t', u', n') < (t, u, n) :\Leftrightarrow (t' < t \vee (t' = t \wedge u' < u) \vee (t' = t \wedge u' = u \wedge n' < n))$.*

Remark 1.7 *Let $t, t', t'' \in \Omega^{\le}_{-1}$ or $t, t', t'' \in \Omega^{\le}_{-1} \times \alpha^{\le}_{-1}$.*

(a) *$t \ll t' \ll t'' \to t \ll t''$.*

(b) $t < t' \ll t'' \to (t \ll t'' \leftrightarrow t \ll t')$.

(c) *If* $f : \underline{O} \to \underline{O}'$, *then* $t < t' \to f(t) < f(t')$ *and* $t \ll t' \to f(t) \ll f(t')$.

Proof:
(a) Assume $t \ll t' \ll t''$.
If $t, t', t'' \in \Omega^{\leq}_{-1}$, then if $t'' = \Omega$, $t < t' \leq \beta$, $t \ll t''$, otherwise $t < t' < t'' < \Omega$.
Case $t = (s, u)$, $t' = (s', u')$, $t'' = (s'', u'')$: $s \leq s' \leq s''$. If $s < s'$ then $u < \beta$, $(s, u) \ll (s'', t'')$, if $s' < s''$, then $u < \beta$ or $u < u' < \beta$, $(s, u) \ll (s'', t'')$, and if $s = s' = s''$, then $u \ll u' \ll u''$, and again the assertion.
(b) Assume $t < t' \ll t''$.
If $t, t', t'' \in \Omega^{\leq}_{-1}$, then if $t'' = \Omega$, $t < t' < \beta$, $t \ll t''$, $t \ll t'$; if $t'' < \Omega$, $t < t' < t''$ and we have $t \ll t''$ and $t \ll t'$.
If $t = (s, u)$, $t' = (s', u')$, $t'' = (s'', u'')$ we have $s \leq s' \leq s''$. If $s < s'$ then $t \ll t' \leftrightarrow u < \beta \leftrightarrow t \ll t''$.
If $s = s' < s''$ then by $t < t' \ll t''$ follows $u < u' < \beta$, $t \ll t''$, $t \ll t'$.
If $s = s' = s''$, then the assertion follows as in the case $t, t', t'' \in \Omega^{\leq}$.
(c): easy.

Definition 1.8 *For* $\underline{O} \in \mathcal{O}$ *we define the semi-formal system* $ID_R(\underline{O})$ *or short* $ID(\underline{O})$, *in which we can interpret* ID_R.

(a) *The symbols of* $ID(\underline{O})$ *are the logical connectives* $\wedge$, $\vee$, $\forall$, $\exists$ *and* $\neg$, *the constant* 0, *the successor function* S, $+$ *and* $\cdot$, *symbols for primitive recursive functions (which are for this analysis not essential),* $=$, $<$ *(for* $=$ *and* $<$ *on natural numbers) further the predicates* $I\!N^s$ $(s \leq \Omega$, *unary),* IA^s $(s \leq \alpha$, *unary) and* $I^{s,t}$ $(s \leq \alpha, t \leq \Omega$, *binary).*
The formulas are built from primeformulas and negated primeformulas by $\wedge$, $\vee$, $\exists$ *and* $\forall$, *the negation being a defined operation using the deMorgan-laws.*
We identify α*-equivalent formulas, eg* $\forall x.A$ *and* $\forall y.A[x/y]$ *are identified, if* $A[x/y]$ *is an allowed substitution.*

(b) *Existential formulas are false arithmetical prime formulas,* $A \vee B$, $\exists x.A$, $I\!N^u(a)$, $IA^u(a)$, $I^{t,u}_\nu(a)$. *Universal formulas are the negation of existential formulas.*

(c) *Let* $N(X, x)$ *be the formula, depending on a unary predicate* X, *defined by* $N(X, x) :\equiv x = 0 \vee \exists y.(y \in X \wedge x = S(y))$.
N *is operator, by which we get the inductive definition of the natural numbers.*
Let $I\!N^{\preceq u}(a) := N(I\!N^u, a)$.
The rules of the system for $I\!N^u$ *will represent the axiom*

(N) $$\bigcup_{u' < u} I\!N^{\preceq u} \subset I\!N^u \subset \bigcup_{u' \ll u} I\!N^{\preceq u'} \quad u \leq \Omega$$

This means $I\!N^u$ *is the iteration of* N u *times,* $I\!N^u = \{x \in \omega | x < u\}$, $I\!N^{\preceq u} = \{x \in \omega | x \leq u\}$ *and further* $I\!N^\Omega \subset \bigcup_{t \prec \beta} I\!N^{\preceq t} = I\!N^\beta$, *especially* $N(I\!N^\beta) \subset I\!N^\Omega \subset I\!N^\beta$, *which is correct if* β *is at least the closure ordinal of* N, *i.e.* $\omega \leq \beta$.

(d) *Let* $Acc(X, x) := x \in \underline{R} \wedge \forall y \prec x.y \in X$.
Let $IA^{\preceq t}(a) := Acc(IA^t, a)$.
If we iterate Acc t *times, we get the elements of* $\underline{R}$ *of order type* $< t$.
The rules of the system for IA^u *will represent the following axiom:*

(IA) $$IA^u = \bigcup_{v < u} IA^{\preceq v} \qquad u < \alpha$$

Note, that these rules are symmetric.
This means IA^u *is the iteration of Acc u times,* $IA^u = \{x \in \underline{R} | \; \|x\| < u\}$ *and* $IA^{\preceq u} = \{x \in \underline{R} | \; \|x\| \leq u\}$.
We define $\|t\| < u :\equiv IA^u(t)$, $\|t\| \leq u :\equiv IA^{\preceq u}(t)$, $\|t\| = u :\equiv \|t\| \leq u \wedge \neg(\|t\| < u)$.

(e) *Let* $\Phi_{\nu,\underline{O}}(X, Y, x)$ *be the result of restricting every quantifier to* $I\!N^\Omega$*, i.e.* $\forall x.\phi(x)$ *is replaced by* $\forall x.x \in I\!N^\Omega \to \phi'(x)$, $\exists x.\phi(x)$ *by* $\exists x.x \in I\!N^\Omega \wedge \phi'(x)$.

$$\Phi^t_{\nu,\underline{O}}(X, x) := (\|\nu\| = t \wedge \Phi_{\nu,\underline{O}}(X, I^{t,0}_{\prec\nu}, x)) \vee (\|\nu\| < t \wedge x \in I)$$

i.e.

$$\Phi^t_{\nu,\underline{O}}(X, x) = \begin{cases} \Phi_{\nu,\underline{O}}(X, I^{t,0}_{\prec\nu}) & \text{if } \|\nu\| = t \\ X & \text{if } \|\nu\| < t \\ \emptyset & \text{if } t < \|\nu\| \end{cases}$$

$I^{\preceq t,u}(a) := I^{\preceq_{\underline{O}} t,u}(a) := \Phi^t_{\nu,\underline{O}}(I^{t,u}_\nu, a)$. *Note, that* $I^{\preceq_{\underline{O}} t,u}$ *depends only on* Ω.
The rules of the system for $I^{t,u}$ *will represent the following axioms:*

$$\text{(I)} \qquad \bigcup_{(t',u')<(t,u)} I^{\preceq_{\underline{O}} t',u'} \subset I^{t,u} \subset \bigcup_{(t',u')\ll(t,u)} I^{\preceq_{\underline{O}} t',u'} \qquad (t, u) \leq (\alpha, \Omega)$$

Therefore we have, if $\|\nu\| = t$, $u < \Omega$, $I^{t,u}_\nu = (\Phi_{\nu,\underline{O}}(\cdot, I^{t,0}_{\prec\nu}))^u$*, the assymetry expresses* $\Phi_{\nu,\underline{O}}(I^{t,\beta}_\nu, I^{t,0}_{\prec\nu}) \subset I^{t,\Omega}_\nu \subset \bigcup_{(t',u')\ll(t,\Omega)} I^{\preceq t',u'} \subset I^{t,\beta}$.
In set theory follows from these axioms for all (t, u) *s.t.* $\|\nu\| < t$*, that* $I^{t,u}_\nu = I^{\|\nu\|,\beta}_\nu$.
Proof by induction on (t, u) *(*$<$ *is naturally a well-ordering for* $t \preceq \alpha$, $u \preceq \Omega$*):* $I^{t,u}_\nu = \bigcup_{(t',u')\ll(t,u)} I^{t',u'}_\nu = \bigcup_{\substack{(t',u')\ll(t,u) \\ (t',u')<(\|\nu\|\Omega)}} I^{t',u'}_\nu \cup \bigcup_{\substack{(t',u')\ll(t,u) \\ (t',u')<(\|\nu\|\Omega)}} I^{t',u'}_\nu \subset I^{\|\nu\|,\Omega}_\nu \cup I^{\|\nu\|,\beta}_\nu \subset I^{\|\nu\|,\beta}_\nu \subset I^{t,u}_\nu$.
For $t < \|\nu\|$ *we have* $I^{t,u}_\nu = \emptyset$.
The rules are correct, if β *is at least the closure ordinal of the* ν*th inductive definition, i.e.* $\omega^{ck}_\nu \leq \beta$*. If* α *is the ordertype of* R*, and* α *is a limit ordinal, we have the condition* $\omega^{ck}_\alpha \leq \beta$.
Girard had a different definition, namely

$$\Phi^t_{\nu,\underline{O}}(X, x) :\equiv Acc(IA^t, \nu) \wedge \Phi_{\nu,\underline{O}}(X, I^{t,0}_{\prec\nu}, x),$$

therefore

$$\Phi_{\nu,\underline{O}}(X, x) = \begin{cases} \Phi_{\nu,\underline{O}}(X, I^{t,0}_{\prec\nu}) & \text{if } \|\nu\| \preceq t \\ \emptyset & \text{otherwise} \end{cases}$$

(f) *Let* $Term_{Cl}$ *be the set of of closed Terms.*

(g) *We define the relation* $A \doteq_{\underline{O}} \bigwedge_{i\in I} A_i$ *or* $A \doteq_{\underline{O}} \bigvee_{i\in I} A_i$ *for* A *being a closed formula in* $ID(\underline{O})$ *as follows (we usually omit the index* $\underline{O}$*):*
(We abbreviate $\bigvee_{i\in\{i'\in J | i<j\}} \cdots$*, by* $\bigvee_{i<j} \cdots \bigvee_{i\in\{i'\in J | i<j\}}$*, where* $J = Ord$*, the class of ordinals, or* $J = Ord \times Ord$*, corresponding to* j*, similar for* $<_{\underline{O}}$, $\ll$, $\bigwedge$.
$A \doteq \bigvee_{i\in\emptyset} A_\iota$ *if* A *is a false arithemtical primeformula.*
$B \vee C \doteq\doteq \bigvee_{i\in\{\underline{0},\underline{1}\}} A_i$ *with* $A_{\underline{0}} := B$, $A_{\underline{1}} := C$ $\qquad \exists x.A \doteq \bigvee_{t\in Term_{Cl}} A[x/t]$,
$N^u(a) \doteq_{\underline{O}} \bigvee_{u'<_{\underline{O}} u} N^{\preceq u'}(a)$, $\qquad IA^u(a) \doteq \bigvee_{u'<u} IA^{\preceq u'}(a)$,
$I^{t,u}_\nu(a) \doteq_{\underline{O}} \bigvee_{(t',u')<_{\underline{O}}(t,u)} I^{\preceq_{\underline{O}} t',u'}_\nu(a)$,
$\neg A \doteq \bigvee_{i\in I} \neg A_i$ *if* $A \doteq \bigwedge_{i\in I} A_i$*, if* A *is a universal formula, except in the following two cases:*
$\neg N^u(a) \doteq_{\underline{O}} \bigwedge_{u'\ll_{\underline{O}} u} \neg N^{\preceq u'}(a)$, $\qquad \neg I^{t,u}_\nu(a) \doteq_{\underline{O}} \bigwedge_{(t',u')\ll_{\underline{O}}(t,u)} I^{\preceq t',u'}_\nu(a)$,

(h) *We have the following list of rules:*

$$(Cut, C) \quad \frac{C \quad \neg C}{}$$

$(\vee, A, j) \quad \frac{A_j}{A} \quad$ *(if* $A \doteq \bigvee_{i\in I} A_i$, $j \in I$*)* $\qquad (\wedge, A) \quad \frac{\cdots A_j \cdots \quad (j\in I)}{A} \quad$ *(if* $A \doteq \bigwedge_{i\in I} A_i$*)*

The premisses are numbered by $J = \{\underline{0}, \underline{1}\}$ *in the case of* (Cut, C), $J = I$ *in the case of* $(\wedge, A)$ *and* $J = \{\underline{0}\}$ *in the case of* $(\vee, A, i)$, *the* J *is called the index set of the rule.*
The main formulas of $(\vee, A, j)$ *and* $(\wedge, A)$ *are* A, *and* (Cut) *has no main formula.*

We define now inductively $\vdash^{\underline{O};\delta} \Gamma$, *by:*
If

$(Rule) \quad \frac{\Delta_i (i\in I)}{\Delta}$

is any rule, $\Delta \subset \Gamma$, *and* $\forall i \in I.\exists\delta' < \delta. \vdash^{\underline{O};\delta'} \Gamma, \Delta_i$ *then* $\vdash^{\underline{O};\delta} \Gamma$ *holds.*

(i) *We can define proofs as formal objects: If* $(Rule)$ *is a rule with index set* I, *if the conclusion is* Δ, *the premisses are* Δ_i, $(P_i)_{i\in I}$ *are proofs of* Γ, Δ, Δ_i, *of height* δ_i, *then* $(\Gamma, (Rule), (P_i)_{i\in I})$ *is a proof of* Γ, Δ *of height* $sup\{\delta_i + 1 | i \in I\}$.

(j) *If* $P = (\Gamma, (Rule), (P_i)_{i\in I}$ *is a proof,* $Rule(P) := (Rule)$.

Definition 1.9 (a) *For* $(t, u, n) \in Deg(\underline{O})$, *we define* $(t, u, n) + 1 := (t, u, n + 1)$.

(b) *We define the degree* $d^o(A) \in Deg(\underline{O})$ *and the predicative degree* $d^p(A) \in Deg(\underline{O})$ *of a formula* A *in* $ID(\underline{O})$ *by:*
$d^o(A) := d^p(A) := (-1, -1, 0)$ *if* A *is a prime arithmetical formula,*
$d^o(I\!N^t(a)) := (-1, t, 0)$.
$d^p(I\!N^t(a)) := (-1, t, 0)$ *if* $t \neq \Omega$, $d^p(I\!N^\Omega(a)) := (-1, -1, -1)$.
$d^o(IA^t(a)) := d^p(IA^t(a)) := (t, 0, 0)$.
$d^o(I^{t,u}_\nu(a)) := (t, u, 0)$.
$d^p(I^{t,u}_\nu(a)) := (t, u, 0)$ *(if* $u \notin \{0, \Omega\}$*),* $d^p(I^{t,0}_\nu(a)) := d^p(I^{t,\Omega}_\nu(a)) := (-1, -1, -1)$.
$d^o(A \vee B) := max\{d^o(A), d^o(B)\} + 1$, $d^p(A \vee B) := max\{d^p(A), d^p(B)\} + 1$,
$d^o(\exists x.A) := d^o(A) + 1$, $d^p(\exists x.A) := d^p(A) + 1$.
$d^p(\neg A) := d^p(A)$, $d^o(\neg A) := d^o(A)$, *if* A *is a universal formula .*
Note, that, if B *is a subformula of* A, *then* $d^o(B) < d^o(A)$.

Definition 1.10 (a) *The degree of a cut with cut formula* A *is* $d^o(A)$, *its predicative cut degree* $d^p(A)$.

(b) *A proof* P *in* $ID(\underline{O})$ *is of degree* d *iff for the degrees of its cuts we have* $d' < d$, *and of predicative cut degree* d' *iff for the predicative degrees of its cuts we have* $d' < d$. *We write* $P \vdash^{\underline{O};\alpha}_{d,d_p} \Gamma$ *for* P *is a proof of* Γ *in* $ID(\underline{O})$ *of degree* d *and predicative degree* d_p.

Definition 1.11 *Given a formula* A *of* $ID(\underline{O})$, d *a degree of the same theory, then* A_r *is the result of replacing every ordinal parameter* l, *s.t.*$\beta < l < \Omega$, *by* Ω.

Definition 1.12 *Assume* $\underline{O} \in \mathcal{O}$

(a) *We define the set of indices for rules as* $\mathcal{I}(\underline{O}) := Term_{Cl} \cup \{\underline{0}, \underline{1}\} \cup \Omega^{\leq} \cup (\alpha^{\leq} \times \Omega^{\leq})$.
$\vec{\mathcal{I}}(\underline{O})$ *is the set of not empty sequences of* $\mathcal{I}(\underline{O})$.

(b) If $P = (\Gamma, \Delta, (Rule), (P_i)_{i \in I})$, then $P[i] := P_i$.
If $A \doteq \bigwedge_{i \in I} B_i$ or $A \doteq \bigvee_{i \in I} B_i$, then $A[i] := B_i$, if $i \in I$.

(c) In the situation above, for some $\vec{i} \in \vec{\mathcal{I}}(\underline{O})$, and P an $ID(\underline{O})$-proof or formula we define $P[\vec{i}]$: If $\vec{i}$ is the list containing the element i, then $P[\vec{i}] :\simeq P[i]$, and if $\vec{i} = i, \vec{j}$, then $P[\vec{i}]$ is defined if $P[i]$ and $P[i][\vec{j}]$ are defined and in this case defined as $P[i][\vec{j}]$.

2 Images of Objects under functions f

Definition 2.1 *Assume $\underline{O}', \underline{O} \in \mathcal{O}$.*
A good function $f : \underline{O}' \to \underline{O}$ is a function $f \in I(\Omega, \Omega')$, s.t. $f(\alpha') = \alpha$, and $f(\beta') = \beta$.

(b) In the following we will define the image under f of indices, degrees, formulas, sequences, rules. In this case we have the usual definition of $f^{-1}(Q)$: $f^{-1}(Q)$ is defined iff $\exists P.f(P) = Q$, and in this case $f^{-1}(Q) = P$. f will always be injective therefore $f^{-1}(Q)$ is uniquely defined, if it is defined.

(c) We will always extend a function $f : \Omega' \to \Omega$ by $f(\Omega') := \Omega$.

(d) If $\underline{O} < \underline{O}'$, then $\iota_{\underline{O},\underline{O}'}$ is the inclusion $\Omega \to \Omega'$.

Definition 2.2 *Assume $\underline{O}', \underline{O} \in \mathcal{O}$ $f : \underline{O}' \to \underline{O}$ good.*

(a) The ordinal parameters in a formula or sequence of $ID(\underline{O})$ are the parameter u in the prime formulas $I\!N^u(a)$, $IA^u(a)$ and t, u in $I_\nu^{t,u}(a)$.

(b) The ordinal parameters in a rule of $ID(\underline{O})$ are the ordinal parameter of A in (Cut, A), $(\wedge, A)$ and i and the ordinal parameter of A in $(\vee, A, i)$.

(c) The image of a rule, formula, sequence under f is the result of replacing the ordinal parameters of it by their image under f. Note, that if the object was of $ID(\underline{O}')$, $f : \underline{O}' \to \underline{O}$, then the image of the object will be an object of $ID(\underline{O})$.

(d) If $i \in \vec{\mathcal{I}}(\underline{O}')$, then we define $\hat{f}(i)$ by $\hat{f}(i) = i$ if $i \in \{\underline{0}, \underline{1}\} \cup Term_{Cl}$, $\hat{f}(i) := f(i)$ for $i \in \Omega + 1$ and $\hat{f}(s,t) := (f(s), f(t))$. We write $f(i)$ for $\hat{f}(i)$ and extend f as well to $\vec{\mathcal{I}}$ by applying f to the elements of the sequence.

(e) $f(-1) := -1$.

(f) If $(t, u, n) \in Deg(\underline{O})$, then $f(t, u, n) := (f(t), f(u), n)$.

(g) For proofs we cannot define the image of a proof under f, since we get too few premisses, but only possibly the inverse image under it:
Assume $Q = (\Gamma, \Delta, (Rule), (P_i)_{i \in I})$, then $f^{-1}(Q)$ is defined iff $\forall i \in range(f).f^{-1}(P_i)$ is defined (f extended as in definition 2.1 (b)). and $f^{-1}(Rule)$, $f^{-1}(\Gamma)$, $f^{-1}(\Delta)$ are defined. In this case $f^{-1}(Q) := (f^{-1}(\Gamma), f^{-1}(\Delta), f^{-1}(Rule), (f^{-1}(P_{f(i)})_{i \in f^{-1}(I)}))$.

Remark 2.3 *(a) If A is a formula of $ID(\underline{O}')$, $f : \Omega' \to \Omega$), then $d^o(f(B)) = f(d^o(B))$.*

(b) Note, that in the following cases, if P is a proof in $ID(\underline{O})$, we have a unique Q s.t. $f^{-1}(Q) = P$, written $f(P) = Q$, in each of the following two cases:
(i) If $\underline{O}' \leq \underline{O}$ and $f = \iota_{\underline{O}'\underline{O}}$ (the inclusion).

(ii) If P is cut-free and all ordinal-parameters $< \Omega$ in the conclusion l are such that $f(l) = l$.
(ii) can be rewritten as:
(ii') If $P \vdash_{0,0} \Gamma$ for some Γ s.t. $f(\Gamma) = \widetilde{\iota}_{\underline{O}\underline{O}'}(\Gamma)$.

Lemma 2.4 *Assume $\underline{O}_1, \underline{O}_2, \underline{O}_3, \underline{O}_4 \in \mathcal{O}$, $f : \underline{O}_1 \to \underline{O}_2$, $h : \underline{O}_2 \to \underline{O}_4$, $g : \underline{O}_1 \to \underline{O}_3$, $k : \underline{O}_3 \to \underline{O}_4$, $h \circ f = k \circ g$.*

(a) If i is an ordinal, index, formula, sequence of $ID(\underline{O}_3)$, $g^{-1}(i)$ is defined, then $h^{-1}(k(i))$ is defined, $h^{-1}(k(i)) = f(g^{-1}(i))$.

(b) If $I \subset \mathcal{I}(\underline{O}_3)$, $f(g^{-1}(I)) \subset h^{-1}(k(I))$.

(c) If Q is a proof in $ID(\underline{O}_3)$, s.t. $f(g^{-1}(Q))$, $h^{-1}(k(Q))$ are proofs, then $f(g^{-1}(Q)) = h^{-1}(k(Q))$.

Proof:
(a) Case $i < \Omega$. By assumption $i = g(j)$, $fg^{-1}(i) = f(j)$, $hf(j) = kg(j) = k(i)$, therefore $h^{-1}(k(i)) = f(j) = fg^{-1}(i)$.
The other cases follow, since the functions just act on the parameters.
(b) by (a).
(c) Proof by induction on the derivations.
Let
$Q =$

$$\frac{\cdots Q_i \vdash \Gamma_i \cdots (i \in I)}{\Gamma}\ (\mathit{Rule})$$

Then
$f(g^{-1}(Q)) =$

$$\frac{\cdots f(g^{-1}(Q_{f^{-1}(g(i))})) \cdots (i \in f(g^{-1}(I)))}{f(g^{-1}(\Gamma))}\ f(g^{-1}(\mathit{Rule}))$$

$h^{-1}k(Q) =$

$$\frac{\cdots h^{-1}(k(Q_{h(k^{-1}(i))})) \cdots (i \in h^{-1}(k(I)))}{h^{-1}(k(\Gamma))}\ h^{-1}(k(\mathit{Rule}))$$

Now $fg^{-1}(\mathit{Rule}) = h^{-1}(k(\mathit{Rule}))$, $f(g^{-1}(I)) \subset h^{-1}(k(I))$. Since the proofs are correct, these sets must be identical. Further, for $i \in f(g^{-1}(I))$, $i = fg^{-1}(j) = h^{-1}k(j)$, therefore $g(f^{-1}(i)) = k^{-1}(h(j))$, $Q_{g(f^{-1}(i))} = Q_{k^{-1}(h(j))}$. Now $h^{-1}(k(Q_{g(f^{-1}(i))}))$, $fg^{-1}(Q_{k^{-1}(h(j))}))$ are defined, by IH therefore equal, further the resulting sequences are equal, and therefore $h^{-1}(k(Q)) = fg^{-1}(Q)$.

Remark 2.5 *Assume $f : \underline{O} \to \underline{O}'$ good, $t, t' \in \Omega^{\leq}_{-1}$ or $t, t' \in \Omega^{\leq}_{-1} \times \alpha^{\leq}_{-1}$ or $t, t' \in \mathit{Deg}(\underline{O})$.*

(a) $t < t' \to f(t) < f(t')$.

(b) $t \ll_{\underline{O}} t' \to f(t) \ll_{\underline{O}'} f(t')$.

3 Categories

Definition 3.1 *(a) Let $\mathcal{C}$ be a category. A category $\mathcal{D}$ is a direct extension of $\mathcal{C}$ if $Ob(\mathcal{D}) \subset Ob(\mathcal{C}) \times A$ for some class A and $Mor(\mathcal{D})((a,b),(a',b')) = Mor(\mathcal{C})(a,a')$ $(a, a' \in \mathcal{C},\, b, b' \in A)$.*

(b) If $\mathcal{D}$ extends a category $\mathcal{C}$, and $\mathcal{E}$ is a category, $L : \mathcal{E} \to \mathcal{C}$ and $M : \mathcal{E} \to \mathcal{D}$, then we say M extends L, if $M(x) = (L(x), a(x))$ for some $a(x) \in A$. We write in this case M_x for $a(x)$.

Definition 3.2 *(a) A finite set K is good, if $\underline{0}, \underline{a} \in K$, where $\underline{0}, \underline{a}$ are some distinguished objects.*

(b) Let ON be the category with $Ob(ON)$ being the class of ordinals and $Mor_{ON}(x,y) := I(x,y) := \{f : x \to y | f \text{ is strictly increasing}\}$.

(c) Let K be good. ON^K is the category with

$$Ob(ON^K) := \{(x,d) | x \text{ Ordinal }, d : K \to x, d(\underline{0}) = 0\}$$

and

$$Mor_{ON^K}((x,d),(y,e)) := J((x,d),(y,e)) := \{f \in I(x,y) | e = f \circ d\}$$

(d) We replace now Girard's definition of a good Category by the following definition of a suitable category, which seems to be the more general principle behind:
A subcategory $\mathcal{C}$ of ON^K for some finite good K is called a suitable category, if the following holds:

(i) $\mathcal{C}$ is a full subcategory of ON^K, e.g. for $(x,d),(y,e) \in \mathcal{C}$, $Mor_{\mathcal{C}}((x,d),(y,e)) = J((x,d),(y,e))$.

(ii) $\mathcal{C}$ is an initial segment of ON^K, i.e. if $(x,d)\mathcal{C}$, $(y,e) \in ON^K$, $J((y,e),(x,d)) \neq \emptyset$, then $(y,e) \in \mathcal{C}$.

(iii) $\mathcal{C} \neq \emptyset$.

(iv) If $(x,d) \in \mathcal{C}$, $x < y$, then $(y,d) \in \mathcal{C}$.

(e) If $\mathcal{C}$, $\mathcal{C}'$ are suitable, $\mathcal{C} \subset ON^K$, $\mathcal{C}' \subset ON^{K'}$, then $\mathcal{C} \leq \mathcal{C}' :\Leftrightarrow K \subset K' \wedge (\forall (x,d) \in \mathcal{C}'.(x, d|K) \in \mathcal{C}) \wedge (\forall (x,d) \in \mathcal{C}.\exists e : K' \to ON.e|K = d \wedge \exists x < y.(y,e) \in \mathcal{C}')$.
(Here $|$ stands for restriction).
Further we define $\mathcal{C} \subset \mathcal{C}' \Leftrightarrow K \subset K' \wedge (\forall (x,d) \in \mathcal{C}'.(x, d|K) \in \mathcal{C})$.

(f) If $\mathcal{C} \subset \mathcal{C}'$, (e.g. $\mathcal{C} \leq \mathcal{C}'$), $\mathcal{C} \subset ON^K$, $\mathcal{C}' \subset ON^{K'}$, F a functor from $\mathcal{C}$ into some category $\mathcal{D}$, then $\theta_{\mathcal{C}\mathcal{C}'}(F)$ is the functor from $\mathcal{C}'$ into $\mathcal{D}$, defined by $\theta_{\mathcal{C},\mathcal{C}'}(F)(x,d') = F(x, d'|K)$, $\theta_{\mathcal{C},\mathcal{C}'}(F)(f) = F(f)$.

(g) $K_0 := \{\underline{0}, \underline{a}\}$, $\mathcal{C}_0 := ON^{K_0}$.
K_d^x is the full subcategory of ON^K s.t. $Ob(K_d^x) = \{(y,e) \in ON^K | \exists x' > x.J((y,e),(x',d)) \neq \emptyset)\}$.

(h) A good functor on a suitable category $\mathcal{C}$ is a functor $\mathcal{C} \to ON$, s.t.
(i) L commutes with direct limits and pull backs
(ii) $L = I + L'$ for some L',
i.e. $L(x,d) = x + L'(x,d)$, if $f \in J((x,d),(y,e))$, then $L(f) = f + L'(f)$, $L(f)(z) = f(z)$ if $z < x$, $L(f)(x+z) = y + L'(f)(z)$.
$L_0 := I$ (identity).

(i) *If L is a good functor on $\mathcal{C}$, an L-parameter is a Functor $l : \mathcal{C} \to ON$ s.t. $l(x,d) \leq L(x,d)$ for $(x,d) \in \mathcal{C}$ and for $f \in J((x,d),(y,e))$, $L(f)(l(x,d)) = l(y,e)$.*
Special L-parameters are $l(x,d) = 0$ (written as 0), $l(x,d) = d(\underline{a})$ (written as $\underline{\alpha}$), $l(x,d) = x$ (written as $\underline{\beta}$) and $l(x,d) = L(x,d)$ (written as $\underline{\Omega}$).
We define, if l,m are L-parameters $l \leq m :\Leftrightarrow \forall(x,d) \in \mathcal{C}.l(x,d) \leq m(x,d)$, $l < m :\Leftrightarrow \forall(x,d) \in \mathcal{C}.l(x,d) < m(x,d)$, and $l+m$ to be the L-parameter, s.t. $(l+m)(x,d) = l(x,d) + m(x,d)$.

Remark 3.3 *(a) K^x_d is suitable.*

(b) If $\mathcal{C}$ is suitable, $\mathcal{C} \subset ON^K$, $(x,d) \in \mathcal{C}$, then, $K^x_d \subset \mathcal{C}$.

Proof:
(a) K^x_d is suitable: $K^x_d \neq \emptyset$, since $(x',d) \in K^x_d$, if $x < x'$ and $rng(d) \subset x'$. If $(y,e) \in K^x_d$, $y < z$, $h \in J((y,e),(x',d)) \neq \emptyset$, $x < x'$, then let $z = y + z'$, $g : z \to x' + z'$, $g(u) := h(u)$ for $u < y$, $g(z+u) := x' + u$, then $g \in J((z,e),(x'+z',d))$, $(z,e) \in K^x_d$. That K^x_d is an initial segment is obvious.
(b) obvious.

Definition 3.4 *Assume L is a good functor, then $L(x,d)$ represents the element $(d(\underline{a}),x,L(x,d)) \in \mathcal{O}$, e.g. $ID(L(x,d)) := ID(d(\underline{a}),x,L(x,d))$, or $\vdash^{L(x,d);\delta} \Gamma$ stands for $\vdash^{(d(\underline{a}),x,L(x,d));\delta} \Gamma$.*

Definition 3.5 *Assume L, M are good functors.*

(a) *A natural transformation between good functors $L = I + L'$ and $M = I + M'$ is good, if $T = I + T'$ (i.e. $T_{x,d}(z) = z$ if $z < x$, $T_{x,d}(x+z) = x + T'_{xd}(z)$).*

(b) *$L \leq M$ iff $M = L + L'$.*

(c) *If $L(x,d) \leq M(x,d)$ for all x,d, then let $\widetilde{E}_{LM}$ represent the family of functions $\widetilde{E}_{LMxd} := \widetilde{\iota}_{L(x,d),M(x,d)} : L(x,d) \to M(x,d)$, $\widetilde{E}_{LMxd}(z) = z$. Note, that by definition, we have $\widetilde{E}_{LMxd}(L(x,d)) = M(x,d)$. $\widetilde{E}_{LM}$ is in general not a natural transformation, in the case $L \leq M$ it is a natural transformation, and denoted by E_{LM}.*

4 Definition of $ID(L)$

Definition 4.1 *The language of $ID(L)$:*

(a) *We define FOR_L as the category extending $\mathcal{C}$ with*

$$Ob(FOR_L) = \{(x,d,A) | (x,d) \in \mathcal{C},\ A \text{ a formula of } ID(L(x,d))\},$$

$$Mor(FOR_L)((x,d,A),(y,e,B)) = \{f \in J((x,d),(y,e)) | L(f)(A) = B\}.$$

(b) *The language of $ID(L)$ is defined as follows: the formulas (called L-formulas) are functors A from $\mathcal{C}$ into FOR_L, extending $Id : \mathcal{C} \to \mathcal{C}$ (i.e. $A(x,d) = (x,d,A_{xd})$ for some $A_{x,d}$, $A(f) = f$).*
It can be easily checked, that these functors commute with pullbacks and direct limits.

(c) *Note that if we define a class FOR'_L of formulas by: every arithmetical prime formula is in FOR'_L, if A, B are in FOR'_L and x is a variable, then $\neg A$, $A \wedge B$, $A \vee B$, $\forall x.A$ and $\exists x.A$ are in FOR'_L, and if l,l' are L-parameters, s, t are terms, then $I\!N^l(a)$, $IA^l(a)$ and $I^{l,l'}(s,t)$ are in FOR'_L. Then FOR'_L are (except $\mathcal{C}$ is empty) all L-formulas, where for $A \in FOR'_L$ A_{xd} is the result of substituting all L-parameters l by $l(x,d)$.*
We will usually write L-formulas as an element of FOR'_L.

(d) *In a similar way we can define $f(\Gamma)$ for sequences and introduce the category SEQ_L of L-sequents in $\mathcal{C}$ as functors from $\mathcal{C}$ into SEQ_L.*
If $\Gamma, \Gamma' \in SEQ_L$ the concatination Γ, Γ' is defined by $(\Gamma, \Gamma')_{xd} = \Gamma_{xd}, \Gamma'_{xd}$.

Definition 4.2 *Definition of the proofs in $ID(L)$, or L-proofs.*

(a) *We define the category of proofs $PROOF_L$, extending SEQ_L as follows: objects are 4-tupels (x, d, Γ, P), s.t. $(x, d) \in \mathcal{C}$, P is a proof of the sequence Γ in $ID(L(x, d))$. Morphisms from (x, d, Γ, P) to (y, e, Γ', Q) are $f \in J((x, d), (y, e))$ such that $f^{-1}(Q) = P$. This forces $f^{-1}(\Gamma') = \Gamma$.*

(b) *An L-proof of a sequence Γ is a functor P from $\mathcal{C}$ into $PROOF_L$ extending Γ (i.e. $P(x, d) = (x, d, \Gamma_{x,d}, P_{x,d})$ with $\Gamma(x, d) = (x, d, \Gamma_{x,d})$, and $P(f) = f$).*
It is obvious, that L-proofs commute with direct limits and pull-backs.

Definition 4.3 (a) *Let Deg_L be the category, extending a K-category $\mathcal{C}$, s.t. $Ob(Deg_L) = \{(x, d, d^o) | (x, d) \in \mathcal{C} \wedge d^o \in Deg(L(x, d))\}$, and $Mor(Deg_L)((x, d, d^o), (y, e, e^o)) = \{f \in J((x, d), (y, e)) | f(d^o) = e^o\}$.*
An L-degree is a functor $D : \mathcal{C} \to Deg_L$, extending $Id : \mathcal{C} \to \mathcal{C}$ (so $D(x, d) = (x, d, D_{xd})$).
By remark 2.3 every L-formula A has a degree $d^o(A)$ in $ID(L)$ defined by $d^o(A)_{x,d} = d^o(A_{x,d})$.
The degree of a formula is therefore $d^o(A)_{xd} = (-1, -1, n)$ or $d^o(A)_{xd} = (-1, l_1(x, d), n)$ or $d^o(A)_{xd} = (l_1(x, d), l_2(x, d), n)$, where l_1 and l_2 are L-parameters, $n \in I\!N$.

Definition 4.4 *A proof P in $ID(L)$ is of degree D if $P_{x,d}$ is of degree D_{xd} for all (x, d) in $\mathcal{C}$.*

Note, that the degree of an L-proof need not exist, since

$$D_{xd} := sup\{t + 1 | t \text{ is the degree of a cut in } P_{x,d}\}$$

need not be the degree of a formula of $ID(L)$.
All proofs, coming from finitary proofs of the first order theory ID have a degree $(\underline{a}, \underline{\Omega}, n)$ some integer n.

Definition 4.5 (a) *We define categories $\mathcal{I}_L$, $\vec{\mathcal{I}}_L$ of L-indices as the category, extending $\mathcal{C}$, s.t. $Ob(\mathcal{I}_L) = \{(x, d, i) | (x, d) \in \mathcal{C} \wedge i \in \mathcal{I}(L(x, d))\}$, $Mor(\mathcal{I}_L)((x, d, i), (y, e, j)) = \{f \in Mor_{\mathcal{C}}((x, d), (y, e)) | f(i) = j\}$, $\vec{\mathcal{I}}_L$ is defined as $\mathcal{I}$ with $\mathcal{I}$ replaced by $\vec{\mathcal{I}}$.*
An L-index is a functor $I : \mathcal{C} \to \vec{\mathcal{I}}$ extending Id.
Indices are now of the form $I_{xd} = \underline{0}$, $I_{xd} = \underline{1}$, $I_{xd} = t$, $I_{xd} = l(x, d)$, $I_{xd} = (u(x, d), l(x, d))$, where u, l are L-parameters and t is a term. We write $I = \underline{0}, \underline{1}, t, l, (t, l)$ in these cases respectively.
If P is an L-proof or L-formula and I an L-index, then we possibly define $P[I]$ by $P[I](x, d) := (x, d, P_{xd}[I_{xd}])$.
It follows, that if $P[I]$ is defined, then $P[I]$ is an L-proof, L-formula.
(Note that by definition I_{xd} is never the empty sequence)

(b) *We define a category $Rule_L$ of L-rules as the category, extending $\mathcal{C}$, s.t. $Ob(\mathcal{I}_L) = \{(x, d, rule) | rule \in Rule(L(x, d))\}$, and $Mor(\mathcal{I}_L)((x, d, rule), (y, e, rule')) = \{f \in Mor_{\mathcal{C}}((x, d), (y, e)) | f(rule) = rule'\}$.*
An L-rule is a functor $Rule : \mathcal{C} \to Rule_L$ extending Id.
If P is an L-proof, than we the functor with $Rule(P)_{xd} := Rule(P_{xd})$ defines an L-rule.
Rules are now of the form $Rule_{xd} = (Cut, C_{xd}), (\wedge, A_{xd}), (\vee, A_{xd}, I_{xd})$, where A, C are L-formulas, I is an L-index, which we will write as $(Cut, C), (\wedge, A), (\vee, A, I)$.

Definition 4.6 *If T is a good transformation, it introduces a natural transformation betweed formulas, sequents,degrees, indices and rules:*
If F is a formula, sequent, proof, degree, index, rule in $ID(L)$, define $T(F)(x,d) := T_{xd}(F_{xd})$. We can define $T(P)$ for proofs P by $T(P)(x,d) := T_{xd}(P_{xd})$ not in general, but in the following cases:
(i) if T is of the form E_{LM}
(ii) if the $P \vdash^{L}_{0,0} \Gamma$ for some Γ s.t. $T(\Gamma) = \widetilde{E}_{LM}(\Gamma)$.
(ii) is esp. fulfilled if
(ii') $P \vdash^{L}_{0,0} E_{L'L}(\Gamma)$, for some L', Γ s.t. $T \circ E_{L'L} = E_{L'M}$, in which case we have $T(P) \vdash^{L}_{0,0} E_{L'M}(\Gamma)$.

Remark 4.7 (a) *If we have $T : L \to M$, and P is an L-proof of Γ of degree D and predicative degree D_p s.t. $T(P)$ is defined, then $T(P)$ is a proof of $T(\Gamma)$ of degree $T(D)$ and predicative degree $T(D_P)$.*

(b) *If we have two good categories $\mathcal{C} < \mathcal{C}'$, $\theta := \theta_{\mathcal{C}\mathcal{C}'}$, then, if P is an L-proof of a sequent S of degree D, predicative degree D_p in $\mathcal{C}$ then $\theta(P)$ is a $\theta(L)$-proof of $\theta(S)$ of degree $\theta(D)$, predicative degree $\theta(D_p)$ in $\mathcal{C}'$.*

5 Double Bar-Induction

We have now to compare proofs and degress of $ID(L)$ and $ID(M)$, which actually depend on the L and M:

Definition 5.1 (a) *We define an ordering on proofs: $P < P'$ iff $P'[I] = P$ for some L-index I.*

(b) *For degrees D, D' of L-proofs we define $D < D' :\Leftrightarrow \forall(x,d) \in \mathcal{C}.D_{xd} < D'_{xd}$,and $D \equiv D' :\Leftrightarrow \forall(x,d) \in \mathcal{C}.D_{xd} = D'_{xd}$.*

(c) *We can compare degrees and proofs in the same category $\mathcal{C}$, but with different functor L as follows: If D is a degree, P a proof of $ID(M)$, D' a degree, P' a proof of $ID(L)$, and $L \le M$, then, with $E := E_{LM}$, we define*
$D < D'$ iff $D < E(D')$.
$D \equiv D'$ iff $D \equiv E(D')$.
$P < P'$ iff $P < E(P')$.

(d) *Using θ, we can compare degrees and proofs of different good categories. If $\mathcal{C}, \mathcal{C}' \le \mathcal{C}''$, then we define, if L is a good functor on $\mathcal{C}$, L' a good functor on $\mathcal{C}'$, D is a degree, P a proof of $ID(L)$, D' a degree, P' a proof of $ID(L')$, and then, with $\theta := \theta_{\mathcal{C}\mathcal{C}''}$, $\theta' := \theta_{\mathcal{C}'\mathcal{C}''}$, we define*
$D < D'$ iff $\theta(D) < \theta'(D')$.
$D \equiv D'$ iff $\theta(D) \equiv \theta'(D')$.
$P < P'$ iff $\theta(P) < \theta'(P')$.
This definition is independant of the choice of $\mathcal{C}''$.
Further we see, that if $D < D'$, then $\theta(D) < \theta(D')$ similar for P.

(e) *We define for pairs (D,P), where P is a proof of degree D: $(D,P) < (D',P') :\Leftrightarrow (D < D' \vee (D \equiv D' \wedge P < P'))$.*

Remark 5.2 *If $L_1 \le L_2 \le L_3 \le \cdots$, then we can define a functor $L = sup(L_n)_{n\in\omega}$ by $L(x,d) = sup\{L_n(x,d) | n \in \omega\}$, and $L(f)(z) := L_n(f)(z)$ if $z < L_n(x,d)$.*
Then $L_n \le L$.

Proof: Obvious.

Theorem 5.3 *Double bar induction on degrees and proofs: The relation $<$ defined in 5.1 is wellfounded.*

Proof:
Assume a strictly decreasing sequence $(K_n, x_n, d_n, \mathcal{C}_n, L_n, D_n, P_n)$, where P_n is a L_n proof of degree D_n on the category $\mathcal{C}_n \subset ON^{K_n}$, $(x_n, d_n) \in \mathcal{C}_n$, $K_n \subset K_{n+1}$, $x_n < x_{n+1}$ and $d_{n+1}|K_n = d_n$.
Define $K := \bigcup K_n$, d defined by $d|K_n = d_n$, (here $|$ stands for restriction), $x := sup\{x_n | n \in \omega\}$, $\mathcal{C} := K_d^x$, $\theta_n := \theta_{\mathcal{C}_n\mathcal{C}}$. then with $P'_n := \theta_n(P_n)$, $D'_n := \theta_n(D_n)$, $L'_n := \theta_n(L_n)$ (P'_n, D'_n) is a strictly decreasing sequence in $\mathcal{C}$ of L'_n proofs and degrees, L'_n is increasing. Let $L := sup(L_n)_{n\in\omega}$, Let $E_n := E_{L'_n,L}$. Then the sequence (D''_n, P''_n) defined by $D''_n := E_n(D'_n)$, $P''_n := E_n(P'_n)$ is a strictly decreasing sequence of pairs (z, t), where z is a degree, t is a well-founded proof-tree, and we have a contradiction.
(Note that $D'_{n+1} < D'_n$, by definition therefore $D'_{n+1} < E_{L'_n,L'_{n+1}}(D'_n)$, therefore $D'_{n+1} = E_{n+1}(D'_{n+1}) < E_{n+1}(E_{L'_n,L'_{n+1}}(D'_{n+1})) = E_n(D'_n)$, similar for P).

6 Local Cutelimination

Definition 6.1 *In the following we will define several operations, which tranform proofs into proofs, depending on certain parameters, written as $Operation_{\underline{O}}(s_1, \ldots, s_n)$, where P is a proof in $ID(\underline{O})$. Such an operation is called functorial, if for good $f : \underline{O}' \to \underline{O}$, if $f^{-1}(s_i)$ is defined for all i (if s_i is a term, variable, then $f^{-1}(s_i) := s_i$), then $f^{-1}(Operation_{\underline{O}}(s_1, \ldots, s_n)) = Operation_{\underline{O}'}(f^{-1}(s_1), \ldots, f^{-1}(s_n))$.*

Lemma 6.2 *(a) We can define an operation $rRed_{\underline{O}}(P; \Gamma; \Gamma')$, s.t. if $P \vdash^{\underline{O};\delta}_{d,d_p} \Gamma$, Γ' is the result of replacing in Γ some parameters $\beta < s < \Omega$ by Ω, then $rRed_{\underline{O}}(P; \Gamma; \Gamma') \vdash^{\underline{O};\delta}_{d,d_p} \Gamma$.*
This operation is functorial.

(b) We can define an operation $\wedge elim_{\underline{O}}(P; \Gamma; A; i)$, s.t. if $P \vdash^{\underline{O};\delta}_{d,d_p} \Gamma, A$, $A \doteq_{\underline{O}} \bigwedge_{i\in I} A_i$ and $i \in I$, then $\wedge elim_{\underline{O}}(P; \Gamma; A; i) \vdash^{\underline{O};\delta}_{d,d_p} \Gamma, A_i$.
This operation is functorial.

(c) We can define an operation $\wedge red_{\underline{O}}(P; \Gamma; A; A'; I')$, s.t. if $P \vdash^{\underline{O};\delta}_{d,d_p} \Gamma, A$, $A \doteq \bigwedge_{i\in I} A_i$, $A' \doteq \bigwedge_{i\in I'} A_i$, $I' \subset I$, then $\wedge red_{\underline{O}}(P; \Gamma; A; A'; I') \vdash\!\vdash^{\underline{O};\delta}_{d,d_p} \Gamma, A'$.
The operation is functorial. In the cases $A = I^{t,u}_\nu(a)$, $A' = I^{t',u'}_\nu(a)$, $(t', u') \ll (t, u)$, or $A = I\!N^u(a)$, $A' = I\!N^{u'}(a)$, $u' \ll u$, we can omit the parameter I'.

(d) We can define an operation $\vee red_{\underline{O}}(P; \Gamma; A; A'; I')$ s.t. if $P \vdash^{\underline{O};\delta}_{d,d_p} \Gamma, A$, $A \doteq \bigvee_{i\in I} A_i$, $A' \doteq \bigvee_{i\in I'} A_i$, $I \subset I'$, then $\vee red_{\underline{O}}(P; \Gamma; A; A'; I') \vdash\!\vdash^{\underline{O};\delta}_{d,d_p} \Gamma, A'$.
The operation is functorial. In the cases $A = I^{t,u}_\nu(a)$, $A' = I^{t',u'}_\nu(a)$, $(t, u) < (t', u')$, or $A = I\!N^u(a)$, $A' = I\!N^{u'}(a)$, $u < u'$, the only cases, we actually use, we don't need to mention the parameter I'.

(e) We can define an operation $Weak_{\underline{O}}(P; \Gamma')$, s.t. if $P \vdash^{\underline{O};\delta}_{d,d_p} \Gamma$, then $Weak_{\underline{O}}(P; \Gamma') \vdash^{\underline{O};\delta}_{d,d_p} \Gamma, \Gamma'$.
The operation is functorial.

(f) We can define an operation $\beta change_{\alpha,\beta',\Omega}(P; \beta)$, s.t. if $P \vdash^{\alpha,\beta',\Omega;\delta}_{d,d_p} \Gamma$, $\alpha < \beta < \beta' < \Omega'$, then $\beta change_{\alpha,\beta',\Omega}(P; \beta) \vdash^{\alpha,\beta,\Omega;\delta}_{d,d_p} \Gamma$.
The operation is functorial.

(g) *We can define an operation* $\Omega change_{\alpha,\beta,\Omega'}(P;\Omega;\Gamma;\Gamma')$, *s.t. if* $P \vdash^{\alpha,\beta,\Omega;\delta}_{d,d_p} \Gamma$, $\alpha < \beta < \Omega < \Omega'$, Γ' *is the result of replacing all, except of some positive occurrences, of* Ω *by* Ω', d', d'_p *is the result of replacing* Ω *by* Ω' *in* d, d_p *then* $\Omega change_{\alpha,\beta,\Omega'}(P;\Omega;\Gamma;\Gamma') \vdash^{\alpha,\beta,\Omega';\delta}_{d',d'_p} \Gamma'$.
The operation is functorial.

Proof: All the following proofs are trivial, we just present them here for completeness.
By recursion on the proofs, let in all cases the proof be

$$\frac{\cdots P_j \vdash \Gamma, \Delta_j \cdots (j \in J)}{\Gamma}\ (Rule)$$

(a) Case $(Rule) = (Cut, A)$: Then $rRed(P, \Gamma, \Gamma') :=$

$$\frac{rRed(P_{\underline{0}};\Gamma, A; \Gamma', A_r) \vdash \Gamma', A_r \qquad rRed(P_{\underline{1}};\Gamma, \neg A; \Gamma', \neg A_r) \vdash \Gamma', \neg A_r}{\Gamma'}\ (Cut, A)$$

Case $(Rule)$ is not a cut:
If $(Rule) = (\wedge, A)$, let $(Rule') = (\wedge, A')$, where A' is the formula, corresponding to A in Γ', $A \doteq \bigwedge_{i\in I} A_i$, $A' \doteq \bigwedge_{i\in I'} A'_i$. Then A'_i corresponds to A_i, and $I' \subset I$; let $J' := I'$.
If $(Rule) = (\vee, A, i)$, let $(Rule') = (\vee, A', i)$, where A' defined as before, $A \doteq \bigwedge_{i\in I} B_i$, $A' \doteq \bigwedge_{i\in I'} B'_i$. Then B'_i corresponds to B_i for $i \in I$, and $I \subset I'$. Let in this case $A_{\underline{0}} := B_i$, $A'_{\underline{0}} := B'_i$, $J' := \{\underline{0}\}$.
Then we define
$rRed(P, \Gamma, \Gamma') :=$

$$\frac{\cdots rRed(P_i;\Gamma, A_i;\Gamma', A'_i) \vdash \Gamma, A'_i \cdots (i \in J')}{\Gamma}\ (Rule')$$

(b) Case $(Rule)$ has not main premisse A. Then
$\wedge elim_{\underline{O}}(P;\Gamma;A;i) :=$

$$\frac{\cdots \wedge elim(P_j;\Gamma,\Delta_j;A;i) \vdash \Gamma, \Delta_j, A_i \cdots (j \in I)}{\Gamma, A_i}\ (Rule)$$

Case $(Rule)$ has main premisse A, then $\wedge elim_{\underline{O}}(P;\Gamma;A;i) := \wedge elim(P_i;\Gamma;A;i)$, and $\wedge elim_{\underline{O}}(P;\Gamma, A_i;A;i) \vdash \Gamma, A_i$.
(c) Case $(Rule)$ has not main premisse A. Then
$\wedge red_{\underline{O}}(P;\Gamma;A;A';I') :=$

$$\frac{\cdots \wedge red(P_j;\Gamma,\Delta_j;A;A';I') \vdash \Gamma, \Delta_j, A' \cdots (j \in J)}{\Gamma, A'}\ (Rule)$$

Case $(Rule)$ has main premisse A, then $(Rule) = (\wedge, A)$,
$\wedge red_{\underline{O}}(P;\Gamma;A;A';I') :=$

$$\frac{\cdots \wedge red(P_j;\Gamma,A_j;A;A';I') \vdash \Gamma, A_j, A' \cdots (j \in I')}{\Gamma, A'}\ (\wedge, A')$$

(d) Case $(Rule)$ has not main premisse A. Then

$\vee red_{\underline{Q}}(P;\Gamma;A;i;I') :=$

$$\frac{\cdots \vee red(P_j;\Gamma,\Delta_j;A;A';I') \vdash \Gamma,\Delta_j,A' \cdots (j \in J)}{\Gamma, A'} \ (Rule)$$

Case $(Rule)$ has main premisse A, then $(Rule) = (\vee, A, i)$, $i \in I'$,
$\vee red_{\underline{Q}}(P;\Gamma;A;A';I') :=$

$$\frac{\vee red(P_{\underline{0}};\Gamma,A_i;A;A';I') \vdash \Gamma,A_i,A'}{\Gamma, A'} \ (\vee, A', i)$$

(e)

$Weak_{\underline{Q}}(P;\Gamma') :=$

$$\frac{\cdots Weak(P_j;\Gamma') \vdash \Gamma,\Delta_j,\Gamma'}{\Gamma,\Gamma'} \ (Rule)$$

(f) If $A \doteq_{\alpha,\beta',\Omega} \bigwedge_{i\in I} A_i$, then $A \doteq_{\alpha,\beta,\Omega} \bigwedge_{i\in I'} A_i$ for some $I' \subset I$, and if $A \doteq_{\alpha,\beta',\Omega} \bigvee_{i\in I} A_i$, then $A \doteq_{\alpha,\beta,\Omega} \bigvee_{i\in I} A_i$.
Therefore we can define
$\beta change(P;\beta) :=$

$$\frac{\cdots rRed(P_i;\Gamma,\Delta_i;\beta) \vdash \Gamma,\Delta_i \cdots (i \in J')}{\Gamma} \ (Rule')$$

where $J' = J$ in case of an $\vee$ or Cut rule, and if $(Rule) = (\wedge, A)$, $A \doteq_{\alpha,\beta',\Omega} \bigwedge_{i\in J} A_i$, then J' defined by $A' \doteq_{\alpha,\beta,\Omega} \bigwedge_{i\in J'} A_i$.
(g) Case $(Rule) = (Cut, A)$, A' be the result of replacing all occurrences of Ω by Ω' in A. Then let $Q := \Omega change(P_{\underline{0}};\Omega;\Gamma,A;\Gamma',A')$, $Q' := \Omega change(P_{\underline{1}};\Omega;\Gamma,\neg A;\Gamma',\neg A')$,
$\Omega change_{\alpha,\beta,\Omega'}(P;\Omega;\Gamma;\Gamma') :=$

$$\frac{Q \vdash \Gamma', A' \qquad Q' \vdash \Gamma', \neg A'}{\Gamma} \ (Cut, A')$$

Case $(Rule)$ is not a cut:
If $(Rule) = (\wedge, A)$, let $(Rule') = (\wedge, A')$, where A' is the formula, corresponding to A in Γ', $A \doteq_{\alpha,\beta,\Omega} \bigwedge_{i\in I} A_i$, $A' \doteq_{\alpha,\beta,\Omega'} \bigwedge_{i\in I} A'_i$. Then A'_i corresponds to A_i. (Note in case $A = \neg I_\nu^{s,u}(a)$ $u < \Omega$, then $I = \{(s',u')|(s' < s \wedge u' < \beta) \vee (s' = s \wedge u' < u)\}$, $A' = A$, $A_{s',u'} = \neg I_\nu^{\preceq_{\alpha,\beta,\Omega} s',u'}$ $A'_{s',u'} = \neg I_\nu^{\preceq_{\alpha,\beta,\Omega'} s',u'}$. If $A = \neg I_\nu^{s,\Omega}(a)$, $I = \{(s',u')|(s' \leq s \wedge u' < \beta)\}$, $A' = \neg I_\nu^{s,\Omega'}(a)$, $A_{s',u'} = \neg I_\nu^{\preceq_{\alpha,\beta,\Omega} s',u'}$ $A'_{s',u'} = \neg I_\nu^{\preceq_{\alpha,\beta,\Omega'} s',u'}$).
If $(Rule) = (\vee, A, i)$, let $(Rule') = (\vee, A', i)$, where A' defined as before, $A \doteq \bigwedge_{i\in I} B_i$, $A' \doteq \bigwedge_{i\in I'} B'_i$. Then B'_i corresponds to B_i for $i \in I$, and $I \subset I'$. Let the $A_{\underline{0}} := B_i$, $A'_{\underline{0}} := B'_i$.

Then we define
$\Omega change(P, \Omega, \Gamma, \Gamma') :=$

$$\frac{\cdots \Omega change(P_i; \Gamma, A_i; \Gamma', A_i') \vdash \Gamma', A_i' \cdots (i \in J)}{\Gamma'} \ (Rule')$$

Lemma 6.3 *We can define an operation* $PredCut_{\underline{Q}}(P; Q; \Gamma; \Gamma'; A)$, *s.t. if* $P \vdash^{Q;\gamma}_{d,d_p} \Gamma, A$, $Q \vdash^{Q;\delta}_{d,d_p} \Gamma', \neg A$, $d^o(A) < d$, $d^p(A) \leq d_p$, $d_p \neq (-1,-1,-1)$, *then*

$$PredCut_{\underline{Q}}(P; Q; \Gamma; \Gamma'; A) \vdash^{Q}_{d,d_p} \Gamma, \Gamma'$$

The operation is functorial.

Proof: By induction on $\gamma \# \delta$.
W.l.o.g. A is existential. If $d^p(A) < d_p$, then
$PredCut(P; Q; \Gamma; \Gamma'; A) :=$

$$\frac{Weak(P; \Gamma') \vdash \Gamma, \Gamma', A \qquad Weak(Q; \Gamma) \vdash \Gamma, \Gamma', \neg A}{\Gamma, \Gamma'} \ (Cut, A)$$

Assume now $d^p(A) = d_p (\neq (-1,-1,-1))$.
Let
$P =$

$$\frac{\cdots P_j \vdash \Gamma, A, \Delta_j \cdots (j \in J)}{\Gamma, A} \ (Rule)$$

Case A is not a main premisse of $(Rule)$. Then
$PredCut_{\underline{Q}}(P; Q; \Gamma; \Gamma'; A) :=$

$$\frac{\cdots PredCut(P_i; Q; \Gamma, \Delta_i; \Gamma'; A) \vdash \Gamma, \Delta_i, \Gamma' \cdots (i \in J)}{\Gamma, \Gamma'} \ (Rule')$$

Case A is the main premisse of $(Rule) = (\bigvee, A, i)$, $(A \not\equiv I^{t,u}_{\nu}(a)$ or $i = (t, u'))$. Then i is an index of $\neg A$ as well. Let
$Q := PredCut(P; Q; \Gamma, A_i; \Gamma'; A)$,
$Q' := Weak(\wedge elim(P; \Gamma; \neg A; i); \Gamma)$,
$PredCut_{\underline{Q}}(P; Q; \Gamma; \Gamma'; A) :=$

$$\frac{Q \vdash \Gamma, \Gamma', A_i \qquad \vdash \Gamma, \Gamma', \neg A_i}{\Gamma, \Gamma'} \ (Cut, A_i)$$

Case otherwise. Therefore A is the main premisse of $(Rule) = (\bigvee, A, i)$, $A \equiv I^{t,u}_{\nu}(a)$, $i = (t', u')$, $t' < t$. Let
$P' := Weak(PredCut(P_{\underline{0}}; Q; \Gamma, I^{\preceq t',u'}_{\nu}(a); \Gamma'; I^{t,u}_{\nu}), I^{t,0}_{\nu}(a))$,
$P'' :=$

$$\frac{P' \vdash \Gamma, \Gamma', I^{\preceq t',u'}_{\nu}(a), I^{t,0}_{\nu}(a)}{\Gamma, \Gamma', I^{t,0}_{\nu}(a)} \ (\vee, I^{t,0}_{\nu}(a), (t', u'))$$

$P''' := Weak(\wedge red(Q;\Gamma';\neg I_\nu^{t,u}(a);\neg I_\nu^{t,0}(a));\Gamma)$
$PredCut(P;Q;\Gamma;\Gamma';A) :=$

$$\frac{P'' \vdash \Gamma,\Gamma',I_\nu^{t,0}(a) \qquad P''' \vdash \Gamma,\Gamma',\neg I_\nu^{t,0}(a)}{\Gamma,\Gamma'}\ (Cut, I_\nu^{t,0}(a))$$

Lemma 6.4 *We can define an operation* $PredCutelim_{\underline{O}}(P,d_p)$*, s.t. if* $P \vdash^{\underline{O}}_{d,d_p} \Gamma$*,* $PredCutelim_{\underline{O}}(P,d_p) \vdash^{\underline{O}}_{d,0} \Gamma$*.*
The operation is functorial.
Proof: Induction on the d_p, side induction on the height of the derivation.
Let
$P =$

$$\frac{\cdots P_j \vdash \Gamma,\Delta_j \cdots (j \in J)}{\Gamma}\ (Rule)$$

Case $(Rule)$ is not a Cut: Then
$PredCutelim(P,d_p) :=$

$$\frac{\cdots PredCutelim(P_j,d_p) \vdash \Gamma,\Delta_j \cdots (j \in J)}{\Gamma}\ (Rule)$$

Case $(Rule) = (Cut, A)$. $Q_i := PredCutelim(P_i)$, $Q_{\underline{0}} \vdash_{d,0} \Gamma, A$, $Q_{\underline{1}} \vdash_{d,0} \Gamma, \neg A$. $Q := PredCut(Q_{\underline{0}}, Q_{\underline{1}}, \Gamma; \Gamma; A) \vdash_{d,d_p(A)} \Gamma$, $d_p(A) < d_p$. $PredCut(P,d_p) := PredCut(Q, d_p(A))$.
Lemma 6.5 *We can define an operation* $ImpredCut_{\alpha,\beta,\Omega''}(P;Q;\Omega;\Omega';\Gamma;\Gamma';A;\widetilde{\Gamma};\widetilde{A})$*, s.t. if we have the following situation:*
$\alpha < \beta < \Omega < \Omega' < \Omega''$.
A is existential, $d_p(A) = (-1,-1,-1)$ *(so* $A \in \{I_\nu^{s,0}(a), I_\nu^{s,\Omega}(a), I\!N^{\Omega}(a)\}$ *for some s),*
$d^o(A) \leq d$*,*
$\widetilde{\Gamma}$ *is the result of replacing in Γ all, except of some positive occurrences, of Ω by Ω'',*
$\widetilde{A}$ *is the result of replacing all occurrences of Ω by Ω'' in A,*
d' *the result of replacing in d Ω by Ω', similar for d_p' and d_p,*
$d_p'' := max\{d', d_p'\}$,
$P \vdash^{\alpha,\beta,\Omega;\gamma}_{d,d_p} \Gamma, A$*,* $Q \vdash^{\alpha,\Omega',\Omega'';\delta}_{d',d_p'} \Gamma', \neg\widetilde{A}$*,*
then
$ImpredCut_{\alpha,\beta,\Omega''}(P;Q;\Omega;\Omega';\Gamma;\Gamma';A;\widetilde{\Gamma};\widetilde{A}) \vdash^{\alpha,\beta,\Omega'';\gamma\#\delta}_{d',d_p''} \widetilde{\Gamma}, \Gamma'$.
The operation is functorial.
Proof: By induction on $\gamma\#\delta$.
Let
$P =$

$$\frac{\cdots P_j \vdash \Gamma, A, \Delta_j \cdots (j \in J)}{\Gamma, A}\ (Rule)$$

Case A is not a main premisse of $(Rule)$. Then
$ImpredCut_{\alpha,\beta,\Omega'}(P;Q;\Omega;\Gamma;\Gamma';A;\widetilde{\Gamma};\widetilde{A}) :=$

$$\frac{\cdots ImpredCut_{\underline{O}}(P;Q;\Omega;\Gamma,\Delta_i;\Gamma';A;\widetilde{\Gamma},\widetilde{\Delta}_i;\widetilde{A}) \vdash \widetilde{\Gamma},\widetilde{\Delta}_i \cdots (i \in J)}{\widetilde{\Gamma}}\ (\widetilde{Rule})$$

where in $\widetilde{\Delta}_i$, we have made the same changes as in $\widetilde{\Gamma}$ in the $\wedge$ and $\vee$ rules, in the case of a Cut $\widetilde{\Delta}$ is the result of replacing in Δ all Ω by Ω'', and $(\widetilde{Rule})$ is the result of making the corresponding changes.
Case A is the main premisse of $(Rule) = (\bigvee, A, i)$, $A \equiv I_\nu^{s,0}(a), I_\nu^{s,\Omega}(a), \mathbb{N}^\Omega(a)$. We consider the case $A \equiv I_\nu^{s,\Omega}(a)$.
$P_{\underline{0}} \vdash \Gamma, I_\nu^{\preceq s',t'}(a), I_\nu^{s,\Omega}(a)$.
$Q' := ImpredCut(P_{\underline{0}}; Q; \Omega; \Gamma; I_\nu^{\preceq_{\alpha,\beta,\Omega} s',t'}(a); \Gamma'; A; \widetilde{\Gamma}; I_\nu^{\preceq_{\alpha,\beta,\Omega''} s't'}(a); \widetilde{A})$,
$Q' \vdash \widetilde{\Gamma}, \Gamma', I_\nu^{\preceq_{\alpha,\beta,\Omega''} s't'}(a)$. $Q'' := \wedge elim(Q; \Gamma'; \neg\widetilde{A}; I_\nu^{\preceq_{\alpha,\Omega',\Omega''} s't'}(a))$
$Q'' \vdash^{\alpha,\Omega',\Omega''} \Gamma', \neg I_\nu^{\preceq_{\alpha,\Omega',\Omega''} s',t'}(a)$.
$Q''' := Weak(\beta change_{\alpha,\Omega',\Omega''}(Q''; \beta), \widetilde{\Gamma})$, $Q''' \vdash \widetilde{\Gamma}, \Gamma', \neg I_\nu^{\preceq_{\alpha,\Omega',\Omega''} s',t'}(a)$ which is the same as $\widetilde{\Gamma}, \Gamma', \neg I_\nu^{\preceq_{\alpha,\beta,\Omega''} s',t'}(a)$,
and the result is now

$$\frac{Q' \vdash \widetilde{\Gamma}, \Gamma', I_\nu^{\preceq_{\alpha,\beta,\Omega''} s't'}(a) \qquad Q''' \vdash \widetilde{\Gamma}, \Gamma', \neg I_\nu^{\preceq_{\alpha,\Omega',\Omega''} s',t'}(a)}{\widetilde{\Gamma}, \Gamma'} \; (Cut, I_\nu^{\preceq s',t'}(a))$$

Lemma 6.6 *(a) For every operation $operation_{\underline{O}}(s_1, \ldots, s_n)$ in Lemma 6.2 (a) - (e) we can define an operation $Operation_L(S_1, \ldots, S_n)$, where S_i are the $ID(L)$ objects, corresponding to the $ID(\underline{O})$-objects s_i, which fulfill the corresponding relations, except that we don't get any ordinal bounds).*
E.g. in 6.2 (c) we have:
If $P \vdash^L_{D,D_p} \Gamma, A$, $A_{xd} \doteq \bigwedge_{i \in I_{xd}} A^i_{xd}$, $A'_{xd} \doteq \bigwedge_{i \in I'_{xd}} A^i_{xd}$, $I'_{xd} \subset I_{xd}$, for all $f \in J((x,d),(y,e))$, $L(f)^{-1}(I'_{ye}) = I'_{xd}$, then
$\wedge red_L(P; \Gamma; A; A'; I') \vdash^L_{D,D_p} \Gamma, A'$.

(b) We can do the same for 6.2 (g): We can define an operation $\Omega change_M(P; L; \Gamma; \Gamma')$ s.t. if $P \vdash^L_{D,D_p} \Gamma$, $L \leq M$ $E := E_{LM}$, Γ' is the result of replacing all, except of some positive occurrences of the L-parameter L by M, then $\Omega change_M(P; L) \vdash_{E(D),E(D_p)} \Gamma'$.
The operation is functorial.

(c) Lemma 6.4 extends as in (a).

(d) Lemma 6.5 extends to an operation on L-proofs as follows:
We can define an operation $ImpredCut_L(P; Q; \Gamma; A; D, D_p)$, s.t. if we have the following situation:
$M := L \circ (L+1)$ ($M(x,d) = L(L(x,d)+1,d)$, $M(f) = L(L(f)+1)$),
$d_p(A) = (-1,-1,-1)$ (so $A \in \{I_\nu^{s,0}(a), I_\nu^{s,\Omega}(a), \mathbb{N}^\Omega(a)\}$),
$d^o(A) \leq D$,
$\Gamma = \Gamma_r$,
$P \vdash^L_{D,D_p} \Gamma, A$, $Q \vdash^L_{D,D_p} \Gamma, \neg A$,
then, with $E := E_{LM}$ we have:
$ImpredCut_L(P; Q; \Gamma; A; D, D_p) \vdash^M_{E(D),0} E(\Gamma)$.

Proof: (a) Define $Operation_L(S_1, \ldots, S_n)_{xd} := operation_{L(x,d)}(S_{1,x,d}, \ldots, S_{n,x,d})$. By the functoriality follows, that we get an L-proof.
(b), (c): similar.
(d). Assume A is existential. Let $P' := PredCut_L(P, D_p)$, $Q' := PredCut_L(Q, D_p)$, $D'_{pxd} := D^o(A_{xd})$. We define
$ImpredCut_L(P; Q; \Gamma; A) :=$

$PredCutelim_{M(x,d)}($
$ImpredCut_{M(x,d)}($
$P'_{xd}; Q'_{L(x,d)+1,d}; L(x,d); L(x,d)+1; \Gamma_{xd}; E(\Gamma_{xd}); A_{xd}; E(\Gamma_{xd}); A_{L(x,d)+1,d}), D'_{p,L(x,d),d})$.
$P' \vdash^{L}_{D,0} \Gamma, A$, $Q' \vdash^{L}_{D,0} \Gamma, \neg A$, $d_0(A) \leq D$, $\Gamma = \Gamma_r$, $E := E_{LM}$, therefore $P'_{xd} \vdash^{L(x,d)}_{D_{xd},0} \Gamma_{xd}, A_{xd}$. Further for the embedding $\iota : x \to L(x,d)+1$ we have $L(\iota) : L(x,d) \to L(L(x,d)+1,d)$. (note that $\iota \circ d = d$. $\Gamma_{L(x,d)+1,d} = L(\iota)(\Gamma_x, d))$. Since $\Gamma = \Gamma_r$, $L(\iota)(z) = E_{xd}(z)$ for $z < x$ or $z = L(x,d)$, follows therefore $\Gamma_{L(x,d)+1,d} = L(\iota)(\Gamma_{xd}) = (E(\Gamma_{xd}))$ which is the result of replacing $L(x,d)$ by $M(x,d)$. A similar relationship we have for $D_{L(x,d)+1,d}$ and D_{xd}. $Q'_{L(x,d)+1,d} \vdash^{d(\underline{a}),L(x,d)+1,M(x,d)}_{E_{xd}(D_{xd})0} E(\Gamma_{xd}), E(\neg A_{xd})$, therefore $ImpredCut_L(P;Q;\Gamma;A;\widetilde{\Gamma})_{xd} \vdash^{d(\underline{a}),x,M(x,d)}_{E(D),0} E(\Gamma_{xd})$.
Further if $f \in J((x,d),(y,e))$, then $M(f)^{-1}(P_{ye}) = L(L(f)+1)^{-1}(P_{ye}) = L(f)^{-1}(P_{ye}) = P_{xd}$, since all parameters in P_{ye} are less than $L(x,d)+1$, and $L = E + L'$ for some L', further $M(f)^{-1}(Q_{L(x,d)+1e}) = L(L(f)+1)^{-1}(Q_{L(x,d)+1,e}) = Q_{L(y,e)+1,d}$, similar for the other parameters, therefore $M(f)^{-1}(ImpredCut_L(P;Q;\Gamma;A))_{ye} = ImpredCut_L(P;Q;\Gamma;A)_{xd}$.

7 Global Cutelimination

Theorem 7.1 (Cutelimination for $ID(L)$) *If Γ is a sequent of $ID(L)$, and Γ is provable in $ID(L)$, then Γ_r is cut-free provable in $ID(L')$ for some good functor L'. Further, if L is s.t. $L(n)$ is finite for all $n \in I\!N$, then this holds for L', too. If L is recursive, then L' is recursive, too. (If L is primitive recursive, L' need not be primitive recursive).*

Proof of the Main Theorem 7.1: We proof the following Lemma:

Lemma 7.2 *We can define operations Λ and N with arguments being a degree D, a good functor L and an L-proof P of degree D, all on a suitable category $\mathcal{C}$, such that if $P \vdash^{L}_{D,0} \Gamma$ then $M := \Lambda_D(L,P)$ is a good functor, $L \leq M$, and $N_D(L,P) \vdash^{M}_{O,O} E_{LM}(\Gamma_r)$.*
Further, if we have: If L is recursive, then $\Lambda_D(L,P)$ is recursive, if $L(n)$ is finite for $n \in \omega$, then the same holds for $\Lambda_D(L,P)$.

Proof: by induction on (D,P).
Assume $\Lambda_{D'}(L',P')$, $N_{D'}(L',P')$ are constructed for $(L',P') < (L,P)$.
Let
$P_{xd} =$

$$\frac{\cdots P^j_{xd} \vdash \Gamma_{xd}, \Delta^j_{xd} \cdots (j \in J_{xd})}{\Gamma_{xd}} \ (Rule_{xd})$$

We have for $f \in J((x,d),(y,e))$,
$L(f)^{-1}(P_{ye}) = P_{xd}$,
$L(f)^{-1}(\Gamma_{ye}) = \Gamma_{xd}$,
$L(f)^{-1}(Rule_{ye}) = Rule_{xd}$,
$L(f)^{-1}(P^{L(f)(j)}_{ye}) = P^j_{xd}$,
$L(f)^{-1}(\Gamma^{L(f)(j)}_{ye}) = \Gamma^j_{xd}$.

Case $(Rule) = (\wedge, A)$, A atomic arithmetical.
$\Lambda_D(L,P) := L$, $N_D(L,P) := P$.

Case $(Rule) = (\vee, A, I)$, $A = N^u_\nu(a)$ and $I = u'$, $\underline{\beta} < u'$ or $I^{t,u}_\nu(a)$, $I = (t',u')$, $\underline{\beta} < u'$. Consider the case $A = I^{t,u}_\nu(a)$.
Let $M' := N_D(L, P^{\underline{0}})$, $Q' := \Lambda_D(L, P^{\underline{0}})$, $Q' \vdash^{M'}_{0,0} E_{LM}(\Gamma_r), I^{\preceq_{M'} t',M'}_\nu(a)$.
Let $\Lambda_D(P,L) := M := M' + 1$,

$Q'' := \Omega change_M(Q'; M'; E_{LM'}(\Gamma_r), I_\nu^{\preceq_{M'}t',M'}(a); E_{LM}(\Gamma_r), I_\nu^{\preceq_M t' M'}(a))$,
$N_D(L,P) :=$

$$\frac{Q'' \vdash^M_{0,0} E_{LM}(\Gamma_r), I_\nu^{\preceq_M t' M'}(a)}{E_{LM}(\Gamma_r)} \quad (\vee, I_\nu^{t,M}(a), (t', M')))$$

Case $Rule = (\vee, A, I)$, but different from the case before. Then let $\Lambda_D(P,L) := M := \Lambda_D(L, P^{\underline{0}})$, $Q' := \Lambda_D(L, P^{\underline{0}})$,
$N_D(P,L) := Q :=$

$$\frac{Q' \vdash^M_{0,0} E_{LM}(\Gamma_r, A_{r,I})}{E_{LM}(\Gamma_r)} \quad (\vee, E_{LM}(A_r), I)$$

Case $(Rule) = (\wedge, A_{\underline{0}} \wedge A_{\underline{1}})$.

$M_i := \Lambda_D(L, P^i)$, $Q^i := N_D(L, P^i)$. We need to unify the functors $M_{\underline{0}}$, $M_{\underline{1}}$:
Let $M_i = L + M_i'$, $M := L + M_{\underline{0}}' + M_{\underline{1}}' (= M_{\underline{0}} + M_{\underline{1}}')$.
$T^{\underline{0}} := E_{M_{\underline{0}},M}$. Let $T^{\underline{1}}_{xd}(z) := z$, if $z < L(x,d)$, $T^{\underline{1}}_{xd}(L(x,d)+z) := M_{\underline{0}}(x,d)+z$, if $z < M_{\underline{1}}'(x,d)$.
Let $R^i := T^i(Q^i)$. $R^{\underline{1}}$ is defined, since $Q^{\underline{1}} \vdash^{M_{\underline{1}}}_{0,0} E_{LM^{\underline{1}}}(\Gamma_r, A_{\underline{1},r})$, $T^{\underline{1}} E_{LM_{\underline{1}}} = E_{LM}$ (see definition 4.6, condition (ii')).
Further, because $T^{\underline{0}} \circ E_{LM_{\underline{0}}} = T^{\underline{1}} \circ E_{LM_{\underline{1}}} = E_{LM}$, we can define, with $E := E_{LM}$, $\Lambda_D(L,P) := M$,
$N_D(L,P) :=$

$$\frac{Q^{\underline{0}} \vdash^M_{0,0} E(\Gamma_r, A_{\underline{0}r}) \qquad\qquad Q^{\underline{1}} \vdash^M_{0,0} E(\Gamma_r, A_{\underline{1}r})}{E(\Gamma_r)} \quad (\wedge, E(A_{\underline{0}} \wedge A_{\underline{1}}))$$

Case $(Rule) = (Cut, C)$.

Since $D_p = 0$, $D_p(C) = (-1,-1,-1)$. Let M, Q^i be defined as in the case before, $\widetilde{M} := M \circ (M+1)$, $\widetilde{P} := ImpredCut_M(Q^{\underline{0}}, Q^{\underline{1}}, \Gamma, C, 0)$. $\widetilde{P} \vdash^{\widetilde{M}}_{D(C),0} E_{M\widetilde{M}}(E_{LM}(\Gamma_r))$, the last sequence is equal to $E_{L\widetilde{M}}(\Gamma_r)$. Now $\Lambda_D(L,P) := \Lambda_{D(C)}(\widetilde{M}, \widetilde{P})$, $N_D(L,P) := N_{D(C)}(\widetilde{M}, \widetilde{P})$.

Case $(Rule) = (\wedge, \forall x.A(x))$.
$M_t := \Lambda_D(L, P^t)$, $Q^t := N_D(L, P^t)$. Let $M_t = L + M_t'$, $M := L + \Sigma_{t \in Term_{Cl}} M_t'$ (where the sum is taking in some ordering $<$ of the closed terms).
$T^t_{xd}(z) := z$, if $z < L(x,d)$, $T^t_{xd}(L(x,d)+z) := \Sigma_{s<t} M_s(x,d) + z$, if $z < M_t(x,d)$.
Let $R^i := T^i(Q^i)$. R^i is defined, since $R^i \circ E_{LM_i} = E_{LM}$, Q^t is a cut-free proof of $E_{LM_t}(\Gamma_{rt})$, see definition 4.6, condition (ii')).
Let $E := E_{LM}$, $\Lambda_D(L,P) := M$,
$N_D(L,P) :=$

$$\frac{\cdots Q^t \vdash^M_{0,0} E(\Gamma_r, A(t)_r) \cdots \quad (t \in Term_{Cl})}{E(\Gamma_r)} \quad (\wedge, E(\forall x.A(x)))$$

Case $(Rule) = (\wedge, A)$, $A \equiv \neg \mathbb{N}^{\widetilde{u}}(a), \neg I A^{\widetilde{u}}(a), \neg I_\nu^{\widetilde{t},\widetilde{u}}(a)$. We consider only the case $A \equiv \neg I_\nu^{\widetilde{t},\widetilde{u}}(a)$.

$P_{xd} =$

$$\cfrac{\cdots P^{t,u}_{xd} \vdash \Gamma_{xd}, \neg I^{\preceq t,u}_{\nu}(a) \cdots ((t,u) \ll_{L(x,d)} (t'(x,d), u'(x,d)))}{\Gamma_{xd}} \quad (\wedge, \neg I^{\widetilde{t}(x,d),\widetilde{u}(x,d)}_{\nu}(a))$$

Let $t'(x,d) := \widetilde{t}(x,d)$, $u'(x,d) := \begin{cases} \widetilde{u}(x,d) & \text{if } \widetilde{u}(x,d) < x \\ L(x,d) & \text{otherwise} \end{cases}$.
Then $I^{u',t'}_{\nu}(a) = (I^{\widetilde{u},\widetilde{t}}_{\nu})_r$.
We want to construct a cut-free proof of $E_{LM}(\Gamma_r)$ and have $I^{t'u'}_{\nu}(a)$ occurrs in Γ_r.
We will use therefore $P^{t,u}_{xd}$ only for $(t,u) \ll (t'(x,d), u'(x,d))$ (note that, if $(t,u) \ll (t'(x,d), u'(x,d))$, then $(t,u) \ll (\widetilde{t}(x,d), \widetilde{u}(x,d))$.
Let $K' := K \uplus \{\underline{2}, \underline{3}\}$, where $\uplus$ stands for disjoint union. We write (x,d,t,u) for $(x,d') \in ON^{K'}$ s.t. $d'|K = d$, $d(\underline{2}) = t$, $d(\underline{3}) = u$. ($|$ stands for restriction)
Let $\mathcal{C}'$ be the full sub-category of $ON^{K'}$ with
$Ob(\mathcal{C}') = \{(x,d,t,u) | (t,u) \ll_{L(x,d)} (t'(x,d), u'(x,d))\}$.
Lemma 7.3 *Assume* $(t'(x,d), u'(x,d)) \neq (0,0)$.

(a) $\mathcal{C}'$ *is a suitable category.*

(b) $\mathcal{C} \leq \mathcal{C}'$.

Proof: (a) We check the properties:
(i) $\mathcal{C}'$ is a full subcategory of $ON^{K'}$.
(ii) If $(y,e,t'',u'') \in \mathcal{C}'$, $f \in J((x,d,t,u),(y,e,t'',u''))$, then $(t'',u'') \ll_{L(y,e)} (l'(y,e), u'(y,e))$, $t'' = f(t)$, $u'' = f(u)$, $l'(y,e) = f(l'(x,d))$, $u'(y,e) = f(u'(x,d))$ $(t,u) \ll_{L(x,d)} (l'(x,d), u'(x,d))$, $(x,d,t,u) \in \mathcal{C}'$, therefore $\mathcal{C}'$ is an initial segment.
(iii) If $(x,d) \in \mathcal{C}$, $x' := max\{1,x\}$, then $(x',d) \in \mathcal{C}$, $(x',d,0,0) \in \mathcal{C}' \neq \emptyset$, since $(t'(x,d), u'(x,d)) \neq (0,0)$.
(iv) If $(x,d,t,u) \in \mathcal{C}'$, $x < y$, then $(y,d) \in \mathcal{C}$. Let $\iota : x \to y$ be the embedding. $(t,u) = L(\iota)(t,u) \ll L(\iota)(t'(x,d), u'(x,d)) = (t'(y,d), u'(y,d))$, $(y,d,t,u) \in \mathcal{C}'$.
(b): If $(x,d,t,u) \in \mathcal{C}'$, we have $(x,d) \in \mathcal{C}$, with $x' := max\{x,1\}$, $(x',d) \in \mathcal{C}$, $(x',d,0,0) \in \mathcal{C}'$.

Definition 7.4 *(a) Let* $\theta := \theta_{\mathcal{C}\mathcal{C}'}$.

(b) $L' := \theta(L)$, $P' := \theta(P)$, $\hat{t} := \theta(t')$, $\hat{u} := \theta(u')$, $D' := \theta(D)$, $\Gamma' := \theta(\Gamma_r)$

(c) Let $T(x,d,t,u) := t$, $U(x,d,t,u) := u$. T, U *are* L'*-terms,* $(T,U) \ll (\hat{t}, \hat{u})$.

(d) Let $P'' := P'[(T,U)]$. $P''_{xdtu} \vdash \Gamma_{xd}, I^{\leq_{L(x,d)} tu}_{\nu}$.

(e) Let $M := \Lambda_{D'}(L', P'')$, $Q := N_{D'}(L', P'')$.
Let $M = L' + M'$.
$Q \vdash^M E_{L'M}(\Gamma'), \neg I^{\preceq_M TU}_{\nu}(a)$.
(This is defined in this way only if $t(x,d) \neq 0$ *or* $u(x,d) \neq 0$*, but otherwise* $\mathcal{C}'$ *is empty, and we can define* M*,* Q *as the only functors on an empty category.)*

(f) Let N *be the good functor* $\mathcal{C} \to ON$,
$N(x,d) = L(x,d) + \Sigma_{(t,u) \ll_{L(x,d)} (t'(x,d), u'(x,d))} M'(x,d,t,u)$*, where the* (t,u) *are taken in increasing order.*

(Note, that if $(t'', u'') < (t, u) \ll (t'(x, d), u'(x, d))$ by Lemma 1.7 (b) we have $(t'', u'') \ll (t'(x, d), u'(x, d)) \leftrightarrow (t'', u'') < (t, u)$, therefore we can order them as will with respect to $\ll$.)
If $f \in J((x, d), (y, e))$, $N(f)(z) := L(f)(z)$ for $z < L(x, d)$,
$N(f)(L(x, d) + (\Sigma_{(t'',u'')\ll(t,u)} M'(x, d, t'', u'')) + z) :=$
$\quad L(y, e) + (\Sigma_{(t'',u'')\ll(f(t),f(u))} M'(y, e, t'', u'')) + M'(f_{tu})(z)$,
where $f_{tu} \in J((x, d, t, u), (y, e, f(t), f(u)))$, $f_{tu}(z) := f(z)$.

(g) Let $N' := \theta(N)$.

(h) We define $V : M \to N'$:
if $z < L(x, d)$, $V_{xdtu}(z) := z$,
if $z < M'(x, d, t, u)$, $V_{xdtu}(L(x, d) + z) := L(x, d) + (\Sigma_{(t'',u'')\ll(t,u)} M'(x, d, t'', u'')) + z$.

Remark 7.5 *(a) $V \circ E_{L'M} = E_{L'N}$.*

(b) V is a natural transformation.

Proof: (a): trivial.
(b): If $f \in J((x, d, t, u), (y, e, t'', u''))$, $f' \in J((x, d), (y, e))$, $f'(z) = f(z)$, then
if $z < L(x, d)$,
$N'(f)(V_{xdtu}(z)) = N(f)(z) = L(f)(z) = V_{yet''u''}(L(f)(z)) = V_{yet''u''}(L'(f)(z))$
$= V_{yet''u''}(M(f)(z))$
and
if $z < M(x, d, t, u)$, then
$N'(f)(V_{xdtu}(L(x, d) + z)) =$
$\quad N(f')(L(x, d) + \Sigma_{(t''',u''')\ll(t,u)} M'(x, d, t''', u''') + z) =$
$\quad L(y, e) + \Sigma_{(t''',u''')\ll(f'(t),f'(u))} M'(y, e, t''', u''') + M'(f'_{tu})(z) =$
$\quad V_{yef'(t)f'(u)}(L(y, e) + M'(f'_{tu})(z)) =$
$\quad V_{yet''u''}(L(y, e) + M'(f)(z)) =$
$\quad V_{yet''u''}(M(f)(L(x, d) + z))$.

Definition 7.6 *Let $R := V(Q)$.*

R can be defined, because $V \circ E_{L'M} = E_{L'N}$ and $Q \vdash^{M}_{0,0} E_{L'M}(\Gamma', \neg I_\nu^{\leq_{L'} TU})$.
We see now $R_{xdtu} \vdash E_{L'N'}(\theta(\Gamma_r))_{xdtu}, \neg I_\nu^{\leq_{N'(x,d,t,u)} tu}(a)$.
$E_{L'N'xdtu}(z) = E_{LNxd}(z)$, $\theta(\Gamma_r)_{xdtu} = (\Gamma_r)_{xd}$.
Therefore $R_{xdtu} \vdash E_{LN}(\Gamma_r)_{xd}, \neg I_\nu^{\leq_{N(x,d)} tu}(a)$.
If $f \in J((x, d), (y, e))$, $(t, u) \ll (t'(x, d), u'(x, d))$, then
$f_{tu} \in J((x, d, t, u), (y, e, f(t), f(u)))$.
Then we have
$R_{xdtu} = V_{xdtu}(Q_{xdtu}) = V_{xdtu}(M(f_{tu})^{-1}(Q_{yef(t)f(u)})) = N'(f_{tu})^{-1}(V_{yef(t)f(u)} Q_{yef(t)f(u)}) = N(f)^{-1}(R_{yef(t)f(u)})$, using $V_{yef(t)f(u)} M(f_{tu}) = N'(f_{tu}) V_{xdtu}$ and Lemma 2.4.
We define now $\Lambda_D(L, P) := N$,
$(N_D(L, P))_{xdtu} :=$

$$\frac{\cdots R_{xdtu} \vdash^{N(x,d)}_{0,0} E_{LN}(\Gamma_r), \neg I_\nu^{\preceq_{N(x,d)} t,u} \cdots}{E_{LN}(\Gamma_r)} \quad \begin{array}{c}((t, u) \ll (t'(x, d), u'(x, d)))\\ (\wedge, I_\nu^{t'(x,d),u'(x,d)}(a))\end{array}$$

and are done.