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Invited Review

Progress with single-item lot-sizing ★

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1. Introduction

The goal in this talk is to look at the history of the single-item uncapacitated lot-sizing (ULS) problem that was apparently well-solved in 1958, and point out a variety of new results that have been obtained in the last few years. For young researchers there is perhaps a moral: the important questions may not have been asked or may change with time, so it is wrong to assume that previous generations have solved all the easy questions and only left them the hard ones! A secondary goal is to indicate the practical reasons for continuing interest in this model.

The outline of the talk is as follows. First we present the simplest multi-item model of practical interest and indicate how various solution algorithms lead naturally to ULS subproblems. We then take a historical viewpoint starting in 1958. In Section 3 (1958–1975) we describe the basic results of Wagner–Whitin on the structure of optimal solutions leading to a dynamic programming algorithm. We then mention briefly some of the important extensions to the model analysed up to the early 70's. In Section 4 (1975–1990) we describe some important reformulations either involving additional variables or the addition of valid inequalities, and indicate also how they can be used in practice. In Section 5 (1990–present) we cite some surprising new results, in particular the improvements in the dynamic programming algorithm and the simplification of the formulations when a "Wagner–Whitin" cost structure is present. We terminate with a brief indication of work in progress on some of the more important extensions to ULS, namely models with capacities and start-up costs and times, as well as a new profit-maximising variant of the ULS model.

2. Multi-item lot-sizing and algorithms

2.1. Models

Multi-item, multi-level, multi-plant capacitated production planning problems are one of the generic industrial optimisation problems. Rather than formulate a detailed model, we consider the simplest nontrivial but practical model, namely a single-machine single-level multi-item problem in which the machine can only produce one item in each period. There are m items or product families, and n periods in the planning horizon, and the data is:

d_t^i the demand for item i in period t (expressed in units of capacity);

p_t^i the unit production cost of item i in period t;

h_t^i the unit storage cost for item i at the end of period t;

f_t^i the fixed set-up cost for item i in period t (incurred when the machine is set-up to produce item i in period t);

C_t^i the production capacity for item i in period t.

To formulate this problem as a mixed integer programming problem, we introduce the following variables:

x_t^i is the amount of item i produced in period t (in units of capacity);

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s_t^i is the inventory (stock) of item i at the end of period t;

y_t^i = 1 if the line is set-up for item i in period t and $y_t^i = 0$ otherwise.

The formulation we obtain is:

$$\min \sum_{i,t} p_t^i x_t^i + \sum_{i,t} h_t^i s_t^i + \sum_{i,t} f_t^i y_t^i, \tag{1}$$

$$\sum_i y_t^i \leqslant 1 \quad \text{for all } t, \tag{2}$$

$$s_{t-1}^i + x_t^i = d_t^i + s_t^i \quad \text{for all } i, t, \tag{3}$$

$$x_t^i \leqslant C_t^i y_t^i \quad \text{for all } i, t, \tag{4}$$

$$x_t^i, s_t^i \geqslant 0, y_t^i \in \{0,1\} \quad \text{for all } i, t. \tag{5}$$

The above can be rewritten in the form:

$$\min \sum_i \sum_t \{p_t^i x_t^i + h_t^i s_t^i + f_t^i y_t^i\},$$

$$\sum_i y_t^i \leqslant 1 \quad \text{for all } t,$$

$$(x^i, s^i, y^i, z^i) \in W^i \quad \text{for all } i,$$

where W^i, described by the constraints (3)-(5), represents the set of feasible solutions for item i.

Apart from using standard branch and bound to solve relatively small instances, three algorithms spring naturally to mind for such problems.

Lagrangian relaxation

Dualising the machine constraint (2) leads to a *Lagrangian dual* lower bound

$$\omega = \max_{\pi \geqslant 0} \sum_{i=1}^m \omega^i + \sum_{t=1}^n \pi_t,$$

where

$$\omega^i = \min_{x^i, s^i, y^i} \left\{ \sum_t (p_t^i x_t^i + h_t^i s_t^i + (f_t^i - \pi_t) y_t^i) : (x^i, s^i, y^i) \in W^i \right\}$$

and $\{\pi_t\}_{t=1}^n$ are dual variables associated with the dualised constraints (2). Thus we need to solve a different ULS problem for each item i.

Column generation

For each item i, a vector $y_q^i \in \{0,1\}^n$ is the characteristic vector of the set-up periods of a feasible production plan with total cost c_q^i. If $\lambda_q^i = 1$ implies that plan y_q^i is chosen for item i, the resulting formulation is

$$\min \sum_i \sum_q c_q^i \lambda_q^i, \tag{6}$$

$$\sum_i \sum_q y_{q,t}^i \lambda_q^i \leqslant 1 \quad \text{for all } t, \tag{7}$$

$$\sum_q \lambda_q^i = 1 \quad \text{for all } i, \tag{8}$$

$$\lambda_q^i \in \{0,1\} \quad \text{for all } i, q. \tag{9}$$

In solving the associated linear programming relaxation by column generation, if $\{\pi_t\}_{t=1}^n$ and $\{\mu^i\}_{i=1}^m$ are dual variables associated with the linking constraints (7) and (8) respectively, we obtain the *column generation subproblem*

$$\zeta^i = \min_{x^i, s^i, y^i} \left\{ \sum_t (p_t^i x_t^i + h_t^i s_t^i + (f_t^i - \pi_t) y_t^i) - \mu^i : (x^i, s^i, y^i) \in W^i \right\}.$$

Note this is the same subproblem as above as $\omega^i = \zeta^i + \mu^i$.

Branch-and-cut, or cutting planes

Given a fractional point $(x^1, s^1, y^1, \ldots, x^m, s^m, y^m)$, the question is to find a valid inequality for the feasible region (2)-(5) cutting off the point. Ignoring the constraints (2), a partial answer is given by checking if the point is valid for constraints (3)-(5), namely the set $W^1 \times \cdots \times W^m$. In other words we need to test, for each item i, whether (x^i, s^i, y^i) can be cut off by a valid inequality for W^i. This is the so-called *Separation Problem for* $\text{conv}(W^i)$.

Thus we see that all three algorithms lead naturally to subproblems involving ULS. In the first two we need to optimise over $\text{conv}(W^i)$, and in the third we need to solve the Separation Problem over $\text{conv}(W^i)$.

Some papers in which these approaches have been applied to multi-item lot-sizing problems include Karmarkar and Schrage (1985), Thizy and Van Wassenhove (1986), Afentakis and Gavish (1986) and Fleischmann (1990) using Lagrangian relaxation, and Cattryse et al. (1990) and Vanderbeck (1994) using column generation. The use of cuts is discussed at the end of the next section.

3. 1958–1975: Basic results for the uncapacitated lot-sizing model

3.1. 1958: Wagner–Whitin

Though Wagner and Whitin did not view ULS as a mixed integer program, ULS is formally the problem:

$$\min\left\{\sum_t (p_t x_t + h_t s_t + f_t y_t) : (x, s, y) \in W\right\},$$

where we have dropped the superscript i, and W is described by:

$$s_{t-1} + x_t = d_t + s_t \quad \text{for all } t, \tag{10}$$

$$(s_0 = s_n = 0), \tag{11}$$

$$x_t \leqslant d_{tn} y_t \quad \text{for all } t, \tag{12}$$

$$x_t, s_t \geqslant 0, \quad 0 \leqslant y_t \leqslant 1 \quad \text{for all } t, \tag{13}$$

$$y_t \text{ integer} \quad \text{for all } t, \tag{14}$$

with $d_{st} = \sum_{i=s}^{t} d_i$ for $s \leqslant t$. Note that we have also assumed for the purposes of this talk that the production capacity $C_t \geqslant d_{tn}$ for all t.

Wagner and Whitin first showed:

Proposition 1. *(i) There exists an optimal solution with $s_{t-1} x_t = 0$ for all t. (We only produce when the entering stock is zero.)*

(ii) There exists an optimal solution in which if $x_t > 0$, then $x_t = d_{t,t+k}$ for some $k \geqslant 0$. (If we produce in period t, we produce to exactly satisfy the demands of a set of consecutive periods starting in t.)

Exploiting property (ii) of the Proposition, they then derived the following dynamic programming algorithm. We consider just the case where $f_t \geqslant 0$ for all t. In this case there is an optimal solution in which $y_t = 1$ if and only if $x_t > 0$.

Let $H(k)$ be the minimum cost of a solution for periods 1 up to k. In such a solution, if t is the last period in which production occurs, then by Proposition 1 $x_t = d_{tk}$ in this solution. So using the property that $s_{t-1} = 0$ and a simple optimality argument, the first part of this solution for periods 1 to $t-1$ must be optimal for periods 1 to $t-1$ and thus has cost $H(t-1)$.

Assuming without loss of generality that $h_t = 0$, this immediately gives the following forward recursion:

$$H(k) = \min_{1 \leqslant t \leqslant k} \{H(t-1) + f_t + p_t d_{tk}\}. \tag{15}$$

Now starting from $H(0) = 0$, calculating $H(k)$ for $k = 1, \ldots, n$ leads to the value $H(n)$ of an optimal solution of ULS. Working back gives a corresponding optimal solution. It is now easy to see that $O(n^2)$ calculations suffice to obtain $H(n)$ and an optimal solution.

3.2. Extensions to model ULS

Zangwill made several important contributions in the 1960s. One important step was to view the set W as the feasible region of a fixed charge network flow problem, see Fig. 1, in which x_t, s_t represent flows and y_t a decision whether or not to open the production arcs.

His other contributions include the introduction of backlogging in the model and the production in series model (Zangwill, 1969), for which he gave $O(n^2)$ and $O(n^4)$ dynamic programming algorithms respectively. With the fixed charge viewpoint (Zangwill, 1968), Proposition 1 generalises easily to these models, and is an immediate consequence of the acyclic structure of basic feasible solutions to flow problems.

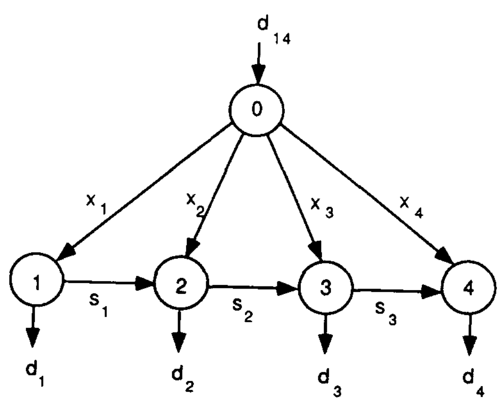

Fig. 1. Uncapacitated lot-sizing network ($n = 4$).

Veinott (1969) generalised further, showing that the structural results are a property of Leontief substitution systems. Florian and Klein (1971) showed how to handle constant production capacities, and Love (1972) showed how nested solutions are obtained for the production in series model with certain classes of objective function.

4. 1975–1987: Reformulations and cuts

4.1. Different formulations

1977: Uncapacitated facility location reformulation

Apparently out of the blue, Krarup and Bilde (1977) developed a reformulation for ULS by introducing a very natural set of additional variables.

Let w_{st} be the fraction of the demand of period t produced in period s ($s \leqslant t$ in this model without backlogging). Now problem ULS can be formulated as (FL):

$$\min \sum_{t=1}^{n} f_t y_t + \sum_{s=1}^{n} \sum_{t=s}^{n} c_s d_t w_{st}, \tag{16}$$

$$\sum_{s=1}^{t} w_{st} = 1 \quad \text{for all } t, \tag{17}$$

$$w_{st} \leqslant d_t y_s \quad \text{for all } 1 \leqslant s \leqslant t \leqslant n, \tag{18}$$

$$w_{st}, y_t \geqslant 0 \quad \text{for all } 1 \leqslant s \leqslant t \leqslant n, \tag{19}$$

$$y_t \leqslant 1 \quad \text{for all } t, \tag{20}$$

$$y_t \text{ integer} \quad \text{for all } t, \tag{21}$$

where constraint (17) says that the demand in period t must be satisfied and (18) that this demand can only be satisfied from production in period s if $y_s = 1$. It is easy to see that FL provides a valid reformulation of ULS. The following results shows that this formulation is as tight as possible, and much better than the original.

Theorem 2. *The linear programming relaxation of FL (16)–(20) always has an optimal solution with y integer.*

1984: Valid inequalities for ULS

The successful work of Crowder, Padberg and Johnson (1983) on solving integer programs with strong cutting planes motivated several researchers to try and extend the approach to MIP. Given the difficulty of solving even very small multi-item lot-sizing problems such as (1)–(5), Barany et al. decided to investigate valid inequalities for W. One possible constraint is:

$$s_{t-1} \geqslant d_t(1 - y_t), \tag{22}$$

or equivalently $\sum_{i=1}^{t-1} x_i + d_t y_t \geqslant d_{1t}$ (using $s_{t-1} = \sum_{i=1}^{t-1} x_i - d_{1,t-1}$). Inequality (22) is valid because when $y_t = 1$, it reduces to $s_{t-1} \geqslant 0$, which holds for all points of W, and when $y_t = 0$, it reduces to $s_{t-1} \geqslant d_t$, which holds because $s_{t-1} + x_t = d_t + s_t \geqslant d_t$ and $y_t = 0$ implies $x_t = 0$. Generalizing this inequality leads to an exponential family of inequalities described in Barany et al. (1984).

Proposition 3. *Let $L = \{1, \dots, l\}$, $l \leqslant n$ and $S \subseteq L$. The (l, S) inequality*

$$\sum_{i \in L \setminus S} x_i + \sum_{i \in S} d_{il} y_i \geqslant d_{1l} \tag{23}$$

is valid for W.

It is also possible to show that these are the only inequalities needed.

Theorem 4. *The constraints* (10)–(13) *and* (23) *describe* $\mathrm{conv}(W)$.

We now have a linear programming formulation (10)–(13), (23) in the original variables (x, y, s) that solves ULS. Unfortunately this formulation contains an exponential number of (l, S) inequalities. As it is impossible to add all the (l, S) inequalities a priori, we can use them in a cutting plane approach.

Separation problem for the (l, S) inequalities

Given (x^*, y^*) satisfying $\sum_{i=1}^{n} x_i^* = d_{1n}$, $y_1^* = 1$, $0 \leqslant y_i^* \leqslant 1$ for $i = 2, \dots, n$, we have to solve the separation problem associated to the (l, S) inequalities. So we wish to test whether any (l, S) inequality is violated.

Fixing l, it suffices to test whether

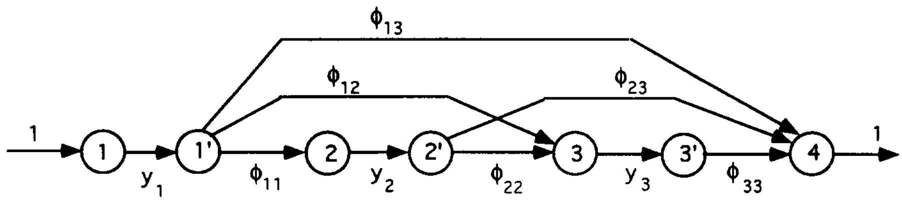

Fig. 2. Shortest path formulation of ULS ($n = 3$).

$$\sum_{i=1}^{l} \min(x_i^*, d_{il}y_i^*) < d_{1l}.$$

If this holds, then letting $S^* = \{i \in L : d_{il}y_i^* \leqslant x_i^*\}$, we see that

$$\sum_{i \in L \setminus S} x_i^* + \sum_{i \in S} d_{il}y_i^* \geqslant d_{1l}$$

is the most violated inequality for the given value of l. On the other hand if there is no violation for any l, $(x^*, y^*) \in \text{conv}(W)$.

This defines an $O(n^2)$ separation algorithm for the (l, S) inequalities.

Shortest path reformulation of ULS

A second reformulation technique developed in Martin (1987) and applied to lot-sizing problems in Eppen and Martin (1987) allows one to transform the dynamic programming algorithm for ULS into a shortest path formulation.

The dynamic programming algorithm essentially tries to construct a succession of least-cost regeneration intervals $[t, k]$ $(t \leqslant k)$, where a regeneration interval $[t, k]$ represents the production of d_{tk} in period t. Defining a variable ϕ_{tk} for all regeneration intervals, problem ULS can be formulated as the following linear program (SP):

$$\min \sum_{t=1}^{n} \sum_{k=t}^{n} c_t d_{tk} \phi_{tk} + \sum_{t=1}^{n} f_t y_t, \tag{24}$$

$$\sum_{t=1}^{n} \phi_{tn} = 1, \tag{25}$$

$$\sum_{i=1}^{t} \phi_{it} - \sum_{l=t+1}^{n} \phi_{t+1,l} = 0 \quad \text{for } 1 \leqslant t \leqslant n-1, \tag{26}$$

$$-\sum_{l=1}^{n} \phi_{1l} = -1, \tag{27}$$

$$\sum_{k=t}^{n} \phi_{tk} \leqslant y_t \quad \text{for all } t, \tag{28}$$

$$\phi_{tk}, y_t \geqslant 0, \quad y_t \leqslant 1 \quad \text{for } 1 \leqslant t \leqslant k \leqslant n, \tag{29}$$

where constraints (25)-(27) represent the possible sequence of regeneration intervals and constraint (28) forces a set-up in period t if a regeneration interval starting in period t is part of the solution. See Fig. 2 for the corresponding shortest path representation.

Here too the formulation is as tight as possible.

Theorem 5. *The linear program SP always has an optimal solution with y integer, and solves ULS.*

4.2. Using the reformulations

The above results can be used to obtain improved formulations for the multi-item model (1)-(5). Essentially constraints (3)-(5) can be replaced by the constraints and additional variables of either the FL or the SP reformulations. In either case the resulting formulation is larger, but much tighter, and its linear programming relaxation is equivalent to the linear program:

$$\min \sum_i \sum_t \{p_t^i x_t^i + h_t^i s_t^i + f_t^i y_t^i\}, \tag{30}$$

$$\sum_i y_t^i \leqslant 1 \quad \text{for all } t, \tag{31}$$

$$(x^i, s^i, y^i, z^i) \in \text{conv}(W^i) \quad \text{for all } i. \tag{32}$$

In practice the SP formulation is preferable to the FL model as it has the same number of variables, but an order of magnitude less constraints.

With cutting planes, there are two options. Either add cuts using the separation algorithm repeatedly for each item. In this case the cutting plane algorithm will terminate with an optimal solution of (30)-(32). The alternative is to choose a small subset of the (l, S) inequalities a priori, add them to the original model (1)-(5) and then solve with a standard MIP branch and bound system. Results using the SP reformulation appear in Eppen and Martin (1987), using a priori cuts in Barany et al. (1984b) and with the separation algorithms in Pochet and Wolsey (1991).

5. 1990–Present

Faster DP algorithms

A first surprise was the independent discovery by Aggarwal and Park (1993), Federgrun and Tsur (1991) and Wagelmans et al. (1992) that problem ULS can be solved in $O(n \log n)$ time either by a relatively minor modification to the original dynamic programming algorithm or using other techniques.

Wagner–Whitin costs

The authors cited above also observed that when ULS has a so-called Wagner–Whitin cost structure $p_{t-1} + h_{t-1} \geqslant p_t$ for all t (ignoring fixed costs, it is always optimal to produce as late as possible), then problem ULS can be solved in $O(n)$.

Given this result, and also the fact that many, if not most, practical instances have such a cost structure, Pochet and Wolsey (1994a) ask whether this cost structure also leads to simpler formulations. Again the results are surprising and remarkably simple.

Using the equality constraints (10) to eliminate the production variables x_t, the objective function $\sum_{t=1}^{n}(p_t x_t + h_t s_t + f_t y_t)$ can be rewritten to within a constant as

$$\min \sum_{t=1}^{n} (f_t y_t + h'_t s_t),$$

where $h'_t = p_t + h_t - p_{t+1}$ for $t = 1, \ldots, n-1$. Thus instances with $p_t = 0$ and $h_t \geqslant 0$ for all t have Wagner–Whitin costs.

Theorem 6. *With Wagner–Whitin costs, the linear program:*

$$\min \sum_t h'_t s_t + \sum_t f_t y_t,$$

$$s_{k-1} \geqslant \sum_{j=k}^{t} d_j \left(1 - \sum_{i=k}^{j} y_i\right) \quad \textit{for } 1 \leqslant k \leqslant t \leqslant n,$$

$$s_t \geqslant 0, \quad 0 \leqslant y_t \leqslant 1 \quad \textit{for all } t$$

solves ULS.

In other words only about $\frac{1}{2}n^2$ (l, S) inequalities are needed out of the $O(2^n)$ needed in the general case.

Work in progress

A profit-maximising ULS model. A practical model under investigation in the cadre of an ESPRIT project contains as subproblem the following simple variant of ULS. Rather than satisfy demand d_t in each period, a quantity up to at most d_t can be sold at a price q_t per unit. Pochet and Raucq (1994) have shown that the dynamic programming algorithm and many of the formulation results cited above can be extended to this model. They also treat lower bounds on stocks which is an important practical consideration.

Reformulations of capacitated lot-sizing models. Most practical applications involve capacity constraints (constant or varying over time) on production of each item per period, and in many cases start-up costs or times when a machine must be set-up to start producing a different item. Many of the results presented for ULS have been extended to the more complicated models, see for example van Hoesel and Kolen (1992), van Hoesel et al. (1994) and Pochet and Wolsey (1993). Constantino (1993, 1995) goes a step further and presents theoretical and computational results for both capacities and start-ups, and initial ideas for handling start-up times.

Final remarks

In such a talk we have necessarily been very selective, and biased. Extensive surveys containing more complete sets of references are Kuik et al. (1994) and Pochet and Wolsey (1994b).

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