Projet Optimisation Globale

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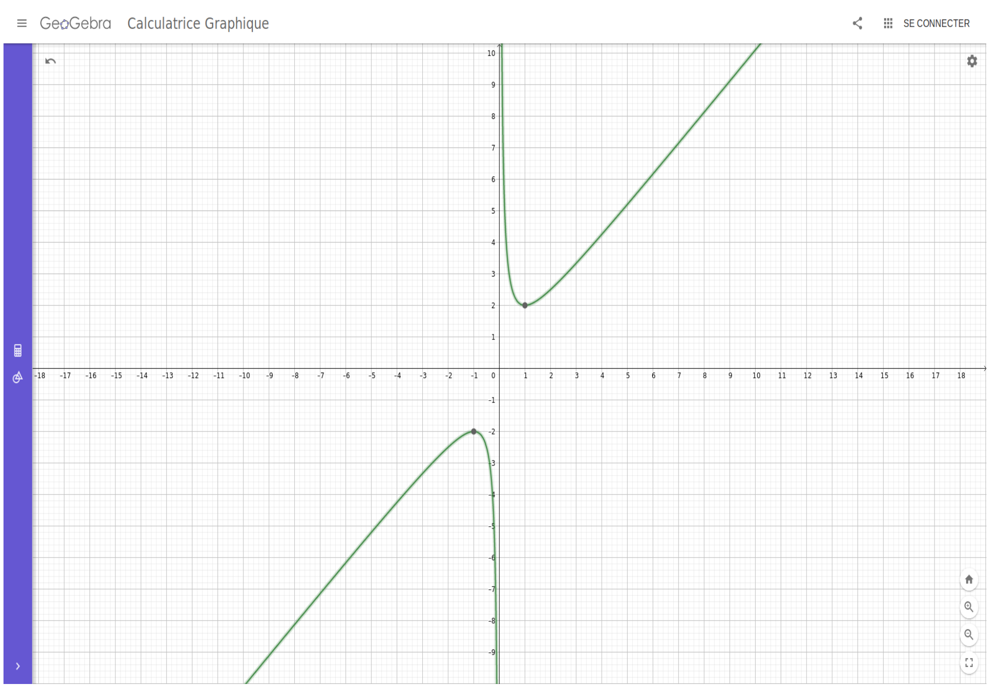
Part I
First part

Let $f(x) = x + \frac{1}{x}$ be a real function over $\mathbf{R}^*$. The graph of f is ploted below



1 Question 1

Let $F(X) = X + \frac{1}{X}$ be a natural extension of f.
Prove that F is not optimal ?

With moore 's theorem

The natural interval extension is not optimal in general and when it will, then its respect Moore theorem which implies that each variables occurs only once. In our problem, variable X occurs 2 times, then F is not optimal.

Comparing exact range and natural interval extension

Let $X_1 = [1, 2]$
range(f, X1) = [2, 2.5]
F(X1) = $X_1 + \frac{1}{X_1}$ = [1.5, 3]
On F(X1), we observe an overestimation of the exact range, then F is not optimal.

2 Question 2

2.1 Implementation of optimal F :

let X = $[inf_x, sup_x]$ given as input

Algorithm 1 : Optimal interval extension F

if $inf_x == sup_x == 0$ **then**
  $F(X) = \emptyset$
else if $inf_x \geq 1$ **then**
  $F(X) = [f(inf_x), f(sup_x)]$
else if $(0 < inf_x \leq 1)$**and**$(sup_x \geq 1)$ **then**
  $F(X) = [2, max(f(inf_x), f(sup_x))]$
else if $(0 < inf_x \leq 1)$**and**$(0 < sup_x \leq 1)$ **then**
  $F(X) = [f(sup_x), f(inf_x)]$
else if $sup_x \leq -1$ **then**
  $F(X) = [f(inf_x), f(sup_x)]$
else if $(-1 \leq sup_x < 0)$**and**$(inf_x \leq -1)$ **then**
  $F(X) = [min(f(inf_x), f(sup_x)), -2]$
else if $(-1 \leq sup_x < 0)$**and**$(-1 \leq inf_x < 0)$ **then**
  $F(X) = [f(sup_x), f(inf_x)]$
else if $(inf_x = 0)$**and**$(sup_x \geq 1)$ **then**
  $F(X) = [2, +\infty]$
else if $(inf_x = 0)$**and**$(sup_x < 1)$ **then**
  $F(X) = [f(sup_x), +\infty]$
else if $(sup_x = 0)$**and**$(inf_x \leq -1)$ **then**
  $F(X) = [-\infty, -2]$
else if $(sup_x = 0)$**and**$(inf_x > -1)$ **then**
  $F(X) = [-\infty, f(inf_x)]$
else
  $\{$Comment : $(inf_x < 0)$**and**$(sup_x > 0)\}$
  $F(X) = [-\infty, +\infty]$
end if

2.2 Monotonicity of the function f :

x	$-\infty$		-1		0		1		$+\infty$
$f(x)$	$-\infty$	$\nearrow$	-2	$\searrow$	$-\infty$ ‖ $+\infty$	$\searrow$	2	$\nearrow$	$+\infty$

Remark : Function f is not defined at 0

3 Question 3 :

Algorithm implementing optimal F' :

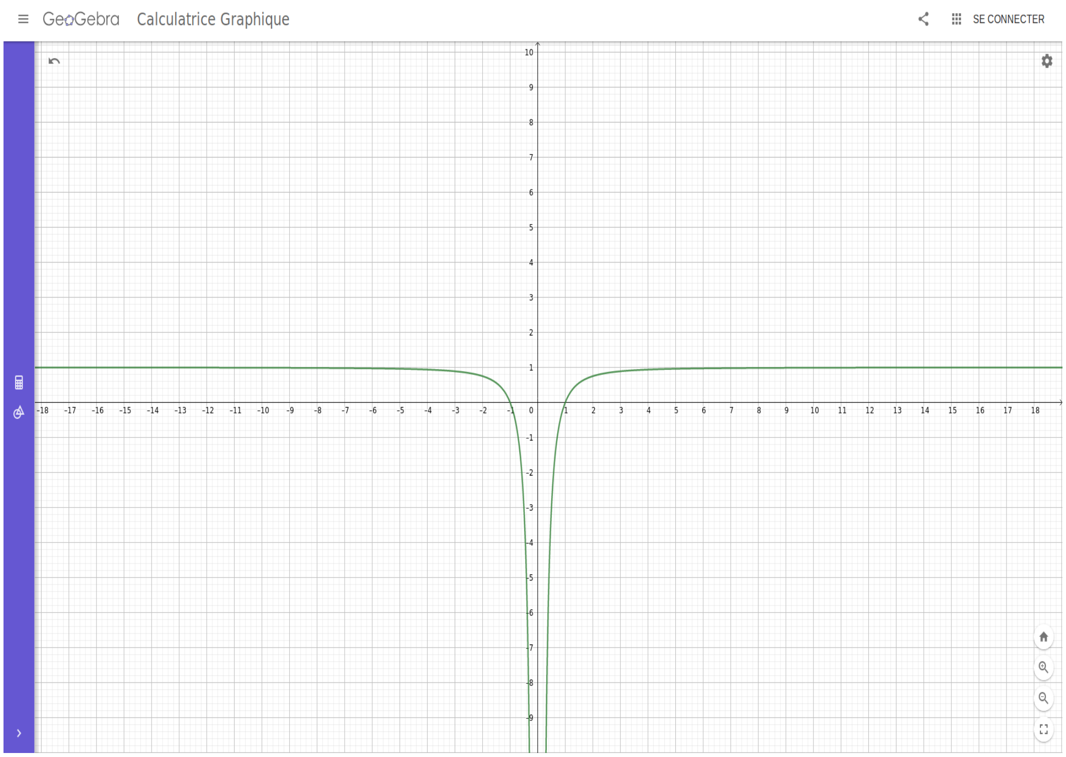
$f'(x) = 1 - \frac{1}{x^2}$ then $F'(X) = 1 - \frac{1}{X^2}$.
Let X $= [inf_x, sup_x]$
F"(x) $= \frac{2}{X^3}$

We can deduce that f' is decreasing in x $\in [\infty, 0]$ and increasing in x $\in [0, \infty]$.
F still not defined at 0.

Algorithm 2 : Optimal interval extension F'

if $inf_x == sup_x == 0$ **then**
    $F'(X) = \emptyset$
else if $sup_x < 0$ **then**
    $F'(X) = [f'(sup_x), f'(inf_x)]$
else if $inf_x > 0$ **then**
    $F'(X) = [f'(inf_x), f'(sup_x)]$
else if $(inf_x = 0)$**and**$(sup_x > 0)$ **then**
    $F'(X) = [-\infty, f'(sup_x)]$
else if $(inf_x < 0)$**and**$(sup_x = 0)$ **then**
    $F'(X) = [-\infty, f'(inf_x)]$
else
    {Comment : $(inf_x < 0)$**and**$(sup_x > 0)$}
    $F'(X) = [-\infty, f'(sup_x)]$
end if

The graph of f' is ploted below :



4 Question 4

Algorithm calculating hull($\{x \in X | (\exists y \in Y) y = f(x)\}$) :

By the property of projection in axis X, we have : X = X $\cap F^{-1}(Y)$.
$f^{-1}(y) = \{\frac{y-\sqrt{y^2-4}}{2}$ ou $\frac{y+\sqrt{y^2-4}}{2}\}$
Let $f_1^{-1}(y) = \frac{y-\sqrt{y^2-4}}{2}$ and $f_2^{-1}(y) = \frac{y+\sqrt{y^2-4}}{2}$

Let Y = $[inf_y, sup_y]$. The hull of the set $\{x \in X | (\exists y \in Y) y = f(x)\}$ given a box X * Y is $(X \cap F^{-1}(Y))$ * Y Where algorithm below is used to find $f^{-1}(Y)$:

Algorithm 3 : Interval extension F^{-1}

if $(inf_y > -2)$**and**$(sup_y < 2)$ **then**
  $F^{-1}(Y) = \emptyset$
else if $-2 < inf_y \leq 2)$**and**$(sup_y > 2)$ **then**
  $F^{-1}(Y) = [f_1^{-1}(sup_y), f_2^{-1}(sup_y)]$
else if $inf_y \geq 2$ **then**
  $F^{-1}(Y) = [f_1^{-1}(sup_y), f_1^{-1}(inf_y)] \cup [f_2^{-1}(inf_y), f_2^{-1}(sup_y)]$
else if $-2 \leq sup_y < 2)$**and**$(inf_y < -2)$ **then**
  $F^{-1}(Y) = [f_1^{-1}(inf_y), f_2^{-1}(inf_y)]$
else if $sup_y \leq -2$ **then**
  $F^{-1}(Y) = [f_1^{-1}(inf_y), f_1^{-1}(sup_y)] \cup [f_2^{-1}(sup_y), f_2^{-1}(inf_y)]$
else
  {Comment : $(inf_y \leq -2)$**and**$(sup_y \geq 2)$}
  $F^{-1}(Y) = [f_1^{-1}(inf_y), f_2^{-1}(inf_y)] \cup [f_1^{-1}(sup_y), f_2^{-1}(sup_y)]$
end if

5 Question 5

The contractor enforcing hull consistency on the equation $y = f(x)$ given a box $X * Y$ is defined as a mapping on boxes
$\Gamma : X * Y \rightarrow (X \cap F^{-1}(Y)) * (Y \cap F(X))$.
prove that it effectively enforces the hull consistency property ?
Hull consistency states that each domain $X\&Y$ is equal to the hull of its projection :

$$\begin{cases} Proj(y = f(x), X * Y, 1) & = X \cap F^{-1}(Y) \\ Proj(y = f(x), X * Y, 2) & = Y \cap F(X) \end{cases}$$

We have :

$$\begin{cases} Proj(y = f(x), X * Y, 1) \subseteq (X \cap F^{-1}(Y)) \\ Proj(y = f(x), X * Y, 2) \subseteq (Y \cap F(X)) \end{cases}$$

Then Γ enforces the hull consistency property.

Part II
Second part

Let $g(x) = ax + \frac{b}{x}$ be a real function over $\mathbf{R}^*$, a $\neq 0$ and b $\neq 0$.
G(X) $= aX + \frac{b}{X}$ with X = $[inf_x, sup_x]$.

6 Question 6

6.1 Implementation of optimal interval extension G

Four cases are possibles :

$$\begin{cases} \textbf{Case (1) :} \ a > 0 \wedge b < 0 \\ \textbf{Case (2) :} \ a > 0 \wedge b > 0 \\ \textbf{Case (3) :} \ a < 0 \wedge b < 0 \\ \textbf{Case (4) :} \ a < 0 \wedge b > 0 \end{cases}$$

- Case (2) and Case (3) are symmetric
  g(x,a,b) = -g(x,-a,-b) $\Rightarrow$ G(X,a,b) = -G(X,-a,-b)
- Case (1) and Case (4) are symmetric
  g(x,a,-b) = -g(x,-a,b) $\Rightarrow$ G(X,a,-b) = -G(X,-a,b)

Algorithm 4 : Case (1) $a > 0 \wedge b < 0$

if $inf_x == sup_x == 0$ **then**
  $G(X) = \emptyset$
else if $sup_x < 0$ **then**
  $G(X) = [g(inf_x), g(sup_x)]$
else if $inf_x > 0$ **then**
  $G(X) = [g(inf_x), g(sup_x)]$
else if $inf_x = 0$ **then**
  $G(X) = [-\infty, g(sup_x)]$
else if $sup_x = 0$ **then**
  $G(X) = [g(inf_x), +\infty]$
else
  $G(X) = [-\infty, +\infty]$
end if

Algorithm 5 : Case (4) $a < 0 \wedge b > 0$

if $inf_x == sup_x == 0$ **then**
  $G(X) = \emptyset$
else if $sup_x < 0$ **then**
  $G(X) = [g(sup_x), g(inf_x)]$
else if $inf_x > 0$ **then**
  $G(X) = [g(sup_x), g(inf_x)]$
else if $inf_x = 0$ **then**
  $G(X) = [g(sup_x), +\infty]$
else if $sup_x = 0$ **then**
  $G(X) = [-\infty, g(inf_x)]$
else
  $G(X) = [-\infty, +\infty]$
end if

Algorithm 6 : Case (2) a$>$ 0 $\wedge$ b > 0

if $inf_x == sup_x == 0$ **then**
  $G(X) = \emptyset$
else if $inf_x \geq \sqrt{\frac{b}{a}}$ **then**
  $G(X) = [g(inf_x), g(sup_x)]$
else if $(0 < inf_x \leq \sqrt{\frac{b}{a}})$**and**$(sup_x \geq \sqrt{\frac{b}{a}})$ **then**
  $G(X) = [g(\sqrt{\frac{b}{a}}), max(g(inf_x), g(sup_x))]$
else if $(0 < inf_x \leq \sqrt{\frac{b}{a}})$**and**$(0 < sup_x \leq \sqrt{\frac{b}{a}})$ **then**
  $G(X) = [g(sup_x), g(inf_x)]$
else if $sup_x \leq -\sqrt{\frac{b}{a}}$ **then**
  $G(X) = [g(inf_x), g(sup_x)]$
else if $(-\sqrt{\frac{b}{a}} \leq sup_x < 0)$**and**$(inf_x \leq -\sqrt{\frac{b}{a}})$ **then**
  $G(X) = [min(g(inf_x), g(sup_x)), g(-\sqrt{\frac{b}{a}})]$
else if $(-\sqrt{\frac{b}{a}} \leq sup_x < 0)$**and**$(-\sqrt{\frac{b}{a}} \leq inf_x < 0)$ **then**
  $G(X) = [g(sup_x), g(inf_x)]$
else if $(inf_x = 0)$**and**$(sup_x \geq \sqrt{\frac{b}{a}})$ **then**
  $G(X) = [g(\sqrt{\frac{b}{a}}), +\infty]$
else if $(inf_x = 0)$**and**$(sup_x < \sqrt{\frac{b}{a}})$ **then**
  $G(X) = [g(sup_x), +\infty]$
else if $(sup_x = 0)$**and**$(inf_x \leq -\sqrt{\frac{b}{a}})$ **then**
  $G(X) = [-\infty, g(-\sqrt{\frac{b}{a}})]$
else if $(sup_x = 0)$**and**$(inf_x > -\sqrt{\frac{b}{a}})$ **then**
  $G(X) = [-\infty, g(inf_x)]$
else
  $G(X) = [-\infty, +\infty]$
end if

Algorithm 7 : **Case (3)** $a < 0 \wedge b < 0$

if $inf_x == sup_x == 0$ **then**
  $G(X) = \emptyset$
else if $inf_x \geq \sqrt{\frac{b}{a}}$ **then**
  $G(X) = [g(sup_x), g(inf_x)]$
else if $(0 < inf_x \leq \sqrt{\frac{b}{a}})$**and**$(sup_x \geq \sqrt{\frac{b}{a}})$ **then**
  $G(X) = [min(g(inf_x), g(sup_x)), g(\sqrt{\frac{b}{a}})]$
else if $(0 < inf_x \leq \sqrt{\frac{b}{a}})$**and**$(0 < sup_x \leq \sqrt{\frac{b}{a}})$ **then**
  $G(X) = [g(inf_x), g(sup_x)]$
else if $sup_x \leq -\sqrt{\frac{b}{a}}$ **then**
  $G(X) = [g(sup_x), g(inf_x)]$
else if $(-\sqrt{\frac{b}{a}} \leq sup_x < 0)$**and**$(inf_x \leq -\sqrt{\frac{b}{a}})$ **then**
  $G(X) = [g(-\sqrt{\frac{b}{a}}), max(g(inf_x), g(sup_x))]$
else if $(-\sqrt{\frac{b}{a}} \leq sup_x < 0)$**and**$(-\sqrt{\frac{b}{a}} \leq inf_x < 0)$ **then**
  $G(X) = [g(inf_x), g(sup_x)]$
else if $(inf_x = 0)$**and**$(sup_x \geq \sqrt{\frac{b}{a}})$ **then**
  $G(X) = [-\infty, g(\sqrt{\frac{b}{a}})]$
else if $(inf_x = 0)$**and**$(sup_x < \sqrt{\frac{b}{a}})$ **then**
  $G(X) = [-\infty, g(sup_x)]$
else if $(sup_x = 0)$**and**$(inf_x \leq -\sqrt{\frac{b}{a}})$ **then**
  $G(X) = [g(-\sqrt{\frac{b}{a}}), +\infty]$
else if $(sup_x = 0)$**and**$(inf_x > -\sqrt{\frac{b}{a}})$ **then**
  $G(X) = [g(inf_x), +\infty]$
else
  $G(X) = [-\infty, +\infty]$
end if

6.2 Monotonicity of the function g

Case (1) : $a > 0 \wedge b < 0$

x	$-\infty$		0		$+\infty$
$f(x)$	$-\infty$	$\nearrow$	$+\infty$ ‖ $-\infty$	$\nearrow$	$+\infty$

Case (4) : $a < 0 \wedge b > 0$

x	$-\infty$		0		$+\infty$
$f(x)$	$+\infty$	$\searrow$	$-\infty$ ‖ $+\infty$	$\searrow$	$-\infty$

Case (2) : $a > 0 \wedge b > 0$

x	$-\infty$		$-\sqrt{\frac{b}{a}}$		0		$\sqrt{\frac{b}{a}}$		$+\infty$
$f(x)$	$-\infty$	$\nearrow$	$f(-\sqrt{\frac{b}{a}})$	$\searrow$	$-\infty$ ‖ $+\infty$	$\searrow$	$f(\sqrt{\frac{b}{a}})$	$\nearrow$	$+\infty$

Case (3) : $a < 0 \wedge b < 0$

x	$-\infty$		$-\sqrt{\frac{b}{a}}$		0		$\sqrt{\frac{b}{a}}$		$+\infty$
$f(x)$	$+\infty$	$\searrow$	$f(-\sqrt{\frac{b}{a}})$	$\nearrow$	$+\infty$ ‖ $-\infty$	$\nearrow$	$f(\sqrt{\frac{b}{a}})$	$\searrow$	$-\infty$

7 Question 7

Implementation of the previous algorithm on machine (C++) and using JAIL Library

The two reals coefficients a & b are implemented as an interval. we have two cases possibles for these implementation :

- When a is **integer**, its interval is set to degenerate interval ;
- Else we set it to $[a - \epsilon, a + \epsilon]$ where ϵ is the maximal error in floating number representation on the run test machine. For example $\epsilon = 1.19209e - 07$ in our case.

Remark : the same thing is done for b. this implementation permits to capture rounding errors when a or b are not integer.

Note also that an empty interval is an interval [c d] with $c > d$.