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BACHELOR THESIS

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Magnetic Transport Along Translationally Invariant Obstacles

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Dedication.

Title: Magnetic Transport Along Translationally Invariant Obstacles

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Abstract: Abstract.

Keywords: two-dimensional quantum systems, magnetic field, non-local perturbations, translational invariance, spectral properties

Contents

List of symbols

$A\vert_\Omega$	Restriction of an operator A to a subspace $\Omega \subset \mathrm{D}(A)$.
$C^k(\Omega,\mathbb{K})$, $C(\Omega,\mathbb{K})$	The space of functions $\Omega \subseteq \mathbb{R} \to \mathbb{K}$ with k continuous derivatives. The space of continuous functions.
$C_0^\infty(\Omega,\mathbb{K})$	The space of C^∞ functions with compact support in Ω.
$\mathrm{D}(A)$	The domain of operator A, usually dense in $\mathcal{H}$.
$D_\nu(w)$	The parabolic cylinder function.
$\mathcal{F}$	The Fourier-Plancherel operator on $L^2(\mathbb{R})$.
${}_1\mathrm{F}_1(\alpha,\beta;z)$	The confluent hypergeometric function of the first kind.
$\mathcal{H}$	A separable Hilbert space.
$H,\ \mathscr{H}(\xi)$	A Hamiltonian operator; a fibre of the Hamiltonian.
$H_n(x)$	The n-th Hermite polynomial.
$L^p(M,\,\mathrm{d}\mu\,,V)$	The space of p-integrable functions from measure space (M,μ) to vector space V. Specifically for $p=2$, a Hilbert space with inner product $(\psi,\phi)_{L^2} = \int_M (\psi,\phi)_V \,\mathrm{d}\mu\,.$
$L^p(\Omega)$	As above, but $M=\Omega\subseteq\mathbb{R}^N$, μ is the Lebesgue measure and $V=\mathbb{C}$.
$L^p_\rho(\Omega)$	A weighted Lebesgue space, a shorthand for $L^p(\Omega \subseteq \mathbb{R}^N,\,\mathrm{d}\mu\,,\mathbb{C})$, where ρ is a real function and $\mu(M) = \int_M \rho\,\mathrm{d}\lambda$ is a rescaling of the Lebesgue measure λ.
$L^1_{\mathrm{loc}}(\Omega)$	The space of functions that are $L^1(K)$ for every compact $K\subset\Omega$.
$\mathbb{N},\ \mathbb{N}_0$	The set of positive integers; the set of non-negative integers.
$\vec{P}, P_x, P_y, P_z$	Momentum operator – a self-adjoint operator, such that $P_x\, f(x,...) = -\mathrm{i}\frac{\partial}{\partial x} f(x,...)\,.$
$\vec{Q}, Q_x, Q_y, Q_z$	Position operator – a self-adjoint operator, such that $Q_x\, f(x,...) = x\, f(x,...)\,.$
$W^{k,p}(\Omega)$	The Sobolev space – the space of integrable functions f, such that $f^{(\alpha)} \in L^p(\Omega)$, where α is a multi-index and $\vert\alpha\vert \le k$.
$\Gamma(z), \Psi(z)$	The gamma function, the digamma function.
μ	A σ-finite measure, usually the Lebesgue measure.
$\sigma(T), \sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{ac}}(T), \sigma_{\mathrm{sc}}(T)$, $\sigma_{\mathrm{disc}}(T), \sigma_{\mathrm{ess}}(T)$	The spectrum of normal operator T; the point, absolutely continuous, singular continuous, discrete, essential spectrum of T. $\sigma(T) = \sigma_{\mathrm{p}} \cup \sigma_{\mathrm{ac}} \cup \sigma_{\mathrm{sc}} = \sigma_{\mathrm{disc}} \cup \sigma_{\mathrm{ess}}\,.$
$\nabla,\ \nabla\times,\ \Delta$	Gradient, rotation, Laplace operator.
$\Delta_{\mathrm{D}}^\Omega, \Delta_{\mathrm{D},A}^\Omega$	The Dirichlet Laplacian, defined on functions from $L^2(\Omega)$ with a Dirichlet boundary condition; a "magnetic" Dirichlet Laplacian given by the vector potential A.

Introduction

1. Formulation & useful concepts

In this chapter we will illustrate what magnetic transport is and give a precise mathematical formulation of the problem. Then we will explain concepts and restate textbook theorems which will be useful later.

1.1 The magnetic Hamiltonian

The simplest example of a quantum system with a magnetic field is the system consisting of a single charged and spinless particle in three-dimensional free space exposed to a homogeneous magnetic field and zero scalar potential. The Hamiltonian that corresponds to this system is:

$$H = (\vec{P} + \vec{A})^2 \,, \quad \vec{B} = \nabla \times \vec{A} = (0, 0, b_0) \,.$$

Here $\vec{P} = -\mathrm{i}\nabla$ is the momentum operator, $\vec{B}$ is the magnetic field, which is constant with magnitude b_0, without loss of generality pointing along the z axis, and $\vec{A}$ is a corresponding vector potential. Notice that we have used nondimensionalization to remove physical units from the Hamiltonian. The spectrum of H is absolutely continuous and the Hamiltonian commutes with P_z, thus it allows the particle to move freely along the z-axis. However, if we restrict the particle to the layer $z = 0$, either physically, or only formally because we are not interested in the movement along z [**EXPAND**], we get a two-dimensional Hamiltonian with infinitely degenerate pure point spectrum, the so-called Landau Hamiltonian:

$$H = (P_x + A_x)^2 + (P_y + A_y)^2 \,.$$

A detailed analysis of this well-known Hamiltonian can be found e.g. in §112 of Landau and Lifshitz [1981]. The pure point spectrum means that the particle is not free to move along x or y, but instead it is "trapped" in some superposition of stationary states. We will investigate perturbations to the Landau Hamiltonian, which cause its spectrum to become continuous and allow the particle to move freely along one axis. These perturbations can be either in the form of a scalar potential, a modification of the magnetic field, or a purely geometric deformation of the layer to which our particle is constrained. We will require all of these perturbations to be translationally invariant, thus constant along one axis – without loss of generality, we choose that they are independent of y and only depend on x.

Throughout this thesis, we will use the Landau gauge:

$$A_x = 0 \,, \quad A_y = \int_0^x B_z(x')\,\mathrm{d}x' \,, \quad A_z = 0 \,.$$

Now we can specify precisely which Hamiltonians we will investigate.

Definition 1 (Potential perturbation)**.** *Let $\Omega \subseteq \mathbb{R}^2$, $\mathcal{H} = L^2(\Omega)$, $b > 0$ and $V \in L^1_{loc}(\mathbb{R})$. A self-adjoint operator on $\mathcal{H}$ given by the equation*

$$H = P_x^2 + \left(P_y + b\,Q_x\right)^2 + V(x)$$

is called the ***Landau Hamiltonian with a potential perturbation****. We will investigate, which boundary conditions and which choices of* V *lead to* $\sigma_{\mathrm{ac}}(H) \neq \emptyset$*. The domain* $\mathrm{D}(H)$ *is determined not only by the asymptotic behaviour of* V*, but also by the boundary conditions imposed on the wave function.*

Definition 2 (Magnetic perturbation)**.** *Let* $b \in C^\infty(\mathbb{R})$*,* $\mathcal{H} = L^2(\mathbb{R}^2)$*, let* $\mathcal{D} = C_0^\infty(\mathbb{R}^2)$ *be the set of* $\mathbb{C}^\infty$ *functions with compact support and* A_y *be a multiplication operator on* $\mathcal{H}$ *given by:*

$$A_y\,\psi(x,y) = \left(\int_0^x b(x')\,\mathrm{d}x'\right)\psi(x,y)\,.$$

Let $\tilde{H} : \mathcal{D} \to \mathcal{H}$ *be an essentially self-adjoint operator given by the equation:*

$$\tilde{H} = P_x^2 + \left(P_y + A_y\right)^2\,,$$

Its closure H *is called the* ***Landau Hamiltonian with a magnetic perturbation****. We will investigate, which choices of* b *lead to* $\sigma_{\mathrm{ac}}(H) \neq \emptyset$*.*

Definition 3 (Geometric perturbation, transl. inv. layer)**.** *Let* $b > 0$ *and* $\ell > 0$*. Let* $\omega : \mathbb{R} \to \mathbb{R}^2$ *be a* C^4*-smooth curve. We define a set* $\Omega' \subset \mathbb{R}^2$ *by*

$$\Omega' = \left\{\; P \in \mathbb{R}^2 \;\mid\; \exists s \in \mathbb{R} \left\|\omega(s) - P\right\| \leq \ell \;\right\},$$

this gives a band of width 2ℓ *around the curve* ω*. Then we define a set* $\Omega \subset \mathbb{R}^3$ *as*

$$\Omega = \left\{\; (x,y,z) \in \mathbb{R}^3 \;\mid\; (x,z) \in \Omega' \;\right\}.$$

We shall call Ω *a* ***translationally invariant layer of width*** 2ℓ ***given by the curve*** ω*. Let us now consider the magnetic Dirichlet Laplacian*

$$\Delta^{\Omega}_{\mathrm{D},A}\psi(x,y,z) = \Delta\psi + 2\mathrm{i}b\,x\,\frac{\partial\psi}{\partial y} - b^2x^2\,\psi$$

defined on functions $\psi \in C^\infty(\Omega)$*, such that* $\psi(x,y,z) = 0$ *on the boundary of* Ω*. The operator* H *which is the closure of* $-\Delta^{\Omega}_{\mathrm{D},A}$ *in* $L^2(\Omega)$ *is called the* ***Landau Hamiltonian with a geometric perturbation****. We will investigate, which choices of* ω *lead to* $\sigma_{\mathrm{ac}}(H) \neq \emptyset$*.*

The Landau Hamiltonian with a translationally invariant magnetic perturbation is also called *"the Iwatsuka Hamiltonian"* by some authors (e.g. Miranda and Popoff [2017] and Hislop and Soccorsi [2015]) after Akira Iwatsuka who studied how perturbations to the Landau Hamiltonian affect its spectrum. In similar spirit, Exner et al. [2018] use the term *"Iwatsuka type effect"* to describe the phenomenon when a particular perturbation changes the spectrum of the Landau Hamiltonian to absolutely continuous. It is exactly this *Iwatsuka type effect* that will be the main focus of this thesis. We will look into more details of Akira Iwatsuka's work in section 2.2.

1.2 Direct integral

The key insight to all three of these problems is that the Hamiltonians in question only depend on the momentum p_y of the particle, and not on its position y. If we were to fix p_y of the particle to a certain value somehow, we could reduce the problem to a one-dimensional operator and solve for each p_y separately. This vague idea can be given a rigorous meaning in terms of the *direct integral*, a generalization of the direct sum.

The following definition is a rephrasing of definitions given in Reed and Simon [1978], pages 280 and 281.

Definition 4 (Direct integral, fibre)**.** *Let $\mathcal{H}'$ be a separable Hilbert space and (M, μ) a measure space. We define a Hilbert space $\mathcal{H}$, which is the space of all square-integrable functions from M to $\mathcal{H}'$:*

$$\mathcal{H} = L^2(M, \mathrm{d}\mu\,, \mathcal{H}')\,.$$

Let $\mathscr{A}$ be a measurable function from M to the self-adjoint operators on $\mathcal{H}'$. Let $f_\psi : M \to \mathbb{R}$ be a function defined by

$$f_\psi(s) = \left\|\mathscr{A}(s)\psi(s)\right\|_{\mathcal{H}'} \quad \textit{for all } \psi \in \mathcal{H}, s \in M \textit{ such that } \psi(s) \in \mathrm{D}(\mathscr{A}(s))\,.$$

We define an operator A on $\mathcal{H}$ by:

$$\begin{gathered}(A\psi)(s) = \mathscr{A}(s)\,\psi(s)\,,\\ \mathrm{D}(A) = \left\{\psi \in \mathcal{H}\;\middle|\;\psi(s) \in \mathrm{D}(\mathscr{A}(s)) \textit{ a.e. } \wedge\ \left\|f_\psi\right\|_{L^2} < \infty\right\}.\end{gathered}$$

Then we shall write

$$\mathcal{H} = \int_M^\oplus \mathcal{H}', \qquad A = \int_M^\oplus \mathscr{A}(s)\,\mathrm{d}s\,.$$

We shall call $\mathcal{H}$ and A ***the direct integral*** *of $\mathcal{H}'$ and $\mathscr{A}$, respectively. Reversely, we shall call $\mathcal{H}'$ a* ***fibre space*** *of $\mathcal{H}$ and $\mathscr{A}(s)$ a* ***fibre*** *of A.*

The concept of a direct integral might initially seem strange to readers who encounter it for their first time. These readers may find it helpful to think of the direct integral as a simple *"rebranding"* of several concepts they already know and understand. For example, a free spin-$\frac{1}{2}$ particle is represented in the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^2)$ of square-integrable functions from the physical space $\mathbb{R}^3$ to the qubit $\mathbb{C}^2$. This space is by definition the direct integral $L^2(\mathbb{R}^3, \mathbb{C}^2) = \int_{\mathbb{R}^3}^\oplus \mathbb{C}^2$, the qubit plays the role of the fibre space here. Another example is related to the fact that a function of two variables can be understood as a function of one variable which returns another function of one variable (programmers call this *currying*). That is exactly the meaning of this direct integral: $L^2(M \times N) \simeq \int_M^\oplus L^2(N) \simeq \int_N^\oplus L^2(M)$. We mentioned that the direct integral is a generalization of the direct sum. To see that this is the case, consider a finite set M together with the counting measure, then $\int_M^\oplus \mathcal{H}' \simeq \bigoplus_M \mathcal{H}'$.

Before we apply the theory of direct integrals to the magnetic Hamiltonian, let us remind the Fourier-Plancherel operator. It is a standard textbook result (see Blank et al. [2008]) that if we take the Fourier transform as an operator on $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$, its closure is a unitary operator on $L^2(\mathbb{R})$. This operator is called the Fourier-Plancherel operator $\mathcal{F}$, it transforms momentum to position $\mathcal{F}P\mathcal{F}^{-1} = Q$, and as a unitary operator, it does not change the spectrum of self-adjoint operators:

$$\sigma(A) = \sigma(\mathcal{F}A\mathcal{F}^{-1}) , \quad \sigma_{\text{disc}}(A) = \sigma_{\text{disc}}(\mathcal{F}A\mathcal{F}^{-1}) , \quad \sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(\mathcal{F}A\mathcal{F}^{-1}) .$$

The theory given so far regards functions of one variable. In this thesis, we will perform a *partial* Fourier transformations on multivariate functions – that is, perform the Fourier transformation on one variable whilst keeping the other variables fixed. We will use a subscript to indicate the variable which is being transformed and the new variable, for example $\mathcal{F}_{y\to\xi} : \psi(x,y) \mapsto \tilde{\psi}(x,\xi)$.

Now we can show, how to express a Landau Hamiltonian with potential and magnetic perturbations in terms of the direct integral:

$$\begin{aligned} H &= \left(\vec{P} + \vec{A}(x)\right)^2 + V(x) \\ &= P_x^2 + (P_y + A_y(x))^2 + V(x) \\ &\simeq \mathcal{F}_{y\to p}\Big(P_x^2 + (P_y + A_y(x))^2 + V(x)\Big)\mathcal{F}_{y\to p}^{-1} \\ &= P_x^2 + (Q_p + A_y(x))^2 + V(x) \\ &= \int_{\mathbb{R}}^{\oplus} \underbrace{{P_x}^2 + (p + A_y(x))^2 + V(x)}_{\mathscr{H}(p)} \;\mathrm{d}p \ , \end{aligned} \tag{1.1}$$

where $P_y\psi(x,y) = \mathrm{i}\frac{\partial}{\partial y}\psi(x,y)$ is a differential operator and $Q_p\psi(x,p) = p\,\psi(x,p)$ is the operator of multiplication by the second coordinate. For every $p \in \mathbb{R}$, $\mathscr{H}(p)$ is a self-adjoint operator on $L^2(\mathbb{R})$. The physical meaning of the parameter p is the particle's momentum in the direction of y and $\mathscr{H}(p)$ is the Hamiltonian for a particle with a fixed y-momentum.

The following theorem is a weakened version of Theorem XIII.85 of Reed and Simon [1978].

Theorem 5 (Spectrum of direct integral)**.** *Let $\lambda \in \mathbb{C}$ and $A = \int_M^{\oplus} \mathscr{A}(s)\,\mathrm{d}s$, as in the previous definition. We define $\Gamma(\lambda)$ as the set of all s, such that λ is an eigenvalue of $\mathscr{A}(s)$, and $\Omega_\varepsilon(\lambda)$ as the set of all s, such that the ε-neighbourhood of λ intersects the spectrum of $\mathscr{A}(s)$ – written symbolically:*

$$\begin{aligned} \Gamma(\lambda) &= \Big\{ \ s \ \mid \ \lambda \textit{ is an eigenvalue of } \mathscr{A}(s) \ \Big\} , \\ \Omega_\varepsilon(\lambda) &= \Big\{ \ s \ \mid \ \sigma(\mathscr{A}(s)) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset \ \Big\} . \end{aligned}$$

Then λ belongs to the spectrum of A if and only if

$$\mu(\ \Omega_\varepsilon(\lambda)\ ) > 0 \qquad \textit{for all } \varepsilon > 0 .$$

Additionally, λ is an eigenvalue of A if and only if

$$\mu(\ \Gamma(\lambda)\ ) > 0 .$$

This means that we can deduce the spectrum of the Hamiltonian H simply by investigating how the spectrum of its fibre $\mathscr{H}(p)$ depends on p. Furthermore, the spectrum of $\mathscr{H}(p)$ typically consists of simple eigenvalues which are particularly convenient to work with.

1.3 Refresher on linear operators

Before we start investigating specific Hamiltonians, let us remind a few textbook theorems regarding self-adjointness and spectral properties of linear operators, which will be useful later. The following definition and the subsequent theorem are from the chapter 4.7 in Blank et al. [2008].

Definition 6 (Deficiency indices)**.** *Let T be a linear operator on $\mathcal{H}$. We define two numbers n_+, $n_- \in \mathbb{N}_0 \cup \{\infty\}$ as follows:*

$$n_\pm(T) = \dim\, \mathrm{Ker}\, \left(T^* \pm \mathrm{i}I\right) ,$$

*where I is the identity operator on $\mathcal{H}$. We call these numbers the **deficiency indices** of T.*

Theorem 7 (Deficiency indices and self-adjoint extensions)**.** *Let T be a closed symmetric operator on $\mathcal{H}$, such that*

$$n_+(T) = n_-(T) < \infty .$$

Then all maximal extensions of T are self-adjoint. Furthermore, if $n_\pm = 0$, then T is already self-adjoint.

The following theorem is given in Weidmann [1980] as Theorem 8.18.

Theorem 8 (Spectrum of self-adjoint extensions)**.** *Let T be a closed symmetric operator on $\mathcal{H}$, such that*

$$n_+(T) = n_-(T) < \infty .$$

Then the essential spectrum of every self-adjoint extension of T is the same. In particular, if one self-adjoint extension of T has a pure discrete spectrum, all of them do.

Finally, a useful theorem about differential operators on L^2 was compiled from the opening of section 8.4 of Weidmann [1980], up to the theorem 8.20 there.

Theorem 9 (Deficiency indices of differential operators)**.** *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ such that $a < b$. Let $p \in C^1(\,(a,b), \mathbb{R}\,)$ be a continuously differentiable real function and $q \in C(\,(a,b), \mathbb{R}\,)$ be a continuous real function. We define L to mean:*

$$L\psi := -(p\,\psi')' + q\,\psi .$$

We define the operator T on $L^2((a,b))$ as following:

$$T\psi = L\psi \quad \textit{for all} \quad \psi \in \mathrm{D}(T) ,$$
$$\mathrm{D}(T) = \left\{ \psi \in W^{2,2}((a,b)) \;\middle|\; L\psi \in L^2((a,b)) \;\wedge\; \mathrm{supp}\,\psi \subset (a,b) \textit{ is compact} \right\}$$

Then $n_-(T) = n_+(T)$.

2. Known results

In this chapter we will restate the results about Landau Hamiltonians with potential, geometric and magnetic perturbations, which have already been proven. Effort was made to unify notation and conventions across the various sources.

2.1 Potential perturbation

2.1.1 Macris et al., 1999

Marcis et al. [1999] investigated the problem of Landau Hamiltonians with a steep (but locally integrable) potential wall along the edge of a half-plane. What follows is a summary of their results.

Definition 10 (Hamiltonian)**.** *Let $\mu \in (0,\infty)$ and $\gamma \in [1,\infty])$. We define the wall potential U:*

$$U(x) = \mu\, x^{\gamma}\, \chi_{\mathbb{R}_+}(x)\,,$$

where $\chi_{\mathbb{R}_+}$ is the characteristic function of $\mathbb{R}_+ \equiv [0,\infty)$. Let $V \in C^1(\mathbb{R}^2,\mathbb{R})$ be a differentiable real function of two variables, such that

$$\sup_{x,y\in\mathbb{R}} \Big|V(x,y)\Big| =: V_0 < \infty\,, \qquad \sup_{x,y\in\mathbb{R}} \Big|\frac{\partial}{\partial x}V(x,y)\Big| =: V_0' < \infty\,.$$

Let $B \in \mathbb{R}$. We define the Hamiltonian H:

$$H = \tfrac{1}{2}\, P_x^2 + \tfrac{1}{2}(P_y - B\, Q_x)^2 + V(x,y) + U(x)\,.$$

The Hamiltonian is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$.

Definition 11 (Auxiliary)**.** *Finally, we define a functional $A(E;U)$, where $E > 0$ and U is as above.*

$$A(E;U) = \sup_{0\le x\le x_0} \left(\frac{U(x)^4}{U'(x)}\right) + 8\int_{x_0}^{\infty} \frac{\sqrt[4]{U(\frac{x}{2})}\, U(x)^4}{\sqrt{2\pi x}\, U'(x)}\, \exp\Big(-x\sqrt{\tfrac{1}{8}\, U(\tfrac{x}{2})}\Big)\, \mathrm{d}x\,,$$

where x_0 is such that $U(\frac{x_0}{2}) = 2E$. And for $n \in \mathbb{N}$, $\delta > 0$ we define a set $\Omega_{n,\delta}$:

$$\Omega_{n,\delta} = \mathcal{U}_\delta\Big(n\, B\Big)\,,$$

where $\mathcal{U}_a(b)$ is the open a-neighbourhood of b.

Theorem 12. *Let $\delta > 0$, such that $\frac{B}{2} > \delta$. If*

$$V_0' < \frac{\left(\frac{B}{2} - \delta - V_0\right)^4}{\sup\limits_{E\in\Omega_{n,\delta}} A(E + V_0; U)}\,,$$

then $\Omega_{n,\delta} \cap \sigma_{\mathrm{p}}(H) = \emptyset$. Furthermore, if

$$V_0' < \frac{\left(\frac{B}{2} - \delta - V_0\right)^4}{\sup\limits_{0\le a\le \frac{B}{2}} \; \sup\limits_{E\in\Omega_{n,\delta}} A(E + a; U)}\,,$$

then $\Omega_{n,\delta} \subset \sigma_{\mathrm{c}}(H) \equiv \sigma_{\mathrm{ac}}(H) \cup \sigma_{\mathrm{sc}}(H)$.

2.1.2 Fröhlich et al., 2000

Fröhlich et al. [2000] investigated the problem of systems constrained to a half-plane $\mathbb{R} \times \mathbb{R}_+$ by either a potential wall (bounded or unbounded), or a Dirichlet boundary condition. The Dirichlet b.c. was also treated for more general subspaces $\Omega \subset \mathbb{R}^2$ – we will not list these, as they were not translationally invariant.

In the case of the steep potential, they used the theory of *Mourre estimates*, introduced in Mourre [1981].

Definition 13 (Conjugate operator)**.** *Let H and A be self-adjoint operators with domains $\mathrm{D}(H)$ and $\mathrm{D}(A)$. Let $\Omega := \mathrm{D}(H) \cap \mathrm{D}(A)$. Then A is called a conjugate operator for H if all of the following conditions apply:*

1. *Ω is a core for H (i.e. $H|_\Omega$ is essentially self-adjoint).*
2. *The unitary group $s \mapsto \mathrm{e}^{\mathrm{i}\,s\,A}$ leaves $\mathrm{D}(H)$ invariant and*
$$\sup_{s<1} \left\| H\, \mathrm{e}^{\mathrm{i}\,s\,A} \right\| < \infty\,.$$
3. *The quadratic form*
$$Q : \Omega \to \mathbb{R}\,, \qquad Q(\psi) = \left\| \sqrt{[H, \mathrm{i}A]}\, \psi \right\|^2$$
is closable and bounded below and its associated self-adjoint operator admits a domain containing $\mathrm{D}(H)$.
4. *Let $B = \sqrt{\big[[H, \mathrm{i}A], \mathrm{i}A\big]}$ and let $|H|$ be the absolute value of H, then*
$$\left\| B\psi \right\|^2 \leq \left\| |H|\, \psi \right\|^2 \qquad \text{for all } \psi \in \mathrm{D}(B)\,.$$

The four conditions for the conjugacy of A are also called the Mourre conditions.

Definition 14 (Hamiltonian with a wall)**.** *Let $H_0 = P_x^2 + (P_y + b\,Q_x)^2$ be the unperturbed Landau Hamiltonian on $\mathbb{R}^2$. Let $\Pi = P_y + b\,Q_x$ be a self-adjoint operator. Let $U(x)$ be a differentiable function that vanishes for $x \leq 0$. Let $V(x, y)$ be a differentiable function. We define the Hamiltonian:*

$$H = H_0 + V(x, y) + U(y).$$

Theorem 15 (Spectrum for an unbounded wall)**.** *We define $\Pi := P_x + b\,Q_y$. Let U and V be such that Π is a conjugate operator for the Hamiltonian H. Furthermore, let there be $\delta > 0$ such that $|V(x, y)| < \delta$ for all x, y and let U be unbounded with $U'(x) \geq 0$ and $\inf_{x \geq \varepsilon} U'(x) > 0$ for all $\varepsilon > 0$. If $E \in \mathbb{C} \setminus \sigma(H_0)$, then there exists some open neighbourhood $\mathcal{U}_a(E)$, such that $\sigma(H) \cap \mathcal{U}_a(E) \subseteq \sigma_{\mathrm{ac}}(H)$.*

Lemma 16 (Sufficient conditions for conjugacy)**.** *Out of the Mourre conditions, 1. holds trivially for H, Π, since C_0^∞ forms a core of the two operators, and 2. is satisfied if for each $s \in \mathbb{R}$ there is some C, such that $U(x+s) \leq C\,U(x)$ uniformly for all x. If $U + V$ is a bound for its own derivatives, then the conditions 3. and 4. are also satisfied.*

Lemma 17 (Bounded wall)**.** *The theorem can be generalized to U which, instead of growing without a bound, levels off at some height E_0, if $U'(x) \geq 0$ still holds.*

Now, we restate the results regarding a half-plane with a Dirichlet boundary condition:

Definition 18 (Hamiltonian with a Dirichlet b.c.)**.** *Let $\Omega = \mathbb{R} \times \mathbb{R}_+$ and let H_0 be a self-adjoint operator on $L^2(\Omega)$ given by:*

$$\Big(H_0\,\psi\Big)(x,y) = -\frac{\partial^2}{\partial x^2}\psi(x,y) + \Big(-\frac{\partial^2}{\partial y^2} + b\,x\Big)^2 \psi(x,y)\,,$$

$$\mathrm{D}(H_0) = \Big\{\ \psi \in W^{2,2}(\Omega) \cap L^2_{x^4}(\Omega)\ \Big|\ \varphi(0,y) = 0\ \Big\}\,.$$

Let V be a bounded real differentiable function. We define the Hamiltonian as $H = H_0 + V$.

Theorem 19 (Spectrum for Dirichlet b.c.)**.**

2.1.3 Combes et al., 2001

(Combes, 2001 ?) studied the case of a particle confined to a strip. [**Find the actual paper.**]

2.2 Magnetic perturbation

Lorem ipsum.

2.2.1 Iwatsuka, 1983

Iwatsuka [1983] proved a very general and important result: a magnetic perturbation which is asymptotically zero will not change the spectrum.

Definition 20 (Hamiltonian with asymptotically constant perturbation)**.** *Let $B(x,y)$ be a smooth real function, such that $B(x,y) \to B_0 \neq 0$ as $\sqrt{x^2+y^2} \to \infty$. Let A_x, A_y be smooth functions satisfying $B(x,y) = \frac{\partial}{\partial x}A_y(x,y) - \frac{\partial}{\partial y}A_x(x,y)$. Let $\tilde{H}$ be the essentially self-adjoint operator on $L^2(\mathbb{R}^2)$ given by*

$$\tilde{H} = \Big(-\mathrm{i}\,\frac{\partial}{\partial x} + A_x\Big)^2 + \Big(-\mathrm{i}\,\frac{\partial}{\partial y} + A_y\Big)^2\,, \qquad \mathrm{D}(\tilde{H}) = C_0^\infty(\mathbb{R}^2)\,.$$

Then the Hamiltonian H is a self-adjoint operator defined as the closure of $\tilde{H}$.

Theorem 21 (Spectrum of H)**.**

$$\sigma_{\mathrm{ess}}(H) = \Big\{(2k+1)B_0\ \Big|\ k \in \mathbb{N}_0\Big\}\,.$$

2.2.2 Iwatsuka, 1985

Iwatsuka [1985] continues the work from Iwatsuka's 1983 paper by studying translationally invariant perturbations which *do not* vanish at infinity.

Definition 22 (Hamiltonian)**.** *Let $A_y(x)$ be a smooth real function, such that $B(x) := \frac{\mathrm{d}}{\mathrm{d}x} A_y(x)$ satisfies the condition $0 < M_- < B(x) < M_+ < \infty$ for all $x \in \mathbb{R}$. Let $\tilde{H}$ be the essentially self-adjoint operator on $L^2(\mathbb{R}^2)$ given by*

$$\tilde{H} = \left(-\mathrm{i}\,\frac{\partial}{\partial x} + A_x\right)^2 + \left(-\mathrm{i}\,\frac{\partial}{\partial y} + A_y\right)^2, \qquad \mathrm{D}(\tilde{H}) = C_0^\infty(\mathbb{R}^2)\,.$$

Then the Hamiltonian H is a self-adjoint operator defined as the closure of $\tilde{H}$.

Theorem 23. *Let $\limsup\limits_{\mathrm{x}\to -\infty} B(x) < \limsup\limits_{\mathrm{x}\to +\infty} B(x)$ or $\limsup\limits_{\mathrm{x}\to +\infty} B(x) < \limsup\limits_{\mathrm{x}\to -\infty} B(x)$. Then the spectrum of H is purely absolutely continuous.*

Theorem 24. *Let $B(x)$ satisfy the following conditions:*

- *There exist real numbers B_0 and R, such that $B(x) = B_0$ for all $|x| > R$.*
- *$B(x)$ is not constant everywhere – there are points $|x|<R$ where $B'(x)\neq 0$.*
- *There is a point $\bar{x}$, such that $B(x)$ is nondecreasing on one side of $\bar{x}$ and nonincreasing on the other side.*[1]

Then the spectrum of H is purely absolutely continuous.[2]

2.3 Hislop et al., 2015

Hislop and Soccorsi [2015]

2.4 Geometric perturbation

REWORK THIS SECTION. A tilted planar layer of fixed width, as well as more general thin layers with translationally invariant bends were studied in (Exner, 2018) and some sufficient conditions for the continuity of spectrum were given.

[1]That means either $B'(x) \le 0$ for $x \le \bar{x}$ and $B'(x) \ge 0$ for $x \ge \bar{x}$ or $B'(x) \ge 0$ for $x \le \bar{x}$ and $B'(x) \le 0$ for $x \ge \bar{x}$. Hence, $B(x) - B_0$ is a bump function.

[2]Note that Theorem 24 does not contradict Theorem 21 – here the perturbation is a bump function but only in the x-direction, in y it extends to infinity, as is the case for every translationally invariant system.

3. Delta potential

In this chapter we will examine the Landau Hamiltonian with a potential perturbation (see definition 1), formally given by the potential $V(x) = \alpha\,\delta_{x_0}$, i.e. the Dirac delta in $x = x_0$ with a coupling constant (*magnitude*) $\alpha \in \mathbb{R}$. Since such a potential is a distribution and not a locally integrable function, the Hamiltonian is rigorously defined as:

$$\Big(H_\alpha \psi\Big)(x,y) = \left(-\frac{\partial^2}{\partial x^2} + \Big(\mathrm{i}\frac{\partial}{\partial y} + bx\Big)^2\right)\, \psi(x,y) \quad \text{a.e.}^1\text{on } (\mathbb{R}\setminus\{x_0\})\times\mathbb{R}$$

with a domain given by the conditions

$$\psi \in W^{1,2}(\mathbb{R}^2)\ \cap\ W^{2,2}\Big((\mathbb{R}\setminus\{x_0\})\times\mathbb{R}\Big)\,,$$
$$\lim_{x\to x_0+}\psi'(x,y)\ -\ \lim_{x\to x_0-}\psi'(x,y) = \alpha \lim_{x\to x_0}\psi(x,y) \quad \text{for a.e. } y\,,^2$$
$$\int_{\mathbb{R}^2} x^2\,\Big|\,\psi(x,y)\Big|^2\,\mathrm{d}x\ \mathrm{d}y\ < \infty\,.$$

Then, by an approach analogous to (1.1), one can show that H_α is unitarily equivalent to a direct integral:

$$H_\alpha \simeq \int_{\mathbb{R}}^{\oplus} \mathscr{H}_\alpha(p)\,\mathrm{d}p\ ,$$

where $\mathscr{H}_\alpha(p)$ is a fibre Hamiltonian satisfying very similar conditions to those of H_α, that is, for almost every $p \in \mathbb{R}$:

$$\Big(\mathscr{H}_\alpha(p)\,\varphi\Big)(x) = -\varphi''(x) + \Big(b\,x + p\Big)^2\varphi(x)\,, \tag{3.1}$$

$$\varphi \in W^{1,2}(\mathbb{R})\ \cap\ W^{2,2}(\mathbb{R}\setminus\{x_0\})\,,$$
$$\lim_{x\to x_0+}\varphi'(x) - \lim_{x\to x_0-}\varphi'(x) = \alpha\lim_{x\to x_0}\varphi(x),$$
$$\int_{\mathbb{R}} x^2\,|\varphi(x)|^2\,\mathrm{d}x\ < \infty\,.$$

Before we start investigating the spectrum, we need to show that the problem is well-posed, i.e. that the Hamiltonian H_α is self-adjoint and bounded from below. Then we will show that the spectrum of $\mathscr{H}_\alpha(p)$ is discrete for every p, and only after that we will investigate the continuity of the spectrum of H_α.

[1]The pointwise equality is to be understood *almost everywhere* with respect to the Lebesgue measure on $\mathbb{R}^2$.

[2]The equality holds for *almost every* y and $\lim_{x\to x_0}$ means the *essential* limit with respect to the Lebesgue measure on $\mathbb{R}$.

3.1 Well-posedness

It is straightforward to check that the fibre Hamiltonian is bounded from below:

$$\begin{aligned}
(\varphi,\, \mathscr{H}_\alpha(p)\,\varphi) &= \int_{\mathbb{R}} \overline{\varphi}(x) \Big(-\varphi''(x) + (bx+p)^2\varphi(x)\Big)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}} \overline{\varphi}\varphi'' + \int_{\mathbb{R}} (bx+p)^2 \,\left|\varphi(x)\right|^2\,\mathrm{d}x \\
&\geq -\int_{\mathbb{R}} \overline{\varphi}\varphi'' = \int_{\mathbb{R}} \overline{\varphi}'\varphi' - [\overline{\varphi}\varphi']_{-\infty}^{x_0} - [\overline{\varphi}\varphi']_{x_0}^{\infty} \\
&= \left\|\varphi'\right\|^2_{L^2(\mathbb{R})} + \overline{\varphi}(x_0)\left(\varphi'(x_0+) - \varphi'(x_0-)\right) \\
&= \left\|\varphi'\right\|^2_{L^2(\mathbb{R})} + \alpha\left|\varphi(x_0)\right|^2 .
\end{aligned}$$

In the last two steps we have used the fact that for $\varphi \in W^{2,2}$ both φ and φ' vanish at infinity, and that $\varphi(x_0+) - \varphi(x_0-) = \alpha\varphi(x_0)$. In case when $\alpha \geq 0$, the right-hand side is non-negative, therefore we can use zero as the lower bound. For $\alpha < 0$ we estimate $|\varphi(x_0)|^2 \leq \|\varphi\|^2_{L^\infty}$ and then use the Sobolev-type inequality

$$\forall a>0\ \exists b>0 :\ \left\|\varphi\right\|^2_{L^\infty} \leq a\left\|\varphi'\right\|^2_{L^2} + b\left\|\varphi\right\|^2_{L^2} ,$$

the proof of which is given in the chapter A.1 of the appendix.

$$\begin{aligned}
(\varphi,\, \mathscr{H}_\alpha(p)\,\varphi) &\geq \left\|\varphi'\right\|^2_{L^2} + \alpha\left|\varphi(x_0)\right|^2 \\
&\geq \left\|\varphi'\right\|^2_{L^2} + \alpha\left\|\varphi\right\|^2_{L^\infty} \\
&\geq \left\|\varphi'\right\|^2_{L^2} + \alpha\left(a\left\|\varphi'\right\|^2_{L^2} + b\left\|\varphi\right\|^2_{L^2}\right) \\
&= \left(1+\alpha a\right)\left\|\varphi'\right\|^2_{L^2} + \alpha b\left\|\varphi\right\|^2_{L^2} .
\end{aligned}$$

By choosing $a \leq |\alpha|^{-1}$, we get

$$(\varphi,\, \mathscr{H}_\alpha(p)\,\varphi) \geq \alpha b\left\|\varphi\right\|^2_{L^2(\mathbb{R})} ,$$

thus we have shown that the fibre Hamiltonian $\mathscr{H}_\alpha(p)$ is bounded from below. And because the bound is independent of p, it is also a lower bound for H_α:

$$\begin{aligned}
\Big(\psi,\, H_\alpha\psi\Big)_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}} \Big(\tilde{\psi}(\cdot,p),\, \mathscr{H}_\alpha(p)\,\tilde{\psi}(\cdot,p)\Big)_{L^2(\mathbb{R})}\,\mathrm{d}p \\
&\geq \int_{\mathbb{R}} \alpha b\left\|\tilde{\psi}(\cdot,p)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}p\, = \alpha b\left\|\psi\right\|^2_{L^2(\mathbb{R}^2)} ,\quad \text{where } \tilde{\psi} = \mathcal{F}_y\psi .
\end{aligned} \tag{3.2}$$

Now we will show that the fibre Hamiltonian is self-adjoint. Let $\varphi \in \mathrm{D}(\mathscr{H}_\alpha(p))$ and ψ from a yet-unknown subset of $\mathcal{H}$.

$$\Big(\mathscr{H}_\alpha(p)\,\varphi,\ \psi\Big) = \int_{\mathbb{R}} -\overline{\varphi}''\psi + \int_{\mathbb{R}}\Big(bx+p\Big)^2\,\overline{\varphi}\,\psi \tag{3.3}$$

$$\begin{aligned}
&= \Big[-\overline{\varphi}'\psi + \overline{\varphi}\psi'\Big]_{-\infty}^{x_0} + \Big[\overline{\varphi}'\psi - \overline{\varphi}\psi'\Big]_{x_0}^{+\infty} + \int_{\mathbb{R}} -\overline{\varphi}\,\psi'' + \int_{\mathbb{R}}\Big(bx+p\Big)^2\overline{\varphi}\psi \\
&= -\overline{\varphi}'(x_0-)\psi(x_0-) + \overline{\varphi}(x_0)\psi'(x_0-) \\
&\quad + \overline{\varphi}'(x_0+)\psi(x_0+) - \overline{\varphi}(x_0)\psi'(x_0+) + \int_{\mathbb{R}}\overline{\varphi}\left(\psi'' + \int_{\mathbb{R}}\Big(bx+p\Big)^2\psi\right) .
\end{aligned}$$

This whole expression has to be equal to (φ, χ) for some $\chi \in \mathcal{H}$ independent of φ. At the second line, we performed an integration by parts, already assuming $\psi \in W^{2,2}(\mathbb{R} \setminus \{x_0\})$. If there was another isolated point $c \in \mathbb{R}$ where ψ weren't twice weakly differentiable, we would get terms $\overline{\varphi}'(c)\big(\psi(c-)-\psi(c+)\big)$ and $\overline{\varphi}(c)\big(\psi'(c-) - \psi'(c+)\big)$ which can't be independent of φ unless $\psi(c-) = \psi(c+)$ and $\psi'(c-) = \psi'(c+)$. However, this would make ψ twice weakly differentiable at c, hence a contradiction. At the third line we simply evaluated the square brackets, making use of the fact that $W^{2,2}$ functions (and their derivatives) vanish at infinity. In order for the entire expression to be independent of φ, the following equation must hold:

$$-\overline{\varphi}'(x_0-)\psi(x_0-) + \overline{\varphi}(x_0)\psi'(x_0-) + \overline{\varphi}'(x_0+)\psi(x_0+) - \overline{\varphi}(x_0)\psi'(x_0+) = 0\,.$$

Substituting $\varphi'(x_0+) - \varphi'(x_0-) = \alpha\varphi(x_0)$ and solving for all φ, we get that $\psi(x_0+) = \psi(x_0-)$ and $\psi'(x_0+) - \psi'(x_0-) = \alpha\psi(x_0+)$. Therefore, ψ must be from $\mathrm{D}(\mathscr{H}_\alpha(p))$ and $\chi = \mathscr{H}_\alpha(p)\psi$. We have shown that $\mathscr{H}_\alpha(p)$ is self-adjoint. And because the direct integral of a self-adjoint operator is also a self-adjoint operator, H_α is self-adjoint, too.

Finally, we will show that the fibre Hamiltonian has a discrete spectrum. The family of operators $\{\mathscr{H}_\alpha(p) \,|\, \alpha \in \mathbb{R}\}$ has a common symmetric restriction:

$$\Omega := \Big\{\varphi \in W^{2,2}(\mathbb{R}) \;\Big|\; x^2\,\varphi(x) \in L^2(\mathbb{R}),\ \varphi(x_0) = 0\Big\}\,, \quad \mathscr{H}_\alpha(p)|_\Omega \text{ is symmetric.}$$

Since fibres $\mathscr{H}_\alpha(p)$ for various values of α only differ in the boundary conditions, the restriction to Ω gives just one operator $h(p) := \mathscr{H}_\alpha(p)|_\Omega$, independent of α. The operator $h(p)$ is closed, we can show it directly from the definition: let $\{\varphi_n\} \subset \Omega$ such that $\varphi_n \to \varphi \in L^2(\mathbb{R})$, then

$$\begin{aligned}
&\lim_{n\to\infty} h(p)\,\varphi_n = \lim_{n\to\infty} \Big(-\varphi_n'' + (bx+p)^2\varphi_n\Big) \in L^2 \\
&\quad\iff \quad \lim_{n\to\infty} \varphi_n'' \in L^2 \;\wedge\; \lim_{n\to\infty} x^2\varphi_n \in L^2 \quad\iff\quad \varphi'' \in L^2 \;\wedge\; x^2\varphi \in L^2\,.
\end{aligned}$$

Furthermore, there is no way for $\varphi(x_0) \neq \varphi_n(x_0) \equiv 0$ without causing $\varphi_n''(x_0)$ to diverge. Therefore, $h(p)\,\varphi_n \to \psi \implies \varphi \in \Omega$. Finally, the requirement $h(p)\,\varphi = \psi$ follows from the fact that both second derivative and multiplication by x^2 are closed operators on their respective domains.

We have shown that $h(p)$ is a closed symmetric operator with many different extensions $\mathscr{H}_\alpha(p)$. We know that at least one of the extensions, the fibre $\mathscr{H}_{\alpha=0}(p)$ of the unperturbed system, has a discrete spectrum. Now, we want to use the theorem 8 to show that the spectrum of all $\mathscr{H}_\alpha(p)$ is discrete. The last premise left to demonstrate is the fact that $n_+(h(p)) = n_-(h(p)) < \infty$. [**To do: simplify this argument.**] Since the deficiency indices (definition 6) are equal to the multiplicity of $\pm\mathrm{i}$ as an eigenvalue of $h(p)^*$, we first need to find the operator $h(p)^*$. Let $\varphi \in \Omega$ and ψ from a yet-unknown set $\Omega' \subset \mathcal{H}$.

$$\begin{aligned}
\big(h(p)\varphi,\,\psi\big) &= \int_{\mathbb{R}} -\overline{\varphi}''\psi + \int_{\mathbb{R}} \Big(bx+p\Big)^2\overline{\varphi}\psi = \int_{\mathbb{R}} \overline{\varphi}\;\overbrace{\Big(\psi'' + \int_{\mathbb{R}}\Big(bx+p\Big)^2\psi\Big)}^{=:\,h(p)^*\psi} + \\
&\quad + \underbrace{\overline{\varphi}(x_0)}_{0}\,\psi'(x_0-) - \underbrace{\overline{\varphi}(x_0)}_{0}\,\psi'(x_0+) + \overline{\varphi}'(x_0)\Big(\psi(x_0+) - \psi(x_0-)\Big)\,.
\end{aligned}$$

We performed an integration by parts, assuming $\Omega' \subseteq W^{2,2}(\mathbb{R}\setminus\{x_0\})$, which can be justified by an argument analogical to that under equation (3.3). Since $\overline{\varphi}'(x_0)$ can take on any value, it must hold that $\psi(x_0+) - \psi(x_0-) = 0$. However, there is no constraint on the values of ψ'. Therefore, the domain of $h(p)^*$ is:

$$\Omega' := \left\{\varphi \in W^{1,2}(\mathbb{R}) \cap W^{2,2}(\mathbb{R}\setminus\{x_0\}) \mid x^2\varphi(x) \in L^2(\mathbb{R})\right\}.$$

The deficiency indices $n_\pm$ are then equal to the number of linearly independent solutions of the ordinary differential equation:

$$h(p)^*\varphi = \pm\mathrm{i}\,\varphi \quad \iff \quad \varphi''(x) = \left((bx+p)^2 \mp \mathrm{i}\right)\varphi(x) \quad \text{where } \varphi \in \Omega' . \tag{3.4}$$

This equation is the generalized parabolic cylinder equation which generally has two solutions. The fact that φ' can have an arbitrarily large jump in x_0 allows us to make both of the solutions obey the growth conditions given by Ω' hence we have $n_+ = n_- = 2$. The details of this calculation will be flashed out in the next section, right under the equation (3.10).

We have shown that the Hamiltonian H_α is self-adjoint and bounded from below for each $\alpha \in \mathbb{R}$, and that the fibre Hamiltonian $\mathscr{H}_\alpha(p)$ has a discrete spectrum for every $\alpha, p \in \mathbb{R}$. In the next section, we will investigate what are the eigenvalues of $\mathscr{H}_\alpha(p)$ and how they depend on p and α.

3.2 Eigenproblem of the fibre Hamiltonian

In order to use utilize the theorem 5 to find the spectrum of H_α, we need to find the eigenvalues of the fibre Hamiltonian for each p. That is, we are looking for a real analytic function $\epsilon(p)$, such that for all $p \in \mathbb{R}$ there exists a $\varphi \in \mathrm{D}(\mathscr{H}_\alpha(p))$ satisfying

$$\mathscr{H}_\alpha(p)\,\varphi = \epsilon(p)\,\varphi\,.$$

As a shorthand, we will often denote $\epsilon(p)$ simply as ϵ. Substituting from (3.1), we get an ordinary differential equation:

$$\begin{gathered}-\varphi''(x) + \left(b^2\,x^2 + 2pb\,x + p^2\right)\varphi(x) = \epsilon\,\varphi(x) \quad \text{on } x \neq x_0\,,\\ \varphi'(x_0+) - \varphi'(x_0-) = \alpha\,\varphi(x_0)\,.\end{gathered}$$

From now on, we shall suppose that $b > 0$; for $b < 0$ one can perform a reflection $x \mapsto -x$ and arrive at the same results. In order to refine this differential equation into the standard form, we change variables $x \mapsto w$ and instead of one function φ on $\mathbb{R}$ we introduce two functions g_-, g_+ on the left and right half-line respectively:

$$w := \sqrt{2b}\left(x + \frac{p}{b}\right), \qquad w_0 := \sqrt{2b}\left(x_0 + \frac{p}{b}\right), \qquad \nu := \frac{\epsilon - b}{2b}\,, \tag{3.5}$$

$$g_- : (-\infty, w_0] \to \mathbb{C}\,, \qquad g_+ : [w_0, +\infty) \to \mathbb{C}\,,$$

$$\varphi(x) = \begin{cases} g_+\left(\sqrt{2b}\,(x + \frac{p}{b})\right) & \text{for } x \geq x_0\,,\\ g_-\left(\sqrt{2b}\,(x + \frac{p}{b})\right) & \text{for } x < x_0\,.\end{cases}$$

Then we arrive at the so-called parabolic cylinder differential equation:

$$g''_{\pm}(w) = \Big(\frac{1}{4}w^2 - \nu - \frac{1}{2}\Big)\, g_{\pm}(w)\,, \tag{3.6}$$

The two functions are then *"glued together"* by the following equations:

$$\begin{aligned} g_+(w_0) - g_-(w_0) &= 0\,, \\ g'_+(w_0) - g'_-(w_0) &= \alpha\,\sqrt{2b}\; g_+(w_0)\,. \end{aligned} \tag{3.7}$$

As stated in Gradshteyn and Ryzhik [2014], the solutions to (3.6) can be expressed as a linear combination of the functions

$$D_\nu(w)\,,\;\; D_\nu(-w)\,,\;\; D_{-\nu-1}(\mathrm{i}w)\,,\;\; D_{-\nu-1}(-\mathrm{i}w)\,, \tag{3.8}$$

where D_ν is a so-called *parabolic cylinder function*, which is a special function that can be expressed in terms of the gamma function Γ and the confluent hypergeometric function ${}_1\mathrm{F}_1$:

$$D_\nu(w) = 2^{\frac{\nu}{2}} \exp\big(-\tfrac{w^2}{4}\big) \left(\frac{\sqrt{\pi}}{\Gamma\big(\frac{1-\nu}{2}\big)}\, {}_1\mathrm{F}_1\Big(-\frac{\nu}{2}, \frac{1}{2}; \frac{w^2}{2}\Big) - \frac{w\,\sqrt{2\pi}}{\Gamma\big(-\frac{\nu}{2}\big)}\, {}_1\mathrm{F}_1\Big(\frac{1-\nu}{2}, \frac{3}{2}; \frac{w^2}{2}\Big)\right). \tag{3.9}$$

Since $1/\Gamma(z)$ is an entire function and $(\alpha, z) \mapsto {}_1\mathrm{F}_1(\alpha, \gamma; z)$ is holomorphic on $\mathbb{C}^2$ for all γ other than non-positive integers, it follows that $(\nu, w) \mapsto D_\nu(w)$ is also holomorphic on $\mathbb{C}^2$.

In the special case when $\nu \in \mathbb{N}_0$, the function D_ν can be expressed using the Hermite polynomials H_n:

$$D_\nu(w) = 2^{\frac{\nu}{2}} \exp\big(-\tfrac{w^2}{4}\big)\, H_\nu\Big(\frac{w}{\sqrt{2}}\Big)$$

The solutions in (3.8) are linearly dependent. For most values of ν, any of the four functions can be expressed as a linear combination of any two others. However, specifically in the case $\nu \in \mathbb{N}_0$ we get $D_\nu(w) = \pm D_\nu(-w)$.

Asymptotic behaviour of the solutions is also given by Gradshteyn and Ryzhik [2014]. As $|w| \to \infty$, the solutions $D_{-\nu-1}(\mathrm{i}w)$ and $D_{-\nu-1}(-\mathrm{i}w)$ grow exponentially. Meanwhile, $D_\nu(w)$ decays exponentially for $w \to +\infty$. Therefore, $D_\nu(w)$ and $D_\nu(-w)$ are better suited for the growth conditions imposed by the domain of $\mathscr{H}_\alpha(p)$. We define $c_{+1}, c_{+2}, c_{-1}, c_{-2} \in \mathbb{C}$, such that

$$g_\pm(w) = c_{\pm 1}\, D_\nu(w) + c_{\pm 2}\, D_\nu(-w)\,. \tag{3.10}$$

Now we can safely return to the eigenproblem of $\mathscr{H}_\alpha(p)$ from where we left it – that is, from the equation (3.10). For $\epsilon \in \mathbb{R}$ can be further shown, that if $\nu \notin \mathbb{N}_0$, the solution $D_\nu(w)$ diverges for $w \to -\infty$. [**Then show it.**] Therefore, $c_{-1} = c_{+2} = 0$ in order for φ to be integrable. On the other hand, for $\nu \in \mathbb{N}_0$, the solutions aren't independent (as discussed above), therefore we can also set $c_{-1} = c_{+2} = 0$ without loss of generality. Applying the gluing equations (3.7), we get:

$$c_{+1}\, D_\nu(w_0) = c_{-2}\, D_\nu(-w_0)$$

$$c_{+1}\,\frac{\mathrm{d}}{\mathrm{d}w} D_\nu(w)\Big|_{w_0} - c_{-2}\,\frac{\mathrm{d}}{\mathrm{d}w} D_\nu(-w)\Big|_{w_0} = \alpha\,\sqrt{2b}\; c_{+1}\, D_\nu(w_0)$$

We substitute using the equality $\frac{\mathrm{d}}{\mathrm{d}w}D_\nu(w) = \frac{w}{2}\,D_\nu(w) - D_{\nu+1}(w)$ from Gradshteyn and Ryzhik [2014] and arrive at the equation:

$$\begin{pmatrix} D_\nu(w_0) & -D_\nu(-w_0) \\ \left(\frac{w_0}{2} - \alpha\sqrt{2b}\right) D_\nu(w_0) - D_{\nu+1}(w_0) & -\frac{w_0}{2}\,D_\nu(-w_0) - D_{\nu+1}(-w_0) \end{pmatrix} \begin{pmatrix} c_{+1} \\ c_{-2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

In order for the equation to have a non-trivial solution, the determinant of the matrix must be zero. Hence, we arrive at the condition:

$$\begin{aligned} 0 &= D_\nu(w_0)\Big(\tfrac{w_0}{2}\,D_\nu(-w_0) - D_{\nu+1}(-w_0)\Big) + D_\nu(-w_0)\Big(\big(\tfrac{w_0}{2} - \alpha\sqrt{2b}\big)D_\nu(w_0) - D_{\nu+1}(w_0)\Big) \\ &= D_\nu(w_0)\,D_{\nu+1}(-w_0) \;+\; D_\nu(-w_0)\,D_{\nu+1}(w_0) \;+\; \alpha\sqrt{2b}\,D_\nu(w_0)\,D_\nu(-w_0)\,. \end{aligned} \tag{3.11}$$

Since we're interested in the allowed values of ν for given w_0 and $\alpha\sqrt{2b} =: a$, this equation effectively defines an implicit function $\nu(a, w_0)$.

3.3 Implicit function for energy levels

Let F be a function of three real variables given by

$$F(a, w, \nu) = D_\nu(w)\,D_{\nu+1}(-w) \;+\; D_\nu(-w)\,D_{\nu+1}(w) \;+\; a\,D_\nu(w)\,D_\nu(-w)\,.$$

We have shown that

$$\epsilon(p) \text{ is an eigenvalue of } \mathscr{H}_\alpha(p) \iff F\left(\alpha\sqrt{2b},\ \sqrt{2b}\left(x_0 + \frac{p}{b}\right),\ \frac{\epsilon(p) + b}{2b}\right) = 0\,.$$

Truly, this is simply the equation (3.11) after the substitution from (3.5). Furthermore, for $\alpha = 0$ the fibre Hamiltonian $\mathscr{H}_\alpha(p)$ reduces to that of a harmonic oscillator. From this fact, it is straightforward to derive the following result:

$$F(0, w, k) = 0 \text{ holds for } k \in \mathbb{N}_0 \text{ and all } w \in \mathbb{R}\,.$$

Moreover, F is analytic in the three variables, as it is a sum of products of entire functions. The implicit function theorem **[Source?]** then tells us that, provided $\frac{\partial}{\partial\nu}F(0, w, k) \neq 0$ for a fixed $k \in \mathbb{N}_0$ (which we will prove soon), there exists an analytic function $\nu(a, w)$, such that $\nu(0, w) = k$ and $F(a, w, \nu(a, w)) = 0$ on the neighbourhood of $a = 0$. Since there is a different implicit function for every k, we will denote them $\nu_k(a, w)$. After changing our variables back to the physical ones and putting $x_0 = 0$ (since it only depends on the choice of origin), we see that the energy levels for a fixed magnitude of the perturbation α are

$$\epsilon_k(p) = b + 2b\,\nu_k\Big(\alpha\sqrt{2b},\ p\sqrt{\tfrac{2}{b}}\Big)\,, \qquad k \in \mathbb{N}_0\,.$$

For an unperturbed system $\alpha = 0$, the allowed energies are the Landau levels $\epsilon_k = b(2k + 1)$. We will show that for $p \to \pm\infty$, the energy approaches those unperturbed levels – in our rescaled coordinates, this is equivalent to $\lim_{w\to\pm\infty} \nu_k(a, w) = k$. To show this is the case, we will investigate the behaviour of $\frac{\partial}{\partial a}\nu_k$ and $\frac{\partial}{\partial w}\nu_k$ as $w \to \pm\infty$. First, let us write down the partial derivatives of F:

$$\frac{\partial}{\partial a}\,F(a,w,\nu) = D_\nu(w)\,D_\nu(-w)$$

$$\frac{\partial}{\partial w}\,F(a,w,\nu) = a\,w\,D_\nu(w)\,D_\nu(-w) + a\Big(D_\nu(w)\,D_{\nu+1}(-w) - D_\nu(-w)\,D_{\nu+1}(w)\Big)$$

In the second equality we used the recursion formulas $\frac{\mathrm{d}}{\mathrm{d}w}D_\nu(w) = \frac{w}{2}\,D_\nu(w) - D_{\nu+1}(w)$ and $\frac{\mathrm{d}}{\mathrm{d}w}D_{\nu+1}(w) = -\frac{w}{2}\,D_{\nu+1}(w) + (\nu+1)\,D_\nu(w)$ from Gradshteyn and Ryzhik [2014]. The last partial derivative of F is a little tougher, therefore we will start with the small bits and build our way up.

$$\Psi(x) := \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x) \quad\Longrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}x}\Gamma(x) = \Psi(x)\,\Gamma(x)$$

$$\begin{aligned}\frac{\partial}{\partial\alpha}\,{}_1\mathrm{F}_1(\alpha,\gamma;z) &= \frac{\partial}{\partial\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n\,z^n}{(\gamma)_n\,n!} = \frac{\partial}{\partial\alpha}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\,\frac{z^n}{(\gamma)_n\,n!}\\ &= \sum_{n=0}^{\infty}\tfrac{\Psi(\alpha+n)\,\Gamma(\alpha+n)\,\Gamma(\alpha)-\Gamma(\alpha+n)\,\Psi(\alpha)\,\Gamma(\alpha)}{\Gamma(\alpha)^2}\,\frac{z^n}{(\gamma)_n\,n!} = \sum_{n=0}^{\infty}\Big(\scriptstyle\Psi(\alpha+n)-\Psi(\alpha)\displaystyle\Big)\,\frac{(\alpha)_n\,z^n}{(\gamma)_n\,n!}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\nu}\,D_\nu(w) &= \frac{\partial}{\partial\nu}\,2^{\frac{\nu}{2}}\,\mathrm{e}^{-\frac{w^2}{4}}\left(\frac{\sqrt{\pi}}{\Gamma\big(\frac{1-\nu}{2}\big)}\,{}_1\mathrm{F}_1\Big(-\frac{\nu}{2},\,\frac{1}{2};\,\frac{w^2}{2}\Big) - \frac{w\sqrt{2\pi}}{\Gamma\big(-\frac{\nu}{2}\big)}\,{}_1\mathrm{F}_1\Big(\frac{1-\nu}{2},\,\frac{3}{2};\,\frac{w^2}{2}\Big)\right)\\ &= \frac{\ln 2}{2}\,D_\nu(w) + 2^{\frac{\nu}{2}}\,\mathrm{e}^{-\frac{w^2}{4}}\Bigg(-\frac{\sqrt{\pi}\,\Psi(\frac{1-\nu}{2})}{2\,\Gamma(\frac{1-\nu}{2})}{}_1\mathrm{F}_1\Big(-\tfrac{\nu}{2},\,\tfrac{1}{2};\,\tfrac{w^2}{2}\Big) + \frac{w\sqrt{2\pi}\,\Psi(-\frac{\nu}{2})}{2\,\Gamma(-\frac{\nu}{2})}{}_1\mathrm{F}_1\Big(\tfrac{1-\nu}{2},\,\tfrac{3}{2};\,\tfrac{w^2}{2}\Big) -\\ &\quad - \frac{\sqrt{\pi}}{2\,\Gamma(\frac{1-\nu}{2})}\sum_{n=0}^{\infty}\Big(\scriptstyle\Psi(-\frac{\nu}{2}+n)-\Psi(-\frac{\nu}{2})\displaystyle\Big)\,\frac{(-\frac{\nu}{2})_n\,(\frac{w^2}{2})^n}{(\frac{1}{2})_n\,n!} + \frac{w\sqrt{2\pi}}{2\,\Gamma(-\frac{\nu}{2})}\sum_{n=0}^{\infty}\Big(\scriptstyle\Psi(\frac{1-\nu}{2}+n)-\Psi(\frac{1-\nu}{2})\displaystyle\Big)\,\frac{(\frac{1-\nu}{2})_n\,(\frac{w^2}{2})^n}{(\frac{3}{2})_n\,n!}\Bigg)\end{aligned}$$

Here, $(a)_n \equiv a(a+1)...(a+n-1)$ is the Pochhammer symbol and $\Psi(x)$ is the digamma function.

Následují poznámky a nedotažené myšlenky. Pravděpodobně se budou hodit odhady:

$$\sum_{n=0}^{\infty}\frac{z^n}{n!\,(\frac{1}{2})_n} = \cosh(2\sqrt{z})$$

$$\sum_{n=0}^{\infty}\frac{z^n}{n!\,(\frac{3}{2})_n} = \frac{1}{2\sqrt{z}}\,\sinh(2\sqrt{z})$$

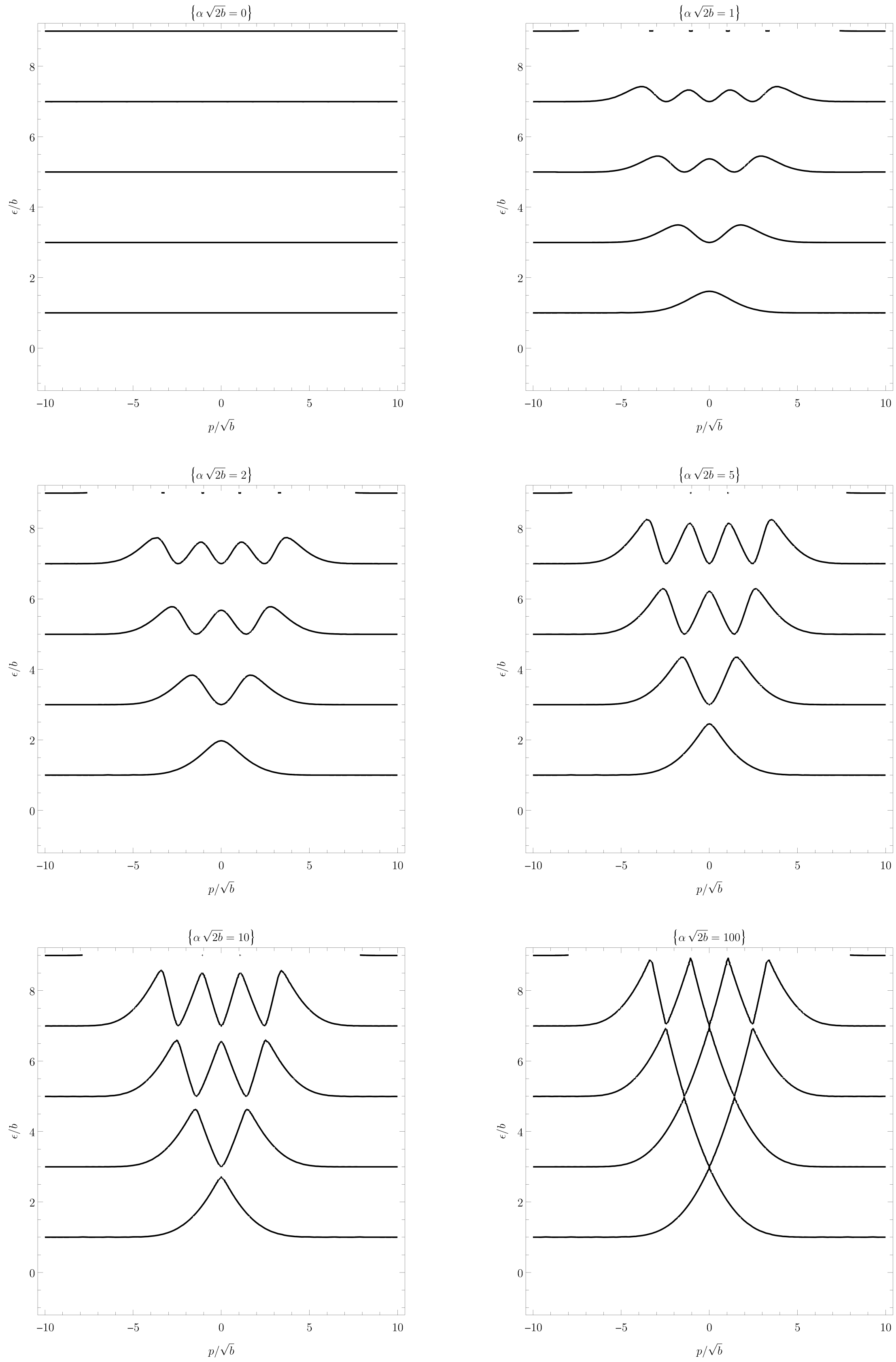

Figure 3.1: The first four energy levels ϵ as a function of the y-momentum p for $\alpha\sqrt{2b} = 0, 1, 2, 5, 10$ and 100 (starting with an unperturbed system, followed by an increasingly repulsive perturbation).

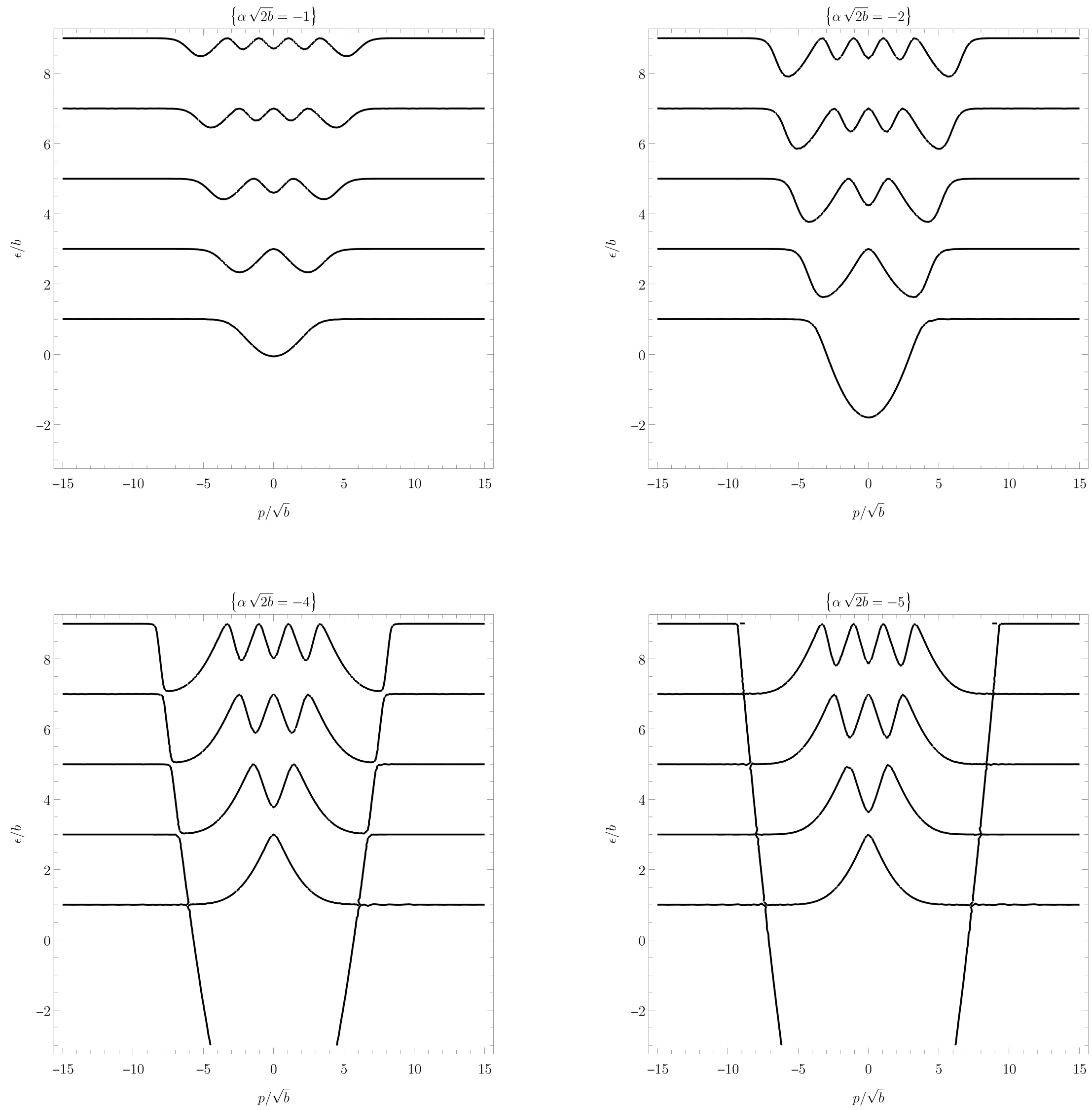

Figure 3.2: The first five energy levels ϵ as a function of the y-momentum p for $\alpha\sqrt{2b} = -1, -2, -4$ and -5 (system with an increasingly attractive perturbation).

4. Half-plane with Robin boundary

In this chapter we will examine the Landau Hamiltonian of a particle confined to a half-plane with a Robin (or *mixed*) boundary condition. Let $\alpha \in \mathbb{R}$ and $\Omega := \mathbb{R} \times \mathbb{R}_+$, where $\mathbb{R}_+ \equiv [0, +\infty)$, then the Hamiltonian is given by[1]

$$\begin{aligned} \Big(H_\alpha \psi\Big)(x,y) &= \left(-\frac{\partial^2}{\partial x^2} + \Big(\mathrm{i}\frac{\partial^2}{\partial y^2} + bx\Big)^2 \right) \psi(x,y) \,, \\ \mathrm{D}(H_\alpha) &= \Big\{ \psi \in W^{2,2}(\Omega) \cap L^2_{x^4}(\Omega) \;\Big|\; \psi(0,y) + \alpha\, \psi'(0,y) = 0 \Big\} \,. \end{aligned} \tag{4.1}$$

Using (1.1), we can once again show that the Hamiltonian is unitarily equivalent to a direct integral $H_\alpha \simeq \int_{\mathbb{R}}^{\oplus} \mathscr{H}_\alpha(p)\, \mathrm{d}p$, where $\mathscr{H}(p)$ is the fiber Hamiltonian given by

$$\begin{aligned} \Big(\mathscr{H}_\alpha(p)\varphi\Big)(x) &= -\varphi''(x) + (p+bx)^2\, \varphi(x) \,, \\ \mathrm{D}(\mathscr{H}_\alpha(p)) &= \Big\{ \varphi \in W^{2,2}(\mathbb{R}_+) \cap L^2_{x^4}(\mathbb{R}_+) \;\Big|\; \varphi(0) + \alpha\, \varphi'(0) = 0 \Big\} \,. \end{aligned} \tag{4.2}$$

When $\alpha = 0$, the problem reduces to the well-known Dirichlet b.c.[2]

4.1 Well-posedness

We will start by showing that the Hamiltonian is bounded from below, starting with the fibre Hamiltonian:

$$\begin{aligned} \Big(\varphi, \mathscr{H}_\alpha(p)\, \varphi\Big) &= -\int_{\mathbb{R}_+} \overline{\varphi}\varphi'' + \overbrace{\int_{\mathbb{R}_+} (bx+p)^2\, |\varphi|^2}^{\geq 0} \\ &\geq -\Big[\overline{\varphi}\varphi'\Big]_0^\infty + \int_{\mathbb{R}_+} |\varphi'|^2 \;=\; \overline{\varphi(0)}\, \varphi'(0) + \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R}_+)} \end{aligned}$$

We have used integration by parts and the fact that for $\varphi \in W^{2,2}$ both φ and φ' vanish at infinity. If $\alpha = 0$, then the boundary condition $\varphi(0) + \alpha\, \varphi'(0) = 0$ dictates that $\varphi(0) = 0$ and the right-hand side is non-negative, hence we have found the lower bound. If $\alpha \neq 0$, we substitute for $\varphi'(0)$ from the boundary condition:

$$\Big(\varphi, \mathscr{H}_\alpha(p)\, \varphi\Big) \geq \overline{\varphi(0)}\, \varphi'(0) + \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R}_+)} = -\frac{1}{\alpha}\Big|\varphi(0)\Big|^2 + \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R}_+)} \,.$$

For $\alpha < 0$, the whole right-hand side is non-negative. For $\alpha > 0$, we have:

$$\Big(\varphi, \mathscr{H}_\alpha(p)\, \varphi\Big) \geq -\frac{1}{\alpha}\Big|\varphi(0)\Big|^2 + \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R}_+)} \geq -\frac{1}{\alpha}\Big\|\varphi\Big\|^2_{L^\infty(\mathbb{R}_+)} + \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R}_+)} \,.$$

[1] The equalities are to be understood in the weak sense again. $(H_\alpha\psi)(x,y) = \ldots$ holds for almost every (x,y) in Ω with respect to the Lebesgue measure. $\psi(0,y) \equiv \lim_{x\to 0} \psi(x,y) = \ldots$ holds for almost every $y \in \mathbb{R}$, and the limit is the *essential* limit wrt. the Lebesgue measure.

[2] b.c. = boundary condition

Now using the lemma 25 from the Appendix, we know that for every $a > 0$ there exists $b > 0$ such that $-\frac{1}{\alpha}\left\|\varphi\right\|^2_{L^\infty} \geq -\frac{a}{\alpha}\left\|\varphi'\right\|^2_{L^2} - \frac{b}{\alpha}\left\|\varphi\right\|^2_{L^2}$. Setting $a = \alpha$, we obtain

$$\Big(\varphi, \mathscr{H}_\alpha(p)\Big) \geq \Big(- \left\|\varphi'\right\|^2_{L^2(\mathbb{R}_+)} - \frac{b}{\alpha}\left\|\varphi\right\|^2_{L^2(\mathbb{R}_+)}\Big) + \left\|\varphi'\right\|^2_{L^2(\mathbb{R}_+)} = -\frac{b}{\alpha}\left\|\varphi\right\|^2_{L^2(\mathbb{R}_+)} .$$

As demonstrated in (3.2) in the previous chapter, the Hamiltonian H_α is therefore also bounded from below with the same lower bound.

Now we will show the self-adjointness, again starting with the fibre Hamiltonian. Let $\varphi \in \mathrm{D}(\mathscr{H}_\alpha(p))$ and $\psi \in M \subset L^2(\mathbb{R}_+)$.

$$\begin{aligned}\Big(\mathscr{H}_\alpha(p)\,\varphi,\ \psi\Big) &= \int_{\mathbb{R}_+} -\overline{\varphi}''\psi + \int_{\mathbb{R}_+}\Big(bx+p\Big)^2\,\overline{\varphi}\,\psi \\ &= -\Big[\overline{\varphi}'\,\psi\Big]_0^\infty + \Big[\overline{\varphi}\,\psi'\Big]_0^\infty + \int_{\mathbb{R}_+} -\overline{\varphi}\,\psi'' + \int_{\mathbb{R}_+}\Big(bx+p\Big)^2\,\overline{\varphi}\,\psi \\ &= \underbrace{-\overline{\varphi}'(0)\,\psi(0) + \overline{\varphi}(0)\,\psi'(0)}_{-\overline{\varphi}'(0)\Big(\psi(0)+\alpha\psi'(0)\Big)} + \int_{\mathbb{R}_+}\overline{\varphi}\,\Big(-\psi'' + (bx+p)^2\psi\Big) .\end{aligned}$$

First, we integrated by parts, assuming that $M \subseteq W^{2,2}(\mathbb{R}_+)$ – if it were not, the result could not be independent of φ, as demonstrated in the previous chapter after (3.3). Then we used the fact that functions from $W^{2,2}$ vanish at infinity, and finally we applied the boundary condition $\varphi'(0) + \alpha\varphi'(0) = 0$. It is clear that ψ must fulfill the same boundary condition in order for the result to be independent of $\varphi'(0)$. Therefore $M = \mathrm{D}(\mathscr{H}_\alpha(p))$ and the fibre Hamiltonian is self-adjoint. The Hamiltonian H, a direct integral of a self-adjoint operator, is hence also self-adjoint.

Lastly, we will show that the spetrum of $\mathscr{H}_\alpha(p)$ is discrete using the theorem 8. We define:

$$\Omega = \Big\{\varphi \in W^{2,2}(\mathbb{R}_+) \cap L^2_{x^4}(\mathbb{R}_+)\ \Big|\ \varphi(0) = \varphi'(0) = 0\Big\} ,$$

then the fibre Hamiltonians for all values of α have a common symmetric restriction:

$$h(p) := \mathscr{H}_\alpha(p)|_\Omega$$

[**Finish this section: show h is closed and $n_+ = n_- < \infty$.**]

4.2 Eigenproblem of the fiber Hamiltonian

We are searching for a function $\epsilon(p)$, such that for every p there exists a $\varphi \in \mathrm{D}(\mathscr{H}_\alpha(p))$ for which

$$\mathscr{H}_\alpha(p)\,\varphi = \epsilon(p)\,\varphi .$$

Substituting from the definition, we get

$$\begin{gathered}-\varphi''(x) + (p+bx)^2\,\varphi(x) = \epsilon\,\varphi(x) , \\ \Big((p+bx)^2 - \epsilon\Big)\,\varphi(x) = \varphi''(x)\end{gathered}$$

This is the parabolic cylinder equation and, as in the previous chapter, its solutions are in the form

$$\varphi(x) = c\, D_\nu(w) + d\, D_\nu(-w)\,, \quad \text{where} \quad w := \sqrt{2b}\Big(x + \frac{p}{b}\Big)\,, \quad \nu := \frac{\epsilon - b}{2b}\,.$$

Except for $\nu \in \mathbb{N}$, the $D_\nu(-w)$ term diverges exponentially, therefore d must be zero in order for $\varphi \in \mathrm{D}(\mathscr{H}_\alpha)$. Now we apply the boundary condition:

$$\begin{aligned}
\varphi(0) &= \alpha\, \varphi'(0)\,, \\
c\, D_\nu(w_0) &= \alpha\, \frac{\mathrm{d}}{\mathrm{d}x}\, c D_\nu(w)\Big|_{w=w_0}\,, \quad \text{where} \quad w_0 = \sqrt{2b}\,(0 + \tfrac{p}{b}) = p\,\sqrt{\tfrac{2}{b}}\,, \\
D_\nu(w_0) &= \alpha\,\sqrt{2b}\,\frac{\mathrm{d}}{\mathrm{d}w}\, D_\nu(w)\Big|_{w=w_0}\,, \\
D_\nu(w_0) &= \alpha\,\sqrt{2b}\,\Big(\frac{w_0}{2} D_\nu(w_0) - D_{\nu+1}(w_0)\Big)\,, \\
\frac{1}{\alpha\sqrt{2b}}\, D_\nu(w_0) &= \frac{p}{\sqrt{2b}} D_\nu(w_0) - D_{\nu+1}(w_0)\,, \quad \text{assuming} \quad \alpha \neq 0\,, \\
\sqrt{2b}\, D_{\nu+1}(w_0) &= \Big(p - \tfrac{1}{\alpha}\Big)\, D_\nu(w_0)\,, \\
\sqrt{2b}\, D_{\nu+1}(p\,\sqrt{\tfrac{2}{b}}) &= \Big(p - \tfrac{1}{\alpha}\Big)\, D_\nu(p\,\sqrt{\tfrac{2}{b}})\,.
\end{aligned} \tag{4.3}$$

The equation (4.3) is the spectral condition, it defines the implicit function $\nu(p)$, which in turn tells us, which values of $\epsilon \equiv (2\nu + 1)\, b$ are in the spectrum of $\mathscr{H}_\alpha$. For $\alpha = 0$ (the Dirichlet b.c.), the spectral condition is $D_\nu(p\,\sqrt{\tfrac{2}{b}}) = 0$.

[Grafy následují na další stránce.]

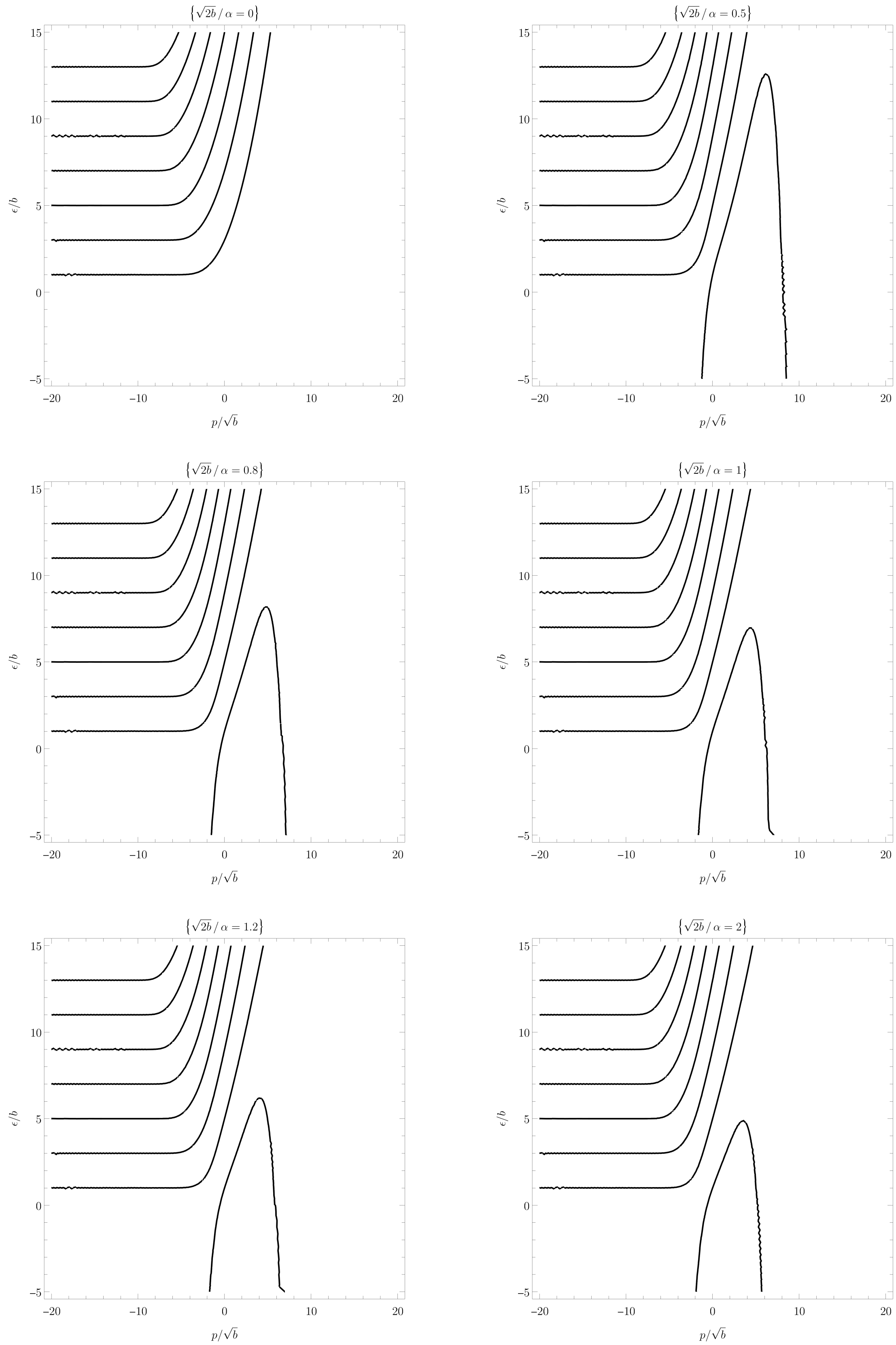

Figure 4.1: The first energy levels ϵ as a function of the y-momentum p for $\alpha\sqrt{2b} = 0.0, 0.5, 0.8, 1.0, 1.2$ and 2.0 (starting with a system with Dirichlet b.c.).

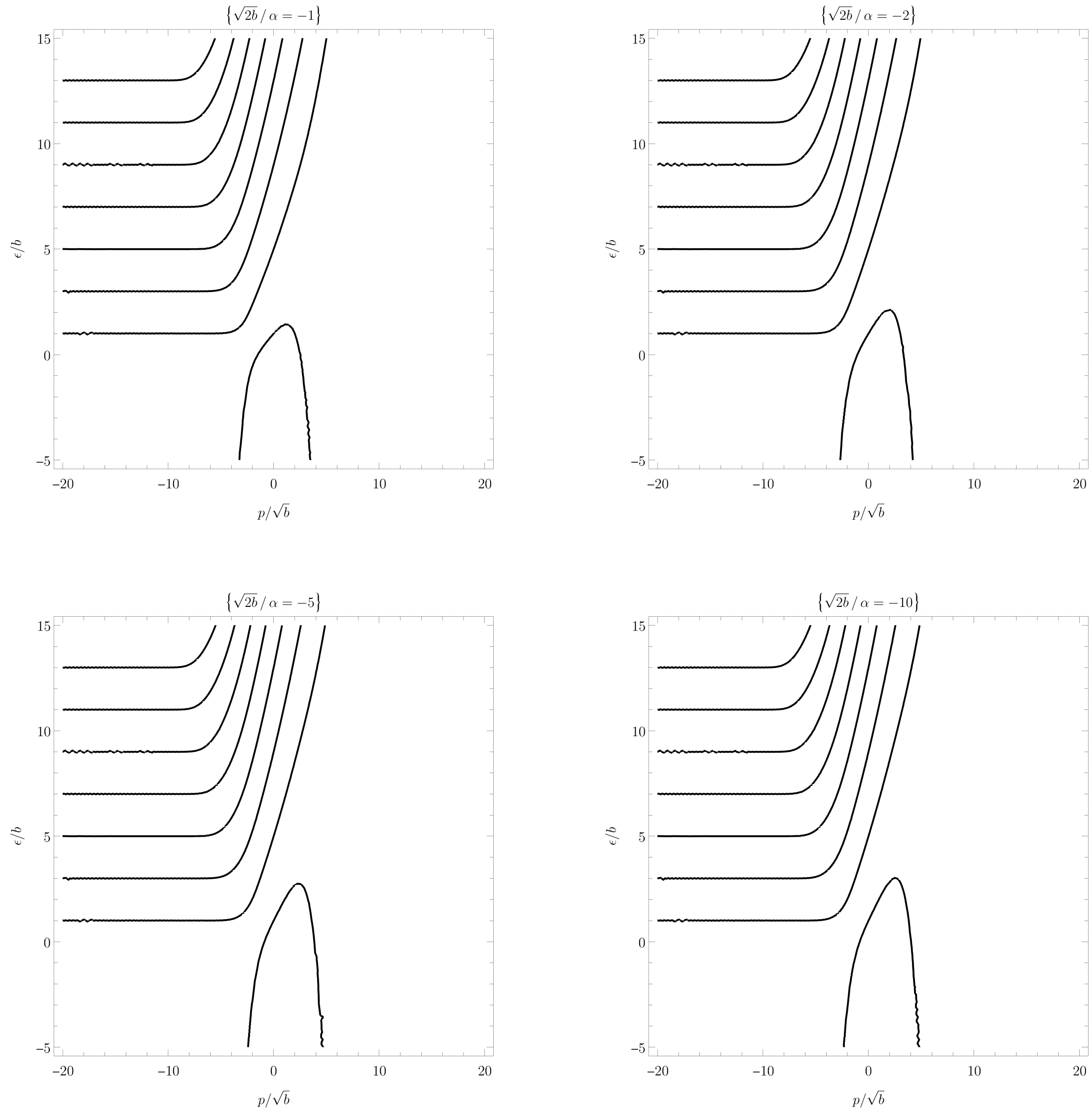

Figure 4.2: The first five energy levels ϵ as a function of the y-momentum p for $\alpha\sqrt{2b} = -1, -2, -5$ and -10.

Conclusion

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List of Figures

A. Appendix

A.1 Sobolev-type inequality

In this section we prove a Sobolev-type inequality analogical to the inequality (7.16) in Blank et al. [2008]. The presented proof is a simple modification of their proof.

Lemma 25. *Let $\psi \in W^{1,2}(\mathbb{R})$, then for every $a > 0$ there exists $b > 0$ such that*

$$\left\|\psi\right\|^n_{L^\infty(\mathbb{R})} \leq a \left\|\psi'\right\|^n_{L^2(\mathbb{R})} + b \left\|\psi\right\|^n_{L^2(\mathbb{R})} , \tag{A.1}$$

where $n \in \{1, 2\}$.

Proof. We define $\phi := \mathcal{F}\psi$. Since, by definition, $\psi' \in L^2(\mathbb{R})$ we have $\mathcal{F}\psi' = T_x\mathcal{F}\psi = T_x\phi \in L^2(\mathbb{R})$, where T_x means the operator of multiplication by the identity function $x \mapsto x$. Next, we utilize the Hölder inequality to find an estimate of $\|\phi\|_{L^1}$.

$$\left\|\phi\right\|_{L^1} = \left\|\tfrac{1}{x+\mathrm{i}}\,(x+\mathrm{i})\,\phi(x)\right\|_{L^1} \leq \underbrace{\left\|\tfrac{1}{x+\mathrm{i}}\right\|_{L^2}}_{=:\,c} \left\|(x+\mathrm{i})\,\phi(x)\right\|_{L^2} \leq c\left(\left\|T_x\,\phi\right\|_{L^2} + \left\|\phi\right\|_{L^2}\right) . \tag{A.2}$$

Since $\phi \in L^1$, it follows that $\psi \in L^\infty$ and

$$\left\|\psi\right\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \left\|\phi\right\|_{L^1} . \tag{A.3}$$

We introduce a scaled function $\phi_r(x) := r\,\phi(rx)$. Scaling changes the norms in the following way:

$$\left\|\phi_r\right\|^2_{L^2} = r\left\|\phi\right\|^2_{L^2} , \qquad \left\|T_x\phi_r\right\|^2_{L^2} = \frac{1}{r}\left\|T_x\right\|^2_{L^2} .$$

By substituting $\phi \mapsto \phi_r$ in (A.2) and combining with (A.3), we get

$$\left\|\psi\right\|_{L^\infty} \leq c\sqrt{r}\left\|T_x\,\phi\right\|_{L^2} + \frac{c}{\sqrt{r}}\left\|\phi\right\|_{L^2} = c\sqrt{r}\left\|\psi'\right\|_{L^2} + \frac{c}{\sqrt{r}}\left\|\psi\right\|_{L^2} .$$

Choosing $r = a^2/c^2$, we have proven the inequality (A.1) for $n = 1$.

To prove $n = 2$, we start with the case $n = 1$ and square both sides of the inequality.

$$\left\|\psi\right\|_{L^\infty} \leq a\left\|\psi'\right\|_{L^2} + b\left\|\psi\right\|_{L^2}$$

$$\left\|\psi\right\|^2_{L^\infty} \leq a\left\|\psi'\right\|^2_{L^2} + 2ab\left\|\psi'\right\|_{L^2}\left\|\psi\right\|_{L^2} + b\left\|\psi\right\|^2_{L^2} \leq 2a\left\|\psi'\right\|^2_{L^2} + 2b\left\|\psi\right\|^2_{L^2}$$

In the last step we have used the fact, that

$$0 \leq \left(a\left\|\psi'\right\|_{L^2} - b\left\|\psi\right\|_{L^2}\right)^2 \quad \Leftrightarrow \quad 2ab\left\|\psi'\right\|_{L^2}\left\|\psi\right\|_{L^2} \leq a\left\|\psi'\right\|^2_{L^2} + b\left\|\psi\right\|^2_{L^2} .$$

Finally, substituting $2a^2 \mapsto a$ and $2b^2 \mapsto b$, we have proven the inequality for $n = 2$. $\square$