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Magnetic Transport Along Translationally Invariant Obstacles

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Dedication.

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Contents

List of symbols

$A\|_\Omega$	Restriction of an operator A to a subspace $\Omega \subset \mathrm{D}(A)$.
$C^k(\Omega,\mathbb{K})$, $C(\Omega,\mathbb{K})$	The space of functions $\Omega \subseteq \mathbb{R} \to \mathbb{K}$ with k continuous derivatives. The space of continuous functions.
$C_0^\infty(\Omega,\mathbb{K})$	The space of C^∞ functions with compact support in Ω.
$\mathrm{D}(A)$	The domain of operator A, usually dense in $\mathcal{H}$.
$D_\nu(w)$	The parabolic cylinder function.
$\mathcal{F}$	The Fourier-Plancherel operator on $L^2(\mathbb{R})$.
${}_1\mathrm{F}_1(\alpha,\beta;z)$	The confluent hypergeometric function of the first kind.
$\mathcal{H}$	A separable Hilbert space.
$H,\ \mathscr{H}(\xi)$	A Hamiltonian operator; a fibre of the Hamiltonian.
$H_n(x)$	The n-th (physicist's) Hermite polynomial.
$L^p(M, \mathrm{d}\mu, V)$	The space of p-integrable functions from measure space (M,μ) to vector space V. Specifically for $p=2$, a Hilbert space with inner product $(\psi,\phi)_{L^2} = \int_M (\psi,\phi)_V \,\mathrm{d}\mu$.
$L^p(\Omega)$	As above, but $M = \Omega \subseteq \mathbb{R}^N$, μ is the Lebesgue measure and $V = \mathbb{C}$.
$L^p_\rho(\Omega)$	A weighted Lebesgue space, a shorthand for $L^p(\Omega \subseteq \mathbb{R}^N, \mathrm{d}\mu, \mathbb{C})$, where ρ is a real function and $\mu(M) = \int_M \rho \,\mathrm{d}\lambda$ is a rescaling of the Lebesgue measure λ.
$L^1_{\mathrm{loc}}(\Omega)$	The space of functions that are $L^1(K)$ for every compact $K \subset \Omega$.
$\mathbb{N},\ \mathbb{N}_0$	The set of positive integers; the set of non-negative integers.
$\vec{P}, P_x, P_y, P_z$	Momentum operator – a self-adjoint operator, such that $P_x\, f(x,\ldots) = -\mathrm{i}\frac{\partial}{\partial x} f(x,\ldots)$.
$\vec{Q}, Q_x, Q_y, Q_z$	Position operator – a self-adjoint operator, such that $Q_x\, f(x,\ldots) = x\, f(x,\ldots)$.
$W^{k,p}(\Omega)$	The Sobolev space – the space of integrable functions f, such that $f^{(\alpha)} \in L^p(\Omega)$, where α is a multi-index and $\|\alpha\| \le k$.
$\Gamma(z)$	The gamma function.
μ	A σ-finite measure, usually the Lebesgue measure.
$\sigma(T), \sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{ac}}(T), \sigma_{\mathrm{sc}}(T)$, $\sigma_{\mathrm{disc}}(T), \sigma_{\mathrm{ess}}(T)$	The spectrum of normal operator T; the point, absolutely continuous, singular continuous, discrete, essential spectrum of T. $\sigma(T) = \sigma_{\mathrm{p}} \cup \sigma_{\mathrm{ac}} \cup \sigma_{\mathrm{sc}} = \sigma_{\mathrm{disc}} \cup \sigma_{\mathrm{ess}}$.
$\nabla,\ \nabla\times,\ \Delta$	Gradient, rotation, Laplace operator.
$\Delta^\Omega_{\mathrm{D}}, \Delta^\Omega_{\mathrm{D},A}$	The Dirichlet Laplacian, defined on functions from $L^2(\Omega)$ with a Dirichlet boundary condition; a "magnetic" Dirichlet Laplacian given by the vector potential A.

Introduction

In this thesis we will investigate the spectral properties of Schrödinger operators of the form $(-\mathrm{i}\nabla + \vec{A}(x))^2 + V(x)$ on $L^2(\Omega)$ where Ω is either (a subset of) $\mathbb{R}^2$, or a thin layer in $\mathbb{R}^3$. These operators are of great interest, as they represent the Hamiltonians in single-particle models of two-dimensional magnetic systems.

Clasically, magnetic field is known to have a localizing effect on charged particles – in a homogeneous field, for example, particles get stuck in circular trajectories due to the Lorentz force. Nonetheless, there are many ways in which such localization can be overcome: if we introduce a potential wall, the particle will bounce off of it, propagating along the wall in one direction in a phenomenon called *skipping*; if we introduce a step in the magnetic field, the radius of circular motion will be different in different parts of the plane leading to so-called *snake orbits* which transport the particle along the step.

All three of these examples translate well into quantum mechanics. In the case of a homogeneous magnetic field, we get the Landau Hamiltonian with a complete set of (localized) eigenstates. A wide range of wall potentials has been shown to lead to transport by Marcis et al. [1999] and Fröhlich et al. [2000]. The quantum-classical correspondence of snake orbits was investigated by Reijniers and Peeters [2000]. However, there numerous other examples of magnetic transport and not all of them have a clear classical analogy. The telltale sign of Hamiltonians which allow for magnetic transport is that they have a non-empty absolute continuous spectrum – that is why we are interested in the spectral properties of such operators.

The work is organized as follows:

1. Formulation & useful concepts

In this chapter we will illustrate what magnetic transport is and give a precise mathematical formulation of the problem. Then we will explain concepts and restate textbook theorems which will be useful later.

1.1 The magnetic Hamiltonian

The simplest example of a quantum system with a magnetic field is the system consisting of a single charged and spinless particle in three-dimensional free space exposed to a homogeneous magnetic field and zero scalar potential. The Hamiltonian that corresponds to this system is:

$$H_{3\mathrm{D}} = (\vec{P} + \vec{A})^2 , \quad \vec{B} = \nabla \times \vec{A} = (0, 0, b_0) .$$

Here $\vec{P} = -\mathrm{i}\nabla$ is the momentum operator, $\vec{B}$ is the magnetic field, which is constant with magnitude b_0, without loss of generality pointing along the z axis, and $\vec{A}$ is a corresponding vector potential. Notice that we have used nondimensionalization to remove physical units from the Hamiltonian. The spectrum of H is absolutely continuous and the Hamiltonian commutes with P_z, thus it allows the particle to move freely along the z axis.

One might wonder what happens if we restrict the particle to the plane $z = 0$. One way to do this is only formally: if the particle is free to move along z, but we are simply not interested in this degree of freedom, we may subtract P_z^2 from the Hamiltonian. Then we get a two-dimensional Hamiltonian with infinitely degenerate pure point spectrum, the so-called *Landau Hamiltonian*:

$$H_{\mathrm{Landau}} = (P_x + A_x)^2 + (P_y + A_y)^2 .$$

The spectrum of H_{Landau} consists of the so-called *Landau levels* $\sigma_{\mathrm{ess}} = \{ (2k+1)\, b_0 \mid k \in \mathbb{N}_0 \}$. A detailed analysis of the Landau Hamiltonian can be found e.g. in §112 of Landau and Lifshitz [1981].

Another way of restricting the system would be physically: we can put two planar walls at $z = \pm\ell$. If the walls are repulsive enough, they can be modelled by a Dirichlet (or more generally a Robin) boundary condition, creating a so-called *quantum waveguide*. Such a system would be described by the Hamiltonian:

$$H = H_{\mathrm{Landau}} \oplus H_{\mathrm{transverse}} ,$$

where $H_{\mathrm{transverse}}$ is the kinetic energy of a particle on a line segment $[-\ell, \ell]$. Since the spectrum of both H_{Landau} and $H_{\mathrm{transverse}}$ is pure point, the spectrum of H is pure point too. A deeper analysis of the concept of quantum waveguides can be found in Exner and Kovařík [2015].

The pure point spectrum of the two Hamiltonians means that the particle is not free to move along x or y, but instead it is “trapped” in some superposition of stationary states. In this thesis, we will investigate perturbations to the Landau Hamiltonian, which cause its spectrum to become continuous and allow the particle to move freely along one axis. These perturbations can be either in the form of a scalar potential, a modification of the magnetic field, or a purely geometric deformation of the layer to which our particle is constrained. We will require all of these perturbations to be translationally invariant, thus constant along one axis – without loss of generality, we choose that they are independent of y and only depend on x.

Throughout this thesis, we will use the Landau gauge:

$$A_x = 0\,, \quad A_y = \int_0^x B_z(x')\,\mathrm{d}x'\,, \quad A_z = 0\,.$$

Now we can specify precisely which Hamiltonians we will investigate.

Definition 1 (Potential perturbation)**.** *Let $\Omega \subseteq \mathbb{R}^2$, $\mathcal{H} = L^2(\Omega)$, $b > 0$ and $V \in L^1_{loc}(\mathbb{R})$. A self-adjoint operator on $\mathcal{H}$ given by the equation*

$$H = P_x^2 + \left(P_y + b\,Q_x\right)^2 + V(x)$$

*is called the **Landau Hamiltonian with a potential perturbation**. We will investigate, which boundary conditions and which choices of V lead to $\sigma_{\mathrm{ac}}(H) \neq \emptyset$. The domain $\mathrm{D}(H)$ is determined not only by the asymptotic behaviour of V, but also by the boundary conditions imposed on the wave function.*

Definition 2 (Magnetic perturbation)**.** *Let $b \in C^\infty(\mathbb{R})$, $\mathcal{H} = L^2(\mathbb{R}^2)$, let $\mathcal{D} = C_0^\infty(\mathbb{R}^2)$ be the set of $\mathbb{C}^\infty$ functions with compact support and A_y be a multiplication operator on $\mathcal{H}$ given by:*

$$A_y\,\psi(x,y) = \left(\int_0^x b(x')\,\mathrm{d}x'\right)\psi(x,y)\,.$$

Let $\tilde{H} : \mathcal{D} \to \mathcal{H}$ be an essentially self-adjoint operator given by the equation:

$$\tilde{H} = P_x^2 + \left(P_y + A_y\right)^2\,,$$

*Its closure H is called the **Landau Hamiltonian with a magnetic perturbation**. We will investigate, which choices of b lead to $\sigma_{\mathrm{ac}}(H) \neq \emptyset$.*

Definition 3 (Geometric perturbation, transl. inv. layer)**.** *Let $b > 0$ and $\ell > 0$. Let $\omega : \mathbb{R} \to \mathbb{R}^2$ be a C^4-smooth curve. We define a set $\Omega' \subset \mathbb{R}^2$ by*

$$\Omega' = \left\{\, P \in \mathbb{R}^2 \;\mid\; \exists s \in \mathbb{R}\, \left\|\omega(s) - P\right\| \leq \ell \,\right\},$$

this gives a band of width 2ℓ around the curve ω. Then we define a set $\Omega \subset \mathbb{R}^3$ as

$$\Omega = \left\{\, (x,y,z) \in \mathbb{R}^3 \;\mid\; (x,z) \in \Omega' \,\right\}.$$

*We shall call Ω a **translationally invariant layer of width** 2ℓ **given by the curve** ω. Let us now consider the magnetic Dirichlet Laplacian*

$$\Delta^\Omega_{\mathrm{D},A}\psi(x,y,z) = \Delta\psi + 2\mathrm{i}b\,x\,\frac{\partial\psi}{\partial y} - b^2x^2\,\psi$$

*defined on functions $\psi \in C^\infty(\Omega)$, such that $\psi(x,y,z) = 0$ on the boundary of Ω. The operator H which is the closure of $-\Delta^\Omega_{\mathrm{D},A}$ in $L^2(\Omega)$ is called the **Landau Hamiltonian with a geometric perturbation**. We will investigate, which choices of ω lead to $\sigma_{\mathrm{ac}}(H) \neq \emptyset$.*

The Landau Hamiltonian with a translationally invariant magnetic perturbation is also called *the Iwatsuka Hamiltonian* by some authors (e.g. Miranda and Popoff [2017] and Hislop and Soccorsi [2015]) after Akira Iwatsuka who studied how perturbations to the Landau Hamiltonian affect its spectrum. In similar spirit, Exner et al. [2018a] use the term *Iwatsuka type effect* to describe the phenomenon when a particular perturbation changes the spectrum of the Landau Hamiltonian to absolutely continuous. It is exactly this Iwatsuka type effect that is the main focus of this thesis. We will look into more details of Akira Iwatsuka's work in section 2.2.

1.2 Direct integral

The key insight to all three of these problems is that the Hamiltonians in question only depend on the momentum p_y of the particle, and not on its position y. If we were to fix p_y of the particle to a certain value somehow, we could reduce the problem to a one-dimensional operator and solve for each p_y separately. This vague idea can be given a rigorous meaning in terms of the *direct integral*, a generalization of the direct sum.

The following definition is a rephrasing of definitions given in Reed and Simon [1978], pages 280 and 281.

Definition 4 (Direct integral, fibre)**.** *Let $\mathcal{H}'$ be a separable Hilbert space and (M, μ) a measure space. We define a Hilbert space $\mathcal{H}$, which is the space of all square-integrable functions from M to $\mathcal{H}'$:*

$$\mathcal{H} = L^2(M, \,\mathrm{d}\mu\,, \mathcal{H}')\,.$$

Let $\mathscr{A}$ be a measurable function from M to the self-adjoint operators on $\mathcal{H}'$. Let $f_\psi : M \to \mathbb{R}$ be a function defined by

$$f_\psi(s) = \left\|\mathscr{A}(s)\psi(s)\right\|_{\mathcal{H}'} \quad \textit{for all } \psi \in \mathcal{H}, s \in M \textit{ such that } \psi(s) \in \mathrm{D}(\mathscr{A}(s))\,.$$

We define an operator A on $\mathcal{H}$ by:

$$(A\psi)(s) = \mathscr{A}(s)\,\psi(s)\,,$$
$$\mathrm{D}(A) = \left\{\psi \in \mathcal{H} \;\middle|\; \psi(s) \in \mathrm{D}(\mathscr{A}(s)) \textit{ a.e. } \wedge \left\|f_\psi\right\|_{L^2} < \infty\right\}.$$

Then we shall write

$$\mathcal{H} = \int_M^{\oplus} \mathcal{H}', \qquad A = \int_M^{\oplus} \mathscr{A}(s)\,\mathrm{d}s\,.$$

*We shall call $\mathcal{H}$ and A **the direct integral** of $\mathcal{H}'$ and $\mathscr{A}$, respectively. Reversely, we shall call $\mathcal{H}'$ a **fibre space** of $\mathcal{H}$ and $\mathscr{A}(s)$ a **fibre** of A.*

The concept of a direct integral might initially seem strange to readers who encounter it for their first time. These readers may find it helpful to think of the direct integral as a simple *"rebranding"* of several concepts they already know and understand. For example, a free spin-$\frac{1}{2}$ particle is represented in the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^2)$ of square-integrable functions from the physical space $\mathbb{R}^3$ to the qubit

$\mathbb{C}^2$. This space is by definition the direct integral $L^2(\mathbb{R}^3, \mathbb{C}^2) = \int_{\mathbb{R}^3}^{\oplus} \mathbb{C}^2$, the qubit plays the role of the fibre space here. Another example is related to the fact that a function of two variables can be understood as a function of one variable which returns another function of one variable (programmers call this *currying*). That is exactly the meaning of this direct integral: $L^2(M \times N) \simeq \int_M^{\oplus} L^2(N) \simeq \int_N^{\oplus} L^2(M)$. We mentioned that the direct integral is a generalization of the direct sum. To see that this is the case, consider a finite set M together with the counting measure, then $\int_M^{\oplus} \mathcal{H}' \simeq \bigoplus_M \mathcal{H}'$.

Before we apply the theory of direct integrals to the magnetic Hamiltonian, let us remind the Fourier-Plancherel operator. It is a standard textbook result (see Blank et al. [2008]) that if we take the Fourier transform as an operator on $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$, its closure is a unitary operator on $L^2(\mathbb{R})$. This operator is called the Fourier-Plancherel operator $\mathcal{F}$, it transforms momentum to position $\mathcal{F}P\mathcal{F}^{-1} = Q$, and as a unitary operator, it does not change the spectrum of self-adjoint operators:

$$\sigma(A) = \sigma(\mathcal{F}A\mathcal{F}^{-1}) \,, \quad \sigma_{\text{disc}}(A) = \sigma_{\text{disc}}(\mathcal{F}A\mathcal{F}^{-1}) \,, \quad \sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(\mathcal{F}A\mathcal{F}^{-1}) \,.$$

The theory given so far regards functions of one variable. In this thesis, we will perform a *partial* Fourier transformations on multivariate functions – that is, perform the Fourier transformation on one variable whilst keeping the other variables fixed. We will use a subscript to indicate the variable which is being transformed and the new variable, for example $\mathcal{F}_{y\to\xi} : \psi(x,y) \mapsto \tilde{\psi}(x,\xi)$.

Now we can show, how to express a Landau Hamiltonian with potential and magnetic perturbations in terms of the direct integral:

$$\begin{aligned} H &= \left(\vec{P} + \vec{A}(x)\right)^2 + V(x) \\ &= P_x^2 + \left(P_y + A_y(x)\right)^2 + V(x) \\ &\simeq \mathcal{F}_{y\to p}\Big(P_x^2 + \left(P_y + A_y(x)\right)^2 + V(x)\Big)\mathcal{F}_{y\to p}^{-1} \\ &= P_x^2 + \left(Q_p + A_y(x)\right)^2 + V(x) \\ &= \int_{\mathbb{R}}^{\oplus} \underbrace{{P_x}^2 + \left(p + A_y(x)\right)^2 + V(x)}_{\mathscr{H}(p)} \; \mathrm{d}p \,, \end{aligned} \tag{1.1}$$

where $P_y\psi(x,y) = -\mathrm{i}\frac{\partial}{\partial y}\psi(x,y)$ is a differential operator and $Q_p\psi(x,p) = p\,\psi(x,p)$ is the operator of multiplication by the second coordinate. For every $p \in \mathbb{R}$, $\mathscr{H}(p)$ is a self-adjoint operator on $L^2(\mathbb{R})$. The physical meaning of the parameter p is the particle's momentum in the direction of y and $\mathscr{H}(p)$ is the Hamiltonian for a particle with a fixed y-momentum.

The following theorem is a weakened version of Theorem XIII.85 of Reed and Simon [1978].

Theorem 5 (Spectrum of direct integral). *Let $\lambda \in \mathbb{C}$ and $A = \int_M^{\oplus} \mathscr{A}(s)\,\mathrm{d}s$, as in the previous definition. We define $\Gamma(\lambda)$ as the set of all s, such that λ is an eigenvalue of $\mathscr{A}(s)$, and $\Omega_\varepsilon(\lambda)$ as the set of all s, such that the ε-neighbourhood* *of λ intersects the spectrum of $\mathscr{A}(s)$ – written symbolically:*

$$\Gamma(\lambda) = \Big\{ \, s \ \mid \ \lambda \text{ is an eigenvalue of } \mathscr{A}(s) \, \Big\} \, ,$$
$$\Omega_\varepsilon(\lambda) = \Big\{ \, s \ \mid \ \sigma(\mathscr{A}(s)) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset \, \Big\} \, .$$

Then λ belongs to the spectrum of A if and only if

$$\mu(\ \Omega_\varepsilon(\lambda)\ ) > 0 \qquad \text{for all } \varepsilon > 0 \, .$$

Additionally, λ is an eigenvalue of A if and only if

$$\mu(\ \Gamma(\lambda)\ ) > 0 \, .$$

This means that we can deduce the spectrum of the Hamiltonian H simply by investigating how the spectrum of its fibre $\mathscr{H}(p)$ depends on p. Furthermore, the spectrum of $\mathscr{H}(p)$ typically consists of simple eigenvalues which are particularly convenient to work with.

1.3 Refresher on linear operators

Before we start investigating specific Hamiltonians, let us remind a few textbook theorems regarding self-adjointness and spectral properties of linear operators, which will be useful later. The following definition and the subsequent theorem are from the chapter 4.7 in Blank et al. [2008].

Definition 6 (Deficiency indices)**.** *Let T be a linear operator on $\mathcal{H}$. We define two numbers n_+, $n_- \in \mathbb{N}_0 \cup \{\infty\}$ as follows:*

$$n_\pm(T) = \dim \, \mathrm{Ker} \, (T^* \pm \mathrm{i}I) \, ,$$

where I is the identity operator on $\mathcal{H}$. We call these numbers the ***deficiency indices*** *of T.*

Theorem 7 (Deficiency indices and self-adjoint extensions)**.** *Let T be a closed symmetric operator on $\mathcal{H}$, such that*

$$n_+(T) = n_-(T) < \infty \, .$$

Then all maximal extensions of T are self-adjoint. Furthermore, if $n_\pm = 0$, then T is already self-adjoint.

The following theorem is given in Weidmann [1980] as Theorem 8.18.

Theorem 8 (Spectrum of self-adjoint extensions)**.** *Let T be a closed symmetric operator on $\mathcal{H}$, such that*

$$n_+(T) = n_-(T) < \infty \, .$$

Then the essential spectrum of every self-adjoint extension of T is the same. In particular, if one self-adjoint extension of T has a pure discrete spectrum, all of them do.

The previous theorems are especially useful in combination with the following theorem about differential operators on L^2, which was compiled from the opening of section 8.4 of Weidmann [1980], up to the theorem 8.20 there.

Theorem 9 (Deficiency indices of differential operators)**.** *Let* $a, b \in \mathbb{R} \cup \{\pm\infty\}$ *such that* $a < b$*. Let* $p \in C^1(\,(a,b), \mathbb{R}\,)$ *be a continuously differentiable real function and* $q \in C(\,(a,b), \mathbb{R}\,)$ *be a continuous real function. We define* L *to mean:*

$$L\psi := -(p\,\psi')' + q\,\psi\,.$$

We define the operator T *on* $L^2((a,b))$ *as following:*

$$T\psi = L\psi \quad \text{for all} \quad \psi \in \mathrm{D}(T)\,,$$
$$\mathrm{D}(T) = \left\{\, \psi \in W^{2,2}((a,b)) \;\middle|\; L\psi \in L^2((a,b)) \;\wedge\; \operatorname{supp}\psi \subset (a,b) \text{ is compact} \,\right\}$$

Then $n_-(T) = n_+(T)$*.*

Finally, we will remind two important bits from Kato [1995]. The following definition is adapted from Chapter VII, section §2.1.

Definition 10 (Holomorphic family of type A)**.** *Let* $s \mapsto T(s)$ *be an operator-valued function from an open set* $\Omega \subset \mathbb{C}$ *to closed operators on* $\mathcal{H}$*. If the domain* $\mathrm{D}(T(s))$ *is identical for all* s *and the map* $s \mapsto T(s)\,\psi$ *is holomorphic for all* $s \in \Omega$ *and* $\psi \in \mathrm{D}(T(s))$*, we say that* $T(s)$ *is a* ***holomorphic family of type A****.*

The following theorem is listed as Theorem 3.9 in Chapter VII, section §3.5 of Kato [1995].

Theorem 11 (On discrete spectrum of type-A holom. operators)**.** *Let* $I \subset \mathbb{R}$ *be an interval on the real axis and let* $T(s)$ *be a holomorphic family of type* A *for* $s \in \Omega \supset I$*. Furthermore, let* $T(s)$ *be self-adjoint for all* $s \in I$ *and let the spectrum of* $T(s)$ *be discrete. Then all eigenvalues of* $T(s)$ *as functions of* s *are holomorphic on* I*.*

2. Known results

In this chapter we will restate the results about Landau Hamiltonians with potential, geometric and magnetic perturbations, which have already been proven. Effort was made to unify notation and conventions across the various sources.

2.1 Potential perturbation

2.1.1 Macris et al., 1999

Marcis et al. [1999] investigated the problem of Landau Hamiltonians with a steep (but locally integrable) potential wall along the edge of a half-plane. What follows is a summary of their results.

Definition 12 (Hamiltonian)**.** *Let $\mu \in (0,\infty)$ and $\gamma \in [1,\infty])$. We define the wall potential U:*

$$U(x) = \mu\, x^{\gamma}\, \chi_{\mathbb{R}_+}(x)\ ,$$

where $\chi_{\mathbb{R}_+}$ is the characteristic function of $\mathbb{R}_+ \equiv [0,\infty)$. Let $V \in C^1(\mathbb{R}^2,\mathbb{R})$ be a differentiable real function of two variables, such that

$$\sup_{x,y\in\mathbb{R}} \Big|V(x,y)\Big| =: V_0 < \infty\ , \qquad \sup_{x,y\in\mathbb{R}} \Big|\frac{\partial}{\partial x} V(x,y)\Big| =: V_0' < \infty\ .$$

Let $B \in \mathbb{R}$. We define the Hamiltonian H:

$$H = \tfrac{1}{2}\, P_x^2 + \tfrac{1}{2}(P_y - B\, Q_x)^2 + V(x,y) + U(x)\ .$$

The Hamiltonian is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$.

Definition 13 (Auxiliary)**.** *Finally, we define a functional $A(E;U)$, where $E > 0$ and U is as above.*

$$A(E;U) = \sup_{0\le x\le x_0} \left(\frac{U(x)^4}{U'(x)}\right) + 8\int_{x_0}^{\infty} \frac{\sqrt[4]{U(\frac{x}{2})}\, U(x)^4}{\sqrt{2\pi x}\, U'(x)}\, \exp\Big(-x\,\sqrt{\tfrac{1}{8}\, U(\tfrac{x}{2})}\Big)\, \mathrm{d}x\ ,$$

where x_0 is such that $U(\frac{x_0}{2}) = 2E$. And for $n \in \mathbb{N}$, $\delta > 0$ we define a set $\Omega_{n,\delta}$:

$$\Omega_{n,\delta} = \mathcal{U}_\delta\Big(n\, B\Big)\ ,$$

where $\mathcal{U}_a(b)$ is the open a-neighbourhood of b.

Theorem 14. *Let $\delta > 0$, such that $\frac{B}{2} > \delta$. If*

$$V_0' < \frac{\left(\frac{B}{2} - \delta - V_0\right)^4}{\sup\limits_{E\in\Omega_{n,\delta}} A(E + V_0; U)}\ ,$$

then $\Omega_{n,\delta} \cap \sigma_{\mathrm{p}}(H) = \emptyset$. Furthermore, if

$$V_0' < \frac{\left(\frac{B}{2} - \delta - V_0\right)^4}{\sup\limits_{0\le a\le \frac{B}{2}}\ \sup\limits_{E\in\Omega_{n,\delta}} A(E + a; U)}\ ,$$

then $\Omega_{n,\delta} \subset \sigma_{\mathrm{c}}(H) \equiv \sigma_{\mathrm{ac}}(H) \cup \sigma_{\mathrm{sc}}(H)$.

2.1.2 Fröhlich et al., 2000

Fröhlich et al. [2000] investigated the problem of systems constrained to a half-plane $\mathbb{R} \times \mathbb{R}_+$ by either a potential wall (bounded or unbounded), or a Dirichlet boundary condition. The Dirichlet b.c. was also treated for more general subspaces $\Omega \subset \mathbb{R}^2$ – we will not list these, as they were not translationally invariant.

In the case of the steep potential, they used the theory of *Mourre estimates*, introduced in Mourre [1981].

Definition 15 (Conjugate operator)**.** *Let H and A be self-adjoint operators with domains $\mathrm{D}(H)$ and $\mathrm{D}(A)$. Let $\Omega := \mathrm{D}(H) \cap \mathrm{D}(A)$. Then A is called a conjugate operator for H if all of the following conditions apply:*

1. *Ω is a core for H (i.e. $H|_\Omega$ is essentially self-adjoint).*
2. *The unitary group $s \mapsto \mathrm{e}^{\mathrm{i}\,s\,A}$ leaves $\mathrm{D}(H)$ invariant and*

$$\sup_{s<1} \left\| H\, \mathrm{e}^{\mathrm{i}\,s\,A} \right\| < \infty\,.$$

3. *The quadratic form*

$$Q : \Omega \to \mathbb{R}\,, \qquad Q(\psi) = \left\| \sqrt{[H, \mathrm{i}A]}\, \psi \right\|^2$$

is closable and bounded below and its associated self-adjoint operator admits a domain containing $\mathrm{D}(H)$.

4. *Let $B = \sqrt{\big[[H, \mathrm{i}A], \mathrm{i}A\big]}$ and let $|H|$ be the absolute value of H, then*

$$\left\| B\psi \right\|^2 \leq \left\| |H|\, \psi \right\|^2 \qquad \textit{for all } \psi \in \mathrm{D}(B)\,.$$

The four conditions for the conjugacy of A are also called the Mourre conditions.

Definition 16 (Hamiltonian with a wall)**.** *Let $H_0 = P_x^2 + (P_y + b\,Q_x)^2$ be the unperturbed Landau Hamiltonian on $\mathbb{R}^2$. Let $\Pi = P_y + b\,Q_x$ be a self-adjoint operator. Let $U(x)$ be a differentiable function that vanishes for $x \leq 0$. Let $V(x, y)$ be a differentiable function. We define the Hamiltonian:*

$$H = H_0 + V(x, y) + U(y).$$

Theorem 17 (Spectrum for an unbounded wall)**.** *We define $\Pi := P_x + b\,Q_y$. Let U and V be such that Π is a conjugate operator for the Hamiltonian H. Furthermore, let there be $\delta > 0$ such that $|V(x, y)| < \delta$ for all x, y and let U be unbounded with $U'(x) \geq 0$ and $\inf_{x \geq \varepsilon} U'(x) > 0$ for all $\varepsilon > 0$. If $E \in \mathbb{C} \setminus \sigma(H_0)$, then there exists some open neighbourhood $\mathcal{U}_a(E)$, such that $\sigma(H) \cap \mathcal{U}_a(E) \subseteq \sigma_{\mathrm{ac}}(H)$.*

Lemma 18 (Sufficient conditions for conjugacy)**.** *Out of the Mourre conditions, 1. holds trivially for H, Π, since C_0^∞ forms a core of the two operators, and 2. is satisfied if for each $s \in \mathbb{R}$ there is some C, such that $U(x+s) \leq C\, U(x)$ uniformly for all x. If $U + V$ is a bound for its own derivatives, then the conditions 3. and 4. are also satisfied.*

Lemma 19 (Bounded wall)**.** *The theorem can be generalized to U which, instead of growing without a bound, levels off at some height E_0, if $U'(x) \geq 0$ still holds.*

Now, we restate the results regarding a half-plane with a Dirichlet boundary condition:

Definition 20 (Hamiltonian with a Dirichlet b.c.)**.** *Let $\Omega = \mathbb{R} \times \mathbb{R}_+$ and let H_0 be a self-adjoint operator on $L^2(\Omega)$ given by:*

$$\Big(H_0\,\psi\Big)(x,y) = -\frac{\partial^2}{\partial x^2}\psi(x,y) + \Big(-\mathrm{i}\,\frac{\partial}{\partial y} + b\,x\Big)^2 \psi(x,y)\,,$$

$$\mathrm{D}(H_0) = \Big\{\ \psi\ \in\ W^{2,2}(\Omega)\ \cap\ L^2_{x^4}(\Omega)\ \Big|\ \varphi(0,y) = 0\ \Big\}\,.$$

Let V be a bounded real differentiable function of two variables. We define the Hamiltonian as $H = H_0 + V$.

Theorem 21 (Spectrum for Dirichlet b.c.)**.** *Let $E \in \mathbb{R}\backslash\{\ (2k{+}1)b \mid k \in \mathbb{Z}\ \}$. For sufficiently small $b^{-1}\left\|V\right\|_{L^\infty(\mathbb{R}^2)}$ the spectrum of H is purely absolutely continuous near E.*

This theorem has been shown to hold for more general $\Omega \subset \mathbb{R}^2$, for more information see the article.

2.1.3 Combes et al., 2001

Combes et al. [2001] studied the case of a particle confined to a strip. [**Finish this.**]

2.1.4 Exner & Kovařík, 2015

In the section 7.2.2 of their textbook on Quantum Waveguides, Exner and Kovařík [2015] provide an example of a system, which classically would not lead to any transport, but whose Hamiltonian operator has a non-empty absolute continuous spectrum nonetheless. The Hamiltonian they investigated can be formally written as

$$H = \Big(P_y + b\,x\Big)^2 + P_x^2 + \sum_{j\in\mathbb{Z}} \alpha\,\delta(x - x_0 - j\,\ell)\,\delta(y)\,,$$

where the potential $V = \sum \alpha\delta(...)\delta(y)$ is a Dirac comb situated on the axis $y = 0$. In classical physics, such a potential would only affect a zero-measure family of initial conditions, therefore the trajectory of particles would *almost surely* be unperturbed. This operator is not translationally invariant in the continuous sense – it is periodic in x – but we list it nonetheless as an interesting example. The rigorous definition of the Hamiltonian operator is:

Definition 22 (Hamiltonian with a Dirac comb)**.** *Let $b, \ell \in \mathbb{R}\backslash\{0\}$ and $\alpha, x_0 \in \mathbb{R}$. We define $\Omega = \mathbb{R}^2 \setminus \{(x_0 + j\,\ell,\ 0) \mid j \in \mathbb{Z}\}$. Then the Hamiltonian is given by:*

$$\Big(H\psi\Big)(x,y) = \left(\Big(-\mathrm{i}\frac{\partial}{\partial y} + b\,x\Big)^2 - \frac{\partial^2}{\partial x^2}\right)\psi(x,y) \quad \textit{for} \quad (x,y) \in \Omega\,,$$

$$\begin{aligned}\mathrm{D}(H) = {} & W^{1,2}(\mathbb{R}^2)\ \cap\ W^{2,2}(\Omega)\ \cap\ L^2_{x^4}(\mathbb{R}^2) \cap \\ & \cap\ \Big\{\psi\ \Big|\ \lim_{x\to x_j+}\psi(x,0) - \lim_{x\to x_j-}\psi(x,0) = \alpha\lim_{x\to x_j}\psi(x,0)\ \ \forall x_j = x_0 + j\ell,\ j{\in}\mathbb{Z}\Big\}\,.\end{aligned}$$

Theorem 23. *The spectrum of H consists of the Landau levels $(2k+1)\,b$ where k is a non-negative integer, and an infinite family of absolutely continuous spectral bands between any two adjacent Landau levels and below b.*

2.2 Magnetic perturbation

2.2.1 Iwatsuka, 1983

Iwatsuka [1983] proved a very general and important result: a magnetic perturbation which is asymptotically zero will not change the spectrum.

Definition 24 (Hamiltonian with asymptotically constant perturbation)**.** *Let $B(x,y)$ be a smooth real function, such that $B(x,y) \to B_0 \neq 0$ as $\sqrt{x^2+y^2} \to \infty$. Let A_x, A_y be smooth functions satisfying $B(x,y) = \frac{\partial}{\partial x} A_y(x,y) - \frac{\partial}{\partial y} A_x(x,y)$. Let $\tilde{H}$ be the essentially self-adjoint operator on $L^2(\mathbb{R}^2)$ given by*

$$\tilde{H} = \left(-\mathrm{i}\,\frac{\partial}{\partial x} + A_x\right)^2 + \left(-\mathrm{i}\,\frac{\partial}{\partial y} + A_y\right)^2, \qquad \mathrm{D}(\tilde{H}) = C_0^\infty(\mathbb{R}^2)\,.$$

Then the Hamiltonian H is a self-adjoint operator defined as the closure of $\tilde{H}$.

Theorem 25 (Spectrum of H)**.**

$$\sigma_{\mathrm{ess}}(H) = \left\{(2k+1)B_0 \;\middle|\; k \in \mathbb{N}_0\right\}.$$

2.2.2 Iwatsuka, 1985

Iwatsuka [1985] continues the work from Iwatsuka's 1983 article by studying translationally invariant perturbations which *do not* vanish at infinity.

Definition 26 (Hamiltonian)**.** *Let $A_y(x)$ be a smooth real function, such that $B(x) := \frac{\mathrm{d}}{\mathrm{d}x} A_y(x)$ satisfies the condition $0 < M_- < B(x) < M_+ < \infty$ for all $x \in \mathbb{R}$. Let $\tilde{H}$ be the essentially self-adjoint operator on $L^2(\mathbb{R}^2)$ given by*

$$\tilde{H} = \left(-\mathrm{i}\,\frac{\partial}{\partial x} + A_x\right)^2 + \left(-\mathrm{i}\,\frac{\partial}{\partial y} + A_y\right)^2, \qquad \mathrm{D}(\tilde{H}) = C_0^\infty(\mathbb{R}^2)\,.$$

Then the Hamiltonian H is a self-adjoint operator defined as the closure of $\tilde{H}$.

Theorem 27. *Let $\limsup_{x\to-\infty} B(x) < \limsup_{x\to+\infty} B(x)$ or $\limsup_{x\to+\infty} B(x) < \limsup_{x\to-\infty} B(x)$. Then the spectrum of H is purely absolutely continuous.*

Theorem 28. *Let $B(x)$ satisfy the following conditions:*

- *There exist real numbers B_0 and R, such that $B(x) = B_0$ for all $|x| > R$.*
- *$B(x)$ is not constant everywhere – there are points $|x| < R$ where $B'(x) \neq 0$.*
- *There is a point $\bar{x}$, such that $B(x)$ is nondecreasing on one side of $\bar{x}$ and nonincreasing on the other side.*[1]

Then the spectrum of H is purely absolutely continuous.[2]

[1] That means either $B'(x) \le 0$ for $x \le \bar{x}$ and $B'(x) \ge 0$ for $x \ge \bar{x}$ or $B'(x) \ge 0$ for $x \le \bar{x}$ and $B'(x) \le 0$ for $x \ge \bar{x}$. Hence, $B(x) - B_0$ is a bump function.

[2] Note that Theorem 28 does not contradict Theorem 25 – here the perturbation is a bump function but only in the x-direction, in y it extends to infinity, as is the case for every translationally invariant system.

2.2.3 Hislop et al., 2015

Hislop and Soccorsi [2015] [**Finish this.**]

2.3 Geometric perturbation

2.3.1 Exner & Kovařík, 2015

Section 7.1.2 of Exner and Kovařík [2015]. [**Finish this.**]

2.3.2 Exner et al., 2018

Exner et al. [2018a] investigated the Landau Hamiltonian on thin layers.

Definition 29. *Let H be the Landau Hamiltonian with a magnetic field of strength b and a geometric perturbation (see Definition 3) which confines the system to a translationally invariant layer Ω of width 2ℓ given by a curve ω, such that κ, the signed curvature of ω, is bounded and that Ω doesn't intersect itself (i.e. $\ell < 1/\|\kappa\|_{L^\infty}$).*

Theorem 30 (One-sided fold)**.** *Let Ω be a one-sided-fold layer, i.e. either $\lim_{s\to+\infty} \omega_x(s) = +\infty$ or $\lim_{s\to-\infty} \omega_x(s) = -\infty$. Let [second part of (9)]. Then the spectrum of H is purely absolutely continuous.*

Theorem 31. *Let Ω be asymptotically flat, i.e. $\dot{\omega}_x(s) = \alpha_\pm$ for large enough positive and negative s, and let $\alpha_+ \neq \alpha_-$. Let ℓ [satisfy bound in IV.3 and IV.4]. Then the spectrum of H is purely absolutely continuous.*

In the following theorem, we will use this notation:

$$\underline{f}_+ := \sup_{a\in\mathbb{R}} \operatorname*{ess\,inf}_{t\in(a,+\infty)} f(t)\,, \qquad \overline{f}_+ := \inf_{a\in\mathbb{R}} \operatorname*{ess\,sup}_{t\in(a,+\infty)} f(t)\,,$$
$$\underline{f}_- := \sup_{a\in\mathbb{R}} \operatorname*{ess\,inf}_{t\in(-\infty,a)} f(t)\,, \qquad \overline{f}_- := \inf_{a\in\mathbb{R}} \operatorname*{ess\,sup}_{t\in(-\infty,a)} f(t)\,.$$

Also, by writing $x(s)$ we will mean $\omega_x(s)$.

Theorem 32. *Let $E \in \mathbb{R}$ and let $\dot{\kappa}, \ddot{\kappa} \in L^\infty$. Furthermore, suppose either one of the following conditions hold:*

1. $\underline{\dot{x}}_\pm > 0$ *and* $\underline{\dot{x}}_+ \geq \overline{\dot{x}}_-$ *and* $\underline{\kappa^2}_+ - \overline{\kappa^2}_- < 4\,b\,(\underline{\dot{x}}_+ - \overline{\dot{x}}_-)$
2. $x(s) = s$ *for* $s \leq 0$ *and* $\dot{x}(s) \geq 0$ *for* $s > 0$ *and* $\underline{\dot{x}}_+ > 0$ *and* $x \neq \mathrm{Id}$.

Then for ℓ sufficiently small, the spectrum of H is absolutely continuous below $E + \left(\frac{\pi}{2\ell}\right)^2$.

[**To do: fill in conditions (9), IV.3 and IV.4**]

3. Delta potential

In this chapter we will examine the Landau Hamiltonian with a potential perturbation (see definition 1), formally given by the potential $V(x) = \alpha\,\delta_{x_0}$, i.e. the Dirac delta in $x = x_0$ with a coupling constant (*magnitude*) $\alpha \in \mathbb{R}$. Since such a potential is a distribution and not a locally integrable function, the Hamiltonian is rigorously defined as follows.

Definition 33 (Landau Hamiltonian with a Dirac delta perturbation)**.**
Let $\alpha, b \in \mathbb{R} \setminus \{0\}$ and $x_0 \in \mathbb{R}$. We define a linear operator H_α on $L^2(\mathbb{R}^2)$

$$\Big(H_\alpha \psi\Big)(x,y) := \left(-\frac{\partial^2}{\partial x^2} + \Big(\mathrm{i}\frac{\partial}{\partial y} + bx\Big)^2\right)\ \psi(x,y) \quad \textit{a.e.}^1 \textit{on } (\mathbb{R}\setminus\{x_0\}) \times \mathbb{R}$$

with a domain given by the conditions

$$\psi \in W^{1,2}(\mathbb{R}^2)\ \cap\ W^{2,2}\Big(\,(\mathbb{R}\setminus\{x_0\})\times\mathbb{R}\,\Big)\,,$$

$$\lim_{x\to x_0+}\frac{\partial}{\partial x}\psi(x,y)\ -\lim_{x\to x_0-}\frac{\partial}{\partial x}\psi(x,y) = \alpha\lim_{x\to x_0}\psi(x,y) \quad \textit{for a.e. } y\,,^2$$

$$\int_{\mathbb{R}^2} x^2\,\Big|\,\psi(x,y)\,\Big|^2\,\mathrm{d}x\ \mathrm{d}y\ < \infty\,.$$

By an approach analogous to (1.1), one can show that H_α is unitarily equivalent to a direct integral:

$$H_\alpha \simeq \int_{\mathbb{R}}^{\oplus} \mathscr{H}_\alpha(p)\,\mathrm{d}p\ ,$$

where $\mathscr{H}_\alpha(p)$ is a fibre Hamiltonian satisfying very similar conditions to those of H_α, that is, for almost every $p \in \mathbb{R}$:

$$\Big(\mathscr{H}_\alpha(p)\,\varphi\Big)(x) = -\varphi''(x) + \Big(b\,x + p\Big)^2\,\varphi(x)\,, \tag{3.1}$$

$$\varphi \in W^{1,2}(\mathbb{R})\ \cap\ W^{2,2}(\mathbb{R}\setminus\{x_0\})\,,$$

$$\lim_{x\to x_0+}\varphi'(x) - \lim_{x\to x_0-}\varphi'(x) = \alpha\lim_{x\to x_0}\varphi(x),$$

$$\int_{\mathbb{R}} x^2\,|\varphi(x)|^2\,\mathrm{d}x\ < \infty\,.$$

Sometimes, we will want to explicitly specify not only α and p, but also the value of x_0. In that case, we will denote the fibre Hamiltonian as $\mathscr{H}_{\alpha,\,x_0}(p)$.

Before we start investigating the spectrum, we need to show that the problem is well-posed, i.e. that the Hamiltonian H_α is self-adjoint and bounded from below. Then we will show that the spectrum of $\mathscr{H}_\alpha(p)$ is discrete for every p, and only after that we will investigate the continuity of the spectrum of H_α.

[1] The pointwise equality is to be understood *almost everywhere* with respect to the Lebesgue measure on $\mathbb{R}^2$.

[2] The equality holds for *almost every y* and $\lim_{x\to x_0}$ means the *essential* limit with respect to the Lebesgue measure on $\mathbb{R}$.

3.1 Well-posedness

It is straightforward to check that the fibre Hamiltonian is bounded from below:

$$\begin{aligned}
(\varphi,\, \mathscr{H}_\alpha(p)\,\varphi) &= \int_{\mathbb{R}} \overline{\varphi}(x) \Big(-\varphi''(x) + (bx+p)^2\varphi(x)\Big)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}} \overline{\varphi}\varphi'' + \int_{\mathbb{R}} (bx+p)^2 \,\Big|\varphi(x)\Big|^2\,\mathrm{d}x \\
&\geq -\int_{\mathbb{R}} \overline{\varphi}\varphi'' = \int_{\mathbb{R}} \overline{\varphi}'\varphi' - [\overline{\varphi}\varphi']_{-\infty}^{x_0} - [\overline{\varphi}\varphi']_{x_0}^{\infty} \\
&= \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R})} + \overline{\varphi}(x_0)\Big(\varphi'(x_0+) - \varphi'(x_0-)\Big) \\
&= \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R})} + \alpha\,\Big|\varphi(x_0)\Big|^2 .
\end{aligned}$$

In the last two steps we have used the fact that for $\varphi \in W^{2,2}$ both φ and φ' vanish at infinity, and that $\varphi(x_0+) - \varphi(x_0-) = \alpha\varphi(x_0)$. In case when $\alpha \geq 0$, the right-hand side is non-negative, therefore we can use zero as the lower bound. For $\alpha < 0$ we estimate $|\varphi(x_0)|^2 \leq \|\varphi\|^2_{L^\infty}$ and then use the Sobolev-type inequality

$$\forall a>0\ \exists b>0 : \ \Big\|\varphi\Big\|^2_{L^\infty} \leq a\,\Big\|\varphi'\Big\|^2_{L^2} + b\,\Big\|\varphi\Big\|^2_{L^2} ,$$

the proof of which is given in the chapter A.1 of the appendix.

$$\begin{aligned}
(\varphi,\, \mathscr{H}_\alpha(p)\,\varphi) &\geq \Big\|\varphi'\Big\|^2_{L^2} + \alpha\,\Big|\varphi(x_0)\Big|^2 \\
&\geq \Big\|\varphi'\Big\|^2_{L^2} + \alpha\,\Big\|\varphi\Big\|^2_{L^\infty} \\
&\geq \Big\|\varphi'\Big\|^2_{L^2} + \alpha\,\Big(a\,\Big\|\varphi'\Big\|^2_{L^2} + b\,\Big\|\varphi\Big\|^2_{L^2}\Big) \\
&= \Big(1+\alpha a\Big)\,\Big\|\varphi'\Big\|^2_{L^2} + \alpha b\,\Big\|\varphi\Big\|^2_{L^2} .
\end{aligned}$$

By choosing $a \leq |\alpha|^{-1}$, we get

$$(\varphi,\, \mathscr{H}_\alpha(p)\,\varphi) \geq \alpha b\,\Big\|\varphi\Big\|^2_{L^2(\mathbb{R})} ,$$

thus we have shown that the fibre Hamiltonian $\mathscr{H}_\alpha(p)$ is bounded from below. And because the bound is independent of p, it is also a lower bound for H_α:

$$\begin{aligned}
\Big(\psi,\, H_\alpha\psi\Big)_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}} \Big(\tilde{\psi}(\cdot,p),\, \mathscr{H}_\alpha(p)\,\tilde{\psi}(\cdot,p)\Big)_{L^2(\mathbb{R})}\,\mathrm{d}p \\
&\geq \int_{\mathbb{R}} \alpha b\,\Big\|\tilde{\psi}(\cdot,p)\Big\|^2_{L^2(\mathbb{R})}\,\mathrm{d}p = \alpha b\,\Big\|\psi\Big\|^2_{L^2(\mathbb{R}^2)} , \quad \text{where } \tilde{\psi} = \mathcal{F}_y\psi .
\end{aligned} \tag{3.2}$$

Now we will show that the fibre Hamiltonian is self-adjoint. Let $\varphi \in \mathrm{D}(\mathscr{H}_\alpha(p))$ and ψ from a yet-unknown subset of $\mathcal{H}$.

$$\Big(\mathscr{H}_\alpha(p)\,\varphi,\, \psi\Big) = \int_{\mathbb{R}} -\overline{\varphi}''\psi + \int_{\mathbb{R}} \Big(bx+p\Big)^2\,\overline{\varphi}\,\psi \tag{3.3}$$

$$\begin{aligned}
&= \Big[-\overline{\varphi}'\psi + \overline{\varphi}\psi'\Big]_{-\infty}^{x_0} + \Big[\overline{\varphi}'\psi - \overline{\varphi}\psi'\Big]_{x_0}^{+\infty} + \int_{\mathbb{R}} -\overline{\varphi}\,\psi'' + \int_{\mathbb{R}} \Big(bx+p\Big)^2\overline{\varphi}\psi \\
&= -\overline{\varphi}'(x_0-)\psi(x_0-) + \overline{\varphi}(x_0)\psi'(x_0-) \\
&\quad + \overline{\varphi}'(x_0+)\psi(x_0+) - \overline{\varphi}(x_0)\psi'(x_0+) + \int_{\mathbb{R}} \overline{\varphi}\,\Big(\psi'' + \int_{\mathbb{R}} \Big(bx+p\Big)^2\psi\Big) .
\end{aligned}$$

This whole expression has to be equal to (φ, χ) for some $\chi \in \mathcal{H}$ independent of φ. At the second line, we performed an integration by parts, already assuming $\psi \in W^{2,2}(\mathbb{R} \setminus \{x_0\})$. If there was another isolated point $c \in \mathbb{R}$ where ψ weren't twice weakly differentiable, we would get terms $\overline{\varphi}'(c)\big(\psi(c-)-\psi(c+)\big)$ and $\overline{\varphi}(c)\big(\psi'(c-) - \psi'(c+)\big)$ which can't be independent of φ unless $\psi(c-) = \psi(c+)$ and $\psi'(c-) = \psi'(c+)$. However, this would make ψ twice weakly differentiable at c, hence a contradiction. At the third line we simply evaluated the square brackets, making use of the fact that $W^{2,2}$ functions (and their derivatives) vanish at infinity. In order for the entire expression to be independent of φ, the following equation must hold:

$$-\overline{\varphi}'(x_0-)\psi(x_0-) + \overline{\varphi}(x_0)\psi'(x_0-) + \overline{\varphi}'(x_0+)\psi(x_0+) - \overline{\varphi}(x_0)\psi'(x_0+) = 0\,.$$

Substituting $\varphi'(x_0+) - \varphi'(x_0-) = \alpha\varphi(x_0)$ and solving for all φ, we get that $\psi(x_0+) = \psi(x_0-)$ and $\psi'(x_0+) - \psi'(x_0-) = \alpha\psi(x_0+)$. Therefore, ψ must be from $\mathrm{D}(\mathscr{H}_\alpha(p))$ and $\chi = \mathscr{H}_\alpha(p)\psi$. We have shown that $\mathscr{H}_\alpha(p)$ is self-adjoint. And because the direct integral of a self-adjoint operator is also a self-adjoint operator, H_α is self-adjoint, too.

Finally, we will show that the fibre Hamiltonian has a discrete spectrum. The family of operators $\{\mathscr{H}_\alpha(p) \,|\, \alpha \in \mathbb{R}\}$ has a common symmetric restriction:

$$\Omega := \Big\{\varphi \in W^{2,2}(\mathbb{R}) \cap L^2_{x^4}(\mathbb{R}) \;\Big|\; \varphi(x_0) = 0\Big\}\,, \quad \mathscr{H}_\alpha(p)|_\Omega \text{ is symmetric.}$$

Since fibres $\mathscr{H}_\alpha(p)$ for various values of α only differ in the boundary conditions, the restriction to Ω gives just one operator $h(p) := \mathscr{H}_\alpha(p)|_\Omega$, independent of α. The operator $h(p)$ is closed, we can show it directly from the definition: let $\{\varphi_n\} \subset \Omega$ such that $\varphi_n \to \varphi \in L^2(\mathbb{R})$, then

$$\begin{aligned}
&\lim_{n\to\infty} h(p)\,\varphi_n = \lim_{n\to\infty} \Big(-\varphi_n'' + (bx+p)^2\varphi_n\Big) \in L^2 \\
&\quad\Longleftrightarrow\quad \lim_{n\to\infty} \varphi_n'' \in L^2 \;\wedge\; \lim_{n\to\infty} x^2\varphi_n \in L^2 \quad\Longleftrightarrow\quad \varphi'' \in L^2 \;\wedge\; x^2\varphi \in L^2\,.
\end{aligned}$$

Furthermore, there is no way for $\varphi(x_0) \neq \varphi_n(x_0) \equiv 0$ without causing $\varphi_n''(x_0)$ to diverge. Therefore, $h(p)\,\varphi_n \to \psi \implies \varphi \in \Omega$. Finally, the requirement $h(p)\,\varphi = \psi$ follows from the fact that both second derivative and multiplication by x^2 are closed operators on their respective domains.

We have shown that $h(p)$ is a closed symmetric operator with many different extensions $\mathscr{H}_\alpha(p)$. We know that at least one of the extensions, the fibre $\mathscr{H}_{\alpha=0}(p)$ of the unperturbed system, has a discrete spectrum. Now, we want to use Theorem 8 to show that the spectrum of all $\mathscr{H}_\alpha(p)$ is discrete. The last premise left to demonstrate is the fact that $n_+(h(p)) = n_-(h(p)) < \infty$. We can use Theorem 9 to show that the deficiency indices are equal. Furthermore, $h(p)^*$ is a second-order differential operator, therefore $n_\pm \leq 2$, hence they are also finite.

We have shown that the Hamiltonian H_α is self-adjoint and bounded from below for each $\alpha \in \mathbb{R}$, and that the fibre Hamiltonian $\mathscr{H}_\alpha(p)$ has a discrete spectrum for every $\alpha, p \in \mathbb{R}$. In the next section, we will investigate what are the eigenvalues of $\mathscr{H}_\alpha(p)$ and how they depend on p and α.

3.2 Eigenproblem of the fibre Hamiltonian

In order to use utilize the theorem 5 to find the spectrum of H_α, we need to find the eigenvalues of the fibre Hamiltonian for each p. That is, we are looking for a real function $\epsilon(p)$, such that for all $p \in \mathbb{R}$ there exists a $\varphi \in \mathrm{D}(\mathscr{H}_\alpha(p))$ satisfying

$$\mathscr{H}_\alpha(p)\,\varphi = \epsilon(p)\,\varphi\,.$$

As a shorthand, we will often denote $\epsilon(p)$ simply as ϵ. Substituting from (3.1), we get an ordinary differential equation:

$$\begin{gathered}-\varphi''(x) + \left(b^2\,x^2 + 2pb\,x + p^2\right)\varphi(x) = \epsilon\,\varphi(x) \quad \text{on } x \neq x_0\,,\\ \varphi'(x_0+) - \varphi'(x_0-) = \alpha\,\varphi(x_0)\,.\end{gathered}$$

From now on, we shall suppose that $b > 0$; for $b < 0$ one can perform a reflection $x \mapsto -x$ and arrive at the same results. In order to refine this differential equation into the standard form, we change variables $x \mapsto w$ and instead of one function φ on $\mathbb{R}$ we introduce two functions g_-, g_+ on the left and right half-line respectively:

$$w := \sqrt{2b}\left(x + \frac{p}{b}\right), \qquad w_0 := \sqrt{2b}\left(x_0 + \frac{p}{b}\right), \qquad \nu := \frac{\epsilon - b}{2b}\,, \tag{3.4}$$

$$g_- : (-\infty, w_0] \to \mathbb{C}\,, \qquad g_+ : [w_0, +\infty) \to \mathbb{C}\,,$$

$$\varphi(x) = \begin{cases} g_+\left(\sqrt{2b}\,(x + \frac{p}{b})\right) & \text{for } x \geq x_0\,,\\ g_-\left(\sqrt{2b}\,(x + \frac{p}{b})\right) & \text{for } x < x_0\,.\end{cases}$$

Then we arrive at the so-called parabolic cylinder differential equation:

$$g''_\pm(w) = \left(\frac{1}{4}w^2 - \nu - \frac{1}{2}\right) g_\pm(w)\,, \tag{3.5}$$

The two functions are then *"glued together"* by the following equations:

$$\begin{aligned} g_+(w_0) - g_-(w_0) &= 0\,,\\ g'_+(w_0) - g'_-(w_0) &= \alpha\sqrt{2b}\;g_+(w_0)\,.\end{aligned} \tag{3.6}$$

As stated in chapter 9.2 of Gradshteyn and Ryzhik [2014], the solutions to (3.5) can be expressed as a linear combination of the functions

$$D_\nu(w)\,,\;\; D_\nu(-w)\,,\;\; D_{-\nu-1}(\mathrm{i}w)\,,\;\; D_{-\nu-1}(-\mathrm{i}w)\,, \tag{3.7}$$

where D_ν is the so-called *parabolic cylinder function*, which is a special function that can be expressed in terms of the gamma function Γ and the confluent hypergeometric function ${}_1\mathrm{F}_1$:

$$D_\nu(w) = 2^{\frac{\nu}{2}}\exp\left(-\tfrac{w^2}{4}\right)\left(\frac{\sqrt{\pi}}{\Gamma\left(\frac{1-\nu}{2}\right)}\,{}_1\mathrm{F}_1\Big(-\frac{\nu}{2},\,\frac{1}{2};\,\frac{w^2}{2}\Big) - \frac{w\sqrt{2\pi}}{\Gamma\left(-\frac{\nu}{2}\right)}\,{}_1\mathrm{F}_1\Big(\frac{1-\nu}{2},\,\frac{3}{2};\,\frac{w^2}{2}\Big)\right). \tag{3.8}$$

Since $1/\Gamma(z)$ is an entire function and $(\alpha, z) \mapsto {}_1\mathrm{F}_1(\alpha, \gamma; z)$ is holomorphic on $\mathbb{C}^2$ for all γ other than non-positive integers, it follows that $(\nu, w) \mapsto D_\nu(w)$ is also holomorphic on $\mathbb{C}^2$.

In the special case when $\nu \in \mathbb{N}_0$, the function D_ν can be expressed using the Hermite polynomials H_n:

$$D_\nu(w) = 2^{-\frac{\nu}{2}} \exp\left(-\tfrac{w^2}{4}\right) H_\nu\Big(\frac{w}{\sqrt{2}}\Big) \tag{3.9}$$

The solutions in (3.7) are linearly dependent. For most values of ν, any of the four functions can be expressed as a linear combination of any two others. However, specifically in the case $\nu \in \mathbb{N}_0$ we get $D_\nu(w) = \pm D_\nu(-w)$.

Asymptotic behaviour of the solutions is also given by Gradshteyn and Ryzhik [2014]. As $|w| \to \infty$, the solutions $D_{-\nu-1}(\mathrm{i}w)$ and $D_{-\nu-1}(-\mathrm{i}w)$ grow exponentially. Meanwhile, $D_\nu(w)$ decays exponentially for $w \to +\infty$. Therefore, $D_\nu(w)$ and $D_\nu(-w)$ are better suited for the growth conditions imposed by the domain of $\mathscr{H}_\alpha(p)$. We define $c_{+1}, c_{+2}, c_{-1}, c_{-2} \in \mathbb{C}$, such that

$$g_\pm(w) = c_{\pm 1}\, D_\nu(w) + c_{\pm 2}\, D_\nu(-w)\,. \tag{3.10}$$

Furthermore, if $\nu \notin \mathbb{N}_0$, the solution $D_\nu(w)$ diverges for $w \to -\infty$, as demonstrated in Lemma 40 in Chapter A.2 of the appendix. Therefore, $c_{-1} = c_{+2} = 0$ in order for φ to be integrable. On the other hand, if $\nu \in \mathbb{N}_0$, the solutions aren't independent (as discussed above), therefore we can also set $c_{-1} = c_{+2} = 0$ without loss of generality. Applying the gluing equations (3.6), we get:

$$c_{+1}\, D_\nu(w_0) = c_{-2}\, D_\nu(-w_0)$$

$$c_{+1}\, \frac{\mathrm{d}}{\mathrm{d}w} D_\nu(w)\Big|_{w_0} - c_{-2}\, \frac{\mathrm{d}}{\mathrm{d}w} D_\nu(-w)\Big|_{w_0} = \alpha\sqrt{2b}\; c_{+1}\, D_\nu(w_0)$$

We substitute using the equality $\frac{\mathrm{d}}{\mathrm{d}w} D_\nu(w) = \frac{w}{2}\, D_\nu(w) - D_{\nu+1}(w)$ from Gradshteyn and Ryzhik [2014] and arrive at the equation:

$$\begin{pmatrix} D_\nu(w_0) & -D_\nu(-w_0) \\ \left(\frac{w_0}{2} - \alpha\sqrt{2b}\right) D_\nu(w_0) - D_{\nu+1}(w_0) & -\frac{w_0}{2}\, D_\nu(-w_0) - D_{\nu+1}(-w_0) \end{pmatrix} \begin{pmatrix} c_{+1} \\ c_{-2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In order for the equation to have a non-trivial solution, the determinant of the matrix must be zero. Hence, we arrive at the condition:

$$\begin{aligned} 0 &= D_\nu(w_0)\Big(\tfrac{w_0}{2}\, D_\nu(-w_0) - D_{\nu+1}(-w_0)\Big) + D_\nu(-w_0)\Big(\big(\tfrac{w_0}{2} - \alpha\sqrt{2b}\big) D_\nu(w_0) - D_{\nu+1}(w_0)\Big) \\ &= D_\nu(w_0)\, D_{\nu+1}(-w_0) \;+\; D_\nu(-w_0)\, D_{\nu+1}(w_0) \;+\; \alpha\sqrt{2b}\, D_\nu(w_0)\, D_\nu(-w_0)\,. \end{aligned} \tag{3.11}$$

Since we're interested in the allowed values of ν for given w_0 and $\alpha\sqrt{2b} =: a$, this equation effectively defines an implicit function $\nu(a, w_0)$.

3.3 Energy levels as analytic functions

Let F be a function of three real variables given by

$$F(a, w, \nu) = D_\nu(w)\, D_{\nu+1}(-w) \;+\; D_\nu(-w)\, D_{\nu+1}(w) \;+\; a\, D_\nu(w)\, D_\nu(-w)\,. \tag{3.12}$$

We have shown that

$$\epsilon(p) \text{ is an eigenvalue of } \mathscr{H}_\alpha(p) \iff F\Big(\alpha\sqrt{2b},\ \sqrt{2b}\left(x_0+\frac{p}{b}\right),\ \frac{\epsilon(p)+b}{2b}\Big) = 0\,. \tag{3.13}$$

Truly, this is simply the equation (3.11) after the substitution from (3.4). The solutions of $F(...)=0$, i.e. the eigenvalues of the fibre Hamiltonian, are plotted in figures 3.1 and 3.2.

For each fixed α, the relation (3.13) defines a set of non-intersecting analytic functions $\big\{\epsilon_0(p), \epsilon_1(p), ...\big\}$ defined on the entire $\mathbb{R}$. This can be demonstrated either from the properties of F via the implicit mapping theorem (listed as 8.6 in Kaup and Kaup [1983]), or from the properties of $\mathscr{H}_\alpha(p)$ via Theorem 11. We choose the latter approach:

$$\mathscr{H}_\alpha(p) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \big(b\,x+p\big)^2 = \mathscr{H}_\alpha(0) + 2bp\,x + p^2$$

The fibre Hamiltonian is clearly holomorphic in p and meets the assumptions of Theorem 11 for all $p \in \mathbb{R}$.

It is also true that the energy levels tend to the unperturbed Landau levels $\epsilon_k(p) \to (2k+1)\,b$ as $p \to \pm\infty$. This follows from the fact that shifting p by a constant amount is equivalent to changing x_0. Or more precisely, the fibre Hamiltonian $\mathscr{H}_{\alpha,\,x_0}(p)$ is unitarily equivalent to $\mathscr{H}_{\alpha,\,x_0+p/b}(0)$. The bound states are concentrated around $x=0$ and decay exponentially away from it. Therefore, moving the point interaction further from $x=0$ will cause it to have a diminishing effect, completely vanishing at infinity.

We want to investigate, whether the functions $\epsilon_k(p)$ touch (or even cross) their respective Landau levels. From (3.13) we know that $\epsilon(p) = (2k+1)\,b$ has a solution precisely if

$$F(a,w,k) = 0 \qquad \text{for some } w \in \mathbb{R}\,.$$

Using (3.9) and the fact that $H_k(-x) = (-1)^k\,H_k(x)$, we get:

$$\begin{aligned}
F(a,w,k) &= D_k(w)\,D_{k+1}(-w) \ +\ D_k(-w)\,D_{k+1}(w) \ +\ a\,D_k(w)\,D_k(-w) \\
&= 2^{-\frac{k}{2}-\frac{k+1}{2}}\,\mathrm{e}^{-\frac{w^2}{4}-\frac{w^2}{4}}\,\Big(H_k\big(\tfrac{w}{\sqrt2}\big)H_{k+1}\big(\tfrac{-w}{\sqrt2}\big) + H_k\big(\tfrac{-w}{\sqrt2}\big)H_{k+1}\big(\tfrac{w}{\sqrt2}\big)\Big) \\
&\quad + a\,2^{-\frac{k}{2}-\frac{k}{2}}\,\mathrm{e}^{-\frac{w^2}{4}-\frac{w^2}{4}}\,H_k\big(\tfrac{w}{\sqrt2}\big)H_k\big(\tfrac{-w}{\sqrt2}\big) \\
&= 2^{-k-\frac12}\,\mathrm{e}^{-\frac{w^2}{2}}\,(-1)^{k+1}\Big(\overbrace{H_k\big(\tfrac{w}{\sqrt2}\big)H_{k+1}\big(\tfrac{w}{\sqrt2}\big) - H_k\big(\tfrac{w}{\sqrt2}\big)H_{k+1}\big(\tfrac{w}{\sqrt2}\big)}^{0}\Big) \\
&\quad + a\,2^{-k}\,\mathrm{e}^{-\frac{w^2}{2}}\,(-1)^k\,H_k\big(\tfrac{w}{\sqrt2}\big)H_k\big(\tfrac{w}{\sqrt2}\big) \\
&= a\,(-1)^k\,2^{-k}\,\mathrm{e}^{-(p+bx_0)^2/b}\,\Big(H_k\big(\tfrac{p+bx_0}{\sqrt b}\big)\Big)^2\,.
\end{aligned}$$

Unsurprisingly, for $a = \alpha\sqrt{2b} = 0$ the expression is zero. This reflects the fact that $\alpha = 0$ is the unperturbed system with constant energy levels $\epsilon_k(p) = (2k+1)b$.

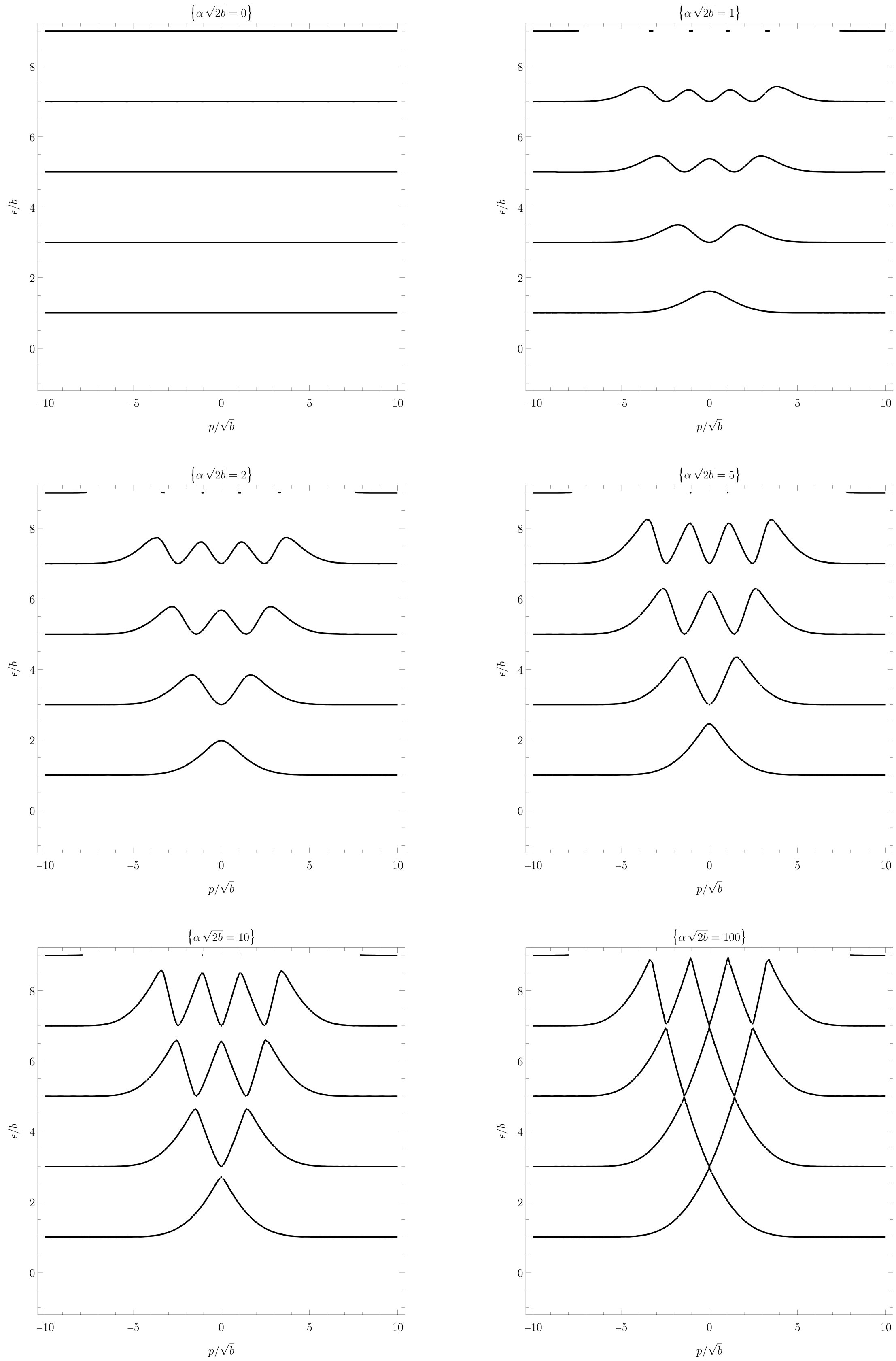

Figure 3.1: The first four energy levels ϵ as a function of the y-momentum p for $\alpha\sqrt{2b} = 0, 1, 2, 5, 10$ and 100 (starting with an unperturbed system, followed by an increasingly repulsive perturbation).

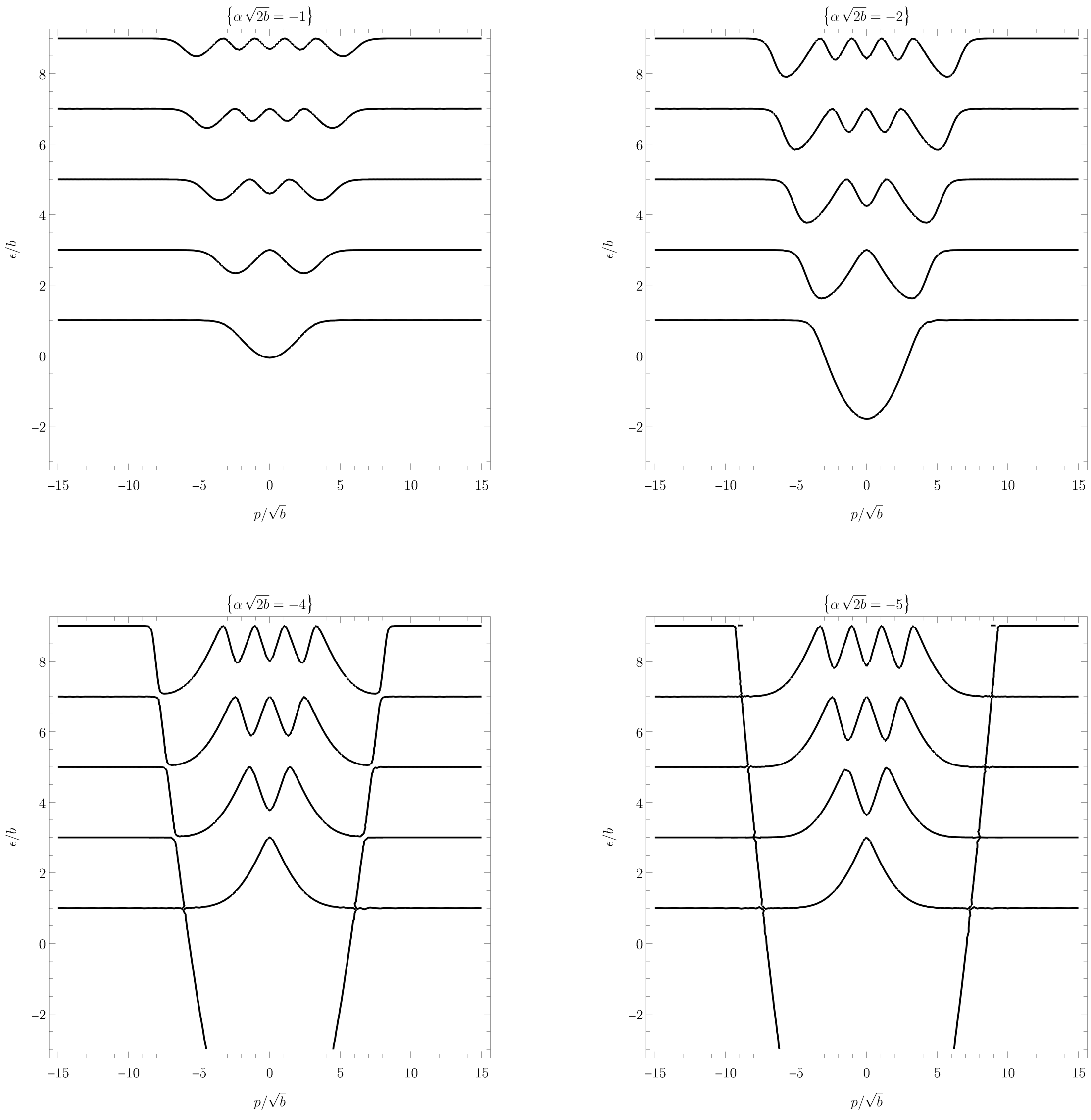

Figure 3.2: The first five energy levels ϵ as a function of the y-momentum p for $\alpha\sqrt{2b} = -1, -2, -4$ and -5 (system with an increasingly attractive perturbation).

For $\alpha \neq 0$, there are several important observations we can make. First, if both a and k are constant, the sign of the expression doesn't change – it is either nonnegative or nonpositive – which means that the functions $\epsilon_k(p)$ never cross the Landau levels.[3] Second, for given $\alpha \neq 0$, either all energy functions are above their Landau level, or all are under it – this follows from the fact that the sign of F alternates when incrementing k. Third, we can tell how many times each $\epsilon_k(p)$ touches the Landau level; such points must satisfy

$$F(a, w = \sqrt{2b}(x_0 + \tfrac{p}{b}),\, k) = a\, (-1)^k\, 2^{-k}\, \mathrm{e}^{-(p+bx_0)^2/b} \left(H_k\left(\tfrac{p+bx_0}{\sqrt{b}}\right)\right)^2 = 0\,.$$

And because the k-th Hermite polynomial H_k has precisely k roots, we know that the k-th Landau level touches an energy function in k distinct points and these points do not depend on the value of α.

Finally, we can make the following observation:

Claim 34. *If $\alpha \neq 0$, then each energy function $\epsilon_k(p)$ is not constant on any interval and its supremum is strictly less than the infimum of ϵ_{k+1}.*

Proof. We have shown that ϵ_k are analytic – an analytic function which is constant on an interval is constant everywhere. However, we know that $\epsilon_k(p) \neq b\,(2k+1)$ almost everywhere, yet $\epsilon_k(p) \to (2k+1)$, hence it is constant nowhere.

The fact that the k-th Landau level touches an energy function k times for any $\alpha \neq 0$, even infinitesimally small, implies that each ϵ_k touches *its own* Landau level. There is no $\epsilon_k(p)$ that touches the $(k \pm 1)$-th Landau level – if there were one, it would have to touch the Landau level exactly at the same point as $\epsilon_{k\pm 1}(p)$, because the touching points do not depend on α. However, this would violate the non-intersection of energy levels. Combined with the fact that all energy functions are above (or below) their Landau levels, this proves that there is a gap between the extrema of adjacent energy functions. □

Claim 35. *If $\alpha > 0$, then $\epsilon_k \geq b\,(2k+1)$, and if $\alpha < 0$, then $\epsilon_k \leq b\,(2k+1)$.*

Proof. We have already shown that all energy functions are either above or below their Landau levels. Now, which is it? This question can be decided using the quadratic form of $\mathscr{H}_\alpha(p)$. Let $\alpha < \beta$, then:

$$\begin{aligned}(\varphi,\, \mathscr{H}_\alpha(p)\,\varphi) &= -\int_{\mathbb{R}} \overline{\varphi}\varphi'' + \int_{\mathbb{R}} (bx+p)^2\, \left|\varphi(x)\right|^2\, \mathrm{d}x \\ &= \alpha\left|\varphi(x_0)\right|^2 + \left\|\varphi'\right\|^2_{L^2(\mathbb{R})} + \left\|\varphi\right\|_{L^2_{(bx+p)^2}(\mathbb{R})} \\ &\leq \beta\left|\varphi(x_0)\right|^2 + \left\|\varphi'\right\|^2_{L^2(\mathbb{R})} + \left\|\varphi\right\|_{L^2_{(bx+p)^2}(\mathbb{R})} = (\varphi,\, \mathscr{H}_\beta(p)\,\varphi)\,.\end{aligned}$$

The quadratic form of $\mathscr{H}_\alpha(p)$ is clearly nondecreasing in α. Therefore, the energy functions will be under the Landau level for α negative and above it for α positive. □

[3] The sign of F clearly flips between the energy levels, it is positive above/below each $\epsilon_k(p)$ and negative on the other side. Therefore, if $\epsilon_k(p)$ were to cross the Landau level, the sign of F would have to change.

Now, we have all we need to characterize the spectrum of H_α using Theorem 5. If $\alpha \neq 0$, the pure point spectrum is clearly empty, because the functions $\epsilon_k(p)$ are not constant, hence the set $\{p \mid \epsilon(p) = E\}$ has measure zero for every E. Furthermore, as H_α is a direct integral of an analytic operator-valued function $\mathscr{H}_\alpha(p)$ whose spectrum is discrete, the singular continuous spectrum of H_α is also empty, as proven by Filonov and Sobolev [2006]. Thus, the spectrum of H_α will be entirely absolute continuous, consisting of disjoint bands spanning from the Landau levels up (for $\alpha > 0$) or down (for $\alpha < 0$).

3.4 Summary

The following theorem summarizes the information obtained about the spectrum of the Landau Hamiltonian with a Dirac delta perturbation:

Theorem 36. *Let α, b and H_α be as in Definition 33. If $\alpha > 0$, then for each $k \in \mathbb{N}_0$ there exists $E_k \in (0,2)$ such that*

$$\sigma(H_\alpha) = \sigma_{\mathrm{ac}}(H_\alpha) = \bigcup_{k \in \mathbb{N}_0} \Big[b\,(2k+1),\ b\,(2k+1+E_k) \Big]\,.$$

If $\alpha < 0$, then there exists $E_0 \in (0,\infty)$ and for each $n \in \mathbb{N}$ there is $E_n \in (0,2)$ such that

$$\sigma(H_\alpha) = \sigma_{\mathrm{ac}}(H_\alpha) = \bigcup_{k \in \mathbb{N}_0} \Big[b\,(2k+1-E_k),\ b\,(2k+1) \Big]\,.$$

4. Half-plane with Robin boundary

In this chapter we will examine the Landau Hamiltonian of a particle confined to a half-plane with a Robin (or *mixed*) boundary condition.

Definition 37 (Landau Hamiltonian in half-plane with Robin boundary)**.**
Let $\alpha \in \mathbb{R}$ *and* $\Omega := \mathbb{R} \times \mathbb{R}_+$*, where* $\mathbb{R}_+ \equiv [0, +\infty)$*. The Hamiltonian is given by*[1]

$$\begin{aligned} \Big(H_\alpha \psi\Big)(x, y) &= \left(-\frac{\partial^2}{\partial x^2} + \Big(-\mathrm{i}\frac{\partial}{\partial y} + bx \Big)^2 \right) \psi(x, y) \,, \\ \mathrm{D}(H_\alpha) &= \Big\{ \psi \in W^{2,2}(\Omega) \cap L^2_{x^4}(\Omega) \;\Big|\; \alpha\, \psi(0, y) + \psi'(0, y) = 0 \Big\} \,. \end{aligned} \tag{4.1}$$

Using (1.1), we can once again show that the Hamiltonian is unitarily equivalent to a direct integral $H_\alpha \simeq \int_{\mathbb{R}}^{\oplus} \mathscr{H}_\alpha(p)\, \mathrm{d}p$, where $\mathscr{H}(p)$ is the fibre Hamiltonian given by

$$\begin{aligned} \Big(\mathscr{H}_\alpha(p) \varphi\Big)(x) &= -\varphi''(x) + (p + bx)^2\, \varphi(x) \,, \\ \mathrm{D}(\mathscr{H}_\alpha(p)) &= \Big\{ \varphi \in W^{2,2}(\mathbb{R}_+) \cap L^2_{x^4}(\mathbb{R}_+) \;\Big|\; \alpha\, \varphi(0) + \varphi'(0) = 0 \Big\} \,. \end{aligned} \tag{4.2}$$

When $\alpha = 0$, the problem reduces to the Neumann boundary condition.

4.1 Well-posedness

We will start by showing that the Hamiltonian is bounded from below, starting with the fibre Hamiltonian:

$$\begin{aligned} \Big(\varphi, \mathscr{H}_\alpha(p)\, \varphi\Big) &= -\int_{\mathbb{R}_+} \overline{\varphi}\varphi'' + \overbrace{\int_{\mathbb{R}_+} (bx + p)^2\, |\varphi|^2}^{\geq 0} \\ &\geq -\Big[\overline{\varphi}\varphi'\Big]_0^\infty + \int_{\mathbb{R}_+} |\varphi'|^2 \;=\; \overline{\varphi(0)}\, \varphi'(0) + \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R}_+)} \end{aligned}$$

We have used integration by parts and the fact that for $\varphi \in W^{2,2}$ both φ and φ' vanish at infinity. We substitute for $\varphi'(0) = -\alpha\, \varphi(0)$ from the boundary condition:

$$\Big(\varphi, \mathscr{H}_\alpha(p)\, \varphi\Big) \geq \overline{\varphi(0)}\, \varphi'(0) + \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R}_+)} = -\alpha\, \Big|\varphi(0)\Big|^2 + \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R}_+)} \,.$$

For $\alpha \leq 0$, the whole right-hand side is non-negative. For $\alpha > 0$, we have:

$$\Big(\varphi, \mathscr{H}_\alpha(p)\, \varphi\Big) \geq -\alpha\, \Big|\varphi(0)\Big|^2 + \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R}_+)} \geq -\alpha \Big\|\varphi\Big\|^2_{L^\infty(\mathbb{R}_+)} + \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R}_+)} \,.$$

Now using the lemma 39 from the Appendix, we know that for every $a > 0$ there exists $b > 0$ such that $-\alpha\Big\|\varphi\Big\|^2_{L^\infty} \geq -a\, \alpha\Big\|\varphi'\Big\|^2_{L^2} - b\, \alpha\Big\|\varphi\Big\|^2_{L^2}$. Setting $a = \alpha^{-1}$, we obtain

$$\Big(\varphi, \mathscr{H}_\alpha(p)\Big) \geq \Big(-\Big\|\varphi'\Big\|^2_{L^2(\mathbb{R}_+)} - b\, \alpha \Big\|\varphi\Big\|^2_{L^2(\mathbb{R}_+)} \Big) + \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R}_+)} = -b\, \alpha\, \Big\|\varphi\Big\|^2_{L^2(\mathbb{R}_+)} \,.$$

[1] The equalities are to be understood in the weak sense again. $(H_\alpha \psi)(x, y) = ...$ holds for almost every (x, y) in Ω with respect to the Lebesgue measure. $\psi(0, y) \equiv \lim_{x \to 0} \psi(x, y) = ...$ holds for almost every $y \in \mathbb{R}$, and the limit is the *essential* limit wrt. the Lebesgue measure.

As demonstrated in (3.2) in the previous chapter, the Hamiltonian H_α is therefore also bounded from below with the same lower bound.

Now we will show the self-adjointness, again starting with the fibre Hamiltonian. Let $\varphi \in \mathrm{D}(\mathscr{H}_\alpha(p))$ and $\psi \in M \subset L^2(\mathbb{R}_+)$.

$$\begin{aligned}\Big(\mathscr{H}_\alpha(p)\,\varphi,\ \psi\Big) &= \int_{\mathbb{R}_+} -\overline{\varphi}''\psi + \int_{\mathbb{R}_+}\Big(bx+p\Big)^2\,\overline{\varphi}\,\psi \\ &= -\Big[\overline{\varphi}'\,\psi\Big]_0^\infty + \Big[\overline{\varphi}\,\psi'\Big]_0^\infty + \int_{\mathbb{R}_+} -\overline{\varphi}\,\psi'' + \int_{\mathbb{R}_+}\Big(bx+p\Big)^2\,\overline{\varphi}\,\psi \\ &= \underbrace{-\overline{\varphi}'(0)\,\psi(0) + \overline{\varphi}(0)\,\psi'(0)}_{-\overline{\varphi}(0)\,\left(\alpha\psi(0)+\psi'(0)\right)} + \int_{\mathbb{R}_+}\overline{\varphi}\,\Big(-\psi'' + (bx+p)^2\psi\Big)\,.\end{aligned}$$

First, we integrated by parts, assuming that $M \subseteq W^{2,2}(\mathbb{R}_+)$ – if it were not, the result could not be independent of φ, as demonstrated in the previous chapter after (3.3). Then we used the fact that functions from $W^{2,2}$ vanish at infinity, and finally we applied the boundary condition $\alpha\varphi(0) + \varphi'(0) = 0$. It is clear that ψ must fulfil the same boundary condition in order for the result to be independent of $\varphi(0)$. Therefore, $M = \mathrm{D}(\mathscr{H}_\alpha(p))$ and the fibre Hamiltonian is self-adjoint. The Hamiltonian H, a direct integral of a self-adjoint operator, is hence also self-adjoint.

Lastly, we will show that the spectrum of $\mathscr{H}_\alpha(p)$ is discrete using the theorem 8. We define:

$$\Omega = \Big\{\varphi \in W^{2,2}(\mathbb{R}_+) \cap L^2_{x^4}(\mathbb{R}_+)\ \Big|\ \varphi(0) = \varphi'(0) = 0\Big\}\,,$$

then the fibre Hamiltonians for all values of α have a common symmetric restriction:

$$h(p) := \mathscr{H}_\alpha(p)|_\Omega\,.$$

The operator $h(p)$ is closed, we show this directly from the definition: let $\{\varphi_n\} \subset \Omega$ such that $\varphi_n \to \varphi \in L^2(\mathbb{R})$, then

$$\begin{aligned}&\lim_{n\to\infty} h(p)\,\varphi_n = \lim_{n\to\infty}\Big(-\varphi_n'' + (bx+p)^2\varphi_n\Big) \in L^2 \\ &\iff\quad \lim_{n\to\infty}\varphi_n'' \in L^2\ \wedge\ \lim_{n\to\infty} x^2\varphi_n \in L^2 \quad\iff\quad \varphi'' \in L^2\ \wedge\ x^2\varphi \in L^2\,.\end{aligned}$$

Furthermore, there is no way for $\varphi(0) \neq \varphi_n(0) \equiv 0$ or $\varphi'(0) \neq \varphi_n'(0) \equiv 0$ without causing $\varphi_n''(0)$ to diverge. Therefore, $h(p)\,\varphi_n \to \psi \implies \varphi \in \Omega$. Finally, the requirement $h(p)\,\varphi = \psi$ follows from the fact that both second derivative and multiplication by x^2 are closed operators on their respective domains.

The deficiency indices of $h(p)$ are equal (from Theorem 9) and finite (as $h(p)$ is a finite-order differential operator).

4.2 Eigenproblem of the fibre Hamiltonian

We are searching for a function $\epsilon(p)$, such that for every p there exists a $\varphi \in \mathrm{D}(\mathscr{H}_\alpha(p))$ for which

$$\mathscr{H}_\alpha(p)\,\varphi = \epsilon(p)\,\varphi\,.$$

Substituting from the definition, we get

$$-\varphi''(x) + (p+bx)^2\,\varphi(x) = \epsilon\,\varphi(x)\,,$$
$$\Big((p+bx)^2 - \epsilon\Big)\,\varphi(x) = \varphi''(x)\,.$$

This is the parabolic cylinder equation and, as in the previous chapter, its solutions are in the form

$$\varphi(x) = c\,D_\nu(w) + d\,D_\nu(-w)\,, \quad \text{where} \quad w := \sqrt{2b}\Big(x + \frac{p}{b}\Big)\,, \quad \nu := \frac{\epsilon - b}{2b}\,.$$

Except for $\nu \in \mathbb{N}$, the $D_\nu(-w)$ term diverges exponentially, see Lemma 40 in the appendix, therefore d must be zero in order for $\varphi \in \mathrm{D}(\mathscr{H}_\alpha)$. Now we apply the boundary condition:

$$\begin{aligned}
\alpha\,\varphi(0) &= -\varphi'(0)\,,\\
\alpha\,c\,D_\nu(w_0) &= -\frac{\mathrm{d}}{\mathrm{d}x}\,c\,D_\nu(w)\Big|_{w=w_0}\,, \quad \text{where} \quad w_0 = \sqrt{2b}\,(0+\tfrac{p}{b}) = p\,\sqrt{\tfrac{2}{b}}\,,\\
\alpha\,D_\nu(w_0) &= -\sqrt{2b}\,\frac{\mathrm{d}}{\mathrm{d}w}\,D_\nu(w)\Big|_{w=w_0}\,,\\
\alpha\,D_\nu(w_0) &= -\sqrt{2b}\,\Big(\frac{w_0}{2}D_\nu(w_0) - D_{\nu+1}(w_0)\Big)\,,\\
\alpha\,D_\nu(w_0) &= -\sqrt{2b}\Big(\frac{p}{\sqrt{2b}}D_\nu(w_0) - D_{\nu+1}(w_0)\Big)\,,\\
\Big(\alpha + p\Big)\,D_\nu(w_0) &= \sqrt{2b}\,D_{\nu+1}(w_0)\,,\\
\Big(\alpha + p\Big)\,D_\nu(p\,\sqrt{\tfrac{2}{b}}) &= \sqrt{2b}\,D_{\nu+1}(p\,\sqrt{\tfrac{2}{b}})\,. \qquad (4.3)
\end{aligned}$$

The equation (4.3) is the spectral condition, it defines the implicit function $\nu(p)$, which in turn tells us, which values of $\epsilon \equiv (2\nu+1)\,b$ are in the spectrum of $\mathscr{H}_\alpha$. For $\alpha = 0$ we get the Neumann b.c. Formally taking $\alpha \to +\infty$, we get the Dirichlet b.c., whose spectral condition is $D_\nu(p\,\sqrt{\tfrac{2}{b}}) = 0$.

4.3 Energy levels as analytic functions

Let F be a function of three real variables given by

$$F(\alpha, p, \nu) = \Big(\alpha + p\Big)\,D_\nu(p\,\sqrt{\tfrac{2}{b}}) - \sqrt{2b}\,D_{\nu+1}(p\,\sqrt{\tfrac{2}{b}})\,. \qquad (4.4)$$

Comparing this with (4.3), we see that:

$$\epsilon(p) \text{ is an eigenvalue of } \mathscr{H}_\alpha(p) \iff F\Big(\alpha,\,p,\,\frac{\epsilon(p)+b}{2b}\Big) = 0\,. \qquad (4.5)$$

The eigenvalues ϵ as functions of p are plotted in Figure 4.1. As was the case in the previous chapter, Theorem 11 implies that the relation (4.5) defines a set of analytic non-intersecting functions $\epsilon_k : \mathbb{R} \to \mathbb{R}$, where $k \in \mathbb{N}_0$.

We are interested in the behaviour of $\epsilon_k(p)$ as $p \to \pm\infty$. The fibre Hamiltonian $\mathscr{H}_\alpha(p)$ with p fixed is unitarily equivalent to a fibre Hamiltonian with $p = 0$

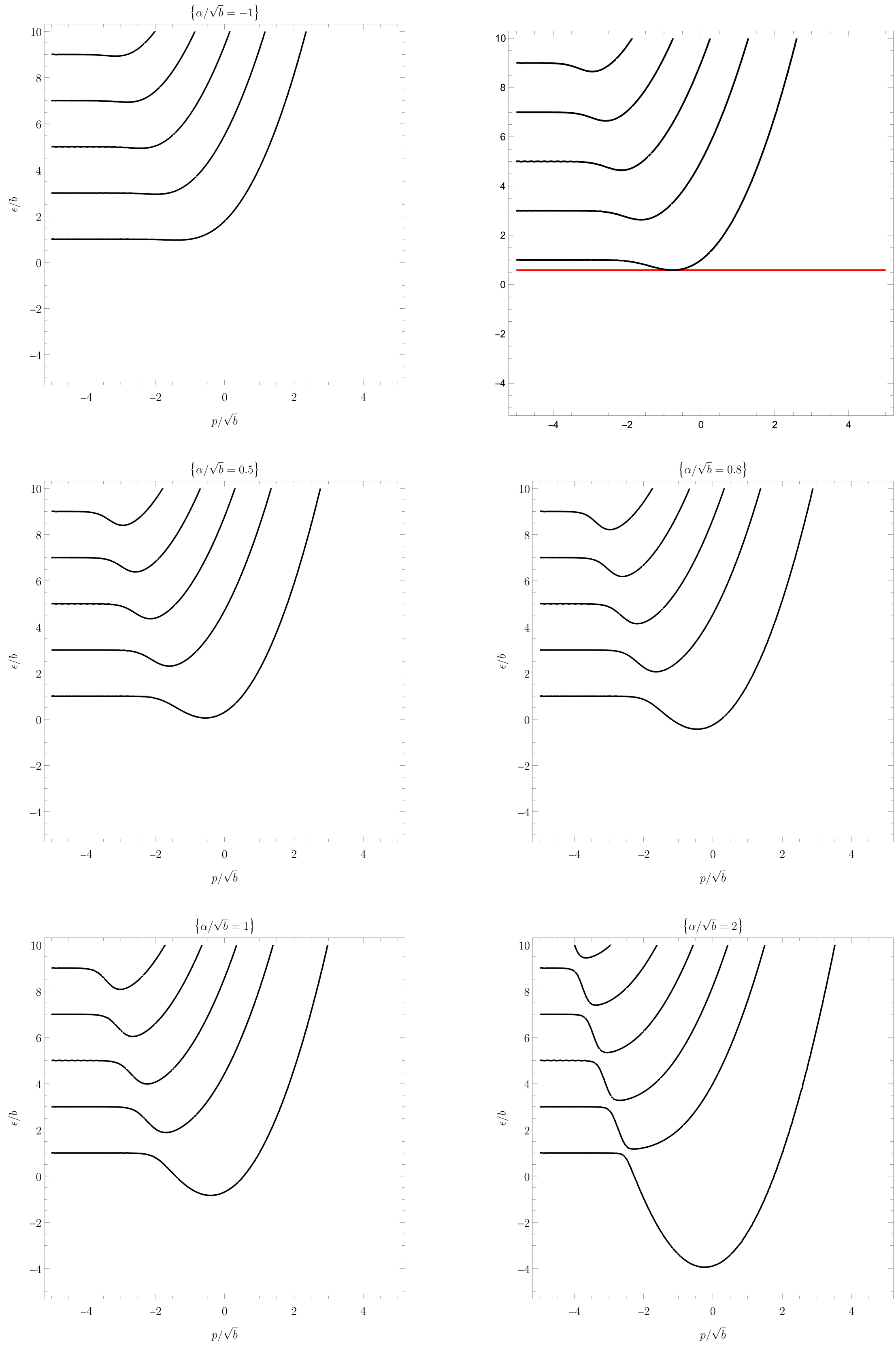

Figure 4.1: The first energy levels ϵ as a function of the y-momentum p for $\alpha\sqrt{b} = -1,\ 0,\ 0.5,\ 0.8,\ 1$ and 2. The red line marks the predicted minimum $\epsilon = \Theta_0\, b$.

translated by p/b, that is $\mathscr{H}_\alpha(p) \simeq T_{p/b}\,\mathscr{H}_\alpha(0)\,T_{-p/b}$, where $T_c\varphi(x) = \varphi(x-c)$. If $p < 0$, this causes the Robin boundary to move to the left, away from the minimum of the effective potential. As $p \to -\infty$, the influence of the boundary decreases and the fibre Hamiltonian approaches that of a one-dimensional harmonic oscillator, hence $\epsilon_k \to b\,(2k+1)$. On the other hand, $p \to +\infty$ moves the boundary to the right, making increasingly larger areas with low effective potential forbidden, hence $\epsilon_k \to +\infty$.

These asymptotics make it clear that all energy levels above the lowest Landau level $\epsilon = b$ are allowed. However, one question still remains: what is the lowest allowed energy level? For the Neumann boundary ($\alpha = 0$), Bonnaillie-Noël [2012] has proven the minimum energy to be

$$\Theta_0 \;:=\; \min_{p\in\mathbb{R}} \frac{\epsilon_0(p)}{b} \;=\; 0.590106125 \pm 10^{-9}\,. \tag{4.6}$$

The constant Θ_0 is called the *de Gennes constant* by some (e.g. Exner et al. [2018b]). For $\alpha > 0$ we will use the fact that the quadratic form of $\mathscr{H}_\alpha(p)$ is nonincreasing in α. Let $\beta > \alpha$, then:

$$\begin{aligned}
\Big(\varphi,\, \mathscr{H}_\alpha(p)\,\varphi\Big) &= -\int_{\mathbb{R}_+} \overline{\varphi}\varphi'' + \int_{\mathbb{R}_+} (bx+p)^2|\varphi|^2 \\
&= -\alpha\,|\varphi(0)|^2 + \Big\|\varphi'\Big\|_{L^2(\mathbb{R}_+)} + \Big\|\varphi\Big\|_{L^2_{(bx+p)^2}(\mathbb{R}_+)} \\
&\geq -\beta\,|\varphi(0)|^2 + \Big\|\varphi'\Big\|_{L^2(\mathbb{R}_+)} + \Big\|\varphi\Big\|_{L^2_{(bx+p)^2}(\mathbb{R}_+)} = \Big(\varphi,\, \mathscr{H}_\beta(p)\,\varphi\Big)\,.
\end{aligned} \tag{4.7}$$

Therefore the minimum energy for $\mathscr{H}_\alpha(p)$, $\alpha > 0$ will be less than $\Theta_0\, b$.

For $\alpha < 0$, on the other hand, the inequality (4.7) implies that the minimum energy will be greater than $\Theta_0\, b$. We will prove that it is still less than the lowest Landau level b. Searching for solutions of $F(\ldots) = 0$ along $\epsilon = b$, we get:

$$\begin{aligned}
0 &= F(\alpha,\, p,\, \nu = 0) \\
0 &= \Big(\alpha + p\Big)\, D_0(p\,\sqrt{\tfrac{2}{b}}) - \sqrt{2b}\, D_1(p\,\sqrt{\tfrac{2}{b}}) \\
0 &= \Big(\alpha + p\Big)\, 2^{-\frac{0}{2}}\, \mathrm{e}^{-\frac{w^2}{4}}\, H_0\Big(\frac{p}{\sqrt{b}}\Big) \;-\; \sqrt{2b}\, 2^{-\frac{1}{2}}\, \mathrm{e}^{-\frac{w^2}{4}}\, H_1\Big(\frac{p}{\sqrt{b}}\Big) \\
0 &= \Big(\alpha + p\Big)\, H_0\Big(\frac{p}{\sqrt{b}}\Big) \;-\; \sqrt{b}\, H_1\Big(\frac{p}{\sqrt{b}}\Big) \\
0 &= \Big(\alpha + p\Big) \;-\; \sqrt{b}\, 2\, \frac{p}{\sqrt{b}} \\
p &= \alpha
\end{aligned}$$

Clearly, the solution always exists. It is straightforward to check that the sign of F changes at $\alpha = p$, indicating that $\epsilon_0(p)$ actually *crosses* the lowest Landau level. And since $\epsilon_0(p \to +\infty) \to +\infty$, the only way it can cross $\epsilon = b$ and still approach it as $p \to -\infty$ is by approaching it *from below*. Thus, we have shown that $\min \epsilon_0(p) < b$.

Our estimate of the minimum energy for different values of α still leaves a lot to be desired. That is why we also computed a numerical approximation using *mpmath* (see Johansson et al. [2013]). A table of the computed values and the code used to compute them is available in Section A.3 of the Appendix.

We have all the information needed to characterize the spectrum of H_α using Theorem 5. Since neither of the energy functions is constant (they are analytic and grow indefinitely), the pure point spectrum of H_α is empty. Applying the result by Filonov and Sobolev [2006], we see that the singular continuous spectrum is empty too. What remains is the absolute continuous spectrum which goes from the energy minimum to infinity.

4.4 Summary

The following theorem summarizes the information we obtained about the spectrum of the Landau Hamiltonian in a half-plane with Robin boundary:

Theorem 38. *Let α, b and H_α be as in Definition 37. If $\alpha > 0$, then there exists $E \in (-\infty, \Theta_0)$ such that $\sigma(H_\alpha) = \sigma_{\mathrm{ac}}(H_\alpha) = [E\, b,\ \infty)$. If $\alpha < 0$, then there exists $E \in (\Theta_0, 1)$ such that $\sigma(H_\alpha) = \sigma_{\mathrm{ac}}(H_\alpha) = [E\, b,\ \infty)$. And finally, if $\alpha = 0$, then $\sigma(H_\alpha) = \sigma_{\mathrm{ac}}(H_\alpha) = [\Theta_0\, b,\ \infty)$. The value of Θ_0 is 0.590106125 ± 10^{-9}.*

Conclusion

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A. Appendix

A.1 Sobolev-type inequality

In this section we prove a Sobolev-type inequality analogical to the inequality (7.16) in Blank et al. [2008]. The presented proof is a simple modification of their proof.

Lemma 39. *Let $\psi \in W^{1,2}(\mathbb{R})$, then for every $a > 0$ there exists $b > 0$ such that*

$$\left\|\psi\right\|^n_{L^\infty(\mathbb{R})} \leq a \left\|\psi'\right\|^n_{L^2(\mathbb{R})} + b \left\|\psi\right\|^n_{L^2(\mathbb{R})} , \tag{A.1}$$

where $n \in \{1, 2\}$.

Proof. We define $\phi := \mathcal{F}\psi$. Since, by definition, $\psi' \in L^2(\mathbb{R})$ we have $\mathcal{F}\psi' = T_x \mathcal{F}\psi = T_x \phi \in L^2(\mathbb{R})$, where T_x means the operator of multiplication by the identity function $x \mapsto x$. Next, we utilize the Hölder inequality to find an estimate of $\|\phi\|_{L^1}$.

$$\left\|\phi\right\|_{L^1} = \left\|\tfrac{1}{x+\mathrm{i}}\,(x+\mathrm{i})\,\phi(x)\right\|_{L^1} \leq \underbrace{\left\|\tfrac{1}{x+\mathrm{i}}\right\|_{L^2}}_{=:\,c} \left\|(x+\mathrm{i})\,\phi(x)\right\|_{L^2} \leq c\left(\left\|T_x\,\phi\right\|_{L^2} + \left\|\phi\right\|_{L^2}\right) . \tag{A.2}$$

Since $\phi \in L^1$, it follows that $\psi \in L^\infty$ and

$$\left\|\psi\right\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \left\|\phi\right\|_{L^1} . \tag{A.3}$$

We introduce a scaled function $\phi_r(x) := r\,\phi(rx)$. Scaling changes the norms in the following way:

$$\left\|\phi_r\right\|^2_{L^2} = r \left\|\phi\right\|^2_{L^2} , \qquad \left\|T_x \phi_r\right\|^2_{L^2} = \frac{1}{r} \left\|T_x\right\|^2_{L^2} .$$

By substituting $\phi \mapsto \phi_r$ in (A.2) and combining with (A.3), we get

$$\left\|\psi\right\|_{L^\infty} \leq c\sqrt{r}\left\|T_x\,\phi\right\|_{L^2} + \frac{c}{\sqrt{r}}\left\|\phi\right\|_{L^2} = c\sqrt{r}\left\|\psi'\right\|_{L^2} + \frac{c}{\sqrt{r}}\left\|\psi\right\|_{L^2} .$$

Choosing $r = a^2/c^2$, we have proven the inequality (A.1) for $n = 1$.

To prove $n = 2$, we start with the case $n = 1$ and square both sides of the inequality.

$$\left\|\psi\right\|_{L^\infty} \leq a \left\|\psi'\right\|_{L^2} + b \left\|\psi\right\|_{L^2}$$

$$\left\|\psi\right\|^2_{L^\infty} \leq a \left\|\psi'\right\|^2_{L^2} + 2ab \left\|\psi'\right\|_{L^2} \left\|\psi\right\|_{L^2} + b \left\|\psi\right\|^2_{L^2} \leq 2a \left\|\psi'\right\|^2_{L^2} + 2b \left\|\psi\right\|^2_{L^2}$$

In the last step we have used the fact, that

$$0 \leq \left(a \left\|\psi'\right\|_{L^2} - b \left\|\psi\right\|_{L^2}\right)^2 \quad \Leftrightarrow \quad 2ab \left\|\psi'\right\|_{L^2} \left\|\psi\right\|_{L^2} \leq a \left\|\psi'\right\|^2_{L^2} + b \left\|\psi\right\|^2_{L^2} .$$

Finally, substituting $2a^2 \mapsto a$ and $2b^2 \mapsto b$, we have proven the inequality for $n = 2$. $\square$

A.2 Asymptotics of $D_\nu(w)$

In this section we will demonstrate the asymptotic behaviour of the parabolic function $D_\nu(w)$, which is needed for chapters 3 and 4.

The Digital Library of Mathematical Functions (see DLMF in Bibliography) lists several useful identities for functions $D_\nu(w)$, $U(a, z)$ and $V(a, z)$ – all three of these functions are called *parabolic cylinder functions*. Equations 12.2.5 and 12.2.15 in DLMF state:

$$D_\nu(w) = U(-\tfrac{1}{2} - \nu,\, w) \qquad \text{for } \nu, w \in \mathbb{C}\,, \tag{A.4}$$

$$U(a,\, -z) = -\sin(\pi\, a)\, U(a,\, z) + \frac{\pi}{\Gamma(\frac{1}{2} + a)}\, V(a,\, z) \qquad \text{for } a, z \in \mathbb{C}\,, \tag{A.5}$$

where $\Gamma(z)$ is the gamma function.

DLMF also lists expansions for large z. Let $a \in \mathbb{C}$ and $z = x\,\mathrm{e}^{\mathrm{i}\varphi}$ where $x \in \mathbb{R}$ and $\varphi \in (-\frac{3\pi}{4}, \frac{3\pi}{4})$, then:

$$U(a,\, z) \sim \mathrm{e}^{-\frac{1}{4}z^2}\, z^{-a-\frac{1}{2}} \sum_{s=0}^{\infty} (-1)^s \frac{(\frac{1}{2} + a)_{2s}}{s!(2z^2)^s} \qquad \text{as } x \to +\infty\,, \tag{A.6}$$

$$V(a,\, z) \sim \sqrt{\tfrac{2}{\pi}}\ \mathrm{e}^{+\frac{1}{4}z^2}\, z^{+a-\frac{1}{2}} \sum_{s=0}^{\infty} \frac{(\frac{1}{2} - a)_{2s}}{s!(2z^2)^s} \qquad \text{as } x \to +\infty\,, \tag{A.7}$$

where $(a)_s$ is the Pochhammer symbol. These two expansions are listed as 12.9.1 and 12.9.2.

Lemma 40.

$$\lim_{w \to -\infty} D_\nu(w) = 0 \qquad \textit{if and only if} \qquad \nu \in \mathbb{N}_0\,.$$

Proof. [1] By substituting (A.4) into (A.5), we get:

$$D_\nu(-w) = \underbrace{-\sin\left(\pi(-\tfrac{1}{2} - \nu)\right)}_{\cos(\pi\,\nu)} U(-\tfrac{1}{2} - \nu,\, w) + \frac{\pi}{\Gamma(-\nu)}\, V(-\tfrac{1}{2} - \nu,\, w)\,.$$

Now, we use the expansions (A.6) and (A.7) with $x = w,\ \varphi = 0$.

$$D_\nu(-w) = \underbrace{\cos(\pi\,\nu)\,\mathrm{e}^{-\frac{1}{4}w^2}\, w^\nu \Big(1 + O(w^{-2})\Big)}_{\to 0} + \frac{\sqrt{2\pi}}{\Gamma(-\nu)}\, \mathrm{e}^{+\frac{1}{4}w^2}\, w^{-1-\nu}\Big(1 + O(w^{-2})\Big),$$

where $O(...)$ is the big-O notation. As $w \to \infty$, the first summand decreases exponentially, while the second summand is zero for $\nu \in \mathbb{N}_0$ and increases exponentially for all other values of ν. □

[1] Thanks to Gergő Nemes for pointing this out to me, see `https://math.stackexchange.com/a/4193412/142487`

A.3 Minimal energies for the Robin boundary condition

In this section we list the numerically computed minimal energies for the Landau Hamiltonian in a half-plane with a Robin boundary.

$\alpha/\sqrt{b}$	$\epsilon/b \pm 0.01$
-2.0	0.99
-1.9	0.99
-1.8	0.99
-1.7	0.99
-1.6	0.99
-1.5	0.99
-1.4	0.98
-1.3	0.98
-1.2	0.97
-1.1	0.97
-1.0	0.96
-0.9	0.94
-0.8	0.93
-0.7	0.91
-0.6	0.88
-0.5	0.85
-0.4	0.81
-0.3	0.77
-0.2	0.72
-0.1	0.66
0.0	0.59
0.1	0.50
0.2	0.41
0.3	0.30
0.4	0.19
0.5	0.05
0.6	-0.09
0.7	-0.26
0.8	-0.43
0.9	-0.63
1.0	-0.84
1.1	-1.07
1.2	-1.31
1.3	-1.58
1.4	-1.86
1.5	-2.16
1.6	-2.48
1.7	-2.82
1.8	-3.18
1.9	-3.55

Now follows the algorithm which was used to compute the energies.

```
import decimal
from mpmath import mp
from itertools import chain

def float_range(start, stop, step):
  while start <= stop:
    yield float(start)
    start += decimal.Decimal(step)



# start searching in the most likely place
# then continue outward
def range_p(step_p: float):
    return chain(
        float_range(-1,  1, step_p),
        float_range( 1,  2, step_p),
        float_range(-2, -1, step_p),
    )


# after reaching this energy, abort
min_e = -5

# F(...) = 0 if e is an eigenvalue
def F(a, p, e) -> float:
    return (
        (a + p)
        * mp.pcfd((e-1)/2, p * mp.sqrt(2))
        -
        mp.sqrt(2)
        * mp.pcfd((e+1)/2, p * mp.sqrt(2))
    )

# checks whether e is is an eigenvalue for any p
def is_allowed_energy(a, e, step_p) -> float:
    for p in range_p(step_p):
        if F(a, p, e) >= 0:
            return True
    return False

# finds the lowest allowed e
def find_minimum_energy(
    a: float, start_e: float,
    step_p: float, step_e: float
) -> float:
    e = start_e
    while is_allowed_energy(a, e, step_p):
```

```
        e -= step_e
        if e < min_e:
            return float("nan")

    return e

# returns a tuple (a, e)
def boundary_minimum_energy_tuple(
    a: float, start_e: float,
    step_p: float, step_e: float
) -> tuple[float, float]:
    return (
        a,
        find_minimum_energy(a, start_e, step_p, step_e)
    )

# prints an array of estimates as CSV
def print_estimates(
    estimates: list[tuple[float, float]],
    prec: int = 1
):
    print('boundary, \t energy', flush=True)
    for (a, e) in estimates:
        print(
            "{0:.1f}, \t {1:.{p}f}".format(a, e, p=prec),
            flush=True
        )


# set up limits and steps
step_e = 0.1
step_p = 0.01
step_a = 0.1
min_a = -2
max_a = 2

estimates = [
    (a, e := find_minimum_energy(a, e, step_p, step_e))
    for a in float_range(min_a, max_a, step_a)
]

print_estimates(estimates)
```