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Magnetic Transport Along Translationally Invariant Obstacles

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Dedication.

Title: Magnetic Transport Along Translationally Invariant Obstacles

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Abstract: Abstract.

Keywords: two-dimensional quantum systems, magnetic field, non-local perturbations, translational invariance, spectral properties

Contents

List of symbols

$C^k(\Omega, \mathbb{K})$	The space of functions $\Omega \subseteq \mathbb{R} \to \mathbb{K}$ with k continuous derivatives.
$C_0^\infty(\Omega, \mathbb{K})$	The space of C^∞ functions with compact support in Ω.
$\mathrm{D}(T)$	The domain of operator T, usually dense in $\mathcal{H}$.
$D_\nu(w)$	The parabolic cylinder function.
$\mathcal{F}$	The Fourier-Plancherel operator on $L^2(\mathbb{R})$.
${}_1\mathrm{F}_1(\alpha, \beta; z)$	The confluent hypergeometric function of the first kind.
$\mathcal{H}$	A separable Hilbert space.
$H,\ \mathscr{H}(\xi)$	A Hamiltonian operator; a fibre of the Hamiltonian.
$H_n(x)$	The n-th Hermite polynomial.
$L^p(M, \mathrm{d}\mu, V)$	The space of p-integrable functions from measure space (M, μ) to vector space V. Specifically for $p = 2$, a Hilbert space with inner product $(\psi, \phi)_{L^2} = \int_M (\psi, \phi)_V \,\mathrm{d}\mu$.
$L^p(\Omega)$	As above, but $M = \Omega \subseteq \mathbb{R}^N$, μ is the Lebesgue measure and $V = \mathbb{C}$.
$L^1_{\mathrm{loc}}(\Omega)$	The space of functions that are $L^1(K)$ for every compact $K \subset \Omega$.
$\mathbb{N},\ \mathbb{N}_0$	The set of positive integers; the set of non-negative integers.
$\vec{P}, P_x, P_y, P_z$	Momentum operator – a self-adjoint operator, such that $P_x\, f(x, ...) = -\mathrm{i}\frac{\partial}{\partial x} f(x, ...)$.
$\vec{Q}, Q_x, Q_y, Q_z$	Position operator – a self-adjoint operator, such that $Q_x\, f(x, ...) = x\, f(x, ...)$.
$W^{k,p}(\Omega)$	The Sobolev space – the space of integrable functions f, such that $f^{(\alpha)} \in L^p(\Omega)$, where α is a multi-index and $\lvert\alpha\rvert \le k$.
$\Gamma(z)$	The gamma function.
μ	A σ-finite measure, usually the Lebesgue measure.
$\sigma(T), \sigma_{\mathrm{p}}(T), \sigma_{\mathrm{ac}}(T), \sigma_{\mathrm{sc}}(T), \sigma_{\mathrm{disc}}(T), \sigma_{\mathrm{ess}}(T)$	The spectrum of normal operator T; the point, absolutely continuous, singular continuous, discrete, essential spectrum of T. $\sigma(T) = \sigma_{\mathrm{p}} \cup \sigma_{\mathrm{ac}} \cup \sigma_{\mathrm{sc}} = \sigma_{\mathrm{disc}} \cup \sigma_{\mathrm{ess}}$.
$\nabla,\ \nabla\times,\ \Delta$	Gradient, rotation, Laplace operator.
$\Delta^\Omega_{\mathrm{D}}, \Delta^\Omega_{\mathrm{D},A}$	The Dirichlet Laplacian, defined on functions from $L^2(\Omega)$ with a Dirichlet boundary condition; a “magnetic” Dirichlet Laplacian given by the vector potential A.

Introduction

1. Formulation & known results

In this chapter we will explain what is the magnetic transport, give a precise mathematical formulation of the problem and restate the known results.

1.1 The magnetic Hamiltonian

The simplest example of a quantum system with a magnetic field is the system consisting of a single charged particle inside a constant homogeneous magnetic field and zero scalar potential. The Hamiltonian that corresponds to this system is:

$$H = (\vec{P} + \vec{A})^2 \,, \quad \vec{B} = \nabla \times \vec{A} = (0, 0, b_0) \,.$$

Here $\vec{P} = -\mathrm{i}\nabla$ is the momentum operator, $\vec{B}$ is the magnetic field (which is constant with magnitude b_0) and $\vec{A}$ is its corresponding vector potential. Notice that we have used nondimensionalization to remove physical units from the Hamiltonian. The spectrum of H is absolutely continuous and the Hamiltonian commutes with P_z, thus it allows the particle to move freely along the z-axis. However, if we restrict the particle to the layer $z = 0$, either physically, or only formally because we are not interested in the movement along z **[EXPAND]**, we get a two-dimensional Hamiltonian with infinitely degenerate discrete spectrum, the so-called Landau Hamiltonian:

$$H = (P_x + A_x)^2 + (P_y + A_y)^2 \,.$$

A detailed analysis of this well-known Hamiltonian can be found e.g. in §112 of Landau and Lifshitz [1981]. The pure point spectrum means that the particle is not free to move along x or y, but instead it is "trapped" in some superposition of stationary states. We will investigate perturbations to the Landau Hamiltonian, which cause its spectrum to become continuous and allow the particle to move freely along the y-axis. These perturbations can be either in the form of a scalar potential, a modification of the magnetic field, or a purely geometric deformation of the layer to which our particle is constrained. We will require all of these perturbations to be translationally invariant, thus constant along one axis – without loss of generality, we choose that they are independent of y and only depend on x.

Throughout this thesis, we will use the Landau gauge:

$$\begin{aligned} A_x &= 0 \,, \\ A_y &= \int_0^x B_z(x') \, \mathrm{d}x' \,, \\ A_z &= 0 \,. \end{aligned}$$

Now we can specify precisely which Hamiltonians we will investigate.

Definition 1 (Potential perturbation)**.** *Let $\Omega \subseteq \mathbb{R}^2$, $\mathcal{H} = L^2(\Omega)$, $\mathrm{D}(H)$ a dense subset of $\mathcal{H}$, $b > 0$ and $V \in L^1_{loc}(\mathbb{R})$. A self-adjoint operator $H : \mathrm{D}(H) \to \mathcal{H}$ given by the equation*

$$H = P_x^2 + \left(P_y + b\, Q_x\right)^2 + V(x) \,,$$

*is called the **Landau Hamiltonian with a potential perturbation**. We will investigate, which choices of* $\mathrm{D}(H)$ *and* V *lead to* $\sigma_{\mathrm{ess}}(H) \neq \emptyset$. *The domain* $\mathrm{D}(H)$ *is determined not only by the asymptotic behaviour of* V, *but also by the boundary conditions imposed on the wave function.*

Definition 2 (Magnetic perturbation)**.** *Let* $b \in C^\infty(\mathbb{R})$, $\mathcal{H} = L^2(\mathbb{R}^2)$ *and* $\mathcal{D} = C_0^\infty(\mathbb{R}^2)$ *be the set of* $\mathbb{C}^\infty$ *functions with compact support. Let* A_y *be a multiplication operator on* $\mathcal{H}$ *given by:*

$$A_y\, \psi(x,y) = \Big(\int_0^x b(x')\, \mathrm{d}x' \Big)\, \psi(x,y)$$

Let $\tilde{H} : \mathcal{D} \to \mathcal{H}$ *be an essentially self-adjoint operator given by the equation:*

$$\tilde{H} = P_x^2 + \big(P_y + A_y\big)^2\,,$$

Its closure H *is called the **Landau Hamiltonian with a magnetic perturbation**. We will investigate, which choices of* b *lead to* $\sigma_{\mathrm{ess}}(H) \neq \emptyset$.

Definition 3 (Geometric perturbation, transl. inv. layer)**.** *Let* $b > 0$, *and* $\ell \in \mathbb{R}$. *Let* $\omega : \mathbb{R} \to \mathbb{R}^2$ *be a* C^4*-smooth curve. We define a set* $\Omega' \subset \mathbb{R}^2$ *by*

$$\Omega' = \Big\{\ P \in \mathbb{R}^2 \ \mid\ \exists s \in \mathbb{R}\ \Big\| \omega(s) - P \Big\| \leq \ell\ \Big\}\,,$$

this gives a band of width 2ℓ *around the curve* ω. *Then we define a set* $\Omega \subset \mathbb{R}^3$ *as*

$$\Omega = \Big\{\ (x,y,z) \in \mathbb{R}^3\ \mid\ (x,z) \in \Omega'\ \Big\}\,.$$

We shall call Ω *a **translationally invariant layer of width*** 2ℓ ***given by the curve*** ω. *Let us now consider the magnetic Dirichlet Laplacian*

$$\Delta^{\Omega}_{\mathrm{D},A}\psi(x,y,z) = \Delta\psi + 2\mathrm{i}b\, x\, \frac{\partial \psi}{\partial y} - b^2 x^2\, \psi$$

defined on functions $\psi \in C^\infty(\Omega)$, *such that* $\psi(x,y,z) = 0$ *on the boundary of* Ω. *The operator* H *which is the closure of* $-\Delta^{\Omega}_{\mathrm{D},A}$ *in* $L^2(\Omega)$ *is called the **Landau Hamiltonian with a geometric perturbation**. We will investigate, which choices of* ω *lead to* $\sigma_{\mathrm{ess}}(H) \neq \emptyset$.

1.2 Direct integral

The key insight to all three of these problems is that the Hamiltonians in question only depend on the momentum p_y of the particle, and not on its position y. If we were to fix p_y of the particle to a certain value somehow, we could reduce the problem to a one-dimensional operator and solve for each p_y separately. This vague idea can be given a rigorous meaning in terms of the *direct integral*.

The following definition is a rephrasing of definitions given in Reed and Simon [1978], pages 280 and 281.

Definition 4 (Direct integral, fibre)**.** *Let $\mathcal{H}'$ be a separable Hilbert space and (M,μ) a measure space. We define a Hilbert space $\mathcal{H}$, which is the space of all square-integrable functions from M to $\mathcal{H}'$:*

$$\mathcal{H} = L^2(M,\, \mathrm{d}\mu\,, \mathcal{H}')\,.$$

Let $\mathscr{A}$ be a measurable function from M to the self-adjoint operators on $\mathcal{H}'$. Let $f_\psi : M \to \mathbb{R}$ be a function defined by

$$f_\psi(s) = \left\|\mathscr{A}(s)\psi(s)\right\|_{\mathcal{H}'} \quad \textit{for all } \psi \in \mathcal{H}, s \in M \textit{ such that } \psi(s) \in \mathrm{D}(\mathscr{A}(s))\,.$$

We define an operator A on $\mathcal{H}$ by:

$$\begin{aligned}(A\psi)(s) &= \mathscr{A}(s)\,\psi(s)\,,\\ \mathrm{D}(A) &= \left\{\psi \in \mathcal{H} \;\middle|\; \psi(s) \in \mathrm{D}(\mathscr{A}(s)) \textit{ a.e.} \wedge \left\|f_\psi\right\|_{L^2} < \infty\right\}.\end{aligned}$$

Then we shall write

$$\mathcal{H} = \int_M^\oplus \mathcal{H}', \qquad A = \int_M^\oplus \mathscr{A}(s)\,\mathrm{d}s\,.$$

*We shall call $\mathcal{H}$ and A **the direct integral** of $\mathcal{H}'$ and $\mathscr{A}$, respectively. Reversely, we shall call $\mathcal{H}'$ a **fibre space** of $\mathcal{H}$ and $\mathscr{A}(s)$ a **fibre** of A.*

The concept of a direct integral might initially seem strange to readers who encounter it for their first time. These readers may find it helpful to think of the direct integral as a simple *"rebranding"* of several concepts they already know and understand. For example, a free spin-$\frac{1}{2}$ particle is represented in the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^2)$ of square-integrable functions from the physical space $\mathbb{R}^3$ to the qubit $\mathbb{C}^2$. This space is by definition the direct integral $L^2(\mathbb{R}^3, \mathbb{C}^2) = \int_{\mathbb{R}^3}^\oplus \mathbb{C}^2$, the qubit plays the role of the fibre space here. Another example is related to the fact that a function of two variables can be understood as a function of one variable which returns another function of one variable (programmers call this *currying*). That is exactly the meaning of this direct integral: $L^2(M \times N) \simeq \int_M^\oplus L^2(N) \simeq \int_N^\oplus L^2(M)$.

Before we apply the theory of direct integrals to the magnetic Hamiltonian, let us remind the Fourier-Plancherel operator. It is a standard textbook result (see Blank et al. [2008]) that if we take the Fourier transform as an operator on $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$, its closure is a unitary operator on $L^2(\mathbb{R})$. This operator is called the Fourier-Plancherel operator $\mathcal{F}$, it transforms momentum to position $\mathcal{F}P\mathcal{F}^{-1} = Q$, and as an isomorphism, it does not change the spectrum of self-adjoint operators:

$$\sigma(A) = \sigma(\mathcal{F}A\mathcal{F}^{-1})\,, \quad \sigma_{\mathrm{disc}}(A) = \sigma_{\mathrm{disc}}(\mathcal{F}A\mathcal{F}^{-1})\,, \quad \sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(\mathcal{F}A\mathcal{F}^{-1})\,.$$

The theory given so far regards functions of one variable. In this thesis, we will perform a *partial* Fourier transformations on multivariate functions – that is, perform the Fourier transformation on one variable whilst keeping the other variables fixed. We will use a subscript to indicate which variable is being transformed, for example $\mathcal{F}_y : \psi(x,y) \mapsto \tilde{\psi}(x,\xi)$.

Now, we can show, how to express a Landau Hamiltonian with potential and magnetic perturbations in terms of the direct integral:

$$
\begin{aligned}
H &= \left(\vec{P} + \vec{A}(x)\right)^2 + V(x) \\
&= P_x^2 + (P_y + A_y(x))^2 + V(x) \\
&\simeq \mathcal{F}_y\left(P_x^2 + (P_y + A_y(x))^2 + V(x)\right)\mathcal{F}_y^{-1} \\
&= P_x^2 + (Q_y + A_y(x))^2 + V(x) \\
&= \int_{\mathbb{R}}^{\oplus} \underbrace{{P_x}^2 + (p + A_y(x))^2 + V(x)}_{\mathscr{H}(p)} \, \mathrm{d}p
\end{aligned}
\tag{1.1}
$$

Where for every $p \in \mathbb{R}$, $\mathscr{H}(p)$ is a self-adjoint operator on $L^2(\mathbb{R})$. The physical meaning of the parameter p is the particle's momentum in the y direction and $\mathscr{H}(p)$ is the Hamiltonian for a particle with a fixed y-momentum. The following theorem is a weakened version of Theorem XIII.85 of Reed and Simon [1978].

Theorem 1 (Spectrum of direct integral)**.** *Let $\lambda \in \mathbb{C}$ and $A = \int_M^{\oplus} \mathscr{A}(s)\,\mathrm{d}s$, as in the previous definition. We define $\Gamma(\lambda)$ as the set of all s, such that λ is an eigenvalue of $\mathscr{A}(s)$, and $\Omega_\varepsilon(\lambda)$ as the set of all s, such that the ε-neighbourhood of λ intersects the spectrum of $\mathscr{A}(s)$ – written symbolically:*

$$
\begin{aligned}
\Gamma(\lambda) &= \left\{\ s \ \mid\ \lambda \textit{ is an eigenvalue of } \mathscr{A}(s)\ \right\}, \\
\Omega_\varepsilon(\lambda) &= \left\{\ s \ \mid\ \sigma(\mathscr{A}(s)) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset\ \right\}.
\end{aligned}
$$

Then λ belongs to the spectrum of A if and only if

$$\mu(\ \Omega_\varepsilon(\lambda)\ ) > 0 \qquad \textit{for all } \varepsilon > 0\,.$$

Additionally, λ is an eigenvalue of A if and only if

$$\mu(\ \Gamma(\lambda)\ ) > 0\,.$$

This means that we can deduce the spectrum of the Hamiltonian H simply by investigating how the spectrum of its fibre $\mathscr{H}(p)$ depends on p. Furthermore, the spectrum of $\mathscr{H}(p)$ typically consists of simple eigenvalues which are particularly convenient to work with.

1.3 Refresher on linear operators

Before we start investigating specific Hamiltonians, let us remind a few textbook theorems regarding self-adjointness and spectral properties of linear operators, which will be useful later. The following definition and the subsequent theorem are from the chapter 4.7 in Blank et al. [2008].

Definition 5 (Deficiency indices)**.** *Let T be a linear operator on $\mathcal{H}$. We define two numbers n_+, $n_- \in \mathbb{N}_0 \cup \{\infty\}$ as follows:*

$$n_\pm(T) = \dim\ \mathrm{Ker}\ (T^* \pm \mathrm{i}I)\,,$$

*where I is the identity operator on $\mathcal{H}$. We call these numbers the **deficiency indices** of T.*

Theorem 2 (Deficiency indices and self-adjoint extensions). *Let T be a closed symmetric operator on $\mathcal{H}$, such that*

$$n_+(T) = n_-(T) < \infty \, .$$

Then all maximal extensions of T are self-adjoint. Furthermore, if $n_\pm = 0$, then T is already self-adjoint.

The following theorem is given in Weidmann [1980] as Theorem 8.18.

Theorem 3 (Spectrum of self-adjoint extensions). *Let T be a closed symmetric operator on $\mathcal{H}$, such that*

$$n_+(T) = n_-(T) < \infty \, .$$

Then the essential spectrum of every self-adjoint extension of T is the same. In particular, if one self-adjoint extension of T has a pure discrete spectrum, all of them do.

1.4 Potential perturbation

REWORK THIS SECTION. A well-studied case of potential barriers is the Hall effect, where a particle is confined to a strip or semi-plane by electrostatic potential (Combes, 2001 ?). The Hall effect on a plane with two different electrostatic potentials, one on each half of the plane, was studied in (Combes, 2005).

1.5 Magnetic perturbation

REWORK THIS SECTION. The case of non-local perturbations (i.e. those which don't disappear at infinity) of the magnetic field were studied by (Iwatsuka, 1985).

1.6 Geometric perturbation

REWORK THIS SECTION. A tilted planar layer of fixed width, as well as more general thin layers with translationally invariant bends were studied in (Exner, 2018) and some sufficient conditions for the continuity of spectrum were given.

2. Delta potential

In this chapter we will examine the Landau Hamiltonian with a potential perturbation (see definition 1), formally given by the potential $V(x) = \alpha\,\delta_{x_0}$, i.e. the Dirac delta in $x = x_0$, scaled by $\alpha \in \mathbb{R}$. Since such a potential is a distribution and not a locally integrable function, the Hamiltonian is rigorously defined as:

$$\Big(H_\alpha \psi\Big)(x,y) = \left(-\frac{\partial^2}{\partial x^2} + \Big(\mathrm{i}\frac{\partial}{\partial y} + bx\Big)^2\right)\ \psi(x,y) \quad \text{a.e.}^1\text{on } (\mathbb{R}\setminus\{x_0\})\times\mathbb{R}$$

with a domain given by the conditions

$$\psi \in W^{1,2}(\mathbb{R}^2)\ \cap\ W^{2,2}\Big(\,(\mathbb{R}\setminus\{x_0\})\times\mathbb{R}\,\Big)\,,$$

$$\lim_{x\to x_0+}\psi'(x,y)\ -\ \lim_{x\to x_0-}\psi'(x,y) = \alpha \lim_{x\to x_0}\psi(x,y) \quad \text{for a.e. } y\,,^2$$

$$\int_{\mathbb{R}^2} x^2\,\Big|\,\psi(x,y)\,\Big|^2\ \mathrm{d}x\ \mathrm{d}y\ < \infty\,.$$

Then, by an approach equivalent to that in (1.1), one can show that H_α is isomorphic to a direct integral:

$$H_\alpha \simeq \int_{\mathbb{R}}^{\oplus} \mathscr{H}_\alpha(p)\,\mathrm{d}p\ ,$$

where $\mathscr{H}_\alpha(p)$ is a fibre Hamiltonian satisfying very similar conditions to those of H_α, that is, for almost every $p \in \mathbb{R}$:

$$\Big(\mathscr{H}_\alpha(p)\,\varphi\Big)(x) = -\varphi''(x) + \Big(b\,x + p\Big)^2\,\varphi(x)\,, \tag{2.1}$$

$$\varphi \in W^{1,2}(\mathbb{R})\ \cap\ W^{2,2}(\mathbb{R}\setminus\{x_0\})\,,$$

$$\lim_{x\to x_0+}\varphi'(x) - \lim_{x\to x_0-}\varphi'(x) = \alpha\ \lim_{x\to x_0}\varphi(x),$$

$$\int_{\mathbb{R}} x^2\,|\varphi(x)|^2\,\mathrm{d}x\ < \infty\,.$$

Before we start investigating the spectrum, we need to show that the problem is well-posed, i.e. that the Hamiltonian H_α is self-adjoint and bounded from below. Then we will show that the spectrum of $\mathscr{H}_\alpha(p)$ is discrete for every p, and only after that we will investigate the continuity of the spectrum of H_α.

[1] The pointwise equality is to be understood *almost everywhere* with respect to the Lebesgue measure on $\mathbb{R}^2$.

[2] The equality holds for *almost every y* and $\lim_{x\to x_0}$ means the *essential* limit with respect to the Lebesgue measure on $\mathbb{R}$.

2.1 Well-posedness

It is straightforward to check that the fibre Hamiltonian is bounded from below:

$$\begin{aligned}(\varphi,\, \mathscr{H}_\alpha(p)\,\varphi) &= \int_{\mathbb{R}} \overline{\varphi}(x)\Big(-\varphi''(x) + (bx+p)^2\varphi(x)\Big)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}} \overline{\varphi}\varphi'' + \int_{\mathbb{R}} (bx+p)^2 \,\Big|\varphi(x)\Big|^2\,\mathrm{d}x \\ &\geq -\int_{\mathbb{R}} \overline{\varphi}\varphi'' = \int_{\mathbb{R}} \overline{\varphi}'\varphi' - [\overline{\varphi}\varphi']_{-\infty}^{x_0} - [\overline{\varphi}\varphi']_{x_0}^{\infty} \\ &= \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R})} + \overline{\varphi}(x_0)\Big(\varphi'(x_0+) - \varphi'(x_0-)\Big) \\ &= \Big\|\varphi'\Big\|^2_{L^2(\mathbb{R})} + \alpha\Big|\varphi(x_0)\Big|^2\end{aligned}$$

In the last two steps we have used the fact that for $\varphi \in W^{2,2}$ both φ and φ' vanish at infinity, and that $\varphi(x_0+) - \varphi(x_0-) = \alpha\varphi(x_0)$. In case when $\alpha \geq 0$, the right-hand side is non-negative, therefore we can use zero as the lower bound. For $\alpha < 0$ we estimate $|\varphi(x_0)|^2 \leq \|\varphi\|^2_{L^\infty}$ and then use the Sobolev-type inequality

$$\forall a>0\ \exists b>0:\ \Big\|\varphi\Big\|^2_{L^\infty} \leq a\Big\|\varphi'\Big\|^2_{L^2} + b\Big\|\varphi\Big\|^2_{L^2},$$

the proof of which is given in the chapter A.1 of the appendix.

$$\begin{aligned}(\varphi,\, \mathscr{H}_\alpha(p)\,\varphi) &\geq \Big\|\varphi'\Big\|^2_{L^2} + \alpha\Big|\varphi(x_0)\Big|^2 \\ &\geq \Big\|\varphi'\Big\|^2_{L^2} + \alpha\Big\|\varphi\Big\|^2_{L^\infty} \\ &\geq \Big\|\varphi'\Big\|^2_{L^2} + \alpha\Big(a\Big\|\varphi'\Big\|^2_{L^2} + b\Big\|\varphi\Big\|^2_{L^2}\Big) \\ &= \Big(1+\alpha a\Big)\Big\|\varphi'\Big\|^2_{L^2} + \alpha b\Big\|\varphi\Big\|^2_{L^2}\end{aligned}$$

By choosing $a \leq |\alpha|^{-1}$, we get

$$(\varphi,\, \mathscr{H}_\alpha(p)\,\varphi) \leq \alpha b\Big\|\varphi\Big\|^2_{L^2(\mathbb{R})},$$

thus we have shown that the fibre Hamiltonian $\mathscr{H}_\alpha(p)$ is bounded from below. And because the bound is independent of p, it is also a lower bound for H_α:

$$\begin{aligned}\Big(\psi,\, H_\alpha\psi\Big)_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}} \Big(\tilde{\psi}(\cdot,p),\, \mathscr{H}_\alpha(p)\,\tilde{\psi}(\cdot,p)\Big)_{L^2(\mathbb{R})}\mathrm{d}p \\ &\leq \int_{\mathbb{R}} \alpha b\Big\|\tilde{\psi}(\cdot,p)\Big\|^2_{L^2(\mathbb{R})}\mathrm{d}p = \alpha b\Big\|\psi\Big\|^2_{L^2(\mathbb{R}^2)},\quad \text{where } \tilde{\psi} = \mathcal{F}_y\psi\,.\end{aligned}$$

Now we will show that the fibre Hamiltonian is self-adjoint. Let $\varphi \in \mathrm{D}(\mathscr{H}_\alpha(p))$ and ψ from a yet-unknown subset of $\mathcal{H}$.

$$\begin{aligned}\Big(\mathscr{H}_\alpha(p)\,\varphi,\, \psi\Big) &= \int_{\mathbb{R}} -\overline{\varphi}''\psi + \int_{\mathbb{R}}\Big(bx+p\Big)^2\overline{\varphi}\,\psi \qquad (2.2) \\ &= \Big[-\overline{\varphi}'\psi + \overline{\varphi}\psi'\Big]_{-\infty}^{x_0} + \Big[\overline{\varphi}'\psi - \overline{\varphi}\psi'\Big]_{x_0}^{+\infty} + \int_{\mathbb{R}} -\overline{\varphi}\,\psi'' + \int_{\mathbb{R}}\Big(bx+p\Big)^2 \\ &= -\overline{\varphi}'(x_0-)\psi(x_0-) + \overline{\varphi}(x_0)\psi'(x_0-) \\ &\quad + \overline{\varphi}'(x_0+)\psi(x_0+) - \overline{\varphi}(x_0)\psi'(x_0+) + \int_{\mathbb{R}}\overline{\varphi}\Big(\psi'' + \int_{\mathbb{R}}\Big(bx+p\Big)^2\psi\Big)\,.\end{aligned}$$

This whole expression has to be equal to (φ, χ) for some $\chi \in \mathcal{H}$ independent of φ. At the second line, we performed an integration by parts, already assuming $\psi \in W^{2,2}(\mathbb{R} \setminus \{x_0\})$. If there was another isolated point $c \in \mathbb{R}$ where ψ weren't twice weakly differentiable, we would get terms $\overline{\varphi}'(c)\Big(\psi(c-)-\psi(c+)\Big)$ and $\overline{\varphi}(c)\Big(\psi'(c-) - \psi'(c+)\Big)$ which can't be independent of φ unless $\psi(c-) = \psi(c+)$ and $\psi'(c-) = \psi'(c+)$. However, this would make ψ twice weakly differentiable at c, hence a contradiction. At the third line we simply evaluated the square brackets, making use of the fact that $W^{2,2}$ functions (and their derivatives) vanish at infinity. In order for the entire expression to be independent of φ, the following equation must hold:

$$-\overline{\varphi}'(x_0-)\psi(x_0-) + \overline{\varphi}(x_0)\psi'(x_0-) + \overline{\varphi}'(x_0+)\psi(x_0+) - \overline{\varphi}(x_0)\psi'(x_0+) = 0\,.$$

Substituting $\varphi'(x_0+) - \varphi'(x_0-) = \alpha\varphi(x_0)$ and solving for all φ, we get that $\psi(x_0+) = \psi(x_0-)$ and $\psi'(x_0+) - \psi'(x_0-) = \alpha\psi(x_0+)$. Therefore, ψ must be from $\mathrm{D}(\mathscr{H}_\alpha(p))$ and $\chi = \mathscr{H}_\alpha(p)\psi$. We have shown that $\mathscr{H}_\alpha(p)$ is self-adjoint. And because the direct integral of a self-adjoint operator is also a self-adjoint operator, H_α is also self-adjoint.

Finally, we will show that the fibre Hamiltonian has a discrete spectrum. The family of operators $\{\mathscr{H}_\alpha(p) \,|\, \alpha \in \mathbb{R}\}$ has a common symmetric restriction:

$$\Omega := \Big\{\varphi \in W^{2,2}(\mathbb{R}) \;\Big|\; x^2\,\varphi(x) \in L^2(\mathbb{R}),\ \varphi(x_0) = 0\Big\}\,, \quad \mathscr{H}_\alpha(p)|_\Omega \text{ is symmetric.}$$

Since fibres $\mathscr{H}_\alpha(p)$ for various values of α only differ in the boundary conditions, the restriction to Ω gives just one operator $h(p) := \mathscr{H}_\alpha(p)|_\Omega$, independent of α. The operator $h(p)$ is closed, we can show it directly from the definition: let $\{\varphi_n\} \subset \Omega$ such that $\varphi_n \to \varphi \in L^2(\mathbb{R})$, then

$$\begin{aligned}
&\lim_{n\to\infty} h(p)\,\varphi_n = \lim_{n\to\infty}\Big(-\varphi_n'' + (bx+p)^2\varphi_n\Big) \in L^2 \\
&\quad\iff\quad \lim_{n\to\infty}\varphi_n'' \in L^2 \,\wedge\, \lim_{n\to\infty} x^2\varphi_n \in L^2 \quad\iff\quad \varphi'' \in L^2 \,\wedge\, x^2\varphi \in L^2\,.
\end{aligned}$$

Furthermore, there is no way for $\varphi(x_0) \neq \varphi_n(x_0) \equiv 0$ without causing $\varphi_n''(x_0)$ to diverge. Therefore, $h(p)\,\varphi_n \to \psi \implies \varphi \in \Omega$. Finally, the requirement $h(p)\,\varphi = \psi$ follows from the fact that both second derivative and multiplication by x^2 are closed operators on their respective domains.

We have shown that $h(p)$ is a closed symmetric operator with many different extensions $\mathscr{H}_\alpha(p)$. We know that at least one of the extensions, the fibre $\mathscr{H}_{\alpha=0}(p)$ of the unperturbed system, has a discrete spectrum. Now, we want to use the theorem 3 to show that the spectrum of all $\mathscr{H}_\alpha(p)$ is discrete. The last premise left to demonstrate is the fact that $n_+(h(p)) = n_-(h(p)) < \infty$. Since the deficiency indices (definition 5) are equal to the multiplicity of $\pm\mathrm{i}$ as an eigenvalue of $h(p)^*$, we first need to find the operator $h(p)^*$. Let $\varphi \in \Omega$ and ψ from a yet-unknown set $\Omega' \subset \mathcal{H}$.

$$\begin{aligned}
\Big(h(p)\varphi,\,\psi\Big) &= \int_\mathbb{R} -\overline{\varphi}''\psi + \int_\mathbb{R}\Big(bx+p\Big)^2\overline{\varphi}\psi = \int_\mathbb{R}\overline{\varphi}\,\overbrace{\Big(\psi'' + \int_\mathbb{R}\Big(bx+p\Big)^2\psi\Big)}^{=:\,h(p)^*\psi} + \\
&\quad + \underbrace{\overline{\varphi}(x_0)}_{0}\,\psi'(x_0-) - \underbrace{\overline{\varphi}(x_0)}_{0}\,\psi'(x_0+) + \overline{\varphi}'(x_0)\Big(\psi(x_0+) - \psi(x_0-)\Big)\,.
\end{aligned}$$

We performed an integration by parts, assuming $\Omega' \subseteq W^{2,2}(\mathbb{R} \setminus \{x_0\})$, which can be justified by an argument analogical to that under equation (2.2). Since $\overline{\varphi}'(x_0)$ can take on any value, it must hold that $\psi(x_0+) - \psi(x_0-) = 0$. However, there is no constraint on the values of ψ'. Therefore, the domain of $h(p)^*$ is:

$$\Omega' := \left\{ \varphi \in W^{1,2}(\mathbb{R}) \cap W^{2,2}(\mathbb{R} \setminus \{x_0\}) \mid x^2 \varphi(x) \in L^2(\mathbb{R}) \right\} .$$

The deficiency indices $n_\pm$ are then equal to the number of linearly independent solutions of the ordinary differential equation:

$$h(p)^* \varphi = \pm \mathrm{i} \quad \Longleftrightarrow \quad \varphi''(x) = \Big((bx + p)^2 \mp \mathrm{i} \Big)^2 \varphi(x) \quad \text{where } \varphi \in \Omega' . \tag{2.3}$$

In the next section, we will investigate how the eigenvalues of $\mathscr{H}_\alpha(p)$ depend on p and α.

2.2 Eigenproblem of the fibre Hamiltonian

In order to use utilize the theorem 1 we need to find the eigenvalues of the fibre Hamiltonian for each p. That is, we are looking for a real analytic function $\epsilon(p)$, such that for all $p \in \mathbb{R}$ there exists a $\varphi \in \mathrm{D}(\mathscr{H}_\alpha(p))$ satisfying

$$\mathscr{H}_\alpha(p)\, \varphi = \epsilon(p)\, \varphi .$$

As a shorthand, we will often denote $\epsilon(p)$ simply as ϵ. Substituting from (2.1), we get an ordinary differential equation:

$$\begin{gathered} -\varphi''(x) + \Big(b^2\, x^2 + 2pb\, x + p^2 \Big)\, \varphi(x) = \epsilon\, \varphi(x) \quad \text{on } x \neq x_0 , \\ \varphi'(x_0+) - \varphi'(x_0-) = \alpha\, \varphi(x_0) . \end{gathered}$$

From now on, we shall suppose that $b > 0$; for $b < 0$ one can perform a reflection $x \mapsto -x$ and arrive at the same results. In order to refine this differential equation into the standard form, we change variables $x \mapsto w$ and instead of one function φ on $\mathbb{R}$ we introduce two functions h_-, h_+ on the left and right half-line respectively:

$$w := \sqrt{2b} \left(x + \frac{p}{b} \right) , \qquad w_0 := \sqrt{2b} \left(x_0 + \frac{p}{b} \right) , \qquad \nu := \frac{\epsilon - b}{2b} , \tag{2.4}$$

$$h_- : (-\infty, w_0] \to \mathbb{C} , \qquad h_+ : [w_0, +\infty) \to \mathbb{C} ,$$

$$\varphi(x) = \begin{cases} h_+\Big(\sqrt{2b}\, (x + \frac{p}{b}) \Big) & \text{for } x \geq x_0 , \\ h_-\Big(\sqrt{2b}\, (x + \frac{p}{b}) \Big) & \text{for } x < x_0 . \end{cases}$$

Then we arrive at the so-called parabolic cylinder differential equation:

$$h_\pm''(w) = \left(\frac{1}{4} w^2 - \nu - \frac{1}{2} \right) h_\pm(w) , \tag{2.5}$$

The two functions are then *"glued together"* by the following equations:

$$\begin{aligned} h_+(w_0) - h_-(w_0) &= 0 , \\ h_+'(w_0) - h_-'(w_0) &= \alpha \sqrt{2b}\, h_+(w_0) . \end{aligned} \tag{2.6}$$

As stated in Gradshteyn and Ryzhik [2014], the solutions to (2.5) can be expressed as a linear combination of the functions

$$D_\nu(w)\,,\quad D_\nu(-w)\,,\quad D_{-\nu-1}(\mathrm{i}w)\,,\quad D_{-\nu-1}(-\mathrm{i}w)\,, \tag{2.7}$$

where D_ν is a so-called *parabolic cylinder function*, which is a special function that can be expressed in terms of the gamma function Γ and the confluent hypergeometric function ${}_1\mathrm{F}_1$:

$$D_\nu(w) = 2^{\frac{\nu}{2}} \exp\left(-\tfrac{w^2}{4}\right) \left(\frac{\sqrt{\pi}}{\Gamma\left(\frac{1-\nu}{2}\right)}\, {}_1\mathrm{F}_1\Big(-\frac{\nu}{2}, \frac{1}{2}; \frac{w^2}{2}\Big) - \frac{w\sqrt{2\pi}}{\Gamma\left(-\frac{\nu}{2}\right)}\, {}_1\mathrm{F}_1\Big(\frac{1-\nu}{2}, \frac{3}{2}; \frac{w^2}{2}\Big)\right). \tag{2.8}$$

Since $1/\Gamma(z)$ is an entire function and $(\alpha, z) \mapsto {}_1\mathrm{F}_1(\alpha, \gamma; z)$ is holomorphic on $\mathbb{C}^2$ for all γ other than non-positive integers, it follows that $(\nu, w) \mapsto D_\nu(w)$ is also holomorphic on $\mathbb{C}^2$.

In the special case when $\nu \in \mathbb{N}_0$, the function D_ν can be expressed using the Hermite polynomials H_n:

$$D_\nu(w) = 2^{\frac{\nu}{2}} \exp\left(-\tfrac{w^2}{4}\right) H_\nu\Big(\frac{w}{\sqrt{2}}\Big)$$

The solutions in (2.7) are linearly dependent. For most values of ν, any of the four functions can be expressed as a linear combination of any two others. However, specifically in the case $\nu \in \mathbb{N}_0$ we get $D_\nu(w) = \pm D_\nu(-w)$.

Asymptotic behaviour of the solutions is also given by Gradshteyn and Ryzhik [2014]. As $|w| \to \infty$, the solutions $D_{-\nu-1}(\mathrm{i}w)$ and $D_{-\nu-1}(-\mathrm{i}w)$ grow exponentially. Meanwhile, $D_\nu(w)$ decays exponentially for $w \to +\infty$. Therefore, $D_\nu(w)$ and $D_\nu(-w)$ are better suited for the growth conditions imposed by the domain of $\mathscr{H}_\alpha(p)$. We define $c_{+1}, c_{+2}, c_{-1}, c_{-2} \in \mathbb{C}$, such that

$$h_\pm = c_{\pm 1}\, D_\nu(w) + c_{\pm 2}\, D_\nu(-w)\,.$$

It can be further shown, that if $\nu \notin \mathbb{N}_0$, the solution $D_\nu(w)$ diverges for $w \to -\infty$. [**Then show it.**] Therefore, $c_{-1} = c_{+2} = 0$ in order for φ to be integrable. On the other hand, for $\nu \in \mathbb{N}_0$, the solutions aren't independent (as discussed above), therefore we can also set $c_{-1} = c_{+2} = 0$ without loss of generality. Applying the gluing equations (2.6), we get:

$$c_{+1}\, D_\nu(w_0) = c_{-2}\, D_\nu(-w_0)$$

$$c_{+1} \frac{\mathrm{d}}{\mathrm{d}w} D_\nu(w)\Big|_{w_0} - c_{-2} \frac{\mathrm{d}}{\mathrm{d}w} D_\nu(-w)\Big|_{w_0} = \alpha\sqrt{2b}\; c_{+1}\, D_\nu(w_0)$$

We substitute using the equality $\frac{\mathrm{d}}{\mathrm{d}w} D_\nu(w) = \frac{w}{2}\, D_\nu(w) - D_{\nu+1}(w)$ from Gradshteyn and Ryzhik [2014] and arrive at the equation:

$$\begin{pmatrix} D_\nu(w_0) & -D_\nu(-w_0) \\ \left(\frac{w_0}{2} - \alpha\sqrt{2b}\right) D_\nu(w_0) - D_{\nu+1}(w_0) & -\frac{w_0}{2}\, D_\nu(-w_0) - D_{\nu+1}(-w_0) \end{pmatrix} \begin{pmatrix} c_{+1} \\ c_{-2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In order for the equation to have a non-trivial solution, the determinant of the matrix must be zero. Hence, we arrive at the condition:

$$
\begin{aligned}
0 &= D_\nu(w_0)\Big(\tfrac{w_0}{2}\, D_\nu(-w_0) - D_{\nu+1}(-w_0)\Big) + D_\nu(-w_0)\Big(\big(\tfrac{w_0}{2} - \alpha\sqrt{2b}\big) D_\nu(w_0) - D_{\nu+1}(w_0)\Big) \\
&= D_\nu(w_0)\, D_{\nu+1}(-w_0) \ + \ D_\nu(-w_0)\, D_{\nu+1}(w_0) \ + \ \alpha\sqrt{2b}\, D_\nu(w_0)\, D_\nu(-w_0)\,.
\end{aligned} \tag{2.9}
$$

Since we're interested in the allowed values of ν for given w_0 and $\alpha\sqrt{2b} =: a$, this equation effectively defines an implicit function $\nu(a, w_0)$.

2.3 Implicit function for energy levels

Let F be a function of three real variables given by

$$
F(a, w, \nu) = D_\nu(w)\, D_{\nu+1}(-w) \ + \ D_\nu(-w)\, D_{\nu+1}(w) \ + \ a\, D_\nu(w)\, D_\nu(-w)\,.
$$

We have shown that

$$
\epsilon(p) \text{ is an eigenvalue of } \mathscr{H}_\alpha(p) \iff F\Big(\alpha\sqrt{2b},\ \sqrt{2b}\,\Big(x_0 + \frac{p}{b}\Big),\ \frac{\epsilon(p) + b}{2b}\Big) = 0\,.
$$

Truly, this is simply the equation (2.9) after the substitution from (2.4). Furthermore, for $\alpha = 0$ the fibre Hamiltonian $\mathscr{H}_\alpha(p)$ reduces to that of a harmonic oscillator. From this fact, it is straightforward to derive the following result:

$$
F(0, w, k) = 0 \text{ holds for } k \in \mathbb{N}_0 \text{ and all } w \in \mathbb{R}\,.
$$

Moreover, F is analytic in the three variables, as it is a sum of products of entire functions. The implicit function theorem then tells us that, provided $\frac{\partial}{\partial\nu} F(0, w, k) \neq 0$ for a fixed $k \in \mathbb{N}_0$ (which we will prove soon), there exists an analytic function $\nu(a, w)$, such that $\nu(0, w) = k$ and $F(a, w, \nu(a, w)) = 0$ on the neighbourhood of $a = 0$. Since there is a different implicit function for every k, we will denote them $\nu_k(a, w)$. After changing our variables back to the physical ones and putting $x_0 = 0$ (since it only depends on the choice of origin), we get that the energy levels for a fixed magnitude of the perturbation α are

$$
\epsilon_k(p) = b + 2b\, \nu_k\Big(\alpha\sqrt{2b},\ p\sqrt{\tfrac{2}{b}}\Big)\,, \qquad k \in \mathbb{N}_0\,.
$$

For an unperturbed system $\alpha = 0$, the allowed energies are the Landau levels $\epsilon_k = b(2k + 1)$. We will to show that for $p \to \pm\infty$, the energy approaches those unperturbed levels – in our rescaled coordinates, this is equivalent to $\lim_{w\to\pm\infty} \nu_k(a, w) = k$. To show this is the case, we will investigate the behaviour of $\frac{\partial}{\partial a}\nu_k$ and $\frac{\partial}{\partial w}\nu_k$ as $w \to \pm\infty$. First, let us write down the partial derivatives of F:

$$
\frac{\partial}{\partial a} F(a, w, \nu) = D_\nu(w)\, D_\nu(-w)
$$

$$
\frac{\partial}{\partial w} F(a, w, \nu) = a\, w\, D_\nu(w)\, D_\nu(-w) + a\Big(D_\nu(w)\, D_{\nu+1}(-w) - D_\nu(-w)\, D_{\nu+1}(w)\Big)
$$

In the second equality we used the recursion formulas $\frac{\mathrm{d}}{\mathrm{d}w} D_\nu(w) = \frac{w}{2}\, D_\nu(w) - D_{\nu+1}(w)$ and $\frac{\mathrm{d}}{\mathrm{d}w} D_{\nu+1}(w) = -\frac{w}{2}\, D_{\nu+1}(w) + (\nu+1)\, D_\nu(w)$ from Gradshteyn and Ryzhik [2014]. The last partial derivative of F is a little tougher, therefore we will start with the small bits and build our way up.

$$\Psi(x) := \frac{\mathrm{d}}{\mathrm{d}x} \ln\Gamma(x) \quad\Longrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}x}\Gamma(x) = \Psi(x)\,\Gamma(x)$$

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\,{}_1\mathrm{F}_1(\alpha,\gamma;z) &= \frac{\partial}{\partial\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n\, z^n}{(\gamma)_n\, n!} = \frac{\partial}{\partial\alpha}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\,\frac{z^n}{(\gamma)_n\, n!} \\
&= \sum_{n=0}^{\infty}\tfrac{\Psi(\alpha+n)\,\Gamma(\alpha+k)\,\Gamma(\alpha)-\Gamma(\alpha+n)\,\Psi(\alpha)\,\Gamma(\alpha)}{\Gamma(\alpha)^2}\,\frac{z^n}{(\gamma)_n\, n!} = \sum_{n=0}^{\infty}\Big(\scriptstyle\Psi(\alpha+n)-\Psi(\alpha)\displaystyle\Big)\,\frac{(\alpha)_n\, z^n}{(\gamma)_n\, n!}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial\nu}\,D_\nu(w) &= \frac{\partial}{\partial\nu}\,2^{\frac{\nu}{2}}\,\mathrm{e}^{-\frac{w^2}{4}}\left(\frac{\sqrt{\pi}}{\Gamma\big(\frac{1-\nu}{2}\big)}\,{}_1\mathrm{F}_1\Big(-\frac{\nu}{2},\,\frac{1}{2};\,\frac{w^2}{2}\Big) - \frac{w\,\sqrt{2\pi}}{\Gamma\big(-\frac{\nu}{2}\big)}\,{}_1\mathrm{F}_1\Big(\frac{1-\nu}{2},\,\frac{3}{2};\,\frac{w^2}{2}\Big)\right) \\
&= \frac{\ln 2}{2}\,D_\nu(w) + 2^{\frac{\nu}{2}}\,\mathrm{e}^{-\frac{w^2}{4}}\left(-\frac{\sqrt{\pi}\,\Psi(\frac{1-\nu}{2})}{2\,\Gamma(\frac{1-\nu}{2})}{}_1\mathrm{F}_1\Big(-\tfrac{\nu}{2},\,\tfrac{1}{2};\,\tfrac{w^2}{2}\Big) + \frac{w\,\sqrt{2\pi}\,\Psi(-\frac{\nu}{2})}{2\,\Gamma(-\frac{\nu}{2})}{}_1\mathrm{F}_1\Big(\tfrac{1-\nu}{2},\,\tfrac{3}{2};\,\tfrac{w^2}{2}\Big) - \right. \\
&\left. - \frac{\sqrt{\pi}}{2\,\Gamma(\frac{1-\nu}{2})}\sum_{n=0}^{\infty}\Big(\scriptstyle\Psi(-\frac{\nu}{2}+n)-\Psi(-\frac{\nu}{2})\displaystyle\Big)\,\frac{(-\frac{\nu}{2})_n\,(\frac{w^2}{2})^n}{(\frac{1}{2})_n\; n!} + \frac{w\,\sqrt{2\pi}}{2\,\Gamma(-\frac{\nu}{2})}\sum_{n=0}^{\infty}\Big(\scriptstyle\Psi(\frac{1-\nu}{2}+n)-\Psi(\frac{1-\nu}{2})\displaystyle\Big)\,\frac{(\frac{1-\nu}{2})_n\,(\frac{w^2}{2})^n}{(\frac{3}{2})_n\; n!}\right)
\end{aligned}$$

Here, $(a)_n \equiv a(a+1)...(a+n-1)$ is the Pochhammer symbol and $\Psi(x)$ is the digamma function.

Následují poznámky a nedotažené myšlenky. Pravděpodobně se budou hodit odhady:

$$\sum_{n=0}^{\infty}\frac{z^n}{n!\,(\frac{1}{2})_n} = \cosh(2\sqrt{z})$$

$$\sum_{n=0}^{\infty}\frac{z^n}{n!\,(\frac{3}{2})_n} = \frac{1}{2\sqrt{z}}\,\sinh(2\sqrt{z})$$

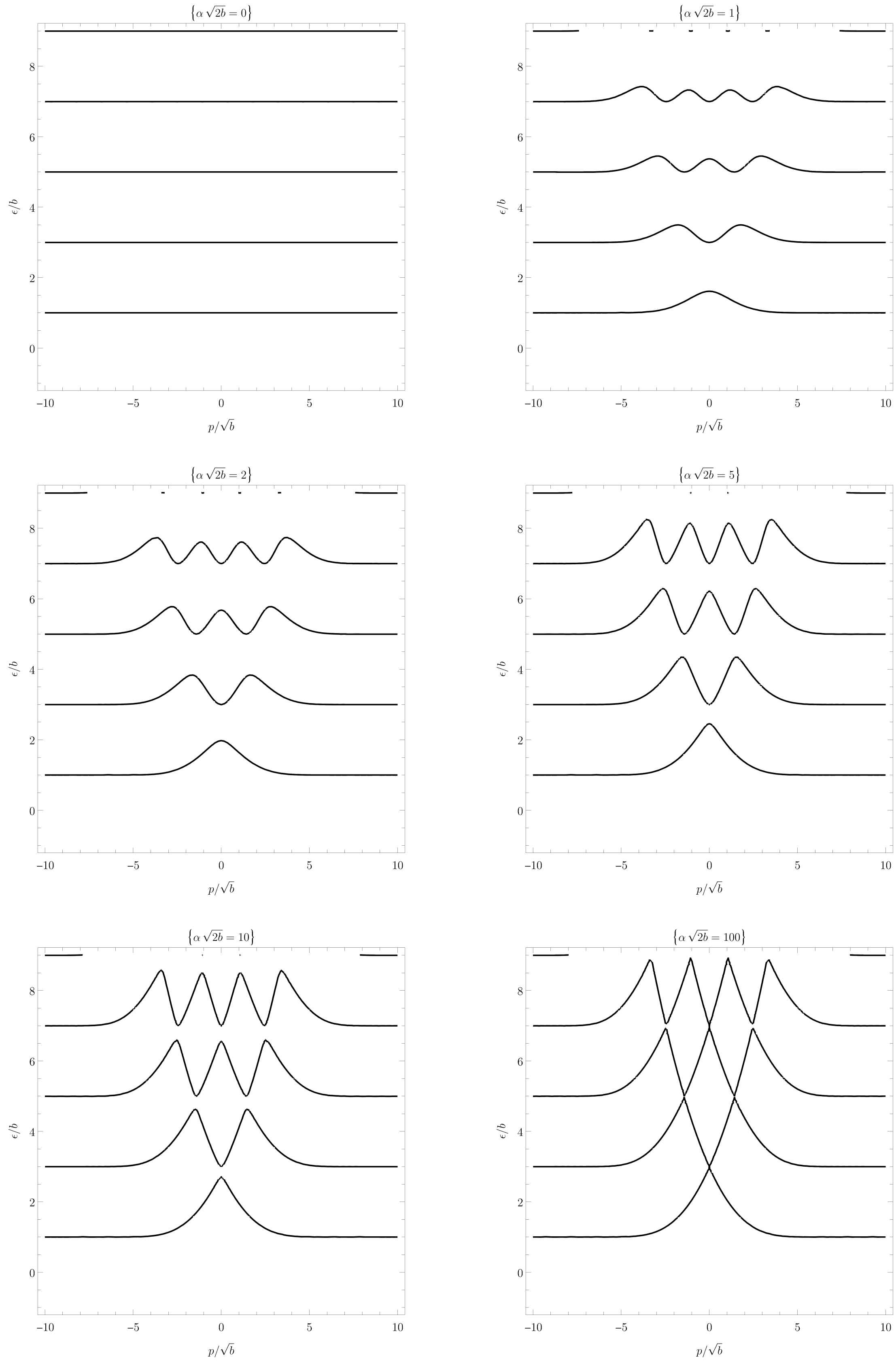

Figure 2.1: The first four energy levels ϵ as a function of the y-momentum p for $\alpha\sqrt{2b} = 0, 1, 2, 5, 10$ and 100 (starting with an unperturbed system, followed by an increasingly repulsive perturbation).

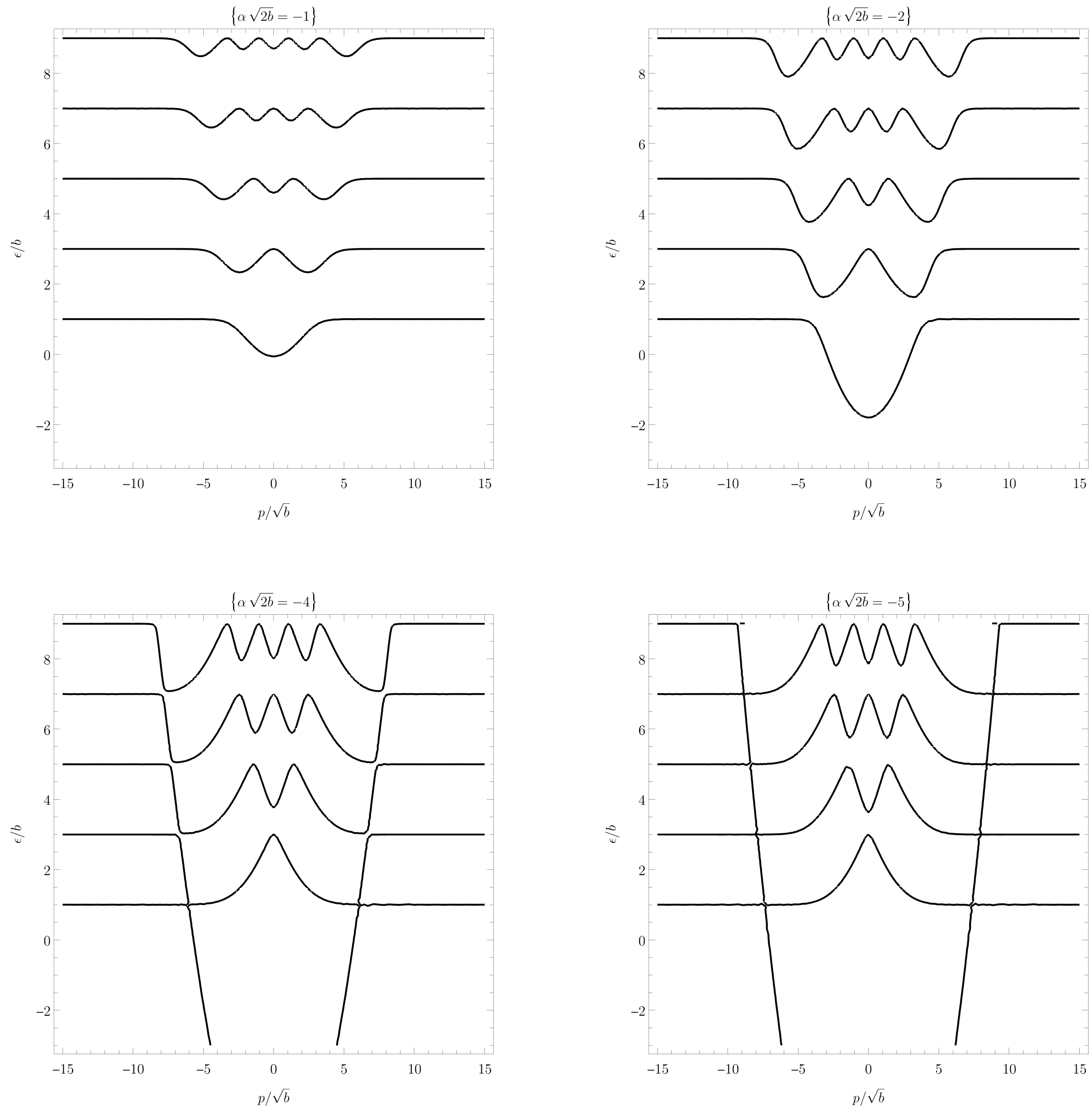

Figure 2.2: The first five energy levels ϵ as a function of the y-momentum p for $\alpha\sqrt{2b} = -1, -2, -4$ and -5 (system with an increasingly attractive perturbation).

3. Half-plane with Robin boundary

Conclusion

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A. Appendix

A.1 Sobolev-type inequality

In this section we prove a Sobolev-type inequality analogical to the inequality (7.16) in Blank et al. [2008]. The presented proof is a simple modification of their proof.

Lemma 4. *Let $\psi \in W^{1,2}(\mathbb{R})$, then for every $a > 0$ there exists $b > 0$ such that*

$$\left\|\psi\right\|^n_{L^\infty(\mathbb{R})} \leq a \left\|\psi'\right\|^n_{L^2(\mathbb{R})} + b \left\|\psi\right\|^n_{L^2(\mathbb{R})}\,, \tag{A.1}$$

where $n \in \{1, 2\}$.

Proof. We define $\phi := \mathcal{F}\psi$. Since, by definition, $\psi' \in L^2(\mathbb{R})$ we have $\mathcal{F}\psi' = T_x\mathcal{F}\psi = T_x\phi \in L^2(\mathbb{R})$, where T_x means the operator of multiplication by the identity function $x \mapsto x$. Next, we utilize the Hölder inequality to find an estimate of $\|\phi\|_{L^1}$.

$$\left\|\phi\right\|_{L^1} = \left\|\tfrac{1}{x+\mathrm{i}}\,(x+\mathrm{i})\,\phi(x)\right\|_{L^1} \leq \underbrace{\left\|\tfrac{1}{x+\mathrm{i}}\right\|_{L^2}}_{=:\,c} \left\|(x+\mathrm{i})\,\phi(x)\right\|_{L^2} \leq\, c\left(\left\|T_x\,\phi\right\|_{L^2} + \left\|\phi\right\|_{L^2}\right). \tag{A.2}$$

Since $\phi \in L^1$, it follows that $\psi \in L^\infty$ and

$$\left\|\psi\right\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}}\left\|\phi\right\|_{L^1}\,. \tag{A.3}$$

We introduce a scaled function $\phi_r(x) := r\,\phi(rx)$. Scaling changes the norms in the following way:

$$\left\|\phi_r\right\|^2_{L^2} = r\left\|\phi\right\|^2_{L^2}\,, \qquad \left\|T_x\phi_r\right\|^2_{L^2} = \frac{1}{r}\left\|T_x\right\|^2_{L^2}\,.$$

By substituting $\phi \mapsto \phi_r$ in (A.2) and combining with (A.3), we get

$$\left\|\psi\right\|_{L^\infty} \leq c\sqrt{r}\left\|T_x\,\phi\right\|_{L^2} + \frac{c}{\sqrt{r}}\left\|\phi\right\|_{L^2} = c\sqrt{r}\left\|\psi'\right\|_{L^2} + \frac{c}{\sqrt{r}}\left\|\psi\right\|_{L^2}\,.$$

Choosing $r = a^2/c^2$, we have proven the inequality (A.1) for $n = 1$.
[**Dodělat $n = 2$ podle BEH nerovnost (4.3) a diskuse pod ní.**] □