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Magnetic Transport Along Translationally Invariant Obstacles

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Dedication.

Title: Magnetic Transport Along Translationally Invariant Obstacles

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Abstract: Abstract.

Keywords: two-dimensional quantum systems, magnetic field, nonlocal perturbations, translational invariance, spectral properties

Contents

List of symbols

$C^k(\Omega, \mathbb{K})$	The space of functions $\Omega \subseteq \mathbb{R} \to \mathbb{K}$ with k continuous derivatives.
$C_0^\infty(\Omega, \mathbb{K})$	The space of C^∞ functions with compact support in Ω.
$\mathrm{D}(T)$	The domain of operator T, usually dense in $\mathcal{H}$.
$D_\nu(w)$	The parabolic cylinder function.
$\mathcal{F}$	The Fourier-Plancherel operator on $L^2(\mathbb{R})$.
${}_1\mathrm{F}_1(\alpha, \beta; z)$	The confluent hypergeometric function of the first kind.
$\mathcal{H}$	A separable Hilbert space.
$H, \mathscr{H}(\xi)$	A Hamiltonian operator; a fiber of the Hamiltonian.
$H_n(x)$	The n-th Hermite polynomial.
$L^p(M, \mathrm{d}\mu, V)$	The space of p-integrable functions from measure space (M, μ) to vector space V. Specifically for $p = 2$, a Hilbert space with inner product $(\psi, \phi)_{L^2} = \int_M (\psi, \phi)_V \, \mathrm{d}\mu$.
$L^p(\Omega)$	As above, but $M = \Omega \subseteq \mathbb{R}^N$, μ is the Lebesgue measure and $V = \mathbb{C}$.
$L^1_{\mathrm{loc}}(\Omega)$	The space of functions that are $L^1(K)$ for every compact $K \subset \Omega$.
$\mathbb{N}, \ \mathbb{N}_0$	The set of positive integers; the set of non-negative integers.
$\vec{P}, P_x, P_y, P_z$	Momentum operator – a self-adjoint operator, such that $P_x \, f(x, ...) = -\mathrm{i}\frac{\partial}{\partial x} f(x, ...)$.
$\vec{Q}, Q_x, Q_y, Q_z$	Position operator – a self-adjoint operator, such that $Q_x \, f(x, ...) = x \, f(x, ...)$.
$W^{k,p}(\Omega)$	The Sobolev space – the space of integrable functions f, such that $f^{(\alpha)} \in L^p(\Omega)$, where α is a multi-index and $\lvert\alpha\rvert \leq k$.
$\Gamma(z)$	The gamma function.
μ	A σ-finite measure, usually the Lebesgue measure.
$\sigma(T), \sigma_{\mathrm{p}}(T),$ $\sigma_{\mathrm{ac}}(T), \sigma_{\mathrm{sc}}(T)$	The spectrum of normal operator T; the point, absolutely continuous, singular continuous spectrum of T
$\nabla, \ \nabla\times, \ \Delta$	Gradient, rotation, Laplace operator.
$\Delta_{\mathrm{D}}^{\Omega}, \Delta_{\mathrm{D},A}^{\Omega}$	The Dirichlet Laplacian, defined on functions from $L^2(\Omega)$ with a Dirichlet boundary condition; a “magnetic” Dirichlet Laplacian given by the vector potential A.

Introduction

1. Formulation & known results

In this chapter we will explain what is the magnetic transport, give a precise mathematical formulation of the problem and restate the known results.

1.1 The magnetic Hamiltonian

The simplest example of a quantum system with a magnetic field is the system consisting of a single charged particle inside a constant homogeneous magnetic field and zero scalar potential. The Hamiltonian that corresponds to this system is:

$$H = (\vec{P} + \vec{A})^2 , \quad \vec{B} = \nabla \times \vec{A} = (0, 0, b_0) .$$

Here $\vec{P} = -\mathrm{i}\nabla$ is the momentum operator, $\vec{B}$ is the magnetic field (which is constant with magnitude b_0) and $\vec{A}$ is its corresponding vector potential. Notice that we have used nondimensionalization to remove physical units from the Hamiltonian. H has absolutely continuous spectrum and commutes with P_z, thus it allows the particle to move freely along the z-axis. However if we restrict the particle to the layer $z = 0$, either physically, or only formally because we are not interested in the movement along z [**EXPAND**], we get a two-dimensional Hamiltonian with infinitely degenerate pure point spectrum, the so-called Landau Hamiltonian:

$$H = (P_x + A_x)^2 + (P_y + A_y)^2 .$$

A detailed analysis of this well-known Hamiltonian can be found eg. in §112 of Landau and Lifshitz [1981]. The pure point spectrum means that the particle is not free to move along x or y, but instead it is "trapped" in some superposition of stationary states. We will investigate perturbations to the Landau Hamiltonian, which cause its spectrum to become continuous and allow the particle to move freely along the y-axis. These perturbations can be either in the form of a scalar potential, a modification of the magnetic field, or a purely geometric deformation of the layer, to which our particle is constrained. We will require all of these perturbations to be translationally invariant, thus constant along one axis – without loss of generality, we choose that they are independent of y and only depend on x.

Throughout this thesis, we will use the Landau gauge:

$$\begin{aligned} A_x &= 0 , \\ A_y &= \int_0^x B_z(x')\,\mathrm{d}x' , \\ A_z &= 0 . \end{aligned}$$

Now we can specify precisely which Hamiltonians we will investigate.

Definition 1 (Potential perturbation)**.** *Let $\mathcal{H} = L^2(\mathbb{R}^2)$, $D(H)$ a* [**???**] *dense subset of $\mathcal{H}$, $b > 0$ and $V \in L^1_{loc}(\mathbb{R})$. A self-adjoint operator $H : D(H) \to \mathcal{H}$ given by the equation*

$$H = P_x^2 + \left(P_y + b\,Q_x\right)^2 + V(x) ,$$

*is called the **Landau Hamiltonian with a potential perturbation**. We will investigate, which choices of $D(H)$ and V lead to $\sigma(H) \neq \sigma_{\mathrm{p}}(H)$.*

Definition 2 (Magnetic perturbation)**.** *Let $b \in C^\infty(\mathbb{R})$, $\mathcal{H} = L^2(\mathbb{R}^2)$ and $\mathcal{D} = C_0^\infty(\mathbb{R}^2)$ be the set of $\mathbb{C}^\infty$ functions with compact support. Let A_y be a multiplication operator on $\mathcal{H}$ given by:*

$$A_y\,\psi(x,y) = \left(\int_0^x b(x')\,\mathrm{d}x' \right) \psi(x,y)$$

Let $\tilde{H} : \mathcal{D} \to \mathcal{H}$ be an essentially self-adjoint operator given by the equation:

$$\tilde{H} = P_x^2 + \left(P_y + A_y\right)^2 ,$$

*Its closure H is called the **Landau Hamiltonian with a magnetic perturbation**. We will investigate, which choices of b lead to $\sigma(H) \neq \sigma_{\mathrm{p}}(H)$.*

Definition 3 (Geometric perturbation, transl. inv. layer)**.** *Let $b > 0$, and $\ell \in \mathbb{R}$. Let $\omega : \mathbb{R} \to \mathbb{R}^2$ be a C^4-smooth curve. We define a set $\Omega' \subset \mathbb{R}^2$ by*

$$\Omega' = \left\{ \; P \in \mathbb{R}^2 \;\; \middle| \;\; \exists s \in \mathbb{R} \left\| \omega(s) - P \right\| \leq \ell \; \right\},$$

this gives a band of width 2ℓ around the curve ω. Then we define a set $\Omega \subset \mathbb{R}^3$ as

$$\Omega = \left\{ \; (x,y,z) \in \mathbb{R}^3 \;\; \middle| \;\; (x,z) \in \Omega' \; \right\}.$$

*We shall call Ω a **translationally invariant layer of width** 2ℓ **given by the curve** ω. Let us now consider the magnetic Dirichlet Laplacian*

$$\Delta^{\Omega}_{\mathrm{D},A}\psi(x,y,z) = \Delta\psi + 2\mathrm{i}b\,x\,\frac{\partial\psi}{\partial y} - b^2x^2\,\psi$$

*defined on functions $\psi \in C^\infty(\Omega)$, such that $\Delta^{\Omega}_{\mathrm{D},A}\psi = 0$ on the boundary of Ω. The operator H which is a closure of $-\Delta^{\Omega}_{\mathrm{D},A}$ is called the **Landau Hamiltonian with a geometric perturbation**. We will investigate, which choices of ω lead to $\sigma(H) \neq \sigma_{\mathrm{p}}(H)$.*

1.2 Direct integral

The key insight to all three of these problems is that the Hamiltonians in question only depend on the momentum p_y of the particle, and not on its position y. If we were to fix p_y of the particle to a certain value somehow, we could reduce the problem to a one-dimensional operator and solve for each p_y separately. This vague idea can be given a rigorous meaning in terms of the *direct integral*.

The following definition is a rephrasing of definitions given in Reed and Simon [1978], pages 280 and 281.

Definition 4 (Direct integral, fiber)**.** *Let $\mathcal{H}'$ be a separable Hilbert space and (M, μ) a measure space. We define a Hilbert space $\mathcal{H}$, which is the space of all square-integrable functions from M to $\mathcal{H}'$:*

$$\mathcal{H} = L^2(M,\, \mathrm{d}\mu\,, \mathcal{H}')\,.$$

Let $\mathscr{A}$ be a measurable function from M to the self-adjoint operators on $\mathcal{H}'$. Let $f_\psi : M \to \mathbb{R}$ be a function defined by

$$f_\psi(s) = \left\|\mathscr{A}(s)\psi(s)\right\|_{\mathcal{H}'} \quad \textit{for all } \psi \in \mathcal{H}, s \in M \textit{ such that } \psi(s) \in \mathrm{D}(\mathscr{A}(s)) \,.$$

We define an operator A on $\mathcal{H}$ by:

$$(A\psi)(s) = \mathscr{A}(s)\,\psi(s)\,,$$
$$\mathrm{D}(A) = \left\{\psi \in \mathcal{H} \,\middle|\, \psi(s) \in \mathrm{D}(\mathscr{A}(s)) \textit{ a.e. } \wedge \left\|f_\psi\right\|_{L^2} < \infty\right\}.$$

Then we shall write

$$\mathcal{H} = \int_M^\oplus \mathcal{H}', \qquad A = \int_M^\oplus \mathscr{A}(s)\,\mathrm{d}s\,.$$

We shall call $\mathcal{H}$ and A the ***the direct integral*** *of $\mathcal{H}'$ and $\mathscr{A}$, respectively. Reversely, we shall call $\mathcal{H}'$ a* ***fiber space*** *of $\mathcal{H}$ and $\mathscr{A}(s)$ a* ***fiber*** *of A.*

Before we apply this to the magnetic Hamiltonian, let us remind the Fourier-Plancherel operator. It is a standard textbook result (see Blank et al. [2008]) that if we take the Fourier transform as an operator on $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$, its closure is a unitary operator on $L^2(\mathbb{R})$. This operator is called the Fourier-Plancharel operator $\mathcal{F}$, it transforms momentum to position $\mathcal{F}P\mathcal{F}^{-1} = Q$, and as an isomorphism, it does not change the spectrum of self-adjoint operators – in particular:

$$\sigma(A) \neq \sigma_{\mathrm{p}}(A) \iff \sigma(\mathcal{F}A\mathcal{F}^{-1}) \neq \sigma_{\mathrm{p}}(\mathcal{F}A\mathcal{F}^{-1})\,.$$

This theory regards functions of one variable. In this thesis, we will often perform a *partial* Fourier transformation on multivariate functions – that is, perform the Fourier transformation on one variable whilst keeping the other variables fixed. We will use subscript to indicate which variable is being transformed, for example $\mathcal{F}_y : (x, y) \mapsto (x, \xi)$.

Now, we can show, how to express a Landau Hamiltonian with potential and magnetic perturbation in terms of the direct integral:

$$\begin{aligned}
H &= \left(\vec{P} + \vec{A}(x)\right)^2 + V(x) \\
&= P_x^2 + (P_y + A_y(x))^2 + V(x) \\
&\simeq \mathcal{F}_y\Big(P_x^2 + (P_y + A_y(x))^2 + V(x)\Big)\mathcal{F}_y^{-1} \\
&= P_x^2 + (Q_y + A_y(x))^2 + V(x) \\
&= \int_{\mathbb{R}}^\oplus \underbrace{{P_x}^2 + (s + A_y(x))^2 + V(x)}_{\mathscr{H}(s)}\ \mathrm{d}s
\end{aligned} \tag{1.1}$$

Where for every s, $\mathscr{H}(s)$ is a self-adjoint operator on $L^2(\mathbb{R})$. The following theorem is a weakened version of Theorem XIII.85 of Reed and Simon [1978].

Theorem 1 (Spectrum of direct integral)**.** *Let $\lambda \in \mathbb{C}$ and $A = \int_M^\oplus \mathscr{A}(s)\,\mathrm{d}s$, as in the previous definition. We define $\Gamma(\lambda)$ as the set of all s, such that λ is an* *eigenvalue of* $\mathscr{A}(s)$*, and* $\Omega_\varepsilon(\lambda)$ *as the set of all* s*, such that the* ε*-neighbourhood of* λ *intersects the spectrum of* $\mathscr{A}(s)$ *– written symbolically:*

$$\Gamma(\lambda) = \left\{ \, s \;\middle|\; \lambda \text{ is an eigenvalue of } \mathscr{A}(s) \, \right\} ,$$
$$\Omega_\varepsilon(\lambda) = \left\{ \, s \;\middle|\; \sigma(\mathscr{A}(s)) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset \, \right\} .$$

Then λ *belongs to the spectrum of* A *if and only if*

$$\mu(\ \Omega_\varepsilon(\lambda)\ ) > 0 \qquad \text{for all } \varepsilon > 0 \, .$$

Additionally, λ *is an eigenvalue of* A *if and only if*

$$\mu(\ \Gamma(\lambda)\ ) > 0 \, .$$

This means that we can deduce the spectrum of the Hamiltonian H simply by investigating how the spectrum of its fiber $\mathscr{H}(s)$ depends on s.

1.3 Potential perturbation

REWORK THIS SECTION. A well-studied case of potential bariers is the Hall effect, where a particle is confined to a strip or semi-plane by electrostatic potential (Combes, 2001 ?). The Hall effect on a plane with two different electrostatic potentials, one on each half of the plane, was studied in (Combes, 2005).

1.4 Magnetic perturbation

REWORK THIS SECTION. The case of non-local perturbations (ie. those which don't disapear at infinity) of the magnetic field were studied by (Iwatsuka, 1985).

1.5 Geometric perturbation

REWORK THIS SECTION. A tilted planar layer of fixed width, as well as more general thin layers with translationally invariant bends were studied in (Exner, 2018) and some sufficient conditions for the continuity of spectrum were given.

2. Delta potential

In this chapter we will examine the Landau Hamiltonian with a potential perturbation (see definition 1), caused by the potential $V(x) = \alpha\, \delta_{x_0}$, ie. the Dirac delta in $x = x_0$, scaled by $\alpha \in \mathbb{R}$. Since such a potential is a distribution and not a locally integrable function, the Hamiltonian is formally given by:

$$\Big(H\psi\Big)(x,y) = \left(-\frac{\partial^2}{\partial x^2} + \Big(\mathrm{i}\frac{\partial}{\partial y} + bx\Big)^2\right)\ \psi(x,y) \quad \text{a.e.}^1\text{on } (\mathbb{R}\setminus\{x_0\})\times\mathbb{R}$$

with a domain given by the conditions

$$\psi \in W^{1,2}(\mathbb{R}^2)\ \cap\ W^{2,2}\Big(\,(\mathbb{R}\setminus\{x_0\})\times\mathbb{R}\,\Big)\,,$$

$$\lim_{x\to x_0+}\psi'(x,y)\ -\ \lim_{x\to x_0-}\psi'(x,y) = \alpha\lim_{x\to x_0}\psi(x,y) \quad \text{for a.e. } y\,,^2$$

$$\int_{\mathbb{R}^2} x^2\,\Big|\,\psi(x,y)\,\Big|\ \mathrm{d}x\ \mathrm{d}y\ < \infty\,.$$

The Hamiltonian is self-adjoint. [**PROVE.**] Then, by an approach equivalent to that in (1.1), one can show that H is isomorphic to a direct integral:

$$H \simeq \int_{\mathbb{R}}^{\oplus} \mathscr{H}(s)\,\mathrm{d}s\ ,$$

where $\mathscr{H}(s)$ is a fiber Hamiltonian satisfying very similar conditions to those of H – that is, for almost every $s \in \mathbb{R}$:

$$\Big(\mathscr{H}(s)\,\varphi\Big)(x) = -\varphi''(x) + \Big(b^2\,x^2 + 2sb\,x + s^2\Big)\,\varphi(x)\,,$$

$$\varphi \in W^{1,2}(\mathbb{R})\ \cap\ W^{2,2}(\mathbb{R}\setminus\{x_0\})\,,$$

$$\lim_{x\to x_0+}\varphi'(x) - \lim_{x\to x_0-}\varphi'(x) = \lim_{x\to x_0}\varphi(x),$$

$$\int_{\mathbb{R}} x^2\,|\varphi(x)|\,\mathrm{d}x\ < \infty\,.$$

In the next section, we will see that $\mathscr{H}(s)$ has a pure point spectrum [**I don't know how to prove this**] and investigate how it depends on s and α.

2.1 The eigenproblem of the fiber Hamiltonian

From now on, we shall suppose that $b > 0$; for $b < 0$ one can do a reflection $x \mapsto -x$ and arrive at the same results. Let $\epsilon \in \mathbb{R}$, we define:

$$w_0 = \sqrt{2b}\,\Big(x_0 + \frac{s}{b}\Big)\,, \qquad \nu = \frac{\epsilon + b}{2b}\,. \tag{2.1}$$

[1] The pointwise equality is to be understood *almost everywhere* with respect to the Lebesgue measure on $\mathbb{R}^2$.

[2] The equality holds for *almost every y* and $\lim_{x\to x_0}$ means the *essential* limit with respect to the Lebesgue measure on $\mathbb{R}$.

Let $h_- \in C^2\big((-\infty, w_0], \mathbb{C}\big)$ and $h_+ \in C^2\big([w_0, +\infty), \mathbb{C}\big)$, such that:

$$h''_\pm(w) = \left(\frac{1}{4}w^2 - \nu - \frac{1}{2}\right) h_\pm(w)\,, \tag{2.2}$$

$$\begin{aligned} h_+(w_0) - h_-(w_0) &= 0\,, \\ h'_+(w_0) - h'_-(w_0) &= \alpha\sqrt{2b}\; h_+(w_0)\,. \end{aligned} \tag{2.3}$$

(If such functions exist for a given ν.) Then, if the function $\varphi : \mathbb{R} \to \mathbb{C}$ given by

$$\varphi(x) = \begin{cases} h_+\Big(\sqrt{2b}\,(x + \frac{s}{b})\Big) & \text{for } x \geq x_0 \\ h_-\Big(\sqrt{2b}\,(x + \frac{s}{b})\Big) & \text{for } x < x_0 \end{cases}$$

is in $\mathrm{D}(\mathscr{H}(s))$, it is an eigenfunction of $\mathscr{H}(s)$ with the eigenvalue ϵ. To check this is really the case, one can simply substitute $w = \sqrt{2b}(x+s/b)$ in the equation (2.2) and see they arrive at the equation $\mathscr{H}(s)\,\varphi(x) = \epsilon\,\varphi(x)$.

As stated in Gradshteyn and Ryzhik [2014], the solutions to (2.2) can be expressed as a linear combination of the functions

$$D_\nu(w)\,,\;\; D_\nu(-w)\,,\;\; D_{-\nu-1}(\mathrm{i}w)\,,\;\; D_{-\nu-1}(-\mathrm{i}w)\,, \tag{2.4}$$

where D_ν is a so-called *parabolic cylinder function*, which is a special function that can be expressed in terms of the gamma function Γ and the confluent hypergeometric function ${}_1\mathrm{F}_1$:

$$D_\nu(w) = 2^{\frac{\nu}{2}} \exp\left(-\tfrac{w^2}{4}\right) \left(\frac{\sqrt{\pi}}{\Gamma\big(\frac{1-\nu}{2}\big)}\, {}_1\mathrm{F}_1\Big(-\frac{\nu}{2}, \frac{1}{2}; \frac{w^2}{2}\Big) - \frac{w\sqrt{2\pi}}{\Gamma\big(-\frac{\nu}{2}\big)}\, {}_1\mathrm{F}_1\Big(\frac{1-\nu}{2}, \frac{3}{2}; \frac{w^2}{2}\Big)\right). \tag{2.5}$$

Since $1/\Gamma(z)$ is an entire function and $(\alpha, z) \mapsto {}_1\mathrm{F}_1(\alpha, \gamma; z)$ is holomorphic on $\mathbb{C}^2$ for all γ other than non-positive integers, it follows that $(\nu, w) \mapsto D_\nu(w)$ is also holomorphic on $\mathbb{C}^2$.

In the special case when $\nu \in \mathbb{N}_0$, the function D_ν can be expressed using the Hermite polynomials H_n:

$$D_\nu(w) = 2^{\frac{\nu}{2}} \exp\left(-\tfrac{w^2}{4}\right) H_\nu\Big(\frac{w}{\sqrt{2}}\Big)$$

The solutions in (2.4) are linearly dependent. For most values of ν, any of the four functions can be expressed as a linear combination of any two others. However, specifically in the case $\nu \in \mathbb{N}_0$ we get $D_\nu(w) = \pm D_\nu(-w)$.

Asymptotic behavior of the solutions is also given by Gradshteyn and Ryzhik [2014]. As $|w| \to \infty$, the solutions $D_{-\nu-1}(\mathrm{i}w)$ and $D_{-\nu-1}(-\mathrm{i}w)$ grow exponentially. Meanwhile $D_\nu(w)$ decays exponentially for $w \to +\infty$. Therefore $D_\nu(w)$ and $D_\nu(-w)$ are better suited for the growth conditions imposed by the domain of $\mathscr{H}(s)$. We define $c_{+1}, c_{+2}, c_{-1}, c_{-2} \in \mathbb{C}$, such that

$$h_\pm = c_{\pm 1}\, D_\nu(w) + c_{\pm 2}\, D_\nu(-w)\,.$$

It can be further shown, that if $\nu \notin \mathbb{N}_0$, the solution $D_\nu(w)$ diverges for $w \to -\infty$. **[Then show it.]** Therefore $c_{-1} = c_{+2} = 0$ in order for φ to be integrable. On the other hand, for $\nu \in \mathbb{N}_0$, the solutions aren't independent (as discussed above), therefore we can also set $c_{-1} = c_{+2} = 0$ without loss of generality. Applying the gluing equations (2.3), we get:

$$c_{+1}\, D_\nu(w_0) = c_{-2}\, D_\nu(-w_0)$$

$$c_{+1}\,\frac{\mathrm{d}}{\mathrm{d}w} D_\nu(w)\Big|_{w_0} - c_{-2}\,\frac{\mathrm{d}}{\mathrm{d}w} D_\nu(-w)\Big|_{w_0} = \alpha\,\sqrt{2b}\; c_{+1}\, D_\nu(w_0)$$

We substitute using the equality $\frac{\mathrm{d}}{\mathrm{d}w} D_\nu(w) = \frac{w}{2}\, D_\nu(w) - D_{\nu+1}(w)$ from Gradshteyn and Ryzhik [2014] and arrive at the equation:

$$\begin{pmatrix} D_\nu(w_0) & -D_\nu(-w_0) \\ \left(\frac{w_0}{2} - \alpha\sqrt{2b}\right) D_\nu(w_0) - D_{\nu+1}(w_0) & -\frac{w_0}{2}\, D_\nu(-w_0) - D_{\nu+1}(-w_0) \end{pmatrix} \begin{pmatrix} c_{+1} \\ c_{-2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In order for the equation to have a non-trivial solution, the determinant of the matrix must be zero. Hence, we arrive at the condition:

$$\begin{aligned} 0 &= D_\nu(w_0)\Big(\tfrac{w_0}{2}\, D_\nu(-w_0) - D_{\nu+1}(-w_0)\Big) + D_\nu(-w_0)\Big(\big(\tfrac{w_0}{2} - \alpha\sqrt{2b}\big) D_\nu(w_0) - D_{\nu+1}(w_0)\Big) \\ &= D_\nu(w_0)\, D_{\nu+1}(-w_0) \;+\; D_\nu(-w_0)\, D_{\nu+1}(w_0) \;+\; \alpha\sqrt{2b}\, D_\nu(w_0)\, D_\nu(-w_0)\,. \end{aligned} \tag{2.6}$$

Since we're interested in the allowed values of ν for given w_0 and $\alpha\,\sqrt{2b} =: a$, this equation effectively defines an implicit function $\nu(a, w_0)$.

2.2 Implicit function for energy levels

Let F be a function of three real variables given by

$$F(a, w, \nu) = D_\nu(w)\, D_{\nu+1}(-w) \;+\; D_\nu(-w)\, D_{\nu+1}(w) \;+\; a\, D_\nu(w)\, D_\nu(-w)\,.$$

We have shown that

$$\epsilon(s) \text{ is an eigenvalue of } \mathscr{H}(s) \iff F\Big(\alpha\,\sqrt{2b},\ \sqrt{2b}\,\Big(x_0 + \frac{s}{b}\Big),\ \frac{\epsilon(s) + b}{2b}\Big) = 0\,.$$

Truly, this is simply the equation (2.6) after the substitution from (2.1). Furthermore, for $\alpha = 0$ the fiber Hamiltonian $\mathscr{H}(s)$ reduces to that of a harmonic oscillator. From this fact, it is straightforward to derive the following result:

$$F(0, w, k) = 0 \text{ holds for } k \in \mathbb{N}_0 \text{ and all } w \in \mathbb{R}\,.$$

Moreover, F is analytic in the three variables, as it is a sum of products of entire functions. The implicit function theorem then tells us that, provided $\frac{\partial}{\partial \nu} F(0, w, k) \neq 0$ for a fixed $k \in \mathbb{N}_0$ (which we will prove soon), there exists an analytic function $\nu(a, w)$, such that $\nu(0, w) = k$ and $F(a, w, \nu(a, w)) = 0$ on the neighbourhood of $a = 0$. Since there is a different implicit function for every k, we will denote them $\nu_k(a, w)$. We will also investigate the behavior of $\frac{\partial}{\partial a}\nu_k$ and $\frac{\partial}{\partial w}\nu_k$ as $w \to \pm\infty$. First, let us write down the partial derivatives of F:

$$\frac{\partial}{\partial a}\, F(a, w, \nu) = D_\nu(w)\, D_\nu(-w)$$

$$\frac{\partial}{\partial w}\, F(a, w, \nu) = a\, w\, D_\nu(w)\, D_\nu(-w) + a\Big(D_\nu(w)\, D_{\nu+1}(-w) - D_\nu(-w)\, D_{\nu+1}(w)\Big)$$

In the second equality we used the recursion formulas $\frac{\mathrm{d}}{\mathrm{d}w} D_\nu(w) = \frac{w}{2}\, D_\nu(w) - D_{\nu+1}(w)$ and $\frac{\mathrm{d}}{\mathrm{d}w} D_{\nu+1}(w) = -\frac{w}{2}\, D_{\nu+1}(w) + (\nu+1)\, D_\nu(w)$ from Gradshteyn and Ryzhik [2014]. The last partial derivative of F is a little tougher, therefore we will start with the small bits and build our way up.

$$\Psi(x) := \frac{\mathrm{d}}{\mathrm{d}x} \ln\Gamma(x) \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\Gamma(x) = \Psi(x)\,\Gamma(x)$$

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\, {}_1\mathrm{F}_1(\alpha,\gamma;z) &= \frac{\partial}{\partial\alpha} \sum_{n=0}^{\infty} \frac{(\alpha)_n\, z^n}{(\gamma)_n\, n!} = \frac{\partial}{\partial\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\, \frac{z^n}{(\gamma)_n\, n!} \\
&= \sum_{n=0}^{\infty} \tfrac{\Psi(\alpha+n)\,\Gamma(\alpha+k)\,\Gamma(\alpha) - \Gamma(\alpha+n)\,\Psi(\alpha)\,\Gamma(\alpha)}{\Gamma(\alpha)^2}\, \frac{z^n}{(\gamma)_n\, n!} = \sum_{n=0}^{\infty} \Big(\scriptstyle\Psi(\alpha+n) - \Psi(\alpha)\displaystyle\Big)\, \frac{(\alpha)_n\, z^n}{(\gamma)_n\, n!}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial\nu}\, D_\nu(w) &= \frac{\partial}{\partial\nu}\, 2^{\frac{\nu}{2}}\, \mathrm{e}^{-\frac{w^2}{4}} \left(\frac{\sqrt{\pi}}{\Gamma\big(\frac{1-\nu}{2}\big)}\, {}_1\mathrm{F}_1\Big(-\frac{\nu}{2}, \frac{1}{2}; \frac{w^2}{2}\Big) - \frac{w\sqrt{2\pi}}{\Gamma\big(-\frac{\nu}{2}\big)}\, {}_1\mathrm{F}_1\Big(\frac{1-\nu}{2}, \frac{3}{2}; \frac{w^2}{2}\Big)\right) \\
&= \frac{\ln 2}{2}\, D_\nu(w) + 2^{\frac{\nu}{2}}\, \mathrm{e}^{-\frac{w^2}{4}} \Bigg(-\frac{\sqrt{\pi}\,\Psi(\frac{1-\nu}{2})}{2\,\Gamma(\frac{1-\nu}{2})}\, {}_1\mathrm{F}_1\Big(-\tfrac{\nu}{2}, \tfrac{1}{2}; \tfrac{w^2}{2}\Big) + \frac{w\sqrt{2\pi}\,\Psi(-\frac{\nu}{2})}{2\,\Gamma(-\frac{\nu}{2})}\, {}_1\mathrm{F}_1\Big(\tfrac{1-\nu}{2}, \tfrac{3}{2}; \tfrac{w^2}{2}\Big) - \\
&\quad - \frac{\sqrt{\pi}}{2\,\Gamma(\frac{1-\nu}{2})} \sum_{n=0}^{\infty} \Big(\scriptstyle\Psi(-\frac{\nu}{2}+n) - \Psi(-\frac{\nu}{2})\displaystyle\Big)\, \frac{(-\frac{\nu}{2})_n\, (\frac{w^2}{2})^n}{(\frac{1}{2})_n\, n!} + \frac{w\sqrt{2\pi}}{2\,\Gamma(-\frac{\nu}{2})} \sum_{n=0}^{\infty} \Big(\scriptstyle\Psi(\frac{1-\nu}{2}+n) - \Psi(\frac{1-\nu}{2})\displaystyle\Big)\, \frac{(\frac{1-\nu}{2})_n\, (\frac{w^2}{2})^n}{(\frac{3}{2})_n\, n!}\Bigg)
\end{aligned}$$

Here, $(a)_n \equiv a(a+1)...(a+n-1)$ is the Pochhammer symbol and $\Psi(x)$ is the digamma function.

Následují poznámky a nedotažené myšlenky. Podle numerických výpočtů to vypadá, že $\frac{\partial}{\partial a} F \to 0$ a $\frac{\partial}{\partial w} F \to 0$ při $w \to \pm\infty$ pro všechny a, ν. Zbývá najít vhodný odhad. Pro $D_\nu(w \to \infty)$ to není problém, ale chování $D_\nu(w \to -\infty)$ je velmi citlivé na ν. Naopak $\frac{\partial}{\partial \nu} F$ jde do plus nebo mínus ∞ a je odražené od nuly kromě bodu $w = 0$. Zde se pravděpodobně budou hodit odhady:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!\,(\frac{1}{2})_n} = \cosh(2\sqrt{z})$$

$$\sum_{n=0}^{\infty} \frac{z^n}{n!\,(\frac{3}{2})_n} = \frac{1}{2\sqrt{z}}\, \sinh(2\sqrt{z})$$

Následují grafy pro $F(a, w, \nu) = 0$ od $a = 0$ do $a = 10$. Bohužel se mi ještě nepodařilo rozchodit Mathematicu na studentskou licenci, takže tohle je jenom z bezplatné webové verze. U kamaráda, který Mathematicu má, jsem to zkoušel až do $a = 100$ a pásy se nikdy nepřekřížily.

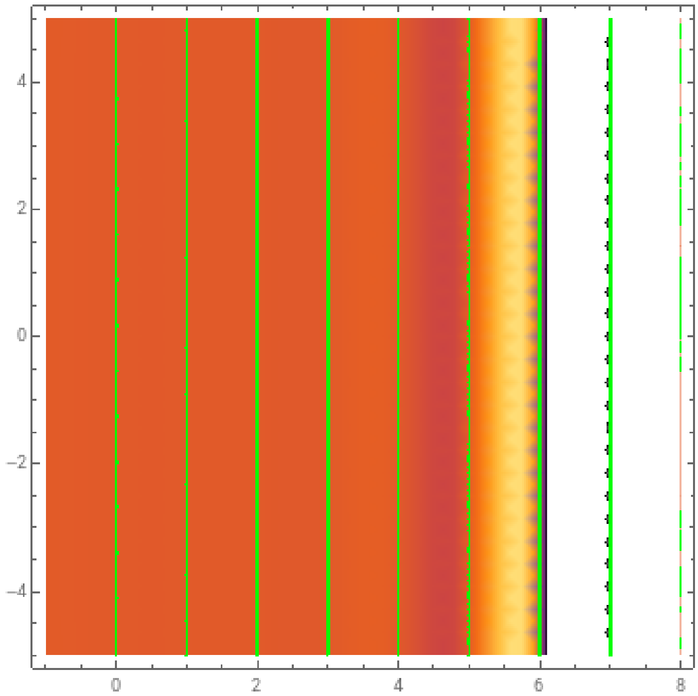

Figure 2.1: $a = 0$

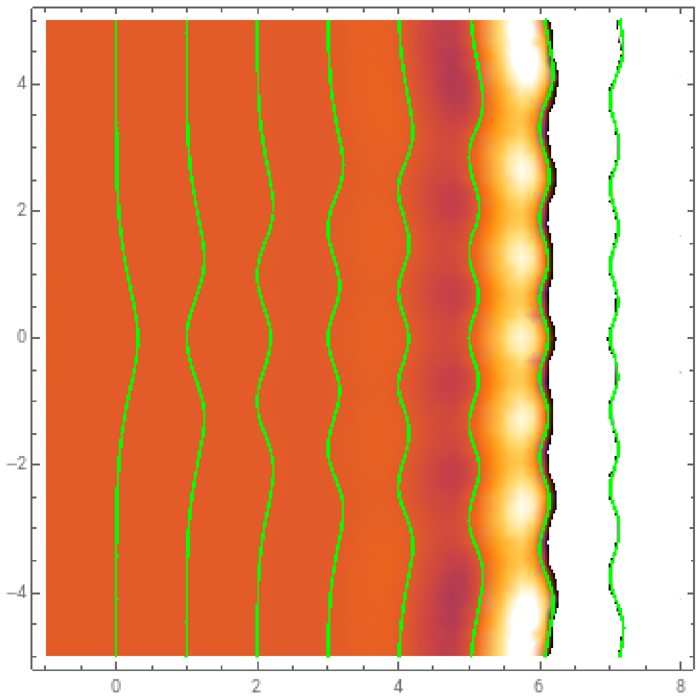

Figure 2.2: $a = 1$

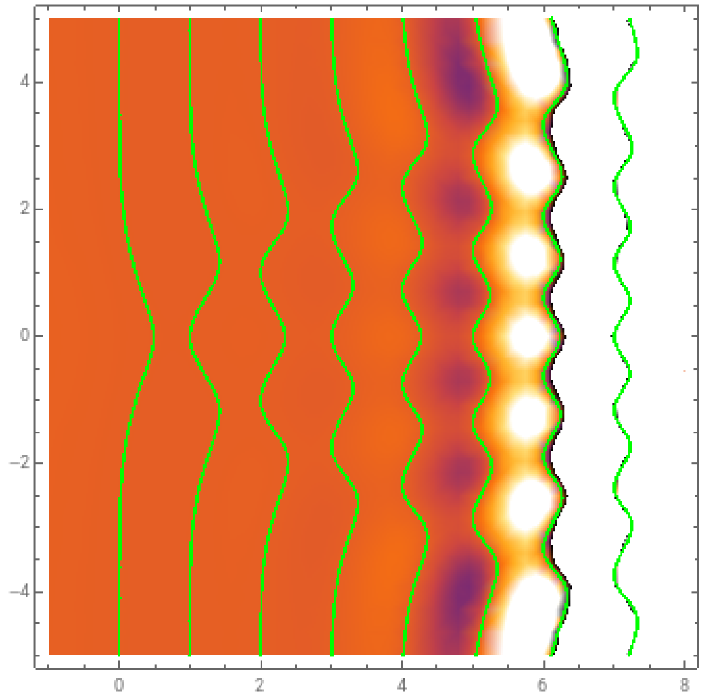

Figure 2.3: $a = 2$

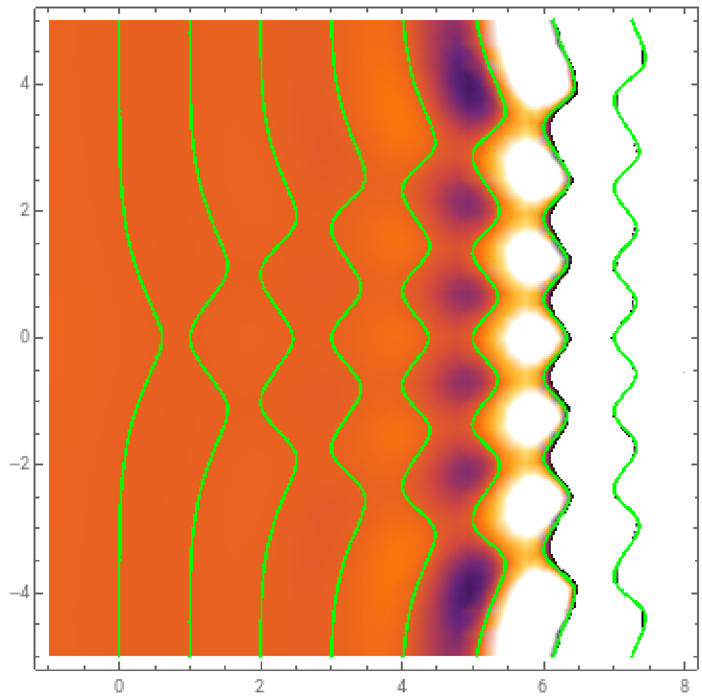

Figure 2.4: $a = 3$

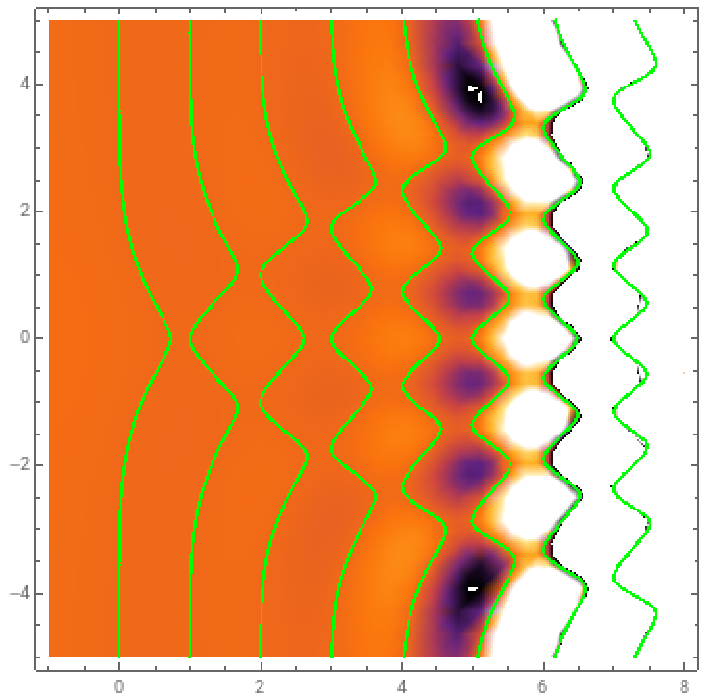

Figure 2.5: $a = 5$

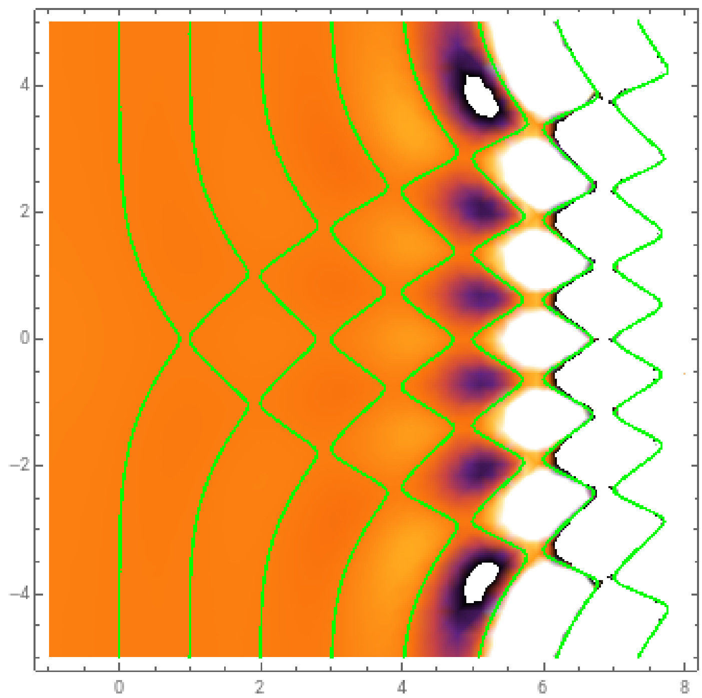

Figure 2.6: $a = 10$

Conclusion

Bibliography

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A. Attachments

A.1 First Attachment