Magnetic Transport Along Translationally Invariant Obstacles

A Bachelor's Thesis Defense

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September 14, 2021

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Its spectrum is $\sigma(H) = \sigma_{\mathrm{p}}(H) = \left\{ (2n+1)\,\|B_0\| \;\middle|\; n \in \mathbb{N}_0 \right\}$.

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  - the layer is smoothly bent and invariant under translation $y \mapsto y + c$
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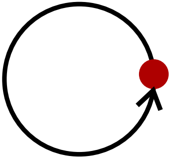
Classically:



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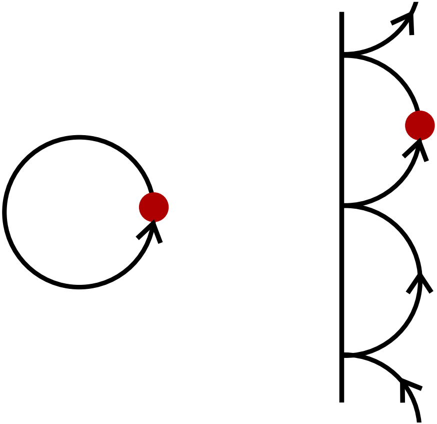

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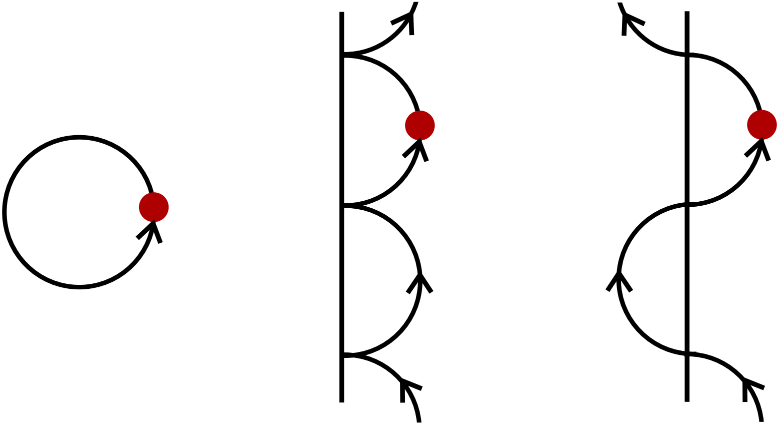

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Quantum Mechanics:

- Spectrum of H is pure point
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  - $\leftrightsquigarrow$ time evolution is trivial
- Spectrum of H is continuous
  - $\leftrightsquigarrow$ there are no stationary states
  - $\leftrightsquigarrow$ **Magnetic Transport!** *(or Iwatsuka type effect)*

Magnetic Transport Along Translationally Invariant Obstacles

The Hamiltonian is either of these:

(a) $H = \left(-\mathrm{i}\vec{\nabla} + \vec{A}\right)^2 + V(x)$ on $L^2(\Omega \subset \mathbb{R}^2)$
(b) $H = \left(-\mathrm{i}\vec{\nabla} + \vec{A}(x)\right)^2$ on $L^2(\Omega \subset \mathbb{R}^2)$
(c) $H = (-\mathrm{i}\vec{\nabla} + \vec{A})$ on $L^2(\Omega)$, Ω being a thin layer in $\mathbb{R}^3$

And we are interested in its pure point / continuous spectrum.

The two parts

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1. Summary of known results
   - Steep potential wall (Macris et al., 1999) and (Fröhlich et al., 2000)
   - Half-plane with Dirichlet boundary (Fröhlich et al., 2000)
   - Bounded magnetic perturbation (Iwatsuka, 1983 and 1985)
   - Layer with one-sided fold, asymptotically flat layer, very thin layer (Exner et al., 2018)

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2. Original work (potential obstacles)
   - Half-plane with Robin boundary
     - $\psi \in L^2([0,\infty))\,, \qquad \alpha\,\psi(0,y) + \partial_x \psi(0,y) = 0$
   - Dirac δ-interaction on a line
     - $\psi \in L^2(\mathbb{R})\,, \qquad \partial_x \psi(0+,y) - \partial_x \psi(0-,y) = \alpha\,\psi(0,y)$

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We choose the Landau gauge:

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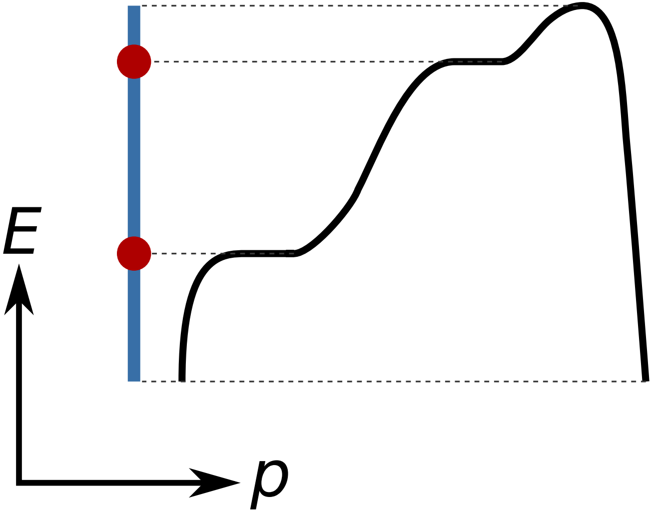
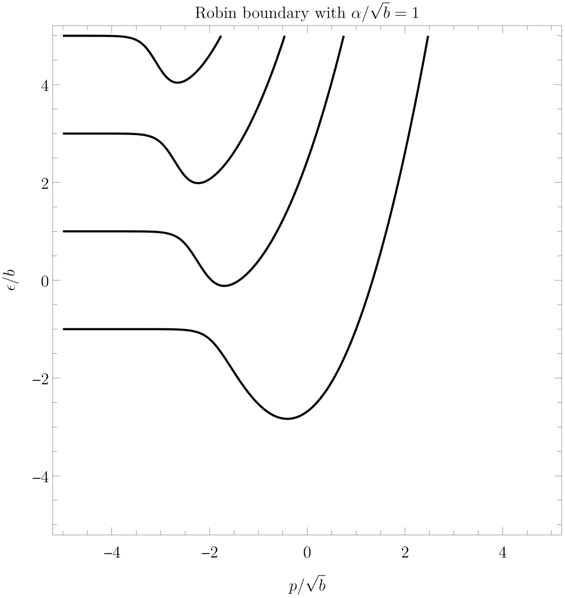
where $\mathscr{H}(p)$ is 1D and (hopefully) has discrete spectrum.

The Spectrum of a Direct Integral



Half-plane with Robin boundary



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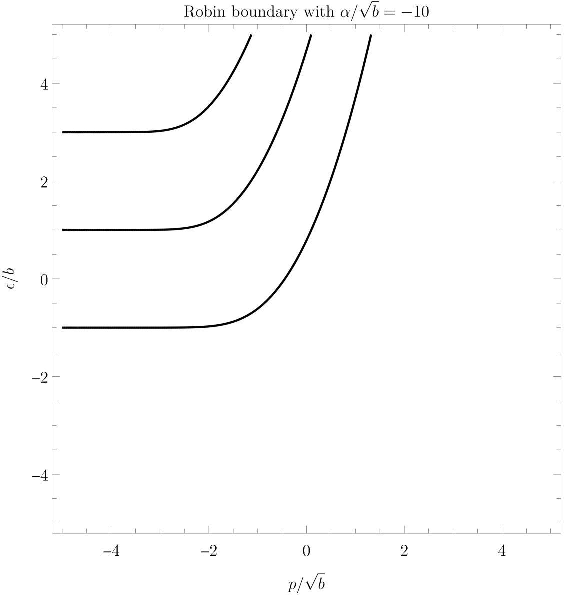

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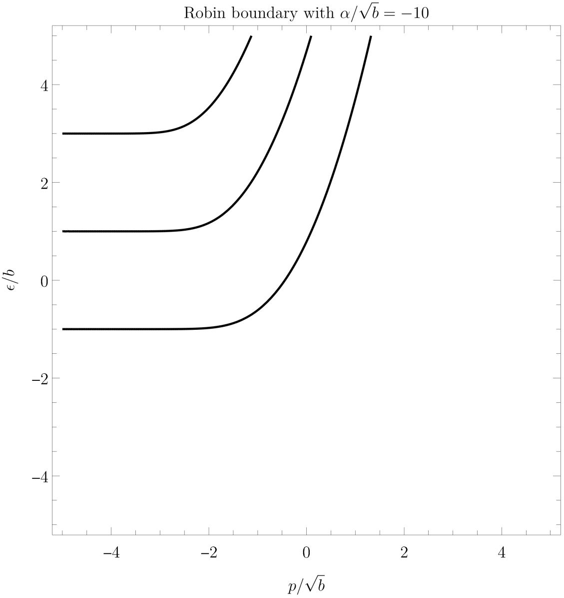
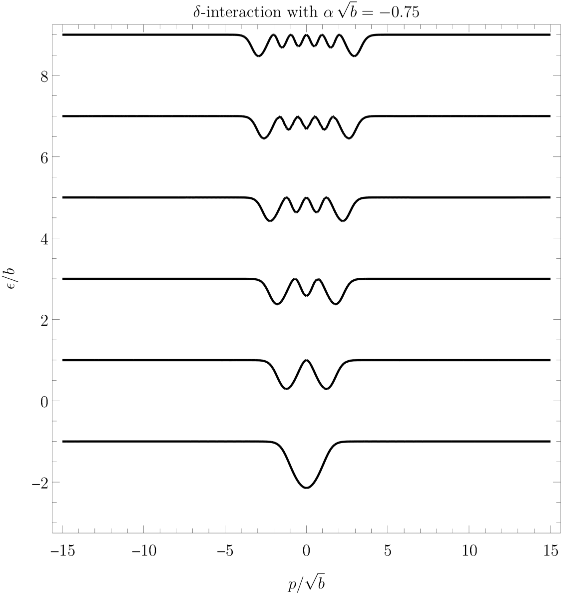


Dirac δ-interaction



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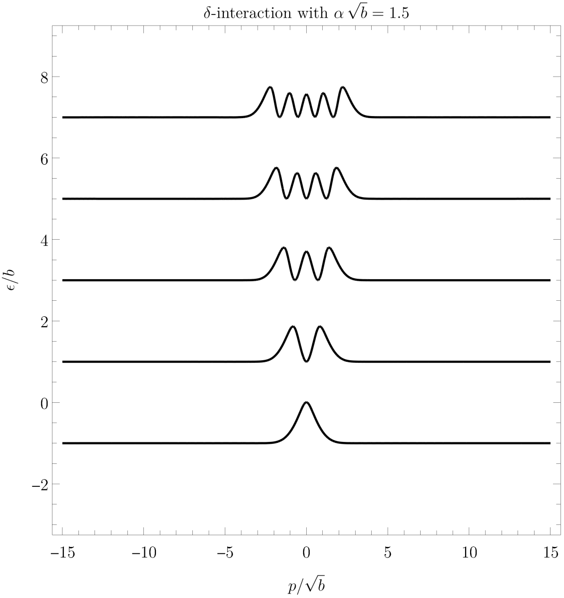

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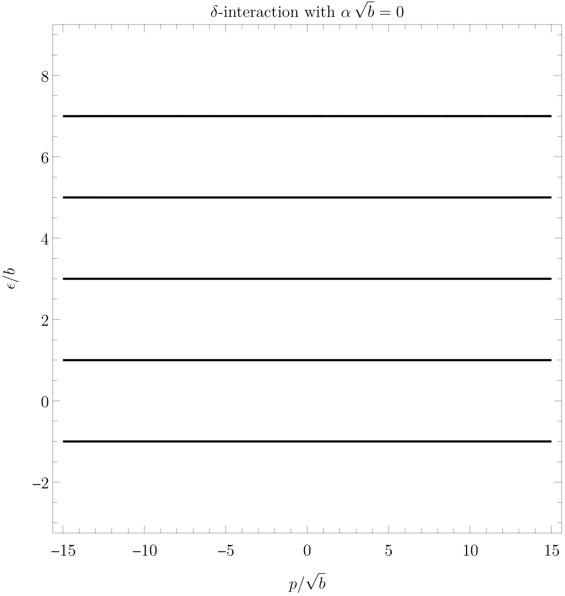

Results

Let H_α be the Landau Hamiltonian in a half-plane with Robin boundary. If $\alpha > 0$, then there exists $E \in (-\infty, \Theta_0)$ such that $\sigma(H_\alpha) = \sigma_{\mathrm{ac.}}(H_\alpha) = [E\, b,\ \infty)$. If $\alpha < 0$, then there exists $E \in (\Theta_0, 1)$ such that $\sigma(H_\alpha) = \sigma_{\mathrm{ac.}}(H_\alpha) = [E\, b,\ \infty)$. And finally, if $\alpha = 0$, then $\sigma(H_\alpha) = \sigma_{\mathrm{ac.}}(H_\alpha) = [\Theta_0\, b,\ \infty)$. Here $\Theta_0 \approx 0.590106125$.

Let H_α be the Landau Hamiltonian in a plane with δ-interaction. If $\alpha > 0$, then for each $k \in \mathbb{N}_0$ there exists $E_k \in (0, 2)$ such that

$$\sigma(H_\alpha) = \sigma_{\mathrm{ac.}}(H_\alpha) = \bigcup_{k \in \mathbb{N}_0} \left[b\,(2k+1),\ b\,(2k+1+E_k)\right].$$

If $\alpha < 0$, then there exists $E_0 \in (0, \infty)$ and for each $n \in \mathbb{N}$ there is $E_n \in (0, 2)$ such that

$$\sigma(H_\alpha) = \sigma_{\mathrm{ac.}}(H_\alpha) = \bigcup_{k \in \mathbb{N}_0} \left[b\,(2k+1-E_k),\ b\,(2k+1)\right].$$

Thank you for your attention!