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BACHELOR THESIS

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**Magnetic Transport Along
Translationally Invariant Obstacles**

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Dedication.

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List of symbols

$C^k(\Omega, \mathbb{K})$	The space of functions $\Omega \subseteq \mathbb{R} \rightarrow \mathbb{K}$ with k continuous derivatives.
$C_0^\infty(\Omega, \mathbb{K})$	The space of C^∞ functions with compact support in Ω .
$D(T)$	The domain of operator T , usually dense in \mathcal{H} .
$D_\nu(w)$	The parabolic cylinder function.
\mathcal{F}	The Fourier-Plancherel operator on $L^2(\mathbb{R})$.
${}_1F_1(\alpha, \beta; z)$	The confluent hypergeometric function of the first kind.
\mathcal{H}	A separable Hilbert space.
$H, \mathcal{H}(\xi)$	A Hamiltonian operator; a fiber of the Hamiltonian.
$L^p(M, d\mu, V)$	The space of p -integrable functions from measure space (M, μ) to vector space V . Specifically for $p = 2$, a Hilbert space with inner product $(\psi, \phi)_{L^2} = \int_M (\psi, \phi)_V d\mu$.
$L^p(\Omega)$	As above, but $M = \Omega \subseteq \mathbb{R}^N$, μ is the Lebesgue measure and $V = \mathbb{C}$.
$L^1_{\text{loc}}(\Omega)$	The space of functions that are $L^1(K)$ for every compact $K \subset \Omega$.
\mathbb{N}, \mathbb{N}_0	The set of positive integers; the set of non-negative integers.
\vec{P}, P_x, P_y, P_z	Momentum operator – a self-adjoint operator, such that $P_x f(x, \dots) = -i \frac{\partial}{\partial x} f(x, \dots)$.
\vec{Q}, Q_x, Q_y, Q_z	Position operator – a self-adjoint operator, such that $Q_x f(x, \dots) = x f(x, \dots)$.
$W^{k,p}(\Omega)$	The Sobolev space – the space of integrable functions f , such that $f^{(\alpha)} \in L^p(\Omega)$, where α is a multi-index and $ \alpha \leq k$.
$\Gamma(z)$	The gamma function.
μ	A σ -finite measure, usually the Lebesgue measure.
$\sigma(T), \sigma_p(T), \sigma_{\text{ac}}(T), \sigma_{\text{sc}}(T)$	The spectrum of normal operator T ; the point, absolutely continuous, singular continuous spectrum of T
$\nabla, \nabla \times, \Delta$	Gradient, rotation, Laplace operator.
$\Delta_D^\Omega, \Delta_{D,A}^\Omega$	The Dirichlet Laplacian, defined on functions from $L^2(\Omega)$ with a Dirichlet boundary condition; a “magnetic” Dirichlet Laplacian given by the vector potential A .

Introduction

1. Formulation & known results

In this chapter we will explain what is the magnetic transport, give a precise mathematical formulation of the problem and restate the known results.

1.1 The magnetic Hamiltonian

The simplest example of a quantum system with a magnetic field is the system consisting of a single charged particle inside a constant homogeneous magnetic field and zero scalar potential. The Hamiltonian that corresponds to this system is:

$$H = (\vec{P} + \vec{A})^2, \quad \vec{B} = \nabla \times \vec{A} = (0, 0, b_0).$$

Here $\vec{P} = -i\nabla$ is the momentum operator, \vec{B} is the magnetic field (which is constant with magnitude b_0) and \vec{A} is its corresponding vector potential. Notice that we have used nondimensionalization to remove physical units from the Hamiltonian. H has absolutely continuous spectrum and commutes with P_z , thus it allows the particle to move freely along the z -axis. However if we restrict the particle to the layer $z = 0$, either physically, or only formally because we are not interested in the movement along z **[EXPAND]**, we get a two-dimensional Hamiltonian with infinitely degenerate pure point spectrum, the so-called Landau Hamiltonian:

$$H = (P_x + A_x)^2 + (P_y + A_y)^2.$$

A detailed analysis of this well-known Hamiltonian can be found eg. in §112 of Landau and Lifshitz [1981]. The pure point spectrum means that the particle is not free to move along x or y , but instead it is “trapped” in some superposition of stationary states. We will investigate perturbations to the Landau Hamiltonian, which cause its spectrum to become continuous and allow the particle to move freely along the y -axis. These perturbations can be either in the form of a scalar potential, a modification of the magnetic field, or a purely geometric deformation of the layer, to which our particle is constrained. We will require all of these perturbations to be translationally invariant, thus constant along one axis – without loss of generality, we choose that they are independent of y and only depend on x .

Throughout this thesis, we will use the Landau gauge:

$$\begin{aligned} A_x &= 0, \\ A_y &= \int_0^x B_z(x') dx', \\ A_z &= 0. \end{aligned}$$

Now we can specify precisely which Hamiltonians we will investigate.

Definition 1 (Potential perturbation). *Let $\mathcal{H} = L^2(\mathbb{R}^2)$, $D(H)$ a [??] dense subset of \mathcal{H} , $b > 0$ and $V \in L^1_{loc}(\mathbb{R})$. A self-adjoint operator $H : D(H) \rightarrow \mathcal{H}$ given by the equation*

$$H = P_x^2 + (P_y + bQ_x)^2 + V(x),$$

is called the **Landau Hamiltonian with a potential perturbation**. We will investigate, which choices of $D(H)$ and V lead to $\sigma(H) \neq \sigma_p(H)$.

Definition 2 (Magnetic perturbation). Let $b \in C^\infty(\mathbb{R})$, $\mathcal{H} = L^2(\mathbb{R}^2)$ and $\mathcal{D} = C_0^\infty(\mathbb{R}^2)$ be the set of C^∞ functions with compact support. Let A_y be a multiplication operator on \mathcal{H} given by:

$$A_y \psi(x, y) = \left(\int_0^x b(x') dx' \right) \psi(x, y)$$

Let $\tilde{H} : \mathcal{D} \rightarrow \mathcal{H}$ be an essentially self-adjoint operator given by the equation:

$$\tilde{H} = P_x^2 + (P_y + A_y)^2,$$

Its closure H is called the **Landau Hamiltonian with a magnetic perturbation**. We will investigate, which choices of b lead to $\sigma(H) \neq \sigma_p(H)$.

Definition 3 (Geometric perturbation, transl. inv. layer). Let $b > 0$, and $\ell \in \mathbb{R}$. Let $\omega : \mathbb{R} \rightarrow \mathbb{R}^2$ be a C^4 -smooth curve. We define a set $\Omega' \subset \mathbb{R}^2$ by

$$\Omega' = \left\{ P \in \mathbb{R}^2 \mid \exists s \in \mathbb{R} \left\| \omega(s) - P \right\| \leq \ell \right\},$$

this gives a band of width 2ℓ around the curve ω . Then we define a set $\Omega \subset \mathbb{R}^3$ as

$$\Omega = \left\{ (x, y, z) \in \mathbb{R}^3 \mid (x, z) \in \Omega' \right\}.$$

We shall call Ω a **translationally invariant layer of width 2ℓ given by the curve ω** . Let us now consider the magnetic Dirichlet Laplacian

$$\Delta_{D,A}^\Omega \psi(x, y, z) = \Delta \psi + 2ibx \frac{\partial \psi}{\partial y} - b^2 x^2 \psi$$

defined on functions $\psi \in C^\infty(\Omega)$, such that $\Delta_{D,A}^\Omega \psi = 0$ on the boundary of Ω . The operator H which is a closure of $-\Delta_{D,A}^\Omega$ is called the **Landau Hamiltonian with a geometric perturbation**. We will investigate, which choices of ω lead to $\sigma(H) \neq \sigma_p(H)$.

1.2 Direct integral

The key insight to all three of these problems is that the Hamiltonians in question only depend on the momentum p_y of the particle, and not on its position y . If we were to fix p_y of the particle to a certain value somehow, we could reduce the problem to a one-dimensional operator and solve for each p_y separately. This vague idea can be given a rigorous meaning in terms of the *direct integral*.

The following definition is a rephrasing of definitions given in Reed and Simon [1978], pages 280 and 281.

Definition 4 (Direct integral, fiber). Let \mathcal{H}' be a separable Hilbert space and (M, μ) a measure space. We define a Hilbert space \mathcal{H} , which is the space of all square-integrable functions from M to \mathcal{H}' :

$$\mathcal{H} = L^2(M, d\mu, \mathcal{H}').$$

Let \mathcal{A} be a measurable function from M to the self-adjoint operators on \mathcal{H}' . Let $f_\psi : M \rightarrow \mathbb{R}$ be a function defined by

$$f_\psi(s) = \left\| \mathcal{A}(s)\psi(s) \right\|_{\mathcal{H}'} \quad \text{for all } \psi \in \mathcal{H}, s \in M \text{ such that } \psi(s) \in D(\mathcal{A}(s)).$$

We define an operator A on \mathcal{H} by:

$$(A\psi)(s) = \mathcal{A}(s)\psi(s), \\ D(A) = \left\{ \psi \in \mathcal{H} \mid \psi(s) \in D(\mathcal{A}(s)) \text{ a.e.} \wedge \left\| f_\psi \right\|_{L^2} < \infty \right\}.$$

Then we shall write

$$\mathcal{H} = \int_M^\oplus \mathcal{H}', \quad A = \int_M^\oplus \mathcal{A}(s) \, ds.$$

We shall call \mathcal{H} and A the **the direct integral** of \mathcal{H}' and \mathcal{A} , respectively. Reversely, we shall call \mathcal{H}' a **fiber space** of \mathcal{H} and $\mathcal{A}(s)$ a **fiber** of A .

Before we apply this to the magnetic Hamiltonian, let us remind the Fourier-Plancherel operator. It is a standard textbook result (see Blank et al. [2008]) that if we take the Fourier transform as an operator on $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$, its closure is a unitary operator on $L^2(\mathbb{R})$. This operator is called the Fourier-Plancherel operator \mathcal{F} , it transforms momentum to position $\mathcal{F}P\mathcal{F}^{-1} = Q$, and as an isomorphism, it does not change the spectrum of self-adjoint operators – in particular:

$$\sigma(A) \neq \sigma_p(A) \iff \sigma(\mathcal{F}A\mathcal{F}^{-1}) \neq \sigma_p(\mathcal{F}A\mathcal{F}^{-1}).$$

This theory regards functions of one variable. In this thesis, we will often perform a *partial* Fourier transformation on multivariate functions – that is, perform the Fourier transformation on one variable whilst keeping the other variables fixed. We will use subscript to indicate which variable is being transformed, for example $\mathcal{F}_y : (x, y) \mapsto (x, \xi)$.

Now, we can show, how to express a Landau Hamiltonian with potential and magnetic perturbation in terms of the direct integral:

$$\begin{aligned} H &= \left(\vec{P} + \vec{A}(x) \right)^2 + V(x) \\ &= P_x^2 + (P_y + A_y(x))^2 + V(x) \\ &\simeq \mathcal{F}_y \left(P_x^2 + (P_y + A_y(x))^2 + V(x) \right) \mathcal{F}_y^{-1} \\ &= P_x^2 + (Q_y + A_y(x))^2 + V(x) \\ &= \int_{\mathbb{R}}^\oplus \underbrace{P_x^2 + (s + A_y(x))^2 + V(x)}_{\mathcal{H}(s)} \, ds \end{aligned} \tag{1.1}$$

Where for every s , $\mathcal{H}(s)$ is a self-adjoint operator on $L^2(\mathbb{R})$. The following theorem is a weakened version of Theorem XIII.85 of Reed and Simon [1978].

Theorem 1 (Spectrum of direct integral). *Let $\lambda \in \mathbb{C}$ and $A = \int_M^\oplus \mathcal{A}(s) \, ds$, as in the previous definition. We define $\Gamma(\lambda)$ as the set of all s , such that λ is an*

eigenvalue of $\mathcal{A}(s)$, and $\Omega_\varepsilon(\lambda)$ as the set of all s , such that the ε -neighbourhood of λ intersects the spectrum of $\mathcal{A}(s)$ – written symbolically:

$$\begin{aligned}\Gamma(\lambda) &= \left\{ s \mid \lambda \text{ is an eigenvalue of } \mathcal{A}(s) \right\}, \\ \Omega_\varepsilon(\lambda) &= \left\{ s \mid \sigma(\mathcal{A}(s)) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset \right\}.\end{aligned}$$

Then λ belongs to the spectrum of A if and only if

$$\mu(\Omega_\varepsilon(\lambda)) > 0 \quad \text{for all } \varepsilon > 0.$$

Additionally, λ is an eigenvalue of A if and only if

$$\mu(\Gamma(\lambda)) > 0.$$

This means that we can deduce the spectrum of the Hamiltonian H simply by investigating how the spectrum of its fiber $\mathcal{H}(s)$ depends on s .

1.3 Potential perturbation

REWORK THIS SECTION. A well-studied case of potential barriers is the Hall effect, where a particle is confined to a strip or semi-plane by electrostatic potential (Combes, 2001 ?). The Hall effect on a plane with two different electrostatic potentials, one on each half of the plane, was studied in (Combes, 2005).

1.4 Magnetic perturbation

REWORK THIS SECTION. The case of non-local perturbations (ie. those which don't disappear at infinity) of the magnetic field were studied by (Iwatsuka, 1985).

1.5 Geometric perturbation

REWORK THIS SECTION. A tilted planar layer of fixed width, as well as more general thin layers with translationally invariant bends were studied in (Exner, 2018) and some sufficient conditions for the continuity of spectrum were given.

2. Delta potential

In this chapter we will examine the Landau Hamiltonian with a potential perturbation (see definition 1), caused by the potential $V(x) = \alpha \delta_{x_0}$, ie. the Dirac delta in $x = x_0$, scaled by $\alpha \in \mathbb{R}$. Since such a potential is a distribution and not a locally integrable function, the Hamiltonian is formally given by:

$$(H\psi)(x, y) = \left(-\frac{\partial^2}{\partial x^2} + \left(i\frac{\partial}{\partial y} + bx \right)^2 \right) \psi(x, y) \quad \text{a.e.}^1 \text{ on } (\mathbb{R} \setminus \{x_0\}) \times \mathbb{R}$$

with a domain given by the conditions

$$\begin{aligned} \psi &\in W^{1,2}(\mathbb{R}^2) \cap W^{2,2}((\mathbb{R} \setminus \{x_0\}) \times \mathbb{R}), \\ \lim_{x \rightarrow x_0+} \psi'(x, y) - \lim_{x \rightarrow x_0-} \psi'(x, y) &= \alpha \lim_{x \rightarrow x_0} \psi(x, y) \quad \text{for a.e. } y,^2 \\ \int_{\mathbb{R}^2} x^2 |\psi(x, y)| \, dx \, dy &< \infty. \end{aligned}$$

The Hamiltonian is self-adjoint. **[PROVE.]** Then, by an approach equivalent to that in (1.1), one can show that H is isomorphic to a direct integral:

$$H \simeq \int_{\mathbb{R}}^{\oplus} \mathcal{H}(s) \, ds,$$

where $\mathcal{H}(s)$ is a fiber Hamiltonian satisfying very similar conditions to those of H – that is, for almost every $s \in \mathbb{R}$:

$$(\mathcal{H}(s)\varphi)(x) = -\varphi''(x) + (b^2 x^2 + 2sbx + s^2)\varphi(x),$$

$$\begin{aligned} \varphi &\in W^{1,2}(\mathbb{R}) \cap W^{2,2}(\mathbb{R} \setminus \{x_0\}), \\ \lim_{x \rightarrow x_0+} \varphi'(x) - \lim_{x \rightarrow x_0-} \varphi'(x) &= \lim_{x \rightarrow x_0} \varphi(x), \\ \int_{\mathbb{R}} x^2 |\varphi(x)| \, dx &< \infty. \end{aligned}$$

In the next section, we will see that $\mathcal{H}(s)$ has a pure point spectrum **[I don't know how to prove this]** and investigate how it depends on s and α .

2.1 The eigenproblem of the fiber Hamiltonian

From now on, we shall suppose that $b > 0$; for $b < 0$ one can do a reflection $x \mapsto -x$ and arrive at the same results. Let $\epsilon \in \mathbb{R}$, we define:

$$w_0 = \sqrt{2b} \left(x_0 + \frac{s}{b} \right), \quad \nu = \frac{\epsilon + b}{2b}.$$

¹The pointwise equality is to be understood *almost everywhere* with respect to the Lebesgue measure on \mathbb{R}^2 .

²The equality holds for *almost every* y and $\lim_{x \rightarrow x_0}$ means the *essential* limit with respect to the Lebesgue measure on \mathbb{R} .

Let $h_- \in C^2((-\infty, w_0], \mathbb{C})$ and $h_+ \in C^2([w_0, +\infty), \mathbb{C})$, such that:

$$h''_{\pm}(w) = \left(\frac{1}{4}w^2 - \nu - \frac{1}{2}\right) h_{\pm}(w), \quad (2.1)$$

$$h_+(w_0) - h_-(w_0) = 0,$$

$$h'_+(w_0) - h'_-(w_0) = \alpha \sqrt{2b}.$$

(If such functions exist for a given ν .) Then, if the function $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ given by

$$\varphi(x) = \begin{cases} h_+(\sqrt{2b}(x + \frac{s}{b})) & \text{for } x \geq x_0 \\ h_-(\sqrt{2b}(x + \frac{s}{b})) & \text{for } x < x_0 \end{cases}$$

is in $D(\mathcal{H}(s))$, it is an eigenfunction of $\mathcal{H}(s)$ with the eigenvalue ϵ . To check this is really the case, one can simply substitute $w = \sqrt{2b}(x + s/b)$ in the equation (2.1) and see they arrive at the equation $\mathcal{H}(s) \varphi(x) = \epsilon \varphi(x)$.

As stated in Gradshteyn and Ryzhik [2014], the solutions to (2.1) can be expressed as a linear combination of the functions

$$D_{\nu}(w), \quad D_{\nu}(-w), \quad D_{-\nu-1}(iw), \quad D_{-\nu-1}(-iw), \quad (2.2)$$

where D_{ν} is a so-called *parabolic cylinder function*, which is a special function that can be expressed in terms of the gamma function Γ and the confluent hypergeometric function ${}_1F_1$:

$$D_{\nu}(w) = 2^{\frac{\nu}{2}} \exp\left(-\frac{w^2}{4}\right) \left(\frac{\sqrt{\pi}}{\Gamma(\frac{1-\nu}{2})} {}_1F_1\left(-\frac{\nu}{2}, \frac{1}{2}; \frac{w^2}{2}\right) - \frac{w\sqrt{2\pi}}{\Gamma(-\frac{\nu}{2})} {}_1F_1\left(\frac{1-\nu}{2}, \frac{3}{2}; \frac{w^2}{2}\right) \right)$$

In the special case when $\nu \in \mathbb{N}_0$, the function D_{ν} can be expressed using the Hermite polynomials H_n :

$$D_{\nu}(w) = 2^{\frac{\nu}{2}} \exp\left(-\frac{w^2}{4}\right) H_{\nu}\left(\frac{w}{\sqrt{2}}\right)$$

The solutions in (2.2) are linearly dependent. For most values of ν , any of the four functions can be expressed as a linear combination of any two others. However, specifically in the case $\nu \in \mathbb{N}_0$ we get $D_{\nu}(w) = \pm D_{\nu}(-w)$.

Asymptotic behavior of the solutions is also given in Gradshteyn and Ryzhik [2014]. As $|w| \rightarrow \infty$, the solutions $D_{-\nu-1}(iw)$ and $D_{-\nu-1}(-iw)$ grow exponentially. Meanwhile $D_{\nu}(w)$ decays exponentially for $w \rightarrow +\infty$. Therefore $D_{\nu}(w)$ and $D_{\nu}(-w)$ are better suited for the growth conditions imposed by the domain of $\mathcal{H}(s)$. We define $c_{+1}, c_{+2}, c_{-1}, c_{-2} \in \mathbb{C}$, such that

$$h_{\pm} = c_{\pm 1} D_{\nu}(w) + c_{\pm 2} D_{\nu}(-w).$$

It can be further shown, that if $\nu \notin \mathbb{N}_0$, the solution $D_{\nu}(w)$ diverges for $w \rightarrow -\infty$. **[Then show it.]** Therefore $c_{-1} = c_{+2} = 0$ in order for φ to be integrable. Applying the gluing equations for h_- , h_+ now, we get:

$$\begin{aligned} c_{+1} D_{\nu}(w_0) &= c_{-2} D_{\nu}(-w_0) \\ c_{+1} \frac{d}{dw} D_{\nu}(w) \Big|_{w_0} - c_{-2} \frac{d}{dw} D_{\nu}(-w) \Big|_{w_0} &= \alpha \sqrt{2b} \end{aligned}$$

2.2 Title of the second subchapter of the second chapter

Conclusion

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List of Figures

List of Tables

A. Attachments

A.1 First Attachment