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1 Introduction

1.1 Basic Mathematical Models; Direction Fields

- A **mathematical model** is a differential equation that describes some physical process. E.g.

$$F = ma = m\frac{dv}{dt}$$
$$F = mg - \gamma v \text{ where } \gamma = \text{drag coefficient}$$

so putting the two equations together...

$$m\frac{dv}{dt} = mg - \gamma v$$

Constructing mathematical models involves recognizing and solving a few steps:

1. Identify independent and dependent variables. Keep in mind time is usually independent.
2. Choose units of measurement
3. Find a basic principal that governs the problem. For example: the rate of change of temperature of a glass of water is proportional to the difference between its temp and the environment's.
4. Express part 3 in terms of the variables in part 1. Make sure units match up. This can sometimes be a system of equations.

- **Direction fields** are formed by evaluating $\frac{dy}{dt} = f(t, y)$ at points in the $y - t$ plane. At each coordinate, a line segment is drawn with the slope of $f(t, y)$.

1.2 Solutions of Some Differential Equations

- The **initial value problem** is the solution of the differential equation, together with an initial condition

- An example of a simple initial value problem

$$\frac{dy}{dt} = ay - b \text{ with } y(0) = y_0$$
$$\frac{dy}{ay-b} = dt$$
$$\int \frac{dy}{ay-b} = \int dt$$
$$\frac{\log ay - b}{a} = t$$
$$e^{\log ay - b} = e^{at}$$
$$y = \frac{b + e^{at}}{a}$$
$$y = \frac{b}{a} + Ce^{at}$$

plug in for the initial condition

$$y(0) = y_0 = \frac{b}{a} + Ce^0$$
$$y_0 = \frac{b}{a} + C$$
$$C = y_0 - \frac{b}{a}$$

so our final equation looks like...

$$y = \frac{b}{a} + (y_0 - \frac{b}{a})e^{at}$$

1.3 Classification of Differential Equations

- **Ordinary and Partial Differential Equations**
  - Ordinary differential equations depend only on a single independent variable, and only *ordinary derivatives* appear in the differential equation.
  - Partial differential equations depend on multiple independent variables. Therefore *partial derivatives* appear in them.
- **Systems of Differential Equations**. This one is easy. If there are multiple unknown functions involved, you are dealing with a system of DEs. An example is the Lotka-Volterra (predator-prey) equations.

- **Order of the Differential Equation**. The order of a differential equation is the order of the highest derivative that appears in the equation. Thus $F(t, y(t), y'(t), y''(t), \ldots, y^n(t))$ is order n. We look at functions like these in the form

$$y^n = f(t, y, y', \ldots, y^{n-1}) \tag{1}$$

- **Linear and Nonlinear Equations**.
  - Linear equations are of the form:

    $$F(t, y, y', y'', \ldots, y^n) = 0$$

    The equation is a linear combination of all the derivatives up to n. Any y term can be multiplied by any function of t still, however.
  - Nonlinear equations, on the other hand, do not satisfy this constraint. Any equation with a product yy', y^2 or a term like $\sin(y)$ are all nonlinear.

1.4 Historical Remarks

Newton is great. Leibniz is also cool. Euler beats them both. The Bernoulli brothers loved their integrals, and a lot of fun was had by all.

2 First Order Differential Equations

This chapter deals with differential equations of the first order:

$$\frac{dy}{dt} = f(t, y) \tag{2}$$

Any differentiable function $y = \phi(t)$ that satisfies this equation for all t in some interval is called a solution.

2.1 Linear Equations; Method of Integrating Factors

The **first order linear equation** can be written in the form:

$$\frac{dy}{dt} + p(t)y = g(t) \tag{3}$$

The method in part 1 cannot solve this generally unless $p(t)$ and $g(t)$ are constants. We need a new method.

- Find an integrating factor μ, and multiply it on both sides:

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t)$$

- We now choose μ such that the left hand side is the derivative of some expression. Recall the product rule:

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y$$

- Now we can just pick a $\mu(t)$ such that

$$\begin{aligned}
\frac{d\mu(t)}{dt} &= \mu(t)p(t) \\
\frac{d\mu(t)}{\mu(t)} &= p(t)dt \\
\int \frac{d\mu(t)}{\mu(t)} &= \int p(t)dt \\
\log \mu(t) &= \int p(t)dt \\
\mu(t) &= e^{\int p(t)dt}
\end{aligned}$$

- Now our equation looks something like this:

$$(\mu(t)y)' = \mu(t)g(t) \rightarrow (e^{\int p(t)dt)}y)' = e^{\int pt(1)dt}g(t)$$

- Solving the equation yields

$$y = \frac{\int \mu(t)g(t)dt + C}{\mu(t)}$$

2.2 Separable Equations

Recall the general first order equation (2) is

$$\frac{dy}{dx} = f(x,y)$$

Note: this is not necessarily linear, and there is no universal method for solving this class of equations. We will consider a subclass of equations in this set. These equations are of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{4}$$

Note that we are simply rewriting equation (2). It is *always* possible to do this, simply by setting $M(x,y) = -f(x,y)$ and $N(x,y) = 1$.
An equation is **separable** if M is a function of x only, and N is a function of y only. Thus,

$$\begin{aligned} M(x) + N(y)\frac{dy}{dx} &= 0 \\ M(x)dx + N(y)dy &= 0 \end{aligned} \tag{5}$$

This can be solved by integrating M and N to find an implicit solution.

$$\int M(x)dx + \int N(y)dy = 0$$

Homogeneous Equations. If the right side of $\frac{dy}{dx} = f(x,y)$ can be expressed as a ratio of $\frac{y}{x}$ only, the equation is said to be homogeneous. These can always be solved with a change of dependent variable.

$$\begin{aligned} \frac{dy}{dx} &= \frac{y-4x}{x-y} \\ &= \frac{y-4x}{x-y}\frac{1/x}{1/x} \\ &= \frac{y/x-4}{1-y/x} \end{aligned}$$

let $v = \frac{y}{x}$ so $y = xv(x)$

$$\begin{aligned} \frac{dy}{dx} &= x\frac{dv}{dx} + v \\ \frac{v-4}{1-v} &= x\frac{dv}{dx} + v \\ x\frac{dv}{dx} &= \frac{v^2-4}{1-v} \end{aligned}$$

which is now separable!

2.3 Modeling with First Order Equations

- Mixing Problems. These problems involve mixing quantities (salt, toxin, etc) into water at a certain rate, and water leaving the system at a certain rate. We usually want to find the amount of salt in the tank at time t.
$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$
Where the rate out $= \frac{rQ}{volume}$. Usually, volume will remain constant.
- Compound interest. Let the value of an investment $= S$ and the interest rate $= r$.
$$\frac{dS}{dt} = rS$$
- Escape velocity. This is a more complex example and is on page 58.

2.4 Differences Between Linear and Nonlinear Equations

The Existence and Uniqueness Theorems
Theorem 2.4.1
If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation
$$y' + p(t)y = g(t)$$
for each t in I, and that also satisfies the initial condition
$$y(t_0) = y_0$$
where y_0 is an arbitrary prescribed initial value. **WARNING**. This only holds for **linear, first order** equations!

Theorem 2.4.2
Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem
$$y' = f(t, y) \text{ and } y(t_0) = y_0$$

2.5 Autonomous Equations and Population Dynamics

An **autonomous** function is a first order differential equation in which the independent variable does not appear explicitly.
$$\frac{dy}{dx} = f(y) \tag{6}$$

- Exponential Growth. A simple hypothesis about population growth: The rate of change of y is proportional to the current value of y.

$$\frac{dy}{dt} = ry$$

  Here $r =$ the **rate of growth or decline**. Solving for the initial value $y(0) = y_0$, we get

$$y = y_0 e^{rt}$$

- Logistic Growth. The exponential model is too ideal. The logistic equation takes into account that the growth *rate* depends on y.

$$\frac{dy}{dt} = (r - ay)y$$

  or, in the equivalent form:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

  where $K = \frac{r}{a}$. r is called the **intrinsic growth rate**, and K is the **carrying capacity**.

Equilibrium Solutions. An equilibrium solution is one such that $\frac{dy}{dt} = 0$. E.g. in the logistic equation

$$\frac{dy}{dx} = r\left(1 - \frac{y}{K}\right)y = 0$$
$$(1 - \frac{y}{K})y = 0$$

So when $y = 0, K$, our solution is at equilibrium. The equilibrium solutions of a general equation can be found by locating the roots of $f(y) = 0$. These are also called **critical points**.

Stability

- Asymptotically stable solution. If the y tends toward the solution regardless of whether it is initially above or below it.

- Unstable solution. The only way this can be a solution is if the initial condition is exactly this.

- Semistable solution. If solutions on one side tend to approach it, and depart from it on the other side.

2.6 Exact Equations and Integrating Factors

Suppose we have a differential equation of the form (4)

$$M(x,y) + N(x,y)y' = 0$$

Further suppose we can find a function ψ such that

$$\frac{\partial \psi}{\partial x}(x,y) = M(x,y)$$

$$\frac{\partial \psi}{\partial y}(x,y) = N(x,y)$$

$$\psi(x,y) = c$$

The differential equation is said to be **exact** if $M_y = N_x$. And if the equation is exact, then:

$$\psi_x(x,y) = M(x,y) \text{ and } \psi_y(x,y) = N(x,y)$$

Therefore, we integrate, holding y constant

$$\psi(x,y) = \int M(x,y)dx = Q(x,y) + h(y)$$

now, by differentiating this in terms of y we can find h

$$\psi_y(x,y) = \frac{\partial Q}{\partial y}(x,y) + h'(y) = N(x,y)$$

$$h'(y) = N(x,y) - \frac{\partial Q}{\partial y}(x,y)$$

Integrating Factors. Sometimes you can convert an inexact equation into an exact equation by multiplying the equation by a suitable integrating factor (recall 2.1). Let μ be such a factor. Our goal is to make the equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

exact. It will only be exact if

$$(\mu M)_y = (\mu N)_x$$

$$\mu_y M + M_y \mu = (\mu_x N + N_x \mu)$$

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

2.7 Numerical Approximations: Euler's Method

Euler's method involves

- Carrying out the linking of tangent lines in a systematic manner
- The resulting linear function approximates the actual solution

the line tangent to the curve at (t_0, y_0) is

$$y = y_0 + f(t_0, y_0)(t - t_0)$$