Constrained Geometric Attitude Control in SO(3)

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***Abstract*— The objective of this project was to find and implement a state of the art controller for attitude control of rigid bodies. The controller's model is found in [1] is the current state of the art and its properties have been extensively investigated using MATLAB simulations.**

I. THE ATTITUDE CONTROL PROBLEM

The attitude control problem is defined is typical of a rigid body system such as a satellite for example which needs to reorient itself for sending signals to the ground or to recharge it's solar panels. Often, the satellite would have an optical sensor or some sensitive component which needs to be shielded from direct exposure to the sun or space debris while it is reorienting itself. Therefore the dynamics of the rigid body is constrained . Furthermore there may be external disturbance moments on the system due to:

1. Changes in Inertia of the system such as when a satellite expends fuel to provide control moments.
2. Torque induced on body due gravity gradients(common in orbital spacecraft due to their non-spherical shapes)
3. Turbulence due to wind and rotor effects in Quadrotors.
4. Uneven mass distribution in system due to loading

The control problem therefore is defined as designing a robust control input u for the above described system taking into account constraints and disturbances.

The first issue related to attitude control comes from Representations. Existing attitude representations suffer from the following [4],[5] :

1. Ambiguity of parametrization: Parametrizations such as Euler angles suffer from the ambiguity problem i.e. there are 24 canonical representations of Euler angles. Similarly quaternions represent 2 antipodal points.
2. Singularities: Euler angles and axis-angle representations suffer from singularities due to gimbal lock(roll and yaw axes aligning) and when the rotation angle is 0 respectively.
3. Failure to capture Dynamics of Rotation: Attitude dynamics evolve on a non-linear manifold whose properties are not captured using locally linear parametrizations such as Euler angles.

These issues result in the a large number of unwanted problems with traditional controllers such as :

1. Non-smooth, Globally stable control law: The existence of singularities prevents a globally stable control law and forces a controller to stay in small part of the attitude space to ensure stability.
2. Failure at large Angular Slews: Linearized controllers based on Euler angles fail at large angular slews because of the Jacobian does not remain constant over the entire range of motion.
3. Unwinding behavior: Quaternion space controllers perform very well but are characterized by 'unwinding' behavior which occurs when the rigid body is close to its desired attitude but slews through large angles nonetheless due to two antipodal points representing the same rotation.
4. Failure due to presence of constraints: Existing linear controllers do not explicitly model constraints and so there is an additional requirement to apriori design trajectories for such systems.

The above issues can be resolved by formulating a non-linear controller on the space of rotation matrices SO(3).The advantage of such a controller is that it avoids singularities/ambiguities and exactly captures the dynamics of rotation. In addition with the use of potential functions a control law can be synthesized which accounts for obstacles, is globally stable and converges exponentially. This report attempts to describe and simulate such as controller .Section 2 describes the control problem, Section 3 proceeds to discuss the controller law and Section 4 shows the results of simulation.

II. DYNAMICS OF SYSTEM

A. *Attitude Dynamics of a Rigid body*

Consider the attitude dynamics of a rigid body. An inertial frame is defined using the standard orthonormal basis vectors *{* [1 0 0], [0 1 0], [0 0 1] *}*. A body fixed frame is defined using the standard orthonormal basis vectors $e_i = \{e_1, e_2, e_3\}$ with $i = 1,2,3$ having the origin at the center of mass of the body and aligned with it's principal directions. The dynamics of the body evolves on the space of rotation matrices SO(3) such that its equations of motion can be described using the following equations.

$$J\dot{\Omega} + \Omega \times J\Omega = \mathrm{u} + \mathrm{W}(\mathrm{R}, \Omega)\Delta$$

$$\dot{R} = R\widehat{\Omega}$$

Where $J \in \mathbb{R}^{3\times3}$ is the Inertia Matrix and $\Omega \in \mathbb{R}^3$ is the angular velocity of the body represented in the body fixed frame. $R \in \mathbb{R}^{3\times3}$ $such\ that\ R^TR = I\ and\ \det(R) = 1$ is the rotation matrix group,SO(3) representing the rotation from body fixed frame to inertial frame. The ^ symbol represents the skew symmetric matrix form of the vector. The external disturbance appears as a moment $\mathrm{W}(\mathrm{R}, \Omega)\Delta$ where W is the structure of the disturbance which is a known function of the attitude and angular velocity. The disturbance itself is represented by Δ the unknown but bounded function of time. This is the typical form of the disturbance moment which appears on the input side of the system.

B. Constraint Dynamics

The attitude inequality constraint for the system can be defined as

$$r^T R^T \mathrm{v} \leq cos\theta$$

Where r is the unit vector representing the pointing direction of say an optical sensor in the body fixed frame and v is the unit vector representing the direction of the debris or bright object towards which the sensor should not be pointing at. θ Represents the minimum angle of separation between r and $R^T \mathrm{v}$ and so the constraint dynamics have been defined.

III. ADAPTIVE CONTROLLER DESIGN

The first step in designing a non-linear controller involves choosing an error function which the controller uses to output the desired torques. The desired properties of this function is the following:

1. It is smooth and continuous
2. It is positive definite
3. Has a minima corresponding to zero error.
4. Does not have local minima.

The attitude error function can be represented as

$$\psi(R, R_d) = A(R, R_d)(1 + C_i(R, R_d))$$

Where

$$A(R, R_d) = \frac{1}{2} trace(G(I - R_d^T R))$$

$$B_i(R, R_d) = 1 - \left(\frac{1}{\alpha}\right) ln\left(\frac{cos\theta_\mathrm{i} - r^T R^T \mathrm{v_i}}{1 + cos\theta_\mathrm{i}}\right)$$

The A term is an attractive potential which attracts the system to the desired attitude R_d and B_i is a repulsive potential which repels the system away from the undesired direction $R^T \mathrm{v_i}$ at an angle θ_i corresponding to the ith 'obstacle' vector. $G \in \mathbb{R}^{3\times3}$ is used to shape the attractive potential while $\alpha \in \mathbb{R}$ can be used to shape the logarithmic repulsive barrier. In this way a controller law can be formulated to navigate the rigid body along the negative gradient of the potential field to the desired attitude as follows

$$u = -k_R e_R - k_\Omega e_\Omega + \Omega \times J\Omega - \mathrm{W}\bar{\Delta}$$

$$\dot{\bar{\Delta}} = k_\Delta W^T (e_\Omega + c e_R)$$

Where

$$\mathrm{e_R} = e_{R_A} \sum_i B_i + \sum_i A(R, R_d) e_{R_{Bi}}$$

$$e_\Omega = \Omega$$

With

$$e_{R_A} = 0.5\left(G R_d^T R - R^T R_d\, G\right)^V$$

$$e_{R_{Bi}} = \frac{(R^T \mathrm{v_i})^\wedge r}{\alpha(r^T R^T \mathrm{v_i} - cos\theta_\mathrm{i})}$$

Where e_{R_A} the error vector is related to attractive potential and $e_{R_{Bi}}$ is the error vector related the $\mathrm{i^{th}}$ repulsive potential. The two combine to give a net rotation error vector $\mathrm{e_R}$ .The angular velocity error is represented as e_Ω. The terms k_R, k_Ω are similar to proportional (P) and Derivative (D) gains respectively.

$\Omega \times J\Omega$ is a feedforward term which is used to cancel out the dynamics of the rigid body. $\bar{\Delta}$ is an estimator which estimates the unknown disturbance Δ and computes a correction term -W. $\bar{\Delta}$ to cancel out the effects of external disturbances . It is similar to an Integral (I) control term which explicitly models the system. k_Δ is the gain associated with the correction torque and c is a weight factor to determine the relative weight of attitude error and angular velocity error.

The stability and exponential convergence properties of the controller can be proven using the lyapunov function

$$V = \frac{1}{2} e_\Omega^T J e_\Omega + k_R \psi + c e_\Omega^T J^T e_\mathrm{R} + \frac{1}{2k_\Delta} e_\Delta^T e_\Delta$$

Where $e_\Delta = \Delta - \bar{\Delta}$ is the estimation error. By proper choice of gains the above lyapunov function is positive definite and decresent. Furthermore its derivative $\dot{V}$ is negative semidefinite. The La-Salle theorem can be used to prove that $||e_\Omega||$ and $||e_\mathrm{R}||$ converge to 0 as $t \to \infty$. A detailed discussion and proof of the above can be found in [1].

IV. SIMULATIONS AND RESULTS

The system has been simulated in matlab. The inertia tensor of the body was chosen to be

$$J = 10^{-3} \begin{bmatrix} 5.5 & 0.06 & -0.03 \\ 0.06 & 5.5 & 0.01 \\ -0.03 & 0.01 & 0.1 \end{bmatrix} kg\, m^2$$

The external disturbance is modeled as the sum of a fixed term and a time varying bounded term as shown below

$$\begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \end{bmatrix} + 0.02 \begin{bmatrix} sin9t \\ cos9t \\ \frac{1}{2}(cos9t + sin9t) \end{bmatrix}$$

The controller parameters and disturbance model are chosen to be

$$G = diag[0.9, 1.1, 1.0], W = I;$$

$$c = 1.0, k_R = 0.4, k_\Omega = 0.296, k_\Delta = 0.5\ \alpha = 15$$

Four Constraint Vectors are defined in the table below.

Constraint Vector	**Angle(θ)**
$(0.174,-0.934,-0.034)^T$	40°
$(0,0.7071,0.7071)^T$	40°
$(-0.853,0.436,-0.286)^T$	40°
$(-0.122,-0.140,-0.983)^T$	20°

The matrix G as described previously can be chosen to increase the gradient of the attractive potential term and α is chosen based on the nature of the obstacle. If the obstacle is diffuse then a small value of α is appropriate to give a small gradient to the repulsive potential while for sharp maneuvers a large α is chosen to give a steep gradient repulsive potential. The gains can be chosen using classical control analysis based on settling time and overshoot specifications. In this simulation r is chosen to be $(1,0,0)^T$, $R_0 = \exp(225 * \frac{pi}{180}\ \widehat{(0,0,1)})$ the desired trajectory is chosen to be $R_d = I, \Omega_d = 0$. The initial estimate of the disturbance is chosen to be $(0.5,0.5,0.5)^T$. Figure 1 and Figure 2 show the evolution of the system through 10 seconds. As expected the system converges to the desired attitude and the disturbance estimate is bounded about the true value of the disturbance. It is seen that the angle constraints are all preserved. The Lyapunov function is positive definite and decresent as predicted and its derivative is negative semidefinite.

Another simulation is done with same conditions as before except that after 4 seconds the desired trajectory is changed to $R_d = R_0, , \Omega_d = 0$. Figure 3 and Figure 4 show the evolution of the system through 10 seconds. Similar conclusions as before can be made but this time there is a sudden spike in the Lyapunov function at t = 4s.This is expected as changing the the desired attitude is akin to adding 'energy' to the system. This does not affect the stability of the system as the Lyapunov function behaves the same way as before (i.e decresent and positive semidefinite) .

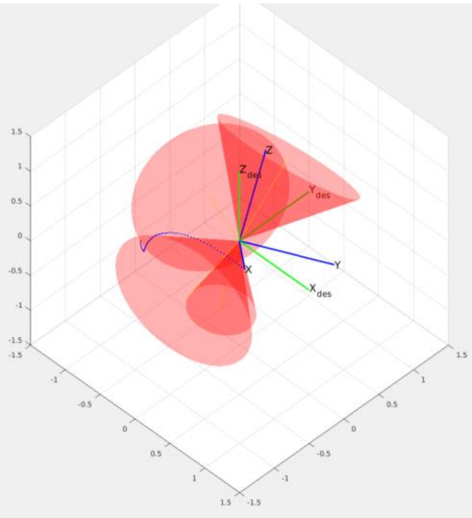

Figure 0. Sample Trajectory of System

A sample snapshot of it's trajectory is shown in Figure 0. The controller has been hence validated.

V. Conclusion

As seen in the simulation there is no need to compute apriori a continuous trajectory for the controller. A series of desired 'keypoint' attitudes (from a planner like RRT) can be fed into the controller and the controller takes the shortest 'path'(minimizing the integral of the first variation of $\psi(R)$) to the desired attitude.It may be necessary to control the rate of convergence using time scaling. This is particularly useful in say the case of a satellite which may be damaged due to large angular accelerations.

A demo of a hexrotor (Flight Control Lab@ George Washington University) working on this controller can be found at
https://www.youtube.com/watch?v=dsmAbwQram4&list=UUREzO49zrgcepGkLW4Qgn2Q&index=7
The controller is seen to be robust and can easily implemented in realtime embedded hardware.

ATTACHED CODE README

The simulation can be observed by running the file runsim.m.The parameters can be changed by changing the quad_params.m file. The sample trajectories can be switched by uncommenting the lines of that particular trajectory in trajectory.m. Alternatively the trajectory.m function can be rewritten to for any desired trajectory. The disturbance model can be edited in the file noise.m.The controller has been implemented in the file controller.m.A helper function inv_hat and hat has been used to convert between skew symmetric form and vector form.

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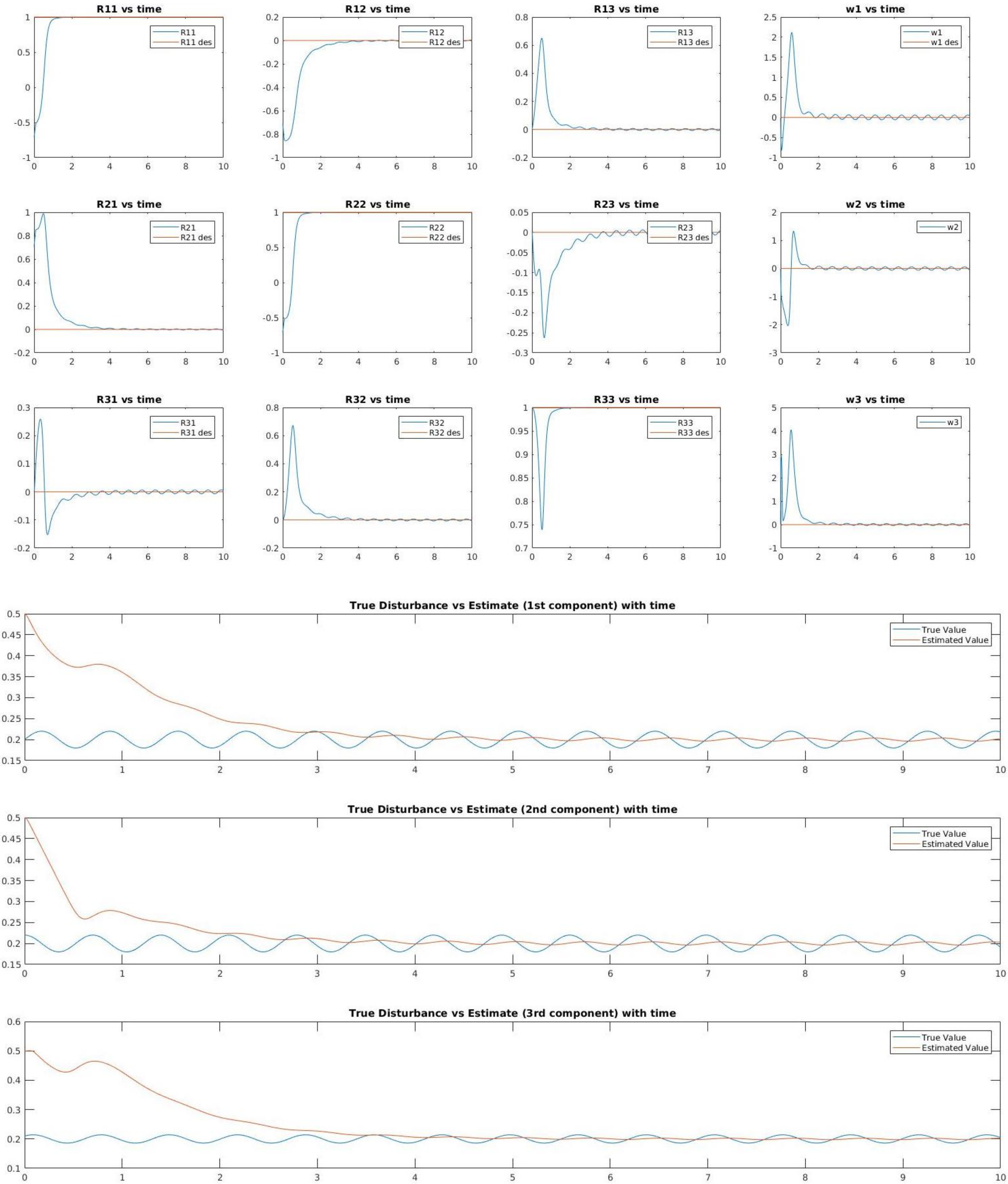

Figure 1 Full State Evolution of the System for 1 Way point

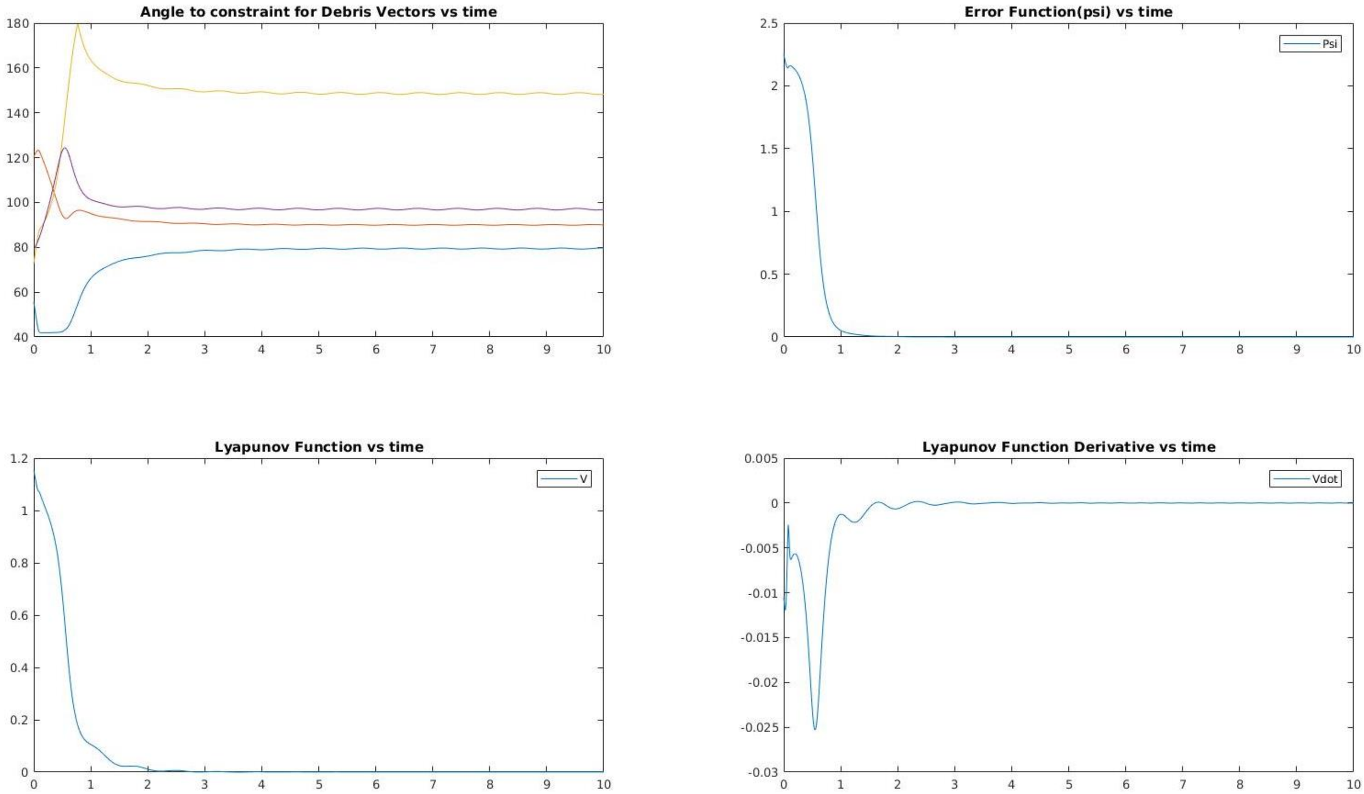

Figure 2 Constraints, Lyapunov and Error State Evolution for System

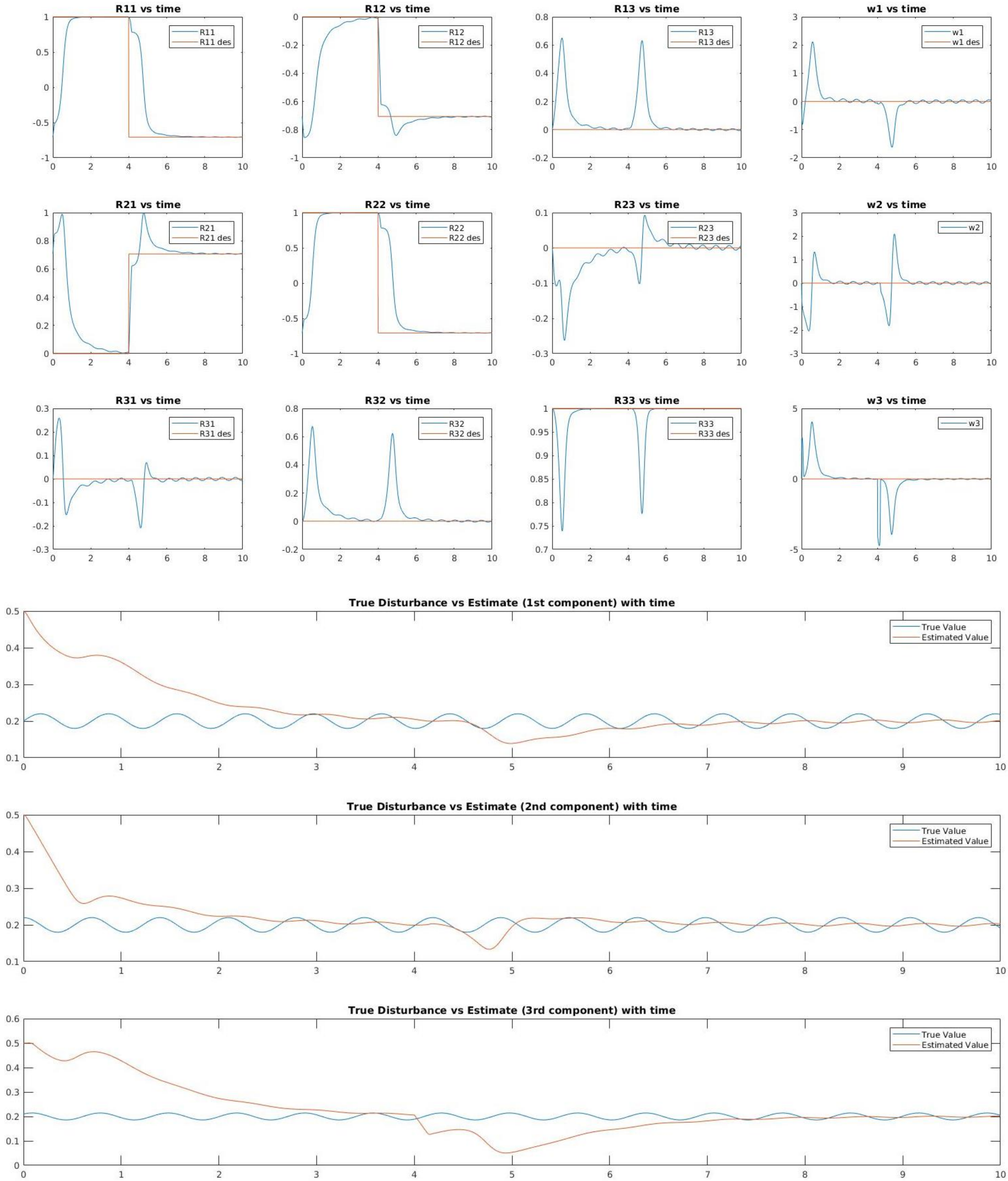

Figure 3 Full State Evolution of the System for 2 Way points

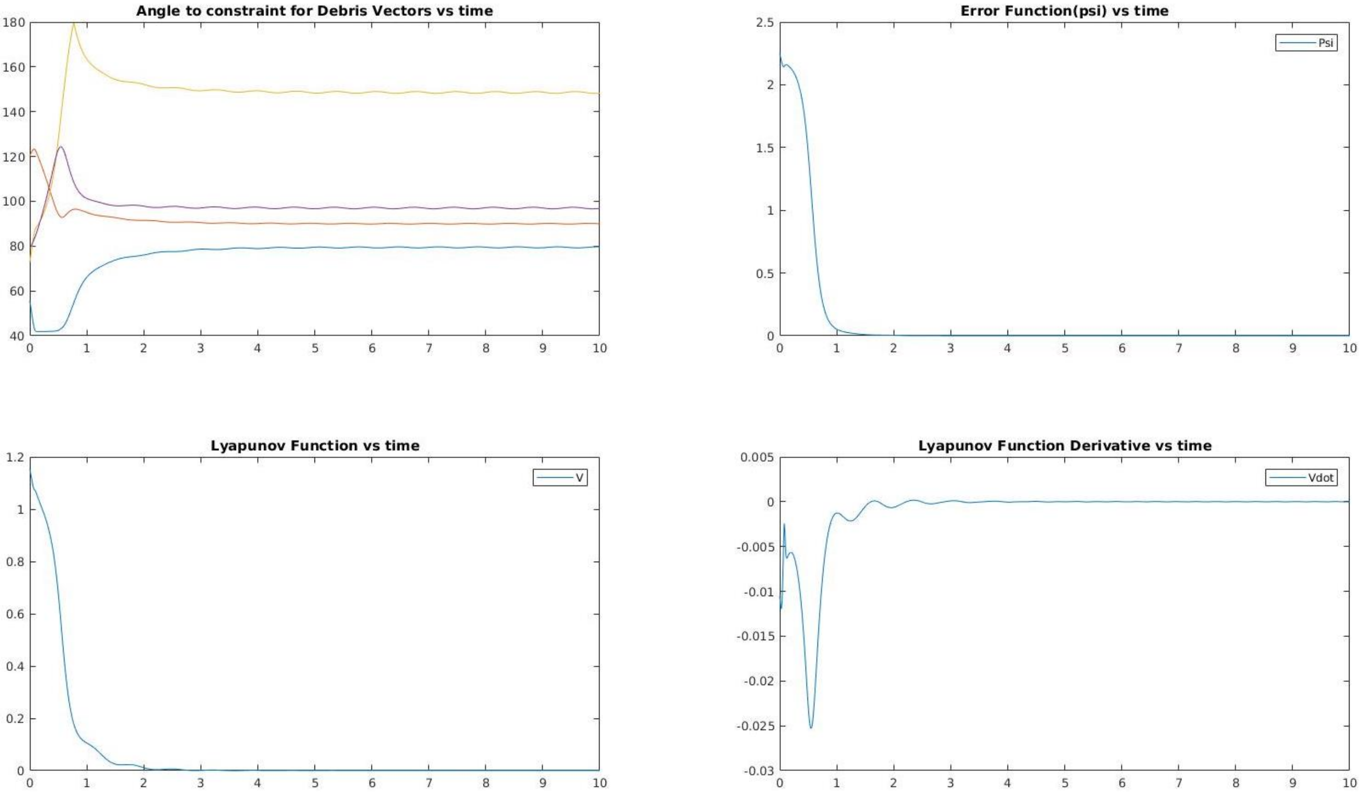

Figure 4 Constraints, Lyapunov and Error State Evolution for System