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CHAPTER 29

Spectral Wave-Current Bottom Boundary Layer Flows

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Abstract

Based on the linearized governing equations, a bottom roughness specified by the equivalent Nikuradse sand grain roughness, k_N, and a time-invariant eddy viscosity analogous to that of Grant and Madsen (1979 and 1986) the solution is obtained for combined wave-current turbulent bottom boundary layer flows with the wave motion specified by its near-bottom orbital velocity directional spectrum. The solution depends on an *a priori* unknown shear velocity, u_{*r}, used to scale the eddy viscosity inside the wave boundary layer. Closure is achieved by requiring the spectral wave-current model to reduce to the Grant-Madsen model in the limit of simple periodic plane waves. To facilitate application of the spectral wave-current model it is used to define the characteristics (near-bottom orbital velocity amplitude, u_{br}, radian frequency, ω_r, and direction of propagation, ϕ_{wr}) of a representative periodic wave which, in the context of combined wave-current bottom boundary layer flows, is equivalent to the wave specified by its directional spectrum. Pertinent formulas needed for application of the model are derived and their use is illustrated by outlining efficient computational procedures for the solution of wave-current interaction for typical specifications of the current.

Introduction

Over the past couple of decades several theoretical models for turbulent bottom boundary layers associated with combined wave-current flows have been proposed. In view of the vastly different levels of sophistication with which these different models represent turbulence, their end result, most

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notably their predicted effect of the presence of waves on the current, is surprisingly similar. Thus, in the absence of solid experimental evidence as to which of the many models is truly *the* best, choosing a particular model for applications becomes a matter of personal preference and convenience. However, since all existing wave-current interaction models have been derived for a wave motion corresponding to a simple periodic plane progressive wave, one additional choice – the choice of simple periodic wave characteristics to represent a wave motion which more realistically is described by its directional spectrum – must be made prior to model applications.

The objective of this study is therefore to derive a simple theoretical model for turbulent wave-current bottom boundary layer flows for a wave motion described by its directional spectrum and to use this model to determine the characteristics of the periodic wave which, in the context of wave-current interaction, is equivalent to the directional sea.

Theoretical Model

The theoretical model for spectral wave-current boundary layer flows is based on the linearized boundary layer equation which, in standard notation, reads

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial z}\left[\nu_t \frac{\partial \mathbf{u}}{\partial z}\right] \tag{1}$$

in which the turbulent eddy viscosity, ν_t , is assumed to be scaled by an *a priori* unknown shear velocity, u_{*_r}, which inside the wave boundary layer, $z < \delta_{wc}$, reflects the combined wave-current flow and outside the wave boundary layer is a function of only the average, *i.e.* the current, shear velocity, u_{*_c}. To keep the analysis relatively simple and because the universally made *assumption* of a *single* roughness scale, the equivalent Nikuradse sand grain roughness, k_N, for currents and waves alone as well as for combined wave-current flows has been experimentally validated (Mathisen and Madsen, 1993) only for this model, we adopt the Grant-Madsen (1979 and 1986) eddy viscosity

$$\nu_t = \begin{cases} \kappa u_{*r} z & \text{for} \quad z < \delta_{wc} \\ \kappa u_{*c} z & \text{for} \quad z > \delta_{wc} \end{cases} \tag{2}$$

Assuming u_{*r} and therefore ν_t to be time-invariant, resolving velocity and pressure into their mean and time-varying components, *i.e.*

$$\mathbf{u} = \bar{\mathbf{u}} + \tilde{\mathbf{u}} = \mathbf{u}_c + \mathbf{u}_w \quad ; \quad p = \bar{p} + \tilde{p} = \bar{p} + \tilde{p}_b \tag{3}$$

and introducing these expressions in the governing equation, two separate equations are obtained. One for the wave motion

$$\frac{\partial(\mathbf{u_w}-\mathbf{u_b})}{\partial t}=\frac{\partial}{\partial z}\left[\nu_t\frac{\partial(\mathbf{u_w}-\mathbf{u_b})}{\partial z}\right] \tag{4}$$

in which the relationship between $\tilde{p}_b$ and the near-bottom velocity $\mathbf{u_b}$, predicted by potential theory, has been invoked; and another equation for the current

$$\nu_t\frac{\partial \mathbf{u_c}}{\partial z}=\tau_c/\rho=u_{*c}^2\{\cos\phi_c,\sin\phi_c\} \tag{5}$$

in which the law-of-the-wall arguments have been used, and ϕ_c denotes the current angle with the x-axis.

Wave Solution

For the wave solution only the eddy viscosity formulation assumed for $z<\delta_{wc}$ is pertinent. Hence, the wave portion of the wave-current problem is completely analogous to the pure wave boundary layer problem discussed and solved by Madsen *et al.* (1988) for a wave motion described by its directional spectrum, *i.e.* the solution to (4) may be written as a sum of wave components, each being the real part of

$$\mathbf{u}_{\mathbf{w}nm}=\left[1-\frac{\ker 2\sqrt{\zeta_n}+i\,\mathrm{kei}2\sqrt{\zeta_n}}{\ker 2\sqrt{\zeta_{no}}+i\,\mathrm{kei}2\sqrt{\zeta_{no}}}\right]\mathbf{u}_{\mathbf{b}nm}e^{i\omega_n t} \tag{6}$$

in which the velocity is related to the directional near-bottom orbital velocity spectrum, $S_{u_b}(\omega,\theta)$, through,

$$\mathbf{u}_{\mathbf{b}nm}=\sqrt{2S_{u_b}(\omega_n,\theta_m)d\theta d\omega}\{\cos\theta_m,\sin\theta_m\} \tag{7}$$

ker and kei denote the zeroth order Kelvin functions and

$$\zeta_n=z\omega_n/(ku_{*r}) \tag{8}$$

with ζ_{no} denoting the value of ζ_n at $z=z_o=k_N/30$ where k_N is the equivalent Nikuradse sand grain roughness of the bottom.

Current Solution

The solution for the current, obtained from (5) and (2), is for $z<\delta_{wc}$

$$\mathbf{u_c}=\frac{u_{*c}u_{*c}}{u_{*r}\kappa}\ell n\frac{z}{z_o}\{\cos\phi_c,\sin\phi_c\} \tag{9}$$

and for $z>\delta_{wc}$

$$\mathbf{u_c}=\frac{u_{*c}}{\kappa}\ell n\frac{z}{z_{oa}}\{\cos\phi_c,\sin\phi_c\} \tag{10}$$

in which z_{oa} – the apparent bottom roughness experienced by the current in presence of waves – is obtained by matching the currrent velocities at $z=\delta_{wc}$, *i.e.*

$$\ell n\ \frac{\delta_{wc}}{z_{oa}}=\frac{u_{*c}}{u_{*r}}\ \ell n\ \frac{\delta_{wc}}{z_o} \tag{11}$$

Closure

In the preceeding analysis we have formally obtained the solution for the near-bottom turbulent flow associated with a wave motion described by its directional frequency spectrum and a superimposed steady current. However, the solution can be evaluated only when the value of the representative wave-current shear velocity, u_{*r}, is known. To determine this unknow, *i.e.* to close the problem, we obtain the bottom shear stress for each wave component from

$$\tau_{\mathbf{w}nm}=\rho\kappa u_{*r}z\frac{\partial\mathbf{u}_{\mathbf{w}nm}}{\partial z}\bigg|_{z=z_o}=$$

$$\rho\kappa u_{*r}\sqrt{\zeta_n}\frac{\partial\mathbf{u}_{\mathbf{w}nm}}{\partial(2\sqrt{\zeta_n})}\bigg|_{\zeta_n=\zeta_{no}}=$$

$$\rho\kappa u_{*r}\sqrt{\zeta_{no}}\ \frac{-\ker'2\sqrt{\zeta_{no}}-i\ \mathrm{kei}'2\sqrt{\zeta_{no}}}{\ker 2\sqrt{\zeta_{no}}\ +i\,\mathrm{kei}2\sqrt{\zeta_{no}}}\ \mathbf{u}_{\mathbf{b}nm}e^{i\omega_n t} \tag{12}$$

in which "prime" denotes the derivative of the zeroth order Kelvin function with respect to its argument, and $\mathbf{u}_{\mathbf{b}nm}$ is given by (7).

This equation may alternatively be interpreted as an expression for the directional spectrum of the wave-associated bottom shear stress

$$S_{\tau_w}(\omega,\theta)=K^2(\omega)S_{u_b}(\omega,\theta) \tag{13}$$

$$K(\omega) = \rho\kappa u_{*r}\sqrt{\zeta_o}\left|\frac{-\mathrm{ker}'2\sqrt{\zeta_o} - i\mathrm{kei}2\sqrt{\zeta_o}}{\mathrm{ker}2\sqrt{\zeta_o} \ + i\mathrm{kei}2\sqrt{\zeta_o}}\right| \tag{14}$$

is a function of ω since

$$\zeta_o = \frac{z_o\omega}{\kappa u_{*r}} = \frac{k_N\omega}{30\kappa u_{*r}} \tag{15}$$

From (13) we may obtain the representative amplitude of the bottom shear stress of a simple harmonic wave with the same variance as the spectral representation, *i.e.* the root-mean-square amplitude,

$$\tau_{wr}^2 = 2\iint S_{\tau_w}(\omega,\theta)d\theta d\omega = \int_0^\infty\int_0^{2\pi} K^2(\omega)\left[2S_{u_b}(\omega,\theta)\right]d\theta d\omega \tag{16}$$

with a direction given by

$$\tan\theta_{wr} = \frac{\iint S_{\tau_w}(\omega,\theta)\sin\theta d\omega d\theta}{\iint S_{\tau_w}(\omega,\theta)\cos\theta d\omega d\theta} \tag{17}$$

In principle (16) and (17) may be evaluated if the bottom roughness, k_N, and the near-bottom orbital velocity spectrum, $S_{u_b}(\omega,\theta)$, are known. This would lead to a representative wave-associated bottom shear stress amplitude vector

$$\boldsymbol{\tau}_{\mathbf{wr}} = \tau_{wr}\{\cos\theta_{wr}, \sin\theta_{wr}\} \tag{18}$$

which depends on the unknown representative wave-current shear velocity, u_{*r}.

Final closure is obtained by requiring the spectral wave-current solution to reduce to that of Grant and Madsen (1986) in the limit of simple periodic waves, *i.e.*

$$u_{*r}^2 = \frac{1}{\rho}\left|\boldsymbol{\tau}_{\mathbf{wr}} + \boldsymbol{\tau}_{\mathbf{c}}\right| \tag{19}$$

The bottom shear stress in (12) is evaluated at $z = z_o = k_N/30$ rather than by taking the limit $z \to 0$, which was used in Madsen *et al.* (1988). For small values of ζ_o which, by (15), is seen to correspond to small values of the bottom roughness, this difference is of negligible importance. However, for larger roughness values and hence larger values of ζ_o this difference has important implications that will be discussed in Appendix A.

The Representative Periodic Wave

Actual application of the theoretical spectral wave-current model as presented in the preceeding section would be extremely cumbersome, particularly since the evaluation of the integrals in (16) and (17) presumes u_{*r} to be known. For this reason a further simplification is achieved by introducing the concept of a representative periodic wave which, in the context of wave-current interaction, is equivalent to the spectral wave representation.

This representative periodic wave is characterized by its near-bottom orbital velocity amplitude, u_{br}, radian frequency, ω_r, and direction of propagation, ϕ_{wr}. To obtain the representative wave characteristics we start by writing (16) in the form

$$\begin{gathered}\tau_{wr}^2 = 2\int_0^\infty K^2(\omega)\int_0^{2\pi} S_{u_b}(\omega,\theta)d\theta d\omega = \\ \int_0^\infty\left[K^2(\omega_r) + \left.\frac{\partial K^2}{\partial\omega}\right|_{\omega=\omega_r}(\omega-\omega_r) + \cdots\right]\int_0^{2\pi} 2S_{u_b}(\omega,\theta)d\theta d\omega\end{gathered} \tag{20}$$

Neglecting higher order terms in the expansion of the transfer function $K^2(\omega)$ around $\omega = \omega_r$, the expression given by (20) reduces to that for a simple periodic wave

$$\tau_{wr} = \tau_{wm} = K(\omega_r)u_{br} \tag{21}$$

in which τ_{wm} is the maximum bottom shear stress of the periodic wave with the representative wave orbital velocity amplitude given by

$$u_{br} = \sqrt{2\iint S_{u_b}(\omega,\theta)d\omega d\theta} \tag{22}$$

and the representative wave radian frequency taken as

$$\omega_r = \frac{\iint \omega S_{u_b}(\omega,\theta)d\omega d\theta}{\iint S_{u_b}(\omega,\theta)d\omega d\theta} \tag{23}$$

Finally, the direction of propagation of the representative periodic wave may be obtained from (17) as

$$\tan\phi_{wr} = \frac{\iint S_{u_b}(\omega,\theta)\sin\theta d\omega d\theta}{\iint S_{u_b}(\omega,\theta)\cos\theta d\omega d\theta} \tag{24}$$

when it is assumed either that $K(\omega)$ is sufficiently accurately representated by $K(\omega_r)$ or that the directional spreading function for $S_{u_b}(\omega,\theta)$ is independent of radian frequency.

For completeness it is noted that the expression for ω_r, (23), differs from that obtained by Madsen *et al.* (1988). Since Madsen *et al.* (1988) based their representative radian frequency on rather intuitive arguments whereas (23) is based on more rigorous considerations the expression given here should be considered the correct definition of ω_r. The effects of small variations of ω_r on the predicted wave-current interaction are, however, insignificant compared to other uncertainties so this point is made primarily to avoid confusion.

Application of Spectral Wave-Current Model

Pertinent Formulas

To illustrate the application of the spectral wave-current model we assume that the current is specified and that the characteristics of the representative periodic wave are known.

We first define the angle between current direction, ϕ_c, and direction of wave propagation, ϕ_{wr}, as

$$\phi_{cw} = \phi_c - \phi_{wr} \tag{25}$$

With this definition, the representative shear velocity is obtained from (19) as

$$u_{*r}^2 = \frac{1}{\rho}\left|\tau_{wr}\{1,0\} + \tau_c\{\cos\phi_{cw}, \sin\phi_{cw}\}\right| = C_\mu u_{*wm}^2 \tag{26}$$

in which $u_{*wm} = \sqrt{\tau_{wr}/\rho}$ is the shear velocity based on the maximum representative wave-associated shear stress,

$$C_\mu = \left(1 + 2\mu\left|\cos\phi_{cw}\right| + \mu^2\right)^{1/2} \tag{27}$$

and

$$\mu = \tau_c/\tau_{wr} = \left(\frac{u_{*c}}{u_{*wm}}\right)^2 \tag{28}$$

expresses the ratio of current and wave bottom shear stresses; a ratio which generally is much smaller than unity and therefore results in values of C_μ, given by (27) , close to unity.

To obtain the maximum wave shear stress, we introduce the wave friction factor concept in the presence of a current through the definition

$$\frac{1}{\rho}\tau_{wr} = u_{*wm}^2 = \frac{1}{2}f_{wc}u_{br}^2 \tag{29}$$

Introducing (14) and (15), with $\omega = \omega_r$, in (21) and using (29) with (26) to replace shear stresses and shear velocities lead to an implicit equation for the wave friction factor in the presence of a current

$$\sqrt{f_{wc}/C_\mu} = \kappa\sqrt{2\zeta_{ro}}\left|\frac{-\mathrm{ker}'2\sqrt{\zeta_{ro}} - i\mathrm{kei}'2\sqrt{\zeta_{ro}}}{\mathrm{ker}2\sqrt{\zeta_{ro}} \quad + i\mathrm{kei}2\sqrt{\zeta_{ro}}}\right| \tag{30}$$

in which

$$\zeta_{ro} = \frac{k_N\omega_r}{30\kappa u_{*r}} = \frac{(k_N\omega_r)/(u_{br}C_\mu)}{(30\kappa/\sqrt{2})\sqrt{f_{wc}/C_\mu}} \tag{31}$$

Written in this form clearly brings out the feature that the wave friction factor, in the presence of a current, is a function of the relative magnitude of the current shear stress, expressed through the factor C_μ defined by (27) and (28), and the bottom roughness, $(C_\mu u_{br}/\omega_r)/k_N = C_\mu A_{br}/k_N$, relative to the representative wave near-bottom orbital excursion amplitude, $A_{br} = u_{br}/\omega_r$, modified by the factor C_μ to account for the presence of a current. It is particularly interesting to note that (30) and (31) reduce to the pure wave case in the absence of any current since then $C_\mu = 1$. Thus, (30) and (31) may be regarded as the generalized wave friction factor relationship valid both in the presence and absence of a current.

To evaluate the wave friction factor as a function of relative roughness, series expansions for zeroth order Kelvin functions and their derivatives, given in Abramowitz and Stegun (1972, Chapter 9), are used to obtain $\sqrt{f_{wc}/C_\mu}$ from (30) with von Karman's constant $\kappa = 0.4$ for a chosen value of ζ_{ro}. Once $\sqrt{f_{wc}/C_\mu}$ is obtained from (30) the corresponding relative roughness, $C_\mu u_{br}/(\omega_r k_N)$, is obtained from (31). The resulting relationships between wave friction factor and relative roughness is shown in Figure 1, and may be approximated by the following explicit formulas

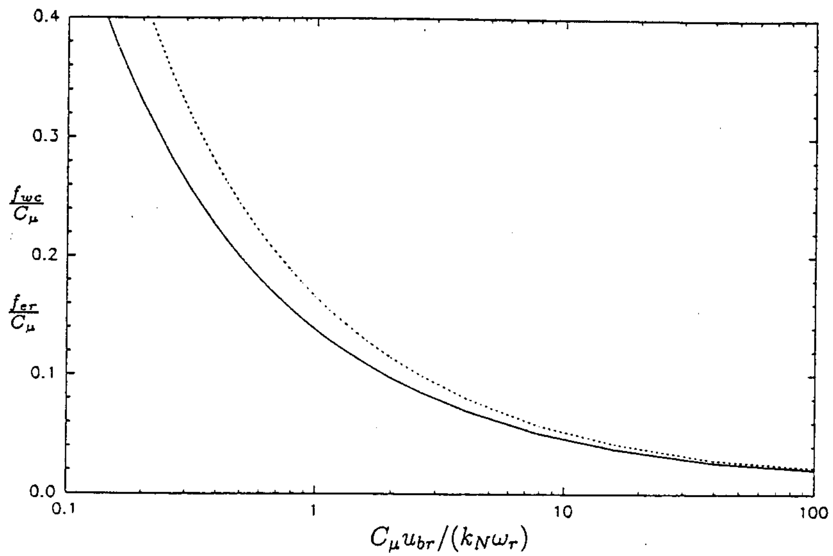

Figure 1: Generalized Friction Factor Diagram for Waves in the Presence of a Current. Wave Friction Factor, f_{wc}, (dashed line), Wave Energy Dissipation Factor, f_{er} (full line), for the Representative Periodic Wave.

$$f_{wc} = C_\mu \exp\left\{7.02\left(\frac{C_\mu u_{br}}{k_N\omega_r}\right)^{-0.078} - 8.82\right\} \tag{32}$$

for $\quad 0.2 < C_\mu u_{br}/(k_N\omega_r) < 10^2$;

and

$$f_{wc} = C_\mu \exp\left\{5.61\left(\frac{C_\mu u_{br}}{k_N\omega_r}\right)^{-0.109} - 7.30\right\} \tag{33}$$

for $\quad 10^2 < C_\mu u_{br}/(k_N\omega_r) < 10^4$.

These expressions are accurate to within about 1% for the indicated ranges of relative roughness and are far superior to the approximate implicit friction factor equations that may be drived from (30) and (31) under the assumption of small values ζ_{ro}, *i.e.* large values of $C_\mu u_{br}/(k_N\omega_r)$. Thus the wave-current friction factor equation given *e.g.* by Madsen *et al.* (1988) is not only far more cumbersome to use but it is also less accurate than (33)! The author is indebted to Prof. Stephen R. McLean (U.C. Santa Barbara, personal communication) for pointing out the advantages of friction factor formulas of this type (originally suggested by Swart, 1974).

The Wave Boundary Layer Thickness, δ_{wc}

The matching level for the current profile, here referred to as the wave boundary layer thickness and denoted by δ_{wc}, has up to this point not been defined in quantitative terms.

Considerations of δ_{wc} as the level required for the wave orbital velocity to approach its free stream value within a certain percentage were used by Grant and Madsen (1979 and 1986) to arrive at

$$\delta_{wc} = \alpha\frac{\kappa u_{*r}}{\omega_r} \tag{34}$$

with α-values in the range of 1 to 2. Madsen and Wikramanayake (1991) concluded from a comparison of current velocity profiles predicted by the Grant-Madsen model with limited experimental result as well as predictions afforded by more sophisticated turbulence closure models that reasonable agreement was obtained for α-values in the range of 1 to 1.5.

If therefore one accepts (34) as the appropriate expression for δ_{wc} one is faced with the problem of predicting a wave boundary layer thickness smaller than the equivalent Nikuradse sand grain roughness, k_N, when the roughness is large. This apparent inconsistency of the wave-current theory may be removed by requiring that δ_{wc} should be at least some fraction of the Nikuradse roughness, k_N. Choosing, somewhat arbitrarily, $\delta_{wc} \geq k_N$ for the current profile matching level leads to a limiting value of (34) given by

$$\frac{k_N\omega_r}{\kappa u_{*r}} = 30\zeta_{ro} \leq \alpha \tag{35}$$

in which (31) was used.

Choosing, in honor of Bill Grant, $\alpha = 2$ use of (35) in (30) and subsequent use of (31) leads to a limiting value of the relative roughness for which δ_{wc} is determined from (34), and results in the following definition:

$$\delta_{wc} = \begin{cases} 2\kappa u_{*r}/\omega_r & \text{for} \quad C_\mu u_{br}/(k_N\omega_r) > 8 \\ k_N & \text{for} \quad C_\mu u_{br}/(k_N\omega_r) < 8 \end{cases} \tag{36}$$

Whereas there is evidence in support of (34) for small relative roughness, the "prediction" of $\delta_{wc} = k_N$ should be regarded as tentative (at best). Since large roughness values, for naturally occuring wave-current flows over a movable bed, generally are associated with ripples and since $k_N \simeq 4\times$ (ripple height) for two-dimensional equilibrium-range ripples, *e.g.* Grant and Madsen (1982), the prediction from (36) of $\delta_{wc} = 4\times$ (physical scale of two-dimensional roughness features) does, however, appear reasonable.

Application

Assuming the bottom roughness ($k_N = 30z_o$) and the representative periodic wave characteristics(u_{br}, ω_r, and ϕ_{wr}) known the method of solution depends on the specification of the current.

Current Specified by u_{*c}^2 and ϕ_c. This specification may physically arise in near-shore waters with a strong wind-driven shore-parallel velocity. In this situation the bottom shear stress may be approximated by the shore-parallel wind stress.

Obtaining ϕ_{cw} from (25), solution is started by initially taking $\mu = \mu^{(o)} = 0$ in (28), to obtain $C_\mu = C_\mu^{(o)} = 1$ from (27). With $C_\mu = C_\mu^{(o)}$ and known wave and bottom roughness conditions $f_{wc} = f_{wc}^{(o)}$ is obtained from (32) or (33), whichever is appropriate. Now, $f_{wc}^{(o)}$ and (29) provide a first estimate of $u_{*wm}^{(o)}$, so that (28) may be revisited to obtain an improved estimate of $\mu = \mu^{(1)}$. With $\mu = \mu^{(1)}$ the procedure is repeated and iteration is terminated when $f_{wc}^{(n+1)} = f_{wc}^{(n)}$ within 1%.

Convergence is generally achieved within a few iterations and from knowledge of u_{*c} and u_{*wm} the represenatative shear velocity, u_{*r}, is obtained from (26). The current velocity profile may now be evaluated from (9) and (10) with δ_{wc} specified by (36) and the apparent bottom roughness, z_{oa}, may be determined from (11).

Current Specified by ϕ_c and $u_c(z_r) = u_{cr}$. This specification of the current corresponds to a current velocity and direction measured at a given elevation, z_r, above the bottom. (It is assumed that $z_r > \delta_{wc}$.)

For this specification of the current the solution procedure is somewhat more cumbersome. Again, after use of (25) the iterations are started by $\mu = \mu^{(o)} = 0$ in (28) and $C_\mu = C_\mu^{(o)} = 1$ from (27) resulting in $f_{wc} = f_{wc}^{(0)}$ from (32) or (33) and then $u_{*wm} = u_{*wm}^{(o)}$ from (29) followed by $u_{*r} = u_{*r}^{(o)}$ from (26). Since it is assumed that $z_r > \delta_{wc}$, the specified current velocity is given by (10), which with the aid of (11) may be written as a quadratic equation in the current shear velocity, u_{*c},

$$u_c(z_r) = u_{cr} = \frac{u_{*c}}{\kappa}\ell n\frac{z_r}{\delta_{wc}} + \frac{u_{*c}^2}{\kappa u_{*r}}\ell n\frac{\delta_{wc}}{z_o} \qquad (37)$$

Since δ_{wc} is given by (36) this equation, with $\kappa = 0.4, z_o = k_N/30$ and the latest value of u_{*r} may be solved to give

$$u_{*c} = \frac{u_{*r}}{2}\frac{\ell n(z_r/\delta_{wc})}{\ell n(\delta_{wc}/z_o)}\left(-1 + \sqrt{1 + \frac{4\kappa\ell n(\delta_{wc}/z_o)}{(\ell n(z_r/\delta_{wc}))^2}\frac{u_{cr}}{u_{*r}}}\right) \qquad (38)$$

For the initial iteration $u_{*r} = u_{*r}^{(o)}$ in (36) and (38) yields the first approximation for the current shear velocity $u_{*c} = u_{*c}^{(o)}$. With $u_{*c} = u_{*c}^{(o)}$ and $u_{*wm} = u_{*wm}^{(o)}$ the value of μ may be updated by use of (28) and the procedure may be repeated until convergence is achieved ($f_{wc}^{(n+1)} = f_{wc}^{(n)}$ within 1%). Again, convergence is generally achieved after a couple of iterations and the current velocity profile is determined from (9) and (10) with the apparent bottom roughness obtained from (11).

Discussion and Conclusions

Based on the linearized governing equations and adopting a simple time-invariant eddy viscosity a solution for combined wave-current turbulent bottom boundary layer flows was obtained for a wave motion specified by its near-bottom orbital velocity directional spectrum, $S_{u_b}(\omega, \theta)$. Closure was achieved by requiring the model to reduce the Grant-Madsen model in the limit of simple periodic plane waves. The spectral wave-current interaction model was used to determine the characteristics of a representative periodic wave which, in the context of combined wave-current bottom boundary layer flows, was equivalent to the directional sea. This representative periodic wave is specified by u_{br}, its near-bottom orbital velocity amplitude,

$$u_{br}^2 = \iint 2S_{u_b}(\omega,\theta)d\omega d\theta,$$

$\omega = \frac{2\pi}{T}$

i.e. the root-mean-square bottom velocity amplitude of the directional sea; ω_r, its radian frequency,

$T_r = \frac{2\pi}{\omega}$

$$\omega_r = \frac{\iint \omega S_{u_b}(\omega,\theta)d\omega d\theta}{\iint S_{u_b}(\omega,\theta)d\omega d\theta},$$

i.e. the mean frequency of the directional sea; and ϕ_{wr}, its direction of propagation,

$$\phi_{wr} = \arctan\frac{\iint S_{u_b}(\omega,\theta)\sin\theta d\omega d\theta}{\iint S_{u_b}(\omega,\theta)\cos\theta d\omega d\theta},$$

i.e. the mean direction of the directional sea.

Although based on the simple Grant-Madsen eddy viscosity formulation it is believed that the representative periodic wave characteristics determined here can be adopted with any wave-current interaction model to determine the effect of a directional sea on the near-bottom flow associated with a superimposed current. This transferability of the present results to any model that is based on a time-invariant eddy viscosity is assured on theoretical grounds and is not likely to seriously affect the practical significance of results obtained from elaborate numerical turbulent-closure models.

The most severe limitation of the spectral wave-current model's ability to predict current velocity profiles in the presence of waves is associated with the uncertainty in the determination of matching level for the current profile segments, *i.e.* the wave boundary layer thickness, δ_{wc}, for large values of the equivalent bottom roughness. This limitation is not unique to the class of models presented here. It is inherent in all theoretical formulations of turbulent boundary layer flows that apply the turbulent no-slip condition at $z = z_o$, and may, for large roughness, lead to the nonsensical result of a boundary layer thickness less than the physical scale of bottom roughness elements. Clearly, when δ_{wc} is not large relative to the physical scale of the bottom roughness, assuming a horizontally uniform flow is a poor assumption. This issue is particularly important in the context of wave-current interaction in the coastal environment where wave-generated bottom bedforms (ripples) create a large bottom roughness. Theoretical and especially experimental studies are required to shed some light on this problem whose existence is acknowledged here by tentatively suggesting $\delta_{wc} \geq k_N$.

In retrospect these characteristics of the representative periodic wave can hardly be considered surprising. The most important wave characteristic in terms of wave-current interaction is the magnitude of the bottom orbital velocity since this has the greatest effect on the bottom shear stress which, in turn, dominates the turbulence intensity within the wave boundary layer. Since the wave-associated bottom shear stress is proportional to the square of the near-bottom orbital velocity amplitude, u_{bm}^2, with the "constant" of proportionality, the wave friction factor, being "essentially constant" it follows directly that the near-bottom velocity of a representative periodic wave should be $(\overline{u^2}_{bm})^{1/2}$, *i.e.* exactly the form of u_{br} that we obtained through laborious, albeit theoretically rigorous, considerations. Despite the "intuitively obvious" nature of our rigorously derived expression for the representative periodic wave's bottom orbital velocity amplitude it is interesting to note that several investigators have chosen to represent a random sea by a "significant" bottom velocity amplitude, *i.e.* greater than our u_{br} by a factor of $\sqrt{2}$. The theoretically rigorous derivation of the representative periodic wave characteristics presented here therefore firmly establishes the *significance* of the root-mean-square near-bottom wave orbital velocity (and the *insignificance* of the "significant" velocity) in the context of combined wave-current bottom boundary layer flows.

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Appendix A: Spectral Wave Energy Dissipation

As mentioned earlier the effect of evaluating the bottom shear stress at $z = z_o(\zeta = \zeta_{ro})$ rather than by taking the limit $z \to 0(\zeta \to 0)$ has pronounced effects for larger roughness values. Thus the wave friction factors predicted by (32) are significantly lower than those obtained if the limit $z \to 0$ were used to define the bottom shear stress. Furthermore, the phase difference, φ_τ, between bottom shear stress and free stream wave orbital velocity, which is important for the computation of wave energy dissipation in the wave bottom boundary layer, is extremely sensitive to the choice of definition , $z = z_o$ or $z \to 0$, used to evaluate the bottom shear stress for large values of the bottom roughness.

The phase difference, φ_τ, is given by the argument of the complex fraction of Kelvin functions and their derivatives in (12), (14) and (30). The phase difference, $\varphi_{\tau r}$, for the representative periodic wave, obtained from (30), may be approximated by the explicit relationship

$$\phi^{\circ}_{\tau r} = 33 - 6.0 \ \log_{10} \frac{C_\mu u_{br}}{k_N \omega_r} \quad \text{for} \quad 0.2 < C_\mu u_{br}/(k_N \omega_r) < 10^3 \tag{39}$$

which is accurate within 1° for the range indicated.

For each wave component the rate of energy dissipation in the bottom boundary layer, D_{nm}, is obtained from Kajiura (1968) with bottom shear stress given by (12) and free stream velocity by (6), evaluated for $\zeta_n \to \infty$. The resulting formula is written in compact form by using (26) and (29) to express u_{*r} and generalizing (30) and (39) for arbitrary choice of ω. The end result is

$$D_{nm} = \overline{\tau_{\mathbf{w}nm} \cdot (\mathbf{u}_{\mathbf{w}nm})_{\zeta_n \to \infty}} = \frac{1}{4}\rho\sqrt{f_{wc}}\sqrt{f_{wc,n}}\cos\varphi_{\tau n} u_{br} \mathrm{u}^2_{\mathrm{b}nm} \tag{40}$$

in which $f_{wc,n}$ and $\varphi_{\tau n}$ are obtained from (32) or (33) and (39), respectively, with ω_r replaced by ω_n and $\mathrm{u}_{\mathrm{b}nm}$ is given by (7).

Madsen *et al.* (1988) used the $z \to 0$ definition to obtain the wave friction factor and neglected the influence of the phase difference, φ_τ, in the evaluation of the energy dissipation rate. Strictly speaking this limits the applicability of their results to the range of relative roughness associated with (33). For this range $f_{wc,n}$ and $\cos\varphi_{\tau n}$ depend weakly on ω_n and may be replaced by their values for $\omega_n = \omega_r$. In this way (40) becomes equivalent to Eq. (26) in Madsen *et al.* (1988) if their "f_{wr}" is replaced by f_{er} = "the representative wave energy dissipation factor" $= f_{wc}\cos\varphi_{\tau r}$. For small bottom roughness $\cos\varphi_{\tau r} \simeq 1$ and $f_{er} = f_{wc}$ makes (40) identical to Eq. (26) of Madsen *et al.*. However, as seen in Figure 1, f_{er} may be significantly different from f_{wc} for large bottom roughness.