

Lesson 14

Concentration Inequalities

A concentration inequality
gives us a probability
band on certain random

variables taking atypically
large or atypically small
values (tail probabilities)

1. Markov's Inequality

Assume X is a non-negative r.v.

with $E[X] < \infty$, then for

any $\alpha > 0$

$$P(X > \alpha) \leq \frac{E[X]}{\alpha}$$

Remark: This inequality is non-trivial when $\alpha > E[X]$.

The image shows a blank sheet of lined paper, likely from a notebook. It is divided into two main sections by a horizontal line. Each section contains ten horizontal lines for writing. The lines are evenly spaced and extend across most of the width of each section. The paper has a slightly aged, off-white color and a thin brown border. The top right corner of the paper is folded over, creating a small triangular flap.

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The image shows two blank, lined pages from a notebook. Each page is enclosed in a brown border and contains ten horizontal lines for writing. The top page has a small notch in the upper right corner. The bottom page is slightly larger and has a small red mark near the bottom center. The pages are otherwise empty of any text or markings.

Question: The "decay rate" is $1/\alpha$ in Markov Inequality.

Can we have a decay rate that is faster than $1/\alpha$?

The answer is ~~YES~~:

2. Chebyshev's Inequality

Assume that X is a r.v. with expectation μ and variance $\sigma^2 < \infty$, then

$$P(|X - \mu| > k\sigma) \leq 1/k^2$$
$$\forall k > 0$$

OR

$$P(|X - \mu| > c) \leq \frac{\sigma^2}{c^2}$$

Proof:

Chebyshev's Inequality
for various k 's

$$k \geq 1 \quad P(|X - \mu| > k\sigma) \leq \frac{1}{k^2}$$

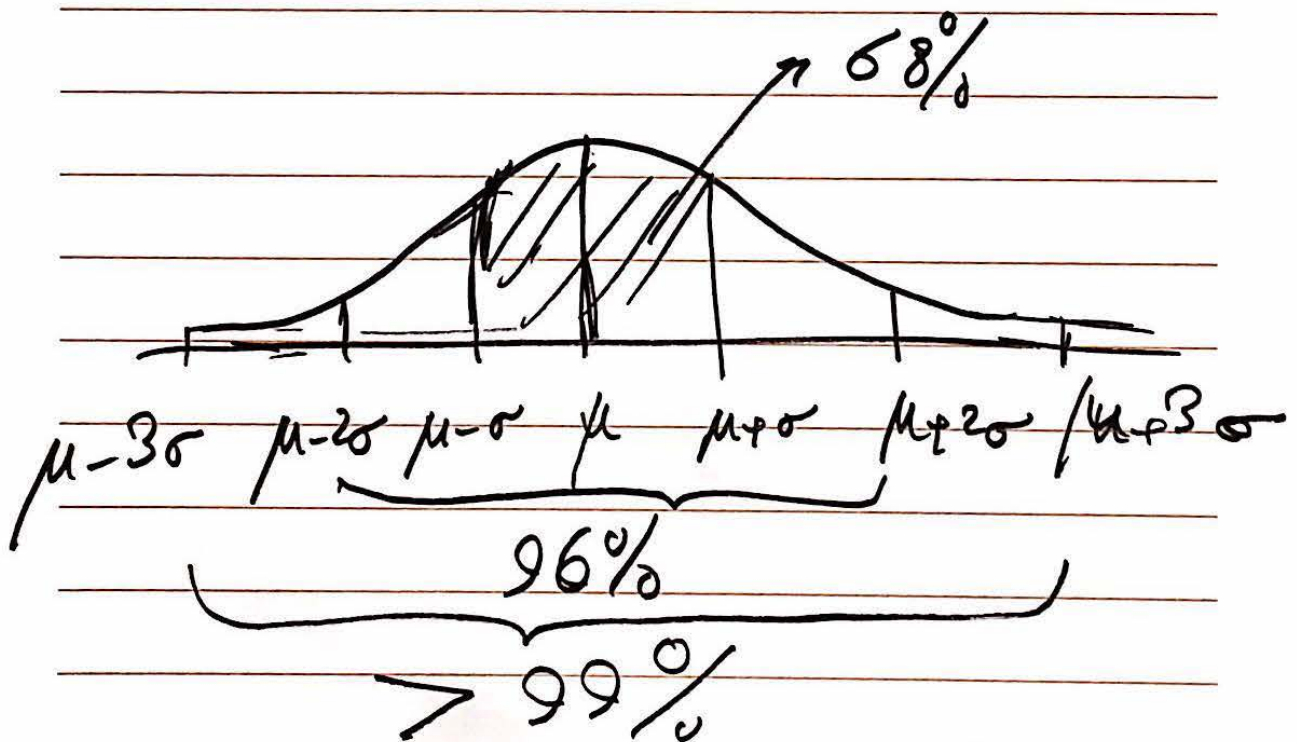
when X is Normal

$$k=2 \quad \mathbb{P}(|X-\mu| > 2\sigma) \leq$$

For Normal

$$k=3 \quad \mathbb{P}(|X-\mu| > 3\sigma) \leq$$

For normal



Chebyshev Inequality yields a bound that decays as fast as $1/k^2$, an improvement over the basic Markov Inequality.

Question: Can we find bounds that decay faster than ~~$1/\text{polynomials}$~~ , i.e., exponentially?
YES!