1. A median of a continuous random variable X is a value m such that $\mathbb{P}(X \le m) = 1/2$ and $\mathbb{P}(X \ge m) = 1/2$. Therefore, if X is continuous, m satisfies $\int_{-\infty}^{m} f_X(x)dx = \int_m^\infty f_X(x)dx$).

   (a) Find the median of the following distributions: (5 pts)

      i. $f_X(x) = 4x^3,\ 0 < x < 1$.

      ii. $f_X(x) = \frac{1}{\pi}\frac{1}{x^2+1}$.

   (b) Show that $\min_a \mathbb{E}[|X-a|] = \mathbb{E}[|X-m|]$. Hint: $\frac{d}{d\theta}\int_{g(\theta)}^{h(\theta)} f(x)dx = h'(\theta)f(h(\theta)) - g'(\theta)f(g(\theta))$. (20 pts)

Let m denote a medium.

(a) (i) 2 pts $P(X \le m) = \int_{-\infty}^{m} 4x^3 dx = m^4 = \frac{1}{2}$,

$P(X \ge m) = 1 - P(X \le m) = 1 - m^4 = \frac{1}{2}$, $m^4 = \frac{1}{2} = 1 - m^4$; $m = (\frac{1}{2})^{\frac{1}{4}}$.

(ii) 3 pts $f_X(x)$ is an even function

Therefore $\int_{-\infty}^{0} f_X(x)\,dx = \int_0^\infty f_X(x)\,dx$,

$P(X \le 0) = P(X \ge 0)$.

As $P(X \le 0) + P(X \ge 0) = 1$ for X is continuous,

$P(X \le 0) = P(X \ge 0) = \frac{1}{2}$, and $m = 0$.

(b) (Fact 1 (Fundamental Thm of Calculus): If $g(x)$ is continuous, then $\frac{d}{d\theta}\int_{-\infty}^{\theta} g(x)\,dx = g(\theta)$ and $\frac{d}{d\theta}\int_\theta^\infty g(x)\,dx = -g(\theta)$.)

$E[|X-a|] = \int_{-\infty}^{a}(a-x)f_X(x)dx + \int_a^\infty (x-a) f_X(x)dx$.

$= \underbrace{a\int_{-\infty}^{a} f_X(x)dx}_{①} - \underbrace{\int_{-\infty}^{a} x f_X(x)dx}_{②} + \underbrace{\int_a^\infty x f_X(x)dx}_{③} - \underbrace{a\int_a^\infty f_X(x)dx}_{④}$ 5 pts

By Fact 1,

$\frac{d}{da}① = a f_X(a) + \int_{-\infty}^{a} f_X(x)\,dx$.

$\frac{d}{da}② = a f_X(a)$,

$\frac{d}{da}③ = -a f_X(a)$,

$\frac{d}{da}④ = \int_a^\infty f_X(x)\,dx - a f_X(a)$. 5 pts

Thus $\frac{d}{da}E[|X-a|] = \int_{-\infty}^{a} f_X(x)dx - \int_a^\infty f_X(x)dx$

If $a=m$, then $\frac{d}{da}\{E[|X-a|]\} = 0$ —(1) 5 pts ↳

$$\frac{d^2}{da^2} E\{|X-a|\} = \frac{d}{da}\left\{\int_{-\infty}^{a} f_X(x)\,dx - \int_{a}^{\infty} f_X(x)\,dx\right\}$$

$$= f_X(a) - (-1) f_X(a) = 2 f_X(a) \geq 0, \ \forall a \in \mathbb{R}. \quad -(2)$$

By (1) and (2), $\min_a E\{|X-a|\} = E\{|X-m|\}$. 5pts

2. Let $f_X(x) = \frac{1}{2}e^{-|x-\alpha|}$, $x \in \mathbb{R}$. Calculate the moment generating function of X, $M_X(s)$. (20 pts)

$$M_X(s) = E[e^{sX}] = \int_{-\infty}^{\infty} e^{sx} \frac{1}{2} e^{-|x-\alpha|} dx$$

$$= \int_{\alpha}^{\infty} e^{sx} \frac{1}{2} e^{-(x-\alpha)} dx + \int_{-\infty}^{\alpha} e^{sx} \frac{1}{2} e^{-(\alpha-x)} dx \quad 5\text{pts}$$

$$= \frac{1}{2} e^{\alpha} \int_{\alpha}^{\infty} e^{(s-1)x} dx + \frac{1}{2} e^{-\alpha} \int_{-\infty}^{\alpha} e^{(s+1)x} dx.$$

$$= \frac{1}{2} e^{\alpha} \left[\frac{e^{(s-1)x}}{(s-1)} \right]_{\alpha}^{\infty} + \frac{1}{2} e^{-\alpha} \left[\frac{e^{(s+1)x}}{(s+1)} \right]_{-\infty}^{\alpha}$$

$$= \begin{cases} \frac{-e^{s\alpha}}{2(s-1)} + \frac{e^{s\alpha}}{2(s+1)}, & \text{when } s-1<0 \text{ and } s+1>0, \quad 10\text{pts} \\ \infty, & \text{otherwise.} \quad 5\text{pts} \end{cases}$$

3. The pdf of a Rayleigh random variable is $f(z) = z\exp(-z^2/2)$, $z > 0$, and the pmf of a Geometric random variable X is $f_X(x) = p(1-p)^{x-1}$, $0 < p < 1$, $x = 1, 2, \ldots$. Assuming that $X_1, X_2, \ldots, X_n$ is an i.i.d Geometric sample and that p has a Rayleigh prior distribution, find the MAP estimate of p. (20 pts)

As X_i's are i.i.d, $g(\underline{x}|p) = \prod_{i=1}^{n} p(1-p)^{x_i-1} = p^n(1-p)^{(\sum_{i=1}^{n} x_i - n)}$ 5 pts

On the other hand, $h(p) = p\, e^{-p^2/2}$ 2 pts

Therefore, $l(p) = \log[h(p)\, g(\underline{x}|p)] = \log[p^{n+1} e^{-\frac{p^2}{2}} (1-p)^{(\sum_{i=1}^{n} x_i - n)}]$.

$$= \frac{n+1}{p} - p - \frac{\sum_{i=1}^{n} x_i - n}{1-p} \quad \text{3 pts}$$

$\frac{\partial}{\partial p} l(p) = 0$, $\frac{(1-p)(n+1) - p^2(1-p) - (\sum_{i=1}^{n} x_i - n)p}{p(1-p)} = 0$

$\boxed{\hat{p} \text{ is the root of } (1-p)(n+1) - p^2(1-p) - (\sum_{i=1}^{n} x_i - n)p = 0.}$ 5 pts

Check $\left[\frac{\partial^2}{\partial p^2} l(p)\right]_{\hat{p}} \le 0$:

$$\frac{\partial^2}{\partial p^2} l(p) = (n+1)(-1)\frac{1}{p^2} - 1 + \left(\sum_{i=1}^{n} x_i - n\right)\frac{-1}{(1-p)^2} \le 0 \quad \forall\, 0 \le p \le 1.$$

$\therefore \left[\frac{\partial^2}{\partial p^2} l(p)\right]_{\hat{p}} \le 0$. 5 pts

4. Let $U_i, i = 1, 2, \ldots$ be independent $U[0,1]$ random variables, and let X be independent from U_i's and have distribution $\mathbb{P}(X = x) = c/x!,\ x = 1, 2, 3, \ldots$.

   (a) Calculate c. (5 pts)

   (b) Find the distribution of $Z = \max(U_1, \ldots, U_X)$. (20 pts)

(a) $1 = \sum_{x=1}^{\infty} P(X=x) = c\{\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\} = c\{e^1 - 1\}$, $\therefore c = \frac{1}{e^1-1}$.

5pts

(b) $P(U_i \le z) = \begin{cases} 0, & \text{if } z < 0, \\ z, & \text{if } z \in [0,1], \\ 1, & \text{if } z > 1, \end{cases}$ for $U_i \sim U[0,1]$. 5 pts

$$P\{Z \le z \mid X = x\} = P\{\max(U_1, U_2, \cdots, U_x) \le z\}$$
$$= P\{U_1 \le z, U_2 \le z, \cdots, U_x \le z\}$$
$$= \prod_{i=1}^{x} P\{U_i \le z\} = \begin{cases} 0, & \text{if } z < 0, \\ z^x, & \text{if } z \in [0,1], \\ 1, & \text{if } z > 1. \end{cases}$$

5 pts

$$P\{Z \le z\} = \sum_{x=1}^{\infty} P\{Z \le z \mid X = x\} P\{X = x\}$$
$$= \begin{cases} \sum_{x=1}^{\infty} 0\, P\{X=x\}, \\ \sum_{x=1}^{\infty} z^x \left(\frac{1}{e^1-1} \frac{1}{x!}\right), \\ \sum_{x=1}^{\infty} 1\, P\{X=x\}, \end{cases} = \begin{cases} 0, & \text{if } z < 0, \\ \frac{1}{e^1-1} \sum_{x=1}^{\infty} z^x \frac{1}{x!}, & \text{if } z \in [0,1], \\ 1, & \text{if } z > 1, \end{cases}$$

where $\frac{1}{e^1-1} \sum_{x=1}^{\infty} z^x \frac{1}{x!} = \frac{1}{e^1-1}\left\{\frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right\} = \frac{e^z - 1}{e^1 - 1}$. 10 pts

■

5. Let X_n and X be integer-valued random variables with probability mass functions $p_n(k) = \mathbb{P}(Xn = k)$ and $p(k) = \mathbb{P}(X = k)$, respectively.

   (a) If $X_n \xrightarrow{d} X$, show that for each k, $\lim_{n\to\infty} p_n(k) = p(k)$. (20 pts)

   (b) If X_n and X are nonnegative, and if for each $k \geq 0$, $\lim_{n\to\infty} p_n(k) = p(k)$, show that $X_n \xrightarrow{d} X$. (10 pts)

(a) ∵ $X_n \xrightarrow{d} X$, $\lim\limits_{n\to\infty} F_{X_n}(k) = F_X(k)$. where $\lim\limits_{n\to\infty} F_{X_n}(k)$ and $F_X(k)$ is continuous, i.e, k is not an integer.

As X_n and X are integere-valued, for any integer h, let ~~define $h^+ = h+\varepsilon$, $h^- = h-\varepsilon$ where~~ $\varepsilon \in (0, 1]$.

Then $\lim\limits_{n\to\infty} F_{X_n}(h+\varepsilon) = F_X(h+\varepsilon)$ and $\lim\limits_{n\to\infty} F_{X_n}(h-\varepsilon) = F_X(h-\varepsilon)$.

$$\lim_{n\to\infty} p_n(h) = \lim_{n\to\infty} F_{X_n}(h+\varepsilon) - \lim_{n\to\infty} F_{X_n}(h-\varepsilon) = F_X(h+\varepsilon) - F_X(h-\varepsilon) = p(h).$$

Therefore $\lim\limits_{n\to\infty} p_n(k) = p(k)$ $\forall$ integer k. 20 pts

(b) $\lim\limits_{n\to\infty} p_n(k) = p(k)$ $\forall k \in \{\mathbb{N} \cup 0\}$. For any positive non integer h,

$$\lim_{n\to\infty} F_{X_n}(h) = \sum_{k=0}^{h^-} \lim_{n\to\infty} p_n(k) = \sum_{k=0}^{h^-} p(k) = F_X(h),$$

where h^- is defined as the largest integer smaller than h.

Thus $\lim\limits_{n\to\infty} F_{X_n}(h) = F_X(h)$ for all non-integer h. —(1)

For any integer m, let $m^- = \underset{\varepsilon\to 0}{\lim}\, m-\varepsilon$, and m^- is not an integer,

we have $\lim\limits_{n\to\infty} F_{X_n}(m^-) = \sum\limits_{k=0}^{M^-} \lim\limits_{n\to\infty} p_n(k) = \sum\limits_{k=0}^{M^-} p(k) = F_X(m^-)$,

where M^- is the largest integer smaller than m^-.

Finally, $\lim\limits_{n\to\infty} F_{X_n}(m) = \lim\limits_{n\to\infty} F_{X_n}(m^-) + p_n(m) = F_X(m^-) + p(m) = F_X(m)$. (2).

By (1) and (2), $X_n \xrightarrow{d} X$. 10 pts