Lesson 3

Review of Set Theory

A set is a collection of objects, which are its elements

(This is not a definition)
("Set" is a primitive concept)

- We show the elements of a set using $\in$

$x \in A$

- A set with no elements is called ~~an~~ empty set or null set $\emptyset$

$x \in \emptyset \quad \forall x$, is false

We can show a set with its elements

$A = \{x_1, x_2, x_3, \dots, x_n\}$

A finite # of elements

Or with a predicate

$A = \{x \mid P(x)\}$

Ex: $\boxed{\{x \in \mathbb{N} \mid x > 5\}}$

Subsets (super sets)

$A \subseteq B \Rightarrow (x \in A \Rightarrow x \in B)$

$A \subseteq B \Leftrightarrow \forall x, x \in A \Rightarrow x \in B$

Equivalence:

$A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B)$

Proper subset hood

$A \subseteq B$ means $x \in A \Rightarrow x \in B$

A can be equal to B

$A \subset B$ means $(x \in A \Rightarrow x \in B)$

$\wedge (A \neq B)$

$\Rightarrow \exists y \in B, y \notin A$

"A: proper subset of B"

Universe of Discourse

The Universe of Discourse Ω (or $\mathcal{U}$) contains all elements that could conceivably be of interest in a particular context.

Ex: In number theory,

$$\Omega = \mathcal{U} = \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}$$

One can then consider all sets of interest as subsets of the Universe of Discourse

Complement of ~~A~~ Set (with respect to Ω).

Complement of A : A^c

$$A^c = \boxed{\{x \in \Omega \mid x \notin A\}}$$

Obviously, $\Omega^c = \boxed{\{x \in \Omega \mid x \notin \Omega\}} = \phi$

Operations on Sets:

Union :

$$A \cup B = \{x \in \Omega \mid x \in A \vee x \in B\}$$

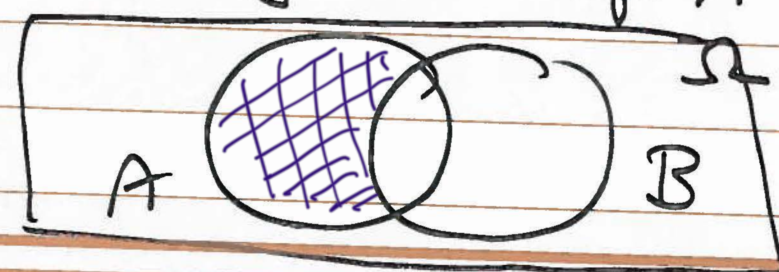
Intersection:

$$A \cap B = \{x \in \Omega \mid x \in A \wedge x \in B\}$$

Set Difference $A \setminus B$ / $A - B$ (or relative complement of B in A

$$A \setminus B = \{x \in A \mid x \notin B\}$$

Venn Diagram of $A \setminus B$



$$A \setminus B = A \cap B^c$$

$$= A \setminus (A \cap B)$$

More generally, assume that

I is an index set, e.g.

$I = \mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

$I = \mathbb{Q} = \{ b/a \mid a, b \in \mathbb{Z}, b \neq 0 \}$

$I = \mathbb{R}$

(possibly infinite)

$$\bigcup_{i \in I} A_i = \{ x \in \Omega \mid \exists_{i \in I} \text{ s.t. } x \in A_i \}$$

$$\bigcap_{i \in I} A_i = \{ x \in \Omega \mid \forall_{i \in I} \text{ s.t. } x \in A_i \}$$

Example: $A_i = [0, i) \quad , i \in \mathbb{N}$

$A_1 = [0,1)$ $A_2 = [0,2)$ $A_3 = [0,3)$

$$\bigcap_{i \in \mathbb{N}} A_i = [0,1)$$

$$\bigcup_{i \in \mathbb{N}} A_i = [0,+\infty) = \mathbb{R}^{\geq 0}$$

Exercise:

$$\bigcap_{i \in \mathbb{N}} (-\infty, 1/i) = (-\infty, 0]$$

$\forall i \in \mathbb{N} \quad 0 \in A_i$

$\Rightarrow 0 \in \bigcap_{i \in \mathbb{N}} A_i$

Exercise:

$$\bigcup_{i \in \mathbb{N}} (-\infty, -1/i] = (-\infty, 0)$$

$0 \notin A_i \ \forall i$

Exercise:

$$\bigcap_{i \in \mathbb{N}} [0, 1/i) = \{0\}$$

$\forall i \ \{0\} \notin A_i$

Exercise:

$$\bigcup_{i \in \mathbb{N}} [-i, i] = (-\infty, \infty) = \mathbb{R}$$

Exercise: $\bigcap_{i \in \mathbb{N}} (-\infty, -i) = \phi$

Disjoint / Mutually Exclusive Sets

Def: A_1, A_2 disjoint $\iff$ $A_1 \cap A_2 = \phi$

Example:

Def: $A_1, A_2, \ldots$ are mutually disjoint

$\Longleftrightarrow A_i \cap A_j = \phi \quad \forall_{i \neq j}$

Def: Partition ; empty set is exclusive

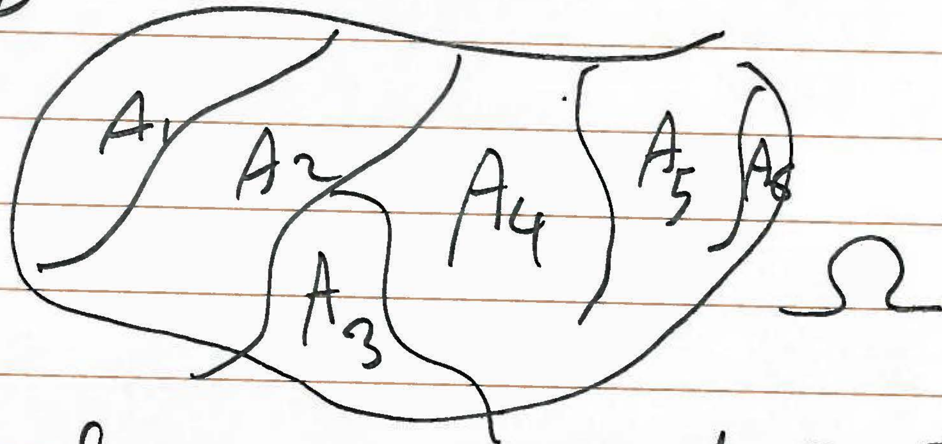
Non-empty $A_1, A_2, \ldots$ Consist a partition of B iff

1) they are mutually exclusive $A_i \cap A_j = \phi \quad \forall i \neq j$ and

2) they are collectively exhaustive $\cup_i A_i = B$

$\Rightarrow$ every member of B is a member of one A_i's member

Example



(A finite number of partitions for Ω)

Exercise: which one is a partition of $\mathbb{R}^{\geq 0}$

a) $A_i = [i, i+1)$ $\quad i \in \{0, 1, 2, \ldots\}$

$A_i \cap A_j = \phi \quad i \neq j$

$\bigcup_i A_i = \mathbb{R}$

b) $A_i = [0, i)$ $\quad i \in \{0, 1, 2, \ldots\}$

$\exists\, i \neq j \quad A_i \cap A_j \neq \phi$: not mutually exclusive

(Some) Properties of Set Operations

Commutativity

$A \cap B = B \cap A$ $\qquad$ $A \cup B = B \cup A$

$p \wedge q \equiv q \wedge p$ $\qquad$ $p \vee q \equiv q \vee p$

Associativity

$A \cap (B \cap C) = (A \cap B) \cap C$ $\qquad$ $A \cup (B \cup C) = (A \cup B) \cup C$

$p \wedge (q \wedge r) \equiv p \wedge q) \wedge r$

Distributivity

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Complement laws

$(A^c)^c = A$

$\neg(\neg p) \equiv p$

$A \cup A^c = \Omega$ $\quad$ $p \vee \neg p \equiv T$

$A \cap A^c = \phi$ $\quad$ $p \wedge \neg p \equiv F$

Identity Laws

$A \cup \emptyset = A$ $\quad$ $A \cap \Omega = A$

$p \vee F \equiv p$ $\quad$ $p \wedge T \equiv p$

Domination Laws

$A \cup \Omega = \Omega$ $\quad$ $A \cap \emptyset = \phi$

$p \vee T \equiv T$ $\quad$ $p \wedge F \equiv F$

Idempotency

$A \cup A = A$ $\quad$ $A \cap A = A$

$p \vee p \equiv p$ $\quad$ $p \wedge p \equiv p$

Absorption Laws

$A \cup (A \cap B) = A$ $\quad$ $p \vee (p \wedge q) \equiv p$

$A \cap (A \cup B) = A$ $\quad$ $p \wedge (p \vee q) \equiv p$

Venn Diagram

De Morgan's Laws

$$(A_1 \cup A_2)^c = A_1^c \cap A_2^c$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$(A_1 \cap A_2)^c = A_1^c \cup A_2^c$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

More Generally, for an index set I

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$$

$$\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$$

Element Chasing

To prove $A = B$, we must prove:

$$\boxed{A \subseteq B \wedge B \subseteq A}$$

Therefore, we must prove:

$$(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A)$$

i.e., we must "chase" elements

Proof of $(A \cap B)^c = A^c \cup B^c$

$x \in (A \cap B)^c \Rightarrow x \notin A \cap B$

$\Rightarrow \neg(x \in A \wedge x \in B)$

De Morgan's $\Rightarrow \neg(x \in A) \vee \neg(x \in B)$

$\Rightarrow x \notin A \vee x \notin B$

$\Rightarrow x \in A^c \vee x \in B^c$

$x \in A^c \cup B^c$

$\therefore (A \cap B)^c \subseteq A^c \cup B^c$

opposite $A^c \cup B^c \subseteq (A \cap B)^c$

$\therefore A^c \cup B^c = (A \cap B)^c$ Q.E.D.

Proof of $\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c$

$x \in \left(\bigcup_{i\in I} A_i\right)^c \Rightarrow \neg\left(x \in \bigcup_{i\in I} A_i\right) \Rightarrow \neg\left(\exists_{i\in I} \text{ s.t. } x_i \in A_i\right)$

$\Rightarrow \left(\forall_{i\in I} \text{ s.t } x_i \notin A_i\right)$

$\Rightarrow \left(\forall_{i\in I} \text{ s.t } x_i \in A_i^c\right)$

$x_i \in \bigcap A_i^c$

General form of Distributivity

$$\left(\bigcap_{i\in I} A_i\right) \cup B = \bigcap_{i\in I} (A_i \cup B)$$

$$\left(\bigcup_{i\in I} A_i\right) \cap B = \bigcup_{i\in I} (A_i \cap B)$$

Try to prove them

Notation: Special Sets

$\mathbb{R}$: Real Numbers

$\mathbb{R}^* = \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$

extended real numbers

$\mathbb{Z}$: Integers $\{\ldots, -4, -3, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{N}$: Natural Numbers or Strictly Positive Integers $\mathbb{N} = \mathbb{Z}^{>0}$

Intervals

$[a,b] = \{x \in \overline{\mathbb{R}} \mid a \leq x \leq b\}$

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$

$(a,b] = \{x \in \mathbb{R} \quad a < x \leq b\}$

Cartesian Product

$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \land a_2 \in A_2\}$

$A_1, A_2, \dots, A_n$

$A_1 \times A_2 \times \dots \times A_n = \prod_{i=1}^{n} A_i$

$= \{(a_1, \dots, a_n) \mid a_i \in A_i \ , i \in \{1, 2, \dots, n\}\}$

More Generally,

$A_1, A_2, \dots$

$\prod_{i=1}^{\infty} A_i$ is the <u>sequence</u>

$\boxed{\{(a_1, a_2, \dots) \mid a_i \in A_i \ , i \in \mathbb{N}\}}$

$A_i = A \Rightarrow \prod_{i=1}^{\infty} A_i = A^{\infty}$

Example: $A = \{0, 1\}$

$$A^{\infty} = \prod_{i=1}^{\infty} A_i \quad , \quad \forall i \quad A_i = \{0, 1\}$$

A^{∞} is the set of all binary sequencies

Family of Subsets of A / Power Set of A

The set of all subsets of A is called the power set of A and is denoted as 2^{A} or $\mathcal{P}_A$ or $\mathcal{P}(A)$.

Example: $A = \{1, 2\}$

$2^A = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

$x \in \phi \Rightarrow x \in A$

$\therefore \phi \subseteq A$

$|A| = 2$

$|2^A| = 2^{|A|} = 2^2 = 4$

Relations

$A_1 \times A_2 \times \cdots \times A_n \times B$

Assume $K = \left(\prod_{i=1}^{n} A_i\right) \times B$

Any subset of K, $R \subseteq K$, is called a relation. (on K)

Example: $A = \{2, 3\}$ $B = \{1, 2\}$

$K = A \times B = \{(2,1), (2,2), (3,1), (3,2)\}$

$R_1 = \phi$ $R_2 = \{(3,1)\}$ $R_3 = \{(2,1), (2,2)\}$ ...

A Relation maps some tuples in $\prod_{i=1}^{n} A_i$ to at least one element in B

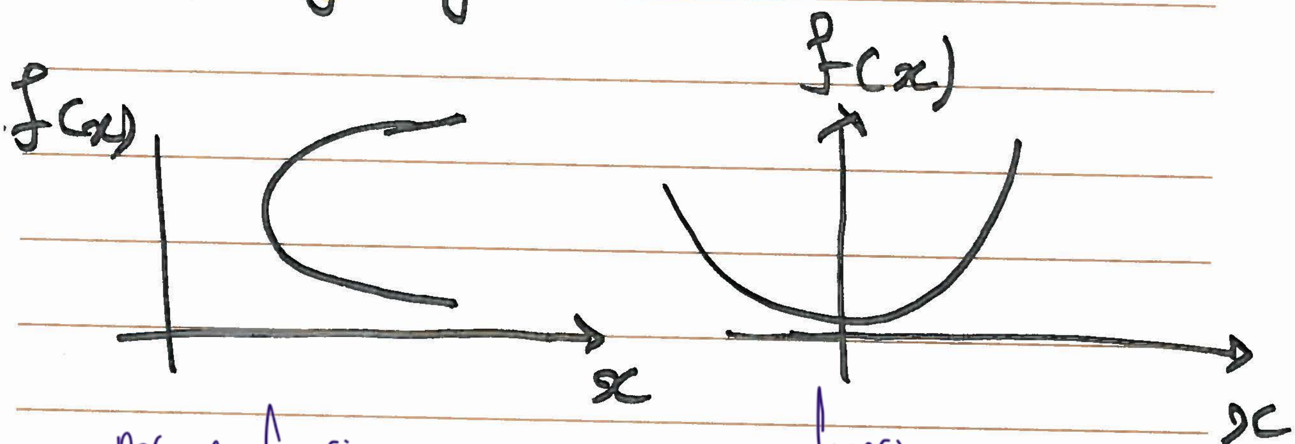
Function:

Def: If a relation maps each n-tuple to a unique member of B, it is called a function

$f \subseteq K$

$(x_1, \dots, x_n, y) \in f \wedge (x_1, \dots, x_n, y') \in f$

$\Longrightarrow y = y'$



not a function   function

We write $f : A_1 \times \dots \times A_n \longrightarrow B$

$\prod_{i=1}^{n} A_i$ is called the $\boxed{\text{domain}}$ of f

B is called the co-domain of f.

For simplicity, we continue with functions of the form $f : A \longrightarrow B$

(pull forward)

Forward Image of $f: A \to B$

Def. $\vec{f}: 2^A \to 2^B$

$\forall S \subseteq A \quad (S \in 2^A)$

$$\vec{f}(S) = \{ y \in B \mid y = f(x), x \in S \}$$

Easier $\vec{f}(S) = \{ f(x) \mid x \in S \} \subseteq B$

The notation $f(S)$ is also used.

Example: $f: \mathbb{R} \to \mathbb{R}$, $y = f(x) = x^2$

$$\vec{f}([1,2]) = \{ f(x) \mid x \in [1,2] \}$$

$$= \{ x^2 \mid x \in [1,2] \} = [1,4]$$

$$\vec{f}([-1,3]) = [0,9]$$

$$\vec{f}([-1,-\tfrac{1}{2}) \cup (\tfrac{1}{2},1]) = (1/4, 1]$$

(pull back)

Inverse Image (pre_Image) of $f: A \to B$

$$f^{\leftarrow}: 2^B \to 2^A$$

$\forall T \subseteq B \; (T \in 2^B)$

$$f^{\leftarrow}(T) = \{x \in A \mid f(x) \in T\}$$

Obviously, $f^{\leftarrow}(T) \subseteq A \; (\in 2^A)$

The notation $f^{-1}(T)$ is also used, but we avoid it because it can be confused with an inverse function

Example : $f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$\overleftarrow{f}([4,9]) = \{x \in \mathbb{R} \mid f(x) \in [4,9]\}$

$= \{x \in \mathbb{R} \mid x^2 \in [4,9]\}$

$= [-3,-2] \cup [2,3]$

$f^{\leftarrow}([0,9]) = [-3,+3]$

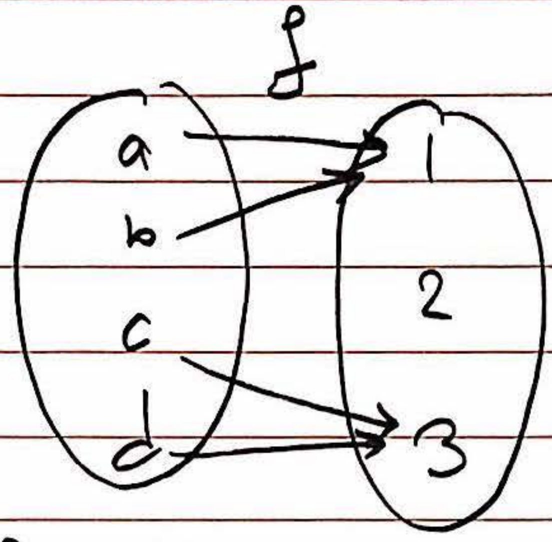
$f^{\leftarrow}([-9,-1]) = \phi$

$f^{\leftarrow}([-9,0]) = \{0\}$

- Important note: forward and inverse images are functions that map sets to sets. In other words, they are set functions

- Remark: It is a very good habit to always show the domain and ~~co~~ domain of a function to avoid ambiguity.

Example:



Is f a function? yes

$f^{\rightarrow}(\{a,b\}) = \{1\}$

$f^{\rightarrow}(\{a,b,c\}) = \{1,3\}$

$f^{\rightarrow}(\emptyset) = \phi$

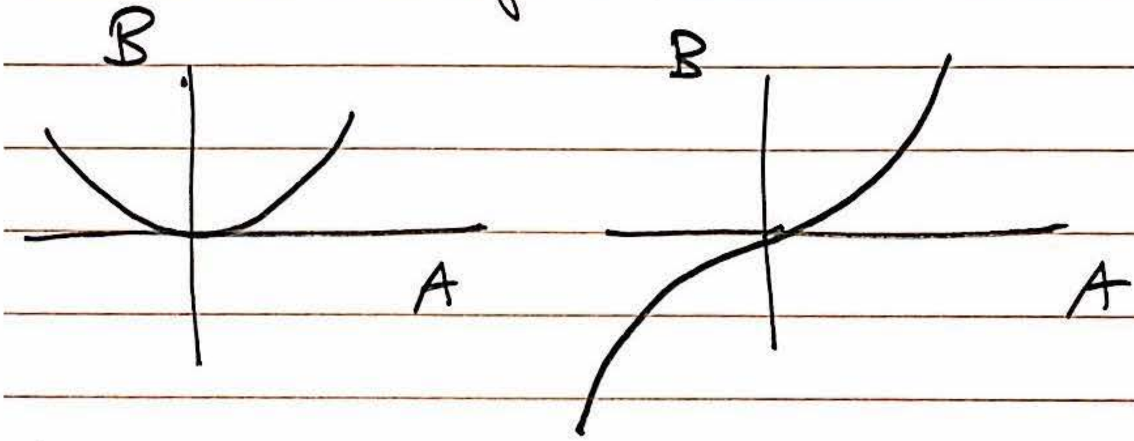
$f^{\leftarrow}(\{1\}) = \{a,b\}$

$f^{\leftarrow}(\{2\}) = \phi$

$f^{\leftarrow}(\{1,2,3\}) = \{a,b,c,d\}$

Def: One-to-one function

A function $f: A \to B$ is one-to-one (1-1) iff it assigns no more than one member of A to each member of B



or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

or $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

One-to-one functions are also called injective

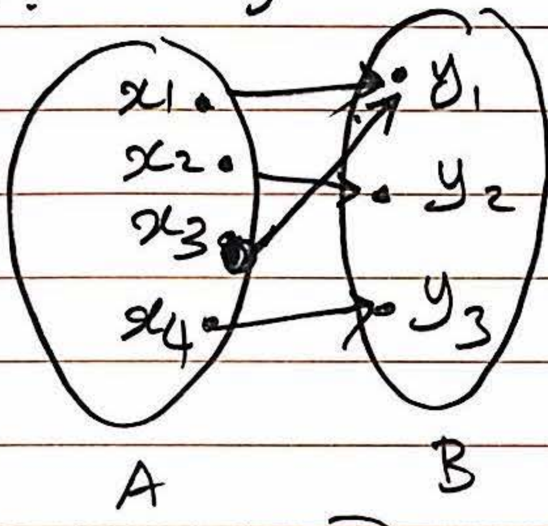
Onto function:

Def: $f: A \to B$ is onto iff $\vec{f}(A) = B$.

Range

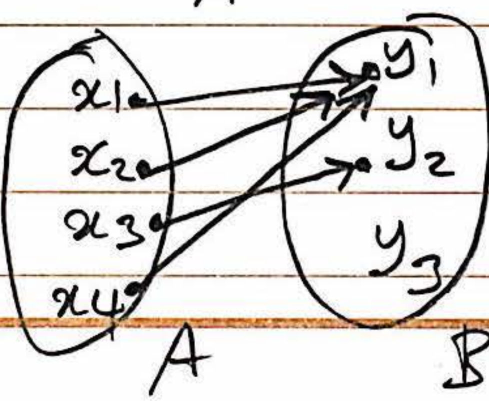
Example: f which function is onto?

a)



A onto B

(b)



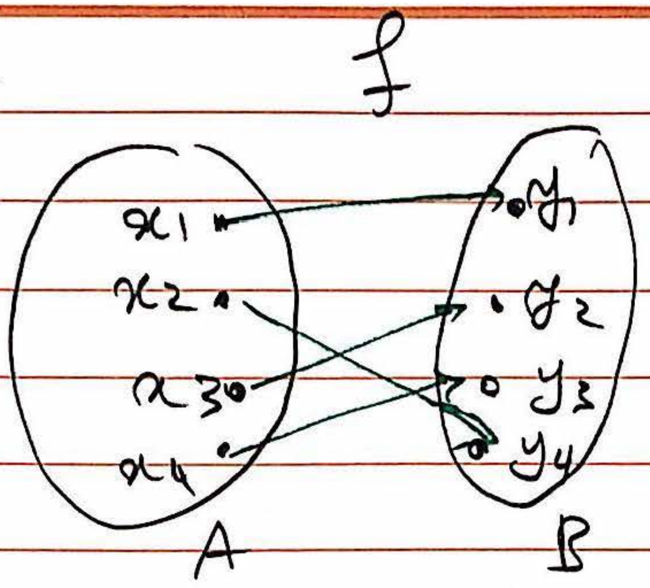
Onto functions are also called surjective

One-to-One Correspondence

one-to-on + onto

Def: $f: A \to B$ is called a one-to-one correspondence iff it is both 1-1 and onto. It is also called bijective.

Example:



bijective

Remark: In a bijective mapping,
one-to-one correspondence
every element in the co-domain has a pre-image and the pre-images are unique.

Thus, we can define the inverse function, $f^{-1}: B \longrightarrow A$, such that $f^{-1}(y) = x$ if $f(x) = y$. Therefore, bijective functions are invertible.

Cardinality

In informal terms, the cardinality of a set is the number of elements in that set.

If one wishes to compare the cardinalities of two finite sets, A and B, they can simply count the number of elements in each set.

But what if the sets contain infinitly many elements?

Def. Two sets A, B are said to be of the same cardinality (equicardinal) iff there exists a one-to-one correspondence between A and B, and we write $|A| = |B|$.

Question: Using the concept of one-to-one and onto functions, define $|A| \geq |B|$ and $|A| \leq |B|$.

$|A| \geq |B| \Leftrightarrow$ there exists an onto function from A to B

$|A| \leq |B| \Leftrightarrow$ there exists a one to one function from A to B

Def: A set A is said to be countably infinite if A and $\mathbb{N}$ have the same cardinality. (Denumerable)

Def: A set is said to be countable if it is either finite or countably infinite.

Example: $\mathbb{N}$ is countably infinite

$f: \mathbb{N} \to \mathbb{N}$

$f(n) = n$; identity function

Example: $\mathbb{Z}$ is countably infinite.

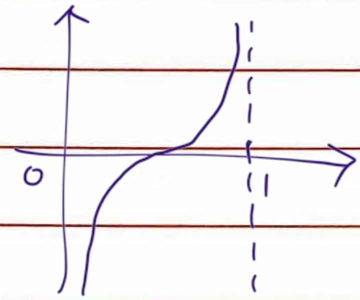
$f : \mathbb{Z} \longrightarrow \mathbb{N}$

$$f(x) = \begin{cases} 2x+1 & x \geq 0 \\ -2x & x < 0 \end{cases}$$

$n = f(x) \in \mathbb{N}$	$x \in \mathbb{Z}$
1	0
3	+1
2	−1
5	+2
4	−2

$|\mathbb{Z}| = |\mathbb{N}|$

Example: Do $(0,1)$ and $\mathbb{R}$ have the same cardinality?



$y = \tan(\pi x - \pi/2)$

Example: $\mathbb{N} \times \mathbb{N}$ is Countable

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	...
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	...
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	...
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	...
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	...
⋮	⋱				

Define $f(i,j)$ equal to the number of pairs visited when (i,j) visited. $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

f is a one-to-one correspondence because it visits all pairs and it visits each pair once.

$\Rightarrow$ $\mathbb{N} \times \mathbb{N}$ is countable

Remark: Using the same argument, one can prove that $A \times B$ is countable when both A and B are countable.

Proposition: $\mathbb{Z} \times \mathbb{N}$ and $\mathbb{Z} \times \mathbb{Z}$ are countable!

$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z},\ b \neq 0 \right\}$$

$\mathbb{Q} \subseteq \mathbb{Z} \times \mathbb{Z}$

is $\boxed{\text{countable}}$

any subset of countable set is either finite or countable

Def: Uncountable set

A is uncountable if it is not countable.

Example: It can be shown that $[0,1)$, $\mathbb{R}$, $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ are uncountable.

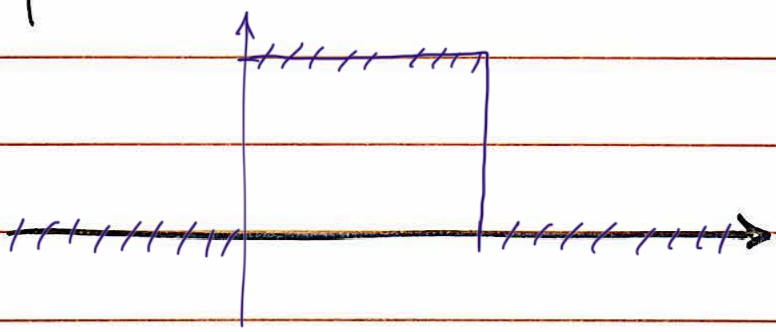
Limit of a sequence of Sets

Def: Indicator function of $A \subseteq \Omega$

$$I_A : \Omega \to \{0, 1\}$$

$$\forall \omega \in \Omega \quad I_A(\omega) = \begin{cases} 0 & \omega \notin A \\ 1 & \omega \in A \end{cases}$$

Example: $\Omega = \mathbb{R}$, $A = [0, 1]$



Def: Assume $A_1, A_2, \ldots$ is a sequence of sets, $\forall i \in \mathbb{N}$, $A_i \subseteq \Omega$.

We say $\lim_{n \to \infty} A_n = A$

iff $\forall \omega \in \Omega$ $\boxed{\lim_{n\to\infty} I_{A_n}(\omega) = I_A(\omega)}$

(i.e. iff the indicator functions converge pointwise)

i.g.) $A_n = [-\frac{1}{n}, 1 + \frac{1}{2n}]$

$\lim_{n\to\infty} I_{A_n}(\omega) = I_{[0,1]}(\omega)$

Example : $A_i = [-1/i, 1/i]$

$$\lim_{i \to \infty} A_i = \{0\}$$

because

$$\forall \omega \in \mathbb{R}, \lim_{i \to \infty} I_{A_i}(\omega) = \begin{cases} 0 & \omega \neq 0 \\ 1 & \omega = 0 \end{cases}$$

$I_{\{0\}}(\omega)$

Theorem:

(a) Suppose that A_n is an increasing sequence of sets ($\forall n,\ A_n \subseteq A_{n+1}$), then $\lim_{n\to\infty} A_n$ exists and is equal to

$\bigcup_{n\in\mathbb{N}} A_n$.

(b) If A_n is a decreasing sequence of sets ($A_n \supseteq A_{n+1}$), then $\lim_{n\to\infty} A_n$ exists and is equal to $\bigcap_{n\in\mathbb{N}} A_n$.

Example: (a) $A_i = [0, i) \quad i \in \mathbb{N}$

Increasing or decreasing?

increasing

$$\lim_{i \to \infty} A_i = \bigcup_{i \in \mathbb{N}} A_i = [0, +\infty] = \mathbb{R}^{\geq 0}$$

(b) $B_i = [i, +\infty) \quad , i \in \mathbb{N}$

Increasing or decreasing?

decreasing

$$\lim_{i \to \infty} B_i = \bigcap_{i \in \mathbb{N}} B_i = \phi$$

Exercise: Show that

$$\vec{f}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \vec{f}(A_i)$$

Element Chasing

$y \in \vec{f}\left(\bigcup_{i\in I} A_i\right) \Leftrightarrow \exists x \in \bigcup_{i\in I} A_i$ s.t. $y = f(x)$

$\Leftrightarrow \exists x, \exists i \in I$, s.t. $x \in A_i$, $y = f(x)$

$\Leftrightarrow \exists i \in I\ \exists x$ s.t. $x \in A_i, y = f(x)$

$\Leftrightarrow \exists i \in I$, $y \in \vec{f}(A_i)$

$\Leftrightarrow y \in \bigcup_{i\in I} \vec{f}(A_i)$

$$\Rightarrow \begin{cases} \vec{f}\left(\bigcup_{i\in I} A_i\right) \subseteq \bigcup_{i\in I} \vec{f}(A_i) \\ \bigcup_{i\in I} \vec{f}(A_i) \subseteq \vec{f}\left(\bigcup_{i\in I} A_i\right) \end{cases}$$

$$\therefore \vec{f}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \vec{f}(A_i)$$

Note.

$\forall u\, \forall v \equiv \forall v\, \forall u$

$\exists u\, \exists v \equiv \exists v\, \exists u$

$\exists u\, \forall v \Rightarrow \forall v\, \exists u$ okay

$\forall u\, \exists v \not\Rightarrow \exists v\, \forall u$