

Lesson 7

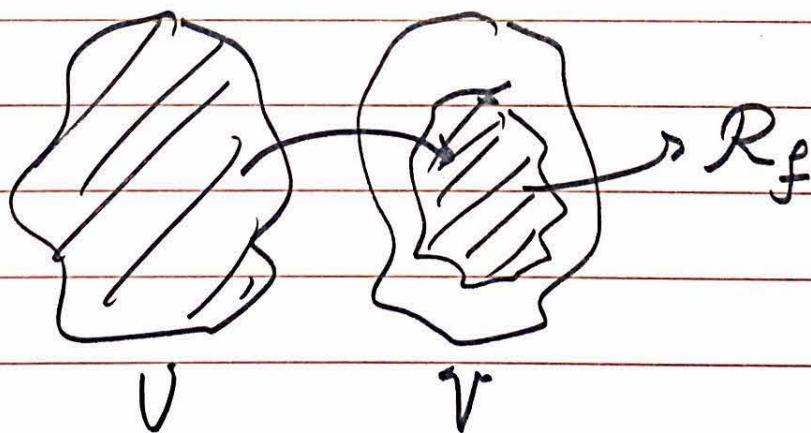
Discrete Random Variables

Definition (Range) :

Assume that $f: U \rightarrow V$ is

a function. The range of

f , R_f , is defined as $\vec{f}(U)$



Def 1 (Discrete Random Variable):

A random variable $X: \Omega \rightarrow \mathbb{R}$

is called discrete if $\vec{x}(\Omega) = R_x$

(i.e. the range of X) is countable.

(In other words, X can take

discrete values x_1, x_2, \dots)

Def 2 (More Precise):

A random variable $X: \Omega \rightarrow \mathbb{R}$

is said to be discrete if

it takes values in a

countable subset of \mathbb{R} with probability 1.

Therefore, there is a countable

set $A = \{x_1, x_2, \dots\}$ such that

$$\underbrace{P(X \in A)}_{\text{event:}} = P_X(A) = 1.$$

This definition does not demand

the range of the random variable

be countable. There might

be a zero probability ~~set~~ event

that is mapped to an uncountable set of \mathbb{R} .

Remark: We can work with both definitions, but Def 1 is slightly easier, because it focuses on the range of the random variable, i.e. the values

that X can take, so

we work with the second

definition.

Def (Probability Mass Function

PMF): If X is a

discrete r.v., the function

$p_X : \mathbb{R} \rightarrow [0, 1]$, defined as

$p_X(x) = P(X=x)$ is called

the pmf of X .

Interestingly, ~~the~~ a discrete random variable

can be completely characterized

by its pmf:

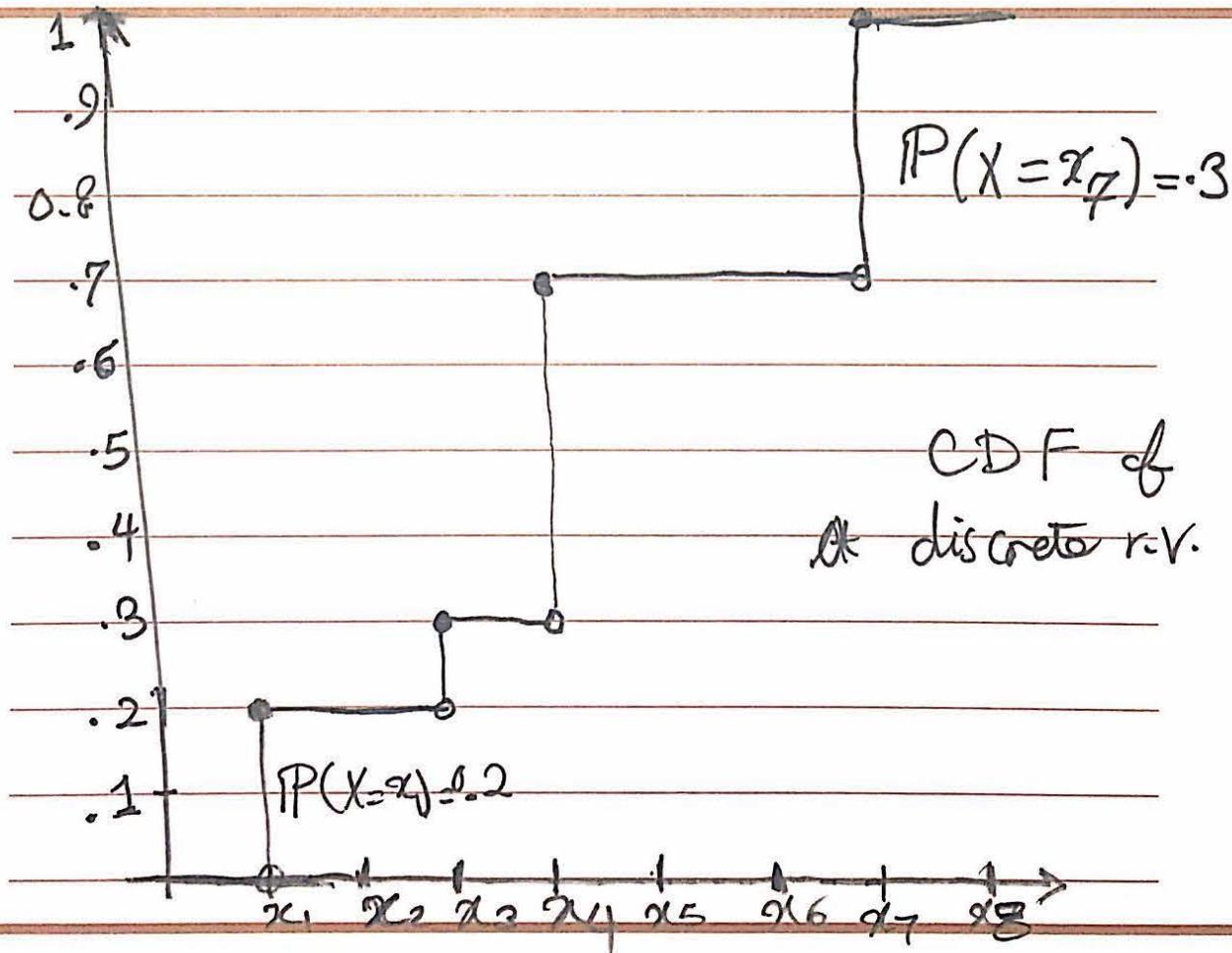
For any Borel Set B ,

$$P(X \in B) = P_X(B) =$$

The cdf of a discrete r.v.
is given by:

$$F_X(x) = P(X \leq x) =$$

$\sum_{-\infty < x_i \leq x}$



Obviously $\sum_x P_X(x) = 1$

Important Discrete Random Variables and Their pmfs

(a) Bernoulli with parameter P

where $0 \leq P \leq 1$

$X \sim \text{Ber}(p)$

or $X \stackrel{d}{=} \text{Ber}(p)$

$$\begin{cases} P_X(1) = \\ P_X(0) = \end{cases}$$

A Bernoulli random variable

models binary events that have
two possible outcomes:
success or failure.

Coin Toss : Success:

Failure:

$$\Omega = [0, 1]$$

Success

Failure

$$x \in A = [0, \frac{1}{2}] \quad x \in (\frac{1}{2}, 1]$$

(I_A is a Bernoulli Random Variable)

(b) Discrete Uniform

$$a, b \in \mathbb{Z}$$

$$a < b$$

$$P_X(x) = \begin{cases} & x = a, a+1, \dots, b \\ & \text{otherwise,} \\ & x \in \mathbb{Z} \end{cases}$$

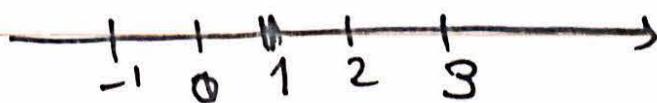
$$X \sim dU(a, b)$$

Example:

$$a = -1$$

$$b = 3$$

$$P_X(x) = \begin{cases} & k = -1, \dots, 3 \\ & \text{otherwise} \end{cases}$$



(c) Binomial with parameters n, p

$n \in \mathbb{N}$, $p \in [0, 1]$

A Binomial r.v. can be seen

as a sequence of n independent

Bernoulli trials with success

probability p . The Binomial

r.v. shows the number of

successes.

The pmf of Binomial r.v.'s is.

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$X \sim \text{Bin}(n, p)$$

Justification.

$$P_X(x) = P(X=x) = P(E_x)$$

↑ number of success

where

$$E_x = \{ \omega \in \Omega \mid \omega \text{ has } \dots \text{ successes} \\ \text{and } \dots \text{ failures} \}$$

From combinatorics, we know

$$|E_x| =$$

$$\text{Thus } P(E_x) =$$

Example: Sara hits 35% of

her free throws in basketball

games. Yesterday she had 7
free throws.

a) What is the probability

that she had between 4 and
7 hits?

(b) What is the probability

that she had at most 5 hits?

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This image shows a single, blank page from a notebook. The page is ruled with horizontal blue lines. It features a thick red border around the entire perimeter. There are no holes or tears in the paper; it is entirely blank and white.

Exercise: The simplest error

detection scheme used in data

communication is parity -

checking. Messages sent consist

of characters, each character

consisting of a number of

0 or 1 bits. In parity checking

a 0 or 1 is appended to

the end of the message to

make the number of 1's even.

1110100 0 keeping # of 1's even
7 bits parity

1100100 1 adding 1 to make
7 bits parity the # of 1's even

The receiver checks the number

of 1's in every character

received and if the result is

odd, it signals an error. Suppose

that each bit is received correctly

with probability $1 - \epsilon_g$ independently

of other bits. What is the
Assuming that the message is in bits
Probability that \oplus

- a) An erratic message is observed
in the receiver side?

$$P(\text{erratic message})$$

$$= P(\text{odd \# of ones})$$

$$= P(\text{odd \# of bit-flips})$$

(b) What is the probability that an erratic message is received and not detected by the parity check?

(d) Geometric with parameter p

We define two kinds of

geometric random variables :

i) $X \sim \text{Geo}(p)$, $p \in (0, 1]$

$$P_X(x) = \quad x \in \mathbb{N}$$

The Geo, random variable

represents the number of independent

Bernoulli trials needed until

the first success is observed

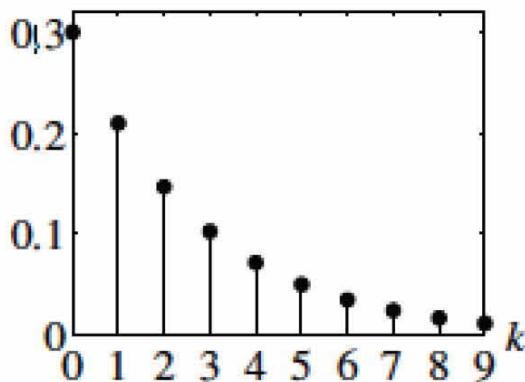
ii) The second type of geometric random variable has the following pmf

$$X \sim \text{Geo}_0(p), \quad p \in (0, 1]$$

$$p_X(x) = \quad x \in \{0, 1, 2, \dots\}$$

The pmf of $\text{Geo}(0.3)$

shown below:



Exercise: Show that when

$X \sim \text{Geo}_0(p)$ or $Y \sim \text{Geo}_1(p)$,

$$\sum_{x=0}^{\infty} P_X(x) = 1 \quad \text{and} \quad \sum_{y=1}^{\infty} P_Y(y) = 1,$$

indeed.

Example: A hacker wants to break into a computer file, which is password protected.

Assume that the passwords only can be alpha-numeric, i.e. only

contain a-z, A-Z and 0-9.

Assume that the hacker tries possible passwords in a random order. Let X be the number of trials required to break into

the file. Determine the pmf

of X : if

a) if unsuccessful passwords
are not eliminated from further
selections

(b) if unsuccessful passwords are
eliminated.

6. a) X has geometric distribution

with $P = \frac{1}{n}$, where $n =$

1

because it is the number

of independent Bernoulli trials

until the first "success"

(correct password)

$P(X = x) =$

(b) Here, the probability of success increases after each failure, so the distribution of X is not Geometric. Since the order is random, the correct

password has the same chance to

appear in each sequence.

Therefore, X has a

distribution

$$P(X=x) =$$

(e) Poisson with parameter $\lambda > 0$

The Poisson random variable is

used to model many different

phenomena such as the

photoelectric effect,

radioactive decay, and computer

traffic arriving at a queue

for transmission.

$$X \sim \text{Pois}(\lambda)$$

$$P_X(x) = \frac{\lambda^x}{x!} e^{-\lambda} \quad x=0,1,2,\dots$$

Recall that

$$e^{\lambda} = \sum_{l=0}^{\infty} \frac{\lambda^l}{l!}$$

Show that the pmf of

a Poisson random variable sums

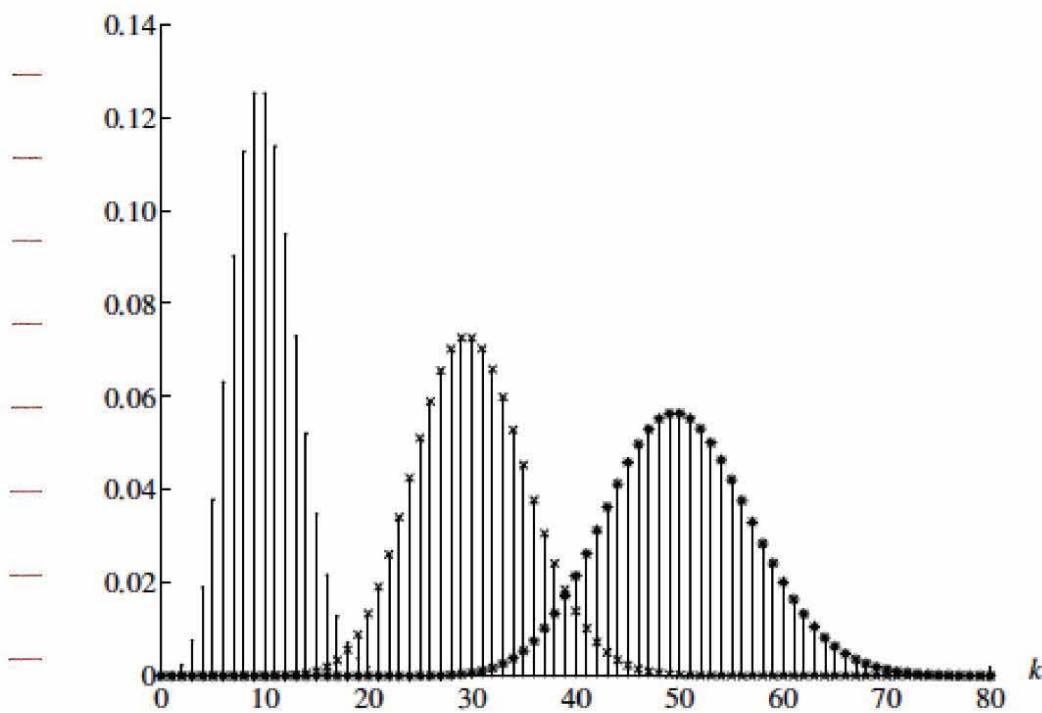
to one for any $\lambda > 0$

The pmf of a $\text{Pois}(\lambda)$

is shown in the following figure

for $\lambda = 10, 30, 50$, from left

to right respectively:



The Poisson Distribution was

introduced by Abraham de Moivre

in 1711 and was further developed

by Simeon Denis Poisson in 1837

The Poisson

Random

Variable represents the

number of events ~~or~~ that occur

in a fixed time interval (or volume,
length etc)

assuming that they occur

independently of the time

Since last event arrived,

and their average rate in
that interval is A

Applications:

- The number of deaths by

horse kicking in the Prussian

army (first application in

reliability engineering (1898)

- birth defects and genetic mutations
- number of errors in a computer code / page
- hair found in McDonald's hamburger
- spread of an endangered animal in Africa
- failure of a machine in one month

Exercise: A life insurance salesman

sells on average 4 life insurance

policies per week. Use

Poisson's distribution to calculate

the probability that in a

given ^{day,} he will sell

a) Some policies

(b) 2 or more policies but

less than 5 policies

A rectangular frame containing ten sets of horizontal lines for handwriting practice. The lines are evenly spaced and extend across the width of the frame.

A rectangular frame containing ten sets of horizontal lines for handwriting practice. The lines are evenly spaced and extend across the width of the frame.

Poisson Distribution as a limit of the binomial

A binomial r.v. with a very small probability of success (p) and very large number of

trials n can be approximated with a Poisson distribution with

$$\lambda = np \text{ i.e.}$$

$$\frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \approx e^{-\lambda} \frac{\lambda^x}{x!}$$

If $n \rightarrow \infty$ $p \rightarrow 0$ $\lambda = np$

Theorem: Assume that $X_n \sim \text{Bin}(n, \frac{\lambda}{n})$

\sqrt{n} . Also, let $X \sim \text{Pois}(\lambda)$.

Then :

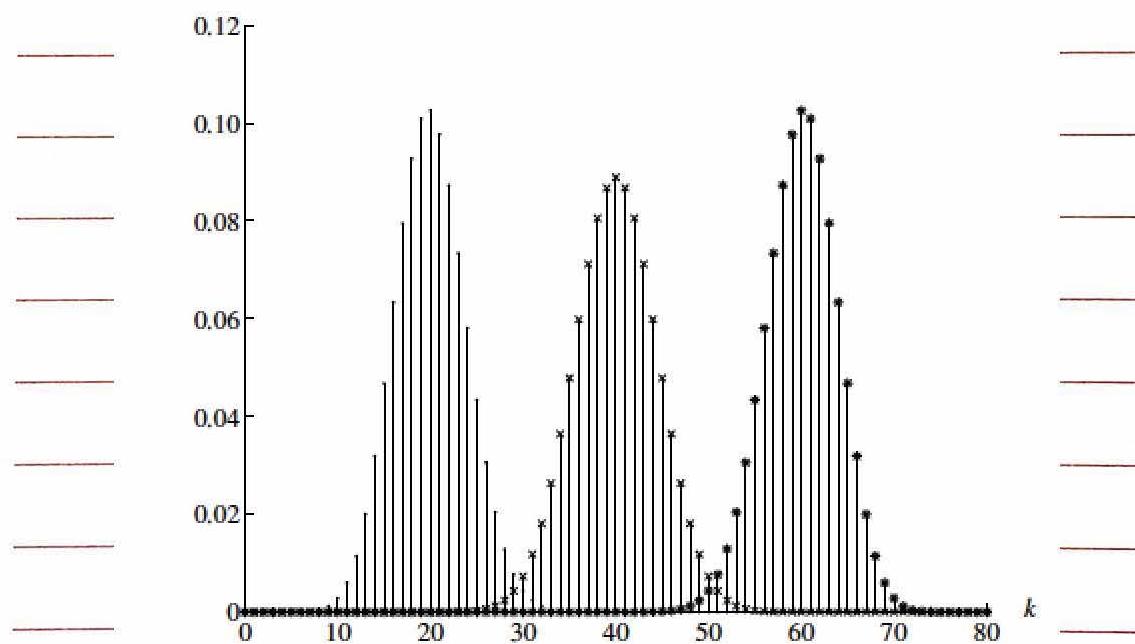
$$\lim_{n \rightarrow \infty} P(X_n = x) = P(X = x) = p_X(x) \quad \forall x \in \{0, 1, 2, \dots\}$$

Proof:

The binomial pmf $p_x(x) = \binom{n}{x} p^x (1-p)^{n-x}$

is shown below for $n=80$ and

$p=.25, .5, \text{ and } 0.75$ from left to right, respectively



Example : Assume that 3% of the tires manufactured in a particular factory are defective. What is the probability of obtaining 5 defective tires from a sample of 50 tires?

Using binomial

$$P(X=5) = \binom{50}{5} (.03)^5 (.97)^{45}$$

$$\approx 0.013074$$

Using Poisson $\lambda = np = 50 \times .03 = 1.5$

$$\mathbb{P}(X=5) = \frac{e^{-1.5} (1.5)^5}{5!}$$

$$= 0.014120$$

(f) Negative Binomial

A negative binomial random variable represents the number of independent Bernoulli trials that take place UNTIL

success is seen k times.

$$X \sim \text{NegB}(k, p)$$

$\underbrace{\text{k-1 successes}}_{\substack{\text{x-1 trials} \\ \text{Binomial}}} \quad \underbrace{\text{k}^{\text{th}} \text{ success}}_{\substack{\text{x}^{\text{th}} \text{ trial}}}$

$$P_X(x) =$$

Remark: A negative binomial trial
can be seen as k (independent)
 Geo_1 trials (looking for the first
success, k times)

Example: What is the probability
that we need 100 coin tosses
before we see the 90th head?

(g) Power Law Distributions:

Power law distributions are distributions

which are proportional to $x^{-\alpha}$

(i) A general power law

distribution is the

Gibrat-Mandelbrot Distribution:

$$P_X(x) = \frac{c}{(x + q)^{\alpha}} \quad x=1, 2, \dots, N$$

Question: What is the value of c ?

(ii) when $N \rightarrow \infty$, the

pmf becomes the Hurwitz

Zeta function $\zeta(\alpha, q)$.

$$P_X(x) = \frac{c}{(cx+q)^{\alpha}} \quad \alpha = 1, 2, \dots$$

$$\alpha > 1$$

For this function to be

a pmf, α must be bigger

than 1. (Why?)

iii) when $N \rightarrow \infty$, $q \approx 0$, we

have the Zeta - distribution

$$X \sim \text{Zeta}(\alpha)$$

$$P_X(x) = \frac{C}{x^\alpha} \quad \alpha = 1, 2, \dots$$

$\alpha > 1$

iv) When $q \neq 0$ and N is finite,

we have the Zipf distribution

$$P_X(x) = \frac{C}{x^\alpha} \quad x = 1, 2, \dots, N$$

($\alpha > 1$ is not needed)

$$X \sim \text{Zipf}(\alpha)$$

Example: If $X \sim \text{Beta}(2)$,

determine the pmf of X

$$P_X(x) = \frac{c}{x^2}$$

$$\sum_{x=1}^{\infty} \frac{c}{x^2} =$$

(v) Another power law

(We use $X \sim \text{Pow}(\alpha)$)

$$P_X(x) = \frac{1}{x^\alpha} - \frac{1}{(x+1)^\alpha} \quad x \in \mathbb{N} \quad \alpha > 0$$

Exercise: Show that

$$\sum_{x=1}^{\infty} \left(\frac{1}{x^\alpha} - \frac{1}{(x+1)^\alpha} \right) = 1 \quad \checkmark \alpha > 0$$

h) Hypergeometric Distribution

This probability distribution

represents a random variable

that gives the number of

successes (x) in n draws,

without replacement, from a

finite population of size N that

contains exactly K successes,

given that each draw is a

success or failure

$$X \sim HG(N, K, n)$$

$$P_X(x) = \frac{\binom{K}{x} \binom{N-K}{n-x}}{\binom{N}{n}}$$

$x \in \mathbb{N}$, for $P_X^{(x)} > 0$, we must have $x \leq \min(K, n)$

$$x \geq \max(0, n+K-N)$$

To prove that $\sum P_X(x) = 1$,

we should use the Vandermonde

Inequality:

Proposition: $\sum_{x=0}^n \binom{K}{x} \binom{N-K}{n-x} = \binom{N}{n}$

Proof.

$$(1+a)^N = (1+a)^K (1+a)^{N-K}$$

Using binomial theorem for the

RHS:

When $N \rightarrow \infty$, the effect of sampling without replacement becomes less and less. One can imagine that drawing samples from the population

N has a success probability

f [] , and it can

be modeled as a

[] distribution with parameters []

Theorem: Assume that

$X_N \sim H G(N, pN, n)$ and

$\rightarrow X \sim \text{Bin}(n, p)$. Then

$$\lim_{N \rightarrow \infty} P(X_N = x) = P(X = x)$$

Proof: Too involved. Will be handed out.

Multinomial Distribution: It is a generalization of the binomial distribution, where instead of two outcomes, each trial has r outcomes with probabilities p_1, p_2, \dots, p_r .

The probability of seeing x_i of outcome i is:

$$P(x_1, x_2, \dots, x_r) = \frac{n!}{x_1! x_2! \cdots x_r!} p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}$$

where $x_1 + x_2 + \cdots + x_r = n$ $p_1 + p_2 + \cdots + p_r = 1$

Example: Rolling a six-sided die n times.

Dealing with Multiple Random Variables

Joint pmfs:

In reality, we deal with collections
of r.v.'s, rather than just a single

r.v.

The relationship between two

discrete r.v.'s X and Y is

captured by their joint pmf.

$$P_{X,Y} : \mathbb{R}^2 \rightarrow [0, 1]$$

$$P_{X,Y}(x, y) = P$$

More generally, if X_1, \dots, X_n

are r.v.'s:

$$P_{X_1, \dots, X_n}(x_1, \dots, x_n) = P$$

Example: 100 independent coin

tosses. Define X = number of

heads and Y = number of tails

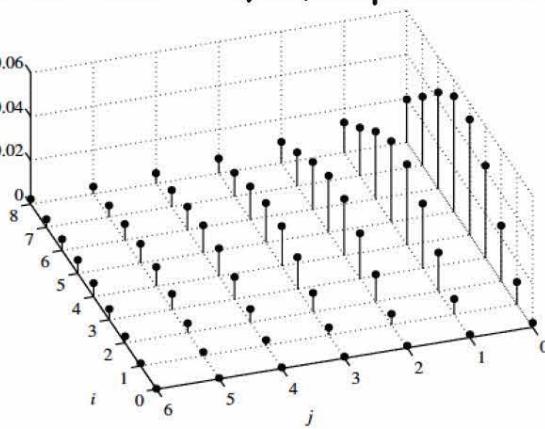
$$P_{X,Y}(1,1) = P$$

$$P_{X,Y}(1,99) =$$

An example of a joint pmf for

two integer valued r.v.'s X, Y

is shown in the following:



Marginal pmfs:

If X, Y are two r.v.'s with joint pmf $P_{X,Y}$, then their individual pmfs P_X and P_Y are called the marginal pmfs, and

can be retrieved using the following equations:

$$P_X(x) =$$

$$P_Y(y) =$$

Reason: The Law of Total
Probability

Conditional pmfs

Let X and Y be two discrete r.v's on (Ω, \mathcal{F}, P) . The conditional pmf of X given Y , $P_{X|Y}$ is defined as:

$$P_{X|Y}(x|y) =$$

given that $P(Y=y)$

Remark : If $P(Y=y) = 0$, we leave $P_{X|Y}(x|y)$

Therefore

$$P_{X_1 \neq Y}(x|y) = \frac{P}{P} =$$

where $P_Y(y) \neq 0$
 > 0

More generally, if X_1, X_2, \dots, X_n
 and Y_1, \dots, Y_m are r.v.'s
 defined on (Ω, \mathcal{F}, P) , we define

$$P_{X_1, \dots, X_n | Y_1, \dots, Y_m}(x_1, x_2, \dots, x_n | y_1, y_2, \dots, y_m)$$

$$= P(X_1=x_1, X_2=x_2, \dots, X_n=x_n | Y_1=y_1, Y_2=y_2, \dots, Y_m=y_m)$$

Example: The joint pmf
of X, Y is given.

$$P_{X,Y}(x,y) = \begin{cases} \frac{2}{n(n+1)} \left(\frac{x}{n+1}\right)^y & x=0, \dots, n-1 \\ & y=0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

Find $P_{Y|X}(y|x)$.

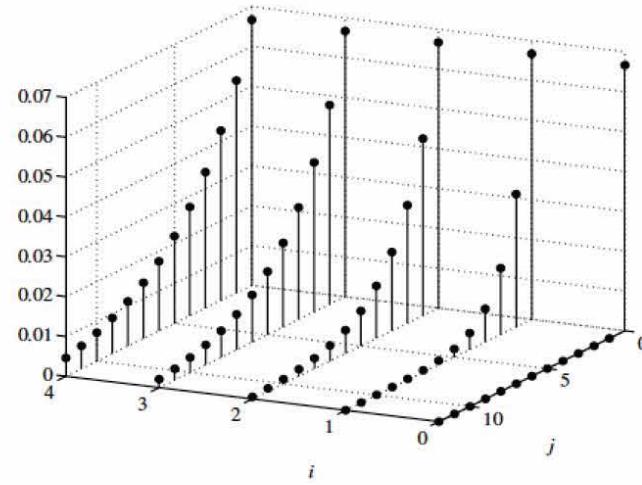


Figure 3.3. Sketch of bivariate probability mass function $p_{XY}(i,j)$ of Example 3.10 with $n = 5$. For fixed i , $p_{XY}(i,j)$ as a function of j is proportional to $p_{Y|X}(j|i)$, which is $\text{geometric}_0(i/(i+1))$. The special case $i = 0$ results in $p_{Y|X}(j|0) \sim \text{geometric}_0(0)$, which corresponds to a constant random variable that takes the value $j = 0$ with probability one.

Exercise: Assume that X

Discrete
is a Uniform R.V., $\text{U}(1, N+1)$,

given that N is a Poisson R.V.

with parameter λ . Find the pmf
of X .

The Law of Total Probability

Let A be an event and
discrete

X a r.r in the probability
space (Ω, \mathcal{F}, P) . Let

$$B_i = \{X = x_i\} = \{\omega \in \Omega \mid X(\omega) = x_i\}$$

where x_i 's are the values

that ~~\in~~ X can take (i.e.

$$\mathcal{R}_X = \bigcup_i \{x_i\}.$$

B_i 's form a partition of
 Ω (why?)

The Law of Total Probability

yields:

$$P(A) = \sum_i P(A|B_i) P(B_i)$$

$$= \sum_i P(A|X=x_i) P(X=x_i)$$

$$= \sum_i P(A|X=x_i) p_X(x_i)$$

Obviously, $A = \{Y=y_i\}$ yields

the formula for marginal probability

mass function of sum r.v. Y .

Ques~~tion~~ How can one

formulate the Baye's Rule

using events $\{X=x_i\}$ and $\{Y=y_j\}$?

The Substitution Law

Assume that X, Y are discrete r.v.'s on $(\mathcal{S}, \mathcal{F}, P)$ and $Z = g(X, Y)$

where $g: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. The pmf of Z

can be calculated as:

$$P(Z=z) = P(g(X, Y) = z)$$

$$= \sum_i P(g(X, Y) = z \mid X = x_i) P(X = x_i)$$

$P_X(x_i)$

→ The substitution law states

that X can be substituted with x_i in the above expression,

i.e.

$$P(Z=z) = \sum_i P(g(x_i, Y) = z | X=x_i) p_{X,Y}(x_i)$$

Proof: Show the substitution

law is correct by using

$$\{ \omega \mid g(X^{(\omega)}, Y^{(\omega)}) = z \} = \cap \{ \omega \mid X^{(\omega)} = x_i \}$$

$$= \{ \omega \mid g(x_i, Y^{(\omega)}) = z \}$$

and apply it to the definition

of Conditional probability.

Independence of Discrete

Random variables.

Recall that X, Y are independent ^{Called}

If $P(X \in B_1, Y \in B_2) = P(X \in B_1)$
 $\times P(Y \in B_2)$.

We also saw that X, Y are

independent iff $F_{X,Y}(x,y) = F_X(x)F_Y(y)$
 $\forall x, y \in \mathbb{R}$

Theorem: Assume that X, Y are discrete r.v.'s on (Ω, \mathcal{F}, P) .

The following statements are equivalent:

(a) X, Y are independent.

(b) $\forall x, y \in \mathbb{R}$, the events

$\{\omega | X(\omega) = x\}$ and $\{\omega | Y(\omega) = y\}$

are independent.

(c) $\forall x, y \in \mathbb{R}$, $P_{X,Y}(x,y) = P_X(x)P_Y(y)$

(d) $\forall x, y \in \mathbb{R}$ if $P_Y(y) > 0$, then $P_{X|Y}(x|y) = P_X(x)$

Many of the equivalences are obvious. We show $(c) \Rightarrow (a)$.

Exercise : Generalize (a)-(d)

in the previous Theorem for
multiple r.v.'s X_1, X_2, \dots, X_n

Example : X_1, X_2, \dots, X_n are
independent $\Leftrightarrow P_{X_1, X_2, \dots, X_n} =$

Theorem: Let X, Y be independent

discrete r.v.'s on (Ω, \mathcal{F}, P) .

Let g, h be real functions,

i.e. $g: \mathbb{R} \rightarrow \mathbb{R}$ and $h: \mathbb{R} \rightarrow \mathbb{R}$.

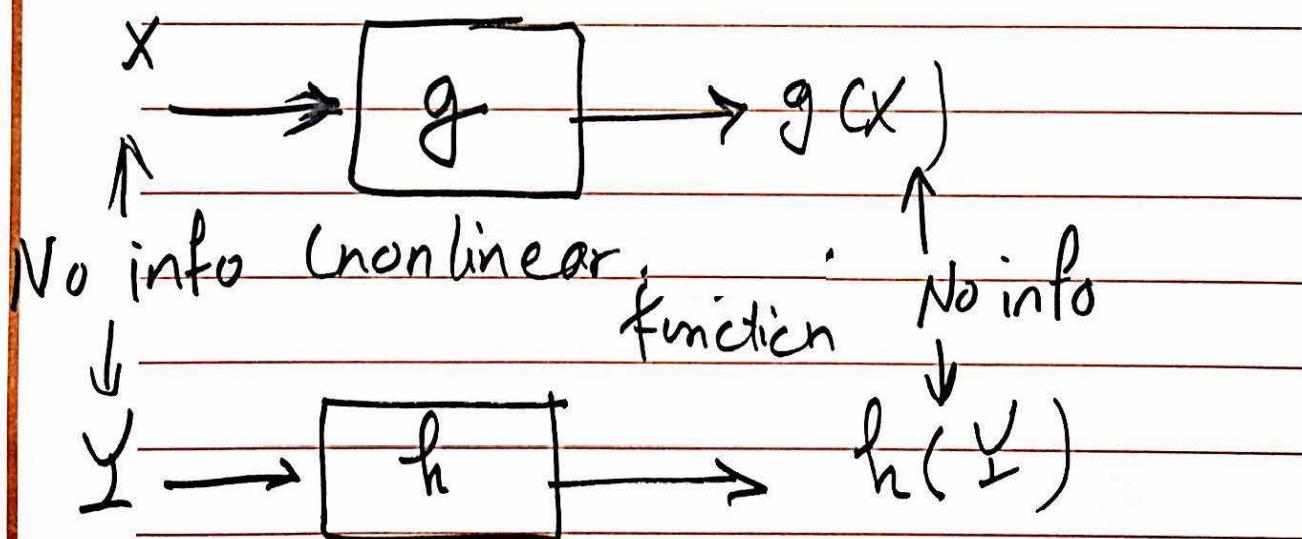
Then, the random variables

$g(X), h(Y)$ are independent.

Proof: Exercise.

Interpretation: If X, Y are independent, they contain no information about each other. Passing them through functions g and h does not

make the results have any information about each other



Question: Why if X and Y are discrete, $g(X)$ and $h(Y)$ are always random variables even if $g(\cdot)$ and $h(\cdot)$ are not continuous?

Example: n independent coin flips

$X = \# \text{ of heads}$

$Y = \# \text{ of tails} = n - X$

$P(\{H\}) = p$

Are X, Y independent?

Intuition: X, Y are []

$$P(X=0) =$$

$$P(Y=0) =$$

$$P(X=0, Y=0) =$$

$$P(X=0, Y=0) \square P(X=0)P(Y=0)$$

Derived Distributions (Functions of Discrete R.V.'s)

Given a discrete random variable

with density P_X , and a function

$g: \mathbb{R} \rightarrow \mathbb{R}$, we are interested

in finding the CDF/PMF of

the random variable $Y = g(X)$.

$$P(Y=y) = P_Y(y) = \sum_{\{x | g(x) = y\}} P_X(x)$$

It is a Borel Set because

Example: Assume that $X \sim \text{Pois}(\lambda)$

and $Y = X^2$. Find the pmf of Y .

$$P_Y(y) = \sum_{x|x^2=y} P_X(x)$$

$$= \begin{cases} & y < 0 \\ & y \geq 0, \quad \boxed{} \end{cases}$$

$$= \left\{ \begin{array}{l} \dots \\ \dots \\ \dots \\ \dots \\ \dots \end{array} \right.$$

$$= \left\{ \begin{array}{l} \dots \\ \dots \\ \dots \\ \dots \\ \dots \end{array} \right.$$

More generally, if X_1, \dots, X_n
 are discrete r.v.'s and $g: \mathbb{R}^n \rightarrow \mathbb{R}$
 is a function, the pmf of
 $\sum g(X_1, X_2, \dots, X_n)$ is obtained

as

$$P(Y=g) = P_Y(g) = \sum_{\{(x_1, x_2, \dots, x_n) \mid g(x_1, x_2, \dots, x_n) = g\}}$$

Sum of Independent Random Variables – Convolution

If X and Y are independent

discrete r.v.'s, the pmf of $X+Y$

can be calculated as:

$$Z = X + Y$$

$$P_Z(z) = P_{X+Y}(z) = P(X+Y=z)$$

$$= \sum_{\{(x,y) | x+y=z\}} P(X=x, Y=y) =$$

(Convolution
Sum)

Remark: The Convolution Sum
also derived
can be by the substitution

law. Prove it using the substitution

law!

Exercise: Assume that

$X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Pois}(\mu)$.

Show that $Z = X + Y$ has

a Poisson distribution with $\mu + \lambda$,

i.e. $Z \sim \text{Pois}(\mu + \lambda)$

Exercise: Assume that X, Y are discrete r.v.'s on (Ω, \mathcal{F}, P) , and are independent. Use the substitution law to find an expression for the pmf of

$$Z = XY.$$

Max and Min Problems

Let X_1, \dots, X_n be independent

Random Variables.

The CDF of PDF of

$Y = \max(X_1, X_2, \dots, X_n)$ is

$$F_Y(y) = P(Y \leq y) = P(\max(X_1, \dots, X_n) \leq y)$$

=

The CDF of $Z = \min(X_1, X_2, \dots, X_n)$

can be calculated as:

$$F_Z(z) = P(Z \leq z) = P(\min(X_1, \dots, X_n) \leq z)$$

Remark: The above argument
for calculating the CDF of
Min and Max of independent
r.v.'s is NOT restricted to
discrete r.v.'s. In the

Case of discrete r.v.'s, the pmf can easily be calculated from the cdf at its points of discontinuity,

$$\begin{aligned} P(X = x_i) &= F_X(x_i) - \lim_{y \rightarrow x_i^-} F_X(x_i) \\ &= \frac{1}{x_i} \end{aligned}$$