LESSON 2

The concept of an event is at the center of probability theory.

An event is modeled by a set.

To understand probability theory, one needs to study set theory and logic.

Aristotelian Logic

A proposition or Statement is a sentence that is either true or false.

(This is not a definition)

A propositional variable represents an arbitrary proposition. We use lower case letters $p, q, r, \ldots$ for propositional variables.

We show the truth and falsehood of propositional variables with 1 and 0 (or T and F) respectively

Logical Connectives

(negation, conjunction, disjunction, implication, double implication or equivalence)

They are defined using truth tables

antecedent consequent

p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \Rightarrow q$	$p \Leftrightarrow q$
1	1	0	1	1	1	1
0	1	1	1	0	1	0
1	0	0	1	0	0	0
0	0	1	0	0	1	1

$p \Rightarrow q$

$p \Leftrightarrow q$

Exercise: Show that

$$p \Rightarrow q \equiv \neg p \vee q$$

p	q	$\neg p$	$\neg p \vee q$	$p \Rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Q.E.D.

$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$

Exercise: Exclusive or (XOR)
= Exclusive disjunction

- It models the "or" that is used in natural languages, which essentially means "either one, but not both nor none"

Construct a truth table for $p \underline{\vee} q$ (exclusive or) and also for $(p \vee q) \wedge \neg (p \wedge q)$ and show that they are equivalent

Example: Prove De Morgan Laws Using Truth Tables;

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Exercise: Show that

$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

(used in proof by contradiction)

p	q	$p \Rightarrow q$	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	0	1

Def: A tautology is a logical expression that is always true for all of the values of the propositional variable

Ex: $p \vee \neg p$

p	$\neg p$	$p \vee \neg p$
1	0	1
0	1	1

Def: A Contradiction is a logical expression that is always false for all of the values of propositional variables

Example: $p \wedge \neg p$

p	$\neg p$	$p \wedge \neg p$
1	0	0
0	1	0

Exercise: Show that

$$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$$

is a tautology, using truth tables

Modus Ponens is a Tautology

$p \Rightarrow q$ | human ⇒ mortal

p | Socrates is a human

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q | Socrates is mortal

$$\boxed{((p \Rightarrow q) \wedge p) \equiv q}$$

Modus Tollens is a Tautology

$p \Rightarrow q$ | GPA=A ⇒ Scholarship

$\neg q$ | no scholarship

$\neg p$ | GPA not A

$$\boxed{((p \Rightarrow q) \wedge \neg q) \equiv \neg p}$$

Two more ~~ta~~ utologies:

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline q \end{array}$$

$$\boxed{((p \vee q) \wedge \neg p) \equiv q}$$

Hypothetical Syllogism

$$\begin{array}{l} p \Rightarrow q \\ q \Rightarrow r \\ \hline p \Rightarrow r \end{array}$$

$$\boxed{[(p \Rightarrow q) \wedge (q \Rightarrow r)] \equiv p \Rightarrow r}$$

Proofs Truth Tables

Predicate logic

Predicate: Loosely speaking, a predicate is a proposition with a variable in it

Proposition | Predicate

Assigning a value to the variable(s) makes the predicate, a proposition

More precisely

Def: A predicate is a boolean-valued function $P: X \longrightarrow \{0, 1\}$

T, F

Logical Quantifiers

Universal Quantifier $\forall$

It represents a universal property, and is interpreted as "given any" or "for all".

Ex: $\forall n \in \mathbb{N},\ 2n = n + n$

Existential Quantifier $\exists$

It is a type of quantifier, which is interpreted as "there exists" or "there is at least one" or "for some"

Ex: $\exists x \in \mathbb{R},\ x^2 = -1$ false

Negation of statements with Logical Quantifiers $\exists$

$\neg(\exists x, p(x)) \equiv \boxed{\forall x, \neg p(x)}$

$\equiv \boxed{\nexists x, p(x)}$

Ex: $\neg(\exists x \in \mathbb{R} \text{ s.t. } x^2 = -1)$

$\equiv \boxed{\nexists x \in \mathbb{R}} \boxed{, \ x^2 = -1}$

$\equiv \boxed{\forall x \in \mathbb{R},\ x^2 \neq -1}$

Negation of $\forall$

$$\neg(\forall x, p(x)) \equiv \exists x, \neg p(x)$$

s.t.

Ex: $\neg(\forall n \in \mathbb{N}, \frac{n}{2} \in \mathbb{N})$

$$\equiv \boxed{\exists n \in \mathbb{N}, \frac{n}{2} \notin \mathbb{N}}$$

$\exists_x \forall_y \Rightarrow \forall_y \exists_x$

$\forall_x \exists_y \not\Rightarrow \exists_y \forall_x$