Lesson 3

Review of Set Theory

A set is a collection of objects, which are its elements

(This is not a definition)

("Set" is a primitive concept)

- We show the elements of a set using $\in$

$x \in A$

- A set with no elements is called an empty set or null set $\emptyset$

$x \in \emptyset \quad \forall x$, is false

We can show a set with its elements

$$A = \{x_1, x_2, x_3, \ldots, x_n\}$$

A finite # of elements

Or with a predicate

$$A = \{x \mid P(x)\}$$

Ex: $\boxed{}$

Subsets (supersets)

Equivalence:

Proper subsethood

$A \subseteq B$ means

$A \subset B$ means

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Universe of Discourse

The Universe of Discourse Ω (or $\mathcal{U}$) contains all elements that could conceivably be of interest in a particular context.

Ex: In number theory,

$$\Omega = \mathcal{U} = \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$$

One can then consider all sets of interest as subsets of the Universe of Discourse

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Complement of A Set (with respect to Ω).

Complement of A : A^c

$$A^c = \boxed{}$$

Obviously, $\Omega^c = \boxed{}$

Operations on sets:

Union :

$$A \cup B =$$

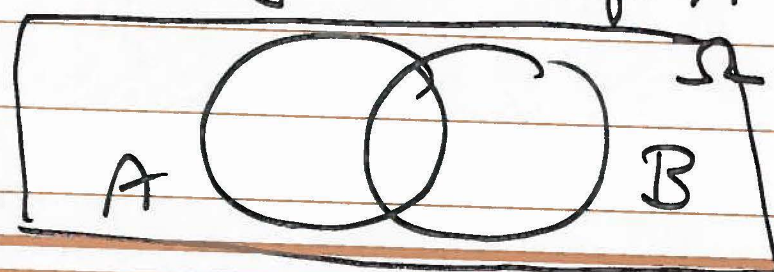
Intersection:

$$A \cap B$$

Set Difference $A \setminus B$ ($A-B$) (or relative complement of B in A

$A \setminus B =$

Venn Diagram of $A \setminus B$



$$A \setminus B = A \cap B^c$$

More generally, assume that

I is an index set, e.g.

$$I = \mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$$

$$I = \mathbb{Q} = \{ \qquad \}$$

$$I = \mathbb{R}$$

(possibly infinite)

$$\bigcup_{i \in I} A_i =$$

$$\bigcap_{i \in I} A_i =$$

Example: $A_i = [0, i)$, $i \in \mathbb{N}$

$A_1 =$ $A_2 = [$ $A_3 =$

$$\bigcap_{i \in \mathbb{N}} A_i =$$

$$\bigcup_{i \in \mathbb{N}} A_i =$$

Exercise:

$$\bigcap_{i \in \mathbb{N}} (-\infty, 1/i) =$$

Exercise:

$$\bigcup_{i\in\mathbb{N}} (-\infty, -1/i] =$$

Exercise:

$$\bigcap_{i\in\mathbb{N}} [0, 1/i)$$

Exercise:

$$\bigcup_{i \in \mathbb{N}} [-i, i]$$

Exercise: $\bigcap_{i \in \mathbb{N}} (-\infty, -i)$

Disjoint / Mutually Exclusive Sets

Def: A_1, A_2 disjoint $\Longleftrightarrow$

Example:

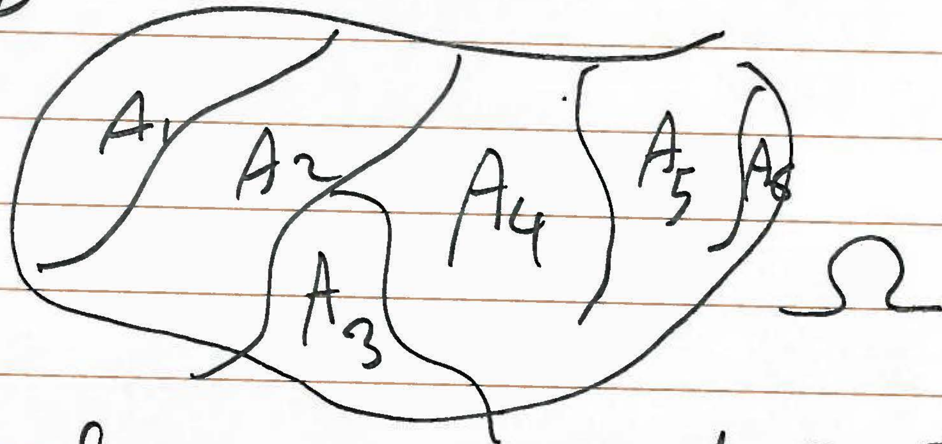
Def: $A_1, A_2, \dots$ are mutually disjoint

$\Longleftrightarrow$

Def: Partition

$A_1, A_2, \dots$ Consist a partition of B iff

Example



(A finite number of partitions for Ω)

Exercise: which one is a partition of $\mathbb{R}^{\geq 0}$

a) $A_i = [i, i+1)$ $\quad i \in \{0, 1, 2, \dots\}$

b) $A_i = [0, i)$ $\quad i \in \{0, 1, 2, \dots\}$

(Some) Properties of Set Operations

Commutativity

$A \cap B =$ $A \cup B =$

Associativity

$A \cap (B \cap C) =$ $A \cup (B \cup C)$

Distributivity

$A \cap (B \cup C) =$

$A \cup (B \cap C) =$

Complement laws

$(A^c)^c =$

$A \cup A^c =$

$A \cap A^c =$

Identity Laws

$A \cup \emptyset =$ $A \cap \Omega =$

Domination Laws

$A \cup \Omega =$ $A \cap \emptyset =$

Idempotency

$A \cup A =$ $A \cap A =$

Absorption Laws

$A \cup (A \cap B) =$

$A \cap (A \cup B) =$

Venn Diagram

De Morgan's Laws

$(A_1 \cup A_2)^c =$

$(A_1 \cap A_2)^c =$

More Generally, for an index set I

$\left(\bigcup_{i \in I} A_i\right)^c =$

$\left(\bigcap_{i \in I} A_i\right)^c =$

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Element Chasing

To prove $A=B$, we must prove:

$\boxed{\qquad\qquad\qquad\qquad\qquad\qquad}$

Therefore, we must prove:

$(x \in A \Rightarrow \qquad\qquad\qquad) \wedge (x \in B \Rightarrow \qquad\qquad\qquad)$

i.e., we must "chase" elements

Proof of $(A \cap B)^c = \qquad A^c \cup B^c$

Proof of $\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c$

General form of Distributivity

$$\left(\bigcap_{i\in I} A_i\right) \cup B =$$

$$\left(\bigcup_{i\in I} A_i\right) \cap B =$$

Notation: Special Sets

$\mathbb{R}$: Real Numbers

$\mathbb{R}^* = \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$

extended real numbers

$\mathbb{Z}$: Integers $\{\ldots, -4, -3, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{N}$: Natural Numbers or Strictly Positive Integers $\mathbb{N} = \mathbb{Z}^{>0}$

Intervals

$[a,b] = \{x \in \overline{\mathbb{R}} \mid a \leq x \leq b\}$

$(a,b) =$

$[a,b) =$

$(a,b] =$

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Cartesian Product

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \land a_2 \in A_2\}$$

$A_1, A_2, \ldots, A_n$

$$A_1 \times A_2 \times \ldots \times A_n = \prod_{i=1}^{n} A_i$$

$=$

More Generally,

$A_1, A_2, \ldots$

$\prod_{i=1}^{\infty} A_i$ is the <u>sequence</u>

$$A_i = A \Rightarrow \prod_{i=1}^{\infty} A_i = A^{\infty}$$

Example: $A = \{0, 1\}$

$$A^{\infty} = \prod_{i=1}^{\infty} A_i \quad , \forall i \quad A_i = \{0, 1\}$$

Family of Subsets of A / Power Set of A

The set of all subsets of A is called the power set of A and is denoted as 2^A or $\mathcal{P}_A$ or $\mathcal{P}(A)$.

Example: $A=\{1,2\}$

$2^A=\{$

$|A|=$

$|2^A|=$

Relations

Assume $K=\left(\prod_{i=1}^{n} A_i\right)\times B$

Any subset of K, $R\subseteq K$, is called a relation.

Example: $A = \{2,3\}$ $B = \{1,2\}$

$K = A \times B =$

$R =$

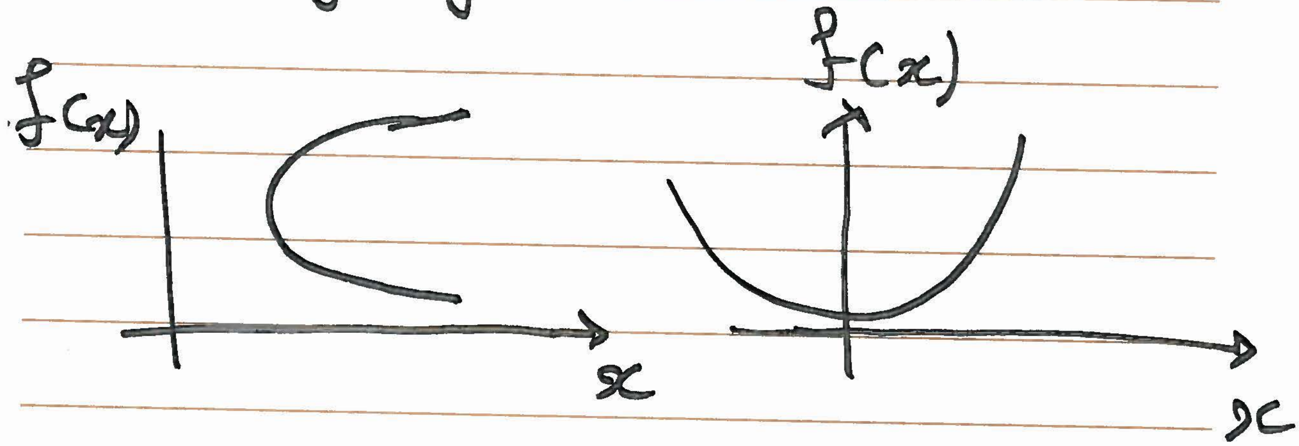
Function:

Def: If a relation maps each n-tuple to a unique member of B, it is called a function

$f \subseteq K$

$(x_1, \dots, x_n, y) \in f \wedge (x_1, \dots, x_n, y') \in f$

$\Longrightarrow y = y'$



We write $f: A_1 \times \dots \times A_n \longrightarrow B$

$\prod_{i=1}^{n} A_i$ is called the $\boxed{\text{domain}}$ of f

B is called the co-domain of f.

For simplicity, we continue with functions of the form $f: A \longrightarrow B$

Forward Image of $f: A \to B$

Def. $\vec{f}: 2^A \to 2^B$

$\forall S \subseteq A \ (S \in 2^A)$

$\vec{f}(S) =$

The notation $f(S)$ is also used.

Example: $y = f(x) = x^2$ $f: \mathbb{R} \to \mathbb{R}$

$\vec{f}([1,2]) = \{f(x) \mid x \in [1,2]\}$

$\vec{f}([-1,3]) =$

$\vec{f}([-1,-\frac{1}{2}) \cup (\frac{1}{2},1]) =$

Inverse Image (pre_Image) of $f: A \to B$

$$f^{\leftarrow}: 2^B \to 2^A$$

$\forall T \subseteq B \ (T \in 2^B)$

$$f^{\leftarrow}(T) = \{x \in A \mid f(x) \in T\}$$

Obviously, $f^{\leftarrow}(T) \subseteq A \ (\in 2^A)$

The notation $f^{-1}(T)$ is also used, but We avoid it because it can be confused with an inverse function

Example : $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = x^2$$

$$\overleftarrow{f}([4,9]) = \{x \in \mathbb{R} \mid f(x) \in [4,9]\}$$

$$\overleftarrow{f}([-9,-1]) =$$

- Important note: forward and inverse images are functions that map sets to sets. In other words, they are set functions

- Remark: It is a very good habit to always show the domain and co-domain of a function to avoid ambiguity.

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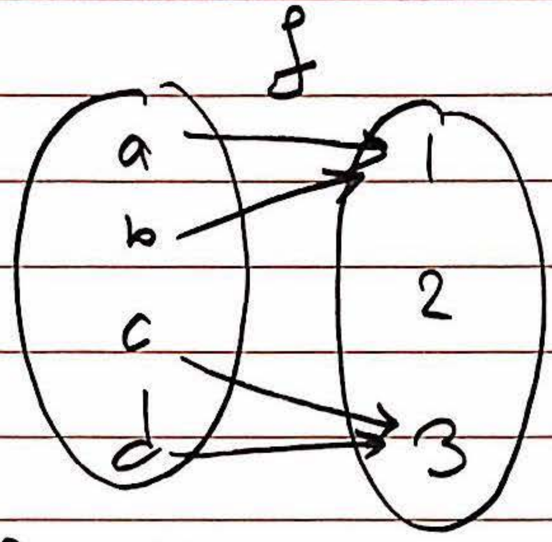
Example:



Is f a function?

$f^{\rightarrow}(\{a,b\}) =$

$f^{\rightarrow}(\{a,b,c\}) =$

$f^{\rightarrow}(\emptyset) =$

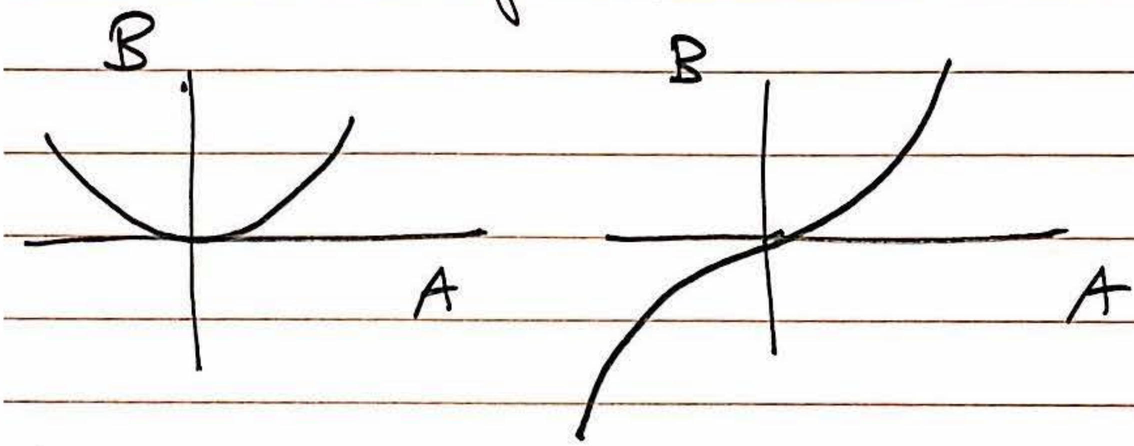
$f^{\leftarrow}(\{1\}) =$

$f^{\leftarrow}(\{2\}) =$

$f^{\leftarrow}(\{1,2,3\}) =$

Def: One-to-one function

A function $f: A \to B$ is one-to-one (1-1) iff it assigns no more than one member of A to each member of B



or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

or $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

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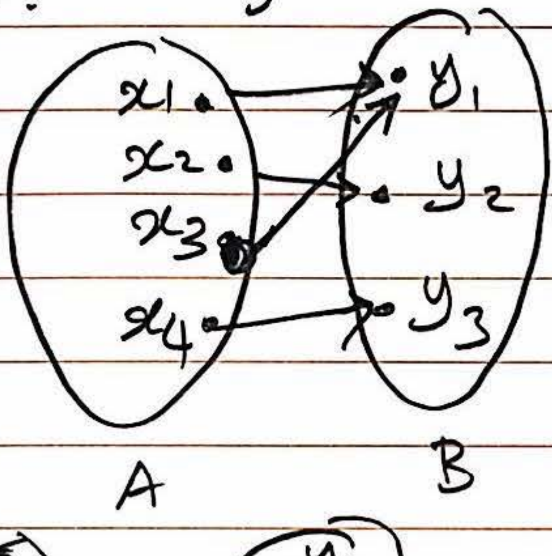
One-to-one functions are also called <u>injective</u>

Onto function:

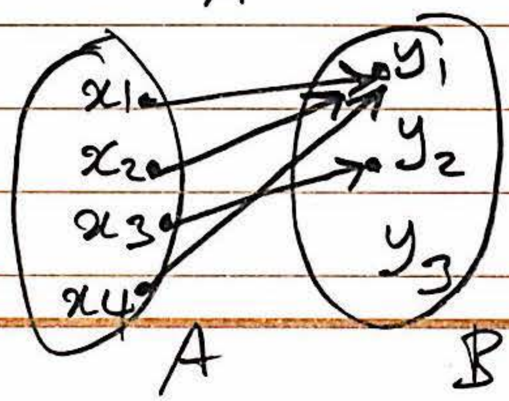
Def: $f: A \to B$ is onto iff $\vec{f}(A) = B$.

Example: f whic function is onto?

a)



(b)



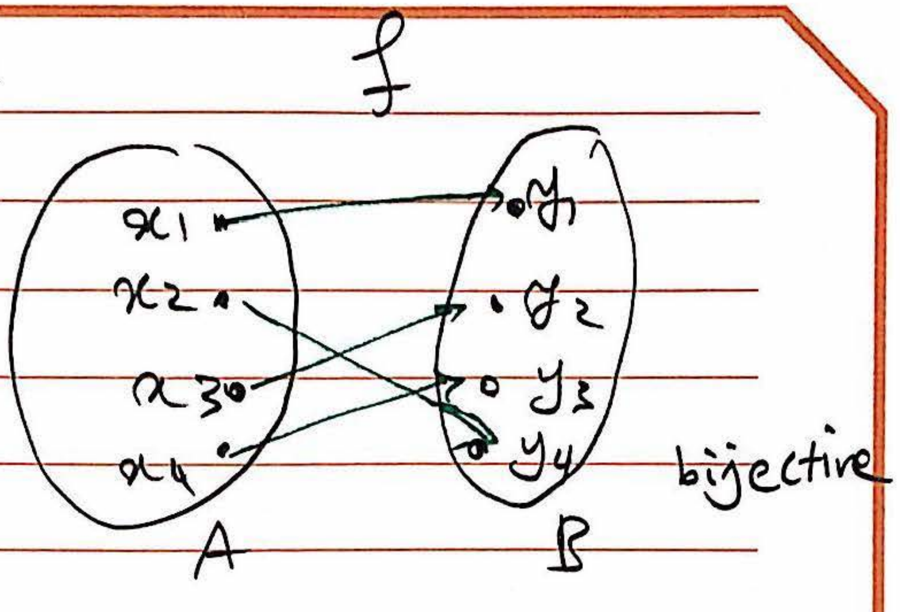
Onto functions are also called surjective

One-to-One Correspondence

Def: $f: A \to B$ is called a one-to-one correspondence

iff it is both 1-1 and onto. It is also called bijective.

Example:



Remark: In a bijective mapping, every element in the co-domain has a pre-image and the pre-images are unique. Thus, we can define the inverse function, $f^{-1}: B \longrightarrow A$, such that $f^{-1}(y) = x$ if $f(x) = y$. Therefore, bijective functions are invertible.

Cardinality

In informal terms, the cardinality of a set is the number of elements in that set. If one wishes to compare the cardinalities of two finite sets, A and B, they can simply count the number of elements in each set. But what if the sets contain infinitly many elements?

Def. Two sets A, B are said to be of the same cardinality (equicardinal) iff there exists a one-to-one correspondence between A and B, and we write $|A| = |B|$.

Question: Using the concept of one-to one and onto functions, define $|A| \geq |B|$ and $|A| \leq |B|$.

Def: A set A is said to be countably infinite if A and $\mathbb{N}$ have the same cardinality.

Def: A set is said to be countable if it is either finite or countably infinite.

Example: $\mathbb{N}$ is Countably infinite

$f:$

Example: $\mathbb{Z}$ is countably infinite.

$f : \mathbb{Z} \longrightarrow \mathbb{N}$

$f(x) = \begin{cases} & \\ & \end{cases}$

$n = f(x) \in \mathbb{N}$	$x \in \mathbb{Z}$
	0
	+1
	−1
	+2
	−2

Example: Do $(0,1)$ and $\mathbb{R}$ have the same cardinality?

Example: $\mathbb{N} \times \mathbb{N}$ is countable

$$
\begin{array}{cccccc}
(1,1) & (2,1) & (3,1) & (4,1) & (5,1) & \cdots \\
(1,2) & (2,2) & (3,2) & (4,2) & (5,2) & \cdots \\
(1,3) & (2,3) & (3,3) & (4,3) & (5,3) & \cdots \\
(1,4) & (2,4) & (3,4) & (4,4) & (5,4) & \cdots \\
(1,5) & (2,5) & (3,5) & (4,5) & (5,5) & \cdots \\
\vdots & \ddots & & & &
\end{array}
$$

Define $f(i,j)$ equal to the number of pairs visited when (i,j) visited. $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

f is a one-to-one correspondence because it visits all pairs and it visits each pair once.

$\Longrightarrow$ $\mathbb{N} \times \mathbb{N}$ is countable

Remark: Using the same argument, one can prove that $A \times B$ is countable when both A and B are countable.

Proposition: $\mathbb{Z} \times \mathbb{N}$ and $\mathbb{Z} \times \mathbb{Z}$ are countable!

$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z},\ b \neq 0 \right\}$$

is $\boxed{}$

Def: Uncountable set

A is uncountable if it is not countable.

Example: It can be shown that $[0,1)$, $\mathbb{R}$, $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$

are uncountable.

Limit of a sequence of Sets

Def: Indicator function of $A \subseteq \Omega$

$$I_A : \Omega \to \{0,1\}$$

$$I_A(\omega) = \left\{ \right.$$

Example: $\Omega = \mathbb{R}$, $A = [0,1]$

Def: Assume $A_1, A_2, \ldots$ is a sequence of sets, $\forall i \in \mathbb{N}$, $A_i \subseteq \Omega$.

We say $\lim_{n \to \infty} A_n = A$

iff

(i.e. iff the indicator functions converge pointwise)

Example : $A_i = [-1/i, 1/i]$

$\lim_{i \to \infty} A_i =$

because

$\forall \omega \in \mathbb{R}, \lim_{i \to \infty} I_{A_i}(\omega) =$

Theorem:

(a) Suppose that A_n is an increasing sequence of sets $(\forall n, A_n \subseteq A_{n+1})$, then $\lim_{n \to \infty} A_n$ exists and is equal to $\bigcup_{n \in \mathbb{N}} A_n$.

(b) If A_n is a decreasing sequence of sets $(A_n \supseteq A_{n+1})$, then $\lim_{n \to \infty} A_n$ exists and is equal to $\bigcap_{n \in \mathbb{N}} A_n$.

Example: (a) $A_i = [0, i)$ $i \in \mathbb{N}$

Increasing or decreasing?

$\lim_{i \to \infty} A_i =$

(b) $B_i = [i, +\infty)$, $i \in \mathbb{N}$

Increasing or decreasing?

$\lim_{i \to \infty} B_i =$

Exercise: Show that

$$f^{\rightarrow}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f^{\rightarrow}(A_i)$$