Lesson 11

Random Vectors

Informally, a vector whose entries are random variables is called a random vector, and a matrix whose elements are random variables is called a random matrix.

Def: Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. A function

$\underline{X} \to \Omega \to \mathbb{R}^n$ is called a random variable, if for all $i \in \{1, \dots, n\}$ and $B \in \mathcal{B}(\mathbb{R})$, $\overleftarrow{X_i}(B)$ is $\mathcal{F}$-measurable, i.e. X_i is a random variable, where X_i is the i^{th} component of $\underline{X}$.

Def: Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. A function $\underline{X} : \Omega \to \mathbb{R}^{n \times m}$ is called a random matrix, if for all

$B \in \mathcal{B}(\mathbb{R})$ and $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$, $X_{ij}^{-1}(B)$ is $\mathcal{F}$-measurable, where X_{ij} is the ij element of $\underline{X}$.

Expectation of A Random Vector

$$\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \Rightarrow \mathbb{E}[\underline{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \vdots \\ \mathbb{E}[X_n] \end{bmatrix}$$

Expectation of A Random Matrix

$$\underline{X} = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & & \\ X_{n1} & \cdots & X_{nm} \end{bmatrix} \Rightarrow \mathbb{E}[\underline{X}] = \begin{bmatrix} \mathbb{E}[X_{11}] & \cdots & \mathbb{E}[X_{1m}] \\ \vdots & & \\ \mathbb{E}[X_{n1}] & \cdots & \mathbb{E}[X_{nm}] \end{bmatrix}$$

in other words,

$$\left(\mathbb{E}[\underline{X}]\right)_{ij} = \mathbb{E}[X_{ij}]$$

Linearity of Expectation

Assume that $A \in \mathbb{R}^{n \times m}$ is a non-random matrix and $\underline{X}$ is a random $m \times p$ matrix. Then

$$\mathbb{E}[A\underline{X}] = A\,\mathbb{E}[\underline{X}]$$

Proof:

More generally:

If $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{p \times q}$, $C \in \mathbb{R}^{n \times q}$,

and X is a random $n \times q$ matrix

$$\mathbb{E}[AXB + C] = A\,\mathbb{E}[X]B + C$$

(Auto) Correlation Matrix

Def: Assume that $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ is a random vector. The Correlation matrix of $\underline{X}$ is defined as:

$$R = \mathbb{E}[\underline{X}\,\underline{X}^T]$$

where $\underline{X}^T$ is the transpose of $\underline{X}$, with $(\underline{X})_{ij} = (\underline{X}^T)_{ji}$

Note that $\underline{X}\,\underline{X}^T$ is:

$$\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Note that the correlation matrix is a symmetric matrix, i.e. $R_{ij} = R_{ji}$. Why?

(Auto) Covariance Matrix

Def: Assume that $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ and its mean is $m = \mathbb{E}[\underline{X}]$. The (auto) covariance matrix of X is defined as $\text{cov}(\underline{X}) = \mathbb{E}[(\underline{X} - m)(\underline{X} - m)^T]$

Exercise: Show that

$$\mathrm{cov}(X) = R - mm^T = \mathbb{E}[XX^T] - mm^T$$

"Theorem": if $C = \mathrm{Cov}(\underline{X})$,

then (a) $C_{ij} = \mathrm{cov}(X_i, X_j)$.

(b) $C_{ii} = \mathrm{Var}(X_i)$
(diagonal elements)

(c) $C^T = C$, i.e. C is symmetric

(d) $i \neq j$, $C_{ij} = 0 \Longleftrightarrow X_i, X_j$ are uncorrelated

Remark: (d) implies that C is a diagonal matrix iff X_i, X_j are uncorrelated for all $i \neq j$

Proof: Exercise.

Exercise: If $Cor(\underline{X}) = C$

Calculate:

(a) $Cor(A\underline{X} + B)$, where A, B are matrices with appropriate dimensions

(b) $Cor(A\underline{X}B + G)$, where A, B, G are matrices with appropriate dimensions.

Positive Definite Matrices

Def: Assume that $C \in \mathbb{R}^{n \times n}$, $C^T = C$ (i.e. C is symmetric). If for any vector $x \in \mathbb{R}^n$, $Q(x) = x^T C x \geq 0$, C is called a <u>positive semi-definite</u> matrix. If for any $x \in \mathbb{R}^n$, $x \neq 0$, $Q(x) = x^T C x > 0$, C is called a <u>positive definite matrix</u>

Lemma: Let $\underline{X}$ be a random vector and $C = Cov(\underline{X})$. Then C is positive semi-definite

Proof:

The Cross-Correlation Matrix

Def: Assume that $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ and $\underline{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix}$ are random vectors. The cross correlation matrix of $\underline{X}$ and $\underline{Y}$, is defined as

$$R_{XY} = \mathbb{E}[XY^T].$$

Cross-Covariance Matrix

Def: Assume that $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ and $\underline{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix}$ are both random vectors and $\mathbb{E}[\underline{X}] = m_X$ and $\mathbb{E}[\underline{Y}] = m_Y$. Their cross-covariance matrix is an $n \times m$ matrix:

$$\mathrm{cov}(\underline{X}, \underline{Y}) = \mathbb{E}\left[(\underline{X} - m_X)(\underline{Y} - m_Y)^T\right]$$

Assume that we stack $\underline{X}$ and $\underline{Y}$ into $\underline{Z} = \begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix}$, which is an $(n+m)$-dimensional vector.

The (auto)-covariance matrix of $\underline{Z}$ is:

$$\mathbb{E}[(\underline{Z} - m_Z)(\underline{Z} - m_Z)^T]$$

$$= \mathbb{E}\left[\left(\begin{bmatrix} \underline{X} - m_X \\ \underline{Y} - m_Y \end{bmatrix} \begin{bmatrix} \underline{X} - m_X \\ \underline{Y} - m_Y \end{bmatrix}^T\right)\right.$$

$=$

Therefore:

$$C_Z = \begin{bmatrix} C_X & C_{XY} \\ C_{YX} & C_Y \end{bmatrix}$$

where C_Z is $(n+m) \times (n+m)$,

$$C_Z \in \mathbb{R}^{(n+m) \times (n+m)}$$

and

$C_X \in \mathbb{R}^{n \times n}$

$C_{XY} \in \mathbb{R}^{n \times m}$

$C_{YX} \in \mathbb{R}^{m \times n}$

$C_Y \in \mathbb{R}^{m \times m}$

Recall that two random variables X and Y are called uncorrelated if $Cov(X, Y) = 0$.

Analogously, two random vectors

$$\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \text{ and } \underline{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$

are called uncorrelated if all of their entries are uncorrelated, i.e. $cov(X_i, Y_j) = 0 \ \forall i \in \{1, 2, \dots, n\}$
$j \in \{1, 2, \dots, m\}$

This means that $C_{XY} \in \mathbb{R}^{n \times m}$ is $0_{n \times m}$.

This is equivalent to the matrix C_Z being "block diagonal."

$$C_Z = \begin{bmatrix} C_X & 0_{n \times m} \\ 0_{m \times n} & C_Y \end{bmatrix}$$