LESSON 2

The concept of an <u>event</u> is at the center of probability theory.

An event is modeled by a set.

To understand probability theory, one needs to study <u>set theory</u> and <u>logic</u>.

Aristotelian Logic

A proposition or statement is a sentence that is either true or false.

(This is not a definition)

A propositional variable represents an arbitrary proposition. We use lower case letters $p, q, r, \ldots$ for propositional variables.

We show the truth and falsehood of propositional variables with 1 and 0 (or T and F) respectively

Logical Connectives

(negation, conjunction, disjunction, implication, double implication or equivalence)

They are defined using truth tables

premise / antecedent

consequent / conclusion

p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \Rightarrow q$	$p \Leftrightarrow q$
1	1	0	1	1	1	1
0	1	1	1	0	1	0
1	0	0	1	0	0	0
0	0	1	0	0	1	1

negation of EXOR

implication

equivalence

$(p \Rightarrow q) \wedge (q \Rightarrow p)$

Exercise: Show that

$$p \Rightarrow q \equiv \neg p \vee q$$

p	q	$\neg p$	$\neg p \vee q$	$p \Rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Q.E.D

$(p \Rightarrow q) \Longleftrightarrow (\neg p \vee q)$ is a tautology

Exercise: Exclusive or (XOR)
= Exclusive disjunction

It models the "or" that is used in natural languages, which essentially means "either one, but not both nor none"

Construct a truth table for $p \veebar q$ (exclusive or) and
$p \oplus q$
also for $(p \vee q) \wedge \neg(p \wedge q)$
and show that they are equivalent

Example: Prove De Morgan Laws Using Truth Tables;

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Exercise

Exercise: Show that

$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

(used in proof by contradiction)

p	q	$p \Rightarrow q$	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	0	1

Def: A tautology is a logical expression that is always true for all of the values of the propositional variable

Ex: $p \vee \neg p$

p	$\neg p$	$p \vee \neg p$
1	0	1
0	1	1

Def: A Contradiction is a logical expression that is always false for all of the values of propositional variables

Example: $p \wedge \neg p$

p	$\neg p$	$p \wedge \neg p$
1	0	0
0	1	0

Exercise: Show that

$$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$$

is a tautology, using truth tables

$2+2=4$

$2+2=3$

$5=4$

$-3 \quad -3$

$2=1$

Modus Ponens is a Tautology

$p \Rightarrow q$ — humans ⇒ mortal

p — Socrates is a human

---

q — Socrates is mortal

$$\boxed{((p \Rightarrow q) \wedge p) \Rightarrow q}$$ tautology

Modus Tollens is a Tautology

$p \Rightarrow q$ — GPA=A ⇒ Scholarship

$\neg q$ — no scholarship

$\neg p$ — GPA not A

$$\boxed{((p \Rightarrow q) \wedge \neg q) \Rightarrow \neg p}$$

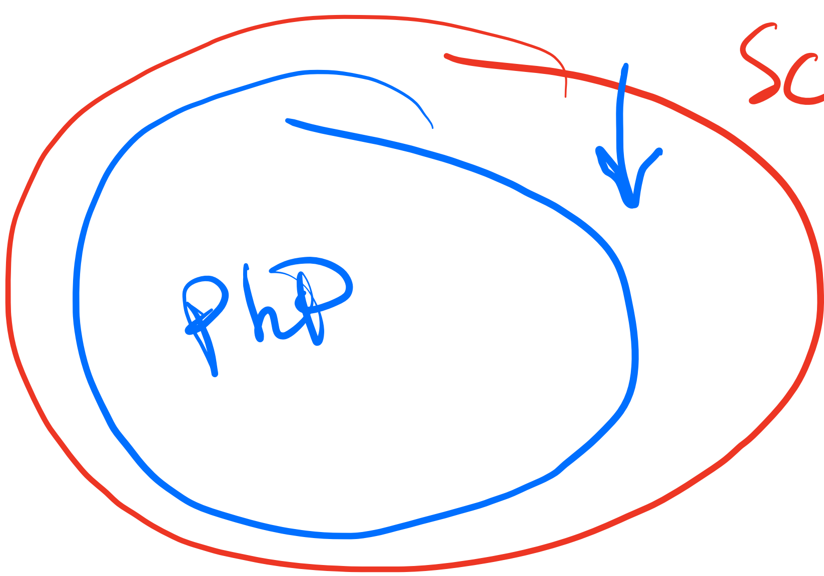
is a tautology

$$\cancel{\begin{array}{l} p \Rightarrow q \\ \neg p \\ \hline \neg q \end{array}}$$

$$\cancel{\begin{array}{l} \text{PhD student} \Rightarrow \text{Scholarship} \\ \neg\, \text{PhD student} \\ \hline \neg\, \text{Scholarship} \end{array}}$$

WRONG ARGUMENT

Scholarship



Two more ~~ta~~ tautologies:

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline q \end{array}$$

$$\boxed{\left[(p \vee q) \wedge \neg p\right] \Rightarrow q}$$

Hypothetical Syllogism

$$\begin{array}{l} p \Rightarrow q \\ q \Rightarrow r \\ \hline p \Rightarrow r \end{array}$$

$$\boxed{\left[(p \Rightarrow q) \wedge (q \Rightarrow r)\right] \Rightarrow (p \Rightarrow r)}$$

Proofs Truth Tables

Predicate Logic

Predicate: Loosely speaking,

a predicate is a proposition

with a variable in it

Proposition	Predicate
$\boxed{1 > 5}$	$\boxed{x > 5}$

Assigning a value to the variable(s)

makes the predicate, a proposition

More precisely

Def: A predicate is a boolean-

valued function $P: X \to \{0, 1\}$

$\qquad\qquad\qquad\qquad T, F$

Logical Quantifiers

Universal Quantifier $\forall$

It represents a universal property, and is interpreted as "given any" or "for all"

Ex: $\forall n \in \mathbb{N}, \quad 2n = \underbrace{n+n}_{P(n)}$

Existential Quantifier $\exists$

It is a type of quantifier, which is interpreted as "there exists" or "there is at least one" or "for some"

Ex: $\exists \underbrace{x \in \mathbb{R}}, \ \underbrace{x^2 = -1}_{p(x)}$

Negation of statements with Logical Quantifiers $\exists$

$$\neg(\exists x, p(x)) \equiv \boxed{\nexists x, p(x)}$$

$$\equiv \boxed{\forall x, \neg p(x)}$$

Ex: $\neg(\exists x \in \mathbb{R} \text{ s. t. } x^2 = -1)$

$$\equiv \boxed{\nexists x \in \mathbb{R} \text{ s.t } x^2 = -1}$$

$$\equiv \boxed{\forall x \in \mathbb{R}, \ x^2 \neq -1}$$

Negation of $\forall$

$$\neg(\forall x, p(x)) \equiv \exists x, \neg p(x)$$

s.t.

Ex: $\neg\left(\underbrace{\forall n \in \mathbb{N}, \frac{n}{2} \in \mathbb{N}}\right)$

$$\equiv \boxed{\exists n \in \mathbb{N}, \frac{n}{2} \notin \mathbb{N}} \quad T$$