

Lesson 5

Random Variables and Their CDFs

It is not always easy to work
with Sample Spaces like

$\{H, T\}^n$, especially when we want
to quantify the properties of
such sample spaces. Therefore,
we work with a Random
Variable.

Loosely speaking, a random variable is a function from the sample space to real numbers.

Def. Assume that (Ω, \mathcal{F}, P) is a probability space. The function $X: \Omega \rightarrow \mathbb{R}$ is a random variable if $\forall c \in \mathbb{R}, X^{-1}((-\infty, c])$ is \mathcal{F} -measurable, i.e. it is an event,

In other words

$$\star ((-\infty, c]) = \{ \omega \in \Omega \mid X(\omega) \in (-\infty, c] \in F \}$$

is an event for all real numbers $c \in \mathbb{R}$.

Example: Consider (Ω, F, P)

where $\Omega = \{\alpha, \beta, \gamma\}$

\Rightarrow F -field generated by $\{\alpha, \beta\}$ $F = \{\emptyset, \{\alpha, \beta\}, \{\gamma\}, \Omega\}$

and $P(\emptyset) = 0, P(\Omega) = 1$

$P(\{\alpha, \beta\}) = \frac{1}{2}, P(\{\gamma\}) = \frac{1}{2}$

Consider $g : \Omega \rightarrow \mathbb{R}$

$$g(\omega) = \begin{cases} 1 & \omega \in \{\alpha, \beta\} \\ 2 & \omega \in \{\gamma\} \end{cases}$$

$$P(g(\omega) = 1) = P(\{\omega | g(\omega) \in \{\alpha, \beta\}\})$$

$$= P(\{\alpha, \beta\}) = 1/2$$

$$P(g(\omega) = 2) = P(\{\omega | g(\omega) \in \{\gamma\}\})$$

$$= P(\{\gamma\}) = 1/2$$

$$P(g(\omega) \leq c) = \begin{cases} P(\emptyset) = 0 & c < 1 \\ P(\{\alpha, \beta\}) = 1/2 & 1 \leq c \leq 2 \\ P(\{\gamma\}) = 1 & c \geq 2 \end{cases}$$

$\leftarrow [(-\infty, c])$

For all values of c , $P()$

is Well-defined / not defined

Next, consider $h: \Omega \rightarrow \mathbb{R}$

$$h(\omega) = \begin{cases} 0 & \omega = \alpha \\ 1 & \omega = \beta \\ 2 & \omega = \gamma \end{cases}$$

$$P(h(\omega) = 0) = P(\{\omega \mid \omega = \alpha\})$$

$$= P(\{\alpha\}) = \text{undefined}$$

$\notin F$

Issue: The σ -field on

which P is defined is not

rich / large enough.

Convention: Upper-case letters

are used for r.v.'s : X, Y, Z,

and lower case letters

are used for numerical

values that they can

take.

$$X(\omega) = X \in \mathbb{R}$$

It is an appropriate practice

to show the dependence of

a random variable on $\omega \in \Omega$:

$$X(\omega), Y(\omega), Z(\omega)$$

Example : Indicator Function
of A Measurable Set.

Let (Ω, \mathcal{F}) be a measurable space and $A \in \mathcal{F}$.

The indicator function of A :

$$I_A : \Omega \rightarrow \{0, 1\}$$

$$I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

is a random variable.

$$X((-\alpha, 2]) = \{\omega \in \Omega \mid X(\omega) \in (-\alpha, 2]\}$$

$$= \{\omega \in \Omega \mid X(\omega) \in \Omega\} = \Omega \in \mathcal{F}$$

$$\overleftarrow{X}((-\infty, -2]) = \{ \omega \in \Omega \mid I_A(\omega) \leq -2 \} = \emptyset$$

$$\overleftarrow{X}((-\infty, 1/2]) = \{ \omega \in \Omega \mid I_A(\omega) \leq 1/2 \} = A \in F$$

In general

$$\overleftarrow{X}((-\infty, c]) = \begin{cases} \emptyset & c < 0 \\ A^c & 0 \leq c < 1 \\ \Omega & c \geq 1 \end{cases}$$

Note: It is common to

show the event

$$\overleftarrow{X}((-\infty, 2]) \text{ as}$$

$$\{ X \leq 2 \} = \{ \omega \in \Omega \mid X(\omega) \leq 2 \}$$

Example: If (Ω, \mathcal{F}) is a measurable space equipped with probability measure P , and $A \notin \mathcal{F}$, I_A , the indicator function of A , is NOT a random variable:

variable:

$$I_A : \Omega \rightarrow \{0, 1\}$$

$$A \notin \mathcal{F} \quad I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

$$\{X \leq Y_2\} = X^{\leftarrow}((-\infty, Y_2])$$

$$= \{\omega \mid X(\omega) \leq Y_2\} = \{\omega \mid I_A \leq Y_2\}$$

$$= A^c \notin \mathcal{F}$$

When $A \in F$, I_A is called

a "Bernoulli Random Variable"

Bernoulli Random Variable

represents an experiment

that has two types of

outcome : Success and failure
 WCA WEA^c

If A occurs (outcome belongs

to A), success occurred

If A^c occurs (outcome belongs
to A^c), doesn't

failure occurred.

$$P(\text{Success}) = P(X=1)$$

$$= P(I_A = 1) = P(\{\omega \mid I_A(\omega) = 1\})$$

$$= P(A)$$

$$P(\text{Failure}) = P(X=0)$$

$$= P(I_A = 0) = P(\{\omega \mid I_A(\omega) = 0\})$$

$$= P(A^c) = 1 - P(A)$$

Example: A function of a r.v.

Assume that X is a r.v.

on (Ω, \mathcal{F}) . Also assume

that $Y: \Omega \rightarrow \mathbb{R}$

$$Y(\omega) = X^3(\omega)$$

Is Y a random variable?

In order to show that Y is

a random variable, we must

show that

$$\leftarrow Y((-\infty, c]) = \{ \omega \mid Y(\omega) \leq c \} \\ = \{ \omega \mid X^3(\omega) \leq c \} \in \mathcal{F}$$

is an event for all $c \in \mathbb{R}$.

$$Y^k((-\infty, c]) = \{w \in \Omega \mid Y(w) \leq c\}$$

$$= \{w \in \Omega \mid X^3(w) \leq c\}$$

$$= \{w \in \Omega \mid X(w) \leq c_1\}$$

$$= X^k((-\infty, c_1])$$

Prove $Y(w) = X^2(w)$

Remark: The term random variable is a misnomer, because a random variable is neither random, nor is it a variable. A random variable, as we observed, is

just a function from the sample space Ω to real numbers \mathbb{R} .

The term "random" actually refers to the randomness in

observing a particular member
(or particular members) of a
sample space Ω , as the
outcome of an experiment.
The random variable maps

the outcome (ω) to a "fixed"
real number, $X(\omega)$.

One can imagine different
members of the sample
space are selected.

randomly, and are mapped into different real numbers, through the random variable.

However, the random variable does not assign real numbers

to the members of the sample space "randomly"

Remark: It is also VERY IMPORTANT

to note that probability is assigned to subsets

of the sample space (events)

So: $P: \mathcal{F} \rightarrow [0, 1]$, but

a random variable assigns a real number to each element of the sample space, therefore

$$X: \Omega \rightarrow \mathbb{R}.$$

Borel Sets and The Probability Law of A Random Variable

Def (σ -field generated by a set)

Let C be any collection of

Subsets of Ω , i.e. $C \subseteq \mathcal{P}(\Omega)$, which

is not necessarily a σ -field.

Define $\sigma(C)$ to be the smallest

σ -field that contains C , which

means that if D is a

$$C = \{A\}, F_1 = \{A, A^c, \emptyset, \Omega\}$$

$$F_1 \subseteq F_2$$

F_1 : σ -field generated by $\{A\}$

$$F_2 = \{A, B, A^c, B^c, \dots, \emptyset, \Omega\}$$

σ -field and $C \subseteq D$, then

$$\sigma(C) \subseteq D.$$

Example : We previously saw

that the smallest σ -field that

contains some $A \subset \Omega$ is:

$$\mathcal{F} = \{\emptyset, A, A^c, \Omega\}.$$

\mathcal{F} is the σ -field generated

by $C = \{A\} \subseteq 2^\Omega$.

It is obvious that any other

σ -field that contains $C = \{A\}$

is a superset of $\sigma(C)$.

$\{\emptyset, A, A^c, \Omega\}$. For example,

if $B \neq A$, the σ -field that

has both B , A as

its members, is a superset

of $\sigma(C) = \{A, A^c, \Omega, \emptyset\}$.

Exercise: Try to enumerate the

members of the σ -field that has ^{minimum}

both A and B as its members.

$$\sigma(\{A, B\}) = \{\emptyset, \Omega, A \cap B, A^c \cap B, A^c \cap B^c\}$$

Borel σ -field:

Def: The Borel σ -field of subsets

of \mathbb{R} , denoted as $\mathcal{B}(\mathbb{R})$ or

simply \mathcal{B} , is a σ -field

generated by all of the intervals

$(-\infty, c]$, $c \in \mathbb{R}$.

Question: Are intervals in the

form $(c, +\infty)$, $c \in \mathbb{R}$ and

(c, d) , $c, d \in \mathbb{R}$ members of \mathcal{B} ?

$$(c, \infty] = [-\infty, c]^c \in \beta(\mathbb{R})$$

$$(-\infty, d) = \bigcup_{n \in \mathbb{N}} (-\infty, d - \frac{1}{n}] \in \beta(\mathbb{R})$$

$$(c, d) = (-\infty, d) \cap (c, \infty) \in \beta(\mathbb{R})$$

It can be proved that all intervals, their countable unions and intersections, and any set that is a result of countable applications of \cup , \cap , and \setminus intervals are in $\beta(\mathbb{R})$.

- Remark: It can be proved that any singleton $\{a\}$ is a Borel Set, $\forall a \in \mathbb{R}$, e.g $\{1\}$, $\{2, 3\}$, $\{\pi\}$ one Borel Sets.
- Any finite/countably infinite

subset of \mathbb{R} is a Borel Set.

- Remark: Borel Sets should not be confused with the Borel σ -Field itself.

Borel Sets are the members of $\mathcal{B}(\mathbb{R})$, the Borel σ -field.

- Proposition: The pre-image

$X^{-1}(B)$ of any Borel Set

$B \in \mathcal{B}(\mathbb{R})$ is an event, if

X is a r.r. on (Ω, \mathcal{F}, P)

(Sketch of Proof): Any

Borel Set is a countable

union of intervals of the form

$(-\infty, c]$ or their complements

$(c, +\infty)$, so.

$$B = \bigcup_n A_n$$

where A_n is either $(-\infty, c_n]$
or $(-\infty, d_n]^c$.

$$\overset{\leftarrow}{X}(B) = \overset{\leftarrow}{X}(\bigcup_{n \in \mathbb{N}} A_n)$$

$$= \bigcup_{n \in \mathbb{N}} \overset{\leftarrow}{X}(A_n) \in \mathcal{F}$$

exact proof needs transfinite induction

The Probability Law of A

Random Variable

Since $\forall B \in \mathcal{B}(R)$, $\overset{\leftarrow}{X}(B)$ is

an event (is \mathcal{F} -measurable),

we can define the

probability of $X^<(B)$.

$$P(X^<(B)) = P(\{\omega \mid X(\omega) \in B\})$$

$$= P_X(B)$$

$$= P(X \in B)$$

Precisely, $(R, \mathcal{B}(R), P_X)$ is

a probability space induced/
created by X , where

$$P_X(B) = P(X^<(B))$$

$$= P(X \in B)$$

Example: Assume that

$$\Omega = \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad P(A) = \frac{|A|}{6}$$

$$\forall \omega \in \Omega, \quad X_1(\omega) = 2 \quad \text{and} \quad X_2(\omega) = \omega^2. \\ X_3(\omega) = \sqrt{\omega}$$

Calculate $P_{X_1}(B)$ and $P_{X_2}(B)$

when:

$$P_{X_3}(B)$$

a) $B = (-\infty, -10]$

$$P_{X_1}(B) = P(X_1(\omega) \in B) = 0$$

$$P_{X_2}(B) = P(X_2(\omega) \in (-\infty, -10])$$

$$= P(\omega \mid \omega^2 \in (-\infty, -10]) = 0$$

$$P_{X_3}(B) = 0$$

$$(b) \quad B = [1.5, 2.5]$$

$$P_{X_1}(B) = P(X(\omega) \in [1.5, 2.5])$$

$$= P(\omega | \omega \in [1.5, 2.5]) = 1$$

$$P_{X_2}(B) = P(\omega | \omega^2 \in [1.5, 2.5]) = 0$$

$$P_{X_3}(B) = P(\omega | \sqrt{\omega} \in [1.5, 2.5]) = 0$$

$$= P(\{3, 4, 5, 6\}) = 4/6$$

$$(c) \quad B = \mathbb{N}$$

$$P_{X_3}(B) = P(\omega | \sqrt{\omega} \in \mathbb{N})$$

$$= P(\{1, 4\}) = 2/6$$

Remark: Borel sets ~~do~~ shed

light on the usefulness of

the concept of a random variable.

A random variable helps creating

a probability space on \mathbb{R} ,

whose σ -field is $\mathcal{B}(\mathbb{R})$, and

its probability measure is

$P_X(\cdot)$. Instead of dealing

with general sample spaces

and σ -fields, by using a

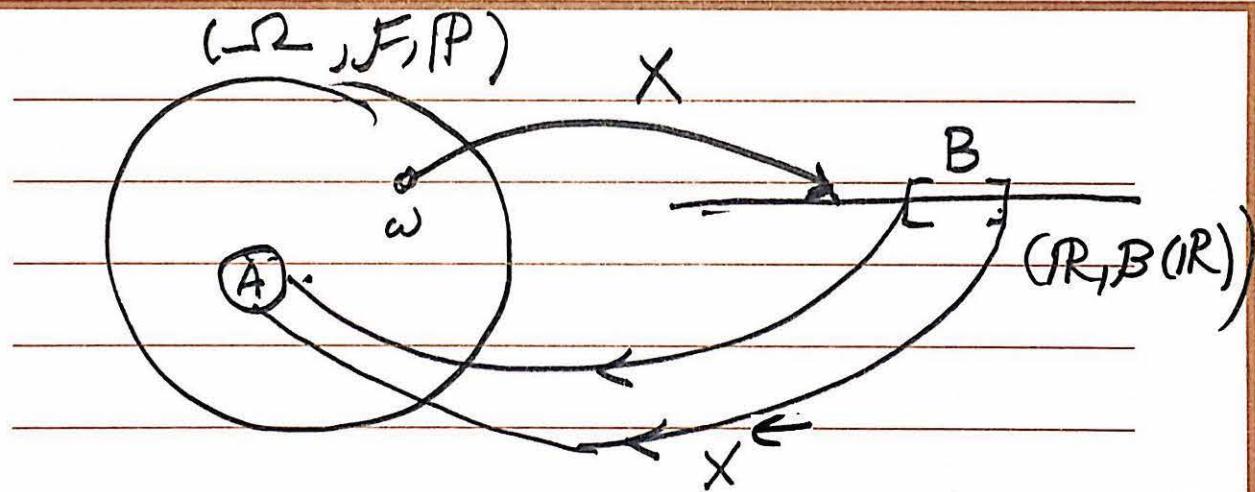
random variable, we can

deal with \mathbb{R} and $\mathcal{B}(\mathbb{R})$,

and compute probabilities

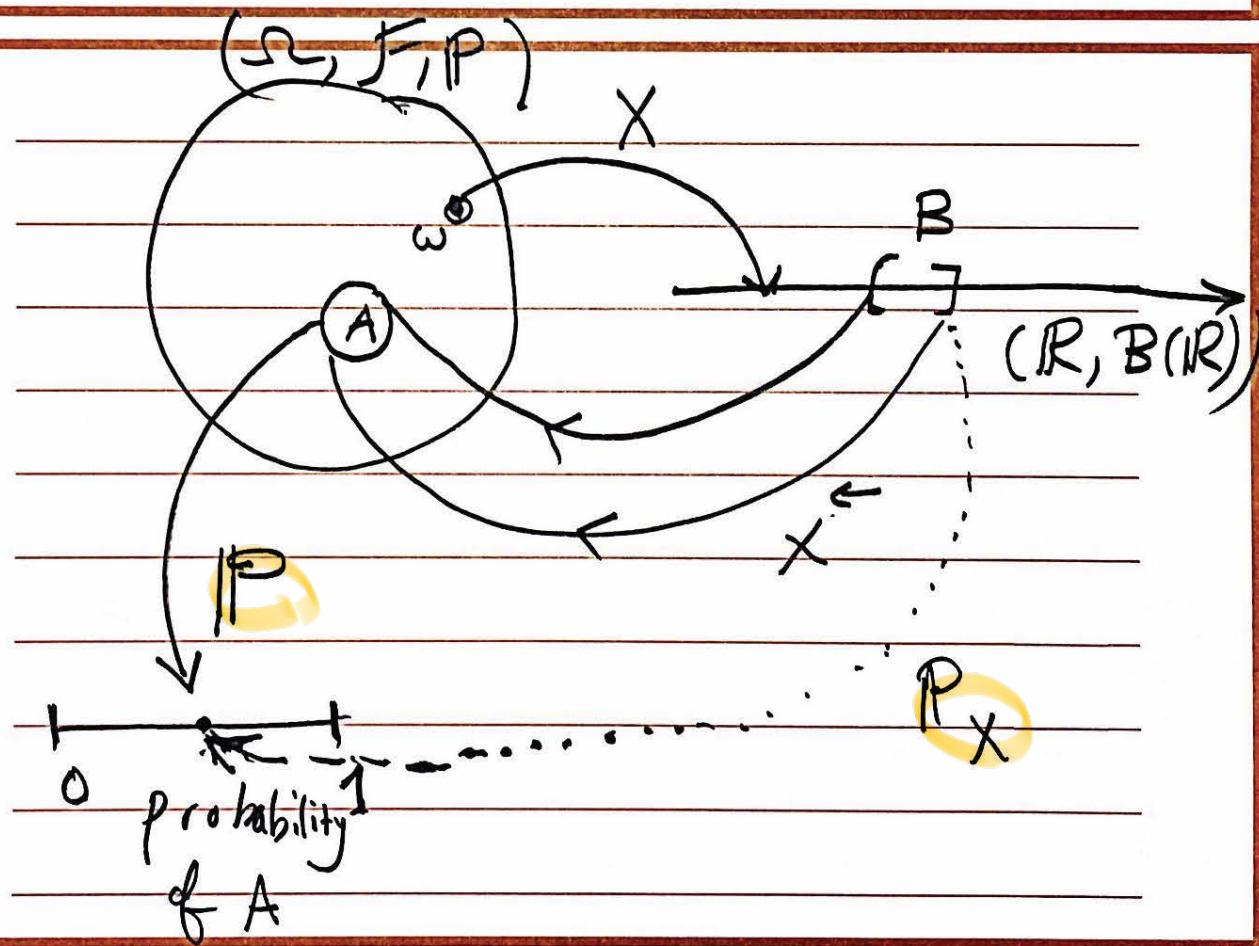
of "numerical" measurable

sets such as intervals.



In this picture, X is a random variable that maps every element ω in the sample

Space to a real number,
 in a way that the pre-image
 of every Borel Set $B \in \mathcal{B}(\mathbb{R})$
 is an event $A \in \mathcal{F}$.



In this picture, $\overset{\leftarrow}{X}$ maps any Borel subset of \mathbb{R} into an event, and the probability measure P maps the event into a number in $[0, 1]$.

Therefore, the probability law of X , P_X , can be viewed as composition of P and $\overset{\leftarrow}{X}$;

$$P_X(B) = P(\overset{\leftarrow}{X}(B)) = P_0 \overset{\leftarrow}{X}(B)$$

Theorem (Construction of r.v.'s)

(a) If (Ω, \mathcal{F}, P) is a probability

space, and $A \in \mathcal{F}$ (is an event)

the corresponding indicator function

of A , $I_A(\omega)$ is a random

variable, where

$$I_A(\omega) = \begin{cases} 0 & \omega \notin A \\ 1 & \omega \in A \end{cases}$$

Bernoulli Random Variable

(b) If $A_1, A_2, \dots, A_n \in F$ and

$\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$, then:

$$X(\omega) = \sum_{i=1}^n \alpha_i I_{A_i}(\omega)$$

is a random variable.

Multinoulli/Categorical

(c) Any continuous real function

is a random variable. So, if $\Omega = \mathbb{R}$,

$X: \mathbb{R} \rightarrow \mathbb{R}$ is a random variable

if ~~X~~ is a continuous function, i.e.

$$\lim_{\omega \rightarrow \omega_0} X(\omega) = X(\omega_0) \quad \forall \omega_0 \in \mathbb{R}$$

(d) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous

function (i.e. $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ $\forall x_0 \in \mathbb{R}$)

then $\tilde{Y} = f(X)$ is also

a random variable

Remark: (d) doesn't mean that

if f is not continuous, $f(X)$

is NECESSARILY~~Y~~ not a random

variable $f(X)$ CAN^{still} be a

random variable if f is not continuous

continuous guarantee random variable

(e) (Functions of Multiple Random Variables) Assume that X_1, \dots, X_n are random variables on (Ω, \mathcal{F}, P) and assume that $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function.

Then, $Y = f(X_1, X_2, \dots, X_n)$ is

also a random variable.

Example : $y = f(x_1, x_2) = x_1 + x_2$

and $y = g(x_1, x_2) = x_1 x_2$ are

continuous functions, so $x_1 + x_2$ and

X_1, X_2 are random variables

if X_1 and X_2 are random variables.

i.g.)

$$Y(\omega) = \min(X_1(\omega), X_2(\omega), \dots, X_n(\omega))$$

$$Z(\omega) = \max(X_1(\omega), X_2(\omega), \dots, X_n(\omega))$$

order statistics

Proof of above statement is beyond the scope of class

The Cumulative Distribution Function (CDF)

Def: The cdf (cumulative distribution function) of a r.v. $X, F_x: \mathbb{R} \rightarrow [0, 1]$ is defined as:

$$F_X(x) = P(\underbrace{X \leq x}_{\text{event}})$$

$$\{X \leq x\} = X^{\leftarrow}((-∞, x])$$

$$= \{ω | X(ω) ∈ (-∞, x]\}$$

always event by our definition of random variable

Ques: Explain why any random

variable has a well-defined

CDF, i.e. $\forall x, F_X(x)$ exists

for all $x ∈ ℝ$.

Lemma : A cdf of X , a random variable defined on (Ω, \mathcal{F}, P) , has the following properties:

(a) $\lim_{x \rightarrow -\infty} F_X(x) = 0$

$$\lim_{x \rightarrow +\infty} F_X(x) = 1$$

(b) $x < y \Rightarrow F_X(x) \leq F_X(y)$

(It is monotonic), \rightarrow
non-decreasing

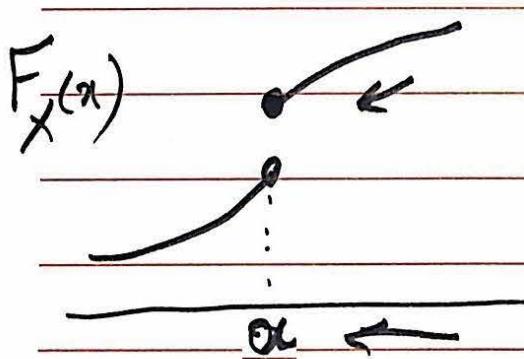
(c) F is right-continuous,

i.e.

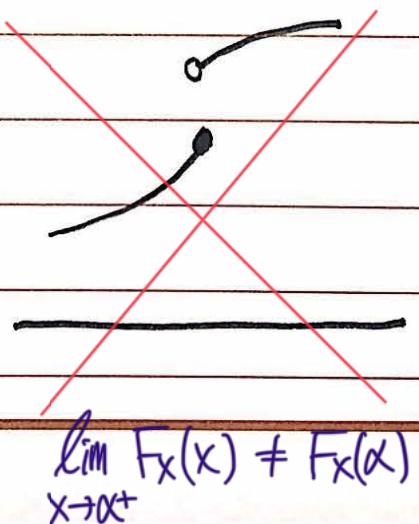
$$\lim_{x \rightarrow a^+} F_X(x) = F_X(a)$$

or

$F(x+h) \rightarrow F(x)$ when $h \rightarrow 0^+$



right continuous function



$\lim_{x \rightarrow a^+} F_X(x) \neq F_X(a)$

Proof: $\lim_{x \rightarrow -\infty} F_x(x) = 0$

(a) Define $B_n = \{\omega \in \Omega \mid X(\omega) \leq -n\}$

$$A_n = (-\infty, n]$$

$$A_1 \supseteq A_2 \supseteq A_3, \dots$$

$$X^{-}(A_1) \supseteq X^{-}(A_2) \dots$$

$$B_1 \supseteq B_2 \dots$$

$$\lim_{n \rightarrow \infty} B_n = \bigcap_{n \in \mathbb{N}} B_n = \emptyset$$

$$P(B_n) = P(\{\omega \mid X(\omega) \leq -n\}) = P(X \leq -n) = F_X(-n)$$

$$\lim_{n \rightarrow \infty} P(B_n) = \lim_{n \rightarrow \infty} F_X(-n)$$

$$\Rightarrow P\left(\lim_{n \rightarrow \infty} B_n\right) = \lim_{n \rightarrow \infty} F_X(-n) = 0$$

To prove $\lim_{x \rightarrow a} F_X(x)$, we similarly need to define an appropriate set of B_n 's. It is left as an exercise.

$$A_n = (-\infty, A]$$

$$\lim_{n \rightarrow \infty} A_n = \mathbb{R}$$

$$B_n = X^{\leftarrow}(A_n)$$

$$P(B_n) = P(X^{\leftarrow}(A_n))$$

$$(b) F_x(x) = P(X \leq x)$$

$$F_x(y) = P(X \leq y) \quad x < y$$

$$\{\omega | X(\omega) < y\} = \{\omega | X(\omega) < x\} \cup \\ \{\omega | X(\omega) \geq x & X(\omega) < y\}$$

$$P(X \leq y) = P(X \leq x) + P(x < X \leq y) \quad \text{disjoint events}$$

$$F_x(y) \geq F_x(x)$$

(c) Again, another appropriate set

of events has to be defined.

Then, one should use continuity

of probability.

$$B_n = \{\omega | X(\omega) \leq a + \frac{1}{n}\}$$

$$A_n = (-\infty, a + \frac{1}{n}]$$

$$A_1 \supseteq A_2 \supseteq A_3 \dots$$

$$B_n = X^{-1}(A_n)$$

$$B_1 \supseteq B_2 \supseteq B_3 \dots$$

$$\lim_{n \rightarrow \infty} B_n = \bigcap_{n \in \mathbb{N}} B_n$$

$$= \{\omega \mid X(\omega) \leq a\}$$

$$\lim_{n \rightarrow \infty} P(B_n) = \lim_{n \rightarrow \infty} F_x(a + \frac{1}{n})$$

$$P(\lim_{n \rightarrow \infty} B_n) = \lim_{n \rightarrow \infty} F_x(a + \frac{1}{n})$$

$$= P(\{\omega \mid X(\omega) \leq a\}) = F_x(a) = \lim_{x \rightarrow a^+} F_x(x)$$

Example (Constant Random

Variable)

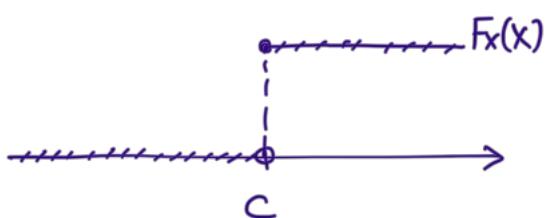
Let $c \in \mathbb{R}$, $X: \Omega \rightarrow \mathbb{R}$

$$X(\omega) = c \quad \forall \omega \in \Omega$$

What is the cdf of X ?

$$F_x(x) = P(X \leq x)$$

$$= \begin{cases} P(\emptyset) = 0, & X < c \\ P(\Omega) = 1, & X \geq c \end{cases}$$



constant random variable : degenerate random variable

Lemma (Properties of cdf).

Let X be a r.v. on $(\mathbb{R}, \mathcal{F}, P)$

and F_X be the cdf of X .

Then:

$$(a) P(X > x) = P(X \in (x, +\infty)) \\ = 1 - P(X \leq x) = 1 - F_X(x)$$

$$(b) P(x < X \leq y) = P(X \in [x, y])$$

$$= F_X(y) - F_X(x)$$

$$(c) P(X = x) = P(X \in \{x\})$$

Borel Set

$$= F_X(x) - \lim_{y \rightarrow x^-} F_X(y)$$

$$\text{For } c \quad B_n = \left\{ x - \frac{1}{n} < x(\omega) \leq x + \frac{1}{n} \right\}$$

Example (Constant Random Variable)

$$X: \Omega \rightarrow \mathbb{R}$$

$$X(\omega) = c$$

What is $P(\{X=c\})$?

$$P(\{X=c\}) = F_X(c) - \lim_{x \rightarrow c^-} F_X(x) = 1$$

Exercise: Assume that X
 is a r.r. on (Ω, \mathcal{F}, P) and
 F_X is its CDF.

Is $G(x) = [1 - F_x(x)] \log(1 - F_x(x)) + F_x(x)$

a Cdf of a random variable?

Check properties of a cdf will be enough

1. $\lim_{x \rightarrow -\infty} G(x) = \lim_{x \rightarrow -\infty} [1 - F_x(x)] \log(1 - F_x(x)) + F_x(x)$
 $= 1 \log 1 + 0 = 0$

2. $\lim_{x \rightarrow \infty} G(x) = \lim_{x \rightarrow \infty} [1 - F_x(x)] \log(1 - F_x(x)) + F_x(x)$
 $= \lim_{u \rightarrow 1^-} (1 - u) \log(1 - u) + 1$

$$= \lim_{z \rightarrow 0^+} z \log z + 1 = \lim_{z \rightarrow 0^+} \frac{\log z}{1/z} + 1$$

L'Hôpital's rule

$$\lim_{z \rightarrow 0^+} \frac{\frac{\ln z}{z}}{\frac{1}{z^2}} \rightarrow +1 = 1$$

$$3. G(x) = (1 - F_x(x)) \log(1 - F_x(x)) + F_x(x)$$

$$h(u) = (1-u) \log(1-u) + u$$

if $h(u)$ is non-decreasing because $F_x(x)$ is non-decreasing
then $G(x)$ is non-decreasing

$$h'(u) = -\log(1-u) - \ln(1-u) + 1 \geq 0$$

$$\begin{array}{c} | \\ f \end{array}$$

$\therefore G(x)$ is non-decreasing w.r.t x

$$\begin{aligned} 4. \lim_{x \rightarrow a^+} G(x) &= \lim_{x \rightarrow a^+} (1 - F_x(x)) \log(1 - F_x(x)) + F_x(x) \\ &= [1 - F_x(a)] \log(1 - F_x(a)) + F_x(a) \\ &= G(a) \end{aligned}$$

because F_x is CDF and right continuous
and \log is continuous

Exercise: Are the following

Statements equivalent

(1) $F_x(a)$ is continuous for
all $a \in \mathbb{R}$

(2) $P\{X=a\} = 0 \quad \forall a \in \mathbb{R}$

$$P(\{X=a\}) = 0 \quad \forall a \in \mathbb{R}$$

$$\Leftrightarrow F_x(a) - \lim_{x \downarrow a^-} F_x(x) = 0$$

$$\Leftrightarrow F_x(a)$$

Exercise: let X be a r.v. with continuous cdf F_X . Find the cdf of the following random variables.

$$(a) Y = X^2 \quad F_Y(y) = P(Y \leq y)$$

$$P(X^2 \leq y) = \begin{cases} 0 & y < 0 \\ P(0 \leq X^2 \leq y) & y \geq 0. \end{cases}$$

$$P(-\sqrt{y} < X \leq \sqrt{y}) = P(-\sqrt{y} \leq Y \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$(b) Y = \sqrt{X}$$

$$F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y)$$

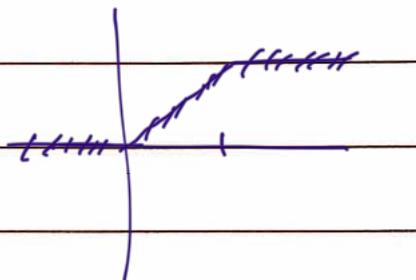
$$= \begin{cases} 0 & y < 0 \\ P(0 < \sqrt{X} \leq y) & y \geq 0 \end{cases}$$

$$P(0 < \sqrt{X} \leq y) = P(0 < X \leq y^2) = F_X(y^2) - F_X(0)$$

(c) $y = F_x(x)$ for simplicity assume F_x is strictly increasing

$$\begin{aligned} F_Y(y) &= P(F_X(x) \leq y) & 0 \leq y \leq 1 \\ &= P(X \leq F_X^{-1}(y)) & ; \text{ increasing fct.} \\ &= F_X(F_X^{-1}(y)) = y \end{aligned}$$

$$F_Y(y) = \begin{cases} y & 0 \leq y \leq 1 \\ 0 & y < 0 \\ 1 & y > 1 \end{cases}$$



(d) $Y = G^{-1}(X)$, where G is a

continuous and strictly increasing

function

$$F_Y(y) = P(Y \leq y) = P(G^{-1}(X) \leq y)$$

$$= P(X \leq G(y)) = F_X(G(y))$$

Exercise :

Let X have the CDF



$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x & 0 \leq x \leq 2 \\ 1 & x > 2 \end{cases}$$

and let $Y = X^2$. Find

$$(a) P(1 \leq X \leq 2)$$

$$= P(1 < X \leq 2)$$

$$= F_X(2) - F_X(1)$$

$$= \frac{1}{2}(2) - \frac{1}{2}(1) = \frac{1}{2}$$

$$(b) P(2X + Y \leq 3)$$

$$= P(2X + X^2 \leq 3)$$

$$= P(X^2 + 2X - 3 \leq 0)$$

$$= P(-3 \leq X \leq 1)$$

$$= F_X(1) - F_X(-3) = 1/2$$

Independence of Random Variables

Recall that the events A, B, C, F

were said to be independent

iff :

$$P(A \cap B) = P(A)P(B)$$

Based on the concept of independence

for events we define independence

for r.v.'s :

Assume

Def: X_1, X_2 are r.v.'s on

(Ω, \mathcal{F}, P) . They are said to be

independent, if for all Borel

sets $B_1, B_2 \in \mathcal{B}(\mathbb{R})$:

$$P(\underbrace{X_1 \in B_1}_{\text{=: } I}, \underbrace{X_2 \in B_2})$$

$$= 1$$

Question: How can we extend the

definition of independence a set

of r.r.'s $\{X_i | i \in I\}$, where

I is an index set?

Def(Joint Cdf).

Assume that X_1 and X_2 are

r.v.'s on (Ω, \mathcal{F}, P) . The

joint cdf of X_1 and X_2 is defined

$$\text{as } F_{X_1, X_2}(x_1, x_2) =$$

More generally, if X_1, X_2, \dots, X_n

are random variables on

(Ω, \mathcal{F}, P) , the joint cdf of

X_1, X_2, \dots, X_n is denoted as

F_{X_1, X_2, \dots, X_n} and is defined as.

$$F_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n)$$

$$= P$$

lemma: X_1, X_2 are

independent if and only if
 $\forall x_1, x_2 \in \mathbb{R}$

$$F_{X_1, X_2}(x_1, x_2) =$$

More generally, X_1, X_2, \dots, X_n are
 independent if and only if

$$F_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) =$$

Remark: The previous lemma states that instead of checking the following equation for all possible Borel sets A, B ,

$$P(X_1 \in A, X_2 \in B) = P(X_1 \in A)P(X_2 \in B)$$

one can check it only for

A 's and B 's in the following

form $A = (-\infty, x_1]$, $B = (-\infty, x_2]$

$$P(X_1 \in A, X_2 \in B) = P(X_1 \in (-\infty, x_1], X_2 \in (-\infty, x_2])$$