

Lesson 3
Review of Set Theory

A **set** is a collection of objects, which are its **elements**

(This is not a definition)
("Set" is a **primitive** concept)

- We show the elements of a set using $\in$

$x \in A$ $\qquad$ $x \notin A$

- A set with no elements is called ~~an~~ empty set or null set $\emptyset$

Aside: Using the null set to construct natural numbers

$$\{\emptyset\} \equiv 0$$

$$\{\emptyset, \{\emptyset\}\} \equiv 1$$

$$\{\emptyset, \{\emptyset, \{\emptyset\}\}\} \equiv 2$$

$$\{\emptyset, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\} \equiv 3$$

$x \in \emptyset$ $\forall x$, is false

We can show a set with its elements

$A = \{x_1, x_2, x_3, \dots, x_n\}$

A finite # of elements

Or with a predicate

$A = \{x \mid P(x)\}$

Ex: $\boxed{\{x \in \mathbb{N} \mid x > 5\}}$

$\{x \mid \varphi(x)\}$ a set

a predicate

$x \notin x$

$\{1, 2, 3\}$

$R = \{x \mid x \notin x\}$

$R \in R \Rightarrow R \notin R \Rightarrow R \in R$

$\Longrightarrow$ ...

ZF(C)

Von-Neumann-Gödel-Bernays

Subsets (super sets)

$(A \subseteq B) \Leftrightarrow (x \in A \Rightarrow x \in B)$

It is equivalent to $B \supseteq A$

Equivalence:

$(A = B) \Leftrightarrow (x \in A \Rightarrow x \in B) \wedge$
$(x \in B \Rightarrow x \in A)$

$(A = B) \Leftrightarrow (x \in A \Leftrightarrow x \in B)$

Proper subset hood

$a \leq b$ $\quad A \subseteq B$ means $x \in A \Rightarrow x \in B$

A can be equal to B

$a < b$ $\quad A \subset B$ means

$(x \in A \Rightarrow x \in B)$

$\wedge \ (A \neq B)$

$\exists y \in B$ s.t $y \notin A$

we call A a proper subset of B

Grubner does not use $\subseteq$ at all. By $\leq$ Grubner actually means $\subseteq$.

Universe of Discourse

The Universe of Discourse Ω (or $\mathcal{U}$) contains all elements that could conceivably be of interest in a particular context.

Ex: In number theory,

$$\Omega = \mathcal{U} = \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}$$

One can then consider all sets of interest as subsets of the Universe of Discourse

Complement of ~~A~~ Set (with respect to Ω).

Complement of A : A^c $\left(A' \text{ or } \overline{A} \right)$

$$A^c = \boxed{\{x \in \Omega \mid x \notin A\}}$$

Obviously, $\Omega^c = \boxed{\{x \in \Omega \mid x \notin \Omega\}} = \emptyset$

Operations on sets:

Union :

$$A \cup B = \{x \in \Omega \mid x \in A \vee x \in B\}$$

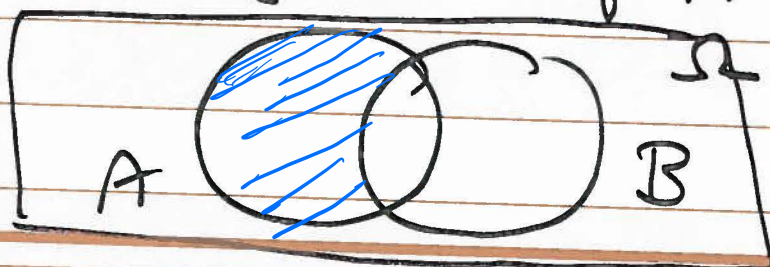
Intersection:

$$A \cap B = \{x \in \Omega \mid x \in A \wedge x \in B\}$$

Set Difference $A \setminus B$ ($A - B$) (or relative complement of B in A)

$$A \setminus B = \{x \in A \mid x \notin B\}$$

Venn Diagram of $A \setminus B$



$$A \setminus B = A \cap B^c$$

$$A \setminus B = A \setminus (A \cap B)$$

More generally, assume that

I is an index set, e.g.

$$I = \mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$$

$$I = \mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z},\ b \neq 0 \right\}$$

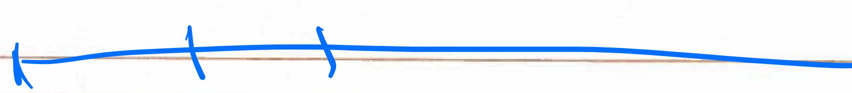
$$I = \mathbb{R}$$

(possibly infinite)

$$\bigcup_{i \in I} A_i = \{ x \in \Omega \mid \exists i \in I \text{ s.t. } x \in A_i \}$$

$$\bigcap_{i \in I} A_i = \{ x \in \Omega \mid \forall i \in I \text{ s.t } x \in A_i \}$$

Example: $\underbrace{A_i = [0, i)}_{}$, $i \in \mathbb{N}$



$A_1 = [0,1) \quad A_2 = [0,2) \qquad A_3 = [0,3)$

$$\bigcap_{i \in \mathbb{N}} A_i = [0, 1)$$

$$\bigcup_{i \in \mathbb{N}} A_i = [0, +\infty) = \mathbb{R}^{\geq 0}$$

Exercise:

$$\bigcap_{i \in \mathbb{N}} (-\infty, 1/i) = (-\infty, 0]$$

$A_1 = (-\infty, 1)$

$A_2 = (-\infty, 1/2)$ $\qquad \forall i \in \mathbb{N}$

$A_3 = (-\infty, 1/3)$ $\qquad 0 \in A_i$

$\Rightarrow 0 \in \bigcap_{i \in \mathbb{N}} A_i$

?

Exercise:

$$\bigcup_{i\in\mathbb{N}} (-\infty, -1/i] = (-\infty, 0)$$

$A_1 = (-\infty, -1]$ $\quad 0 \notin A_i \;\; \forall i$

$A_2 = (-\infty, -1/2]$

$A_3 = (-\infty, -1/3]$

Exercise:

$$\bigcap_{i\in\mathbb{N}} [0, 1/i) = \{0\}$$

$A_1 = [0, 1)$

$A_2 = [0, 1/2)$

$A_3 = [0, 1/3)$

Exercise:

$$\bigcup_{i \in \mathbb{N}} [-i, i] = (-\infty, +\infty) = \mathbb{R}$$

Exercise: $\bigcap_{i \in \mathbb{N}} (-\infty, -i) = \emptyset$

Disjoint / Mutually Exclusive Sets

Def: A_1, A_2 disjoint $\overset{\Delta}{\Longleftrightarrow}$ $A_1 \cap A_2 = \emptyset$

Example: $\Omega = \mathbb{R}$

$\{2\}$ and $[3, +\infty)$ are disjoint

$[-5, 9)$ and $[16, 19]$ are disjoint

Def: $A_1, A_2, \ldots$ are mutually disjoint

$\iff A_i \cap A_j = \emptyset \quad \forall i \neq j$

Def: Partition (empty set is excluded)

(Non-empty) $A_1, A_2, \ldots$ Consist a partition of B iff

1) They are mutually exclusive

$A_i \cap A_j = \emptyset \quad \forall i \neq j$ and

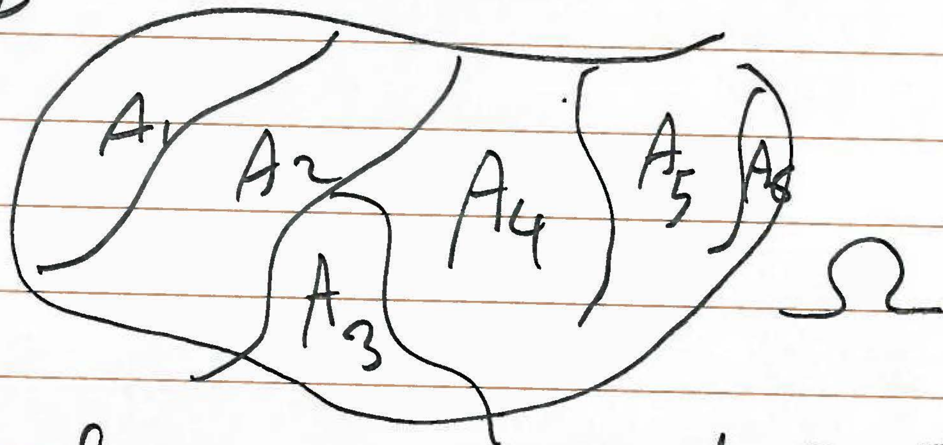
2) They are collectively exhaustive

$\bigcup_i A_i = B$

i.e.

Every member of B is a member of one and only one of A_i's.

Example



(A finite number of partitions for Ω)

Exercise: which one is a partition of $\mathbb{R}^{\geq 0}$

a) $A_i = [i, i+1)$ $\quad i \in \underbrace{\{0, 1, 2, \dots\}}$

✓ $A_i \cap A_j = \varnothing \quad i \neq j$ $\quad \mathbb{Z}^{\geq 0} = \mathbb{I} = \mathbb{N} \cup \{0\}$

$\bigcup A_i = \mathbb{R}^{\geq 0}$

b) $A_i = [0, i)$ $\quad i \in \{0, 1, 2, \dots\}$

Collectively exhaustive $\bigcup A_i = \mathbb{R}^{\geq 0}$

Not mutually exclusive ✗

(Some) Properties of Set Operations

Commutativity

$A \cap B = B \cap A$

$p \wedge q \equiv q \wedge p$

$A \cup B = B \cup A$

$p \vee q \equiv q \vee p$

Associativity

$A \cap (B \cap C) = (A \cap B) \cap C$

$p \wedge (q \wedge r) = (p \wedge q) \wedge r$

$A \cup (B \cup C) = (A \cup B) \cup C$

$p \vee (q \vee r) = (p \vee q) \vee r$

Distributivity

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Complement laws

$(A^c)^c = A$

$\neg(\neg p) = p$

$x \in \Omega$ always true

$x \in \emptyset$ always false

$A \cup A^c = \Omega$ $\quad p \vee \neg p \equiv T$

$A \cap A^c = \emptyset$ $\quad p \wedge \neg p \equiv F$

Identity Laws

$A \cup \emptyset = A$ $\quad A \cap \Omega = A$

$p \vee F \equiv p$ $\quad p \wedge T \equiv p$

Domination Laws

$A \cup \Omega = \Omega$ $\quad A \cap \emptyset = \emptyset$

$p \vee T \equiv T$ $\quad p \wedge F \equiv F$

Idempotency

$A \cup A = A$ $\quad A \cap A = A$

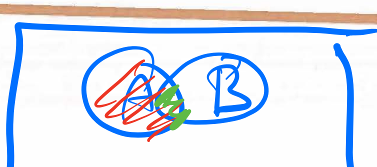
$p \vee p \equiv p$ $\quad p \wedge p \equiv p$

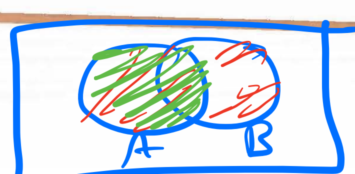
Absorption Laws

$A \cup (A \cap B) = A$ $\quad p \vee (p \wedge q) \equiv p$

$A \cap (A \cup B) = A$ $\quad p \wedge (p \vee q) \equiv p$

Venn Diagram





De Morgan's Laws

$$(A_1 \cup A_2)^c = A_1^c \cap A_2^c$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$(A_1 \cap A_2)^c = A_1^c \cup A_2^c$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

More Generally, for an index set I

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$$

$$\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$$

Element Chasing

To prove $A=B$, we must prove:

$$\boxed{A \subseteq B \wedge B \subseteq A}$$

Therefore, we must prove:

$(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A)$

i.e., we must "chase" elements

Proof of $(A \cap B)^c = A^c \cup B^c$

$A^c \cup B^c \subseteq (A \cap B)^c$

$\Leftarrow x \in (A \cap B)^c \Rightarrow x \notin A \cap B$

$\Rightarrow \neg (x \in A \wedge x \in B)$

De Morgan's

$\Leftrightarrow \neg x \in A \vee \neg x \in B$

$\Leftrightarrow x \notin A \vee x \notin B$

$\Leftrightarrow x \in A^c \vee x \in B^c$

$\Leftrightarrow x \in A^c \cup B^c \Rightarrow (A \cap B)^c \subseteq A^c \cup B^c$

Q.E.D.

So, we proved that

$(A \cap B)^c \subseteq A^c \cup B^c$

To prove that $(A^c \cup B^c) \subseteq (A \cap B)^c$

we should use element chasing again in the opposite direction, so

$(A \cap B)^c = A^c \cup B^c$

Proof of $\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c$

$x \in \left(\bigcup_{i \in I} A_i \right)^c \Leftrightarrow \neg \left(x \in \bigcup_{i \in I} A_i \right)$

$\Leftrightarrow \neg (\exists i \in I \text{ s.t. } x \in A_i)$

$\Leftrightarrow \forall i \in I \quad x \notin A_i$

$\Leftrightarrow \forall i \in I \quad x \in A_i^c$

$\Leftrightarrow x \in \bigcap A_i^c$

General form of Distributivity

$$\left(\bigcap_{i\in I} A_i\right) \cup B = \bigcap_{i\in I} (A_i \cup B)$$

$$\left(\bigcup_{i\in I} A_i\right) \cap B = \bigcup_{i\in I} (A_i \cap B)$$

(Try to prove them)

Notation: Special Sets

$\mathbb{R}$: Real Numbers

$\mathbb{R}^* = \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$

extended real numbers

$\mathbb{Z}$: Integers $\{\dots, -4, -3, -2, -1, 0, 1, 2, \dots\}$

$\mathbb{N}$: Natural Numbers or Strictly Positive Integers $\mathbb{N} = \mathbb{Z}^{>0}$

Intervals

$[a,b] = \{x \in \overline{\mathbb{R}} \mid a \leq x \leq b\}$

$(a,b) = \{ \quad a < x < b\}$

$[a,b) = \{ \quad a \leq x < b\}$

$(a,b] = \{ \quad a < x \leq b\}$

Cartesian Product

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \land a_2 \in A_2\}$$

$A_1, A_2, \ldots, A_n$

$$A_1 \times A_2 \times \cdots \times A_n = \prod_{i=1}^{n} A_i$$

$$= \{(a_1, \ldots, a_n) \mid a_i \in A_i,\ i \in \{1, 2, \ldots, n\}\}$$

More Generally,

$A_1, A_2, \ldots$

$\prod_{i=1}^{\infty} A_i$ is the set of sequences

$$\boxed{\{(a_1, a_2, \ldots) \mid a_i \in A_i,\ i \in \mathbb{N}\}}$$

$$A_i = A \Rightarrow \prod_{i=1}^{\infty} A_i = A^{\infty}$$

Example: $\underbrace{A=\{0,1\}}$

$$A^{\infty}=\prod_{i=1}^{\infty} A_i \quad , \forall i \quad A_i=\{0,1\}$$

A^{∞} is the set of all binary sequences

$010001110101\ldots$

Family of Subsets of A / Power Set of A

The set of all subsets of A is called the power set of A and is denoted as 2^A or $\cancel{\mathcal{P}_A}$ or $\cancel{\mathcal{P}(A)}$.

$\cancel{\mathcal{P}_A}$

Example: $A=\{1,2\}$

$$2^A=\{\emptyset,\{1\},\{2\},\{1,2\}\}$$

$$\underbrace{\overset{\text{False}}{x\in\emptyset} \Rightarrow x\in A}_{\text{Always true}} \qquad \emptyset\subseteq A$$

Cardinality = # of elements of A

$$|A|=2$$

$$|2^A|=2^{|A|}=2^2=4$$

Relations

$A_1\times A_2\times\cdots\times A_n\times B$

Assume $K=\left(\prod_{i=1}^{n}A_i\right)\times B$

Any subset of K, $R\subseteq K$, is called a relation. (on K)

Example: $A = \{2, 3\}$ $B = \{1, 2\}$

$K = A \times B = \{(2,1), (2,2), (3,1), (3,2)\}$

$R_1 = \emptyset$

$R_2 = \{(3,1)\}$

$R_3 = \{(2,1), (2,2)\}$

A relation maps some elements/tuples in $\prod_{i=1}^{n} A_i$ to at least one element in B.

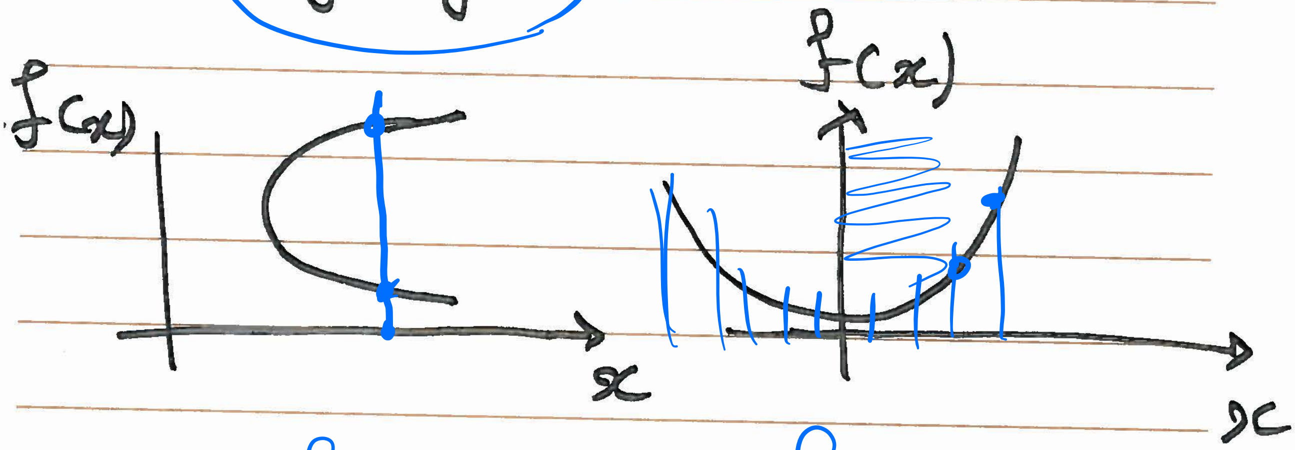
Function:

Def: If a relation maps each n-tuple to a unique member of B, it is called a function

$f \subseteq K$

$(x_1, \dots, x_n, y) \in f \wedge (x_1, \dots, x_n, y') \in f$

$\Rightarrow y = y'$



not a function function

We write $f : A_1 \times \dots \times A_n \to B$

$\prod_{i=1}^{n} A_i$ is called the $\boxed{\text{domain}}$ of f

B is called the co-domain of f.

For simplicity, we continue with functions of the form

$f : A \to B$

Pull-forward

Forward Image of $f: A \to B$

Def. $\vec{f}: \underbrace{2^A}\to\underbrace{2^B}$

√ $S \subseteq A$ $(S \in 2^A)$

$$\vec{f}(S) = \{y \in B \mid y = f(x),\ x \in S\}$$

Easier $\vec{f}(S) = \{f(x) \mid x \in S\} \subseteq B$

$\in 2^B$

The notation $f(S)$ is also used.

$f: \mathbb{R} \to \mathbb{R}$

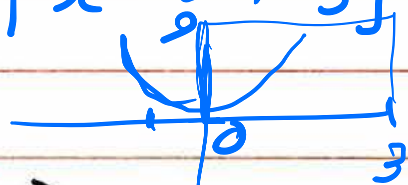
Example: $y = f(x) = x^2$

$$\vec{f}([1,2]) = \{f(x) \mid x \in [1,2]\}$$

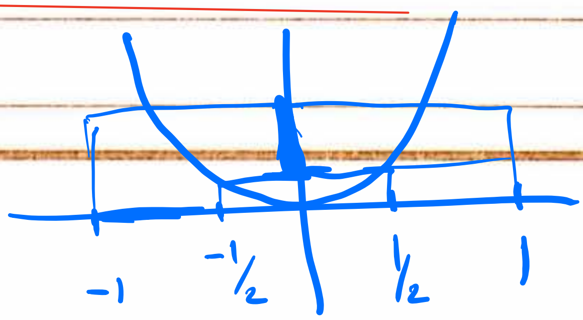
$$= \{x^2 \mid x \in [1,2]\} = [1,4]$$

$$\vec{f}([-1,3]) = \{x^2 \mid x \in [-1,3]\}$$

$$= [0,9]$$



$$\vec{f}([-1,-\tfrac{1}{2}) \cup (\tfrac{1}{2},1]) = (\tfrac{1}{4},1]$$



Pull-back

Inverse Image (pre-Image)

of $f: A \to B$

$$f^{\leftarrow} : 2^B \to 2^A$$

$\forall T \subseteq B \ (T \in 2^B)$

$$f^{\leftarrow}(T) = \{x \in A \mid f(x) \in T\}$$

Obviously, $f^{\leftarrow}(T) \subseteq A \ (\in 2^A)$

The notation $f^{-1}(T)$ is also used, but we avoid it because it can be confused with an inverse function

Example : $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^2$$

$$\overleftarrow{f}([4,9]) = \{x \in \mathbb{R} \mid f(x) \in [4,9]\}$$

$$= \{x \in \mathbb{R} \mid x^2 \in [4,9]\}$$

$$= [-3,-2] \cup [2,3]$$

$$\overleftarrow{f}([0,9]) = [-3,+3]$$

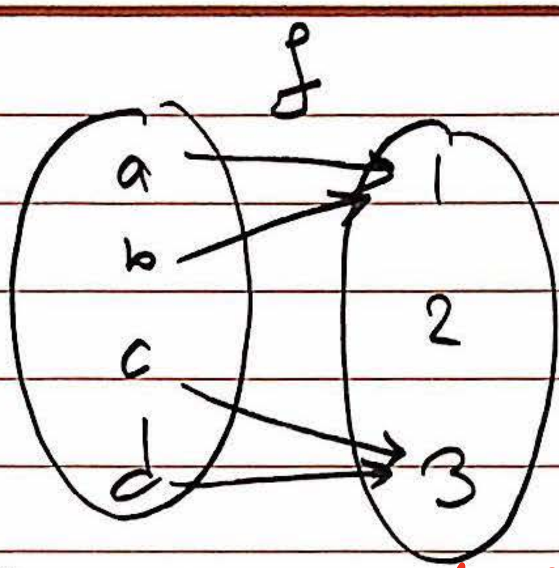
$$\overleftarrow{f}([-9,-1]) = \emptyset$$

$$\overleftarrow{f}([-9,0]) = \{0\}$$

- Important note: forward and inverse images are functions that map sets to sets. In other words, they are set functions

- Remark: It is a very good habit to always show the domain and co-domain of a function to avoid ambiguity.

Example:



Is f a function? Yes.

$f^{\rightarrow}(\{a,b\}) = \{1\}$

$f^{\rightarrow}(\{a,b,c\}) = \{1,3\}$

$f^{\rightarrow}(\emptyset) = \emptyset$

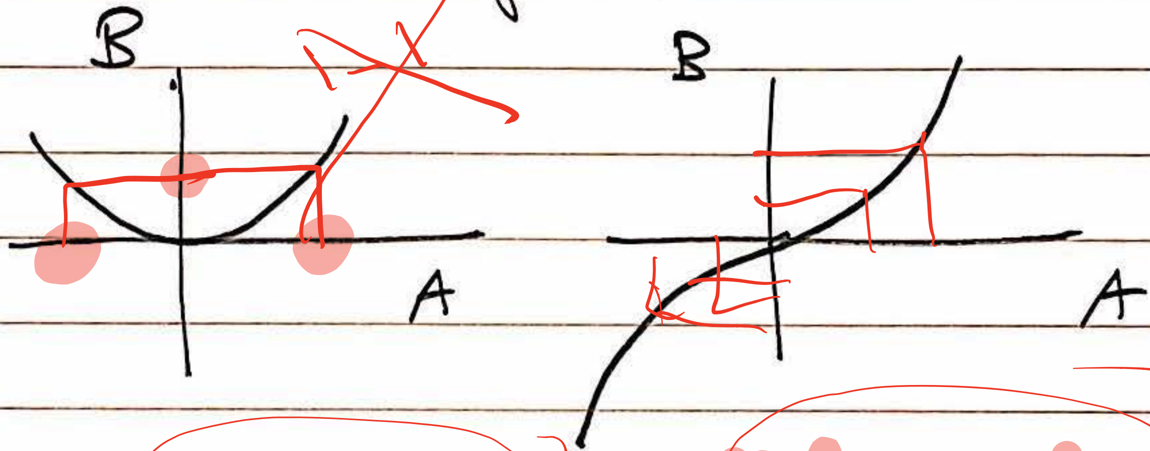
$f^{\leftarrow}(\{1\}) = \{a,b\}$

$f^{\leftarrow}(\{2\}) = \{\emptyset\}$

$f^{\leftarrow}(\{1,2,3\}) = \{a,b,c,d\}$

Def: One-to-one function

A function $f: A \to B$ is one-to-one (1-1) iff it assigns no more than one member of A to each member of B



or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

or $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

One-to-one functions are also called **injective**

Onto function:

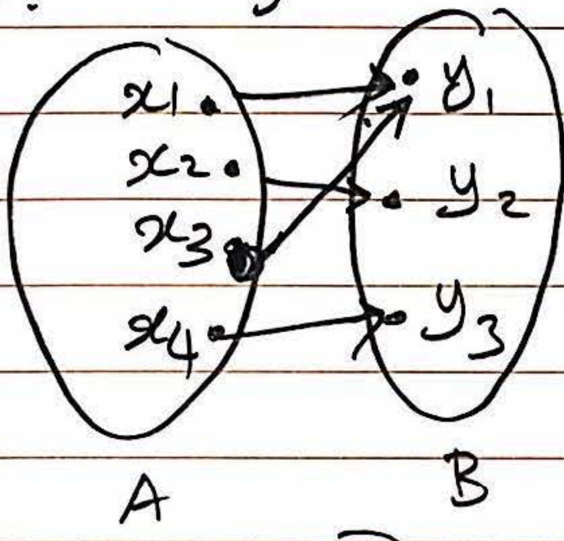
Def: $f: A \to B$ is onto iff

$\underbrace{\vec{f}(A)}_{\text{Range}} = B.$

Example: f

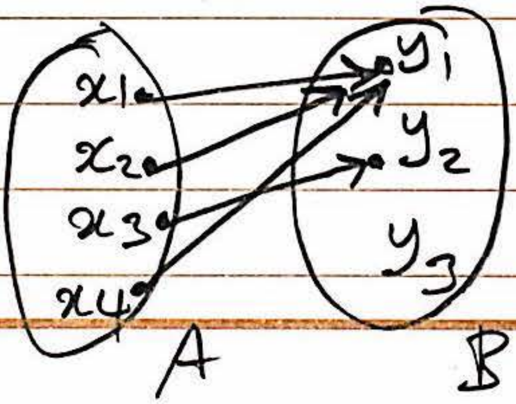
whic function is onto?

a)



onto

(b)



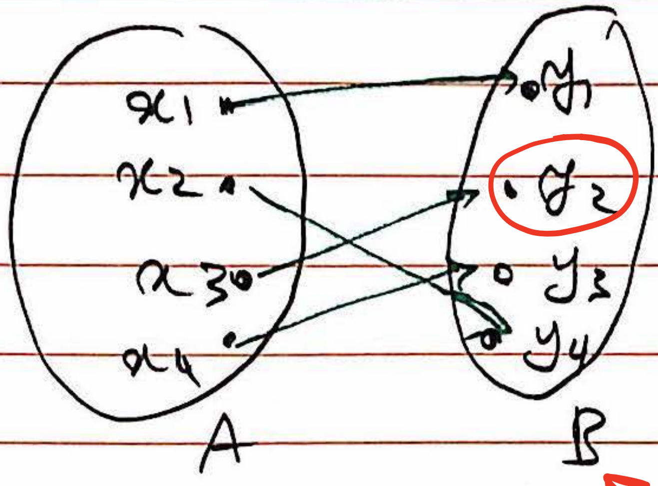
No

Onto functions are also called surjective

One-to-One Correspondence

Def: $f: A \to B$ is called a one-to-one correspondence iff it is both 1-1 and onto. It is also called bijective.

Example: f



bijective

Remark: In a bijective mapping, one-to-one correspondence every element in the co-domain has a pre-image and the pre-images are unique. elements of A

Thus, we can define the inverse function, $f^{-1}: B \longrightarrow A$, such that $f^{-1}(y) = x$ if $f(x) = y$. Therefore, bijective functions are invertible.

Cardinality

In informal terms, the cardinality of a set is the number of elements in that set. If one wishes to compare the cardinalities of two finite sets, A and B, they can simply count the number of elements in each set. But what if the sets contain infinitly many elements?

George Cantor

Def. Two sets A, B are said to be of the same cardinality (equicardinal) iff there exists a one-to-one correspondence between A and B, and we write $|A| = |B|$.

Question: Using the concept of one-to-one and onto functions, define $|A| \geq |B|$ and $|A| \leq |B|$.

$|A| \geq |B| \iff$ There exists an onto function from A to B

$|A| \leq |B| \iff$ There exists a one to one function from A to B

Def: A set A is said to be countably infinite if A and $\mathbb{N}$ have the same cardinality. (Denumerable)

Def: A set is said to be countable if it is either finite or countably infinite.
denumerable

Example: $\mathbb{N}$ is Countably infinite

$f: \mathbb{N} \to \mathbb{N}$

$f(n) = n$

Example: $\mathbb{Z}$ is countably infinite.

$f: \mathbb{Z} \longrightarrow \mathbb{N}$

$$f(x) = \begin{cases} 2x+1 & x \geq 0 \\ -2x & x < 0 \end{cases}$$

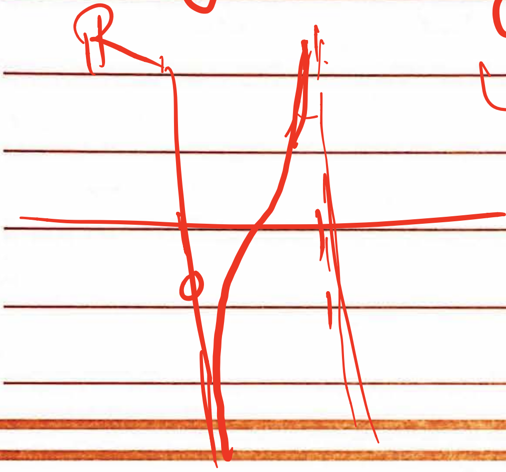
$n = f(x) \in \mathbb{N}$	$x \in \mathbb{Z}$
1	0
3	+1
2	-1
5	+2
4	-2

Strange fact:

$|\mathbb{Z}| = |\mathbb{N}|$

Example: Do $(0,1)$ and $\mathbb{R}$ have the same cardinality?

$y =$ $\quad$ $y = \tan\left(\pi x - \frac{\pi}{2}\right)$



Example: $\mathbb{N} \times \mathbb{N}$ is Countable

$(1,1)$ $(2,1)$ $(3,1)$ $(4,1)$ $(5,1)\dots$
$(1,2)$ $(2,2)$ $(3,2)$ $(4,2)$ $(5,2)\dots$
$(1,3)$ $(2,3)$ $(3,3)$ $(4,3)$ $(5,3)\dots$
$(1,4)$ $(2,4)$ $(3,4)$ $(4,4)$ $(5,4)\dots$
$(1,5)$ $(2,5)$ $(3,5)$ $(4,5)$ $(5,5)\dots$
$\vdots$ $\quad$ $\vdots$

Define $f(i,j)$ equal to the number of pairs visited when (i,j) visited. $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

f is a one-to-one correspondence because it visits all pairs and it visits each pair once.

$\Longrightarrow$ $\mathbb{N} \times \mathbb{N}$ is countable

Remark: Using the same argument, one can prove that $A \times B$ is countable when both A and B are countable.

Proposition: $\mathbb{Z} \times \mathbb{N}$ and $\mathbb{Z} \times \mathbb{Z}$ are countable!

$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z},\ b \neq 0 \right\}$$

$\mathbb{Q} \subseteq \mathbb{Z} \times \mathbb{Z}$

is $\boxed{\text{countable}}$

any subset of a countable set is either finite or countable

Def: Uncountable set

A is uncountable if it is not countable.

Example: It can be shown that $[0,1)$, $\mathbb{R}$, $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ are uncountable.

Cantor's diagonal argument (See Grubner)

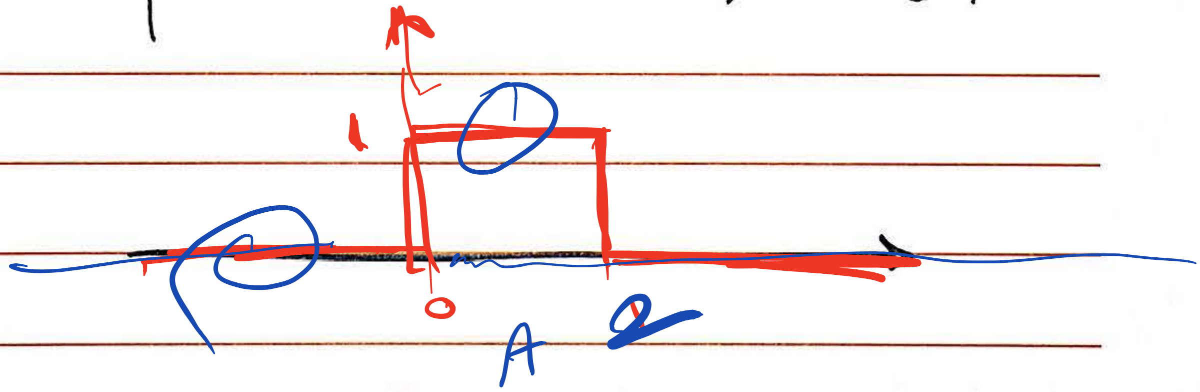
Limit of a sequence of Sets

Def: Indicator function of $A \subseteq \Omega$

$$I_A : \Omega \to \{0, 1\}$$

$$\forall \omega \in \Omega \quad I_A(\omega) = \begin{cases} 0 & \omega \notin A \\ 1 & \omega \in A \end{cases}$$

Example: $\Omega = \mathbb{R}$, $A = [0, 2]$



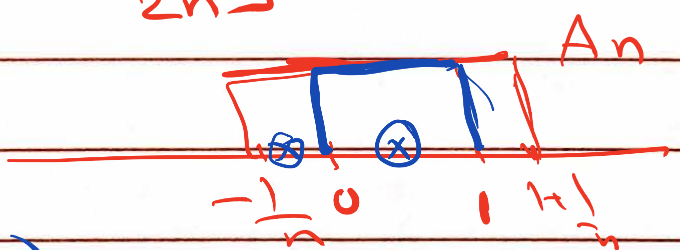
Def: Assume $A_1, A_2, \ldots$ is a sequence of sets, $\forall i \in \mathbb{N}$, $A_i \subseteq \Omega$.

We say $\lim\limits_{n \to \infty} A_n = A$

iff $\forall \omega \in \Omega$ $\lim\limits_{n\to\infty} \boxed{I_{A_n}(\omega) = I_A(\omega)}$

(i.e. iff the indicator functions converge pointwise)

$A_n = \left[-\frac{1}{n}, 1 + \frac{1}{2n}\right]$



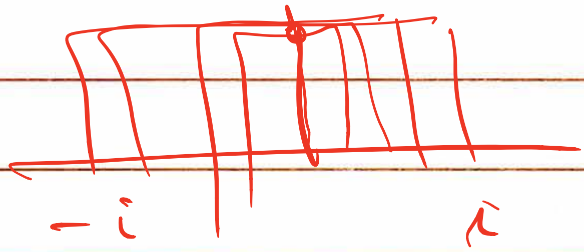
$\forall \omega \in \mathbb{R}$ $\lim\limits_{n\to\infty} I_{A_n}(\omega) = I_{[0,1]}(\omega)$ $\qquad \lim\limits_{n\to\infty} A_n = [0,1]$

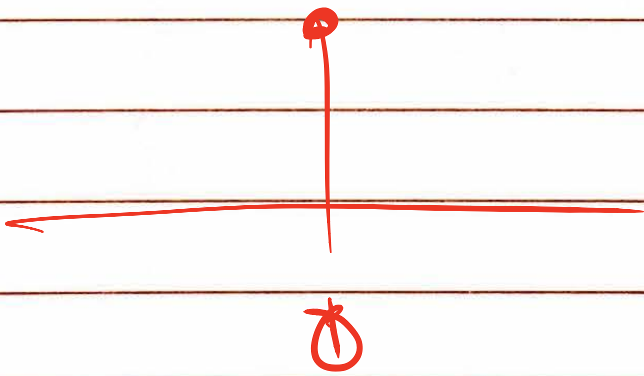
Example : $A_i = [-1/i, 1/i]$

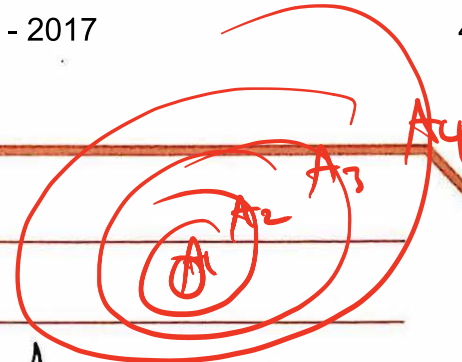
$$\lim_{i\to\infty} A_i = \{0\}$$

because

$$\forall \omega \in \mathbb{R}, \lim_{i\to\infty} I_{A_i}(\omega) = \begin{cases} 0 & \omega \neq 0 \\ 1 & \omega = 0 \end{cases} = I_{\{0\}}(\omega)$$

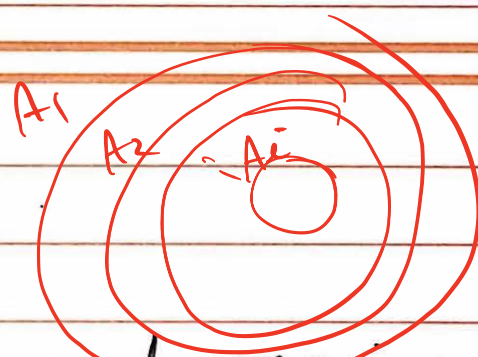






Theorem:

(a) Suppose that A_n is an increasing sequence of sets ($\forall n$, $A_n \subseteq A_{n+1}$), then $\lim_{n \to \infty} A_n$ exists and is equal to $\bigcup_{n \in \mathbb{N}} A_n$.



(b) If A_n is a decreasing sequence of sets ($A_n \supseteq A_{n+1}$), then $\lim_{n \to \infty} A_n$ exists and is equal to $\bigcap_{n \in \mathbb{N}} A_n$.

Example: (a) $A_i = [0, i) \quad i \in \mathbb{N}$

Increasing or decreasing?

Increasing

$$\lim_{i \to \infty} A_i = \bigcup_{i \in \mathbb{N}} A_i = [0, +\infty) = \mathbb{R}^{\geq 0}$$

(b) $B_i = [i, +\infty) \quad , \quad i \in \mathbb{N}$

Increasing or decreasing?

decreasing

$$\lim_{i \to \infty} B_i = \bigcap_{i \in \mathbb{N}} B_i = \emptyset$$

Exercise: Show that

$$\vec{f}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \vec{f}(A_i)$$

Element chasing

$y \in \vec{f}\left(\bigcup_{i\in I} A_i\right) \Leftrightarrow \underbrace{\exists x \in \bigcup_{i\in I} A_i}$

s.t $y = f(x) \Leftrightarrow$

$\underbrace{\exists x, \exists i \in I}_{\text{Can switch}}$, s.t. $x \in A_i$, $y = f(x)$

$\Leftrightarrow \exists i \in I, \underbrace{\exists x \text{ s.t. } x \in A_i, y = f(x)}$

$\Leftrightarrow \underbrace{\exists i \in I, y \in \underbrace{\vec{f}(A_i)}_{B_i}}$

$\Leftrightarrow y \in \bigcup_{i\in I} \vec{f}(A_i)$

$\Rightarrow \begin{cases} \vec{f}\left(\bigcup_{i\in I} A_i\right) \subseteq \bigcup_{i\in I} \vec{f}(A_i) \\ \bigcup_{i\in I} \vec{f}(A_i) \subseteq \vec{f}\left(\bigcup_{i\in I} A_i\right) \end{cases}$

$$\Rightarrow \bigcup_{i\in I} \vec{f}(A_i) = \vec{f}\left(\bigcup_{i\in I} A_i\right)$$

Note (HW)

$$\forall u \; \forall v \equiv \forall v \; \forall u$$

$$\exists u \; \exists v \equiv \exists v \; \exists u$$

For $\forall$ and $\exists$

Love predicate $L(x,y)$: x loves y

$$\exists x \; \forall y \; L(x,y)$$

(Everybody loves Raymond)

does this imply

$\forall y \; \exists x \; L(x,y)$? Yes

Is the reverse true?

$$\forall x \; \exists y \quad L(x,y)$$

Does this mean

$$\exists y \; \forall x \quad L(x,y) \; ?$$

No

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moral of the story

$$\exists \; \forall \Longrightarrow \forall \; \exists$$

But

$$\forall \; \exists \not\Longrightarrow \exists \; \forall$$