Lesson 7 _ Supplement
A Review of Sequences and Series and Their Limits

We saw the definition of the limit of a sequence

$\lim_{n\to\infty} a_n = L \iff \forall \varepsilon > 0, \exists N_0 \in \mathbb{N}$ s.t

$n \geq N_0 \Rightarrow |a_n - L| < \varepsilon$

Def: Partial Sums: for a sequence $\{a_k\}$, $S_n = \sum_{k=1}^{n} a_k$ is called a partial sum of a_k's.

S_n is a sequence, so we can define $\lim_{n\to\infty} S_n$:

Def: The limit $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ is called an infinite series and is shown as $\sum_{k=1}^{\infty} a_k$.

Example: $a_k = q^k$

$$S_n = \sum_{k=1}^{n} q^k$$

$$\Rightarrow S_n = q + q^2 + \cdots + q^n$$

$$- \quad qS_n = q^2 + q^3 + \cdots + q^{n+1}$$

$$(1-q)S_n = q - q^{n+1}$$

$$\Rightarrow S_n = \frac{q - q^{n+1}}{1-q}$$

Also, one can show that

$$\sum_{k=0}^{n} q^k = \frac{1-q^{k+1}}{1-q}$$

So, for $S_n = \frac{q-q^{n+1}}{1-q}$, $\lim_{n\to\infty} S_n$

$$= \frac{q}{1-q} - \frac{\lim_{n\to\infty} q^{n+1}}{1-q} = \begin{cases} 0 & |q|<1 \\ \infty & |q|\geq 1 \end{cases}$$

$$\Rightarrow \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} q^k = \frac{q}{1-q} \quad \text{if } |q|<1$$

Def (Absolute Convergence):

$\sum_{k=0}^{\infty} a_k$ is said to converge

absolutely, iff $\sum_{k=0}^{\infty} |a_k|$ converges

Def (Conditional Convergence):

$\sum_{k=0}^{\infty} a_k$ is said to converge conditionally

if it converges, but doesn't converge

absolutely

Tests for Convergence

(a) The Ratio Test:

Assume that we want to test

whether $\sum_{k=0}^{\infty} a_k$ converges.

Let $L = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|$

$L<1 \Rightarrow \sum_{k=1}^{\infty} a_k$ converges absolutely

$L>1 \Rightarrow \sum_{k=1}^{\infty} a_k$ diverges

$L=1 \Rightarrow$ The test is silent.

(b) . Test for the convergence of $\sum_{k=1}^{\infty} 1/k^p$.

convergence $\Longleftrightarrow p>1$

divergence $\Longleftrightarrow p \leq 1$

(16) Alternating Series Test

$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges if a_k

is positive and decreasing, i.e.

$a_k \geq a_{k+1} > 0$ and

$\lim_{k \to \infty} a_k = 0$

Example

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges to $\log_e 2$,

although it doesn't absolutely

converge.

Example: Does $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converge?

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{(n+1)}}}{\frac{n!}{n^n}} = \frac{n^n (n+1)!}{(n+1)^{n+1}\, n!}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{n+1-1}{n+1}\right)^n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n$$

$n+1=t$ $$= \lim_{t\to\infty}\left(1-\frac{1}{t}\right)^{t-1}$$

$$= \lim_{t\to\infty}\left(1-\frac{1}{t}\right)^t \lim_{t\to\infty}\left(1-\frac{1}{t}\right)^{-1}$$

$$= e^{-1}\times 1 = e^{-1} < 1$$

Therefore, the series is convergent.

Remark : There is a hierarchy of convergence

$Q_p(n) \quad r^n \quad n! \quad n^n$

$\longrightarrow$

Faster convergence to ∞

where Q_p is a polynomial of degree p

If a_n in $\sum a_n$ involves division of a sequence with a faster convergence rate to ∞ by a sequence with a slower convergence to ∞, ~~is~~ the series is divergent, for example $\sum_{n=1}^{\infty} \frac{n^2}{n^2+n+3}$

is divergent.

Otherwise, it is convergent, for example $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ is convergent.

Power series:

Power series involve finite or infinite sums of polynomials.

For example

MacLaurin Series for $f(x)$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$$

Taylor series for $f(x)$ around $x = x_0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)} x_0}{n!} (x - x_0)^n$$

Power series usually converge for specific values of x, called region of convergence (ROC).

Example: Find the ROC of the series $\sum_{k=1}^{\infty} \frac{x^k}{k^2+5}$

To find the ROC, we can use the ratio test

$$\lim_{k\to\infty} \left| \frac{\frac{x^{k+1}}{(k+1)^2+5}}{\frac{x^k}{k^2+5}} \right| = \lim_{k\to\infty} |x| \underbrace{\left| \frac{k^2+5}{(k+1)^2+5} \right|}_{1} < 1$$

$\Rightarrow$ The series converges if $|x| < 1$

(It converges for $x=1$ as well)

(It also converges for $x=-1$ by the alternating series test)

$$\Rightarrow \text{ROC} = \{x \mid x \in [-1,1]\}$$

A very important Taylor/Maclaurin expansion is that of e^x

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Remark: Two-sided series can also be defined if sequences are defined as functions of integers $a_k: \mathbb{Z} \to \mathbb{R}$.

Def: The partial sum S_{N_1,N_2} a sequence $a_k: \mathbb{Z} \to \mathbb{R}$ is defined as

$$S_{N_1,N_2} = \sum_{k=-N_1}^{N_2} a_k$$

Def: We define $\sum_{k=-\infty}^{+\infty} a_k$ as $\lim_{N_1,N_2 \to \infty} S_{N_1,N_2}$.

Example: Does $\sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{1}{k}$ converge?

A naive (and incorrect) answer to this question would be

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} 1/k = [1+(-1)] + \left(\frac{1}{2}-\frac{1}{2}\right) + \left(\frac{1}{3}-1/3\right) + \cdots = 0$$

However, this series does not converge, because the limit depends on the path on which N_1 and N_2 tend to infinity:

$$N_1 = N_2 \Rightarrow \lim_{N_1, N_2 \to \infty} \sum_{-N_1}^{N_2} 1/k = 0$$

$$N_2 = 2N_1 \quad \sum_{-N_1}^{N_2} 1/k = \sum_{-N_1}^{2N_1} 1/k$$

$$\lim_{N_1, N_2 \to \infty} \sum_{-N_1}^{N_2} 1/k = \lim_{N_1 \to \infty} \sum_{-N_1}^{2N_1} 1/k = \infty$$

Therefore, one must note that $\lim_{N \to \infty} \sum_{k=-N}^{N} a_k$ is different from

$$\sum_{k=-\infty}^{+\infty} a_k .$$