

## LESSON 2

The concept of an event is at the center of probability theory.

An event is modeled by a set.

To understand probability theory, one needs to study set theory and logic.

## Aristotelian Logic

A proposition or Statement

is a sentence that is either true or false.

(This is not a definition)

A propositional variable represents,

an arbitrary proposition we

use lower case letters p, q, r, ...

for propositional variables.

We show the truth and falsehood of propositional variables with 1 and 0 (or T and F) respectively

## Logical Connectives

(negation, conjunction, disjunction, implication, double implication or equivalence)

They are defined using truth tables

$P$	$q$	$\neg P$	$P \vee q$	$P \wedge q$	$P \Rightarrow q$	$P \Leftrightarrow q$
1	1	0	1	1	1	1
0	1	1	1	0	1	0
1	0	0	1	0	0	0
0	0	1	0	0	1	1

$$P \Rightarrow q$$

$$P \Leftrightarrow q$$

Exercise : Show that

$$P \Rightarrow q \equiv \neg P \vee q$$

$P$	$q$	$\neg P$	$P \vee q$	$P \Rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Q.E.D.

$$(P \Rightarrow q) \Leftrightarrow (\neg P \vee q)$$

Exercise: Exclusive or (XOR)  
= Exclusive Disjunction

◎

It models the "or" that is

used in natural languages,

which essentially means

"either one, but not both nor none"

Construct a truth table for

$p \vee q$  (exclusive or) and

also for  $(p \vee q) \wedge \neg(p \wedge q)$

and show that they are equivalent

Example : Prove De Morgan Laws

Using Truth Tables;

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

P	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Exercise: Show that

$$P \Rightarrow Q \equiv \neg Q \rightarrow \neg P$$

(used in proof by contradiction)

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	0	1

Def: A tautology is a logical expression that is always true for all of the values of the propositional variable

P	$\neg P$	$P \vee \neg P$
1	0	1
0	1	1

Def: A Contradiction is a logical expression that is always false for all of the values of propositional variables

Example:  $p \wedge \neg p$

$p$	$\neg p$	$p \wedge \neg p$
1	0	0
0	1	0
0	0	0

Exercise : Show that

$$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$$

is a tautology, using truth tables

## Modus Ponens is a Tautology

$$p \Rightarrow q$$

$$\begin{array}{c} p \\ \hline \end{array}$$

$$\begin{array}{c} q \\ \hline \end{array}$$

human  $\Rightarrow$  mortal

Socrates is a human

Socrates is mortal

$$\boxed{((p \Rightarrow q) \wedge p) \equiv q}$$

## Modus Tollens is a Tautology

$$p \Rightarrow q$$

$$\neg q \quad \cancel{\text{GPA}} \cancel{\Rightarrow} \cancel{\text{Scholarship}}$$

$$\begin{array}{c} \neg q \\ \hline \end{array}$$

GPA = A  $\Rightarrow$  Scholarship

no scholarship

GPA not A

$$\boxed{((p \Rightarrow q) \wedge \neg q) \equiv \neg p}$$

Two more tautologies.

## Disjunctive Syllogism

$$P \vee q$$

$$\neg P$$

$$\underline{q}$$

$$((P \vee q) \wedge \neg P) = q$$

## Hypothetical Syllogism

$$P \Rightarrow q$$

$$q \Rightarrow r$$

$$\underline{P \Rightarrow r}$$

$$[(P \Rightarrow q) \wedge (q \Rightarrow r)] \equiv P \Rightarrow r$$

## Proofs Truth Tables

Predicate : Loosely speaking,

a predicate is a proposition

with a variable in it

Proposition



Predicate



Assigning a value to the variable(s)

makes the predicate, a proposition  
More precisely

Def: A predicate is a boolean-

valued function  $P: X \rightarrow \{0, 1\}$   
 $T, F$

# Logical Quantifiers

## Universal Quantifier $\forall$

It represents a universal property, and is interpreted as "given any" or "for all".

$$\text{Ex: } \forall n \in \mathbb{N}, 2n = n + n$$

## Existential Quantifier $\exists$

It is a type of quantifier, which is interpreted as

- "there exists" or
- "there is at least one"

or "for some"

Ex:  $\exists x \in \mathbb{R}, x^2 = -1$  false

Negation of statements with

Logical Quantifiers  $\exists$

$$\neg (\exists x, p(x)) \equiv [\forall x, \neg p(x)]$$

$\equiv$

$$\# x, p(x)$$

Ex:  $\neg (\exists x \in \mathbb{R} \text{ s.t. } x^2 = -1)$

$\equiv$

$$\# x \in \mathbb{R}, x^2 = -1$$

$\equiv$

$$\forall x \in \mathbb{R}, x^2 \neq -1$$

Negation of  $\forall$

$$\neg(\forall x, p(x)) \equiv \exists x, \neg p(x)$$

↑  
s.t.

$$\text{Ex: } \neg(\forall n \in \mathbb{N}, \frac{n}{2} \in \mathbb{N})$$

$$\equiv \boxed{\exists n \in \mathbb{N}, \frac{n}{2} \notin \mathbb{N}}$$