1. Assume that $\Omega = [0, 1/2]$ and $\mathbb{P}(B) = \int_B 2d\omega$ for any Borel subset[1] of Ω. Calculate $\mathbb{P}(B|A)$ when $B = [0, 1/8]$ and $A = [1/16, 1/3)$. Are A and B independent?

[1] Borel sets are countable unions and intersections of intervals.

Solution:[2]

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

But $A \cap B = [1/16, 1/3] \cap [0, 1/8] = [1/16, 1/8]$. Therefore, $\mathbb{P}(A \cap B) = \int_{1/16}^{1/8} 2d\omega = 1/8$. On the other hand, $\mathbb{P}(A) = \int_{1/16}^{1/3} 2d\omega = 13/24$ and $\mathbb{P}(B) = \int_{0}^{1/8} 2d\omega = 1/4$. Because $\mathbb{P}(A \cap B) \neq \mathbb{P}(A)\mathbb{P}(B)$, they are not independent.

[2]Important Note: Posting the course material to online forums or sharing it with other students is strictly prohibited. Instances will be reported to USC officials as academic dishonesty for disciplinary action.

2. Amy and Baichuan have $2k + 1$ coins, each with probability of a head equal to $1/2$. Baichuan tosses $k + 1$ coins, while Amy tosses the remaining k coins. Show that the probability that after all the coins have been tossed, Baichuan will have gotten more heads than Amy is $1/2$.

Solution:

Let B be the event that Baichuan tossed more heads. Let D be the event that after each has tossed k of their coins, Baichuan has more heads than Amy, let E be the event that under the same conditions, Amy has more heads than Baichuan, and let F be the event that they have the same number of heads. Since the coins are fair, we have $\mathbb{P}(D) = \mathbb{P}(E)$, and also $\mathbb{P}(F) = 1 - \mathbb{P}(D) - \mathbb{P}(E)$. Furthermore, we see that

$$\begin{aligned}
\mathbb{P}(B|D) &= 1 \\
\mathbb{P}(B|E) &= 0 \\
\mathbb{P}(B|F) &= \frac{1}{2}
\end{aligned}$$

Now we have, using the theorem of total probability,

$$\begin{aligned}
\mathbb{P}(B) &= \mathbb{P}(D)\mathbb{P}(B|D) + \mathbb{P}(E)\mathbb{P}(B|E) + \mathbb{P}(F)\mathbb{P}(B|F) \\
&= \mathbb{P}(D) + \frac{1}{2}\mathbb{P}(F) = \mathbb{P}(D) + \frac{1}{2}(1 - \mathbb{P}(D) - \mathbb{P}(E)) \\
&= \frac{1}{2} + \frac{1}{2}(\mathbb{P}(D) - \mathbb{P}(E)) = \frac{1}{2}
\end{aligned}$$

as required. What is happening here is that Amy's probability of more heads than Baichuan is less than 1/2, so Baichuan has an advantage. However, the probability of equal number of heads is positive, and when added to Amy's probability of more heads, it gives 1/2.

3. 98% of all patients with a certain type of cancer in a hospital survive. However, 10% of all of those patients are given chemotherapy, and when chemotherapy is performed, the patient survives 90% of the time. If the randomly chosen patient with that type of cancer does not receive chemotherapy, what is the probability that the patient survives?

Solution:

Let S be the event that a patient survives and let C be the event of receiving chemotherapy. Then:

$$\begin{aligned}\mathbb{P}(S|C^c) &= \frac{\mathbb{P}(S\cap C^c)}{\mathbb{P}(C^c)} = \frac{\mathbb{P}(S)-\mathbb{P}(S\cap C)}{\mathbb{P}(C^c)} = \frac{\mathbb{P}(S)-\mathbb{P}(S|C)\mathbb{P}(C)}{1-\mathbb{P}(C)} = \frac{0.98-0.9\times 0.1}{1-0.1}\\ &= \frac{89}{90}\end{aligned}$$

4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A_1, A_2, \ldots$ be events.

   (a) Using mathematical induction, show that $\mathbb{P}(\cup_{i=1}^{n} A_i) = \mathbb{P}(A_1) + \mathbb{P}(A_2 \cap A_1^c) + \ldots + \mathbb{P}(A_n \cap A_{n-1}^c \cap \ldots \cap A_2^c \cap A_1^c)$.

   (b) Show that $\mathbb{P}(\cup_{i\in\mathbb{N}} A_i) = \mathbb{P}(A_1) + \sum_{i=2}^{\infty} \mathbb{P}(A_i \cap A_{i-1}^c \cap \ldots \cap A_1^c)$.

Solution:

(a) Left to the students 😄.

(b) Apply continuity of probability to (4a). 😎

5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A, B \in \mathcal{F}$ be events. Define the odds of A as

$$\mathbb{O}(A) = \frac{\mathbb{P}(A)}{\mathbb{P}(A^c)}$$

and the odds in favor of A given B as:

$$\mathbb{O}(A|B) = \frac{\mathbb{P}(A|B)}{\mathbb{P}(A^c|B)}$$

and the likelihood ratio of B given A as:

$$\mathbb{L}(B|A) = \frac{\mathbb{P}(B|A)}{\mathbb{P}(B|A^c)}$$

Show that: $\mathbb{O}(A|B) = \mathbb{L}(B|A)\mathbb{O}(A)$.

Solution:

$$\frac{\mathbb{P}(A|B)}{\mathbb{P}(A^c|B)} = \frac{\mathbb{P}(A \cap B)/\mathbb{P}(A)}{\mathbb{P}(A^c \cap B)/\mathbb{P}(A)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A^c \cap B)} = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A^c)\mathbb{P}(A^c)} = \mathbb{L}(B|A)\mathbb{O}(A)$$

6. Assume that $X : \Omega \rightarrow \mathbb{R}$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Directly show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is an affine function of the form $f(x) = ax + b,\ a, b \in \mathbb{R}^{\geq 0}$), then $Y = f(X)$ is also a random vaiable.

Solution:

It is obvious that $Y : \Omega \to \mathbb{R}$. We only need to show that $\{\omega | Y(\omega) \leq c\} \in \mathcal{F}, \forall c \in \mathbb{R}$ (i.e., it is an event):

If $a = 0$, $Y(\omega) = b$, so $\{\omega | Y(\omega) \leq c\}$ is either $\emptyset$ or Ω, which are both events.

If $a > 0$,

$$\{\omega | Y(\omega) \leq c\} = \{\omega | aX(\omega) + b \leq c\} = \{\omega | aX(\omega) \leq c - b\} = \{\omega | X(\omega) \leq (c - b)/a\}$$

$\forall c \in \mathbb{R}, (c - b)/a \equiv d \in \mathbb{R}$, so $\{\omega | Y(\omega) \leq c\} = \{\omega | X(\omega) \leq d\}$, which is an event, because X is a random variable.