Lesson 6

Combinatorics:

Principles of Counting

(without counting!)

Probability calculations often involve counting, especially when

we have equally likely outcomes and finite sets. In such situations, the probability of an event is just the "size" (cardinality) of that event relative to the size (cardinality) of the sample space i.e.:

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} \quad \forall A \subseteq \Omega$$

In this lesson, we review the concepts of combinatorics, for measuring the "size" of countable events.

The Multiplication ~~rule~~ Principle

Consider a two-stage experiment.

the first stage can have m results $a_1, a_2, \ldots, a_m$ and the second stage can have n results $b_1, b_2, \ldots b_n$. The number of possible ways to perform the first and the second stage sequentially is the number of possible (a_i, b_j) pairs, which is $\boxed{mn}$

More formally:

If $\Omega = A \times B$, then

$$|\Omega| = |A \times B| = |A||B|$$

More generally

If a process has r stages and stage i has n_i possible results, the number of all possible results is

$$\boxed{n_1 n_2 n_3 \cdots n_r} = \prod_{i=1}^{r} n_i$$

In more formal terms, if

$\Omega = A_1 \times A_2 \times \cdots \times A_r$, then

$$|\Omega| = |A_1 \times A_2 \times \cdots \times A_r| = |A_1||A_2|\cdots|A_r|$$

$$= \prod_{i=1}^{r} |A_i|$$

If $\forall i, A_i = A$, $|\Omega| = |A^k| = |A|^k$

Example: Telephone numbers are 7 digits in Springfield. The first digit cannot be 1 or 0. How many different telephone numbers can we have?

Solution:

$$\underline{8}\times\underline{10}\times\underline{10}\times\underline{10}\times\underline{10}\times\underline{10}\times\underline{10} = 8\times 10^6$$

Example: What is the number of subsets of a set with n elements?

$$\underline{2}\times\underline{2}\times\underline{2}\times\underline{\cdots}\cdots\times\underline{2}\times\underline{2} = 2^n$$

$\emptyset$ 0 0 0 0 0 0

Ω 1 1 1 1 1 1

The multiplication principle helps us understand ordered sampling with replacement.

Example: Assume that we have n objects, and we would like to sample k objects from them with replacement. Each time, we draw an object, make a note of it, put the object back ~~and~~ and select the next object.

The number of different sequences of k objects that can be formed is:

$$\underbrace{n \times n \times \cdots \times n}_{k \text{ times}} = n^k$$

Ordered Sampling with replacement

Permutations: when k objects are selected from a plurality of n objects, and <u>order matters</u>, k-permutations are constructed.

In other words, k-permutations are subsequences of length k, selected from sequences of length n

k places $\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \cdots\ \underline{\quad}$

$$n \quad n-1 \quad n-2 \quad \cdots \quad n-k+1 = \cancel{n(n-1)\cdots(n-k+1)}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)(n-k-1)\cdots 1}{(n-k)(n-k-1)\cdots 1} = \frac{n!}{(n-k)!}$$

In other words, permutations can be considered as results of ordered sampling without replacement

$$\Omega = A \times (A - \{\omega_1\}) \times (A - \{\omega_1, \omega_2\}) \times \cdots \times A - \{\omega_1, \ldots, \omega_{k-1}\}$$

$$|\Omega| = \frac{n!}{(n-k)!}$$

Example: Assume that we would like to enumerate all possible binary sequences that can be created using (0, 0, 1, 1).

$n = 4$

$k = 3$

$\omega_1 =$ $\qquad$ $\omega_2 =$

$\omega_3 =$ $\qquad$ $\omega_4 =$

Observe that $\square$ is a rearrangement of $\square$ and $\square$ is a rearrangement of $\square$

Combinations: when we select k objects from n objects, and order does not matter for us, we are considering combinations.

"Order does not matter." This
no replacement

means that all re-arrangements of a sequence of length k should be considered the same sequence.

Question: How many

rearrangements of a sequence

of length k exist?

$\_\_ \; \_\_ \; \_\_ \; \_\_ \; \ldots \; \_\_$

k places $K \quad K-1 \quad K-2 \quad \ldots \quad 1 = K!$

k objects

number of permutations

$= \frac{k!}{(k-k)!} = \frac{k!}{0!} = K!$

Therefore, the number of

combinations of length k

from n objects is

$\cancel{P^n_k}$ $\cancel{{}^nP_k}$ $\cancel{C^k_n}$ $\cancel{C^n_k}$ $\binom{n}{k} = \frac{\#\text{ permutations}}{k!} = \frac{n!}{k!.(n-k)!}$

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Combinations are useful in understanding "unordered sampling without replacement"

$\binom{n}{k}$ is also called the binomial coefficient, because it appears in the binomial theorem:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

The coefficients $\binom{n}{k}$ can be read from the so-called

Khayyam-Pascal Triangle

1 $n=0$

1 1 $n=1$

$k=0$ $k=1$ $k=2$

1 2 1 $n=2$

$k=0$ $k=1$ $k=2$ $k=3$

1 3 3 1 $n=3$

1 4 6 4 1 $n=4$

The triangle works based on the following formula:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

The binomial theorem can be proved by induction.

Example: We have 5 novels, 3 Sci-fi books, 4 magazines, and 8 college textbooks. In how many different ways can they be arranged so that books of the same genre are placed together?

$\underline{5!}$ $\underline{3!}$ $\underline{4!}$ $\underline{8!}$

novel Scifi mag coll. ← one instance

4 genres

answer: $4! \times 5! \times 3! \times 4! \times 8!$

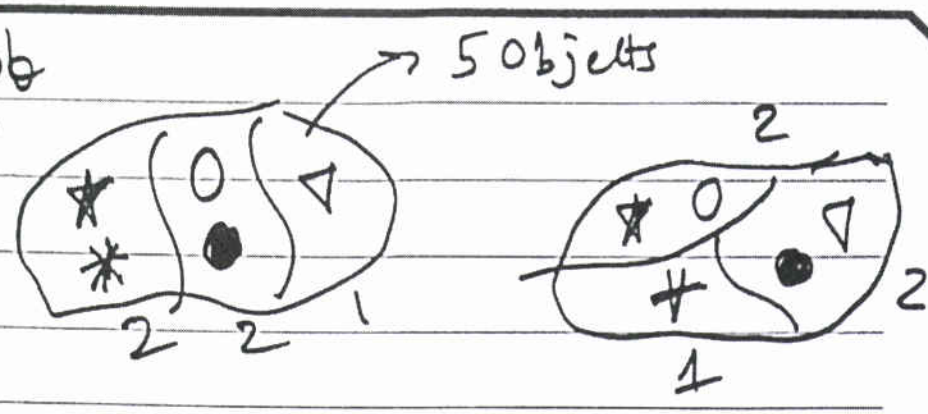
Partitions A combination is a partition of n objects into k and $n-k$ objects.

We can generalize this discussion to ~~making~~ ~~groups~~ dividing n objects into r partitions, given that the i^{th} partition has n_i elements.

Obviously

$$n_1 + n_2 + \cdots + n_r = n$$

Example



Number of such partitions is

$$\boxed{\frac{n!}{n_1!\, n_2! \cdots n_r!}}$$

you can prove this by induction on

We also use the notation:

$$\begin{pmatrix} n \\ n_1\ n_2\ \cdots n_r \end{pmatrix}$$

Interestingly, ~~the~~

$$\binom{n}{n_1 \; n_2 \; \cdots \; n_r}$$

is called a "multinomial coefficient," because it appears in the multi-nomial theorem

$$(a_1+a_2)^n = \sum_{k} \binom{n}{k, n-k} a_1^k a_2^{n-k} \qquad \binom{n}{k}$$

$$(a_1+a_2+\cdots+a_m)^n = \sum_{k_1, k_2, \ldots, k_m}^{n} \binom{n}{k_1, \ldots, k_m} a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m}$$

$k_1+k_2+\cdots+k_m = n$

Example (Anagrams)

How many different words (letter sequences) can be obtained by rearranging the letters in the word

RAT
TAR

~~[illegible]~~ TATTOO?

3 1 2

T A O

$$\frac{6!}{3!\,2!\,1!} = \frac{6 \times 5 \times 4 \times \cancel{3!}}{\cancel{3!} \times 2! \times 1!} = 60$$