23 Sep, 2025

Independence of Random Variables

X_1, X_2 are random variables on $(\Omega, \mathcal{F}, P)$

X_1, X_2 are independence iff all Borel Set $B_1, B_2 \in \mathcal{B}(\mathbb{R})$

$$P(X_1 \in B_1, X \in B_2) = P(X_1 \in B_1) P(X_2 \in B)$$

$\{X_i \mid i \in I\}$ are independent

iff $\forall J \subseteq I$, $|J| < \infty$

$$P\left(\bigcap_{i \in J} X_i^{\leftarrow}(B_i)\right) = \prod_{i \in J} P(X_i^{\leftarrow}(B_i))$$

Joint CDF

X_1, X_2 are random variables on $(\Omega, \mathcal{F}, P)$

the joint cdf of X_1 and X_2

$$F_{X_1 X_2}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)$$

$$= P(\{\omega \mid X_1(\omega) \in x_1\} \cap \{\omega \mid X_2(\omega) \in x_2\})$$

$$F_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \cdots X_n \leq x_n)$$

Lemma: X_1, X_2 are independent iff $F_{X_1 X_2}(x_1, x_2) = F_{X_1}(x_1) F_{X_2}(x_2)$

Principle of Counting : probability of event is size (cardinality) of that event

The Multiplication Principal if $\Omega = A \times B \Rightarrow |\Omega| = |A||B|$

$$\Omega = \prod_{i=1}^{n} A_i \Rightarrow |\Omega| = \prod_{i=1}^{n} |A_i|$$

Permutation ; ordered sample without replacement

$${}_nP_k = \frac{n!}{(n-k)!}$$

Combination ; unordered sample without replacement

$${}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

i.g.) binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(n-k)!\,(k-1)!} + \frac{(n-1)!}{(n-k-1)!\,k!}$$

$$= \frac{k(n-1)! + (n-k)(n-1)!}{(n-k)!\,k!} = \frac{n\cdot(n-1)!}{(n-k)!\,k!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

Partitions ; combination is partition of objects into k and $n-k$ objects

$\Rightarrow$ dividing n object into r partitions

ith partition size : n_i

$n_1 + n_2 + \cdots + n_r = n$

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n_r}{n_r} = \binom{n}{n_1 n_2 \cdots n_r} = \frac{n!}{n_1!\,n_2!\cdots n_r!}$$; multinomial coefficient

Discrete Random Variables

Range assume that function $f: U \to V$
the range of f R_f is define as $f^{\to}(U)$

Discrete Random Variable : random variable $X: \Omega \to \mathbb{R}$ is called discrete
if $X^{\to}(\Omega) = R_X$ is countable

Probability Mass Function (PMF)
if X is a discrete r.v. , the function
$P_X : \mathbb{R} \to [0,1]$ define as $P_X(x) = P(X=x)$ is pmf

A discrete random variable can be characterized by its pmf

for any Borel Set B , $P(X \in B) = P_X(B) = P(X \in B \cap R_X) = \sum_{x \in B \cap R_X} P(X=x) = \sum_{x \in B \cap R_X} P_X(x)$

The CDF of a discrete random variable

$$F_X(x) = P(X \leq x) = \sum_{y \leq x} P_X(y) = \sum_{y \leq x,\, y \in R_y} P_X(y)$$

satisfied $P(X=x) = F_X(x) - \lim_{y \to x^-} F_X(y)$

Exercise: $\sum_{x \in \mathbb{R}} P_X(x) = 1$

$= P(X \in \mathbb{R}) = P(X^{\leftarrow}(\mathbb{R})) = P(\Omega) = 1$

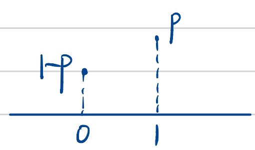
Important Discrete Random Variables and their PMF

a) Bernoulli with parameter p where $0 \le p \le 1$

$X \sim Ber(p)$ $\quad P_X(1) = p$, $\quad P_X(0) = 1-p$

i.g.) $I_A = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$

pmf



b) Discrete Uniform

$a, b \in \mathbb{Z}$ $\quad a < b$

$$P_X(x) = \begin{cases} \frac{1}{b-a+1} & x = a, a+1, \cdots, b-1, b \\ 0 & \text{otherwise } x \in \mathbb{Z} \end{cases}$$

$X \sim dU(a, b)$

c) Binomial with parameter n, p

$n \in \mathbb{N}$, $p \in [0, 1]$

pmf : $P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$

$X \sim Bin(n, p)$