~~Home~~work 2

DSCI 564
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1.

a) $f^{\rightarrow}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f^{\rightarrow}(A_i)$

i) $y \in f^{\rightarrow}(\bigcup_{i\in I} A_i) \Rightarrow \exists\, x \in \bigcup_{i\in I} A_i$ s.t. $y = f(x)$

$\Rightarrow \exists\, x,\ \exists\, i\in I$ s.t. $x \in A_i,\ y = f(x)$

$\Rightarrow \exists\, i\in I\ \exists\, x$ s.t. $x\in A_i,\ y = f(x)$

$\Rightarrow \exists\, i \in I,\ y \in f^{\rightarrow}(A_i)$

$\Rightarrow y \in \bigcup_{i\in I} f^{\rightarrow}(A_i)$

$\therefore f^{\rightarrow}(\bigcup_{i\in I} A_i) \subseteq \bigcup_{i\in I} f^{\rightarrow}(A_i)$

ii) $y \in \bigcup_{i\in I} f^{\rightarrow}(A_i) \Rightarrow \exists\, i\in I,\ y \in f^{\rightarrow}(A_i)$

$\Rightarrow \exists\, i\in I,\ \exists\, x$ s.t. $x \in A_i,\ y = f(x)$

$\Rightarrow \exists\, x,\ \exists\, i\in I$ s.t. $x\in A_i,\ y = f(x)$

$\Rightarrow \exists\, x,\ x \in \bigcup_{i\in I} A_i,\ y = f(x)$

$\Rightarrow y \in f^{\rightarrow}(\bigcup_{i\in I} A_i)$

$\therefore \bigcup_{i\in I} f^{\rightarrow}(A_i) \subseteq f^{\rightarrow}(\bigcup_{i\in I} A_i)$

$\Rightarrow \bigcup_{i\in I} f^{\rightarrow}(A_i) = f^{\rightarrow}(\bigcup_{i\in I} A_i)$

b) $f^{\to}(\bigcap_{i\in\mathbb{N}} A_i) \subseteq \bigcap_{i\in I} f^{\to}(A_i)$; if f is one-to-one , equality occurs

$y \in f^{\to}(\bigcap_{i\in\mathbb{N}} A_i) \Rightarrow \exists\, x \in \bigcap_{i\in\mathbb{N}} A_i$ s.t. $y=f(x)$

$\Rightarrow \exists x ,\ \forall i\in\mathbb{N} ,\ x\in A_i ,\ y=f(x)$

$\Rightarrow \forall i\in\mathbb{N} ,\ \exists x ,\ x\in A_i ,\ y=f(x)$

$\Rightarrow \forall i\in\mathbb{N} ,\ y\in f^{\to}(A_i)$

$\Rightarrow y \in \bigcap_{i\in\mathbb{N}} f^{\to}(A_i)$

$\therefore f^{\to}(\bigcap_{i\in\mathbb{N}} A_i) \subseteq \bigcap_{i\in\mathbb{N}} f^{\to}(A_i)$

c) $f^{\leftarrow}(\bigcup_{i\in I} A_i) = \bigcup_{i\in\mathbb{N}} f^{\leftarrow}(A_i)$

i) $x \in f^{\leftarrow}(\bigcup_{i\in I} A_i) \Rightarrow \exists\, y \in \bigcup_{i\in I} A_i$ s.t. $y=f(x)$

$\Rightarrow \exists y ,\ \exists_{i\in I}$, s.t. $y\in A_i ,\ y=f(x)$

$\Rightarrow \exists_{i\in I} ,\ \exists y$, s.t. $y\in A_i ,\ y=f(x)$

$\Rightarrow \exists\, i\in I$, s.t. $x\in f^{\leftarrow}(A_i)$

$\Rightarrow x \in \bigcup_{i\in I} f^{\leftarrow}(A_i)$

$\therefore f^{\leftarrow}(\bigcup_{i\in I} A_i) \subseteq \bigcup_{i\in I} f^{\leftarrow}(A_i)$

ii) $x \in \bigcup_{i\in I} f^{\leftarrow}(A_i) \Rightarrow \exists\, i\in I ,\ x\in f^{\leftarrow}(A_i)$

$\Rightarrow \exists\, i\in I\ \exists\, y\in A_i$, s.t $y=f(x)$

$\Rightarrow \exists y\ \exists_{i\in I} ,\ y\in A_i ,\ y=f(x)$

$\Rightarrow \exists y ,\ y\in \bigcup_{i\in I} A_i ,\ y=f(x)$

$\Rightarrow x \in f^{\leftarrow}(\bigcup_{i\in I} A_i)$

$\therefore \bigcup_{i\in I} f^{\leftarrow}(A_i) \subseteq f^{\leftarrow}(\bigcup_{i\in I} A_i)$

$\Rightarrow f^{\leftarrow}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f^{\leftarrow}(A_i)$

$f^{\leftarrow}(A^c) = (f^{\leftarrow}(A))^c$

i) $x \in f^{\leftarrow}(A^c) \Rightarrow \exists\, y \in A^c,\ y = f(x)$

$\Rightarrow \exists\, y \notin A,\ y = f(x)$

$\Rightarrow x \notin f^{\leftarrow}(A)$

$\Rightarrow x \in (f^{\leftarrow}(A))^c$

$\therefore\ f^{\leftarrow}(A^c) \subseteq (f^{\leftarrow}(A))^c$

ii) $x \in (f^{\leftarrow}(A))^c \Rightarrow \exists\, y \notin A,\ y = f(x)$

$\Rightarrow \exists\, y \in A^c,\ y = f(x)$

$\Rightarrow x \in f^{\leftarrow}(A^c)$

$\therefore\ (f^{\leftarrow}(A))^c \subseteq f^{\leftarrow}(A^c)$

$\Rightarrow f^{\leftarrow}(A^c) = (f^{\leftarrow}(A))^c$

2. F_1 and F_2 are σ-fields of subsets of Ω

Show $F_1 \cap F_2$ is also σ-fields of subsets of Ω

i) Closure under complement

$A \in F_1 \cap F_2 \Rightarrow (A \in F_1) \wedge (A \in F_2)$

$\Rightarrow (A^c \in F_1) \wedge (A^c \in F_2)$ $\because$ F_1, F_2 are σ-fields

$\Rightarrow A^c \in F_1 \cap F_2$

ii) closure under countable union

$A_1, A_2, \cdots \in F_1 \cap F_2 \Rightarrow (A_1 \cdots \in F_1) \wedge (A_1 \cdots \in F_2)$

$\Rightarrow (\bigcup_{i \in I} A_i \in F_1) \wedge (\bigcup_{i \in I} A_i \in F_2)$

$\Rightarrow \bigcup_{i \in I} A_i \in F_1 \cap F_2$

3. $A, B \in \mathcal{F} \subseteq 2^{\Omega}$. Show $A \cap B \in \mathcal{F}$, $A \setminus B \in \mathcal{F}$, and
$A \triangle B = (A \setminus B) \cup (B \setminus A) \in \mathcal{F}$

a) $A \cap B \in \mathcal{F}$

$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

$B \in \mathcal{F} \Rightarrow B^c \in \mathcal{F}$

$\therefore (A^c \cup B^c) \in \mathcal{F} \Rightarrow (A^c \cup B^c)^c \in \mathcal{F}$

$\Rightarrow A \cap B \in \mathcal{F}$ Q.E.D

b) $A \setminus B \in \mathcal{F}$

$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

$\therefore A^c \cup B \in \mathcal{F} \quad \because A^c, B \in \mathcal{F}$

$\Rightarrow (A^c \cup B)^c \in \mathcal{F} \Rightarrow A \cap B^c \in \mathcal{F}$

$\Rightarrow A \setminus B \in \mathcal{F}$ QED.

c) $(A \setminus B) \cup (B \setminus A) \in \mathcal{F}$

$B \in \mathcal{F} \Rightarrow B^c \in \mathcal{F}$

$A \cup B^c \in \mathcal{F} \Rightarrow (A \cup B^c)^c \in \mathcal{F} \Rightarrow A^c \cap B \in \mathcal{F}$

$\Rightarrow B \setminus A \in \mathcal{F}$

$\therefore (A \setminus B) \cup (B \setminus A) \in \mathcal{F}$

Show the formula

$$P((A\cap B^c)\cup(A^c\cap B)) = P(A)+P(B)-2P(A\cap B)$$

$$P((A\cap B^c)\cup(A^c\cap B)) = P(A\cap B^c)+P(A^c\cap B) \quad \because (A\cap B^c)\cap(A^c\cap B)=\phi$$

from $P(A)=P(A\cap B)+P(A\cap B^c)$, $P(B)=P(A\cap B)+P(A^c\cap B)$

$$P((A\cap B^c)\cup(A^c\cap B)) = P(A)-P(A\cap B)+P(B)-P(A\cap B)$$

$$= P(A)+P(B)-2P(A\cap B)$$

Q.E.D.

5. Gubner Chapter 1.24

$\Omega = \{(\alpha,\beta) \mid \alpha,\beta\in\{a,b,\dots,z\},\ \alpha\neq\beta\}$

$|\Omega| = 26\times 25$

i) Vowel → Consonant

$|A| = 5\times 21$

$P(A) = \frac{5\times 21}{26\times 25} = \frac{21}{130}$

ii) Consonant → Vowel

$|A| = 21\times 5$

$P(A) = \frac{5\times 21}{26\times 25} = \frac{21}{130}$

$P(A) = \frac{21}{130}+\frac{21}{130} = \frac{21}{65}$

iii) Vowel → vowel

$|A| = 5\times 4$

$P(A) = \frac{5\times 4}{26\times 25} = \frac{2}{65}$

6. Gubner Chapter 1, 29

$P(A)$. $P(B)$ and $P(A\cup B)$ are known. Express below in terms of these probabilities

a) $P(A\cap B) = P(A)+P(B)-P(A\cup B)$

b) $P(A\cap B^c) = P(A)-P(A\cap B) \quad \because P(A)=P(A\cap B^c)+P(A\cap B)$

$= P(A)-P(A)-P(B)+P(A\cup B)$

$= P(A\cup B)-P(B)$

c) $P(B\cup(A\cap B^c)) = P(B)+P(A\cap B^c)-P(B\cap A\cap B^c)$

$= P(B)+P(A\cap B^c) \quad \because P(A\cap B\cap B^c)=P(\phi)=0.$

$= P(B)+P(A\cup B)-P(B)=P(A\cup B)$

d) $P(A^c\cap B^c) = 1-P(A\cup B) \quad \because A^c\cap B^c=(A\cup B)^c, \quad P(A^c)=1-P(A)$

7. Gubner Chapter 1.30.

Ω sample space equipped with two probability measures, P_1 and P_2

$0\le\lambda\le1$ show $P(A) := \lambda P_1(A)+(1-\lambda)P_2(A)$.

then P satisfied four axioms of a probability measure

i) Non-Negativity

$\lambda P_1(A)+(1-\lambda)P_2(A) \ge 0$

$\because 0\le\lambda\le1 \Rightarrow 0\le 1-\lambda\le1$

$0\le P_1(A)\le1, \quad 0\le P_2(A)\le1$

Normalization

$$P(\Omega) = \lambda P_1(\Omega) + (1-\lambda)P_2(\Omega) = \lambda + (1-\lambda) = 1$$

$\because$ P_1, P_2 are probability measure of sample space Ω

iii) Countable additivity

$A_1, A_2, \cdots \in \mathcal{F}$ $\quad A_i \cap A_j = \phi \ \forall i \neq j$

$$P\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \lambda P_1\left(\bigcup_{i\in\mathbb{N}} A_i\right) + (1-\lambda)P_2\left(\bigcup_{i\in\mathbb{N}} A_i\right)$$

$$= \lambda \sum_{i=1}^{\infty} P_1(A_i) + (1-\lambda)\sum_{i=1}^{\infty} P_2(A_i) \quad \because \ P_1\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i=1}^{\infty} P_1(A_i), \ P_2\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i=1}^{\infty} P_2(A_i)$$

$$= \sum_{i=1}^{\infty}\left[\lambda P_1(A_i) + (1-\lambda)P_2(A_i)\right]$$

iv) Empty set axiom

$$P(\phi) = \lambda P_1(\phi) + (1-\lambda)P_2(\phi) = 0 \quad \because \ P_1(\phi) = 0, \ P_2(\phi) = 0.$$

8. Dobrow 1.41

$$\left(\frac{1}{2}\right)^4 \cdot {}_4C_1 = \frac{1}{4}$$

9. Monte Carlo Simulation

i) Simulate a trial

ii) Determine success

iii) Replication