# Lesson 5 - Appendix

## More on Multiple Random Variables

In the real world, we deal with multiple random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

It is essential to remember that the realizations of multiple random variables that are defined on the same probability space are governed by the same underlying randomness in experiments namely randomness in selecting $\omega \in \Omega$.

As an example, one can imagine that the sample space is the set of all residents of Los Angeles, $X$ can be a random variable that assigns a person's income to a person, and $Y$ can be a random variable that assigns a person's age to them.

It is reasonable to think that there is some interdependence between X and Y. A person's age usually says something (but not everything of course) about a person's income!

Now, an important question arises:

We saw that for every Borel set $B \in \mathcal{B}(\mathbb{R})$, $X^{\leftarrow}(B)$ is an event.

The question is if for the function $(X(\cdot), Y(\cdot)) : \Omega \to \mathbb{R}^2$, which

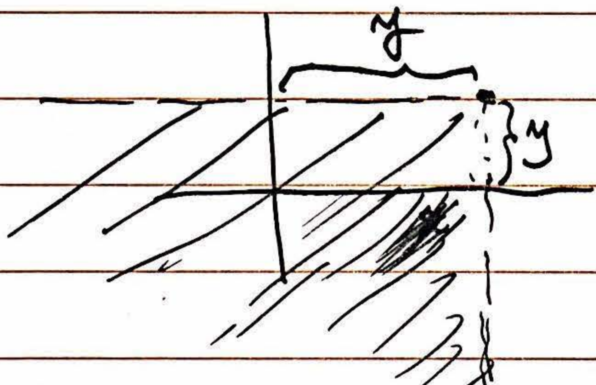
can be seen as a multi-output function or a vector-valued function, the same concept can be envisioned.

To answer this question, one has to

define a Borel $\sigma$-field on $\mathbb{R}^2$, $\mathcal{B}(\mathbb{R}^2)$.

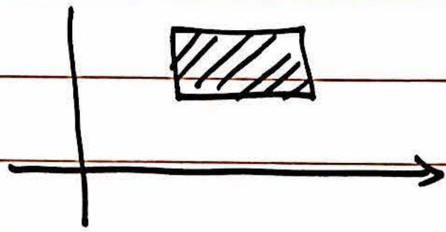
Def: (Borel $\sigma$-field on $\mathbb{R}^2$) Assume that $C = \{(-\infty, x] \times (-\infty, y] \mid x, y \in \mathbb{R}\}$, i.e. the set of all possible sets in the form $(-\infty, x] \times (-\infty, y]$




The Borel $\sigma$-field on $\mathbb{R}^2$, $\mathcal{B}(\mathbb{R}^2)$ is the $\sigma$-field generated by

$C$, i.e. $\sigma(C)$. $\blacksquare$

Similar to $\mathcal{B}(\mathbb{R})$, $\mathcal{B}(\mathbb{R}^2)$ can be constructed using rectangles in $\mathbb{R}^2$



i.e. it consists of countable unions of rectangles in $\mathbb{R}^2$.

Also, any countable subset of $\mathbb{R}^2$, e.g. $\mathbb{N} \times \mathbb{Z}$, is a member of $\mathcal{B}(\mathbb{R}^2)$, i.e. is a Borel set.

Similar to the one dimensional case, we have the following theorem:

Theorem: Let $X$ and $Y$ be two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

Then $(X(\cdot), Y(\cdot)) : \Omega \to \mathbb{R}^2$ is $\mathcal{F}$-measurable, i.e., the pre-images of Borel sets on $\mathbb{R}^2$ under $(X(\cdot), Y(\cdot))$ are events.

Since the pre-images of Borel sets on $\mathbb{R}^2$ are events, we can assign probabilities to them. This leads us to the definition of the joint probability law:

Def: The joint probability law of the random variables $X$ and $Y$ is defined as:

$$\mathbb{P}_{X,Y}(B) = \mathbb{P}(\{\underset{\in\Omega}{\omega} \mid (X(\omega), Y(\omega)) \in B\})$$

$\forall B \in \mathcal{B}(\mathbb{R}^2)$

In particular when $B = (-\infty, x] \times (-\infty, y]$, $\mathbb{P}_{X,Y}$ defines the joint CDF of $X$ and $Y$.

Def.: Let $X$ and $Y$ be two random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The joint cdf of $X$ and $Y$ is defined as follows:

$$F_{X,Y}(x,y) = \mathbb{P}_{X,Y}\big((-\infty, x] \times (-\infty, y]\big)$$

$$= \mathbb{P}\big(\{\omega \mid X(\omega) \leq x, Y(\omega) \leq y\}\big) \quad \forall x, y \in \mathbb{R}$$

Properties of joint CDF:

1. $\lim\limits_{\substack{x \to \infty \\ y \to \infty}} F_{X,Y}(x,y) = 1$

$\lim\limits_{\substack{x \to -\infty \\ y \to -\infty}} F_{X,Y}(x,y) = 0$

2. Monotonocity: $x_1 \leq x_2$, $y_1 \leq y_2$

$\Rightarrow F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)$

3. $F_{X,Y}$ is right continuous, i.e.

$\lim\limits_{\substack{u \to 0^+ \\ v \to 0^+}} F_{X,Y}(x+u, y+v) = F_{X,Y}(x,y)$

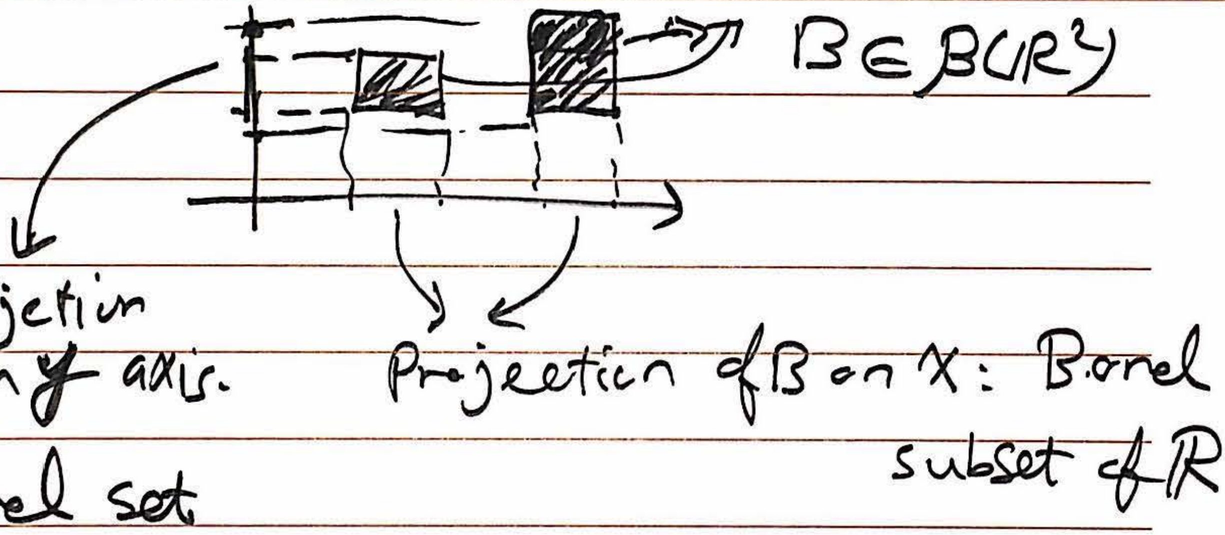
$\forall x, y \in \mathbb{R}$

4. $\lim_{y \to \infty} F_{X,Y}(x,y) = F_X(x)$

$\lim_{x \to \infty} F_{X,Y}(x,y) = F_Y(y)$

Proof: left as exercise. Define appropriate sets of events.

Question: Since Borel subsets of $\mathbb{R}^2$ are countable unions of rectangles, are their projections on the axes also Borel sets?




Answer: The answer to this question intuitively seems to be positive. However, in general, the projection of $B \in \mathcal{B}(\mathbb{R}^2)$ on $\mathbb{R}$, ~~is~~ can be a non-Borel set.

(To see an example of a

non-Borel set, see the handout on Non-measurable sets)

Interestingly, Lebesgue, who is famous in measure theory (a general theory that makes the foundations of probability theory), mistakenly answered this question "yes" in a paper he wrote in 1905, and called it "obvious"!

His mistake was discovered by Suslin, and gave rise to the so-called descriptive set theory.