Sep 4, 2025

## Logical Quantifiers

- Universal Quantifier $\forall$ : any, for all
- Existential Quantifier $\exists$ : exists, at least one

Negation of statements with Logical Quantifier

Negation of $\exists$

$\neg(\exists x, p(x)) \equiv (\nexists x, p(x)) \equiv (\forall x, \neg p(x))$

ig) $\neg(\exists x \in \mathbb{R}$ s.t. $x^2 = -1)$

$\equiv (\nexists x \in \mathbb{R}$ s.t. $x^2 = -1)$

$\equiv (\forall x \in \mathbb{R}$ s.t. $x^2 \neq -1)$

Negation of $\forall$

$\neg(\forall x, p(x)) \equiv (\exists x, \neg p(x))$

ig) $\neg(\forall n \in \mathbb{N}, \frac{n}{2} \in \mathbb{N})$

$\equiv (\exists n \in \mathbb{N}, \frac{n}{2} \notin \mathbb{N})$

## Set Theory

element of set : $x \in A$

set with no element : empty set / null set $\phi$

$\Rightarrow x \in \phi$ then $\forall x$, is false

$A = \{x_1, x_2, x_3, \cdots, x_n\}$ : finite number of elements

$A = \{x \mid p(x)\}$ : with predicate

Subsets (Super sets) $A \subseteq B \Rightarrow (x \in B \Rightarrow x \in A)$

Equivalent $A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B)$

Proper Subset hood

$A \subseteq B$ $x \in A \Rightarrow x \in B$ A can be equal to B

$A \subset B$ $(x \in A \Rightarrow x \in B) \wedge (A \neq B)$

The universe of discourse $\Omega$ : contains all elements that could conceivably be of interest in a particular context to

Complement of a set $A^c = \{x \in \Omega \mid x \notin A\}$

$\Omega^c = \{x \in \Omega \mid x \notin \Omega\} = \phi$

Operations on sets

Union : $A \cup B = \{x \in \Omega \mid x \in A \vee x \in B\}$

Intersection : $A \cap B = \{x \in \Omega \mid x \in A \wedge x \in B\}$

Set difference $A \setminus B = \{x \in A \mid x \notin B\}$

$A \setminus B = A \cap B^c = A \setminus (A \cap B)$

with index set $I = \mathbb{N}$

$\bigcup_{i \in I} A_i = \{x \in \Omega \mid \exists_{i \in I} \text{ s.t } x \in A_i\}$

$\bigcap_{i \in I} A_i = \{x \in \Omega \mid \forall_{i \in I} \text{ s.t. } x \in A_i\}$

Disjoint/Mutually Exclusive sets

, $A_1 \cap A_2 = \phi$ , $A_i \cap A_j = \phi$ , $\forall_{i \neq j}$

Partition : $A_1, A_2, \cdots$ consist a partition of B iff

1) they are mutually exclusive $A_i \cap A_j = \phi$ , $\forall_{i \neq j}$

2) $\bigcup_{i \in I} A_i = B$

## Properties of Set Operations

| | | |
|---|---|---|
| Commutative | $A \cap B = B \cap A$ | $p \wedge q \equiv q \wedge p$ |
| | $A \cup B = B \cup A$ | $p \vee q \equiv q \vee p$ |
| Associativity | $A \cap (B \cap C) = (A \cap B) \cap C$ | $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ |
| | $A \cup (B \cup C) = (A \cup B) \cup C$ | $p \vee (q \vee r) \equiv (p \vee q) \vee r$ |
| Distributivity | $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ |
| | $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ |
| Complement Laws | $(A^c)^c = A$ | $\neg(\neg p) \equiv p$ |
| | $A \cup A^c = \Omega$ | $p \vee \neg p \equiv T$ |
| | $A \cap A^c = \phi$ | $p \wedge \neg p \equiv F$ |
| Identity Laws | $A \cup \phi = A$ | $p \vee F \equiv p$ |
| | $A \cap \Omega = A$ | $p \wedge T \equiv p$ |
| Domination Laws | $A \cup \Omega = \Omega$ | $p \vee T \equiv T$ |
| | $A \cap \phi = \phi$ | $p \wedge F \equiv F$ |
| Idempotency | $A \cup A = A$ | $p \vee p \equiv p$ |
| | $A \cap A = A$ | $p \wedge p \equiv p$ |
| Absorption Laws | $A \cup (A \cap B) = A$ | $p \vee (p \wedge q) \equiv p$ |
| | $A \cap (A \cup B) = A$ | $p \wedge (p \vee q) \equiv p$ |
| De Morgan's Laws | $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$ | $\neg(p \vee q) \equiv \neg p \wedge \neg q$ |
| | $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$ | $\neg(p \wedge q) \equiv \neg p \vee \neg q$ |
| | $\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$ | |
| | $\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$ | |

## Element Chasing

to prove $A=B$ , prove $(A\subseteq B)\wedge(B\subseteq A)$

$\Rightarrow$ prove $(x\in A\Rightarrow x\in B)\wedge(x\in B\Rightarrow x\in A)$

i.g.) proof of $(A\cap B)^c = A^c\cup B^c$

$$
\begin{aligned}
x\in(A\cap B)^c &\Rightarrow x\notin A\cap B \\
&\Rightarrow \neg(x\in A\wedge x\in B) \\
&\Rightarrow \neg(x\in A)\vee\neg(x\in B) \\
&\Rightarrow (x\notin A)\vee(x\notin B) \\
&\Rightarrow (x\in A^c)\vee(x\in B^c) \\
&\Rightarrow x\in A^c\cup B^c
\end{aligned}
$$

$$
\begin{aligned}
x\in A^c\cup B^c &\Rightarrow x\in A^c\vee x\in B^c \\
&\Rightarrow x\notin A\vee x\notin B \\
&\Rightarrow \neg(x\in A)\vee\neg(x\in B) \\
&\Rightarrow \neg(x\in A\wedge x\in B) \\
&\Rightarrow \neg(x\in A\cap B) \\
&\Rightarrow x\notin A\cap B \\
&\Rightarrow x\in(A\cap B)^c
\end{aligned}
$$

$\therefore A^c\cup B^c=(A\cap B)^c$ Q.E.D.

i.g.) proof of $\left(\bigcup_{i\in I}A_i\right)^c=\bigcap_{i\in I}A_i^c$

$$
\begin{aligned}
x\in\left(\bigcup_{i\in I}A_i\right)^c &\Rightarrow \neg\left(x\in\bigcup_{i\in I}A_i\right)\Rightarrow\neg(\exists_{i\in I}\ s.t.\ x\in A_i) \\
&\Rightarrow (\forall_{i\in I}\ s.t.\ x\notin A_i) \\
&\Rightarrow x\in\bigcap_{i\in I}A_i^c
\end{aligned}
$$

$$
\begin{aligned}
x\in\bigcap_{i\in I}A_i^c &\Rightarrow \forall_{i\in I}\ s.t.\ x\notin A_i \\
&\Rightarrow \neg(\exists_{i\in I}\ s.t.\ x\in A_i) \\
&\Rightarrow \neg\left(x\in\bigcup_{i\in I}A_i\right) \\
&\Rightarrow x\in\left(\bigcup_{i\in I}A_i\right)^c
\end{aligned}
$$

$\therefore\left(\bigcup_{i\in I}A_i\right)^c=\bigcap_{i\in I}A_i^c$

General form of Distributivity

$$\left(\bigcap_{i\in I} A_i\right)\cup B = \bigcap_{i\in I}(A_i \cup B)$$

$$\left(\bigcup_{i\in I} A_i\right)\cap B = \bigcup_{i\in I}(A_i \cap B)$$

Special Sets

$\mathbb{R}$ : real numbers

$\mathbb{R}^* = \overline{\mathbb{R}}$ extemeded real numbers

$\mathbb{Z}$ : Integers

$\mathbb{N}$ : Natural Numbers

Intervals

Cartesian Product

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \wedge a_2 \in A_2\}$$

$$A_1 \times A_2 \times \cdots \times A_n = \prod_{i=1}^{n} A_i = \{(a_1, a_2, \cdots, a_n) \mid a_i \in A_i,\ i \in \{1, 2, \cdots, n\}\}$$

Family of Subsets of A : the set of all subsets of $A$ : $2^A$

$|2^A| = 2^{|A|}$

Relations

$$K = \left(\prod_{i=1}^{n} A_i\right) \times B$$

any subset of K. $R \subseteq K$ is called relation

Function

if a relation maps each n-tuple to a unique member of B

$(x_1, x_2, \cdots x_n, y) \in f \wedge (x_1, \cdots, x_n, y') \in f$

$\Rightarrow y = y'$

$\prod_{i=1}^{n} A_i$ : domain of $f$

$B$ : co-domain of $f$

$f : A \to B$

Forward Image of $f: A \to B$

$f^{\rightarrow}: 2^A \to 2^B$

$\forall S \subseteq A \ (S \in 2^A)$

$f^{\rightarrow}(S) = \{y \in B \mid y = f(x), x \in S\}$

Inverse Image

$f^{\leftarrow}: 2^B \to 2^A$

$\forall T \subseteq B \ (T \in 2^B)$

$f^{\leftarrow}(T) = \{x \in A \mid f(x) \in T\}$