

## Lesson 6 Combinatorics

### Principles of Counting

(without counting!)

Probability calculations often involve

counting, especially when

we have equally likely outcomes

and finite sets. In such

situations, the probability of

an event is just the "size"

(cardinality) of that event

relative to the size

(cardinality) of the sample

space i.e.:

$$P(A) = \frac{|A|}{|\Omega|} \quad \forall A \subset \Omega$$

In this lesson, we review the

concepts of combinatorics, for

measuring the "size" of countable

events.

The Multiplication Principle

Consider a two-stage experiment.

the first stage can have

$m$  results  $a_1, a_2, \dots, a_m$

and the second stage can

have  $n$  results  $b_1, b_2, \dots, b_n$ .

The number of possible ways

to perform the first and the

second stage sequentially is

the number of possible  $(a_i, b_j)$

pairs, which is  $\boxed{\phantom{00}}$

More formally:

If  $\Omega = A \times B$ , then

$$|\Omega| = |A \times B| =$$

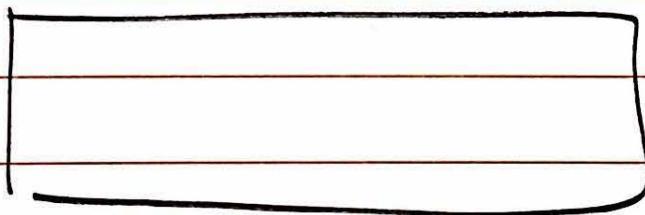
More generally,

If a process has  $r$  stages

and stage  $i$  has  $n_i$

possible results, the number

of all possible results is



In more formal terms, if

$\Omega = A_1 \times A_2 \times \cdots \times A_r$ , then

$$|\Omega| = |A_1 \times A_2 \times \cdots \times A_r| =$$

If  $\forall i, A_i = A$ ,  $|\Omega| = |A^k| =$

Example : Telephone numbers

are 7 digits in Springfield.

The first digit cannot be

1 or 0. How many different

telephone numbers can we have?

Solution:



Example: What is the number  
of subsets of a set with  $n$   
elements?



$\emptyset$

The multiplication principle helps us understand ordered sampling with replacement.

Example: Assume that we have  $n$  objects, and we would like to

sample  $k$  objects from them with replacement. Each time, we draw an object, make a note of it, put the object back and select the next object.

The number of different sequences of  $k$  objects that can be formed is:

$$\underbrace{n \times n \times \cdots \times n}_{k \text{ times.}} = n^k$$

Permutations: ~~are~~ when  $k$  objects are selected from a plurality of  $n$  objects, and order matters,  $k$ -permutations are constructed.

In other words,  $k$ -permutations

are subsequences of length  $k$ , selected

from sequences of length  $n$

$\underbrace{\text{places}}_k$      $\underline{\quad} \quad \underline{\quad} \quad \dots \quad \underline{\quad}$   
 $n \quad n-1 \quad n-2 \quad \dots \quad n-k+1$

In other words, permutations can

be considered as results of ordered

sampling without replacement

$$\Omega = A \times (A - \{\omega_1\}) \times (A - \{\omega_1, \omega_2\}) \times \dots \times (A - \{\omega_1, \dots, \omega_{k-1}\})$$

$$|\Omega| =$$

Example : Assume that

we would like to enumerate  
all possible <sup>binary</sup> sequences ~~sequences~~ that

can be created using  $(0, 0, 1, 1)$ .

$$m = 4$$

$$k = 3$$

$$\omega_1 =$$

$$\omega_2 =$$

$$\omega_3 =$$

$$\omega_4 =$$

Observe that  $\square$  is a

rearrangement of  $\square$  and

$\square$  is a rearrangement of  $\square$

Combinations: When we select ~~any~~

k objects from m objects, and

order does not matter for us,

we are considering combinations.

\* Order does not matter." This

means that all rearrangements

of a sequence of length

k should be considered

the same sequence.

Question: How many

rearrangements of a sequence  
of length  $k$  exist?

$k$  places  
 $k$  objects

number of permutations

Therefore, the number of

combinations of length  $k$

from  $n$  objects is

$$\binom{n}{k} = \frac{\# \text{ permutations}}{=} =$$

Combinations are useful

in understanding "unordered

sampling without replacement"

$\binom{n}{k}$  is also called the

binomial coefficient, because it

appears in the binomial theorem:

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2$$

$$+ \dots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n$$

The coefficients  $\binom{n}{k}$  can

be read from the so-called

# Khayyam-Pascal Triangle

The triangle works based on  
the following formula:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

The binomial theorem can be  
proved by induction.

Example: We have 5 novels,

3 Sci-fi books, 4 magazines,

and 8 college textbooks.

In how many different ways

can they be arranged so that

books of the same genre

are placed together?

Partitions: A combination is

a partition of  $m$  objects

into  $k$  and  $m-k$  objects.

We can generalize this discussion

to

dividing

$m$  objects into  $r$  partitions,

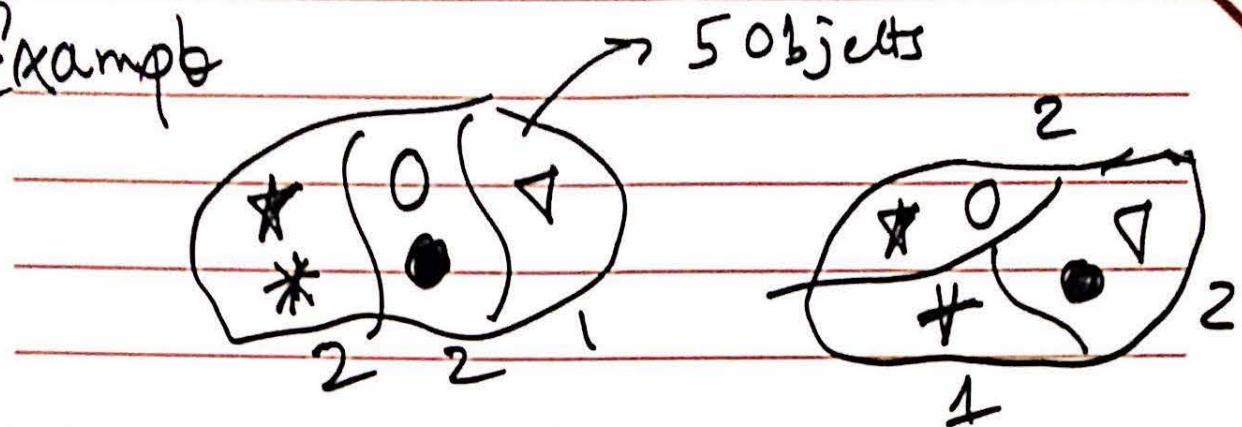
given that the  $i^{\text{th}}$  partition

has  $n_i$  elements.

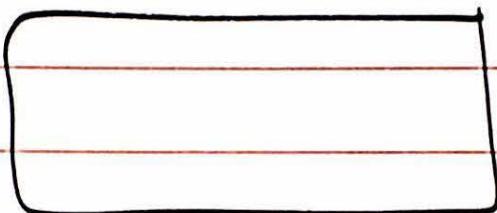
Obviously

$$n_1 + n_2 + \dots + n_r =$$

Example



Number of such partitions is



We <sup>also</sup> use the notation:

$$\binom{n}{n_1, n_2, \dots, n_r}$$

Interestingly,

$$\binom{n}{n_1 \ n_2 \ \dots \ n_r}$$

is called a "multinomial

coefficient," because it appears

in the multinomial theorem

$$(a_1 + a_2)^n = \sum_k \binom{n}{k, n-k} a_1^k a_2^{n-k}$$

$$(a_1 + a_2 + \dots + a_m)^n = \sum_{k_1, k_2, \dots, k_m} \binom{n}{k_1, \dots, k_m} a_1^{k_1} a_2^{k_2} \dots a_m^{k_m}$$

## Example (Anagrams)

How many different words  
(letter sequences) can be  
obtained by rearranging  
the letters in the word

"TATTOO?