Lesson 6

Combinatorics:

Principles of Counting

(without counting!)

Probability calculations often involve counting, especially when

we have equally likely outcomes and finite sets. In such situations, the probability of an event is just the "size" (cardinality) of that event relative to the size (cardinality) of the sample space, i.e.:

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} \quad \forall A \subseteq \Omega$$

In this lesson, we review the concepts of combinatorics, for measuring the "size" of countable events.

The Multiplication Principle

Consider a two-stage experiment.

the first stage can have

m results $a_1, a_2, \ldots, a_m$

and the second stage can

have n results $b_1, b_2, \ldots b_n$.

The number of possible ways

to perform the first and the

second stage sequentially is

the number of possible (a_i, b_j)

pairs, which is $\boxed{m \cdot n}$

More formally:

If $\Omega = A \times B$, then

$$|\Omega| = |A \times B| = |A||B|$$

if $|A| < \infty$, $|B| < \infty$

More generally

If a process has r stages and stage i has n_i possible results, the number of all possible results is

$$\boxed{n_1 \cdot n_2 \cdot \ldots \cdot n_r = \prod_{i=1}^{r} n_i}$$

In more formal terms, if

$\Omega = A_1 \times A_2 \times \cdots \times A_r$, then

$|\Omega| = |A_1 \times A_2 \times \cdots \times A_r| =$

$|A_1||A_2||A_3| \cdots |A_r|$

If $\forall i, A_i = A$, $|\Omega| = |A^k| = |A|^k$

Example: Telephone numbers are 7 digits in Springfield. The first digit cannot be 1 or 0. How many different telephone numbers can we have?

Solution:

$\underset{8}{\_}\ \underset{10}{\_}\ \underset{10}{\_}\ \underset{\cdots}{\_}\ \_\ \_\ \underset{10}{\_} = 8\times10^6$

Example: What is the number of subsets of a set with n elements?

$\_\ \_\ \_\ \_\ \cdots\ \_\ \_$

$\emptyset$

$|2^{\Omega}| = 2^{|\Omega|}$

1

The multiplication principle helps us understand ordered sampling with replacement.

Example: Assume that we have n objects, and we would like to sample k objects from them with replacement. Each time, we draw an object, make a note of it, put the object back and select the next object.

The number of different sequences of k objects that can be formed is:

$$\underbrace{n \times n \times \cdots \times n}_{k \text{ times.}} = n^k$$

Permutations: when k objects are selected from a plurality of n objects, and order matters, k-permutations are constructed.

In other words, k-permutations

are subsequences of length k, selected

from sequences of length n

k places $\_\ \_\ \_\ \ldots\ \_$

$n \quad n-1 \quad n-2 \quad \cdots \quad n-k+1 = \frac{n!}{(n-k)!}$

In other words, permutations can

be considered as results of <u>ordered</u>

<u>sampling without replacement</u>

$$\Omega = A \times (A - \{\omega_1\}) \times (A - \{\omega_1, \omega_2\}) \times \cdots \times A - \{\omega_1, \ldots, \omega_{k-1}\}$$

$$|\Omega| = \frac{n!}{(n-k)!}$$

Example : Assume that we would like to enumerate all possible binary sequences that can be created using $(0,0,1,1)$.

$n = 4$

$k = 3$

$\omega_1 =$ $\omega_2 =$

$\omega_3 =$ $\omega_4 =$

Observe that $\square$ is a rearrangement of $\square$ and $\square$ is a rearrangement of $\square$

Combinations: when we select

k objects from n objects, and

order does not matter for us,

we are considering combinations.

* "Order does not matter." This

means that all re-arrangements

of a sequence of length

k should be considered

the same sequence.

Question: How many

rearrangements of a sequence of length k exist?

$\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad} \cdots \underline{\quad}$

k places $k \quad k-1 \quad k-2 \qquad\qquad 1 = k!$

k objects $\qquad \dfrac{k!}{(k-k)!} = k!$

number of permutations

Therefore, the number of combinations of length k from n objects is

$$\binom{n}{k} = \frac{\#\text{ permutations}}{k!} = \frac{n!}{(n-k)!} \cdot \frac{1}{k!} = \binom{n}{k}$$

Combinations are useful in understanding "unordered sampling without replacement"

$\binom{n}{k}$ is also called the binomial coefficient, because it appears in the binomial theorem:

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

The coefficients $\binom{n}{k}$ can be read from the so-called

proof by mathmetical induction

Khayyam- Pascal Triangle

$$
\begin{array}{ccccccccccc}
 & & & & 1 & & & & & & n=0 \\
 & & & 1 & & 1 & & & & & n=1 \\
 & & 1 & & 2 & & 1 & & & & n=2 \\
 & 1 & & 3 & & 3 & & 1 & & & n=3 \\
1 & & 4 & & 6 & & 4 & & 1 & & n=4 \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & &
\end{array}
$$

~~Pas~~ The triangle works based on the following formula:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

The binomial theorem can be proved by induction.

${}_nC_k = {}_{n-1}C_{k-1} + {}_{n-1}C_k$

Example: We have 5 novels, 3 Sci-fi books, 4 magazines, and 8 college textbooks. In how many different ways can they be arranged so that books of the same genre are placed together?

4 genre : $5!\ 3!\ 4!\ 3!$

$\llcorner$ $4!$ $\Rightarrow 5!3!4!3!4!$

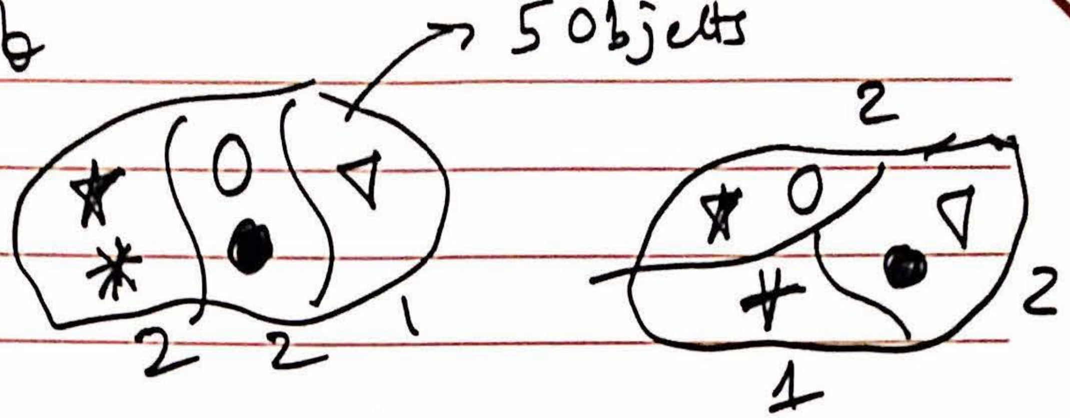
Partitions: A combination is a partition of n objects into k and $n-k$ objects.

We can generalize this discussion to dividing n objects into r partitions, given that the i^{th} partition has n_i elements.

Obviously

$$n_1 + n_2 + \cdots + n_r = n$$

Example



Number of such partitions is

$$\boxed{\frac{n!}{n_1!\, n_2! \cdots n_r!}}$$

prove by induction on r

We also use the notation:

$$\binom{n}{n_1\ n_2\ \cdots\ n_r}$$

Interestingly,

$$\binom{n}{n_1 \; n_2 \; \cdots \; n_r}$$

is called a "multinomial coefficient," because it appears in the multi-nomial theorem

$$(a_1 + a_2)^n = \sum_k \binom{n}{k, n-k} a_1^k a_2^{n-k}$$

$$(a_1 + a_2 + \cdots + a_m)^n = \sum_{k_1, k_2, \ldots, k_m}^{n} \binom{n}{k_1, \ldots, k_m} a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m}$$

Example (Anagrams)

How many different words (letter sequences) can be obtained by rearranging the letters in the word

" TATTOO?

$$\frac{6!}{3!\,2!}$$