~~Home~~work 1

DSCI 564

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1. $\{A_i \mid i \in I\}$. show De Morgan's Laws.

a) $(\bigcup_{i\in I} A_i)^c = \bigcap_{i\in I} A_i^c$

i) $x \in (\bigcup_{i\in I} A_i)^c \Rightarrow \neg(x \in \bigcup_{i\in I} A_i) \Rightarrow \neg(\exists_{i\in I}, x \in A_i) \Rightarrow (\forall_{i\in I}, x \notin A_i)$

$\Rightarrow (\forall_{i\in I}, x \in A_i^c) \Rightarrow x \in \bigcap_{i\in I} A_i^c$

ii) $x \in \bigcap_{i\in I} A_i^c \Rightarrow (\forall_{i\in I}, x \in A_i^c) \Rightarrow \neg(\exists_{i\in I}, x \notin A_i^c) \Rightarrow \neg(\exists_{i\in I}, x \in A_i)$

$\Rightarrow \neg(x \in \bigcup_{i\in I} A_i) \Rightarrow x \in (\bigcup_{i\in I} A_i)^c$

$\therefore (\bigcup_{i\in I} A_i)^c = \bigcap_{i\in I} A_i^c$

b) $(\bigcap_{i\in I} A_i)^c = \bigcup_{i\in I} A_i^c$

i) $x \in (\bigcap_{i\in I} A_i)^c \Rightarrow \neg(x \in \bigcap_{i\in I} A_i) \Rightarrow \neg(\forall_{i\in I}, x \in A_i) \Rightarrow (\exists_{i\in I}, x \notin A_i)$

$\Rightarrow (\exists_{i\in I}, x \in A_i^c) \Rightarrow x \in \bigcup_{i\in I} A_i^c$

ii) $x \in \bigcup_{i\in I} A_i^c \Rightarrow (\exists_{i\in I}, x \in A_i^c) \Rightarrow \neg(\forall_{i\in I}, x \notin A_i^c) \Rightarrow \neg(\forall_{i\in I}, x \in A_i)$

$\Rightarrow \neg(x \in \bigcap_{i\in I} A_i) \Rightarrow x \in (\bigcap_{i\in I} A_i)^c$

$\therefore (\bigcap_{i\in I} A_i)^c = \bigcup_{i\in I} A_i^c$

Q.E.D.

2. $A \cup (A \cup B)^c = A \cup (A^c \cap B^c)$ $\because$ De Morgan's Laws.

$= (A \cup A^c) \cap (A \cup B^c) = \Omega \cap (A \cup B^c)$

$= A \cup B^c$

$\therefore (A \cup B^c = \phi) \Rightarrow (A \cup (A \cup B)^c = \phi)$

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3.

a) $A \subseteq B \Rightarrow (x \in A \Rightarrow x \in B)$

i) $(x \in A \Rightarrow x \in B) \Rightarrow (x \in B \Rightarrow x \in A \cup B) \Rightarrow B \subseteq A \cup B$.

ii) $(x \in A \Rightarrow x \in B) \Rightarrow (x \in A \cup B \Rightarrow x \in B) \Rightarrow A \cup B \subseteq B$

$\therefore \quad A \subseteq B \Rightarrow (A \cup B = B)$

b) $(A \cup B = B) \Rightarrow (x \in A \Rightarrow (x \in A) \vee (x \in B)) \Rightarrow (x \in A \Rightarrow x \in A \cup B)$

$\Rightarrow (x \in A \Rightarrow x \in B)$

$\therefore \quad (A \cup B = B) \Rightarrow A \subseteq B$

Therefore $\quad A \subseteq B \Leftrightarrow A \cup B = B$

4. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

a) $(x,y) \in A \times (B \cup C) \Rightarrow (x \in A) \wedge (y \in B \vee y \in C)$

$\Rightarrow [(x \in A) \wedge (y \in B)] \vee [(x \in A) \wedge (y \in C)]$

$\Rightarrow ((x,y) \in A \times B) \vee ((x,y) \in A \times C)$

$\Rightarrow (x,y) \in (A \times B) \cup (A \times C)$

b) $(x,y) \in (A \times B) \cup (A \times C) \Rightarrow (x,y) \in (A \times B) \vee (x,y) \in (A \times C)$

$\Rightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$

$\Rightarrow (x \in A) \wedge (y \in B \vee y \in C) \Rightarrow (x \in A) \wedge (y \in B \cup C)$

$\Rightarrow (x,y) \in A \times (B \cup C)$

$\therefore \quad A \times (B \cup C) = (A \times B) \cup (A \times C)$

5. Show if $A \subseteq B$ and $C \subset D$, then $(A \times C) \subset (B \times D)$

$A \subset B \Rightarrow (x \in A \Rightarrow x \in B) \wedge (A \neq B)$

$C \subset D \Rightarrow (y \in C \Rightarrow y \in D) \wedge (C \neq D)$

Suppose $(x, y) \in A \times C$

$\Rightarrow (x \in A) \wedge (y \in C)$

$\Rightarrow (x \in B) \wedge (y \in D) \wedge (A \neq B) \wedge (C \neq D) \quad \because (A \subset B) \wedge (C \subset D)$

$\Rightarrow [(x, y) \in (B \times D)] \wedge (A \times C) \neq (B \times D)$

$\therefore (A \subset B) \wedge (C \subset D) \Rightarrow (A \times C) \subset (B \times D)$

6. $A_1, \cdots A_m$ and $B_1, \cdots, B_n$ be partitions of a set Ω

$(\forall_{i \neq j} A_i \cap A_j = \emptyset) \wedge (\bigcup_{i \in I} A_i = \Omega)$

show collection of sets $A_i \cap B_j$; $i = 1, \cdots, m$, $j = 1, \cdots n$ is also a partition

$\forall_{(i,j) \neq (i',j')} (A_i \cap B_j) \cap (A_{i'} \cap B_{j'}) = \forall_{(i,j) \neq (i',j')} (A_i \cap A_{i'}) \cap (B_j \cap B_{j'}) = \emptyset$

$\bigcup_{\substack{i \in \{1, \cdots m\} \\ j \in \{1, \cdots n\}}} (A_i \cap B_j) = \bigcup_{i \in \{1, \cdots m\}} A_i \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = \bigcup_{i \in \{1, \cdots m\}} A_i \cap \Omega = \bigcup_{i \in \{1, \cdots m\}} A_i = \Omega$

$\therefore$ collection of sets $A_i \cap B_j$; $i = 1, \cdots m$, $j = 1, \cdots n$ is a partition

7. $\Omega := \{1, \cdots, 6\}$

if $p(\omega) = 2p(\omega-1)$ for $\omega = 2, \cdots, 6$ and if $\sum_{\omega=1}^{6} p(\omega) = 1$

, $p(\omega) = 2^{\omega-1}/63$.

if $p(\omega) = 2p(\omega-1)$ for $\omega = 2, \dots, 6$

we can express.

$$p(\omega) = 2^{\omega-1} \cdot p(1)$$

from $\sum_{\omega=1}^{6} p(\omega) = 1 \Rightarrow p(1) \frac{2^6-1}{2-1} = 1$

$$\therefore \quad p(1) = \frac{1}{63}$$

$$p(\omega) = 2^{\omega-1} \cdot \frac{1}{63}$$

8.

The finite union bound

$$P\left(\bigcup_{n=1}^{N} F_n\right) \leq \sum_{n=1}^{N} P(F_n)$$

$N=2 \quad p(F_1 \cup F_2) \leq P(F_1) + P(F_2)$

$$\because \quad p(F_1 \cup F_2) = \frac{|F_1 \cup F_2|}{|\Omega|} = \frac{|F_1| + |F_2| - |F_1 \cap F_2|}{|\Omega|} \leq \frac{|F_1| + |F_2|}{|\Omega|} = P(F_1) + P(F_2)$$

assume $N > 2 \quad p\left(\bigcup_{n=1}^{N} F_n\right) \leq \sum_{n=1}^{N} p(F_n)$

$$p\left(\bigcup_{n=1}^{N+1} F_n\right) = p\left(\bigcup_{n=1}^{N} F_n \cup F_{N+1}\right) = \frac{\left|\bigcup_{n=1}^{N} F_n \cup F_{N+1}\right|}{|\Omega|} \leq \frac{\left|\bigcup_{n=1}^{N} F_n\right| + |F_{N+1}|}{|\Omega|} = p\left(\bigcup_{n=1}^{N} F_n\right) + p(F_{N+1})$$

$$\leq \sum_{n=1}^{N} p(F_n) + p(F_{N+1}) = \sum_{n=1}^{N+1} p(F_n)$$

$$\therefore \quad p\left(\bigcup_{n=1}^{N} F_n\right) \leq \sum_{n=1}^{N} p(F_n)$$

9. The Infinite Union Bound (Boole's inequality)

$$P\left(\bigcup_{n=1}^{\infty} F_n\right) \le \sum_{n=1}^{\infty} P(F_n)$$

$$\lim_{N\to\infty} P\left(\bigcup_{n=1}^{N} F_n\right) \le \lim_{N\to\infty} \sum_{n=1}^{N} P(F_n) \quad \because P\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{N\to\infty} P\left(\bigcup_{n=1}^{N} F_n\right) \text{ and}$$

$$P\left(\bigcup_{n=1}^{N} F_n\right) \le \sum_{n=1}^{N} P(F_n)$$

10.

let A : first select card and see one side is blue

B : first selected other side color is blue (select card has both blue)

found $P(B|A)$

$P(A) = 1/2 \qquad P(B) = 1/3$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{P(B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}$$