1. Show the following for forward and inverse images of functions. Assume that I is an arbitrary (not necessarily finite or countably infinite) index set.[1] (20 pts)

   (a) $f^{\rightarrow}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f^{\rightarrow}(A_i)$

   (b) $f^{\rightarrow}(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} f^{\rightarrow}(A_i)$ (if f is one-to-one, equality occurs)[2]

   (c) $f^{\leftarrow}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f^{\leftarrow}(A_i)$

   (d) $f^{\leftarrow}(A^c) = (f^{\leftarrow}(A))^c$

2. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be σ-fields of subsets of Ω. Show that the intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ is also a σ-fields of subsets of Ω. (10 pts)

3. Let $A, B \in \mathcal{F} \subseteq 2^{\Omega}$, where $\mathcal{F}$ is a σ-field. Show that $A \cap B \in \mathcal{F}$, $A\backslash B \in \mathcal{F}$, and $A\Delta B = (A\backslash B) \cup (B\backslash A) \in \mathcal{F}$. (15 pts)

4. Show the formula $\mathbb{P}\left((A \cap B^c) \cup (A^c \cap B)\right) = \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)$, which gives the probability that exactly one of the events A and B will occur. [Compare with the formula $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$, which gives the probability that at least one of the events A and B will occur. Also, do you see any anaolgy with XOR in logic?] (15 pts)

5. Gubner Chapter 1, 24. (15 pts)

6. Gubner Chapter 1, 29. (15 pts)

7. Gubner Chapter 1, 30. (15 pts)

8. Dobrow 1.41. (15 pts)

9. Reading Assignment (useful for problem 8): Dobrow, p. 23-25.

10. (Extra practice. Not graded and you don't need to submit the solutions: Bertsekas and Tsitsiklis: 1.5, 1.9. Grimmet and Stirzaker: 1.2.5, 1.3.1, 1.3.2, 1.3.5, 1.3.7, Dobrow: 1.43, 1.45)

[1] Important Note: Posting the homework and its solutions to online forums or sharing it with other students is strictly prohibited. Instances will be reported to USC officials as academic dishonesty for disciplinary action.

[2] In order to see why in general $f^{\rightarrow}(\bigcap_{i\in I} A_i) \not\supseteq \bigcap_{i\in I} f^{\rightarrow}(A_i)$, one needs to know the following theorem in predicate calculus: $\exists y\ \forall x\ p(x, y)$ implies $\forall x\ \exists y\ p(x, y)$ but not conversely; One can change the order of $\exists\ \forall$, but cannot do that for $\forall\ \exists$. A standard way to illustrate this is with the love predicate: xLy iff x loves y: If there is someone (Raymond) whom everyone loves then everyone loves someone–but not conversely!