Sep 11, 2025

General Baye's Rule

assume that $(\Omega, \mathcal{F}, P)$ and $A, B_i \in \mathcal{F}$ $\forall i \in \mathbb{N}$

$\forall i \in \mathbb{N}$, $P(B_i) > 0$

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)} = \frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{\infty} P(A|B_i)P(B_i)}$$

Multiplication Rule

$(\Omega, \mathcal{F}, P)$ be a probability space and $A_1, A_2, \cdots, A_n$ be events

$$P(\bigcap_{i=1}^{n} A_i) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cdots \cap A_{n-1})$$

$$= P(A_1)\prod_{i=2}^{n} P(A_i|A_1 \cap A_2 \cdots \cap A_{i-1}) \qquad , P(\bigcap_{j=1}^{\ell} A_j) \neq 0$$

Proof (induction)

$n=1$ $P(A_1) = P(A_1)$

$n=2$ $P(A_1 \cap A_2) = P(A_1)P(A_2|A_1)$

assume $P(\bigcap_{i=1}^{k} A_i) = P(A_1)\prod_{i=2}^{k} P(A_i|A_1 \cap A_2 \cdots A_{i-1})$

show that $P(\bigcap_{i=1}^{k+1} A_i) = P(A_1)\prod_{i=2}^{k+1} P(A_i|A_1 \cap A_2 \cdots \cap A_{i-1})$

$$P(\bigcap_{i=1}^{k+1} A_i) = P(\bigcap_{i=1}^{k} A_i \cap A_{k+1}) = P(\bigcap_{i=1}^{k} A_i)\, P(A_{k+1}|\bigcap_{i=1}^{k} A_i)$$

$$= P(A_1)\prod_{i=2}^{k} P(A_i|A_1 \cap A_2 \cdots A_{i-1}) \cdot P(A_{k+1}|\bigcap_{i=1}^{k} A_i)$$

$$= P(A_1)\prod_{i=2}^{k+1} P(A_{k+1}|\bigcap_{i=1}^{k} A_i)$$

General Multiplication Rule

$A_1, A_2, \cdots$ where $A_i \in \mathcal{F}$ $\forall i \in \mathbb{N}$

$$P(\bigcap_{i=1}^{\infty} A_i) = P(A_1)\prod_{i=2}^{\infty} P(A_i|A_1 \cap \cdots \cap A_{i-1})$$

provided that $P(\bigcap_{j=1}^{\ell} A_j) \neq 0$ $\forall \ell \in \mathbb{N}$

Proof

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)\prod_{i=2}^{n} P(A_i \mid A_1 \cap A_2 \cap \cdots \cap A_{i-1})$$

$$\lim_{n\to\infty} P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)\prod_{i=2}^{\infty} P(A_i \mid A_1 \cap A_2 \cdots \cap A_{i-1})$$

continuity of probability

$$= P\left(\lim_{n\to\infty}\bigcap_{i=1}^{n} A_i\right) = P\left(\bigcap_{i\in\mathbb{N}} A_i\right)$$

Independence

$P(A\cap B) = P(A)\cdot P(B)$

and if $P(B)>0$

$$P(A\mid B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

; event independent from itself

$P(A) = P(A\cap A) = P(A)^2$

$P(A)=0$ or $P(A)=1$

Example $(\Omega, \mathcal{F}, P)$ probability space

$A, B \in \mathcal{F}$ and $0 < P(B) < 1$

if A, B are independent

show $P(A\mid B) = P(A\mid B^c)$

$A = (A\cap B)\cup(A\cap B^c)$

$\rightarrow P(A) = P(A\cap B) + P(A\cap B^c)$ $\because (A\cap B)\cap(A\cap B^c) = (A\cap A)\cap(A\cap B^c) = \phi$

$= P(A\mid B)\cdot P(B) + P(A\mid B^c)P(B^c)$

$P(A) = P(A\mid B)P(B) + P(A\mid B^c)(1-P(B))$

$P(A) - P(A)P(B) = P(A\mid B^c)(1-P(B))$

$\because 1 > P(B) > 0 \Rightarrow 1-P(B) > 0$

$P(A) = P(A\mid B^c)$

$\therefore P(A) = P(A\mid B) = P(A\mid B^c)$

Exercise show A^c is independent from B and B^c

$\Rightarrow$ σ-field generated by A is independent from σ-field generated by B

$F_A = \{\phi, A, A^c, \Omega\}$

$F_B = \{\phi, B, B^c, \Omega\}$

every member of F_A is independent from every member of F_B

Conditional Independence

assume (Ω, F, P) is a probability space and C is F-measurable

$C \in F \quad , P(C) > 0$

$A, B \in F$

$$P(A \cap B | C) = P(A|C) P(B|C)$$

Independence for more than two events

$\{A_i \in F \mid i \in I\}$

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i) \quad \forall J \subseteq I, \ |J| < \infty$$

Pairwise Independence

$\{A_i \in F \mid i \in I\}$

$$P(A_i \cap A_j) = P(A_i) P(A_j) \quad \forall i \neq j$$

Independence $\Rightarrow$ pairwise independence

if $\{A_i | i \in I\}$ are independence

$\{A_i, A_j | i \neq j\}$, $\{A_i, A_j, A_k | i, j, k$ are distinct$\}$ $, \cdots$ must be independent

Exercise

mutually exclusive events B_1, B_2, B_3 and C

on the probability space (Ω, F, P)

with $P(B_1) = P(B_2) = P(B_3) = p$ and $P(C) = q$

where $3p + q \leq 1$

assume that $p = -q + \sqrt{q}$ and show that the events

$B_1 \cup C$, $B_2 \cup C$, and $B_3 \cup C$ are pairwise independent

pairwise independent

$P((B_i \cup C) \cap (B_j \cup C)) = P(B_i \cup C) P(B_j \cup C) = [P(B_i) + P(C)][P(B_j) + P(C)] = (p+q)^2 = q$

$= P((B_i \cap B_j) \cup C) = P(C) = q$

Independent

$P((B_1 \cup C) \cap (B_2 \cup C) \cap (B_3 \cup C)) = P((B_1 \cap B_2 \cap B_3) \cup C) = P(C) = q$

$= P(B_1 \cup C) P(B_2 \cup C) P(B_3 \cup C) = (p+q)^3 = q\sqrt{q}$

$q = q\sqrt{q}$ if $q = 0$ or $q = 1$

$q > 0 \Rightarrow 3p + q \leq 1$

$\therefore q < 1$

$\therefore$ both not acceptable

The Borel-Cantelli Lemmas

event that occurs infinitely often

(Ω, F, P) probability space and $A_i \in F$, $\forall i \in \mathbb{N}$

define $B_1 = \bigcup_{i=1}^{\infty} A_i$

$B_2 = \bigcup_{i=2}^{\infty} A_i$

$\vdots$

$B_n = \bigcup_{i=n}^{\infty} A_i$

$A_{i.o.}$: event that occurs infinitely often

$x \in A_{i.o.}$

$\forall n, x \in B_n \Rightarrow x \in \bigcap_{i=1}^{\infty} B_i$

$\Rightarrow A_{i.o.} \subseteq \bigcap_{i=1}^{\infty} B_i = \bigcap_{i=1}^{\infty} \bigcup_{i=n}^{\infty} A_n$

Lemma : Borel-Cantelli

$$A_{i.o.} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$$

1) $\sum_{n=1}^{\infty} P(A_i) < \infty \Rightarrow P(A_{i.o.}) = 0$

2) $\sum_{n=1}^{\infty} P(A_i) = \infty$ and A_i's are independent

$\Rightarrow P(A_{i.o.}) = 1$

Proof : $A_{i.o.} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$

$A_{i.o.} \subseteq B_n$

$\Rightarrow P(A_{i.o.}) \leq P(B_n) = P(\bigcup_{i=n}^{\infty} A_i) \leq \sum_{i=n}^{\infty} P(A_i)$

$\lim_{n\to\infty} P(A_{i.o.}) \leq \lim_{n\to\infty} \sum_{i=n}^{\infty} P(A_i)$

Example: Erdös – Rényi

$A_i = \{H \text{ occurs in the } i^{th} \text{ coin toss}\}$

1) Assume that $P(A_i) = 1/i$

$\Rightarrow \sum_{i=1}^{\infty} P(A_i) = \infty$

then $P(A_{i.o.}) = 1$

2) $P(A_i) = (1/2)^i$

$\sum_{i=1}^{\infty} P(A_i) = 1$

then $P(A_{i.o.}) = 0$