1. Let X and Y be bivariate normal random variables with the density function: (20 pts)

$$f_{X,Y}(x,y) = \frac{1}{\pi\sqrt{3}} \exp\left\{-\frac{2}{3}(x^2 - xy + y^2)\right\}$$

Show that X and $Z = (2Y - X)/\sqrt{3}$ are independent $N(0,1)$ random variables, and show that $\mathbb{P}(X > 0, Y > 0) = 1/3$.

2. Suppose that there are a fixed number of N light bulbs in a box. The lifetime of bulb n (in months) has the exponential distribution with rate parameter n. A bulb is selected at random from the box and tested.

   (a) Find the probability that the selected bulb will last more than one month. (10 pts)

   (b) Given that the bulb lasts more than one month, find the conditional probability mass function of the bulb number. (10 pts)

3. Let X and Y be jointly (bivariate) normal, with $\text{Var}(X) = \text{Var}(Y)$. Show that the two random variables $X + Y$ and $X - Y$ are independent. (10 pts)

4. Let X and Y be jointly normal random variables with parameters $\mu_X = 0, \sigma_X^2 = 1$ and $\mu_Y = -1, \sigma_Y^2 = 4$ and $\rho = -1/2$. Using the normal cdf table:

   (a) Find $\mathbb{P}(X + Y > 0)$ (5 pts)

   (b) Find the constant a if we know $aX + Y$ and $X + 2Y$ are independent. (10 pts)

   (c) Find $\mathbb{P}(X + Y > 0 | X + Y > -3)$. You can use normal cdf tables. (10 pts)

5. Gubner Chapter 7, Problem 57 (10 pts)

6. Gubner Chapter 8, Problem 10 (10 pts)

7. Gubner Chapter 8, Problem 11 (10 pts)

8. Simulate problem 2.

9. Simulate $\mathbb{P}(X + Y > 0 | X + Y > -1)$ in problem 4c.

10. (Extra Practice) Bertsekas and Tsitsiklis: 3.31, 3.34, 3.35. Grimmet and Stirzaker: 4.6.10. Hwei Hsu-Schaum's Outline of Probability, Random Variables and Random Processes: Solved problems in chapter 3.