

Lesson 3

Review of Set Theory

A set is a collection of objects, which are its elements
(This is not a definition)
("Set" is a primitive concept)

- We show the elements of a set using \in
 $x \in A$
- A set with no elements is called an empty set or null set \emptyset

$x \in \emptyset \rightarrow x$, is false

We can show a set with its elements

$$A = \{x_1, x_2, x_3, \dots, x_n\}$$

A finite # of elements

Or with a predicate

$$A = \{x \mid P(x)\}$$

Ex:

$$\{x \in \mathbb{N} \mid x > 5^3\}$$

Subsets (super sets)

$$A \subseteq B \Rightarrow (x \in A \Rightarrow x \in B)$$

Equivalence:

$$A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B)$$

Proper subset hood

$A \subseteq B$ means $x \in A \Rightarrow x \in B$
 A can be equal to B

$A \subset B$ means
 $(x \in A \Rightarrow x \in B)$
 $\wedge (A \neq B)$

$$\Rightarrow \exists y \in B, y \notin A$$

"A: proper subset of B"

Universe of Discourse

The Universe of Discourse Ω (or \mathcal{U})

Contains all elements that

could conceivably be of interest

in a particular context.

Ex: In number theory,

$$\Omega = \mathcal{U} = \mathbb{Z} = \{-\dots, -3, -2, -1, 0, 1, 2, \dots\}$$

One can then consider

all sets of interest as subsets

of the Universe of Discourse

Complement of A Set

(with respect to Ω).

Complement of A : A^c

$$A^c = \{x \in \Omega \mid x \notin A\}$$

$$\text{Obviously, } \Omega^c = \{x \in \Omega \mid x \notin \Omega\} = \emptyset$$

Operations on Sets:

Union :

$$A \cup B = \{x \in \Omega \mid x \in A \vee x \in B\}$$

Intersection:

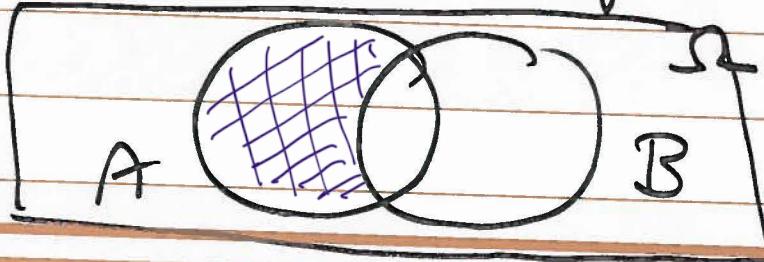
$$A \cap B = \{x \in \Omega \mid x \in A \wedge x \in B\}$$

Set Difference $A \setminus B$ (or relative
 $A - B$)

Complement of B in A

$$A \setminus B = \{x \in A \mid x \notin B\}$$

Venn Diagram of $A \setminus B$



$$A \setminus B = A \cap B^c$$

$$= A \setminus (A \cap B)$$

More generally, assume that

I is an index set, e.g.

$$I = \mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$$

$$I = \mathbb{Q} = \left\{ \frac{b}{a} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$$

$$I = \mathbb{R} \quad (\text{possibly infinite})$$

$$\bigcup_{i \in I} A_i = \{x \in \Omega \mid \exists_{i \in I} \text{ st. } x \in A_i\}$$

$$\bigcap_{i \in I} A_i = \{x \in \Omega \mid \forall_{i \in I} \text{ st. } x \in A_i\}$$

Example: $A_i = [0, i)$, $i \in \mathbb{N}$

$$A_1 = [0, 1) \quad A_2 = [0, 2) \quad A_3 = [0, 3)$$

$$\bigcap_{i \in \mathbb{N}} A_i = [0, 1)$$

$$\bigcup_{i \in \mathbb{N}} A_i = [0, +\infty) = \mathbb{R}^{>0}$$

Exercise:

$$\bigcap_{i \in \mathbb{N}} (-\infty, \frac{1}{i}) = (-\infty, 0]$$

$$\forall i \in \mathbb{N} \quad 0 \in A_i$$

$$\Rightarrow 0 \in \bigcap_{i \in \mathbb{N}} A_i$$

Exercise :

$$\bigcup_{i \in \mathbb{N}} (-\infty, -1/i] = (-\infty, 0)$$

$$0 \notin A_i \quad \forall i$$

Exercise :

$$\bigcap_{i \in \mathbb{N}} [0, 1/i) = \{0\}$$

$$\forall i \quad \{0\} \notin A_i$$

Exercise :

$$\bigcup_{i \in \mathbb{N}} [-i, i] = (-\infty, \infty) = \mathbb{R}$$

Exercise: $\bigcap_{i \in \mathbb{N}} (-\alpha, -i) = \emptyset$

Disjoint / Mutually Exclusive Sets

Def: A_1, A_2 disjoint $\Leftrightarrow A_1 \cap A_2 = \emptyset$

Example:

Def: A_1, A_2, \dots are mutually disjoint

$$\Leftrightarrow A_i \cap A_j = \emptyset \quad \forall i \neq j$$

Def: Partition; empty set is exclusive

Non-empty

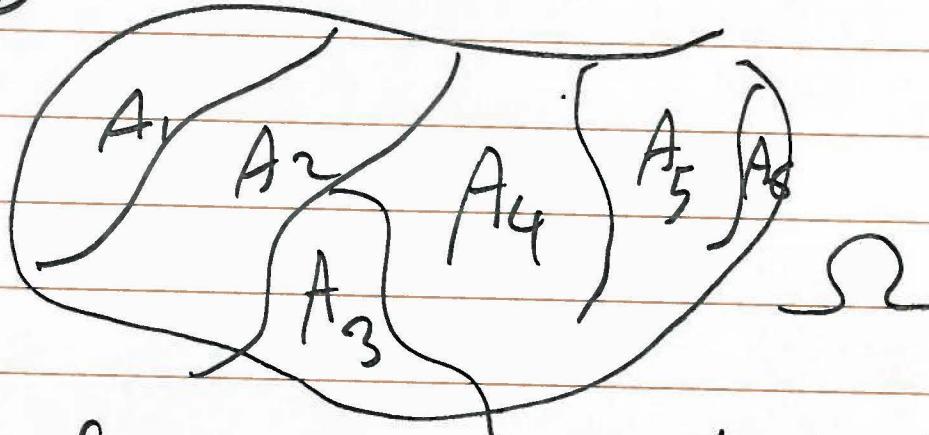
A_1, A_2, \dots consist a partition

of B iff

- 1) they are mutually exclusive $A_i \cap A_j = \emptyset \quad \forall i \neq j$ and
- 2) they are collectively exhaustive $\bigcup_i A_i = B$

\Rightarrow every member of B is a member of one A_i 's member

Example



(A finite number of partitions
for Ω)

Exercise : Which one is

a partition of $\mathbb{R}^{>0}$

a) $A_i = [i, i+1] \quad i \in \{0, 1, 2, \dots\}$

$$A_i \cap A_j = \emptyset \quad i \neq j$$

$$\bigcup A_i = \mathbb{R}$$

b) $A_i = [0, i] \quad i \in \{0, 1, 2, \dots\}$

$\exists i \neq j \quad A_i \cap A_j \neq \emptyset$: not mutually exclusive

(Some) Properties of Set Operations

Commutativity

$$A \cap B = B \cap A$$

$$A \cup B = B \cup A$$

$$p \wedge q = q \wedge p$$

$$p \vee q = q \vee p$$

Associativity

$$A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$p \wedge (q \wedge r) = (p \wedge q) \wedge r$$

Distributivity

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

Complement laws

$$(A^c)^c = A$$

$$\neg(\neg p) = p$$

$$A \cup A^c = \Omega$$

$$P \vee \neg P \equiv T$$

$$A \cap A^c = \emptyset$$

$$P \wedge \neg P \equiv F$$

Identity Laws

$$A \cup \emptyset = A$$

$$P \vee F \equiv P$$

$$A \cap \Omega = A$$

$$P \wedge T \equiv P$$

Domination Laws

$$A \cup \Omega = \Omega$$

$$P \vee T \equiv T$$

$$A \cap \emptyset = \emptyset$$

$$P \wedge F \equiv F$$

Idempotency

$$A \cup A = A$$

$$P \vee P \equiv P$$

$$A \cap A = A$$

$$P \wedge P \equiv P$$

Absorption Laws

$$A \cup (A \cap B) = A$$

$$P \vee (P \wedge q) \equiv P$$

$$A \cap (A \cup B) = A$$

Venn Diagram

$$P \wedge (P \vee q) \equiv P$$

De Morgan's Laws

$$(A_1 \cup A_2)^c = A_1^c \cap A_2^c$$

$$\neg(p \vee q) = \neg p \wedge \neg q$$

$$(A_1 \cap A_2)^c = A_1^c \cup A_2^c$$

$$\neg(p \wedge q) = \neg p \vee \neg q$$

More Generally, for an index set I

$$(\bigcup_{i \in I} A_i)^c = \bigcap_{i \in I} A_i^c$$

$$(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} A_i^c$$

Element Chasing

To prove $A = B$, we must prove:

$$A \subseteq B \wedge B \subseteq A$$

Therefore, we must prove:

$(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A)$
 i.e., we must "chase" elements

Proof of $(A \cap B)^c = A^c \cup B^c$

$$x \in (A \cap B)^c \Rightarrow x \notin A \cap B$$

$$\Rightarrow \neg(x \in A \wedge x \in B)$$

$$\text{De Morgan's} \Rightarrow \neg(x \in A) \vee \neg(x \in B)$$

$$\Rightarrow x \notin A \vee x \notin B$$

$$\Rightarrow x \in A^c \vee x \in B^c$$

$$x \in A^c \cup B^c$$

$$\therefore (A \cap B)^c \subseteq A^c \cup B^c$$

$$\text{opposite } A^c \cup B^c \subseteq (A \cap B)^c$$

$$\therefore A^c \cup B^c = (A \cap B)^c \quad \text{Q.E.D.}$$

Proof of $(\bigcup_{i \in I} A_i)^c = \bigcap_{i \in I} A_i^c$

$$x \in (\bigcup_{i \in I} A_i)^c \Rightarrow \neg(x \in \bigcup_{i \in I} A_i) \Rightarrow \neg(\exists_{i \in I} \text{ s.t. } x \in A_i)$$

$$\Rightarrow (\forall_{i \in I} \text{ s.t. } x \notin A_i)$$

$$\Rightarrow (\forall_{i \in I} \text{ s.t. } x \in A_i^c)$$

$$x \in \bigcap_{i \in I} A_i^c$$

General form of

Distributivity

$$\left(\bigcap_{i \in I} A_i \right) \cup B = \bigcup_{i \in I} (A_i \cup B)$$

$$\left(\bigcup_{i \in I} A_i \right) \cap B = \bigcap_{i \in I} (A_i \cap B)$$

Try to prove them

Notation: Special Sets

\mathbb{R} : Real Numbers

$$\mathbb{R}^* = \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$$

extended real numbers

\mathbb{Z} : Integers $\{-\dots, -4, -3, -2, -1, 0, 1, 2, \dots\}$

\mathbb{N} : Natural Numbers or
Strictly Positive Integers $\mathbb{N} = \mathbb{Z}^{>0}$

Intervals

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

Cartesian Product

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \wedge a_2 \in A_2\}$$

A_1, A_2, \dots, A_n

$$A_1 \times A_2 \times \dots \times A_n = \prod_{i=1}^n A_i$$

$$= \{(a_1, \dots, a_n) \mid a_i \in A_i, i \in \{1, 2, \dots, n\}\}$$

More Generally,

A_1, A_2, \dots

$\prod_{i=1}^{\infty} A_i$ is the sequence

$$\{(a_1, a_2, \dots) \mid a_i \in A_i, i \in \mathbb{N}\}$$

$$A_i = A \Rightarrow \prod_{i=1}^{\infty} A_i = A^{\infty}$$

Example: $A = \{0, 1\}$

$$A^\infty = \prod_{i=1}^{\infty} A_i, \quad \forall i \quad A_i = \{0, 1\}$$

A^∞ is the set of all binary sequences

Family of Subsets of A / Power Set

of A

The set of all subsets of A

is called the power set of A

and is denoted as 2^A or P_A or $P(A)$.

Example: $A = \{1, 2\}$

$$2^A = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

$$\begin{aligned} x \in \emptyset &\Rightarrow x \notin A \\ \therefore \emptyset &\subseteq A \end{aligned}$$

$$|A| = 2$$

$$|2^A| = 2^{|A|} = 2^2 = 4$$

Relations

Assume $K = (\prod_{i=1}^n A_i) \times B$

$$A_1 \times A_2 \times \cdots \times A_n \times B$$

Any subset of K , $R \subseteq K$, is

Called a relation. (on K)

Example: $A = \{2, 3\}$ $B = \{1, 2\}$

$$K = A \times B = \{(2, 1), (2, 2), (3, 1), (3, 2)\}$$

$$R_1 = \emptyset \quad R_2 = \{(3, 1)\} \quad R_3 = \{(2, 1), (2, 2)\} \dots$$

A Relation Maps some tuples in $\prod_{i=1}^n A_i$ to at least one element in B

Function :

Def: If a relation maps each

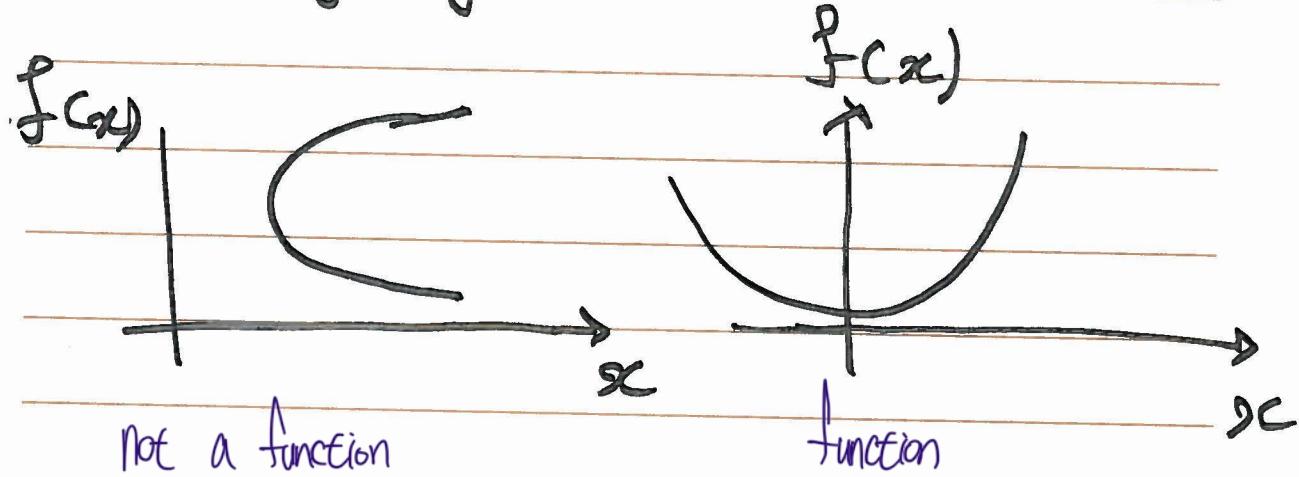
n-tuple to a unique member

of B, it is called a function

$$f \subseteq K$$

$$(x_1, \dots, x_n, y) \in f \wedge (x_1, \dots, x_n, y') \in f$$

$$\Rightarrow y = y'$$



We write $f: A_1 \times \dots \times A_n \rightarrow B$

$\prod_{i=1}^n A_i$ is called the domain of f

B is called the codomain of f .

For simplicity, we continue

with functions of the form

$f: A \rightarrow B$

(full forward)
Forward Image of $f: A \rightarrow B$

Def: $\vec{f}: 2^A \rightarrow 2^B$

$\checkmark S \subseteq A \quad (S \in 2^A)$

$$\vec{f}(S) = \{y \in B \mid y = f(x), x \in S\}$$

Easier $\vec{f}(S) = \{f(x) \mid x \in S\} \subseteq B$

The notation $\vec{f}(S)$ is also used.

Example: $f: \mathbb{R} \rightarrow \mathbb{R}, y = f(x) = x^2$

$$\vec{f}([1, 2]) = \{f(x) \mid x \in [1, 2]\}$$

$$= \{x^2 \mid x \in [1, 2]\} = [1, 4]$$

$$\vec{f}([-1, 3]) = [0, 9]$$

$$\vec{f}([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]) = [1/4, 1]$$

(pull back)

Inverse Image (pre-Image)of $f: A \rightarrow B$

$$f^{\leftarrow} : 2^B \rightarrow 2^A$$

$$\forall T \subseteq B (T \in 2^B)$$

$$f^{\leftarrow}(T) = \{x \in A \mid f(x) \in T\}$$

Obviously, $f^{\leftarrow}(T) \subseteq A (\in 2^A)$

The notation $f^{-1}(T)$ is also used, but we avoid it because it can be confused with an inverse function

Example : $f: \mathbb{R} \rightarrow \mathbb{R}$
 $f(x) = x^2$

$$f^{-1}([4, 9]) = \{x \in \mathbb{R} \mid f(x) \in [4, 9]\}$$

$$= \{x \in \mathbb{R} \mid x^2 \in [4, 9]\}$$

$$= [-3, -2] \cup [2, 3]$$

$$f^{-1}([0, 9]) = [-3, +3]$$

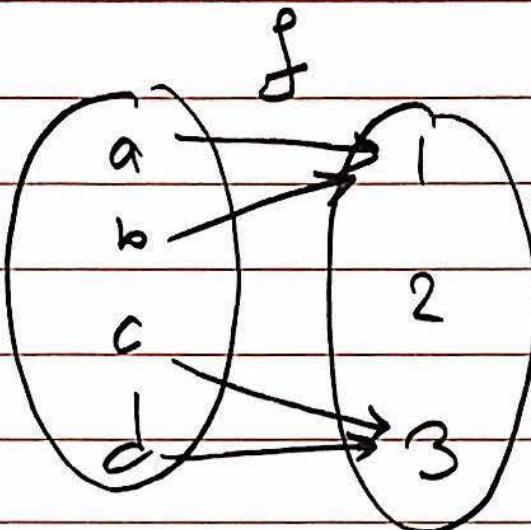
$$f^{-1}([-9, -1]) = \emptyset$$

$$f^{-1}([-9, 0]) = \{0\}$$

— Important note: forward and inverse images are functions that map sets to sets. In other words, they are set functions.

— Remark: It is a very good habit to always show the domain and co-domain of a function to avoid ambiguity.

Example:



Is f a function? yes

$$f^{\rightarrow}(\{a, b\}) = \{1\}$$

$$f^{\rightarrow}(\{a, b, c\}) = \{1, 3\}$$

$$f^{\rightarrow}(\emptyset) = \emptyset$$

$$f^{\leftarrow}(\{1\}) = \{a, b\}$$

$$f^{\leftarrow}(\{2\}) = \emptyset$$

$$f^{\leftarrow}(\{1, 2, 3\}) = \{a, b, c, d\}$$

Def: One-to-one function

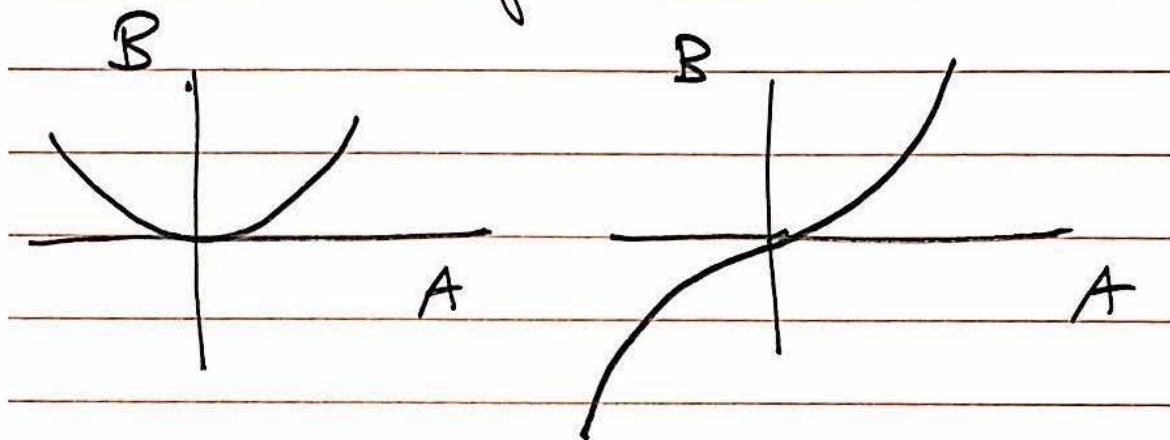
A function $f: A \rightarrow B$ is

one-to-one (1-1) iff it

assigns no more than one

member of A to each

member of B



or

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

or

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

One-to-one functions are

also called injective

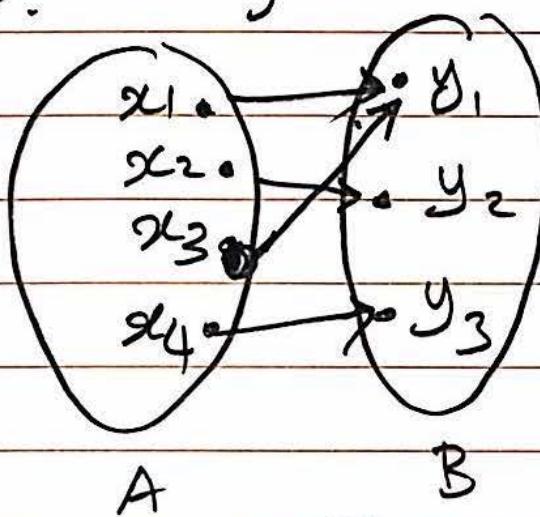
Onto functions:

Def: $f: A \rightarrow B$ is onto iff
 $\overrightarrow{f}(A) = B$.

Range

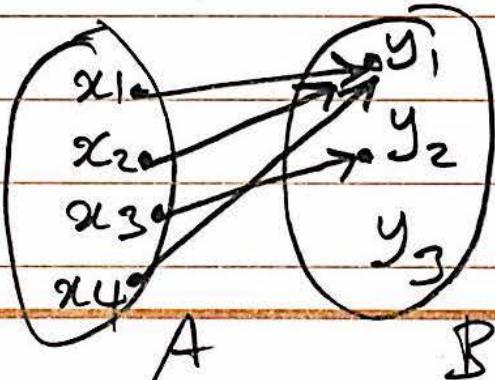
Example: f which function is onto?

a)



A onto B

(b)



One functions are also
called surjective

One-to-One Correspondence

one-to-one + onto

Def: $f: A \rightarrow B$ is called

a one-to-one correspondence

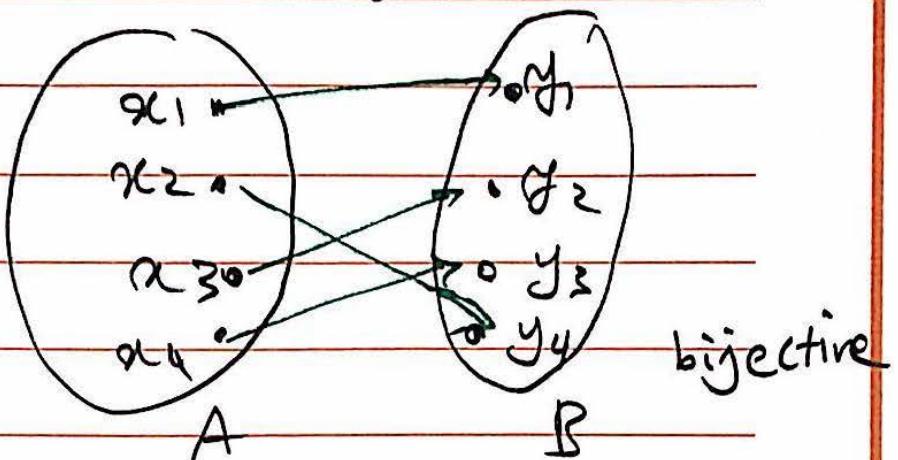
iff it is both 1-1 and

onto. It is also called

bijection.

Example :

f



bijective

Remark: In a bijective mapping,
one-to-one correspondence

every element in the co-domain

has a pre-image and the

pre-images ~~is~~ are unique.

Thus, we can define the inverse

function, $f^{-1}: B \rightarrow A$, such that

$f^{-1}(y) = x$ if $f(x) = y$. Therefore,
bijective functions are invertible.

Cardinality

In informal terms, the cardinality of a set is the number of elements in that set.

If one wishes to compare the cardinalities of two finite

Sets, A and B, they can simply count the number of elements in each set.

But what if the sets contain infinitely many elements?

Def. Two sets A, B are said

to be of the same

cardinality (equicardinal)

iff there exists a one-to-one

correspondence between A and B ,

and \Leftrightarrow we write $|A|=|B|$.

Question: Using the concept of

one-to-one and onto functions,

define $|A| \geq |B|$ and $|A| \leq |B|$

$|A| \geq |B| \Leftrightarrow$ there exists an onto function from A to B

$|A| \leq |B| \Leftrightarrow$ there exists a one-to-one function from A to B

Def: A set A is said to be

Countably infinite if A and \mathbb{N} have the same cardinality. (Denumerable)

Def: A set is said to be

Countable if it is either finite or countably infinite.

Example: \mathbb{N} is Countably infinite

$$f: \mathbb{N} \rightarrow \mathbb{N}$$

$$f(n) = n \text{ : identity function}$$

Example: \mathbb{Z} is countably infinite.

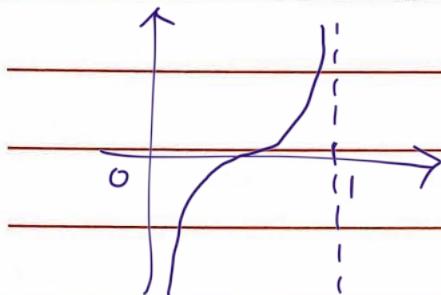
$$f: \mathbb{Z} \rightarrow \mathbb{N}$$

$$f(x) = \begin{cases} 2x+1 & x \geq 0 \\ -2x & x < 0 \end{cases}$$

$$n = f(x) \in \mathbb{N} \quad x \in \mathbb{Z}$$

$n = f(x) \in \mathbb{N}$	$x \in \mathbb{Z}$	$ Z = \mathbb{N} $
1	0	
3	+1	
2	-1	
5	+2	
4	-2	

Example: Do $(0, 1)$ and \mathbb{R} have the same cardinality?



$$y = \tan(\pi x - \pi/2)$$

Example: $\mathbb{N} \times \mathbb{N}$ is Countable

$(1, 1)$	$(2, 1)$	$(3, 1)$	$(4, 1)$	$(5, 1)$...
$(1, 2)$	$(2, 2)$	$(3, 2)$	$(4, 2)$	$(5, 2)$...
$(1, 3)$	$(2, 3)$	$(3, 3)$	$(4, 3)$	$(5, 3)$...
$(1, 4)$	$(2, 4)$	$(3, 4)$	$(4, 4)$	$(5, 4)$...
$(1, 5)$	$(2, 5)$	$(3, 5)$	$(4, 5)$	$(5, 5)$...
:	:				

Define $f(i,j)$ equal to the number of pairs visited when (i,j) visited. $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$

f is a one-to-one correspondence because it

visits all pairs and it visits each pair once.

$\Rightarrow \mathbb{N} \times \mathbb{N}$ is countable

Remark: Using the same argument, one can prove that $A \times B$ is countable when both A and B are countable.

Proposition: $\mathbb{Z} \times \mathbb{N}$ and $\mathbb{Z} \times \mathbb{Z}$ are countable!

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$$

$$\mathbb{Q} \subseteq \mathbb{Z} \times \mathbb{Z}$$

is

countable

any subset of countable set is either finite or countable

Def: Uncountable set

A is uncountable if it is
not countable.

Example: It can be shown

that \mathbb{R} , $\mathbb{B}^c = \mathbb{R} \setminus \mathbb{Q}$

are uncountable.

Limit of a sequence of Sets

Def: Indicator function of A ⊂ Ω

$$I_A : \Omega \rightarrow \{0, 1\}$$

$$\forall \omega \in \Omega \quad I_A(\omega) = \begin{cases} 0 & \omega \notin A \\ 1 & \omega \in A \end{cases}$$

Example: $\Omega = \mathbb{R}$, $A = [0, 1]$



Def: Assume A_1, A_2, \dots is

a sequence of sets, $\forall i \in \mathbb{N}$,

$$A_i \subseteq \Omega.$$

We say $\lim_{n \rightarrow \infty} A_n = A$

iff $\forall \omega \in \Omega$ $\lim_{n \rightarrow \infty} I_{A_n}(\omega) = I_A(\omega)$

(i.e. iff the indicator functions

converge pointwise)

i.g.) $A_n = [-\frac{1}{n}, 1 + \frac{1}{2n}]$

$$\lim_{n \rightarrow \infty} I_{A_n}(\omega) = I_{[0,1]}(\omega)$$

Example : $A_i = [-\frac{1}{i}, \frac{1}{i}]$

$$\lim_{i \rightarrow \infty} A_i = \{0\}$$

because

$$\forall \omega \in \mathbb{R} \lim_{i \rightarrow \infty} I_{A_i}(\omega) = \begin{cases} 0 & \omega \neq 0 \\ 1 & \omega = 0 \end{cases}$$

$$I_{\{0\}}(\omega)$$

Theorem :

(a) Suppose that A_n is an increasing sequence of sets

$(\forall n \in \mathbb{N} \ A_n \subseteq A_{n+1})$, then $\lim_{n \rightarrow \infty} A_n$ exists and is equal to

$$\bigcup_{n \in \mathbb{N}} A_n.$$

(b) If A_n is a decreasing

sequence of sets $(A_n \supseteq A_{n+1})$,

then $\lim_{n \rightarrow \infty} A_n$ exists and

is equal to $\bigcap_{n \in \mathbb{N}} A_n$.

Example: (a) $A_i = [0, i] \quad i \in \mathbb{N}$

Increasing or decreasing?
increasing

$$\lim_{i \rightarrow \infty} A_i = \bigcup_{i \in \mathbb{N}} A_i = [0, +\infty] = \mathbb{R}^{>0}$$

(b) $B_i = [i, +\infty), \quad i \in \mathbb{N}$

Increasing or decreasing?
decreasing

$$\lim_{i \rightarrow \infty} B_i = \bigcap_{i \in \mathbb{N}} B_i = \emptyset$$

Exercise : Show that

$$f \rightarrow (\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i)$$

Element Chasing

$$y \in f(\bigcup_{i \in I} A_i) \Leftrightarrow \exists x \in \bigcup_{i \in I} A_i \text{ st. } y = f(x)$$

$$\Leftrightarrow \exists x, \exists i \in I, \text{ st. } x \in A_i, y = f(x)$$

$$\Leftrightarrow \exists i \in I \exists x \text{ st. } x \in A_i, y = f(x)$$

$$\Leftrightarrow \exists i \in I, y \in f(A_i)$$

$$\Leftrightarrow y \in \bigcup_{i \in I} f(A_i)$$

$$\Rightarrow \left\{ \begin{array}{l} f(\bigcup_{i \in I} A_i) \subseteq \bigcup_{i \in I} f(A_i) \\ \bigcup_{i \in I} f(A_i) \subseteq f(\bigcup_{i \in I} A_i) \end{array} \right.$$

$$\bigcup_{i \in I} f(A_i) \subseteq f(\bigcup_{i \in I} A_i)$$

$$\therefore f(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i)$$

Note.

$$\forall u \forall v \equiv \forall v \forall u$$

$$\exists u \exists v \equiv \exists v \exists u$$

$$\exists u \forall v \Rightarrow \forall v \exists u \text{ day}$$

$$\forall u \exists v \not\Rightarrow \exists v \forall u$$