1. A player throws a fair die and simultaneously flips a fair coin. If the coin lands heads, then she wins twice, and if tails, then one-half of the value that appears on the die. Determine her expected winnings.[1]

[1]Important Note: Posting the sample exam and its solutions to online forums or sharing it with other students is strictly prohibited. Instances will be reported to USC officials as academic dishonesty for disciplinary action.

Solution:

Let X denote the value on the die, let Y be 1 if the coin lands heads and 0 if the coin lands tails, and let $g(X, Y)$ denote the winnings. Then her expected winnings are $\mathbb{E}[\text{winnings}] = \sum_{x=1}^{6} \sum_{y=0}^{1} g(x, y)p(x, y) = \sum_{x=1}^{6} (1/2xp(x, 1) + 2xp(x, 0)) = \sum_{x=1}^{6} (2x.1/12 + 1/2x.1/12) = 5/24 \sum_{x=1}^{6} x = 35/8$

2. There are $n+1$ participants in a game. Each person independently is a winner with probability p. The winners share a total prize of 1 unit. Let A denote a particular player, and let X denote the amount that is received by A.

   (a) Compute $\mathbb{E}[Y]$, the expected total prize shared by the players.

   (b) Show that $\mathbb{E}[X] = \frac{1-(1-p)^{(n+1)}}{n+1}$.

   (c) Compute $\mathbb{E}[X]$ by conditioning on whether A is a winner, and conclude that $\mathbb{E}[(1+Z)^{-1}] = \frac{1-(1-p)^{(n+1)}}{(n+1)p}$, where $Z \sim \text{Bin}(n,p)$

Solution:

(a) Let Y be the total prize shared by the players. If there is no winner, then $Y = 0$. If there is at least one winner, then $Y = 1$. The probability that there is no winner is $(1-p)^{n+1}$. Therefore the the expected total prize shared by the players is $\mathbb{E}[Y] = \mathbb{P}(Y = 1) = 1 - (1-p)^{n+1}$.

(b) Label the players 1 through $n + 1$ and let Y_i denote the prize of the i^{th} player. Note $Y = \sum_{i=1}^{n+1} Y_i$ is the total prize shared by the players. By part (2a), $\mathbb{E}[Y] = 1-(1-p)^{n+1}$. Since the players win independently and with the same probability, $\mathbb{E}[Y_i]$ is independent of i. In particular, $\mathbb{E}[Y_i] = \mathbb{E}[X]$ is the expected prize of A. Therefore:
$1-(1-p)^{n+1} = \mathbb{E}[Y] = \sum_{i=1}^{n+1} \mathbb{E}[Y_i] = (n+1)\mathbb{E}[X]$. Therefore, $\mathbb{E}[X] = \frac{1-(1-p)^{n+1}}{n+1}$.

(c) Let $I = 1$ if A wins and $I = 0$ otherwise. Then $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|I]]$. If $I = 0$, then $\mathbb{E}[X|I] = 0$ since A can only get a prize if A wins. If $I = 1$, then $\mathbb{E}[X|I] = (1+Z)^{-1}$, where Z denotes the number of winners excluding I. Note that Z is a binomial random variable with parameters n and p. Therefore:

$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|I]] = \mathbb{E}[X|I = 0]\mathbb{P}(I = 0) + \mathbb{E}[X|I = 1]\mathbb{P}(I = 1) = 0 + \mathbb{E}[(1+B)^{-1}]p = \mathbb{E}[(1+B)^{-1}]p$

So using part (2b), we get: $\mathbb{E}[(1+B)^{-1}] = \mathbb{E}[X]/p = \frac{1-(1-p)^{n+1}}{(n+1)p}$

3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables, with $X_i \sim \text{Ber}(p)$ and $Y = X_1 + X_2 + \cdots + X_n$. Determine $\mathbb{E}[X_1|Y]$.

Solution:

$$\mathbb{E}[X_1|Y=y] = 0 \times \mathbb{P}(X_1=0|Y=y) + 1 \times \mathbb{P}(X_1=1|Y=y)$$

but

$$\begin{aligned}
\mathbb{P}(X_1=1|Y=y) &= \frac{\mathbb{P}(X_1=1, Y=y)}{\mathbb{P}(Y=y)} \\
&= \frac{\mathbb{P}(X_1=1, X_2+\cdots, X_n=y-1)}{\mathbb{P}(Y=y)} \\
&= \frac{p\begin{pmatrix} n-1 \\ y-1 \end{pmatrix} p^{y-1}(1-p)^{n-y}}{p\begin{pmatrix} n \\ y \end{pmatrix} p^{y}(1-p)^{n-y}} \\
&= \frac{y}{n}
\end{aligned}$$

Therefore,
$\mathbb{E}[X_1|Y=y] = 0 \times \mathbb{P}(X_1=0|Y=y) + 1 \times \mathbb{P}(X_1=1|Y=y) = \frac{y}{n}$.
Therefore $\mathbb{E}[X_1|Y] = \frac{Y}{n}$.

4. An elevator operates in a building with m floors. One day, n people get into the elevator, and each of them chooses to go to a floor selected uniformly at random from 1 to m.

   (a) What is the probability that exactly one person gets out at the i^{th} floor? Give your answer in terms of n.

   (b) What is the expected number of floors in which exactly one person gets out? Hint: let Y_i be 1 if exactly one person gets out on floor i, and 0 otherwise. Then use linearity of expectation.

Solution:

(a) Let $\{X_i = 1\}$ denote the event that exactly one person gets out on the i^{th} floor. X_i is a binomial random variable with probability of success $\frac{1}{m}$.Then,

$$\mathbb{P}(\{X_i = 1\}) = n\left(\frac{1}{m}\right)\left(\frac{m-1}{m}\right)^{n-1}$$

(b) Let $Y_i = 1$ when exactly one person gets out at the i^{th} floor, and $Y_i = 0$ otherwise. Then, $\mathbb{E}[Y_i] = \mathbb{P}(Y_i = 1)$, but $Y_i = 1$ when $X_i = 1$. The number of floors in which exactly one person gets out is $Y = \sum_{i=1}^{m} Y_i$. Therefore, to get $\mathbb{E}[Y]$, we can simply sum over $\mathbb{E}[Y_i]$ for all i floors:

$$\mathbb{E}[Y] = \sum_{i=1}^{m} \mathbb{E}[Y_i] = \sum_{i=1}^{m} \mathbb{P}(Y_i = 1) = \sum_{i=1}^{m} \mathbb{P}(X_i = 1)$$
$$= \sum_{i=1}^{m} n\left(\frac{1}{m}\right)\left(\frac{m-1}{m}\right)^{n-1} = mn\left(\frac{1}{m}\right)\left(\frac{m-1}{m}\right)^{n-1} = n\left(\frac{m-1}{m}\right)^{n-1}$$

5. Show that a X $\sim$ Geo_1 random variabe is memoryless, i.e. $\mathbb{P}(X > r + s|X > s) = \mathbb{P}(X > r)$.

Solution: Left to the student 😄