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One-to-one function (injective)

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

or

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Onto function (Surjective)

$f: A \to B$ is onto iff $f^{\to}(A) = B$

One-to-one Correspondence (bijective)

$f: A \to B$ is called one-to-one correspondence iff both one-to-one and onto

Bijective function can define the inverse function

$f^{-1}: B \to A$; invertible

Cardinality

number of elements in that set

Two sets A,B are said to be of the same cardinality (equicardinal) iff there exists one-to-one correspondence b/w A and B

$|A| = |B|$

$|A| \geq |B| \Leftrightarrow$ exists onto function from A to B

$|A| \leq |B| \Leftrightarrow$ " one-to-one function "

Countably infinite (Denumerable)

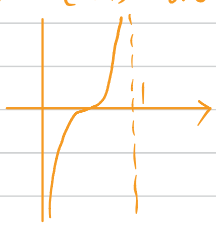
if set A and $\mathbb{N}$ have the same cardinality

i.g.) $\mathbb{Z}$ is countably infinite

$f: \mathbb{Z} \to \mathbb{N}$

$$f(x) = \begin{cases} 2x+1 & x \geq 0 \\ -2x & x < 0 \end{cases}$$

i.g.) $(0,1)$ and $\mathbb{R}$ have same cardinality?



$y = \tan(\pi x - \pi/2)$

$f: (0,1) \Rightarrow \mathbb{R}$

$\Rightarrow$ $A \times B$ is countable when both A and B are countable

; $\mathbb{Z} \times \mathbb{N}$ and $\mathbb{Z} \times \mathbb{Z}$ are countable

$\mathbb{Q} = \{ a/b \mid a, b \in \mathbb{Z}, b \neq 0 \}$ is countable

Uncountable set

: not countable

i.g.) $[0,1)$, $\mathbb{R}$, $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$

Indicator function

$I_A : \Omega \to \{0, 1\}$

$$\forall w \in \Omega, \quad I_A(w) = \begin{cases} 1 & w \in A \\ 0 & w \notin A \end{cases}$$

Assume $A_1, A_2, \cdots$ is a sequence of sets, $\forall i \in \mathbb{N}$

$A_i \subseteq \Omega$

$\lim_{n\to\infty} A_n = A$ iff $\forall \omega \in \Omega$, $\lim_{n\to\infty} I_{A_n}(\omega) = I_A(\omega)$

a) increasing sequence of sets

$\forall n$, $A_n \subseteq A_{n+1}$ then $\lim_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n$

b) decreasing sequence of sets

$\forall n$, $A_n \supseteq A_{n+1}$ then $\lim_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} A_n$

Exercise: Show that $f^{\rightarrow}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f^{\rightarrow}(A_i)$

element chasing

$y \in f^{\rightarrow}(\bigcup_{i\in I} A_i) \Rightarrow \exists x \in \bigcup_{i\in I} A_i$ s.t. $f(x) = y$

$\Rightarrow \exists x, \exists i \in I$ st $x \in A_i$, $y = f(x)$

$\Rightarrow \exists i \in I, \exists x$ st $x \in A_i$, $y = f(x)$

$\Rightarrow \exists i \in I$, $y \in f^{\rightarrow}(A_i)$

$\Rightarrow y \in \bigcup_{i\in I} f^{\rightarrow}(A_i)$

$y \in \bigcup_{i\in I} f^{\rightarrow}(A_i) \Rightarrow \exists i \in I$ $y \in f^{\rightarrow}(A_i)$

$\Rightarrow \exists i \in I$ $\exists x$, $x \in A_i$, $y = f(x)$

$\Rightarrow \exists x$ $\exists i \in I$, $x \in A_i$, $y = f(x)$

$\Rightarrow \exists x$ s.t. $x \in \bigcup_{i\in I} A_i$, $y = f(x)$

$\Rightarrow y \in f^{\rightarrow}(\bigcup_{i\in I} A_i)$

$\therefore f^{\rightarrow}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f^{\rightarrow}(A_i)$

Probability	probable $\rightarrow$ possible possible $\not\rightarrow$ probable $\mathbb{P}$(something possible) may be 0 : ratio of favorable outcomes and the total number of possible outcomes if all outcomes are equally likely $\mathbb{P}(A) = \frac{n_A}{N}$

Relative Frequency Definition $P(A) = \lim_{N \to \infty} \frac{n_A}{N}$

Axiomatic	Definition : countable additive function
Sample Space	Set of all possible outcomes of an experiments , Ω depending on what observer is interested in Ω can be different
Event	subset of the sample space which probabilities will be assigned

σ-field of events	a subset of 2^{Ω}, $\mathcal{F} \subseteq 2^{\Omega}$ is called a σ-field iff 1) $\phi \in \mathcal{F}$ 2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ 3) $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$

i.g.) smallest σ-field for any Ω

$$\mathcal{F} = \{\phi, \Omega\}$$

i.g.) $A \subseteq \Omega$ $\quad \mathcal{F} = \{\phi, A, A^c, \Omega\}$

Exercise : Prove that if $A_1, A_2, \cdots \in \mathcal{F}$ then the countable intersection of A_i's $\bigcap_{i \in I} A_i \in \mathcal{F}$

$$A_i \in \mathcal{F} \Rightarrow A_i^c \in \mathcal{F}$$
$$\Rightarrow \bigcup_{i \in I} A_i^c \in \mathcal{F} \Rightarrow \left(\bigcup_{i \in I} A_i^c\right)^c \in \mathcal{F}$$
$$\Rightarrow \bigcap_{i \in I} A_i \in \mathcal{F}$$

Exercise : using axioms (2), (3) show that ϕ, Ω also in $\mathcal{F}$

$$A_i = \begin{cases} A & i \in \phi \\ A^c & i \in \notin \end{cases}$$

$$A \in \mathcal{F}, A^c \in \mathcal{F} \Rightarrow A_i \in \mathcal{F}, \forall i \in \mathbb{N}$$
$$\Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F} \Rightarrow A \cup A^c \cup A \cdots = \Omega \in \mathcal{F}$$
$$\Rightarrow \Omega^c \in \mathcal{F} \Rightarrow \phi \in \mathcal{F}$$

Lemma : if $A_1, A_2, \cdots A_n$ is a finite sequence of event

$\bigcup_{i=1}^{n} A_i \in F$

define $B_i = \begin{cases} A_i & 1 \le i \le n \\ \phi & i > n \end{cases}$

$\bigcup_{i \in N} B_i \in F \quad \because B_i \in F$

$\Rightarrow \bigcup_{i=1}^{n} A_i \in F$

Probability Measure (Kolmogorov)

probability measure on (Ω, F) is a function

$P: F \to [0,1]$

1) Probability of Ω

$P(\Omega) = 1$

2) Probability of countable union of disjoint events (countable additivity property)

$A_1, A_2, \cdots \in F \quad \{A\}_{i \in \mathbb{N}}$

$A_i \cap A_j = \phi \quad i \neq j$ mutually exclusive : disjoint

$P(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{i=1}^{\infty} P(A_i)$

$(\Omega, F, \mathbb{P})$ a probability space

Lemma

1) additivity for a finite number of events

$A_1, A_2, \cdots, A_n \in \mathcal{F}$, $A_i \cap A_j = \phi \quad \forall i \neq j$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

2) Probability of complement

$A \in \mathcal{F} \Rightarrow P(A^c) = 1 - P(A)$

3) Preservation of order

$A \subseteq B \Rightarrow P(A) \leq P(B)$

4) Boole's Inequality (Union Bound)

$A_1, A_2, \cdots \in \mathcal{F}$

$$\Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

5) Inclusion - Exclusion

$A_1, A_2 \in \mathcal{F}$

$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

6) Inclusion - Exclusion (General form)

$A_1, A_2, \cdots, A_n \in \mathcal{F}$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

Proof

1) Finite sequence of disjoint

$$B_i = \begin{cases} A_i & 0 \le i \le n \\ \phi & i > n \end{cases}$$

$$\bigcup_{i \in \mathbb{N}} B_i = \bigcup_{i=1}^{n} A_i$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i \in \mathbb{N}} B_i\right) = \sum_{i=1}^{\infty} P(B_i) = \sum_{i=1}^{n} P(A_i)$$

2) $P(A) = 1 - P(A^c)$

$A \cup A^c = \Omega \qquad A \cap A^c = \phi$

$$P(A \cup A^c) = P(A) + P(A^c) = 1$$

$$\therefore P(A^c) = 1 - P(A)$$

3) $A \subseteq B \Rightarrow P(A) \le P(B)$

$A \subseteq B \Rightarrow B = A \cup (B \setminus A)$

$$A \cup (B \cap A^c) = (A \cup B) \cap (A \cup A^c)$$
$$= A \cup B$$

$A \cap (B \setminus A) = \phi$

$$\therefore P(B) = P(A \cup (B \setminus A)) = P(A) + P(B \setminus A)$$

$P(B) \ge P(A) \quad \because P(B \setminus A)$

4) Boole's Inequality

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} P(A_i)$$

$B_1 = A_1$

$B_2 = A_2 \setminus A_1$

$B_3 = A_3 \setminus (A_1 \cup A_2)$

$\vdots$

$B_i \cap B_j = \phi \quad \forall i \ne j$

$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$

$B_i \subseteq A_i$

$$\therefore P\left(\bigcup_{i \in \mathbb{N}} A_i\right) = P\left(\bigcup_{i \in \mathbb{N}} B_i\right) = \sum_{i=1}^{\infty} P(B_i) \le \sum_{i=1}^{\infty} P(A_i)$$

$$P\left(\bigcup_{i \in \mathbb{N}} A_i\right) \le \sum_{i=1}^{\infty} P(A_i)$$

5) Inclusive Exclusive

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$P(A_1 \cup A_2) = P(A_1 \cup (A_2 \setminus A_1))$$

$$= P(A_1) + P(A_2 \setminus A_1) \quad \because \ A_1 \cap (A_2 \setminus A_1) = \phi$$

$$= P(A_1) + P(A_2 \setminus (A_1 \cap A_2))$$

$$= P(A_1) + P(A_2) - P(A_1 \cap A_2) \quad \because \ A_2 \supseteq A_1 \cap A_2$$

Continuity of Probability Measure

$\forall i \in \mathbb{N}$, $A_i \in \mathcal{F}$ then

$$P\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \lim_{n \to \infty} P\left(\bigcup_{i=1}^{n} A_i\right)$$

limit of a sequence converge to L as $n \to \infty$

$$\lim_{n \to \infty} a_n = L \quad \text{iff} \quad \forall \varepsilon > 0, \exists N_0 \in \mathbb{N} \quad \text{s.t} \quad n \geq N_0 \Rightarrow |a_n - L| < \varepsilon$$

Proof: $P\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \lim_{n \to \infty} P\left(\bigcup_{i=1}^{n} A_i\right)$

define $B_1 = A_1$
$B_2 = A_2 \setminus A_1$
$B_3 = A_3 \setminus (A_1 \cup A_2)$ $\qquad B_i \cap B_j = \phi \quad i \neq j$
$\vdots$

$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i, \quad \bigcup_{i \in \mathbb{N}} B_i = \bigcup_{i \in \mathbb{N}} A_i$$

$$P\left(\bigcup_{i\in\mathbb{N}} A_i\right) = P\left(\bigcup_{i\in\mathbb{N}} B_i\right) = \sum_{i=1}^{\infty} P(B_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} B_i\right)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right)$$

Corollary: if A_i' in a sequence of noted events

a) if A_i is increasing

$A_i \subseteq A_{i+1} \quad \forall i \in \mathbb{N}$

$$\Rightarrow P\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=0}^{n} A_i\right) = \lim_{n\to\infty} P(A_n)$$

b) if A_i is decreasing

$A_i \supseteq A_{i+1} \quad \forall i \in \mathbb{N}$

$$P\left(\bigcap_{i\in\mathbb{N}} A_i\right) = \lim_{n\to\infty} P\left(\bigcap_{i=1}^{n} A_i\right) = \lim_{n\to\infty} P(A_n)$$

Conditional Probability

$(\Omega, \mathcal{F}, P)$ be probability space and the event $B \in \mathcal{F}$ be such that $P(B) > 0$

for any event $A \in \mathcal{F}$, the conditional probability

$$P(A|B) := \frac{P(A\cap B)}{P(B)}$$

two dices are thrown
given that first die shows 3,
probability that the sum exceed 6?

$\Omega = \{1,2,3,4,5,6\}$

$F = 2^{\Omega}$

$P(A) = \frac{|A|}{|\Omega|} \quad \forall A \in F$

A: event that the total number exceeds 6

$A = \{(a,b) \mid a,b \in \Omega,\ a+b > 6\}$

B: event that the first die shows 3

$B = \{(3,1), (3,2), \cdots, (3,6)\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3/36}{6/36} = 1/2$$

Conditional Probability is a Probability Measure

assume (Ω, F, P) is a probability space

and $B \in F \wedge P(B) > 0$

$\forall A \in F$: $P_B(A) = P(A|B)$ is satisfied Kolmogorov's axioms

Proof

1) $0 \le P_B(A) \le 1$

$A \cap B \subseteq B$

$\Rightarrow 0 \le P(A \cap B) \le P(B)$

$\Rightarrow 0 \le \frac{P(A \cap B)}{P(B)} \le 1$

2) $P_B(\Omega) = 1$

$\frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

3) Countable Additivity

disjoint events $A_1, A_2 \cdots$

$$P_B(\bigcup_{i\in\mathbb{N}} A_i) = \sum_{i=1}^{\infty} P_B(A_i)$$

$$P_B(\bigcup_{i\in\mathbb{N}} A_i) = \frac{P([\bigcup_{i\in\mathbb{N}} A_i]\cap B)}{P(B)} = \frac{P(\bigcup_{i\in\mathbb{N}}(A_i\cap B))}{P(B)}$$

$$\because (A_i\cap B)\cap(A_j\cap B)=\phi \quad \forall i\neq j$$

$$P_B(\bigcup_{i\in\mathbb{N}} A_i) = \frac{\sum_{i=1}^{\infty} P(A_i\cap B)}{P(B)}$$

$$= \sum_{i=1}^{\infty} P_B(A_i)$$

Exercise: assume (Ω, F, P) is a probability space and $B\in F$, $P(B)>0$

a) show that the reduced σ-field $F\cap B = \{B\cap A \mid A\in F\}$ is a σ-field itself

$F\cap B$ is a sigma field

assume $C = A\cap B$, $A\in F$, $B\in F$

show $C\in F\cap B \Rightarrow C^c \Rightarrow F\cap B$

$$B\setminus C = B\setminus(A\cap B) = B\cap(A\cap B)^c$$

$\because (A\cap B)^c\in F$

$\Rightarrow B\cap(A\cap B)^c \in F\cap B$

assume $C_1, C_2, \cdots \in F\cap B$

$$\bigcup_{i\in\mathbb{N}} C_i = \bigcup_{i\in\mathbb{N}}(A_i\cap B) = \bigcup_{i\in\mathbb{N}} A_i \cap B$$

$\because \bigcup_{i\in\mathbb{N}} A_i \in F$

$$\bigcup_{i\in\mathbb{N}} C_i \in F\cap B$$

b) show that $(B \cap \Omega, B \cap \mathcal{F}, P_B(A))$ is also a probability space

$(B, \mathcal{F}_B, P_B(A))$

$$P_B(\Omega_B) = P_B(B) = \frac{P(B \cap B)}{P(B)} = 1$$

assume $C_1, C_2, \cdots \in \mathcal{F}_B$

$C_i \cap C_j = \phi \quad \forall i \neq j$

$$P\left(\bigcup_{i \in \mathbb{N}} C_i\right) = P\left(\bigcup_{i \in \mathbb{N}} (A_i \cap B)\right)$$

$$= \sum_{i=1}^{\infty} P(A_i \cap B)$$

$$\therefore P_B\left(\bigcup_{i \in \mathbb{N}} C_i\right) = \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} P_B(C_i)$$

The Law of Total Probability

assume $(\Omega, \mathcal{F}, P)$ is a probability space and $\exists A, B \in \mathcal{F}$ s.t. $0 < P(B) < 1$

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

proof

$$A = A \cap \Omega = A \cap (B \cup B^c)$$

$$= (A \cap B) \cup (A \cap B^c)$$

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

$$= P(A|B) \cdot P(B) + P(A|B^c) P(B^c)$$

General Total Probability	assume that $(\Omega, \mathcal{F}, P)$ is a probability space and $B_1, B_2, \cdots$ is a partition of Ω such that $P(B_i) > 0$ then $P(A) = \sum_{i=1}^{\infty} P(A\|B_i)P(B_i)$ $A = A \cap \Omega = A \cap (\bigcup_{i \in \mathbb{N}} B_i)$ $= \bigcup_{i \in \mathbb{N}} (A \cap B_i)$ $\Rightarrow P(A) = \sum_{i=1}^{\infty} P(A \cap B_i)$ $= \sum_{i=1}^{\infty} P(A\|B_i)P(B_i)$
The Baye's Rule	assume that $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$ and $P(A) > 0$ and $P(B) > 0$ $P(B\|A) = \frac{P(A\|B) \cdot P(B)}{P(A)}$ $= \frac{P(A\|B) \cdot P(B)}{P(A\|B) \cdot P(B) + P(A\|B^c) \cdot P(B^c)}$ i.g.) $P(B_1) = 0.25$ $P(A\|B_1) = 0.3$ $P(B_2) = 0.25$ $P(A\|B_2) = 0.7$ $P(B_3) = 0.5$ $P(A\|B_3) = 0.2$ $P(A) = P(A\|B_1)P(B_1) + P(A\|B_2)P(B_2) + P(A\|B_3) \cdot P(B_3)$ $P(B_3\|A) = \frac{P(A\|B_3) \cdot P(B_3)}{P(A\|B_1)P(B_1) + P(A\|B_2)P(B_2) + P(A\|B_3)P(B_3)}$