Lesson 14
Concentration Inequalities

A concentration inequality gives us a probability band on certain random variables taking atypically large or atypically small values (tail probabilities)

1. Markov's Inequality

Assume X is a non-negative r.v. with $\mathbb{E}[X]<\infty$, then for any $\alpha>0$

$$\mathbb{P}(X>\alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$$

Remark: This inequality is non-trivial when $\alpha > \mathbb{E}[X]$.

Question: The "decay rate" is $1/\alpha$ in Markov Inequality. Can we have a decay rate that is faster than $1/\alpha$? The answer is YES:

2. Chebyshev's Inequality

Assume that X is a r.v. with expectation μ and variance $\sigma^2 < \infty$, then

$$\mathbb{P}(|X-\mu| > k\sigma) \leq 1/k^2 \quad \forall k > 0$$

OR

$$\mathbb{P}(|X-\mu| > c) \leq \frac{\sigma^2}{c^2}$$

Proof:

Chebyshev's Inequality
for Various k's

$k=1 \quad \mathbb{P}(|X-\mu| > \sigma) \leqslant$

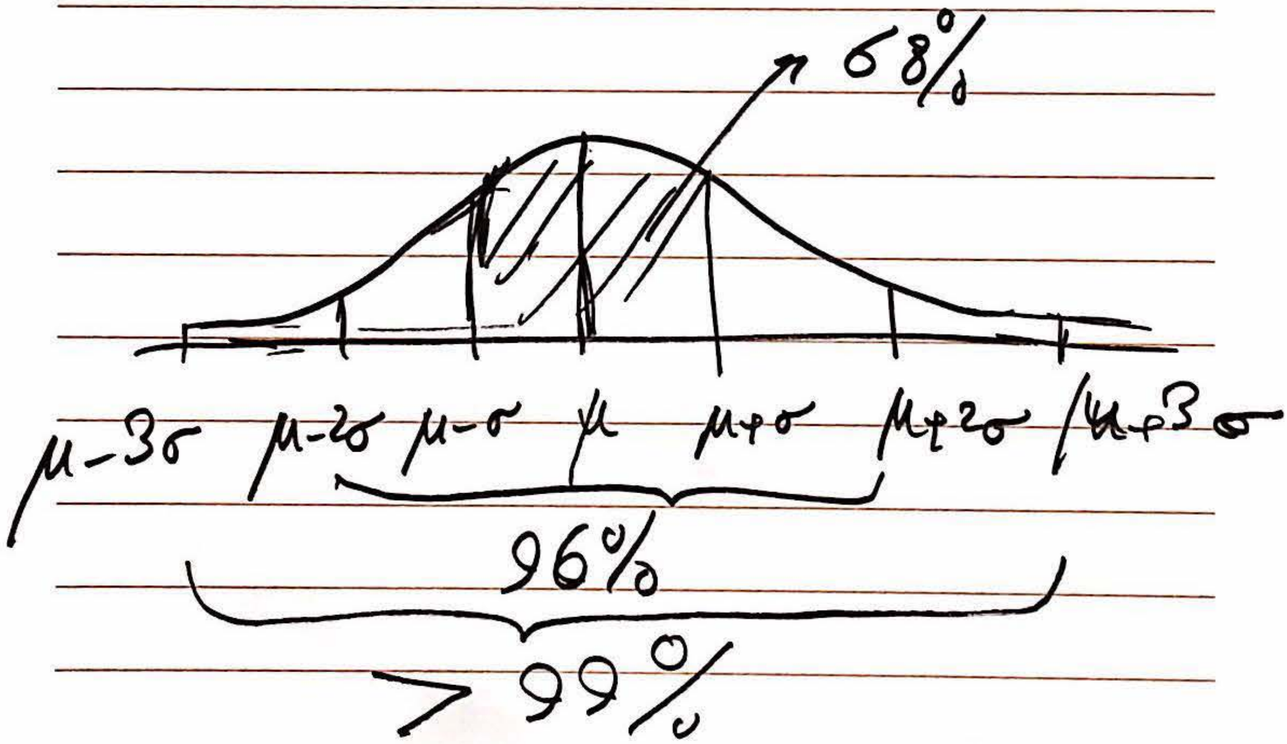
when X is Normal

$k=2$ $\mathbb{P}(|X-\mu| > 2\sigma) \leq$

For Normal

$k=3$ $\mathbb{P}(|X-\mu| > 3\sigma) \leq$

For normal



Chebyshev Inequality yields a bound that decays as fast as $1/k^2$, an improvement over the basic Markov Inequality.

Question: Can we find bounds that decay faster than $1/$polynomias, i.e., exponentially?

YES!