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Construction of Random Variables

a) (Ω, F, P) is a probability space and $A \in F$

corresponding indicator function of A. $I_A(\omega)$ is random variable

$$I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Bernoulli Random Variable

b) $A_1, A_2, \cdots A_n \in F$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$

$X(\omega) = \sum_{i=1}^{n} \alpha_i I_{A_i}(\omega)$ is a random variable

Multinoulli / Categorical random variable

c) Any continous real function is a random variable

So if $\Omega = \mathbb{R}$, $X: \mathbb{R} \to \mathbb{R}$ is random variable

if X is a continous function

$$\lim_{\omega \to \omega_0} X(\omega) = X(\omega_0) \quad \forall \omega \in \mathbb{R}$$

d) If $f: \mathbb{R} \to \mathbb{R}$ be a continous function

then $Y = f(X)$ is also a random variable

e) Assume that $X_1, \dots X_n$ are random variables on (Ω, F, P) and assume that $f: \mathbb{R}^n \to \mathbb{R}$ is a continous function

Then $Y = f(X_1, X_2, \cdots X_n)$ is also random variable

i.g.) $y = f(x_1, x_2) = X_1 + X_2$ and $y = f(X_1, X_2) = X_1 X_2$ are continous functions

so $X_1 + X_2$, $X_1 X_2$ are random variables

Cumulative Distribution Function (CDF)

cdf of a random variable X. $F_X : \mathbb{R} \to [0,1]$

$F_X(x) = P(X \le x)$

$= \{X \le x\} = X^{\leftarrow}((-\infty, x])$

$= \{\omega \mid X(\omega) \in (-\infty, x]\}$

Lemma CDF properties

a) $\lim_{x \to -\infty} F_X(x) = 0$

Proof) define $B_n = \{\omega \in \Omega \mid X(\omega) \le -n\}$

$A_n = (-\infty, -n]$; $A_1 \supseteq A_2 \supseteq A_3 \cdots$

$X^{\leftarrow}(A_n)$: $X^{\leftarrow}(A_1) \supseteq X^{\leftarrow}(A_2) \cdots$

$\Rightarrow B_1 \supseteq B_2 \supseteq B_3 \cdots$

$\lim_{n \to \infty} B_n = \lim_{n \to \infty} \bigcap_{i=1}^{n} B_n = \phi$

b) $\lim_{x \to \infty} F_X(x) = 1$

proof) define $B_n = \{\omega \in \Omega \mid X(\omega) \le n\}$

$A_n = (-\infty, n]$: $A_1 \subseteq A_2 \subseteq A_3 \cdots$

$X^{\leftarrow}(A_n)$: $X^{\leftarrow}(A_1) \subseteq X^{\leftarrow}(A_2) \cdots$

$\Rightarrow B_1 \subseteq B_2 \subseteq B_3 \cdots$

$\lim_{n \to \infty} B_n = \lim_{n \to \infty} \bigcup_{i=1}^{n} B_n = \mathbb{R}$

c) $\lim_{x \to \alpha^+} F_X(x) = F_X(\alpha)$; right continuity proof)

define $A_n = (-\infty, a+\frac{1}{n})$, $B_n = \{\omega \mid X(\omega) \le a+\frac{1}{n}\}$

$$A_1 \supseteq A_2 \supseteq A_3 \cdots$$

$$B_n = X^{\leftarrow}(A_n) \;;\; B_1 \supseteq B_2 \supseteq B_3 \cdots$$

$$\lim_{n\to\infty} B_n = \bigcap_{i\in\mathbb{N}} B_n = \{\omega \mid X(\omega) \le a\}$$

$$\Rightarrow P(\lim_{n\to\infty} B_n) = \lim_{n\to\infty} F_X(a+1/n) = F_X(a)$$

$$= \lim_{x\to a+} F_X(a)$$

Other properties of CDF

a) $P(X > x) = P(x \in (x,\infty))$

$$= 1 - P(X \le x) = 1 - F_X(x)$$

b) $P(x < X \le y) = P(X \in (x, y])$

$$= F_X(x) - F_X(y)$$

c) $P(X = x) = P(X \in \{x\}) = F_X(x) - \lim_{y\to x-} F_X(x)$

i.g.) assume X is a r.v. on (Ω, F, P) and F_X is its CDF

Is $G(x) = [1-F_X(x)]\log(1-F_X(x)) + F_X(x)$

a cdf of a random variable?

1. $\lim_{x\to-\infty} G(x) = (1-0)\log(1-0) + 0 = 0$

2. $\lim_{x\to\infty} G(x) = \lim_{u\to 0} u\log u + 1 = 1$

3. from $G(x)$, $F_X(x)$ is increasing function (non-decreasing)

$h(u) = (1-u)\log(1-u) + u$

$h'(u) \geq 0$ $\therefore$ $G(x)$ is non-decreasing w.r.t. x

4. $\lim_{x\to a^+} G(x) = [1-F_X(a)]\log(1-F_X(a)) + F_X(a)$

Exercise

a) $F_X(a)$ is continuous for all $a \in \mathbb{R}$

b) $P(\{X=a\}) = 0$ $\forall a \in \mathbb{R}$

$a \equiv b$

$P(\{X=a\}) = 0$ $\forall a \in \mathbb{R}$

$\Leftrightarrow F_X(a) - \lim_{x\to a^-} F_X(x) = 0$

$\Leftrightarrow F_X(a) = \lim_{x\to a^+} F_X(x) = \lim_{x\to a^-} F_X(x)$

Exercise : X be a r.v. with CDF F_X

find cdf of following random variables

a) $Y = X^2$

$$F_Y(y) = P(Y \le y)$$

$$= P(X^2 \le y) = \begin{cases} 0 & y < 0 \\ P(0 \le X^2 \le y) & y \ge 0 \end{cases}$$

$$\rightarrow P(-\sqrt{y} \le X \le \sqrt{y}) = P(-\sqrt{y} < X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

b) $Y = \sqrt{X}$

$$F_Y(y) = P(Y \le y)$$

$$= P(\sqrt{X} \le y) = \begin{cases} 0 & , \ y < 0 \\ P(0 \le \sqrt{X} \le y) & , \ y \ge 0 \end{cases}$$

$$P(0 \le \sqrt{X} \le y) = P(0 < X \le y^2)$$

$$= F_X(y^2) - F_X(0)$$

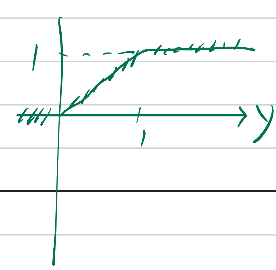
c) $Y = F_X(x)$: for simplication assume F_X is strictly increasing

$$F_Y(y) = P(Y \le y)$$

$$= P(F_X(x) \le y) \quad \text{with inverse function } F_X^{-1}$$

$$= P(x \le F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y$$

$$\therefore F_Y(y) = \begin{cases} 0 & y < 0 \\ y & 0 \le y < 1 \\ 1 & y > 1 \end{cases}$$



d) $Y = G^{-1}(X)$, where G is a continuous and strictly increasing function

$F_Y(y) = P(Y \leq y) = P(G^{-1}(X) \leq y)$

$= P(X \leq G(y)) = F_X(G(y))$

Exercise:

$$F_X(x) = \begin{cases} 0 & x<0 \\ 1/2\, x & 0 \leq x < 2 \\ 1 & x > 2 \end{cases}$$

Let $Y = X^2$ Find

a) $P(1 \leq X < 2) = P(1 < X \leq 2)$

$= F_X(2) - F_X(1) = 1 - 1/2 = 1/2$

b) $P(2X + Y \leq 3) = P(X^2 + 2X - 3 \leq 0)$

$= P((X+3)(X-1) \leq 0)$

$= P(-3 \leq X \leq 1) = P(-3 < X \leq 1)$

$= F_X(1) - F_X(-3) = 1/2$