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Random variables

function from the sample space to real number

$X : \Omega \to \mathbb{R}$ is a random variable

if $\forall c \in \mathbb{R}$, $X^{\leftarrow}((-\infty, c])$ in $\mathcal{F}$-measurable

$X^{\leftarrow}((-\infty, c]) = \{ \omega \in \Omega \mid X(\omega) \in (-\infty, c] \in \mathcal{F} \}$

ig.) $(\Omega, \mathcal{F}, P)$ where $\Omega = \{\alpha, \beta, \gamma\}$

$\mathcal{F} = \{ \phi, \{\alpha, \beta\}, \{\gamma\}, \Omega \}$

and $P(\phi) = 0$ $P(\Omega) = 1$

$P(\{\alpha, \beta\}) = 1/2$ $P(\{\gamma\}) = 1/2$

consider $g : \Omega \to \mathbb{R}$

$$g(\omega) = \begin{cases} 1 & \omega \in \{\alpha, \beta\} \\ 2 & \omega \in \{\gamma\} \end{cases}$$

$P(g(\omega) = 1) = P(\{ \omega \mid g(\omega) \in \{\alpha, \beta\} \})$

$P(\{\alpha, \beta\}) = 1/2$

$P(g(\omega) = 2) = P(\{ \omega \mid g(\omega) \in \{\gamma\} \})$

$= P(\{\gamma\}) = 1/2$

$$\therefore \underset{g^{\leftarrow}((-\infty, c])}{P(g(\omega) \le c)} = \begin{cases} P(\phi) = 0 & c < 1 \\ P(\{\alpha, \beta\}) = 1/2 & 1 \le c < 2 \\ P(\Omega) = 1 & c > 2 \end{cases}$$

i.g.) $h: \Omega \to \mathbb{R}$

$$h(\omega) = \begin{cases} 0 & \omega = \alpha \\ 1 & \omega = \beta \\ 2 & \omega = \gamma \end{cases}$$

$P(h(\omega)=0) = P(\{\omega \mid \omega = \alpha\})$

$= P(\{\alpha\})$: undefined

σ-field on which P is not defined rich/large enough

$X(\omega) = x \in \mathbb{R}$

$Y(\omega) = y \in \mathbb{R}$

i.g.) Indicator function of A measurable set

(Ω, F) a measurable space
and $A \in F$

$I_A : \Omega \to \{0, 1\}$

$$I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$ is a random variable

$X^{\leftarrow}((-\infty, 2]) = \{\omega \in \Omega \mid X(\omega) \in (-\infty, 2]\}$

$= \{\omega \in \Omega \mid X(\omega) \in \Omega\} = \Omega \in F$

$X^{\leftarrow}((-\infty, -2]) = \phi$

$X^{\leftarrow}((-\infty, 1/2]) = \{\omega \in \Omega \mid I_A(\omega) \leq 1/2\}$

$= \{\omega \in \Omega \mid \omega \notin A\} = A^c \in F$

$$\Rightarrow X^{\leftarrow}((-\infty, c]) = \begin{cases} \phi & c < 0 \\ A^c & 0 \leq c < 1 \\ \Omega & c \geq 1 \end{cases}$$

$X^{\leftarrow}((-\infty, 2])$ as

$$\{X \le 2\} = \{\omega \in \Omega \mid X(\omega) \le 2\}$$

i.g.) (Ω, F) is a measurable space

and $A \notin F$

I_A is not a random variable

$$I_A : \Omega \to \{0, 1\}$$

$$I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

$$X^{\leftarrow}((-\infty, 1/2]) = \{X \le 1/2\} = \{\omega \in \Omega \mid I_A(\omega) \le 1/2\}$$
$$= A^c \notin F$$

When $A \in F$

I_A is called Bernoulli Random Variable

Exercise

function of random variable X on (Ω, F)

$$Y : \Omega \to \mathbb{R}$$
$$Y(\omega) = X^3(\omega)$$

show that Y a random variable

need to show

$$Y^{\leftarrow}((-\infty, c]) = \{\omega \in \Omega \mid Y(\omega) \le c\}$$
$$= \{\omega \in \Omega \mid X^3(\omega) \le c\}$$
$$= \{\omega \in \Omega \mid X(\omega) \le c'\}$$
$$= X^{\leftarrow}((-\infty, c']) \in F$$

Borel Sets

$\sigma(\mathcal{C})$: smallest σ-field that contains $\mathcal{C}$

$\sigma(\{A, B\}) = \{\phi, \Omega, A\cap B, A\cap B^c, A^c\cap B, A^c\cap B^c)$

Borel σ-field

$\mathcal{B}(\mathbb{R})$ σ-field generated by all of the intervals $(-\infty, c]$, $c \in \mathbb{R}$

i.g.)

$(c, +\infty) = (-\infty, c]^c \in \mathcal{B}(\mathbb{R})$

$(-\infty, d) = \bigcup_{n\in\mathbb{N}} (-\infty, d-\frac{1}{n}] \in \mathcal{B}(\mathbb{R})$

$(c, d) = (-\infty, d) \cap (c, \infty) \in \mathcal{B}(\mathbb{R})$

Any finite / countably infinite subset of $\mathbb{R}$ is a Borel Set

Borel Sets are the members of $\mathcal{B}(\mathbb{R})$, the Borel σ-field

$X^{\leftarrow}(B)$ of any Borel Set $B \in \mathcal{B}(\mathbb{R})$

is an event, if X is an random variable

$B = \bigcup_n A_n$ where A_n is either $(-\infty, c_n]$ or $(-\infty, c_n]^c$

$X^{\leftarrow}(B) = X^{\leftarrow}(\bigcup_{n\in\mathbb{N}} A_n) = \bigcup_{n\in\mathbb{N}} X^{\leftarrow}(A_n) \in \mathcal{F}$

Probability Laws of a Random Variable

$\forall B \in \mathcal{B}(\mathbb{R})$, $X^{\leftarrow}(B)$ is an event ($\mathcal{F}$-measurable)

Probability of $X^{\leftarrow}(B)$

$$P(X^{\leftarrow}(B)) = P(\{\omega \mid X(\omega) \in B\})$$
$$= P_X(B)$$

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$ is a probability space induced by X

where $P_X(B) = P(X^{\leftarrow}(B)) = P(X \in B)$

i.g.)

$\Omega = \{1,2,3,4,5,6\}$ and $P(A) = \frac{|A|}{6}$

$\forall \omega \in \Omega$, $X_1(\omega) = 2$ and $X_2(\omega) = \omega^2$, $X_3(\omega) = \sqrt{\omega}$

calculate $P_{X_1}(B)$, $P_{X_2}(B)$ and $P_{X_3}(B)$

a) $B = (-\infty, -10]$

$P_{X_1}(B) = P(X_1^{\leftarrow}(B)) = P(X_1(\omega) \in B) = 0$

$P_{X_2}(B) = P(X_2^{\leftarrow}(B)) = P(X_2(\omega) \in B) = 0$

$P_{X_3}(B) = P(X_3^{\leftarrow}(B)) = P(X_3(\omega) \in B) = 0$

b) $B = [3/2, 5/2]$

$P_{X_1}(B) = P(X_1^{\leftarrow}(B)) = P(X_1(\omega) \in B) = P(\Omega) = 1$

$P_{X_2}(B) = P(X_2^{\leftarrow}(B)) = P(X_2(\omega) \in B) = 0$

$P_{X_3}(B) = P(X_3^{\leftarrow}(B)) = P(X_3(\omega) \in B) = 0$

c) $B = \mathbb{N}$

$P_{X_1}(B) = P(X_1^{\leftarrow}(B)) = P(X_1(\omega) \in B) = 1$

$P_{X_2}(B) = P(X_2^{\leftarrow}(B)) = P(X_2(\omega) \in B) = 1$

$P_{X_3}(B) = P(X_3^{\leftarrow}(B)) = P(X_3(\omega) \in B) = P(\{1, 4\}) = 1/3$

Borel Sets shed lights on undefined of the concept of a random variable

random variable help creating a probability space on $\mathbb{R}$