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A Review of Results on Sums of Random Variables

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Abstract Sums of random variables arise naturally in wireless communications and related areas. Here, we provide a review of the known results on sums of exponential, gamma, lognormal, Rayleigh and Weibull random variables. A discussion is provided of two applications. We expect that this review could serve as a useful reference and help to advance further research in this area.

Keywords Exponential distribution · Gamma distribution · Lognormal distribution · Rayleigh distribution · Sums · Weibull distribution

1 Introduction

Sums of random variables have a pivotal role in wireless communications and related areas. As explained below, sums of exponential random variables, sums of gamma random variables, sums of lognormal random variables and the sums of Rayleigh random variables all have roles in wireless communications.

Sums of Exponential Random Variables The spatial diversity is an efficient solution to combat the detrimental effects of channels such as shadowing and deep fading by deploying multiple antennas at both transmitter and receiver (Proakis [1]). An alternative form, called cooperative diversity (Laneman [2]), was exploited to overcome some scenarios where wireless mobiles may be unable to support multiple antennas due to size or other constraints. Thus, it allows single-antenna mobiles to gain several benefits of transmit diversity. While evaluating the performance of such diversity schemes, one usually deals with the problem of finding the probability density function (pdf) of a sum of exponential random variables.

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Sums of Gamma Random Variables The wide versatility, experimental validity, and analytical tractability of the Nakagami distribution (Nakagami [3]) has made it a very popular fading model for performance analysis investigations in two important topics of wireless communications, namely: 1) diversity schemes and 2) cochannel interference in cellular mobile radio systems. Many of these performance analysis problems require determination of the statistics of the sum of the squared envelopes of Nakagami faded signals or, equivalently, the sum of gamma variates since the square of a Nakagami variate follows a gamma distribution (Nakagami [3]).

Sums of Lognormal Random Variables Electric current fluctuations in tunnel junctions tend to exhibit a lognormal distribution (Romeo et al. [4]). The problem of the sum of current fluctuations can be modeled as the sum of lognormal distributed random variables. The lognormal distribution is also a widely used fading model, with applications ranging from the traditional cellular systems to the emerging IEEE 802.15.3a wireless personal area network (Foerster [5]). To determine the outage and bit-error-rate performance of these systems (Liu [6]), one often needs the pdf of a sum of lognormal random variables.

Sums of Rayleigh Random Variables The distribution of a sum of Rayleigh random variables is required for the calculation of the error bounds for coding on generalized mobile satellite fading channels (Divsalar and Simon [7]). Another important application is related to the performance analysis of equal-gain combining receivers, in which the received faded signals are equally weighted, cophased, and summed to produce the output signal. Furthermore, in the scientific field of radar receivers, the decision level for a preassigned false-alarm probability requires determining of the cumulative distribution function (cdf) of such sums (Marcum [8]), while they can be useful in other important applications which are related to signal detection and linear equalizing, as well as intersymbol interference and phase jitter analysis.

The aim of this paper is to review the known results for sums of exponential random variables (Sect. 2), sums of gamma random variables (Sect. 3), sums of lognormal random variables (Sect. 4), sums of Rayleigh random variables (Sect. 5) and the sums of Weibull random variables (Sect. 6). The results give various expressions for the pdf and the cdf of the sums. Possible applications of the results to two problems in wireless communications are discussed in Sect. 7.

2 Sums of Exponential RVs

Let $X_m, m = 1, 2, \ldots, M$ be independent exponential random variables with expected values $1/\lambda_m$, $m = 1, 2, \ldots, M$ and let $Z = X_1 + X_2 + \cdots + X_M$. If the λ_m's are distinct then it is known that (see Ross [9]) the pdf of Z is:

$$f_Z(z) = \prod_{i=1}^{M} \lambda_i \sum_{j=1}^{M} \frac{\exp(-\lambda_j z)}{\prod_{k=1,k\neq j}^{M} (\lambda_k - \lambda_j)} \tag{1}$$

for $z > 0$. Khuong and Kong [10] considered the case that some of the λ_m may not be distinct. Suppose K of the X_m's have the same expected value, say $1/\lambda_e$, and the remaining $N = M - K$ have distinct expected values. Then, the pdf of Z is given by

$$f_Z(z) = \left(\sum_{n=1}^{N} E_n \lambda_n \exp(-\lambda_n z) + \sum_{k=1}^{K} A_k \frac{z^{k-1} \lambda_e^k \exp(-\lambda_e z)}{\Gamma(k)} \right) U(z), \tag{2}$$

where

$$E_n = \left(\frac{1}{1-\lambda_n/\lambda_e}\right)^K \prod_{u=1,u\neq n}^{N} \frac{1}{1-\lambda_n/\lambda_e},$$

$$C_{uv} = \frac{1}{(1-B_u/\lambda_e)^v},$$

$$D_u = \left(\frac{1}{1-B_u/\lambda_e}\right)^K \prod_{n=1}^{N} \frac{1}{1-B_u/\lambda_n} - \sum_{n=1}^{N} \frac{E_n}{1-B_u/\lambda_n},$$

C is a $K \times K$ matrix with the elements $\{C_{uv}\}$, D is a $K \times 1$ vector with the elements $\{D_u\}$, $A = C^{-1}D$, $U(\cdot)$ is the unit step function, and $\{B_p\}$ satisfy the equations

$$\left(1-\frac{B_p}{\lambda_e}\right)^{-K} \prod_{n=1}^{N} \left(1-\frac{B_p}{\lambda_n}\right)^{-1} = \sum_{n=1}^{N} E_n \left(1-\frac{B_p}{\lambda_n}\right)^{-1} + \sum_{k=1}^{K} A_k \left(1-\frac{B_p}{\lambda_e}\right)^{-k}$$

for $p = 1, 2, \ldots, K$. Note that (1) is a particular case of (2) for $K = 0$. Amari and Misra [11] considered the most general case where there are a distinct λ's with $\lambda_1 = \cdots = \lambda_{r_1} = \beta_1$, $\lambda_{r_1} = \cdots = \lambda_{r_1+r_2} = \beta_2, \ldots, \lambda_{r_1+r_2+\cdots+r_{a-1}+1} = \cdots = \lambda_{r_1+r_2+\cdots+r_a} = \beta_a$, where $r_i \geq 1$ and the sum of all r_i is M. Amari and Misra [11] showed that the cdf of Z can be expressed as

$$F(z) = 1 - \left(\prod_{j=1}^{a} \beta_j^{r_j}\right) \sum_{k=1}^{a} \sum_{l=1}^{r_k} \frac{\Psi_{k,l}(-\beta_k) z^{r_k-1} \exp(-\beta_k z)}{(r_k-l)!(l-1)!},$$

where

$$\Psi_{k,l}(t) = -\frac{\partial^{l-1}}{\partial t^{l-1}} \left\{ \prod_{j=0,\, j\neq k}^{a} (\beta_j + t)^{-r_j} \right\}.$$

Under the setting of Amari and Misra [11], one can also rewrite Z as $Z = Y_1 + Y_2 + \cdots + Y_a$, where Y_i, $i = 1, 2, \ldots, a$ are independent gamma random variables with parameters (r_i, β_i), $i = 1, 2, \ldots, a$. Note that Z is a sum of independent gamma random variables and so the results from Sect. 3 can also be applied.

3 Sums of Gamma RVs

Let Y_i, $i = 1, 2, \ldots, N$ be independent gamma random variables with parameters $(\alpha_i, 1/\beta_i)$, $i = 1, 2, \ldots, N$ and let $Z = Y_1 + Y_2 + \cdots + Y_N$. Various representations for the pdf of Z can be found in [12–20]. These representations are expressed in terms of infinite series, infinite integrals and known special functions. Moschopoulos [12] showed that the pdf of Z can be expressed as

$$f_Z(z) = \prod_{n=1}^{N} \left(\frac{\beta_1}{\beta_n}\right)^{\alpha_n} \sum_{k=0}^{\infty} \frac{\delta_k z^{\sum_{n=1}^{N}\alpha_n + k - 1} \exp(-z/\beta_1)}{\beta_1^{\sum_{n=1}^{N}\alpha_n + k} \Gamma\left(\sum_{n=1}^{N}\alpha_n + k\right)} \qquad (3)$$

for $z > 0$, where $\beta_1 = \min \beta_n$ and the coefficients δ_k satisfy the recurrence relations

$$\delta_0 = 1$$

and

$$\delta_{k+1} = \frac{1}{k+1} \sum_{i=1}^{k+1} \left[\sum_{j=1}^{N} \alpha_j \left(1 - \frac{\beta_1}{\beta_j} \right)^i \right] \delta_{k+1-i}.$$

Alouini et al. [13] revisited this result and provided a generalization for the case of correlated gamma random variables: if Y_i and Y_j have the correlation coefficient ρ_{ij} and if $\alpha_i = \alpha$ for all i then (3) generalizes to

$$f_Z(z) = \prod_{n=1}^{N} \left(\frac{\lambda_1}{\lambda_n} \right)^{\alpha} \sum_{k=0}^{\infty} \frac{\delta_k z^{N\alpha+k-1} \exp(-z/\lambda_1)}{\lambda_1^{N\alpha+k} \Gamma(N\alpha + k)} \tag{4}$$

for $z > 0$, where $\lambda_1 = \min \lambda_n$, $\{\lambda_n\}$ are the eigenvalues of the matrix $A = DC$, where D is the $N \times N$ diagonal matrix with the entries $\{\beta_n\}$ and C is the $N \times N$ positive definite matrix defined by

$$C = \begin{bmatrix} 1 & \sqrt{\rho_{12}} & \cdots & \sqrt{\rho_{1N}} \\ \sqrt{\rho_{21}} & 1 & \cdots & \sqrt{\rho_{2N}} \\ \vdots & \vdots & \cdots & \vdots \\ \sqrt{\rho_{N1}} & \cdots & \cdots & 1 \end{bmatrix}$$

and the coefficients δ_k satisfy the recurrence relations

$$\delta_0 = 1$$

and

$$\delta_{k+1} = \frac{\alpha}{k+1} \sum_{i=1}^{k+1} \left[\sum_{j=1}^{N} \left(1 - \frac{\lambda_1}{\lambda_j} \right)^i \right] \delta_{k+1-i}.$$

Provost [14, 15] provided a form similar to (3) given by

$$f_Z(z) = \sum_{k=0}^{\infty} \frac{\theta_k z^{\alpha+k-1} \exp(-z)}{\Gamma(\alpha + k)} \tag{5}$$

for some α and θ_k. Efthymoglou and Aalo [16] derived a single integral representation for the pdf of Z given by

$$f_Z(z) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos\left(\sum_{k=1}^{N} \alpha_k \arctan(\beta_k t) - zt\right)}{\prod_{k=1}^{N} (1 + t^2 \beta_k^2)^{\alpha_k/2}} dt$$

for $z > 0$. Note that the integrand takes an elementary form. Most recently, Aalo et al. [17] provided the compact form

$$f_Z(z) = \frac{\beta_1^{-\alpha_1} \cdots \beta_n^{-\alpha_n} z^{\sum_{k=1}^{n} \alpha_k - 1}}{\Gamma\left(\sum_{k=1}^{n} \alpha_k\right)} \Phi_2^n \left(\alpha_1, \ldots, \alpha_n; \sum_{k=1}^{n} \alpha_k; -\frac{z}{\beta_1}, \ldots, -\frac{z}{\beta_n} \right), \tag{6}$$

where $\Phi_2^n(\cdots)$ denotes the confluent Lauricella multivariate hypergeometric function. Most of the expressions given above can be obtained as particular cases of (6). For other known representations on sums of gamma random variables, we refer the readers to the books by Mathai [18], Mathai and Saxena [19] and Springer [20].

4 Sums of Lognormal RVs

Let Y_i, $i = 1, 2, \ldots, N$ be independent lognormal random variables with common mean μ and variances σ_i^2, $i = 1, 2, \ldots, N$ and let $Z = Y_1 + Y_2 + \cdots + Y_N$. Unfortunately, closed form expressions such as those above do not exist for Z. However, various authors have proposed approximations for the distribution of Z. The most commonly used approximation is that Z also has a lognormal distribution. For instance, Beaulieu and Rajwani [21] showed that the cdf of $\log Z$ can be approximated by $\Phi(a_0 - a_1 z^{-a_2/\lambda})$, where a_0, a_1, a_2 are some suitable constants, $\lambda = (1/10) \log 10$ and $\Phi(\cdot)$ denotes the cdf of the standard normal distribution. More recently, other approximations have been suggested. Lam and Le-Ngoc [22] proposed approximating the distribution of Z by the log shifted gamma distribution given by the pdf:

$$f_Z(z) = \frac{(10 \log_{10} z - \delta)^{\alpha-1}}{\lambda \beta^\alpha \Gamma(\alpha) z} \exp\left(-\frac{10 \log_{10} z - \delta}{\beta}\right)$$

for $z \geq 10^{\delta/10}$, where $\lambda = (1/10) \log 10$, α controls the shape, β is for scaling and δ is the offset parameter. Zhao and Ding [23] suggested a least squares approximation for the distribution of Z. Liu et al. [24] suggested approximating the distribution of Z by that of the mixture of two lognormal distributions. Zhang and Song [25] proposed approximating the distribution of $X = 10 \log_{10} Z$ by a Pearson family distribution with the selection of the particular model based on the moment ratios

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3},$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

and

$$\kappa = \frac{\beta_1 (\beta_2 + 3)^3}{4(2\beta_2 - 3\beta_1 - 6)(4\beta_2 - 3\beta_1)},$$

where $\mu = E(X)$, $\mu_2 = E(X - \mu)^2$, $\mu_3 = E(X - \mu)^3$ and $\mu_4 = E(X - \mu)^4$. Liu et al. [26] considered approximating the distribution of Z by that of power lognormal distributions defined by the cdf

$$F(z) = \left\{\Phi\left(\frac{\log z - \mu}{\sigma}\right)\right\}^\lambda$$

for $z > 0$, $-\infty < \mu < \infty$, $\sigma > 0$ and $\lambda > 0$.

Pratesi et al. [27] considered approximations for linear combinations of correlated lognormal random variables: $Z = A_1 Y_1 + A_2 Y_2 + \cdots + A_N Y_N$, where A_i are themselves random variables. It was assumed that A_i are independent among themselves and are independent of the Y_is. Pratesi et al. [27] proposed several approximations for Z. A Gaussian approximation proposed for $\log Z$ has the mean and standard deviation specified by:

$$\mu = 2 \log(M_1) - \frac{1}{2} \log(M_2)$$

and

$$\sigma = \sqrt{-2\log(M_1) + \log(M_2)},$$

where

$$M_1 = \sum_{i=1}^{N} E(A_i) \exp\left(\mu_i + \frac{\sigma_i^2}{2}\right)$$

and

$$M_2 = \sum_{i=1}^{N} E(A_i^2) \exp(2\mu_i + 2\sigma_i^2)$$
$$+ \sum_{i,j=1, i\neq j}^{N} E(A_i) E(A_j) \exp\left(\mu_i + \mu_j + \frac{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j}{2} t\right),$$

where ρ_{ij} denotes the correlation coefficient between $\log Y_i$ and $\log Y_j$.

Some authors have also developed bounds for the distribution of sums of independent lognormal random variables. For example, Ben Slimane [28] showed that the cdf of Z can be bounded as:

$$F_Z(z) \geq 1 - \int_{-\infty}^{\log z} \int_{-\infty}^{\log(z-\exp(y))/(N-1)} g(x, y)\, dx dy$$

and

$$F_Z(z) \leq 1 - \int_{-\infty}^{\log z} \int_{-\infty}^{\log(z-\exp(y))/(N-1)} h(x, y)\, dx dy,$$

where

$$g(x, y) = \frac{\partial^2 G(x, y)}{\partial y \partial x},$$

$$h(x, y) = \frac{\partial^2 H(x, y)}{\partial y \partial x},$$

$$G(x, y) = \begin{cases} \prod_{i=1}^{N} \left\{1 - \Phi\left(\frac{y-\mu_i}{\sigma_i}\right)\right\}, & \text{if } y \leq x, \\ \prod_{i=1}^{N} \left\{1 - \Phi\left(\frac{y-\mu_i}{\sigma_i}\right)\right\} - \prod_{i=1}^{N} \left\{\Phi\left(\frac{y-\mu_i}{\sigma_i}\right) - \Phi\left(\frac{x-\mu_i}{\sigma_i}\right)\right\}, & \text{if } y > x, \end{cases}$$

and

$$H(x, y) = \begin{cases} \prod_{i=1}^{N} \left\{1 - \Phi\left(\frac{y-\mu_i}{\sigma_i}\right)\right\}, & \text{if } y \leq x, \\ \left[1 - N + \sum_{i=1}^{N} \frac{\Phi\left(\frac{y-\mu_i}{\sigma_i}\right)}{\Phi\left(\frac{x-\mu_i}{\sigma_i}\right)}\right] \prod_{i=1}^{N} \left\{1 - \Phi\left(\frac{x-\mu_i}{\sigma_i}\right)\right\}, & \text{if } y > x. \end{cases}$$

5 Sums of Rayleigh RVs

Let Y_i, $i = 1, 2, \ldots, N$ be independent Rayleigh random variables specified by the pdf

$$f_i(y) = y \exp\left(-\frac{y^2}{2}\right)$$

for $y > 0$. Let $Z = Y_1 + Y_2 + \cdots + Y_N$. As in the lognormal case, no closed form expressions are available for the distribution of Z. Only some approximations and bounds have been known. A widely used approximation due to Schwartz et al. [29] is:

$$F_Z(z) \approx 1 - \exp\left(-\frac{t^2}{2b}\right) \sum_{k=0}^{N-1} \frac{t^{2k}}{(2b)^k k!}, \tag{7}$$

where $t = z/\sqrt{N}$, $b = N^{-1}\sigma^2((2N-1)!!)^{1/N}$ and $(2N-1)!! = (2N-1) \cdot (2N-3) \cdots 3 \cdot 1$. Hu and Beaulieu [30] provided the following refinement of (7):

$$F_Z(z) \approx 1 - \exp\left(-\frac{t^2}{2b}\right) \sum_{k=0}^{N-1} \frac{t^{2k}}{(2b)^k k!} - t \frac{a_0 (t-a_2)^{2N-1}}{2^{N-1}(b/a_1)^N (N-1)!} \exp\left\{-\frac{a_1 (t-a_2)^2}{2b}\right\}$$

for some suitable constants a_0, a_1 and a_2. Karagiannidis et al. [31] established the bound:

$$F_Z(z) \leq \frac{z^{2N}}{N^{2N}} G_{1,N+1}^{N,1}\left(\frac{z^{2N}}{N^{2N}} \middle| \begin{matrix} 0 \\ 0, 0, \ldots, 0, -1 \end{matrix}\right),$$

where $G_{1,N+1}^{N,1}(\cdots)$ is a Meijer G-function, see Sect. 9.3 in Gradshteyn and Ryzhik [32]. For other results on sums of Rayleigh random variables, see Beaulieu [33].

6 Sums of Weibull RVs

Unfortunately, no results (not even approximations) have been known for sums of Weibull random variables. It is expected that this review could help to motivate some work for this case.

7 Application

Here, we discuss two examples from wireless communications where the results reviewed could be applied.

Example 1 Sums of lognormal random variables occur in many problems in wireless communications because signal shadowing is well modeled by the lognormal distribution. Consider the computation of associated bit error probabilities (BEPs). They can be written in the form $R = \Pr(X_1 + X_2 + \cdots + X_L < X_1^{'} + X_2^{'} + \cdots + X_L^{'})$, where L is the number of fading channels. In the simplest form, suppose $X_1, X_2, \ldots, X_L$ are independent lognormal random variables with common parameters $\mu = 0$ and σ^2. Suppose too $X_1^{'}, X_2^{'}, \ldots, X_L^{'}$ are independent lognormal random variables independent of $X_1, X_2, \ldots, X_L$ with common parameters $\mu = 0$ and $\sigma = 1$. As mentioned in Sect. 4, a widely used assumption in the determination

Table 1 Estimates of σ for $R = 10^{-2}$

L	σ	L	σ
2	4.852	12	4.604
3	4.795	13	4.593
4	4.756	14	4.582
5	4.725	15	4.573
6	4.700	16	4.563
7	4.679	17	4.554
8	4.661	18	4.546
9	4.645	19	4.538
10	4.630	20	4.531
11	4.617		

of the sum of lognormal random variables is that it can be approximated by another lognormal random variable. Thus, $Z = X_1 + X_2 + \cdots + X_L$ can be approximated by a lognormal random variable with the parameters

$$\sigma_* = \sqrt{\log\left[1 + \frac{\exp(\sigma^2) - 1}{L}\right]}$$

and

$$\mu_* = \log L + \frac{\sigma^2 - \sigma_*^2}{2}.$$

Likewise, $Z^{'} = X_1^{'} + X_2^{'} + \cdots + X_L^{'}$ can be approximated by a lognormal random variable with the parameters

$$\sigma_{**} = \sqrt{\log\left[1 + \frac{e - 1}{L}\right]}$$

and

$$\mu_{**} = \log L + \frac{1 - \sigma_{**}^2}{2}.$$

Since a lognormal random variable is an exponent of a normal random variable, the associated BEP can be computed by using the well-known fact that $\log Z^{'} - \log Z$ also has the normal distribution. We obtain

$$R = \Phi\left(\frac{\mu_* - \mu_{**}}{\sqrt{\sigma_*^2 + \sigma_{**}^2}}\right),$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution. Table 1 provides estimates of the design parameter σ for given values of the BEP and L.

Example 2 Sums of Rayleigh random variables occur extensively in the evaluation of equal gain combining systems when determining the BEP, see Brennan [34]. The BEP can be writ-

Table 2 Estimates of b

L	$R=10^{-2}$	$R=10^{-3}$	$R=10^{-4}$	$R=10^{-5}$	$R=10^{-6}$
2	15.977	53.493	172.052	546.962	1730.790
3	8.466	20.030	44.907	98.491	213.926
4	6.029	12.046	22.706	41.603	75.237
5	4.849	8.754	14.900	24.606	40.003
6	4.155	7.005	11.142	17.270	26.193
7	3.698	5.930	8.989	13.339	19.243
8	3.372	5.205	7.607	10.852	15.062
9	3.128	4.683	6.650	9.093	12.248
10	2.938	4.290	5.951	7.793	10.276
11	2.785	3.983	5.419	6.861	8.936
12	2.659	3.735	4.908	6.256	8.089
13	2.554	3.532	4.662	5.895	7.542
14	2.464	3.362	4.342	5.642	7.110
15	2.386	3.217	4.149	5.409	6.712
16	2.318	3.092	3.950	5.170	6.328
17	2.258	2.983	3.778	4.926	5.957
18	2.205	2.888	3.629	4.681	5.603
19	2.157	2.803	3.497	4.441	5.268
20	2.114	2.727	3.380	4.208	4.952

ten as $R = \Pr(X_1 + X_2 + \cdots + X_L < X_1^{'} + X_2^{'} + \cdots + X_L^{'})$, where L is the number of fading channels and $X_1, X_2, \ldots, X_L$, $X_1^{'}, X_2^{'}, \ldots, X_L^{'}$ are independent Rayleigh random variables. Using the approximation suggested by (7), one can rewrite $R = \Pr(Z < Z^{'})$, where $Z = X_1 + X_2 + \cdots + X_L$ has the pdf

$$f(z) = \frac{2z^{2L-1}\exp\{-z^2/b\}}{b^L(L-1)!}$$

and $Z^{'} = X_1^{'} + X_2^{'} + \cdots + X_L^{'}$ has the pdf

$$f(z) = \frac{2z^{2L-1}\exp\{-z^2\}}{(L-1)!}.$$

Table 2 provides estimates of the design parameter b for given values of the BEP and L.

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References

1. Proakis, J.G.: Digital Communications, 4th edn. McGraw-Hill, New York (2001)
2. Laneman, J.N.: Limiting analysis of outage probabilities for diversity schemes in fading channels. In: Proceedings of GLOBECOM 2003, vol. 3, pp. 1242–1246 (December 2003)
3. Nakagami, M.: The m-distribution general formula of intensity distribution of rapid fading. In: Statistical Methods in Radio Wave Propagation, pp. 3–36. Pergamon, Oxford (1960)

4. Romeo, M., Da Costa, V., Bardou, F.: Broad distribution effects in sums of lognormal random variables. Eur. Phys. J. B **32**, 513–525 (2003)
5. Foerster, J.: UWB channel modeling sub-committee report final. IEEE P802.15 Working Group for Wireless Personal Area Networks (February 2003)
6. Liu, H.: Error performance of a pulse amplitude and position modulated ultra-wideband system over lognormal fading channels. IEEE Commun. Lett. **7**, 531–533 (2003)
7. Divsalar, D., Simon, M.: Trelis coded modulation for 4800-9600 bps transmission over a fading mobile satellite channel. IEEE J. Sel. Areas Commun. **5**, 162–175 (1997)
8. Marcum, J.I.: A statistical theory of target detection by pulsed radar. IRE Trans. Inf. Theory **IT-6**, 59–267 (1960)
9. Ross, S.M.: Probability Models, 8th edn. Academic, Amsterdam (2003)
10. Khuong, H.V., Kong, H.-Y.: General expression for pdf of a sum of independent exponential random variables. IEEE Commun. Lett. **10**, 159–161 (2006)
11. Amari, S.V., Misra, R.B.: Closed-form expressions for distribution of sum of exponential random variables. IEEE Trans. Reliab. **46**, 519–522 (1997)
12. Moschopoulos, P.G.: The distribution of the sum of independent gamma random variables. Ann. Inst. Stat. Math. **37**, 541–544 (1985)
13. Alouini, M.-S., Abdi, A., Kaveh, M.: Sum of gamma variates and performance of wireless communication systems over Nakagami-fading channels. IEEE Trans. Veh. Technol. **50**, 1471–1480 (2001)
14. Provost, S.B.: The exact density of a general linear combination of gamma variates. Metron **46**, 61–69 (1988)
15. Provost, S.B.: On sums of independent gamma random variables. Statistics **20**, 583–591 (1989)
16. Efthymoglou, G., Aalo, V.: Performance of RAKE receivers in Nakagami fading channel with arbitrary fading parameters. Electron. Lett. **31**, 1610–1612 (1995)
17. Aalo, V.A., Piboongungon, T., Efthymoglou, G.P.: Another look at the performance of MRC schemes in Nakagami-m fading channels with arbitrary parameters. IEEE Trans. Commun. **53**, 2002–2005 (2005)
18. Mathai, A.M.: A Handbook of Generalized Special Functions for Statistical and Physical Sciences. Oxford University Press, New York (1993)
19. Mathai, A.M., Saxena, R.K.: The H-function with Applications in Statistics and Other Disciplines. Wiley, New York (1978)
20. Springer, M.D.: The Algebra of Random Variables. Wiley, New York (1979)
21. Beaulieu, N.C., Rajwani, F.: Highly accurate simple closed-form approximations to lognormal sum distributions and densities. IEEE Commun. Lett. **8**, 709–711 (2004)
22. Lam, C.-L.J., Le-Ngoc, T.: Estimation of typical sum of lognormal random variables using log shifted gamma approximation. IEEE Commun. Lett. **10**, 234–235 (2006)
23. Zhao, L., Ding, J.: Least squares approximations to lognormal sum distributions. IEEE Trans. Veh. Technol. **56**, 991–997 (2007)
24. Liu, Z., Almhana, J., Wang, F., McGorman, R.: Mixture lognormal approximations to lognormal sum distributions. IEEE Commun. Lett. **11**, 711–713 (2007)
25. Zhang, Q.T., Song, S.H.: A systematic procedure for accurately approximating lognormal-sum distributions. IEEE Trans. Veh. Technol. **57**, 663–666 (2008)
26. Liu, Z.Z., Almhana, J.J., McGorman, R.R.: Approximating lognormal sum distributions with power lognormal distributions. IEEE Trans. Veh. Technol. (in press)
27. Pratesi, M., Santiccu, F., Graziosi, F.: Generalized moment matching for the linear combination of lognormal RVs: application to outage analysis in wireless systems. IEEE Trans. Wirel. Commun. **5**, 1122–1132 (2006)
28. Ben Slimane, S.: Bounds on the distribution of a sum of independent lognormal random variables. IEEE Trans. Commun. **49**, 975–978 (2001)
29. Hu, J., Beaulieu, N.C.: Accurate simple closed-form approximations to Rayleigh sum distributions and densities. IEEE Commun. Lett. **9**, 109–111 (2005)
30. Schwartz, M., Bennett, W.R., Stein, S.: Communication Systems and Techniques. McGraw-Hill, New York (1966)
31. Karagiannidis, G.K., Tsiftsis, T.A., Sagias, N.C.: A closed-form upper-bound for the distribution of the weighted sum of Rayleigh variates. IEEE Commun. Lett. **9**, 589–591 (2005)
32. Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products, 6th edn. Academic, San Diego (2000)
33. Beaulieu, N.C.: An infinite series for the computation of the complementary probability distribution function of a sum of independent random variables and its application to the sum of Rayleigh random variables. IEEE Trans. Commun. **38**, 1463–1474 (1990)
34. Brennan, D.G.: Linear diversity combining techniques. Proc. IRE **47**, 1075–1102 (1959)