EE 503

Non-measurable Sets

Def: A binary relation R on a set A is said to be an equivalence relation if

$\forall x, y, z \in A$ it is

Reflexive: xRx

Symmetric $xRy \Rightarrow yRx$

Transitive: $xRy \wedge yRz \Rightarrow xRz$

Examples:

$A = \{1, 2, 3, 4\}$ $R = \{(1,1)\ (2,2\} \subseteq A^2$

$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \text{ is even}\}$

$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \text{ is a multiple of } 6\}$

Exercise: A: set of USC students

Which one is an equivalence relation?

$x R_1 y \Rightarrow x, y$ live in the same dorm

$x R_2 y \Rightarrow x, y$ have a course in common

Equivalence Classes

Def: Given an equivalence relation R on A and an element $x \in A$, the equivalence class of x is

$$C(x) = \{ y \in A \mid xRy)$$

It is the set of all members of A that have a relation with x.

Example: $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (2,2), (3,3), (4,4)\}$

what is $C(1)$?

$C(1) = \{1\}$

$R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a-b \text{ is even}\}$

$C(0)$ = set of all even integers

$C(1)$ = set of all odd integers

Theorem: Given an equivalence relation on A, the collection of all equivalence classes form a partition of A.

Let $\Omega = [0,1]$. The question is that if we can build a probability measure that uses 2^Ω as a σ-field. The answer to this question

is NO. 2^Ω is too large, and there are subsets of Ω ~~such~~ ~~that~~ that create problems and contradiction if they are assigned probabilities

We start with the intuitive concept of a <u>uniform</u> probability measure. Such a measure on $\Omega = [0,1]$ must intuitively have the following properties:

(a) $\forall\, 0 \leq a \leq b \leq 1$ we want

$$\mathbb{P}([a,b]) = \mathbb{P}([a,b))$$
$$= \mathbb{P}((a,b]) = \mathbb{P}((a,b))$$
$$= b-a$$

(b) $\mathbb{P}$ must be shift-invariant, i.e.

$\mathbb{P}([0,.5]) = \mathbb{P}([0.25, 0.75])$

$= \mathbb{P}([.75, 1] \cup [0, .25])$

In general $\forall A \subset [0,1]$

and $0 \leq r \leq 1$, define

the following shift operator:

$A \oplus r = \{a+r \mid a \in A, a+r \leq 1\}$

$\cup \{a+r-1 \mid a \in A, a+r > 1\}$

(for example $[0.7, 0.9] \oplus 0.2$

$= [.9, 1] \cup [0, 0.1]$)

We want $\mathbb{P}$ to be shift invariant, i.e.

$$\mathbb{P}(A \oplus r) = \mathbb{P}(A)$$

(c) We want $\mathbb{P}$ to be countably additive:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

for disjoint subsets $A_1, A_2, \ldots$ of $[0,1]$

The question was if the definition of $\mathbb{P}$ is extendable to all subsets of $[0,1]$.

We show that such an extension would result in contradiction.

Suppose $\mathbb{P}$ is defined for all subsets of $[0,1]$

Define the following equivalence relation R on $[0,1]$

$xRy \iff x-y$ is rational $(\in \mathbb{Q})$

R partitions $[0,1]$ into disjoint equivalence classes.

Let H be a subset of $[0,1]$ consisting exactly one element from each equivalence classes.

let $0 \in H$, then $1 \notin H$ (because $1-0 \in \mathbb{Q}$ and we have exactly one element from each equivalence class)

[Note: We can build such a set by axiom of choice in set theory, See wikipedia]

Claim: The sets $H \oplus r$

, $r \in \mathbb{Q} \cap [0,1]$ are disjoint

Proof: If $r_1 \neq r_2$ and $H \oplus r_1$

and $H \oplus r_2$ share the point

$h_1 + r_1 = h_2 + r_2$ with $h_1, h_2 \in H$,

then h_1, h_2 differ by a

rational number and are

equivalent. If $h_1 = h_2 \Rightarrow$

$r_1 = r_2$, which is a contradiction.

If $h_1 \neq h_2$, it means H contains two unequal members of

the same equivalence class, which

contradicts the definition of

H.

Therefore $H \oplus r$'s are

disjoint $\forall r \in \mathbb{Q} \cap [0,1]$

Also, $H \oplus r$'s form a partition

of $[0,1]$. So, $[0,1) = \bigcup_{r \in [0,1] \cap \mathbb{Q}} H \oplus r$

By countable additivity, we have

$$\mathbb{P}([0,1]) = \mathbb{P}\Big(\bigcup_{r \in [0,1] \cap \mathbb{Q}} H \oplus r$$

$$\Rightarrow 1 = \sum_r \mathbb{P}(H \oplus r) = \sum_r \mathbb{P}(H)$$

Since $r \in \mathbb{Q} \cap [0,1]$, any value for

$\mathbb{P}(H)$ results in a contradiction

H is non-measurable

and has to be excluded

from probability calculations.