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Geomagnetism and Schmidt quasi-normalization

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SUMMARY

Spherical harmonic analysis of the main magnetic field of the Earth and its daily variations is the numerical determination of coefficients of solid spherical harmonics in the mathematical expressions used for the magnetic scalar potential of fields of internal and external origin. The coefficients are determined from vector components of the field and their purpose is to represent the vector field, not to reconstruct the magnetic scalar potential. An alternative interpretation of the spherical harmonic analysis is presented: namely the determination of the coefficients of a series representation of the magnetic vector field on a spherical surface in orthonormal real vector spherical harmonics, which correspond to the internal and external fields, and an additional non-potential toroidal field. The numerical values of the coefficients of an orthonormal vector spherical harmonic series have a direct physical significance, which is not obscured by some arbitrary normalization of the vector spherical harmonics. Therefore, we propose a Schmidt vector normalization to be used in conjunction with the Schmidt quasi-normalization of associated Legendre functions. A property of orthonormalized functions is that the standard deviations of the coefficients determined by the method of least squares from ideal data, which are uniformly accurate and uniformly globally distributed, are constant for all coefficients. The real vector spherical harmonic analysis of the geomagnetic field is extended to a spherical shell and conditions that restrict the radial dependence of the vector spherical harmonic coefficients are examined. In particular, two hypotheses for the current systems deriving from the non-potential toroidal component of the magnetic field over the surface of a sphere are presented, namely, Earth–air currents and field-aligned currents.

Key words: geomagnetic potential, normalization, spherical harmonics, vector spherical harmonics.

1 INTRODUCTION

In the current-free region at the surface of the Earth, magnetic fields of internal and external origin are represented by the gradients of linear combinations of solid spherical harmonics based on associated Legendre functions of integer degree and order (see Section 2). This classical solid spherical harmonic representation of the magnetic field of the Earth in terms of the geomagnetic scalar potential is reviewed in Section 4. From his extensive numerical work with associated Legendre functions and their derivatives, Schmidt (1917, p. 281) proposed his normalization of associated Legendre functions as described in Section 3. We shall refer to it as Schmidt quasi-normalization. The Proceedings of the 1939 Washington Assembly of the Association of Terrestrial Magnetism and Electricity of the International Union of Geodesy and Geophysics (Goldie, A.H.R. & Joyce, J.W., eds, 1940) records the following resolution:

'*Normalized Spherical Harmonics.*— The Association recommends that the normalized spherical harmonics of Adolf Schmidt should be generally used in geophysical research.'

The resolution had the great merit of defining exactly what the Schmidt quasi-normalized functions are, thereby avoiding a number of difficulties, such as those that arise when authors provide only the mean square value of their chosen functions and do not indicate the phase factor, $(-1)^m$. Acceptance of the IAGA resolution by the geomagnetic community has meant that the coefficients for the geomagnetic potential derived by different researchers can be compared directly.

An additional non-potential field can be used to resolve inconsistencies between the gradient expressions for northwards and eastwards components of the internal and external fields over the surface of the Earth. Such a non-potential field is associated with an electrical current system, which contradicts the basic assumption of a current-free region at the surface of the Earth. Schmidt (1898) obtained spherical harmonic coefficients for the non-potential field of the order of 10 nT, corresponding to radial current densities of the order of 1000 pA m^{-2}. Price &

Chapman (1928) comment that such a result is inconsistent with the direct measurements of the atmospheric electric potential gradient and the ionization of the air, which indicate a vertical current density of the order of 3 pA m^{-2}. Schmidt (1939) concluded that the non-potential field is just a consequence of remaining errors in the data. Tinsley (2000), analysing the global electric circuit, notes that the ionosphere–earth potential difference gives rise to a vertical current density J_z, which in clear weather can be measured in the range 1–4 pA m^{-2}. Yoshida (2001) discussed the controversy over vertical electrical currents in geomagnetic research and concluded that the non-zero values for the vertical electrical current in ground based magnetic data generally come from the inaccurate calculation of curl $\boldsymbol{H}$ from mesh-point values. Usually, in the analysis of surface data the non-potential field is assumed to be zero. However, for satellite magnetic data gathered in regions in which field-aligned or other currents may be flowing, the non-potential field may be non-zero and must be included in the magnetic field representation.

In other contexts, such as nuclear physics, for example Blatt & Weisskopf (1952), the three types of vector field (internal, external and non-potential) over the surface of a sphere are known as vector spherical harmonics. In this paper, we develop the theory of the vector spherical harmonic analysis of the magnetic field of the Earth in the presence of electrical currents.

Vector spherical harmonics are orthogonal and complete under integration over the surface of the sphere. Their orthogonality and normalization are derived in Section 5 using a novel argument. Schmidt quasi-normalized coefficients determined from magnetic data are only used for determination and representation of the internal, external and non-potential vector fields. However, vector spherical harmonic fields corresponding to internal, external and non-potential fields, based on Schmidt normalized functions and their derivatives, although orthogonal, are not orthonormal. In a representation of the magnetic field based on orthonormal vector fields, the numerical values of computed coefficients do not require normalization factors. Such a representation has other advantages; for example, the magnetic energy is the sum of the squares of the coefficients without the need for different normalization factors for each coefficient. As shown in Section 6, the coefficients of Schmidt quasi-normalized functions, determined for internal, external and non-potential fields must therefore be multiplied by $\sqrt{n+1}$, $\sqrt{n}$ and $\sqrt{n(n+1)/(2n+1)}$, respectively, if the vector spherical harmonic fields are to be orthonormal. This renormalization could be called Schmidt vector normalization to distinguish it from the Schmidt quasi-normalization used only for the associated Legendre functions. The relationship of the real vector spherical harmonics to standard complex vector spherical harmonics is given in Section 7.

In Section 8, we extend the real vector spherical harmonic analysis of magnetic fields from a single spherical surface to a spherical shell containing electrical currents. The classical spherical harmonic analysis of geomagnetic data with the assumption of curl-free magnetic flux density and its extension to include a non-potential toroidal field must in general be further extended to include a non-potential poloidal field. Without additional hypotheses, the radial dependence of the fields cannot be determined. We examine the no toroidal current assumption (NTC), that there is no toroidal current system flowing in the spherical shell over which the data is being analysed, although a poloidal current system is permitted. This poloidal current system is represented graphically by contours of an Earth–air current system, meaning contours of the vector potential function. It is shown that the hypotheses of either a purely radial field or a field-aligned current are sufficient to determine the radial dependence of the non-potential toroidal field. The real vector spherical harmonic representation on a sphere is given in component form in Section 9. The effect of normalization on the standard deviations of the coefficients is discussed in Section 10.

2 FERRERS NORMALIZATION OF ASSOCIATED LEGENDRE FUNCTIONS

Associated Legendre functions $P_{n,m}(\mu)$ were defined by Ferrers (1877):

$$P_{n,m}(\mu) = \frac{1}{2^n n!}(1-\mu^2)^{m/2}\left(\frac{d}{d\mu}\right)^{n+m}(\mu^2-1)^n,\ m \le n. \tag{2.1}$$

In the case $m = 0$, the Ferrers associated Legendre function reduces to a Legendre polynomial, $P_n(\mu)$:

$$P_{n,0}(\mu) = \frac{1}{2^n n!}\left(\frac{d}{d\mu}\right)^{n}(\mu^2-1)^n = P_n(\mu). \tag{2.2}$$

The associated Legendre functions of (2.1), in Ferrers normalization, can also be given as

$$P_{n,m}(\mu) = (1-\mu^2)^{m/2}\left(\frac{d}{d\mu}\right)^{m} P_n(\mu). \tag{2.3}$$

The parameter $\mu = \cos\theta$ is less than 1 when θ is the colatitude, but in some contexts, such as radial dependence in oblate spheroidal harmonics, a function $P_{n,m}(ir/c)$ is required where the argument ir/c is large and complex.

Leibnitz's theorem for multiple derivatives of a product, can be applied to the associated Legendre function defined in (2.1), to show that

$$P_{n,-m}(\mu) = (-1)^m \frac{(n-m)!}{(n+m)!} P_{n,m}(\mu),\ m \ge 0. \tag{2.4}$$

Associated Legendre functions of the same order m,

$P_{m,m}(\mu),\ P_{m+1,m}(\mu),\ P_{m+2,m}(\mu),\ \ldots,$

are orthogonal under integration with respect to μ, from $\mu = -1$ to $\mu = 1$:

$$\int_{-1}^{1} P_{n,m}(\mu) P_{N,m}(\mu) d\mu = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_n^N. \tag{2.5}$$

The Kronecker delta, δ_n^N, is 1 when $n = N$ and is zero otherwise. Eq. (2.5) is valid when $m = 0$ and

$$\int_{-1}^{1} P_n(\mu) P_N(\mu)\, d\mu = \frac{2}{2n+1}\delta_n^N. \tag{2.6}$$

When trigonometric forms for dependence on longitude are used, the $2n + 1$ independent surface spherical harmonics are

$$\begin{aligned} &P_{n,m}(\cos\theta)\cos m\phi,\ \ m = 0, 1, \ldots, n,\\ &P_{n,m}(\cos\theta)\sin m\phi,\ \ m = 1, 2, \ldots, n. \end{aligned} \tag{2.7}$$

They are defined for positive integer values of m only and they are orthogonal under integration over the surface of a sphere. Thus, when $m \neq 0$ and $M \neq 0$,

$$\begin{aligned} &\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} [P_{n,m}(\cos\theta)\cos m\phi][P_{N,M}(\cos\theta)\cos M\phi]\sin\theta\, d\theta\, d\phi\\ &\quad = \frac{1}{2}\int_0^{\pi}[P_{n,m}(\cos\theta)P_{N,M}(\cos\theta)]\sin\theta\, d\theta \cdot \frac{1}{2\pi}\int_0^{2\pi}\cos m\phi\cos M\phi\, d\phi\\ &\quad = \frac{1}{2}\cdot\frac{1}{2n+1}\cdot\frac{(n+m)!}{(n-m)!}\delta_n^N\delta_m^M, \quad m \neq 0 \text{ and } M \neq 0, \end{aligned} \tag{2.8}$$

where δ_n^N and δ_m^M are Kronecker deltas. Similarly,

$$\begin{aligned} &\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} [P_{n,m}(\cos\theta)\sin m\phi][P_{N,M}(\cos\theta)\sin M\phi]\sin\theta\, d\theta\, d\phi\\ &\quad = \frac{1}{2}\cdot\frac{1}{2n+1}\cdot\frac{(n+m)!}{(n-m)!}\delta_n^N\delta_m^M, \quad m \neq 0 \text{ and } M \neq 0. \end{aligned} \tag{2.9}$$

In the special case when $m = 0$ and $M = 0$ (excluded from eq. 2.8), then, from eqs (2.2) and (2.6), the product integral (2.8) reduces to the orthogonality of the Legendre polynomials:

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}[P_{n,0}(\cos\theta)P_{N,0}(\cos\theta)]\sin\theta\, d\theta\, d\phi = \frac{1}{2n+1}\delta_n^N. \tag{2.10}$$

Also, for all m and M, integration with respect to ϕ gives

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}[P_{n,m}(\cos\theta)\sin m\phi][P_{N,M}(\cos\theta)\cos M\phi]\sin\theta\, d\theta\, d\phi = 0. \tag{2.11}$$

Therefore, given a function $f(\theta, \phi)$ over the surface of sphere, to be represented as a sum of surface spherical harmonics, with Ferrers normalization, we write

$$f(\theta,\phi) = \sum_{n=1}^{N}\sum_{m=0}^{n}(a_{n,m}\cos m\phi + b_{n,m}\sin m\phi)P_{n,m}(\cos\theta), \tag{2.12}$$

where the coefficients $a_{n,m}$ and $b_{n,m}$ are determined by integration, or estimated by a numerical approximation to the integration, by

$$\begin{aligned} a_{n,m} &= \frac{2(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)P_{n,m}(\cos\theta)\cos m\phi\sin\theta\, d\theta\, d\phi,\\ b_{n,m} &= \frac{2(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)P_{n,m}(\cos\theta)\sin m\phi\sin\theta\, d\theta\, d\phi,\\ a_{n,0} &= (2n+1)\int_0^{\pi} f(\theta,\phi)P_{n,0}(\cos\theta)\sin\theta\, d\theta, \quad b_{n,0} = 0. \end{aligned} \tag{2.13}$$

From eq. (2.5) with $n = N$ it can be seen that the root-mean-square amplitudes of the Ferrers normalized functions vary wildly from $\sqrt{2/(2n+1)}$ for $P_{n,0}$ to $\sqrt{2(2n)!/(2n+1)}$ for $P_{n,n}$. Ferrers normalized functions are therefore unsuitable for numerical work, producing coefficients $a_{n,m}$ and $b_{n,m}$, which have a wide range of numerical values depending on degree n and order m resulting entirely from the mathematical normalization of the associated Legendre functions. Schmidt quasi-normalization of associated Legendre functions was introduced in order that all such functions had the same normalization as the Legendre polynomials. This damped down the variations of numerical coefficients $a_{n,m}$ and $b_{n,m}$ required when representing scalar functions $f(\theta, \phi)$. However, as will be shown below, when Schmidt quasi-normalization is used in representations of vector functions, an extra factor must be applied, depending on whether the vector is of the internal, external or non-potential type. This does not require a renormalization of the associated Legendre function; rather, it is an extra factor, which is different for the three different types of field. It will be referred to here as Schmidt vector normalization.

3 SCHMIDT QUASI-NORMALIZATION OF SURFACE SPHERICAL HARMONICS

In an encyclopedia article on geomagnetism, Schmidt (1917, p. 281) defined what are now called Schmidt quasi-normalized functions, $P_n^m(\cos\theta)$:

$$\left.\begin{aligned} P_n^0(\mu) &= \frac{1}{2^n n!}\left(\frac{d}{d\mu}\right)^n (\mu^2-1)^n \equiv P_n(\mu) \\ &\text{and, when } m \neq 0, \\ P_n^m(\mu) &= \sqrt{2\frac{(n-m)!}{(n+m)!}}\,P_{n,m}(\mu) \\ &= \sqrt{2\frac{(n-m)!}{(n+m)!}}(1-\mu^2)^{m/2}\left(\frac{d}{d\mu}\right)^m P_n(\mu). \end{aligned}\right] \tag{3.1}$$

The expressions for Schmidt quasi-normalized functions given in eq. (3.1) can be written as one expression valid for m, such that $|m| \leq n$, as

$$P_n^m(\mu) = \sqrt{\left(2-\delta_m^0\right)\frac{(n-m)!}{(n+m)!}}\,P_{n,m}(\mu),\ |m| \leq n. \tag{3.2}$$

Schmidt further defined real surface spherical harmonics, $C_n^m(\theta,\phi)$ and $S_n^m(\theta,\phi)$:

$$\begin{aligned} C_n^m(\theta,\phi) &= P_n^m(\cos\theta)\cos m\phi, \quad \text{for} \quad m = 0, 1, 2, \ldots, n, \\ S_n^m(\theta,\phi) &= P_n^m(\cos\theta)\sin m\phi, \quad \text{for} \quad m = 1, 2, \ldots, n. \end{aligned} \tag{3.3}$$

The Schmidt quasi-normalized functions $P_n^m(\cos\theta)$, as defined in eq. (3.2), are intended for use only with trigonometric dependence on east longitude ϕ, in the form of surface spherical harmonics, $C_n^m(\theta,\phi)$ and $S_n^m(\theta,\phi)$, and for this reason the additional factor of 2 (not used in the Ferrers normalization) is included under the square root sign in the definition of $P_n^m(\mu)$ in eq. (3.1). In place of eqs (2.8)–(2.11) we now have, when $m \neq 0$ and $M \neq 0$,

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} C_n^m(\theta,\phi)C_N^M(\theta,\phi)\sin\theta\,d\theta\,d\phi = \frac{1}{2n+1}\delta_n^N\delta_m^M, \tag{3.4}$$

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} S_n^m(\theta,\phi)S_N^M(\theta,\phi)\sin\theta\,d\theta\,d\phi = \frac{1}{2n+1}\delta_n^N\delta_m^M, \tag{3.5}$$

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} C_n^m(\theta,\phi)S_N^M(\theta,\phi)\sin\theta\,d\theta\,d\phi = 0, \quad \text{all } n,\ N,\ m \text{ and } M. \tag{3.6}$$

When $m = M = 0$, eq. (3.4) reduces to the form given in eq. (2.10).

Eqs (2.10), (3.4), (3.5) and (3.6) show that the $2n+1$ Schmidt quasi-normalized surface spherical harmonics of degree n have the same mean square value, namely $1/(2n+1)$, over the surface of a sphere as the Legendre polynomial of degree n. The functions, and therefore the computed numerical coefficients of these functions, will be independent of order m, but will depend upon degree n.

Therefore, if a function $f(\theta,\phi)$ is to be represented as a linear combination of Schmidt quasi-normalized functions,

$$\begin{aligned} f(\theta,\phi) &= \sum_{n=1}^{N}\sum_{m=0}^{n}\left(a_n^m\cos m\phi + b_n^m\sin m\phi\right)P_n^m(\cos\theta) \\ &= \sum_{n=1}^{N}\sum_{m=0}^{n}\left[a_n^m C_n^m(\theta,\phi) + b_n^m S_n^m(\theta,\phi)\right], \end{aligned} \tag{3.7}$$

then the linear combination coefficients are given by

$$\begin{aligned} a_n^m &= (2n+1)\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)P_n^m(\cos\theta)\cos m\phi\sin\theta\,d\theta\,d\phi, \\ b_n^m &= (2n+1)\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)P_n^m(\cos\theta)\sin m\phi\sin\theta\,d\theta\,d\phi, \\ a_n^0 &= (2n+1)\int_0^{\pi} f(\theta,\phi)\sin\theta\,d\theta, \quad b_n^0 = 0. \end{aligned} \tag{3.8}$$

In earlier work, Schmidt made use of fully normalized functions in surface spherical harmonics, which he denoted $R_n^m(\theta,\phi)$, but decided against their use in favour of the normalization given in eq. (3.1). No reason was given, but one could surmise that Schmidt quasi-normalized functions kept the values required for the radial component of the field, which has a factor $(n+1)$, in a range better suited to numerical work done using logarithms.

4 GAUSS–SCHMIDT ANALYSIS AND SOLID SPHERICAL HARMONICS

The Gauss theory of the mathematical analysis of the magnetic field of the Earth and its variations is given here, with Schmidt's name appended because of his researches on the non-potential field, which is represented in terms of Earth–air currents, in the same way that the potential fields are interpreted in terms of equivalent, toroidal, current systems.

The magnetic flux density $\boldsymbol{B}$ has no divergence and is therefore described as solenoidal. By Ampère's law,

$$\nabla\times\boldsymbol{H}+\frac{\partial\boldsymbol{D}}{\partial t}=\boldsymbol{J}, \tag{4.1}$$

and in the absence of the displacement current $\partial\boldsymbol{D}/\partial t$ and of electrical current density $\boldsymbol{J}$, the magnetic field strength $\boldsymbol{H}$ is curl-free, $\nabla\times\boldsymbol{H}=\boldsymbol{0}$, and $\boldsymbol{H}$ is represented by the gradient of a scalar potential, $\boldsymbol{H}=-\nabla V$. In a region of uniform magnetic susceptibility, μ, the magnetic flux density $\boldsymbol{B}=\mu\boldsymbol{H}$ and therefore $\boldsymbol{B}$ is curl-free, $\nabla\times\ \boldsymbol{B}=\boldsymbol{0}$, and hence, $\boldsymbol{B}$ can be represented by the gradient of a potential function, $\boldsymbol{B}=-\mu\nabla V$.

For the magnetic flux density $\boldsymbol{B}$ to satisfy $\nabla\cdot\ \boldsymbol{B}=0$, the potential V to be used in the analysis of magnetic field components must satisfy Laplace's equation, for which reason it is said to be harmonic. Solution of Laplace's equation in spherical polars leads to solid spherical harmonics with radial dependence $1/r^{n+1}$,

$$\frac{1}{r^{n+1}}P_n^m(\cos\theta)\cos m\phi,\ \frac{1}{r^{n+1}}P_n^m(\cos\theta)\sin m\phi, \tag{4.2}$$

and with radial dependence r^n,

$$r^n P_n^m(\cos\theta)\cos m\phi,\ r^n P_n^m(\cos\theta)\sin m\phi. \tag{4.3}$$

Series of solid spherical harmonics (4.2) are appropriate in regions that exclude the origin, $r=0$, and are used to represent the magnetic field potential outside a reference sphere, often chosen to be the sphere of minimum radius enclosing the sources. Therefore, they are associated with magnetic fields of internal origin. Similarly, series of solid spherical harmonics (4.3) are appropriate inside a reference sphere and are associated with external magnetic fields whose origin is outside the reference sphere.

Relative to a reference sphere of radius a, the expression used for the potential of magnetic field of internal origin is

$$\begin{aligned}V_i(r,\theta,\phi)&=a\sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+1}\sum_{m=0}^{n}\left(g_{ni}^m\cos m\phi+h_{ni}^m\sin m\phi\right)P_n^m(\cos\theta),\ r\geq a,\\&=a\sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+1}\sum_{m=0}^{n}\left[g_{ni}^m C_m^m(\theta,\phi)+h_{ni}^m S_n^m(\theta,\phi)\right],\qquad r\geq a.\end{aligned} \tag{4.4}$$

The initial factor a in eq. (4.4) is chosen so that the coefficients g_{ni}^m and h_{ni}^m will have units of magnetic flux density, teslas. In geomagnetism, the relevant value of a is the mean radius of the Earth, 6371 km. Following the IAGA resolution (Goldie & Joyce 1940), the associated Legendre functions $P_n^m(\cos\theta)\cos m\phi$ and $P_n^m(\cos\theta)\sin m\phi$ are Schmidt quasi-normalized functions. The geomagnetic field of internal origin will be denoted by $\boldsymbol{B}_i(r,\theta,\phi)$ and

$$\begin{aligned}\boldsymbol{B}_i(r,\theta,\phi)&=-\nabla V_i(r,\theta,\phi)\\&=-\frac{\partial V_i}{\partial r}\boldsymbol{e}_r-\frac{1}{r}\frac{\partial V_i}{\partial\theta}\boldsymbol{e}_\theta-\frac{1}{r\sin\theta}\frac{\partial V_i}{\partial\phi}\boldsymbol{e}_\phi,\end{aligned} \tag{4.5}$$

in which $\boldsymbol{e}_r,\boldsymbol{e}_\theta,\boldsymbol{e}_\phi$ are unit vectors in spherical polars, being in the direction of gradients of coordinates, namely $\nabla r,\nabla\theta,\nabla\phi$. The corresponding vector field in the region $r\geq a$ is

$$\boldsymbol{B}_i(r,\theta,\phi)=\sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left[g_{ni}^m\boldsymbol{G}_{ni}^m(\theta,\phi)+h_{ni}^m\boldsymbol{H}_{ni}^m(\theta,\phi)\right]P_n^m(\cos\theta),\ r\geq a \tag{4.6}$$

and, over the sphere $r=a$, the internal field reduces to

$$\boldsymbol{B}_i(a,\theta,\phi)=\sum_{n=1}^{N}\sum_{m=0}^{n}\left[g_{ni}^m\boldsymbol{G}_{ni}^m(\theta,\phi)+h_{ni}^m\boldsymbol{H}_{ni}^m(\theta,\phi)\right], \tag{4.7}$$

where the spherical polar components of the internal vector harmonics $\boldsymbol{G}_{ni}^m(\theta,\phi)$ and $\boldsymbol{H}_{ni}^m(\theta,\phi)$ are

$$\begin{aligned}\boldsymbol{G}_{ni}^m(\theta,\phi)&=(n+1)P_n^m\cos m\phi\boldsymbol{e}_r-\frac{dP_n^m}{d\theta}\cos m\phi\boldsymbol{e}_\theta+\frac{m}{\sin\theta}P_n^m\sin m\phi\,\boldsymbol{e}_\phi,\\\boldsymbol{H}_{ni}^m(\theta,\phi)&=(n+1)P_n^m\sin m\phi\boldsymbol{e}_r-\frac{dP_n^m}{d\theta}\sin m\phi\boldsymbol{e}_\theta-\frac{m}{\sin\theta}P_n^m\cos m\phi\,\boldsymbol{e}_\phi.\end{aligned} \tag{4.8}$$

Eq. (4.7) is the basis for the statement that the coefficients g_{ni}^m and h_{ni}^m are applied to the vector fields $\boldsymbol{G}_{ni}^m(\theta,\phi)$ and $\boldsymbol{H}_{ni}^m(\theta,\phi)$, respectively, and it is therefore appropriate that these vector fields should be normalized to have a unit mean square value over the surface of the sphere.

The use of the scalar components of eq. (4.8) in numerical analyses may give the impression that the coefficients of a scalar potential are being sought.

Similarly, the field of external origin is denoted by $\boldsymbol{B}_e(r, \theta, \phi)$, where

$$\begin{aligned}\boldsymbol{B}_e(r,\theta,\phi) &= -\nabla V_e(r,\theta,\phi)\\ &= -\frac{\partial V_e}{\partial r}\boldsymbol{e}_r - \frac{1}{r}\frac{\partial V_e}{\partial\theta}\boldsymbol{e}_\theta - \frac{1}{r\sin\theta}\frac{\partial V_e}{\partial\phi}\boldsymbol{e}_\phi.\end{aligned} \tag{4.9}$$

The field of external origin is an essential part of the analysis of magnetic daily variations and disturbances and it is convenient to assume that it is valid within a sphere of radius b, representing the lower limit of the ionosphere or magnetosphere, where $b > a$. The magnetic potential $V_e\,(r, \theta, \phi)$ for vector fields of external origin is a linear combination of solid spherical harmonics as in eq. (4.3):

$$\begin{aligned}V_e(r,\theta,\phi) &= b\sum_{n=1}^{N}\left(\frac{r}{b}\right)^n\sum_{m=0}^{n}\left[g_{ne}^m\cos m\phi + h_{ne}^m\sin m\phi\right]P_n^m(\cos\theta), \quad r\le b\\ &= b\sum_{n=1}^{N}\left(\frac{r}{b}\right)^n\sum_{m=0}^{n}\left[g_{ne}^m C_n^m(\theta,\phi) + h_{ne}^m S_n^m(\theta,\phi)\right], \quad r\le b.\end{aligned} \tag{4.10}$$

The external field $\boldsymbol{B}_e(r, \theta, \phi) = -\nabla V_e\,(r, \theta, \phi)$, in the region $r \le b$ is

$$\boldsymbol{B}_e(r,\theta,\phi) = \sum_{n=1}^{N}\left(\frac{r}{b}\right)^{n-1}\sum_{m=0}^{n}\left[g_{ne}^m\boldsymbol{G}_{ne}^m(\theta,\phi) + h_{ne}^m\boldsymbol{H}_{ne}^m(\theta,\phi)\right] \tag{4.11}$$

and, at the surface of the sphere $r = a$, the external field is

$$\boldsymbol{B}_e(a,\theta,\phi) = \sum_{n=1}^{N}\left(\frac{a}{b}\right)^{n-1}\sum_{m=0}^{n}\left[g_{ne}^m\mathbf{G}_{ne}^m(\theta,\phi) + h_{ne}^m\boldsymbol{H}_{ne}^m(\theta,\phi)\right]. \tag{4.12}$$

It is the standard practice to use the expression

$$\boldsymbol{B}_e(a,\theta,\phi) = \sum_{n=1}^{N}\sum_{m=0}^{n}\left[g_{ne}^m\boldsymbol{G}_{ne}^m(\theta,\phi) + h_{ne}^m\boldsymbol{H}_{ne}^m(\theta,\phi)\right] \tag{4.13}$$

in place of eq. (4.12) and to apply the factor $(\frac{b}{a})^{n-1}$ to the computed coefficients at the conclusion of the calculation, as part of the determination of an equivalent current function, meaning an equivalent toroidal current system. The spherical polar components of the external vector spherical harmonics, $\boldsymbol{G}_{ne}^m(\theta, \phi)$ and $\boldsymbol{H}_{ne}^m(\theta, \phi)$, are

$$\begin{aligned}\boldsymbol{G}_{ne}^m(\theta,\phi) &= -nP_n^m\cos m\phi\,\boldsymbol{e}_r - \frac{dP_n^m}{d\theta}\cos m\phi\,\boldsymbol{e}_\theta + \frac{m}{\sin\theta}P_n^m\sin m\phi\,\boldsymbol{e}_\phi,\\ \boldsymbol{H}_{ne}^m(\theta,\phi) &= -nP_n^m\sin m\phi\,\boldsymbol{e}_r - \frac{dP_n^m}{d\theta}\sin m\phi\,\boldsymbol{e}_\theta - \frac{m}{\sin\theta}P_n^m\cos m\phi\,\boldsymbol{e}_\phi.\end{aligned} \tag{4.14}$$

Therefore the coefficients g_{ne}^m and h_{ne}^m are applied to the vector fields $\boldsymbol{G}_{ne}^m(\theta, \phi)$ and $\boldsymbol{H}_{ne}^m(\theta, \phi)$, respectively, and it is appropriate that these vector fields should be normalized to have a unit mean square value over the surface of the unit sphere.

The representation of the main magnetic field of the Earth, its daily variations and its disturbance variations in terms of fields of internal and external origin using solid spherical harmonic functions was extended by Schmidt (1898) to include a non-potential field. The purpose of the non-potential field is to represent any stream-function part of the magnetic field over the surface of a sphere that cannot be represented as the gradient of a scalar potential. The three fields (internal, external and non-potential), over the surface of a sphere, correspond to the three types of vector spherical harmonics, orthogonal under integration over the surface of a sphere (see Section 5).

The $\boldsymbol{e}_\theta$ components in eqs (4.8) and (4.14), of $\boldsymbol{G}_{ni}^m$ and $\boldsymbol{G}_{ne}^m$ (and of $\boldsymbol{H}_{ni}^m$ and $\boldsymbol{H}_{ne}^m$), have the same mathematical coefficients and, similarly, the same $\boldsymbol{e}_\phi$ components. Therefore, these vector fields cannot deal with the case where the horizontal field components are not represented entirely by the gradients of scalar potentials. To deal with this problem, Schmidt (1917, p. 281) suggested using a non-potential field, $\boldsymbol{B}_v\,(r, \theta, \phi)$, which has a vector potential, $\mathbf{r}V_v(r, \theta, \phi)$. Thus,

$$\begin{aligned}\boldsymbol{B}_v(r,\theta,\phi) &= \frac{1}{\sin\theta}\frac{\partial V_v}{\partial\phi}\boldsymbol{e}_\theta - \frac{\partial V_v}{\partial\theta}\boldsymbol{e}_\phi\\ &= -\boldsymbol{r}\times\nabla V_v(r,\theta,\phi)\\ &= \nabla\times\left[\boldsymbol{r}V_v(r,\theta,\phi)\right].\end{aligned} \tag{4.15}$$

For the sake of simplicity in the next few equations, we consider the function $V_v\,(r, \theta, \phi)$ with just a single term:

$$\begin{aligned}V_v(r,\theta,\phi) &= g_{nv}^m\frac{q_n^m(r)}{q_n^m(a)}P_n^m(\cos\theta)\cos m\phi,\\ \boldsymbol{B}_v(r,\theta,\phi) &= g_{nv}^m\frac{q_n^m(r)}{q_n^m(a)}\left[-\frac{m}{\sin\theta}P_n^m(\cos\theta)\sin m\phi\boldsymbol{e}_\theta - \frac{dP_n^m}{d\theta}\cos m\phi\boldsymbol{e}_\phi\right].\end{aligned} \tag{4.16}$$

From $\boldsymbol{B}_v\ (r, \theta, \phi)$ in the curl form (4.15), it will be clear that the divergence, $\nabla\cdot\boldsymbol{B}_v(r, \theta, \phi)$, is zero, regardless of the expression used for the radial dependence $q_n^m(r)$ of $V_v(r, \theta, \phi)$. The associated current system is, say $\boldsymbol{J}_v(r, \theta, \phi)$, where

$$\boldsymbol{J}_v(r, \theta, \phi) = \frac{1}{\mu_0}\nabla\times\boldsymbol{B}_v(r, \theta, \phi)$$
$$= \frac{g_{nv}^m}{\mu_0 r q_n^m(a)}\left\{\boldsymbol{e}_r q_n^m(r) n(n+1) P_n^m(\cos\theta)\cos m\phi + \frac{d}{dr}\left[r q_n^m(r)\right]\left(\boldsymbol{e}_\theta\frac{\partial}{\partial\theta} + \frac{\boldsymbol{e}_\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right) P_n^m(\cos\theta)\cos m\phi\right\}. \tag{4.17}$$

One is free to choose whatever expression is relevant for $q_n^m(r)$. The choice

$$q_n^m(r) = \frac{1}{r} \tag{4.18}$$

leads to the radial current system, often called Earth–air, in which exactly half of the current will flow in the reverse direction from the air to Earth. Therefore, over the reference sphere $r = a$, eq. (4.17) reduces to

$$\boldsymbol{J}_v(r, \theta, \phi) = \frac{g_{nv}^m}{\mu_0 a} n(n+1) P_n^m(\cos\theta)\cos m\phi\, \boldsymbol{e}_r. \tag{4.19}$$

Noting that the permeability of free space μ_0 is $4\pi\times 10^{-7}$ henrys per metre, where the unit of inductance, the henry, is one weber per ampere, the unit of magnetic flux density, the tesla, is one weber per square metre and the mean radius of the Earth is $a = 6371000$ m, the radial current density $\boldsymbol{J}_v(r, \theta, \phi)$ is

$$\boldsymbol{J}_v(r, \theta, \phi) = \frac{10000}{80060} n(n+1)\left(g_{nv}^m\right)_{\text{teslas}} P_n^m(\cos\theta)\cos m\phi,\ \text{A m}^{-2}. \tag{4.20}$$

Now

$$1\text{A m}^{-2} = 10^6\ \text{A km}^{-2} = 10^{12}\ \text{pA m}^{-2},$$

and, therefore,

$$\boldsymbol{J}_v(r, \theta, \phi) = \frac{10000}{80060}\times 10^3 n(n+1)\left(g_{nv}^m\right)_{\text{nT}} P_n^m(\cos\theta)\cos m\phi,\ \text{pA m}^{-2}$$
$$= 124.9 n(n+1)\left(g_{nv}^m\right)_{\text{nT}} P_n^m(\cos\theta)\cos m\phi,\ \text{pA m}^{-2}. \tag{4.21}$$

Therefore the figure of 1–4 pA m^{-2}, quoted by Tinsley (2000) for the global electric circuit is at least 2 orders of magnitude less than the current that would derive from a non-potential field of some tens of nanoteslas. The global electric circuit current cannot be distinguished from random noise using geomagnetic data. One is free to choose other expressions for $q_n^m(r)$ that will lead to non-potential fields coming from poloidal current systems that have both radial and horizontal components.

For the function $V_v(r, \theta, \phi)$, it is convenient to use

$$V_v(r, \theta, \phi) = \sum_{n=1}^{N}\sum_{m=0}^{n}\left[g_{nv}^m(r)\cos m\phi + h_{nv}^m(r)\sin m\phi\right] P_n^m(\cos\theta)$$
$$= \sum_{n=1}^{N}\sum_{m=0}^{n}\left[g_{nv}^m(r) C_n^m(\theta, \phi) + h_{nv}^m(r) S_n^m(\theta, \phi)\right]. \tag{4.22}$$

As shown in eq. (4.15), the condition that the magnetic flux density $\boldsymbol{B}$ has no divergence does not place any restriction on the radial functions $g_{nv}^m(r)$ and $h_{nv}^m(r)$. The non-potential field $\boldsymbol{B}_v(r, \theta, \phi)$ is

$$\boldsymbol{B}_v(r, \theta, \phi) = \sum_{n=1}^{N}\sum_{m=0}^{n}\left[g_{nv}^m(r)\boldsymbol{G}_{nv}^m(\theta, \phi) + h_{nv}^m(r)\boldsymbol{H}_{nv}^m(\theta, \phi)\right], \tag{4.23}$$

where the vector fields $\boldsymbol{G}_{n\mathrm{v}}^m(\theta, \phi)$ and $\boldsymbol{H}_{n\mathrm{v}}^m(\theta, \phi)$ have spherical polar components

$$\boldsymbol{G}_{nv}^m(\theta, \phi) = -\frac{m}{\sin\theta} P_n^m \sin m\phi\, \boldsymbol{e}_\theta - \frac{dP_n^m}{d\theta}\cos m\phi\, \boldsymbol{e}_\phi,$$
$$\boldsymbol{H}_{nv}^m(\theta, \phi) = \frac{m}{\sin\theta} P_n^m \cos m\phi\, \boldsymbol{e}_\theta - \frac{dP_n^m}{d\theta}\sin m\phi\, \boldsymbol{e}_\phi. \tag{4.24}$$

Over the surface of a sphere $r = a$, with $g_{nv}^m\ (a) = g_{nv}^m$ and $h_{nv}^m\ (a) = h_{nv}^m$, the non-potential field is

$$\boldsymbol{B}_v(a, \theta, \phi) = \sum_{n=1}^{N}\sum_{m=0}^{n}\left[g_{nv}^m \boldsymbol{G}_{nv}^m(\theta, \phi) + h_{nv}^m \boldsymbol{H}_{nv}^m(\theta, \phi)\right], \tag{4.25}$$

The coefficients g_{nv}^m and $h_{n\mathrm{v}}^m$ are applied to the vector fields $\boldsymbol{G}_{nv}^m(\theta, \phi)$ and $\boldsymbol{H}_{nv}^m(\theta, \phi)$, respectively, and it is appropriate that these vector fields should be normalized to have unit mean square values over the surface of a unit sphere.

This non-potential field is a toroidal field. Because a poloidal field can also be a non-potential field, in order to be precise, we will refer to the non-potential field of eq. (4.25) as the non-potential toroidal field and use this nomenclature in the following sections.

The equations above describe the theoretical basis for the spherical harmonic analysis of the geomagnetic main field and daily variation field. The radial dependence of the internal and external fields is given in eqs (4.6) and (4.11).

When the non-potential toroidal field is included, the assumption of a current-free region (used to derive radial dependence of internal and external fields from gradients of a potential function) may be replaced by the assumption that no toroidal current system flows over the surface for which the data is available. This may not be true for satellite magnetic data, gathered in the region between the ionosphere and the magnetopause.

A further consideration arises with satellite magnetic data, especially if the orbit is non-spherical or it is supplemented by other data, e.g. observatory data. The magnetic field representation must be valid for the volume of a spherical shell, not just a spherical surface. In the following, we develop the representation and analysis of the magnetic field in terms of real vector spherical harmonics and investigate the radial dependence under various assumptions.

5 ORTHOGONALITY OF VECTOR SPHERICAL HARMONICS

5.1 First surface integrals

To derive the orthogonality properties of vector spherical harmonics, two different sets of surface integrals are required. For the first set, we commence with the vector $\boldsymbol{A}_1(\theta, \phi)$ given by

$$\boldsymbol{A}_1(\theta, \phi) = C(\theta, \phi)\nabla D(\theta, \phi), \tag{5.1}$$

where

$$\begin{cases} C(\theta, \phi) = P_n^m(\cos\theta)\cos m\phi, \\ D(\theta, \phi) = P_N^M(\cos\theta)\cos M\phi. \end{cases} \tag{5.2}$$

The Laplacian of $D(\theta, \phi)$ is

$$\nabla^2 D(\theta, \phi) = -N(N+1)P_N^M(\cos\theta)\cos M\phi. \tag{5.3}$$

The vector $\boldsymbol{A}_1(\theta, \phi)$ is therefore

$$\begin{aligned} \boldsymbol{A}_1(\theta, \phi) &= P_n^m(\cos\theta)\cos m\phi\nabla\left[P_N^M(\cos\theta)\cos M\phi\right] \\ &= \frac{1}{r}P_n^m(\cos\theta)\cos m\phi\left[\frac{dP_N^M}{d\theta}\cos M\phi\,\boldsymbol{e}_\theta - \frac{M}{\sin\theta}P_N^M(\cos\theta)\sin M\phi\,\boldsymbol{e}_\phi\right] \end{aligned} \tag{5.4}$$

and it has no radial component.

The divergence of $\boldsymbol{A}_1(\theta, \phi)$ is obtained using

$$\nabla\cdot(C\nabla D) = \nabla C\cdot\nabla D + C\nabla^2 D, \tag{5.5}$$

so that

$$\nabla\cdot\boldsymbol{A}_1 = \frac{1}{r^2}\left[\frac{dP_n^m}{d\theta}\frac{dP_N^M}{d\theta}\cos m\phi\cos M\phi + \frac{mM}{\sin^2\theta}P_n^m P_N^M\sin m\phi\sin M\phi - N(N+1)P_n^m P_N^M\cos m\phi\cos M\phi\right]. \tag{5.6}$$

Gauss's theorem for the integration of $\nabla\cdot\,\boldsymbol{A}_1$ throughout a closed sphere is

$$\underset{\substack{\text{closed}\\\text{sphere}}}{\iiint}(\nabla\cdot\boldsymbol{A}_1)\,dv = \underset{\substack{\text{spherical}\\\text{surface}}}{\iint}\boldsymbol{A}_1\cdot d\boldsymbol{S}, \tag{5.7}$$

in which $d\boldsymbol{S}$ is parallel to the unit radius vector and is normal to the spherical surface. The vector $\boldsymbol{A}_1(\theta, \phi)$ defined in eq. (5.1) has no radial component and, therefore, the spherical surface integral on the right of eq. (5.7) is zero and eq. (5.7) reduces to

$$\underset{\substack{\text{closed}\\\text{sphere}}}{\iiint}(\nabla\cdot\boldsymbol{A}_1)\,dv = 0. \tag{5.8}$$

Because the element of volume $dv = r^2\sin\theta\,d\theta\,d\phi$ in spherical polars and because of the factor $1/r^2$ in $\nabla\cdot\,\boldsymbol{A}_1$, in eq. (5.6), the integration with respect to r in the volume integral in eq. (5.8) reduces to $\int_0^R dr = R$. From eqs (5.6) and (5.8),

$$\frac{1}{4\pi}\iint\left[-N(N+1)P_n^m P_N^M\cos m\phi\cos M\phi + \frac{dP_n^m}{d\theta}\frac{dP_N^M}{d\theta}\cos m\phi\cos M\phi + \frac{mM}{\sin^2\theta}P_n^m P_N^M\sin m\phi\sin M\phi\right]\sin\theta\,d\theta\,d\phi = 0, \tag{5.9}$$

where the integration is over the complete spherical surface.

There appears to be a singularity at the poles, $\theta = 0$ and $\theta = \pi$, in the surface integral in eq. (5.9), because of the factor $1/\sin\theta$ in the integrand of

$$\frac{1}{4\pi}\int\int\left(\frac{mM}{\sin\theta}P_n^m P_N^M \sin m\phi \sin M\phi\right) d\phi\, d\phi. \tag{5.10}$$

The surface integral (5.10) is zero when either $m = 0$ or $M = 0$ and in the other cases, $m \geq 1$ and $M \geq 1$, the functions $P_n^m(\cos\theta)$ and $P_N^M(\cos\theta)$ have factors $\sin^m\theta$ and $\sin^M\theta$, respectively, which effectively remove the apparent singularity due to the factor $1/\sin\theta$.

From eq. (5.9), and from the orthogonality of Schmidt normalized surface spherical harmonics in eq. (3.4),

$$\begin{aligned}\frac{1}{4\pi}\iint\left(\frac{dP_n^m}{d\theta}\frac{dP_N^M}{d\theta}\cos m\phi\cos M\phi + \frac{mM}{\sin^2\theta}P_n^m P_N^M \sin m\phi\sin M\phi\right)\sin\theta\, d\theta\, d\phi \\ = N(N+1)\frac{1}{4\pi}\iint P_n^m P_N^M \cos m\phi \cos M\phi \sin\theta\, d\theta\, d\phi, \\ = \frac{n(n+1)}{2n+1}\delta_n^N\delta_m^M. \end{aligned} \tag{5.11}$$

5.2 Second surface integral

To derive the second set of surface integrals, we commence with a vector $\boldsymbol{A}_2(\theta, \phi)$ defined by

$$\begin{aligned}\boldsymbol{A}_2(\theta,\phi) &= \nabla\times[C(\theta,\phi)\nabla D(\theta,\phi)] \\ &= \nabla C(\theta,\phi)\times\nabla D(\theta,\phi) \\ &= \frac{\boldsymbol{e}_r}{r^2\sin\theta}\left(\frac{\partial C}{\partial\theta}\frac{\partial D}{\partial\phi} - \frac{\partial C}{\partial\phi}\frac{\partial D}{\partial\theta}\right).\end{aligned} \tag{5.12}$$

Using

$$\begin{cases} C(\theta,\phi) = P_n^m(\cos\theta)\cos m\phi, \\ D(\theta,\phi) = P_N^M(\cos\theta)\sin M\phi, \end{cases} \tag{5.13}$$

the vector $\boldsymbol{A}_2(\theta, \phi)$ is therefore

$$\boldsymbol{A}_2(\theta,\phi) = \frac{\boldsymbol{e}_r}{r^2\sin\theta}\left(M\frac{dP_n^m}{d\theta}P_N^M\cos m\phi\cos M\phi + mP_n^m\frac{dP_N^M}{d\theta}\sin m\phi\sin M\phi\right) \tag{5.14}$$

and from the curl form in eq. (5.12), $\boldsymbol{A}_2(\theta, \phi)$ has no divergence:

$$\nabla\cdot\boldsymbol{A}_2 = 0. \tag{5.15}$$

In contrast to the vector $\boldsymbol{A}_1(\theta, \phi)$ of eq. (5.4), which has no radial component and a non-zero divergence, the vector $\boldsymbol{A}_2(\theta, \phi)$ therefore has only a radial component and has no divergence.

Gauss's theorem (5.7), using $\boldsymbol{A}_2(\theta, \phi)$, reduces to

$$\iint_{\substack{\text{spherical}\\ \text{surface}}} \boldsymbol{A}_2\cdot d\boldsymbol{S} = 0, \tag{5.16}$$

giving directly that

$$\frac{1}{4\pi}\iint\left(M\frac{dP_n^m}{d\theta}P_N^M\cos m\phi\cos M\phi + mP_n^m\frac{dP_N^M}{d\theta}\sin m\phi\sin M\phi\right)d\theta\, d\phi = 0. \tag{5.17}$$

5.3 Orthogonalities within type of vector spherical harmonics

Vector fields $\boldsymbol{G}_{ni}^m$, $\boldsymbol{H}_{ni}^m$ are defined in eq. (4.8). Vector fields $\boldsymbol{G}_{ne}^m$, $\boldsymbol{H}_{ne}^m$ are defined in eq. (4.14). Vector fields $\boldsymbol{G}_{n\mathrm{v}}^m$, $\boldsymbol{H}_{n\mathrm{v}}^m$ are defined in eq. (4.24).

Note that integrals of some scalar products of $\boldsymbol{G}$ and $\boldsymbol{H}$ components of the internal field will be zero on integration over the entire spherical surface simply by integration with respect to ϕ. Thus,

$$\frac{1}{4\pi}\iint\boldsymbol{G}_{ni}^m(\theta,\phi)\cdot\boldsymbol{H}_{Ni}^M(\theta,\phi)\sin\theta\, d\theta\, d\phi = 0, \tag{5.18}$$

$$\frac{1}{4\pi}\iint\boldsymbol{G}_{ne}^m(\theta,\phi)\cdot\boldsymbol{H}_{Ne}^M(\theta,\phi)\sin\theta\, d\theta\, d\phi = 0 \tag{5.19}$$

and

$$\frac{1}{4\pi}\iint\boldsymbol{G}_{nv}^m(\theta,\phi)\cdot\boldsymbol{H}_{Nv}^M(\theta,\phi)\sin\theta\, d\theta\, d\phi = 0. \tag{5.20}$$

By eq. (5.11), the $\boldsymbol{G}_{ni}^m(\theta, \phi)$ internal field vectors are orthogonal,

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{G}_{ni}^m(\theta, \phi)\cdot \boldsymbol{G}_{Ni}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = \left[\frac{(n+1)^2}{2n+1} + \frac{n(n+1)}{2n+1}\right]\delta_n^N\delta_m^M$$
$$= (n+1)\delta_n^N\delta_m^M,$$

with a corresponding result for the $\boldsymbol{H}_{ni}^m(\theta, \phi)$ components of the internal field

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{H}_{ni}^m(\theta, \phi)\cdot \boldsymbol{H}_{Ni}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = (n+1)\delta_n^N\delta_m^M. \tag{5.21}$$

Also, by eq. (5.11), the $\boldsymbol{G}_{ne}^m(\theta, \phi)$ external field vectors are orthogonal,

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{G}_{ne}^m(\theta, \phi)\cdot \boldsymbol{G}_{Ne}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = \left[\frac{n^2}{2n+1} + \frac{n(n+1)}{2n+1}\right]\delta_n^N\delta_m^M$$
$$= n\delta_n^N\delta_m^M,$$

with a corresponding result for the $\boldsymbol{H}_{ne}^m(\theta, \phi)$ components of the external field

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{H}_{ne}^m(\theta, \phi)\cdot \boldsymbol{H}_{Ne}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = n\delta_n^N\delta_m^M. \tag{5.22}$$

By eq. (5.11), the $\boldsymbol{G}_{nv}^m(\theta, \phi)$ components of the non-potential toroidal field are orthogonal,

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{G}_{nv}^m(\theta, \phi)\cdot \boldsymbol{G}_{Nv}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = \frac{n(n+1)}{2n+1}\delta_n^N\delta_m^M,$$

with a corresponding result for the $\boldsymbol{H}_{nv}^m(\theta, \phi)$ components of the non-potential toroidal field,

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{H}_{nv}^m(\theta, \phi)\cdot \boldsymbol{H}_{Nv}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = \frac{n(n+1)}{2n+1}\delta_n^N\delta_m^M. \tag{5.23}$$

From eqs (5.21), (5.22) and (5.23), respectively, the mean square values of the internal, external and non-potential toroidal fields are

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \left[\boldsymbol{G}_{ni}^m(\theta, \phi)\right]^2\sin\theta\, d\theta\, d\phi = (n+1),\ \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \left[\boldsymbol{H}_{ni}^m(\theta, \phi)\right]^2\sin\theta\, d\theta\, d\phi = (n+1), \tag{5.24}$$

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \left[\boldsymbol{G}_{ne}^m(\theta, \phi)\right]^2\sin\theta\, d\theta\, d\phi = n,\ \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \left[\boldsymbol{H}_{ne}^m(\theta, \phi)\right]^2\sin\theta\, d\theta\, d\phi = n, \tag{5.25}$$

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \left[\boldsymbol{G}_{nv}^m(\theta, \phi)\right]^2\sin\theta\, d\theta\, d\phi = \frac{n(n+1)}{2n+1},\ \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \left[\boldsymbol{H}_{nv}^m(\theta, \phi)\right]^2\sin\theta\, d\theta\, d\phi = \frac{n(n+1)}{2n+1}. \tag{5.26}$$

The result (5.24) is used in deriving the Lowes–Mauersberger spectrum, (Lowes 1966).

5.4 Orthogonalities between types of vector spherical harmonic

The three different types of $\boldsymbol{G}$ field are orthogonal within their own type but also between types. Thus from eq. (5.9), or by eq. (5.17) or by the orthogonality of the trigonometric functions on integration with respect to ϕ from 0 to 2π:
internal and external fields,

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{G}_{ni}^m(\theta, \phi)\cdot \boldsymbol{G}_{Ne}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = 0,\ \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{G}_{ni}^m(\theta, \phi)\cdot \boldsymbol{H}_{Ne}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = 0,$$
$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{H}_{ni}^m(\theta, \phi)\cdot \boldsymbol{G}_{Ne}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = 0,\ \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{H}_{ni}^m(\theta, \phi)\cdot \boldsymbol{H}_{Ne}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = 0;$$

internal and non-potential fields,

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{G}_{ni}^m(\theta, \phi)\cdot \boldsymbol{G}_{Nv}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = 0,\ \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{G}_{ni}^m(\theta, \phi)\cdot \boldsymbol{H}_{Nv}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = 0,$$
$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{H}_{ni}^m(\theta, \phi)\cdot \boldsymbol{G}_{Nv}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = 0,\ \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{H}_{ni}^m(\theta, \phi)\cdot \boldsymbol{H}_{Nv}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = 0;$$

external and non-potential fields,

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{G}_{ne}^m(\theta, \phi)\cdot \boldsymbol{G}_{Nv}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = 0,\ \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{G}_{ne}^m(\theta, \phi)\cdot \boldsymbol{H}_{Nv}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = 0,$$
$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{H}_{ne}^m(\theta, \phi)\cdot \boldsymbol{G}_{Nv}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = 0,\ \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \boldsymbol{H}_{ne}^m(\theta, \phi)\cdot \boldsymbol{H}_{Nv}^M(\theta, \phi)\sin\theta\, d\theta\, d\phi = 0.$$

Therefore, any finite linear combination of internal vector fields $\boldsymbol{G}^m_{ni}(\theta,\phi)$ and $\boldsymbol{H}^m_{ni}(\theta,\phi)$ is orthogonal to any finite linear combination of external vector fields $\boldsymbol{G}^m_{ne}(\theta,\phi)$ and $\boldsymbol{H}^m_{ne}(\theta,\phi)$, under integration over the surface of a sphere.

6 ORTHONORMAL VECTOR SPHERICAL HARMONICS

From eqs (5.24), (5.25) and (5.26) of the previous section, fully normalized vector fields with unit mean square value are

$$\begin{aligned}\frac{1}{\sqrt{n+1}}\boldsymbol{G}^m_{ni} \;\text{ and }\; \frac{1}{\sqrt{n+1}}\boldsymbol{H}^m_{ni},\\ \frac{1}{\sqrt{n}}\boldsymbol{G}^m_{ne} \;\text{ and }\; \frac{1}{\sqrt{n}}\boldsymbol{H}^m_{ne},\\ \sqrt{\frac{2n+1}{n(n+1)}}\boldsymbol{G}^m_{ne} \;\text{ and }\; \sqrt{\frac{2n+1}{n(n+1)}}\boldsymbol{H}^m_{nv}.\end{aligned} \tag{6.1}$$

From eqs (4.6) and (6.1), we may write the internal part of the main field as

$$\boldsymbol{B}_i(r,\theta,\phi) = \sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left[\left(\sqrt{n+1}\,g^m_{ni}\right)\frac{\boldsymbol{G}^m_{ni}(\theta,\phi)}{\sqrt{n+1}} + \left(\sqrt{n+1}\,h^m_{ni}\right)\frac{\boldsymbol{H}^m_{ni}(\theta,\phi)}{\sqrt{n+1}}\right],$$

showing that for work with orthonormalized vector fields:

(i) internal field coefficients, g^m_{ni}, h^m_{ni}, determined using Schmidt quasi-normalized scalar functions should be multiplied by $\sqrt{n+1}$; similarly
(ii) external field coefficients, g^m_{ne}, h^m_{ne}, should be multiplied by $\sqrt{n}$; and
(iii) non-potential toroidal field coefficients, g^m_{nv}, h^m_{nv}, should be multiplied by $\sqrt{n(n+1)/(2n+1)}$.

7 COMPLEX VECTOR SPHERICAL HARMONICS

The orthogonality of the internal, external and non-potential fields under integration over the surface of a sphere is not often used directly in spherical harmonic analyses of the magnetic field of the Earth or its variations, but the orthogonality of the equivalent vector spherical harmonics is well known in the theory of the rotation group in three dimensions. Studies of vector spherical harmonics are usually given in terms of complex surface spherical harmonics, $Y^m_n(\theta,\phi)$, complex reference vectors, $\boldsymbol{e}_1, \boldsymbol{e}_0, \boldsymbol{e}_{-1}$, and particular recurrence relations for surface spherical harmonics using $3-j$ vector coupling coefficients, e.g. James (1976). In this paper, we give the analysis in terms of real variables only, with trigonometric expressions for dependence on longitude, because real variable expressions will be more familiar to those to whom this research is directed.

In terms of the Ferrers normalized functions (2.1), the complex variable form $Y^m_n(\theta,\phi)$ of the surface spherical harmonic is

$$Y^m_n(\theta,\phi) = (-1)^m\sqrt{(2n+1)\frac{(n-m)!}{(n+m)!}}\,P_{n,m}(\cos\theta)e^{im\phi}, \tag{7.1}$$

where a phase factor $(-1)^m$ is included, following Condon & Shortley (1967). In terms of Schmidt normalized functions,

$$Y^m_n(\theta,\phi) = (-1)^m\sqrt{\frac{2n+1}{2-\delta^0_m}}\,P^m_n(\cos\theta)e^{im\phi}. \tag{7.2}$$

Complex vector spherical harmonics are defined as follows, Blatt & Weisskopf (1952):

$$\begin{aligned}\boldsymbol{Y}^m_{n,n+1}(\theta,\phi) &= \frac{1}{\sqrt{(n+1)(2n+1)}}r^{n+2}\nabla\left[\frac{1}{r^{n+1}}Y^m_n(\theta,\phi)\right],\\ \boldsymbol{Y}^m_{n,n}(\theta,\phi) &= -\frac{i}{\sqrt{n(n+1)}}\boldsymbol{r}\times\nabla Y^m_n(\theta,\phi),\\ \boldsymbol{Y}^m_{n,n-1}(\theta,\phi) &= \frac{1}{\sqrt{n(2n+1)}}\frac{1}{r^{n-1}}\nabla\left[r^n Y^m_n(\theta,\phi)\right].\end{aligned} \tag{7.3}$$

The vector spherical harmonics $\boldsymbol{Y}^m_{n,n+1}(\theta,\phi)$, $\boldsymbol{Y}^m_{n,n}(\theta,\phi)$ and $\boldsymbol{Y}^m_{n,n-1}(\theta,\phi)$ under rotation of the reference frame transform like the scalar surface spherical harmonics $Y^m_{n+1}(\theta,\phi)$, $Y^m_n(\theta,\phi)$ and $Y^m_{n-1}(\theta,\phi)$, respectively, but are derived from $Y^m_n(\theta,\phi)$. Their properties of orthogonality and completeness follow directly from the manner of their construction.

From eqs (4.8), (4.14) and (4.24) using Schmidt-normalized functions $P^m_n(\cos\theta)$,

$$\begin{aligned}\boldsymbol{G}^m_{ni}(\theta,\phi) + i\,\boldsymbol{H}^m_{ni}(\theta,\phi) &= -r^{n+2}\nabla\left[\frac{1}{r^{n+1}}P^m_n(\cos\theta)e^{im\phi}\right],\\ \boldsymbol{G}^m_{ne}(\theta,\phi) + i\,\boldsymbol{H}^m_{ne}(\theta,\phi) &= -\frac{1}{r^{n-1}}\nabla\left[r^n P^m_n(\cos\theta)e^{im\phi}\right],\\ \boldsymbol{G}^m_{nv}(\theta,\phi) + i\,\boldsymbol{H}^m_{nv}(\theta,\phi) &= -\boldsymbol{r}\times\nabla\left[P^m_n(\cos\theta)e^{im\phi}\right].\end{aligned} \tag{7.4}$$

Therefore, from eqs (7.3) and (7.4), using eq. (7.2),

$$\begin{aligned}
\boldsymbol{G}^m_{ni}(\theta,\phi) + i\,\boldsymbol{H}^m_{ni}(\theta,\phi) &= (-1)^{m+1}\sqrt{(2-\delta^0_m)(n+1)}\;\boldsymbol{Y}^m_{n,n+1}(\theta,\phi),\\
\boldsymbol{G}^m_{ne}(\theta,\phi) + i\,\boldsymbol{H}^m_{ne}(\theta,\phi) &= (-1)^{m+1}\sqrt{(2-\delta^0_m)n}\;\boldsymbol{Y}^m_{n,n-1}(\theta,\phi),\\
\boldsymbol{G}^m_{nv}(\theta,\phi) + i\,\boldsymbol{H}^m_{nv}(\theta,\phi) &= (-1)^{m+1}\sqrt{(2-\delta^0_m)\frac{n(n+1)}{2n+1}}\;i\,\boldsymbol{Y}^m_{n,n}(\theta,\phi).
\end{aligned} \tag{7.5}$$

Therefore, Gauss–Schmidt analysis of geomagnetic data over the surface of a sphere is, in fact, analysis by means of vector spherical harmonics.

8 RADIAL DEPENDENCE

The standard Gauss–Schmidt analysis of geomagnetic data over the surface of a reference sphere has been given in Section 4, and the orthogonality of the internal, external and non-potential toroidal fields under integration over the surface of the reference sphere established in Section 5. In Sections 6 and 7, it has been shown that the Gauss–Schmidt analysis of magnetic data over the surface of a sphere is equivalent to a representation of the magnetic field by means of orthonormal vector spherical harmonics.

For the analysis of data throughout a spherical shell, the radial dependence of the vector spherical harmonic coefficients in the magnetic field representation has to be determined subject to the basic assumption that the vector field has no divergence. Full determination of the radial dependence of the coefficients requires hypotheses concerning the nature of the electrical current systems throughout the shell. The coefficients, $g^m_{ni}, g^m_{ne}, g^m_{nv}$, and $h^m_{ni}, h^m{}_{ne}, h^m_{nv}$, which are constants when determined over a single sphere, will generally have a radial dependence when determined from magnetic field values throughout a thick spherical shell, such as from a combination of ground and satellite data. It is therefore appropriate to denote them $g^m_{ni}(r)$, $g^m_{ne}(r)$, $g^m_{nv}(r)$ and $h^m_{ni}(r)$, $h^m_{ne}(r)$, $h^m_{nv}(r)$.

Thus, to determine a model for the magnetic flux density of the Earth, $\boldsymbol{B}(r,\theta,\phi)$, throughout a spherical shell, we start with the representation

$$\boldsymbol{B}(r,\theta,\phi) = \sum_{n=1}^{N}\sum_{m=0}^{n}\left[g^m_{ni}(r)\boldsymbol{G}^m_{ni}(\theta,\phi) + h^m_{ni}(r)\boldsymbol{H}^m_{ni}(\theta,\phi) + g^m_{ne}(r)\boldsymbol{G}^m_{ne}(\theta,\phi) + h^m_{ne}(r)\boldsymbol{H}^m_{ne}(\theta,\phi) + g^m_{nv}(r)\boldsymbol{G}^m_{nv}(\theta,\phi) + h^m_{nv}(r)\boldsymbol{H}^m_{nv}(\theta,\phi)\right]. \tag{8.1}$$

The summation over n has been truncated at $n = N$, but any sufficiently smooth vector field can be represented by a series of the form (8.1) in the limit as $N \to \infty$, because vector spherical harmonics are complete. The non-potential toroidal field based on eq. (4.22) is given by $g^m_{nv}(r)\;\boldsymbol{G}^m_{nv}(\theta,\phi) + h^m_{nv}\;(r)\;\boldsymbol{H}^m_{nv}(\theta,\phi)$, but because of the as yet undetermined variation with radius, the other terms in eq. (8.1) may also be non-potential fields. Such non-potential poloidal fields were not included in Schmidt's extension of Gauss's analysis. We now examine ways to restrict the radial dependence of the coefficients in eq. (8.1).

8.1 The divergence of B

The classical vector harmonic fields defined by eqs (4.6), (4.11) and (4.23) all have zero divergence. However, this is not necessarily the case for the first two when the more general radial dependence of eq. (8.1) is introduced. The divergences are

$$\nabla\cdot\left[g^m_{ni}(r)\boldsymbol{G}^m_{ni}(\theta,\phi)\right] = (n+1)\left[\frac{g^m_{ni}(r)}{dr} + \frac{n+2}{r}g^m_{ni}(r)\right]P^m_n(\cos\theta)\cos m\phi, \tag{8.2}$$

$$\nabla\cdot\left[g^m_{ne}(r)\boldsymbol{G}^m_{ne}(\theta,\phi)\right] = -n\left[\frac{dg^m_{ne}(r)}{dr} - \frac{n-1}{r}g^m_{ne}(r)\right]P^m_n(\cos\theta)\cos m\phi, \tag{8.3}$$

$$\nabla\cdot\left[g^m_{nv}(r)\boldsymbol{G}^m_{nv}(\theta,\phi)\right] = 0, \tag{8.4}$$

with corresponding results for $\boldsymbol{H}^m_{ni}(\theta,\phi)$, $\boldsymbol{H}^m_{ne}(\theta,\phi)$ and $\boldsymbol{H}^m_{nv}(\theta,\phi)$.

The sum of all the contributions must be divergence-free everywhere, and therefore it follows from eqs (8.2) and (8.3) that the coefficient of $P^m_n(\cos\theta)\cos m\phi$ in the expression for the divergence of $\boldsymbol{B}$ must be zero, and therefore

$$(n+1)\left[\frac{dg^m_{ni}(r)}{dr} + \frac{n+2}{r}g^m_{ni}(r)\right] - n\left[\frac{dg^m_{ne}(r)}{dr} - \frac{n-1}{r}g^m_{ne}(r)\right] = 0, \tag{8.5}$$

and similarly for the coefficient of $P^m_n(\cos\theta)\sin m\phi$,

$$(n+1)\left[\frac{dh^m_{ni}(r)}{dr} + \frac{n+2}{r}h^m_{ni}(r)\right] - n\left[\frac{dh^m_{ne}(r)}{dr} - \frac{n-1}{r}h^m_{ne}(r)\right] = 0. \tag{8.6}$$

The eq. (8.5) is not sufficient to specify the two independent functions $g^m_{ni}(r)$, $g^m_{ne}(r)$ and similarly eq. (8.6) is not sufficient to specify $h^m_{ni}(r)$, $h^m_{ne}(r)$. The sources of the magnetic flux density need to be known or their geometry specified.

From eq. (8.4), the functions $g^m_{nv}(r)$, $h^m_{nv}(r)$, for the non-potential toroidal component of the field, are not constrained by the mathematical condition of zero divergence.

8.2 The curl of $\boldsymbol{B}$

The Schmidt non-potential extension to the Gauss internal–external analysis of magnetic field values is essential, although incomplete, for the analysis of satellite magnetic data gathered in the region between the ionosphere and magnetosphere, where electrical current systems are known to flow, particularly field-aligned currents. Hence, there are mathematical and physical reasons for including the non-potential toroidal field and the non-potential poloidal field in the analysis of satellite magnetic data.

The currents associated with the representation (8.1) are given by its curl, because, according to Ampère's Law, in the absence of displacement currents,

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \tag{8.7}$$

where $\boldsymbol{J}$ is the current density. Of course, an arbitrary curl-free potential field could be added to the magnetic field, so that

$$\nabla \times (\boldsymbol{B} + \nabla V) = \mu_0 \boldsymbol{J}.$$

Therefore, given magnetic flux density $\boldsymbol{B}$, the current density $\boldsymbol{J}$ is known, but given the current density $\boldsymbol{J}$, the magnetic flux density is known only to within a curl-free gradient field. Vector algebra gives

$$\nabla \times \left[g_{ni}^m(r)\boldsymbol{G}_{ni}^m(\theta,\phi)\right] = \left[\frac{dg_{ni}^m(r)}{dr} + \frac{n+2}{r}g_{ni}^m(r)\right]\left(-\frac{m}{\sin\theta}P_n^m \sin m\phi \boldsymbol{e}_\theta - \frac{dP_n^m}{d\theta}\cos m\phi \boldsymbol{e}_\phi\right), \tag{8.8}$$

$$\nabla \times \left[g_{ne}^m(r)\boldsymbol{G}_{ne}^m(\theta,\phi)\right] = \left[\frac{dg_{ne}^m(r)}{dr} - \frac{n-1}{r}g_{ne}^m(r)\right]\left(-\frac{m}{\sin\theta}P_n^m \sin m\phi \boldsymbol{e}_\theta - \frac{dP_n^m}{d\theta}\cos m\phi \boldsymbol{e}_\phi\right), \tag{8.9}$$

$$\nabla \times \left[g_{nv}^m(r)\boldsymbol{G}_{nv}^m(\theta,\phi)\right] = n(n+1)g_{nv}^m(r)\frac{1}{r}P_n^m \cos m\phi \boldsymbol{e}_r - \frac{1}{r}\frac{d}{dr}\left[rg_{nv}^m(r)\right]\left(-\frac{dP_n^m}{d\theta}\cos m\phi \boldsymbol{e}_\theta + \frac{m}{\sin\theta}P_n^m \sin m\phi \boldsymbol{e}_\phi\right), \tag{8.10}$$

with analogous results for $\boldsymbol{H}_{ni}^m(\theta,\phi)$, $\boldsymbol{H}_{ne}^m(\theta,\phi)$ and $\boldsymbol{H}_{nv}^m(\theta,\phi)$.

Eqs (8.8)–(8.10) show that, apart from curl-free arbitrary scalar potential fields, the non-potential toroidal fields $g_{nv}^m(r)\ \boldsymbol{G}_{nv}^m(\theta,\phi)$ and $h_{nv}^m(r)\ \boldsymbol{H}_{nv}^m(\theta,\phi)$ are the only magnetic fields arising from an electrical current system having a radial component. The current systems associated with the non-potential toroidal field coefficients g_{nv}^m and h_{nv}^m, called the non-potential field in the Gauss–Schmidt analysis, can be seen from the right hand side of eq. (8.10) to be poloidal with $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\phi$ components and are not restricted to radial components only.

Given only spherical harmonic coefficients for the field determined over a single spherical surface, Schmidt chose a/r for the functions $g_{nv}^m(r)$, $h_{nv}^m(r)$, in which case

$$\frac{d}{dr}\left[rg_{nv}^m(r)\right] = 0, \qquad \frac{d}{dr}\left[rh_{nv}^m(r)\right] = 0,$$

leading directly to a purely radial (Earth–air) current system. The choice of (radial) Earth–air currents to represent the non-potential toroidal field is not the only possible choice.

Eqs (8.8) and (8.9) are for toroidal current systems, whilst eq. (8.10) is for a poloidal current system. From the definition (4.24) of $\boldsymbol{G}_{nv}^m$, it follows that the results of eqs (8.8) and (8.9) can be written

$$\begin{aligned}\nabla \times \left[g_{ni}^m(r)\boldsymbol{G}_{ni}^m(\theta,\phi)\right] &= \left[\frac{dg_{ni}^m(r)}{dr} + \frac{n+2}{r}g_{ni}^m(r)\right]\boldsymbol{G}_{nv}^m(\theta,\phi)\\ &= \mu_0 j_{nvi}^m(r)\boldsymbol{G}_{nv}^m(\theta,\phi),\end{aligned} \tag{8.11}$$

$$\begin{aligned}\nabla \times \left[g_{ne}^m(r)\boldsymbol{G}_{ne}^m(\theta,\phi)\right] &= \left[\frac{dg_{ne}^m(r)}{dr} - \frac{n-1}{r}g_{ne}^m(r)\right]\boldsymbol{G}_{nv}^m(\theta,\phi)\\ &= \mu_0 j_{nve}^m(r)\boldsymbol{G}_{nv}^m(\theta,\phi).\end{aligned} \tag{8.12}$$

Eqs (8.11) and (8.12) show the relationships

$$j_{nvi}^m(r) = \frac{1}{\mu_0 r^{n+1}}\frac{d}{dr}\left[r^{n+2}g_{ni}^m(r)\right],\ j_{nve}^m(r) = \frac{r^{n-1}}{\mu_0}\frac{d}{dr}\left[\frac{1}{r^{n-1}}g_{ne}^m(r)\right].$$

From eqs (4.8) and (4.14), we can combine $\boldsymbol{G}_{ni}^m(\theta,\phi)$ and $\boldsymbol{G}_{ne}^m(\theta,\phi)$ to give purely radial and purely tangential fields:

$$\begin{aligned}P_n^m \cos m\phi \boldsymbol{e}_r &= \frac{1}{2n+1}\left[\boldsymbol{G}_{ni}^m(\theta,\phi) - \boldsymbol{G}_{ne}^m(\theta,\phi)\right]\\ -\frac{dP_n^m}{d\theta}\cos m\phi \boldsymbol{e}_\theta + \frac{m}{\sin\theta}P_n^m \sin m\phi \boldsymbol{e}_\phi &= \frac{1}{2n+1}\left[n\boldsymbol{G}_{ni}^m(\theta,\phi) + (n+1)\boldsymbol{G}_{ne}^m(\theta,\phi)\right].\end{aligned} \tag{8.13}$$

From eq. (8.13), it follows that eq. (8.10) for the curl of the non-potential toroidal field can be written in terms of internal and external vector spherical harmonics as

$$\nabla \times \left[g_{nv}^m(r)\boldsymbol{G}_{nv}^m(\theta,\phi)\right] = -\frac{n}{2n+1}\left[\frac{dg_{nv}^m(r)}{dr} - \frac{n}{r}g_{nv}^m(r)\right]\boldsymbol{G}_{ni}^m(\theta,\phi) - \frac{n+1}{2n+1}\left[\frac{dg_{nv}^m(r)}{dr} + \frac{n+1}{r}g_{nv}^m(r)\right]\boldsymbol{G}_{ne}^m(\theta,\phi), \tag{8.14}$$

with corresponding results for $\boldsymbol{H}_{nv}^m(\theta,\phi)$, $\boldsymbol{H}_{ni}^m(\theta,\phi)$ and $\boldsymbol{H}_{ne}^m(\theta,\phi)$.

From eqs (8.11) to (8.14), the curl of the magnetic field $\boldsymbol{B}(r, \theta, \phi)$ defined by eq. (8.1) is

$$\nabla \times \boldsymbol{B}(r,\theta,\phi) = \sum_{n=1}^{N}\sum_{m=0}^{n}\left[\left\{\left[\frac{dg_{ni}^m(r)}{dr} + \frac{n+2}{r} g_{ni}^m(r)\right] + \left[\frac{dg_{ne}^m(r)}{dr} - \frac{n-1}{r} g_{ne}^m(r)\right]\right\} \boldsymbol{G}_{nv}^m(\theta,\phi) \right.$$
$$\left. - \frac{n}{2n+1}\left[\frac{dg_{nv}^m(r)}{dr} - \frac{n}{r} g_{nv}^m(r)\right] \boldsymbol{G}_{ni}^m(\theta,\phi) - \frac{n+1}{2n+1}\left[\frac{dg_{nv}^m(r)}{dr} + \frac{n+1}{r} g_{nv}^m(r)\right] \boldsymbol{G}_{ne}^m(\theta,\phi)\right]$$
$$+ \text{corresponding terms in} \boldsymbol{H}_{nv}^m,\ \boldsymbol{H}_{ni}^m, \text{ and } \boldsymbol{H}_{ne}^m. \tag{8.15}$$

Thus, $\nabla \times \boldsymbol{B}$ consists of a toroidal electrical current from internal and external fields, and a poloidal electrical current from the non-potential fields.

To determine the radial dependence of the coefficients, we assume that there is no toroidal current system flowing in the shell throughout which data are being analyzed. This no toroidal current assumption (NTC) eliminates the non-potential poloidal field from coefficients $g_{ni}^m(r)$, $h_{ni}^m(r)$ and $g_{ne}^m(r)$, $h_{ne}^m(r)$. The radial dependence of the non-potential toroidal field from $g_{nv}^m(r)$, $h_{nv}^m(r)$ is then determined for two simple hypotheses concerning the poloidal currents: (PC1) purely radial Earth–air currents; or (PC2) field-aligned currents. Other hypotheses are possible but are not considered here.

8.3 (NTC) No toroidal electrical current, Gauss–Schmidt analysis

In the absence of displacement current, $\nabla \times \boldsymbol{H} = \boldsymbol{J}$ and therefore the electrical current density $\boldsymbol{J}$ has no divergence. Consequently, the electrical current system has poloidal and toroidal components.

The basic assumption made in the standard Gauss–Schmidt analysis is that there is no toroidal current system over the spherical surfaces for which data is available. This excludes from the analysis data gathered in those regions such as the ionosphere or magnetopause. The assumption of no toroidal current system, requires that the coefficients of $\boldsymbol{G}_{nv}^m(\theta,\phi)$ and $\boldsymbol{H}_{nv}^m(\theta,\phi)$ in eq. (8.15) are zero, which in turn requires that internal and external field coefficients are constrained by

$$\left[\frac{dg_{ni}^m(r)}{dr} + \frac{n+2}{r} g_{ni}^m(r)\right] + \left[\frac{dg_{ne}^m(r)}{dr} - \frac{n-1}{r} g_{ne}^m(r)\right] = 0, \tag{8.16}$$

with an analogous equation for $h_{ni}^m(r)$, $h_{ne}^m(r)$. Eq. (8.15) reduces to

$$\nabla \times \boldsymbol{B}(r,\theta,\phi) = \sum_{n=1}^{N}\sum_{m=0}^{n}\left\{-\frac{n}{2n+1}\left[\frac{dg_{nv}^m(r)}{dr} - \frac{n}{r} g_{nv}^m(r)\right] \boldsymbol{G}_{ni}^m(\theta,\phi) - \frac{n+1}{2n+1}\left[\frac{dg_{nv}^m(r)}{dr} + \frac{n+1}{r} g_{nv}^m(r)\right] \boldsymbol{G}_{ne}^m(\theta,\phi)\right\}$$
$$+ \text{corresponding terms in } \boldsymbol{H}_{ni}^m \text{ and } \boldsymbol{H}_{ne}^m. \tag{8.17}$$

Eq. (8.5) for a solenoidal magnetic field, together with eq. (8.16), require that two separate equations must be satisfied, namely

$$\frac{dg_{ni}^m(r)}{dr} + \frac{n+2}{r} g_{ni}^m(r) = 0 \text{ and } \frac{dg_{ne}^m(r)}{dr} - \frac{n-1}{r} g_{ne}^m(r) = 0, \tag{8.18}$$

with corresponding equations for $h_{ni}^m(r)$, $h_{ne}^m(r)$. Solving the eqs (8.18), with the condition at a reference sphere $r = a$, that

$$g_{ni}^m(a) = g_{ni}^m \text{ and } g_{ne}^m(a) = g_{ne}^m, \tag{8.19}$$

the fields of internal and external origin have the well-known radial dependence,

$$g_{ni}^m(r) = \left(\frac{a}{r}\right)^{n+2} g_{ni}^m, \text{ and } g_{ne}^m(r) = \left(\frac{r}{a}\right)^{n-1} g_{ne}^m, \tag{8.20}$$

where g_{ni}^m, g_{ne}^m and h_{ni}^m, h_{ne}^m are constants, which is the basic mathematical form for the Gauss–Schmidt analysis dealing systematically with the non-potential field.

The non-potential toroidal magnetic flux density $\boldsymbol{B}_v(r, \theta, \phi)$ is defined by

$$\boldsymbol{B}_v(r,\theta,\phi) = \sum_{n=1}^{N}\sum_{m=0}^{n}\left[g_{nv}^m(r)\boldsymbol{G}_{nv}^m(\theta,\phi) + h_{nv}^m(r)\boldsymbol{H}_{nv}^m(\theta,\phi)\right].$$

The hypothesis that there is no poloidal current requires that the coefficients of $\boldsymbol{G}_{ni}^m(\theta,\phi)$ and $\boldsymbol{G}_{ne}^m(\theta,\phi)$ in eq. (8.17) be zero, i.e.

$$\begin{aligned} &\frac{dg_{nv}^m(r)}{dr} - \frac{n}{r} g_{nv}^m(r) = 0 \quad \text{and} \quad \frac{dg_{nv}^m(r)}{dr} + \frac{n+1}{r} g_{nv}^m(r) = 0, \\ &\frac{dh_{nv}^m(r)}{dr} - \frac{n}{r} h_{nv}^m(r) = 0 \quad \text{and} \quad \frac{dh_{nv}^m(r)}{dr} + \frac{n+1}{r} h_{nv}^m(r) = 0, \end{aligned} \tag{8.21}$$

holds on any surface $r =$ constant. These equations are satisfied when

$$g_{nv}^m(R) = 0,\ \left.\frac{dg_{nv}^m}{dr}\right|_{r=R} = 0 \quad \text{and} \quad h_{nv}^m(R) = 0,\ \left.\frac{dh_{nv}^m}{dr}\right|_{r=R} = 0, \tag{8.22}$$

so that not only are the non-potential coefficients g_{nv}^m and h_{nv}^m both required to be zero over the reference sphere $r = R$, but so also are their radial gradients.

8.4 (PC1) Radial Earth–air currents

As noted in eq. (8.4), the functions $g_{nv}^m(r)$ and $h_{nv}^m(r)$ giving the radial dependence of the non-potential toroidal field are not constrained by the requirement that magnetic flux density should be divergence-free. With the choice

$$g_{nv}^m(r) = g_{nv}^m \frac{a}{r} \quad \text{and} \quad h_{nv}^m(r) = h_{nv}^m \frac{a}{r}, \tag{8.23}$$

in which g_{nv}^m and h_{nv}^m are numerical coefficients in units of teslas determined from spherical harmonic analysis over the surface of a sphere $r = a$, the expression (8.10) reduces to a radial component only:

$$\nabla \times \left[g_{nv}^m(r)\boldsymbol{G}_{nv}^m(\theta,\phi)\right] = n(n+1)g_{nv}^m \frac{a}{r^2} P_n^m(\cos\theta)\cos m\phi\, \boldsymbol{e}_r. \tag{8.24}$$

By eqs (8.8), (8.9) and (8.20),

$$\nabla \times \boldsymbol{B}_i(r,\theta,\phi) = \boldsymbol{0} \quad \text{and} \quad \nabla \times \boldsymbol{B}_e(r,\theta,\phi) = \boldsymbol{0},$$

and the corresponding hypothetical radial current system is, by eq. (8.10),

$$\begin{aligned}\mu_0 \boldsymbol{J}(r,\theta,\phi) &= \nabla \times \left[\boldsymbol{B}_i(r,\theta,\phi) + \boldsymbol{B}_e(r,\theta,\phi) + \boldsymbol{B}_v(r,\theta,\phi)\right]\\ &= \nabla \times \left\{\frac{a}{r}\sum_{n=1}^{N}\sum_{m=0}^{n}\left[g_{nv}^m \boldsymbol{G}_{nv}^m(\theta,\phi) + h_{nv}^m \boldsymbol{H}_{nv}^m(\theta,\phi)\right]\right\}\\ &= \frac{a}{r^2}\sum_{n=1}^{N}\sum_{m=0}^{n} n(n+1)\left(g_{nv}^m \cos m\phi + h_{nv}^m \sin m\phi\right) P_n^m(\cos\theta)\,\boldsymbol{e}_r\end{aligned} \tag{8.25}$$

and, over a particular shell, $r = a$,

$$\boldsymbol{J}(a,\theta,\phi) = \frac{1}{\mu_0 a}\sum_{n=1}^{N}\sum_{m=0}^{n} n(n+1)\left(g_{nv}^m \cos m\phi + h_{nv}^m \sin m\phi\right) P_n^m(\cos\theta)\,\boldsymbol{e}_r. \tag{8.26}$$

Eq. (8.26) is used to provide a scalar representation of the radial current flow across the surface of the sphere, with positive values for a radially outwards current, negative values for a radially inwards current.

8.5 (PC2) Field-aligned currents

A second example where the radial dependence of the toroidal magnetic field is fixed occurs with the hypothesis that the currents are aligned to a magnetic field of either internal or external type.

For magnetic data observed at satellite altitudes, a hypothesis of field-aligned currents may be useful. The hypothesis assumes a non-potential field $\boldsymbol{B}_v(r,\theta,\phi)$ corresponding to a current system $(\frac{1}{\mu_0})\nabla \times \boldsymbol{B}_v(r,\theta,\phi)$ that is parallel to a field of internal origin, assuming of course, that any field of external origin is negligibly small. Thus,

$$\nabla \times \boldsymbol{B}_v = \frac{k(r,\theta,\phi)}{a}\boldsymbol{B}, \tag{8.27}$$

where the magnetic field along which the non-potential field current system is aligned is $\boldsymbol{B}(r,\theta,\phi)$, where

$$\boldsymbol{B}(r,\theta,\phi) = \sum_{n=1}^{N}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^{n+2}\left[G_n^m \boldsymbol{G}_{ni}^m(\theta,\phi) + H_n^m \boldsymbol{H}_{ni}^m(\theta,\phi)\right]. \tag{8.28}$$

For simplicity, we consider only the idealised case in which $k(r,\theta,\phi)$ is a constant, k say. By eq. (4.23), for $\boldsymbol{B}_v(r,\theta,\phi)$, and eq. (8.14),

$$\nabla \times \boldsymbol{B}_v(r,\theta,\phi) = \sum_{n=1}^{N}\sum_{m=0}^{N} -\frac{n}{2n+1}\left[\frac{dg_{nv}^m(r)}{dr} - \frac{n}{r}g_{nv}^m(r)\right]\boldsymbol{G}_{ni}^m(\theta,\phi) - \frac{n+1}{2n+1}\left[\frac{dg_{nv}^m(r)}{dr} + \frac{n+1}{r}g_{nv}^m(r)\right]\boldsymbol{G}_{ne}^m(\theta,\phi). \tag{8.29}$$

However, by eqs (8.27) and (8.28),

$$\nabla \times \boldsymbol{B}_v(r,\theta,\phi) = \frac{k}{a}\sum_{n=1}^{N}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^{n+2}\left[G_n^m \boldsymbol{G}_{ni}^m(\theta,\phi) + H_n^m \boldsymbol{H}_{ni}^m(\theta,\phi)\right]. \tag{8.30}$$

Comparing coefficients in eqs (8.29) and (8.30),

$$-\frac{n}{2n+1}\left[\frac{dg_{nv}^n(r)}{dr} - \frac{n}{r}g_{nv}^m(r)\right] = \frac{k}{a}\left(\frac{a}{r}\right)^{n+2} G_n^m, \tag{8.31}$$

$$-\frac{n+1}{2n+1}\left[\frac{dg_{nv}^n(r)}{dr} + \frac{n+1}{r}g_{nv}^m(r)\right] = 0. \tag{8.32}$$

From eq. (8.32), it follows that

$$g_{nv}^m(r) = C\frac{1}{r^{n+1}}, \tag{8.33}$$

where C is a constant. Substituting eq. (8.33) into eq. (8.31) gives $C = (k/n)a^{n+1}G_n^m$ and therefore

$$g_{nv}^m(r) = \frac{k}{n}\left(\frac{a}{r}\right)^{n+1} G_n^m. \tag{8.34}$$

Eq. (8.34) shows that the coefficients G_n^m, H_n^m of a field of internal origin, to which the currents are aligned, will be based on the non-potential field coefficients $g_{nv}^m(a)$, $h_{nv}^m(a)$, respectively,

$$G_n^m = \frac{n}{k} g_{nv}^m(a), \quad H_n^m = \frac{n}{k} h_{nv}^m(a), \tag{8.35}$$

but that the radial dependence of the coefficients $g_{nv}^m(r)$ will be of the form given in eq. (8.34).

9 SPHERICAL HARMONIC ANALYSIS

Over any sphere, magnetic flux density measurements can be represented as a sum of vector spherical harmonics and if the chosen sphere is the Earth, then the fields are the usual internal, external and non-potential fields. Thus, apart from the usual random error term, we write

$$\boldsymbol{B}(a,\theta,\phi) = \sum_{n=1}^{N}\sum_{m=0}^{n}\left\{\left[g_{ni}^m \boldsymbol{G}_{ni}^m(\theta,\phi) + h_{ni}^m \boldsymbol{H}_{ni}^m(\theta,\phi)\right] + \left[g_{ne}^m \boldsymbol{G}_{ne}^m(\theta,\phi) + h_{ne}^m \boldsymbol{H}_{ne}^m(\theta,\phi)\right] + \left[g_{nv}^m \boldsymbol{G}_{nv}^m(\theta,\phi) + h_{nv}^m \boldsymbol{H}_{nv}^m(\theta,\phi)\right]\right\}. \tag{9.1}$$

Magnetic field components in spherical polars, corresponding to north, east and radially downwards, are denoted X, Y and Z, where

$$X = -\boldsymbol{B}\cdot\boldsymbol{e}_\theta, \quad Y = \boldsymbol{B}\cdot\boldsymbol{e}_\phi, \quad Z = -\boldsymbol{B}\cdot\boldsymbol{e}_r. \tag{9.2}$$

These are the geocentric, not geographic definitions of X, Y and Z. Spherical harmonic analyses of data over a sphere, corresponding to a reference sphere, $r = a$, requires the analysis of $X(a, \theta, \phi)$, $Y(a, \theta, \phi)$ and $Z(a, \theta, \phi)$ whose theoretical expressions are as follows:

$$X(a,\theta,\phi) = \sum_{n=1}^{N}\sum_{m=0}^{n}\left[\left(g_{ni}^m + g_{ne}^m\right)\cos m\phi + \left(h_{ni}^m + h_{ne}^m\right)\sin m\phi\right]\frac{dP_n^m}{d\theta} + \left(g_{nv}^m \sin m\phi - h_{nv}^m \cos m\phi\right)\frac{m}{\sin\theta}P_n^m, \tag{9.3}$$

$$Y(a,\theta,\phi) = \sum_{n=1}^{N}\sum_{m=0}^{n}\left[\left(g_{ni}^m + g_{ne}^m\right)\sin m\phi - \left(h_{ni}^m + h_{ne}^m\right)\cos m\phi\right]\frac{m}{\sin\theta}P_n^m - \left(g_{nv}^m \cos m\phi + h_{nv}^m \sin m\phi\right)\frac{dP_n^m}{d\theta}, \tag{9.4}$$

$$Z(a,\theta,\phi) = \sum_{n=1}^{N}\sum_{m=0}^{n}\left[-(n+1)\left(g_{ni}^m \cos m\phi + h_{ni}^m \sin m\phi\right)P_n^m + n\left(g_{ne}^m \cos m\phi + h_{ne}^m \sin m\phi\right)P_n^m\right]. \tag{9.5}$$

The analysis of $X(a, \theta, \phi)$ and $Y(a, \theta, \phi)$ alone gives only the sums of internal and external coefficients $(g_{ni}^m + g_{ne}^m)$ and $(h_{ni}^m + h_{ne}^m)$, but can be expanded to also give the non-potential toroidal field coefficients $g_{nv}^m(a)$ and $h_{nv}^m(a)$, if required. Including the analysis of the radial component of the field $Z(a, \theta, \phi)$ allows separation of the coefficients g_{ni}^m, h_{ni}^m for the internal field and g_{ne}^m, h_{ne}^m for the external field. A hypothesis concerning the non-potential toroidal field, such as Earth–air radial currents or field-aligned currents, is made after the coefficients g_{nv}^m and h_{nv}^m have been determined.

If it were possible, determination of the non-potential field coefficients for different values of r, would give an indication of the radial dependence of the non-potential field coefficients, and could help to distinguish between the Earth–air radial current hypothesis and the field-aligned current hypothesis.

Another method of spherical harmonic analysis of magnetic data over a reference sphere is to use fixed Cartesian field components (defined in terms of a Cartesian coordinate system at the origin) in place of spherical polar components. Surface spherical harmonic coefficients are determined for each of the three Cartesian components and, because of the orthogonality of the vector spherical harmonics corresponding to internal, external and non-potential fields, the required coefficients can be determined independently from simple combinations of the surface spherical harmonics coefficients, e.g. Winch (1968).

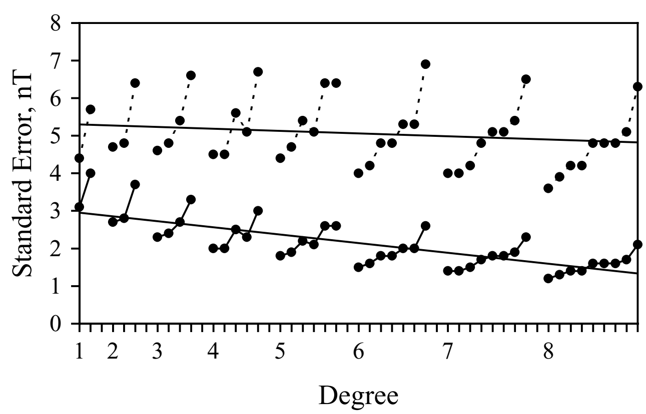

Figure 1. Standard errors for internal field coefficients g_n^m epoch 1960, by Langel & Estes (1987). Standard errors are shown in the lower curve, together with a linear trend. The upper curve is for the same standard errors multiplied by $\sqrt{n+1}$. The linear trend for the upper graph shows a smaller rate of decrease with degree n than the lower graph.

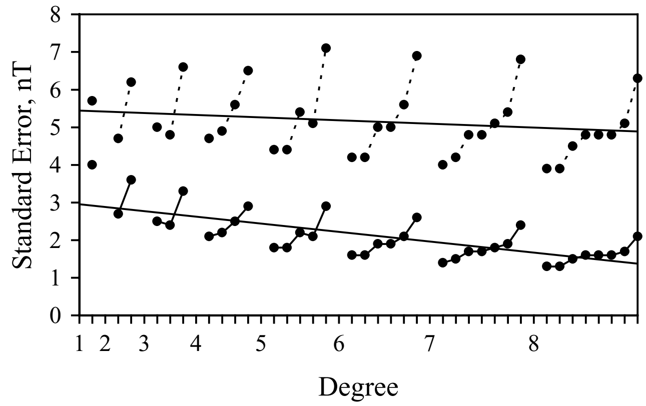

Figure 2. Standard errors for internal field coefficients h_n^m epoch 1960, by Langel & Estes (1987). Standard errors are shown in the lower curve, together with a linear trend. The upper curve is for the same standard errors multiplied by $\sqrt{n+1}$. The linear trend for the upper graph shows a smaller rate of decrease with degree n than the lower graph.

10 STANDARD DEVIATIONS OF THE COEFFICIENTS

The Schmidt-normalized main field coefficients g_n^m and h_n^m all tend to become smaller with increasing degree, with the dipole term, of degree 1, being the largest. This could of course be entirely the result of the physics of the magnetism of the Earth, but then one can easily see that the standard deviations of the same coefficients (when they are given, for example, Langel & Estes 1987, as standard errors) also diminish with increasing degree. By determining coefficients for the three orthonormal vector fields, with the equations of condition suitably weighted to provide errors with a zero mean and common variance, the standard deviations of the coefficients, assuming residuals are independent and distributed with a zero mean and common variance, will be the same for all coefficients.

Figs 1 and 2 show the effect of applying the weights determined above, to the standard errors of Langel & Estes (1987). Fig. 1 is for the g_n^m standard errors and Fig. 2 is for the corresponding h_n^m standard errors, both figures showing a solid line of best fit to the coefficients. The lower, solid curves in each figure are for standard deviations, as published. The unwanted decrease in magnitude of the standard errors as the degree increases is shown by the slope of the line of best fit to the standard errors. The upper, dashed curves are for the same standard errors, but with weights applied as described above. The slope of the line of best fit to the weighted standard errors proposed here is smaller than that for the unweighted standard errors.

The weighting proposed here, to produce normalized coefficients and a constant standard error, does tend to amplify the increase of standard error with order m, within each degree n, that occurs in the Langel & Estes (1987) standard error coefficients. Lowes & Olsen (2004) have shown that the increase of standard deviation with order m, within each degree n, is seen in the analysis of total intensity data alone and may therefore arise from the inclusion of total intensity values in the Langel & Estes (1987) basic data set. However, the reduction in slope of the line of best fit to the standard deviations that occurs for weighted coefficients shows that the decrease in magnitude of the standard deviations with increasing order is solely the result of the non-orthonormal form of the mathematical functions chosen for the analysis and is not the result of any physical cause.

Numerical coefficients should be given for functions whose weight is in some sense equal to 1. That the coefficients as computed at present diminish rather too quickly with increasing degree is confirmed by this study of the standard deviations.

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NOTE ADDED IN PROOF

Schmidt (1899) defined Schmidt quasi-normalized functions in order to simplify the addition theorem for associated Legendre functions.

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