lab1

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1 Introduction

In this lab is implementing NN and back propagation

Lab request:

- Write a simple neural networks without framework (e.g. Tensorflow, PyTorch)
- Only use Numpy and other standard lib
- NN with two hidden layers
- Plot your comparison figure that show the predict result and ground truth

1.1 Implementation parameters

- X,$\hat{y}$: Data
- x_1,x_2 : NN inputs
- y : NN output
- $L(\theta)$: Lost function (MSE $E(|\hat{y} - y|^2)$)
- W : weight matrix
- σ : activation function (sigmoid $\frac{1}{1+e^{-x}}$)

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.colors import LinearSegmentedColormap
```

1.2 Dataset

Two data generator are in this lab.

- Linear
- XOR

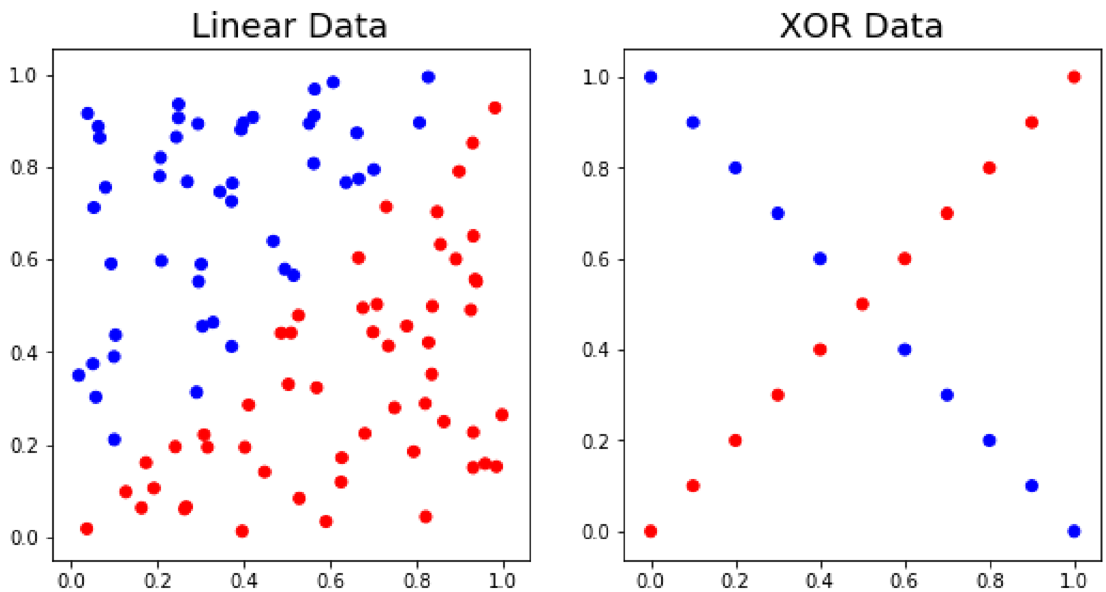
Target y is 0 or 1 like one class classification.

In [2]:
```
def show_result(x, y, pred_y):
    pred_y = np.round(pred_y)
    cm = LinearSegmentedColormap.from_list(
        'mymap', [(1, 0, 0), (0, 0, 1)], N=2)
```

```
    plt.figure(figsize=(10,5))
    plt.subplot(1,2,1)
    plt.title('Ground truth', fontsize=18)
    plt.scatter(x[:,0], x[:,1], c=y[:,0], cmap=cm)

    plt.subplot(1,2,2)
    plt.title('Predict result', fontsize=18)
    plt.scatter(x[:,0], x[:,1], c=pred_y[:,0], cmap=cm)

    plt.show()

def show_data(xs, ys, ts):
    cm = LinearSegmentedColormap.from_list(
        'mymap', [(1, 0, 0), (0, 0, 1)], N=2)
    n = len(xs)
    plt.figure(figsize=(5*n, 5))
    for i, x, y, t in zip(range(n), xs, ys, ts):
        y = np.round(y)
        plt.subplot(1,n, i+1)
        plt.title(t, fontsize=18)
        plt.scatter(x[:,0], x[:,1], c=y[:,0], cmap=cm)

    plt.show()
```

```
In [3]: def generate_linear(n=100):
    pts = np.random.uniform(0, 1, (n, 2))
    inputs = []
    labels = []
    for pt in pts:
        inputs.append([pt[0], pt[1]])
        distance = (pt[0] - pt[1]) / 1.414
        if pt[0] > pt[1]:
            labels.append(0)
        else:
            labels.append(1)
    return np.array(inputs), np.array(labels).reshape(n, 1)

def generate_XOR_easy(n=11):
    inputs = []
    labels = []
    step = 1/(n-1)
    for i in range(n):
        inputs.append([step*i, step*i])
        labels.append(0)

        if i == int((n-1)/2):
            continue
```

```
            inputs.append([step*i, 1 - step*i])
            labels.append(1)

        return np.array(inputs), np.array(labels).reshape(n*2 - 1,1)

    x1, y1 = generate_linear()
    x2, y2 = generate_XOR_easy()
    show_data([x1,x2], [y1,y2], ['Linear Data', 'XOR Data'])
```



2 Experiment setups

2.1 Activation function σ (Sigmoid)

I use Sigmoid function as my activation function.

Derivation of sigmoid is follow as:

$$\sigma(x) = \frac{1}{1+e^{-x}}$$

$$\sigma'(x) = \frac{d(1+e^{-x})^{-1}}{dx}$$

$$= -(1+e^{-x})^2 \frac{d}{dx}(1+e^{-x})$$

$$= -(1+e^{-x})(1+e^{-x})(-e^{-x})$$

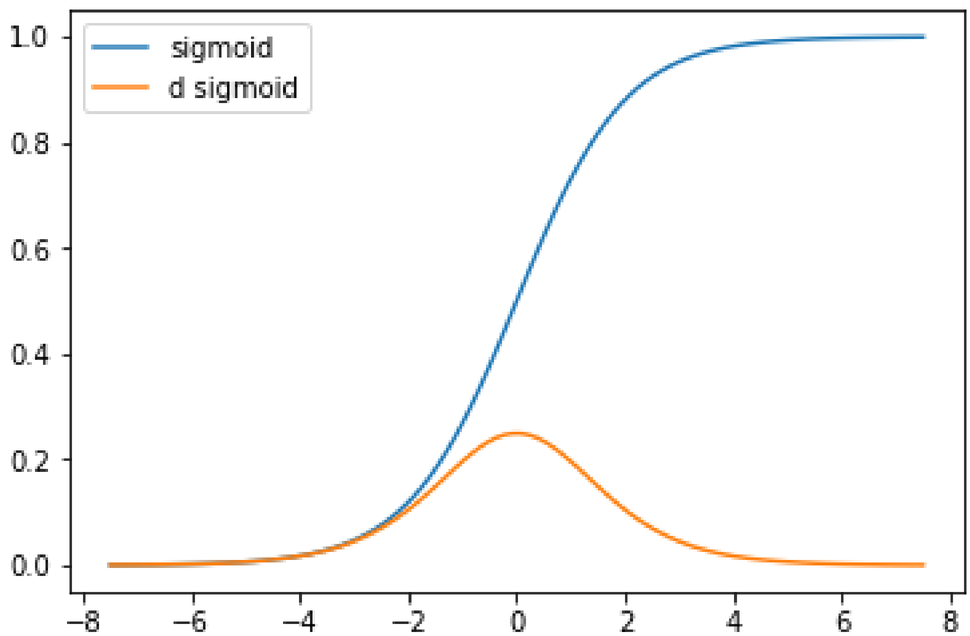
$$= \sigma(x)(1-\sigma(x))$$

The implementation reference from TAs.

```
In [4]: def sigmoid(x):
            return 1.0 / (1.0 + np.exp(-x))

        def derivative_sigmoid(x):
            return np.multiply(x, 1.0 - x)
```

```
In [5]: x = np.linspace(-7.5, 7.5, 100)
        plt.plot(x, sigmoid(x), label='sigmoid')
        plt.plot(x, derivative_sigmoid(sigmoid(x)), label='d sigmoid')
        plt.legend()
```

```
Out[5]: <matplotlib.legend.Legend at 0x7f9e6523b240>
```



2.2 Loss function $L(\theta)$ (MSE)

I use MSE (Mean Square Error) as my loss function.

Derivation of MSE is follow as:

$$L(y, \hat{y}) = MSE(y, \hat{(y)}) = E((y - \hat{y})^2) = \frac{\sum (y - \hat{y})^2}{N}$$

$$L'(y, \hat{y}) = \frac{\partial E((y - \hat{y})^2)}{\partial y}$$

$$= \frac{1}{N}(\frac{\partial(y-\hat{y})^2}{\partial y})$$

$$= \frac{1}{N}(2(y-\hat{y})\frac{\partial(y-\hat{y})}{\partial y})$$

$$= \frac{2}{N}(y-\hat{y})$$

```
In [6]: def loss(y, y_hat):
            return np.mean((y - y_hat)**2)

        def derivative_loss(y, y_hat):
            return (y - y_hat)*(2/y.shape[0])
```

2.3 Neural network

I divide neural network for some part.

2.3.1 Neural Unit

The input x vector get output y scalar through neural unit.
$z = w^T x + b, y = \sigma(z)$
In order to multily output y, extend neural unit as neural layer.

2.3.2 Neural Layer

One neural unit can output one scalar. In order to output N scalar in this layer, we put N units in layer.
Explain some parameter in layer:
w : weight matrix

- size is (input_size + 1, output_size)
- initialize w in layer's `__init__`
- combine bias in w

x : input vector

- size is (data_size, input_size)
- x' automatically extend one columns for bias when `forward`

$z : z = x'w$

- size is (data_size, output_size)

$y : y = \sigma(z)$

- network output when output layer

- next layer input when hidden layer

$\frac{\partial C}{\partial w}, \frac{\partial z}{\partial w}, \frac{\partial C}{\partial z}$: gradient matrix

- there are stored into layer parameter
- use to update w when call `update`

And then see how to compute gradient from cost by using backpropagation

2.4 Backpropagation

At first, all weight parameters in network are randomly initialized. The objective is minimize cost C from loss function $L(\theta)$.

We use gradient descent to update the network's weights. But $\frac{\partial C}{\partial w}$ is hard to compute.
Because of that, we use chain rules.

$$\frac{\partial C}{\partial w} = \frac{\partial z}{\partial w}\frac{\partial C}{\partial z}$$

2.4.1 Forward

$$\frac{\partial z}{\partial w} = \frac{\partial x'w}{\partial w} = x'$$

We record $\frac{\partial z}{\partial w}$ as `forward_gradient` when call `forward` function.
The matrix size = (data_size, input_size+1).

2.4.2 Backward

$$\frac{\partial C}{\partial z} = \frac{\partial y}{\partial z}\frac{\partial C}{\partial y}$$

We can get $\frac{\partial y}{\partial z}$ by:

$$y = \sigma(z), \frac{\partial y}{\partial z} = \sigma'(z)$$

To obtain $\frac{\partial C}{\partial y}$ consider two cases.

- output layer:

C is come from $L(\theta)$ and y is network output and $\hat{y}$ is groundtruth.

$$C = L(y, \hat{y})\frac{\partial C}{\partial y} = L'(y, \hat{y})$$

We compute derivative loss function and then use it as backward input.

- hidden layer:

In this case, $\frac{\partial C}{\partial y}$ is more diffcult than other.

The output y of current layer is input for next layer. We assume that $\frac{\partial C}{\partial z_{next}}$ already know.

$$\frac{\partial C}{\partial y_{this}} = \frac{\partial z_{next}}{\partial y_{this}} \frac{\partial C}{\partial z_{next}}$$

$$\frac{\partial z_{next}}{\partial y_{this}} = w_{next}^T, z_{next} = y_{this} w_{next}$$

Finally, we first compute output layer and then send parameters to previous layer. Thus we can compute $\frac{\partial C}{\partial z}$ every layer.

2.5 Gradient Descent

Now we have $\frac{\partial C}{\partial w}$ and use it to update our network weights w.

We put a new hyperparameter called learning rate η to decide the learning should be performed.

$$w = w - \eta \Delta w$$

2.6 implementation

To modulize the code, I design a python class called `layer`. `layer` initializes all weights when create python class. Every `layer` need two parameter `input_size` and `output_size`.

The `layer` class has following functions:

- `forward` function input x and get output y.
- `backward` function input $\frac{\partial C}{\partial y}$ and get output $\frac{\partial C}{\partial x}$
- `update` function use gradient to update layer's weights

In [7]:
```
class layer():
    def __init__(self, input_size, output_size):
        self.w = np.random.normal(0, 1, (input_size+1, output_size))

    def forward(self, x):
        x = np.append(x, np.ones((x.shape[0],1)), axis=1)
        self.forward_gradient = x
        self.y = sigmoid(np.matmul(x, self.w))
        return self.y

    def backward(self, derivative_C):
        self.backward_gradient = np.multiply(
            derivative_sigmoid(self.y),
            derivative_C
        )
        return np.matmul(self.backward_gradient, self.w[:-1].T)

    def update(self, learning_rate):
```

```
            self.gradient = np.matmul(
                self.forward_gradient.T,
                self.backward_gradient
            )
            self.w -= learning_rate*self.gradient
            return self.gradient
```

To combine multi layers become Neural Network, I design a python class called `NN`. `NN` will create layers by `size` when create it.

The `NN` class has following functions:

- `forward` function positive sequence call all layer's `forward`, return final result
- `backward` function reverse call all layer's `backward`, return final result
- `update` function call all layer's `update`

```
In [8]: class NN():
            def __init__(self, sizes, learning_rate = 0.1):
                self.learning_rate = learning_rate
                sizes2 = sizes[1:] + [0]
                self.l = []
                for a,b in zip(sizes, sizes2):
                    if (a+1)*b == 0:
                        continue
                    self.l += [layer(a,b)]

            def forward(self, x):
                _x = x
                for l in self.l:
                    _x = l.forward(_x)
                return _x

            def backward(self, dC):
                _dC = dC
                for l in self.l[::-1]:
                    _dC = l.backward(_dC)

            def update(self):
                gradients = []
                for l in self.l:
                    gradients += [l.update(self.learning_rate)]
                return gradients
```

3 Results of your testing

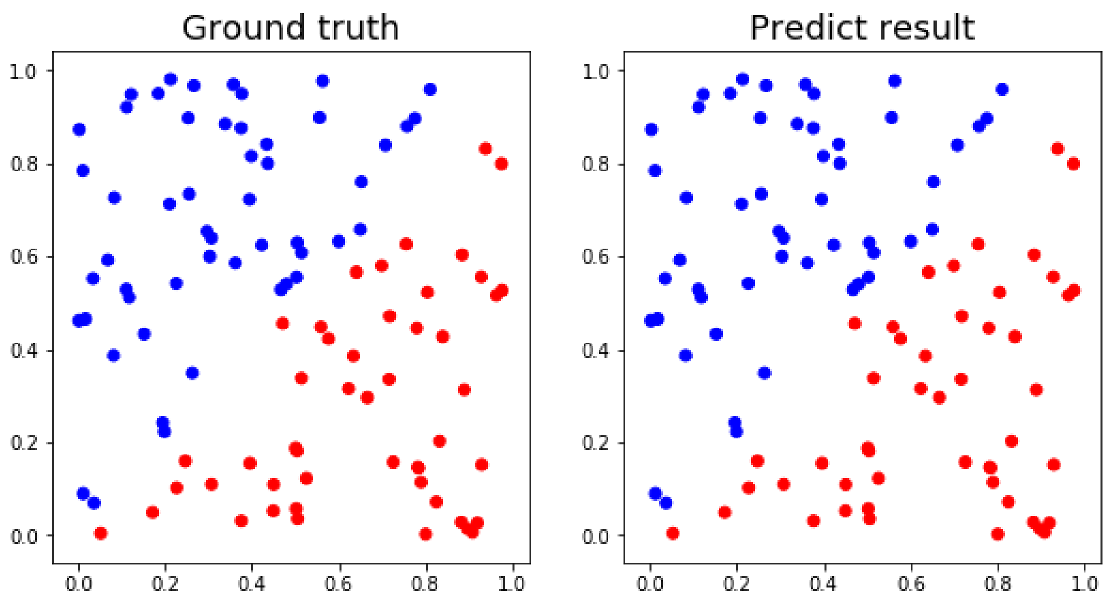
```
In [9]: nn_linear = NN([2,4,4,1], 1)
        nn_XOR = NN([2,4,4,1], 1)
```

```python
epoch_count = 10000
loss_threshold = 0.005
linear_stop = False
XOR_stop = False
x_linear, y_linear = generate_linear()
x_XOR, y_XOR = generate_XOR_easy()
for i in range(epoch_count):
    if not linear_stop:
        y = nn_linear.forward(x_linear)
        loss_linear = loss(y, y_linear)
        nn_linear.backward(derivative_loss(y, y_linear))
        nn_linear.update()

        if loss_linear < loss_threshold:
            print('linear is covergence')
            linear_stop = True

    if not XOR_stop:
        y = nn_XOR.forward(x_XOR)
        loss_XOR = loss(y, y_XOR)
        nn_XOR.backward(derivative_loss(y, y_XOR))
        nn_XOR.update()

        if loss_XOR < loss_threshold:
            print('XOR is covergence')
            XOR_stop = True

    if i%200 == 0 or (linear_stop and XOR_stop):
        print(
            '[{:4d}] linear loss : {:.4f} \t XOR loss : {:.4f}'.format(
                i, loss_linear, loss_XOR))

    if linear_stop and XOR_stop:
        break
```

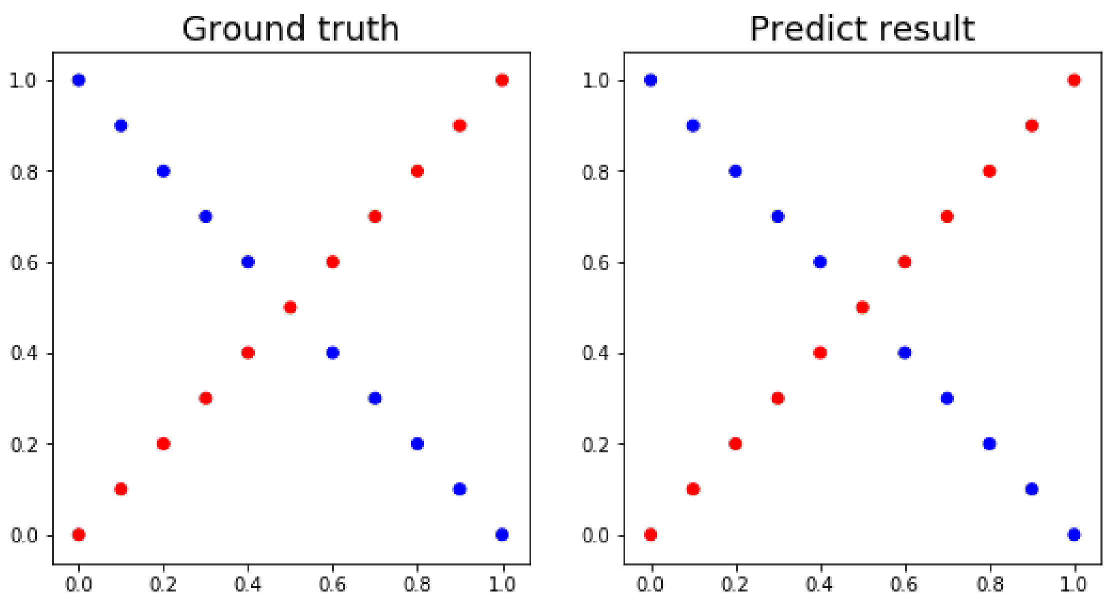
```
[   0] linear loss : 0.3659          XOR loss : 0.2676
[ 200] linear loss : 0.0854          XOR loss : 0.2495
[ 400] linear loss : 0.0302          XOR loss : 0.2495
[ 600] linear loss : 0.0185          XOR loss : 0.2495
[ 800] linear loss : 0.0135          XOR loss : 0.2495
[1000] linear loss : 0.0108          XOR loss : 0.2495
[1200] linear loss : 0.0091          XOR loss : 0.2495
[1400] linear loss : 0.0080          XOR loss : 0.2495
[1600] linear loss : 0.0071          XOR loss : 0.2494
[1800] linear loss : 0.0065          XOR loss : 0.2494
[2000] linear loss : 0.0060          XOR loss : 0.2494
[2200] linear loss : 0.0056          XOR loss : 0.2494
[2400] linear loss : 0.0053          XOR loss : 0.2494
```

```
linear is covergence
[2600] linear loss : 0.0050          XOR loss : 0.2494
[2800] linear loss : 0.0050          XOR loss : 0.2494
[3000] linear loss : 0.0050          XOR loss : 0.2494
[3200] linear loss : 0.0050          XOR loss : 0.2493
[3400] linear loss : 0.0050          XOR loss : 0.2493
[3600] linear loss : 0.0050          XOR loss : 0.2493
[3800] linear loss : 0.0050          XOR loss : 0.2492
[4000] linear loss : 0.0050          XOR loss : 0.2492
[4200] linear loss : 0.0050          XOR loss : 0.2491
[4400] linear loss : 0.0050          XOR loss : 0.2489
[4600] linear loss : 0.0050          XOR loss : 0.2485
[4800] linear loss : 0.0050          XOR loss : 0.2474
[5000] linear loss : 0.0050          XOR loss : 0.2441
[5200] linear loss : 0.0050          XOR loss : 0.2318
[5400] linear loss : 0.0050          XOR loss : 0.2054
[5600] linear loss : 0.0050          XOR loss : 0.1176
[5800] linear loss : 0.0050          XOR loss : 0.0527
[6000] linear loss : 0.0050          XOR loss : 0.0260
[6200] linear loss : 0.0050          XOR loss : 0.0139
[6400] linear loss : 0.0050          XOR loss : 0.0085
[6600] linear loss : 0.0050          XOR loss : 0.0058
XOR is covergence
[6695] linear loss : 0.0050          XOR loss : 0.0050
```

```
In [10]: y1 = nn_linear.forward(x_linear)
         show_result(x_linear, y_linear, y1)
         print('linear test loss : ', loss(y1, y_linear))
         y2 = nn_XOR.forward(x_XOR)
         show_result(x_XOR, y_XOR, y2)
         print('XOR test loss : ', loss(y2, y_XOR))
         print('\n linear test result : \n',y1)
         print('\n XOR test result : \n',y2)
```




```
linear test loss :  0.004997369973721243
```




```
XOR test loss :  0.004985120154166471

 linear test result :
 [[0.99853721]
```

```
[0.00964757]
[0.99905499]
[0.00260557]
[0.99884936]
[0.78789472]
[0.99880066]
[0.00413509]
[0.96854202]
[0.00158948]
[0.99879866]
[0.99906358]
[0.00248632]
[0.00135504]
[0.00401452]
[0.00125463]
[0.99881775]
[0.99893572]
[0.95260893]
[0.00312461]
[0.9990106 ]
[0.01052388]
[0.81454154]
[0.94125602]
[0.03221529]
[0.00119163]
[0.01460374]
[0.00210475]
[0.99886686]
[0.9988201 ]
[0.98402049]
[0.00128568]
[0.00199836]
[0.8961942 ]
[0.48680878]
[0.99318262]
[0.9983295 ]
[0.00116607]
[0.00120535]
[0.00116337]
[0.00415573]
[0.93963421]
[0.02561305]
[0.00236981]
[0.9990605 ]
[0.92167761]
[0.00231055]
[0.99056011]
[0.00659865]
```

```
[0.00121181]
[0.99894374]
[0.9986065 ]
[0.99893035]
[0.00128212]
[0.71270123]
[0.99874477]
[0.99905126]
[0.99890611]
[0.17191886]
[0.99813572]
[0.00117567]
[0.99897263]
[0.00135423]
[0.98879932]
[0.99843313]
[0.99893673]
[0.00162977]
[0.10532662]
[0.00129369]
[0.03828102]
[0.00191669]
[0.00216004]
[0.99899087]
[0.99906756]
[0.97649256]
[0.85697512]
[0.00177605]
[0.00151797]
[0.99852217]
[0.98878003]
[0.99867797]
[0.99830764]
[0.99901554]
[0.00116779]
[0.0017601 ]
[0.00163557]
[0.02383062]
[0.99905395]
[0.99864406]
[0.99661099]
[0.003183  ]
[0.99726702]
[0.02426193]
[0.00197721]
[0.00175934]
[0.99904011]
[0.04570059]
```

```
[0.06377572]
[0.99902116]
[0.9883871 ]]

XOR test result :
[[0.06382205]
[0.98383577]
[0.06382626]
[0.98321181]
[0.06404211]
[0.98135972]
[0.06443143]
[0.97342466]
[0.06496537]
[0.83582484]
[0.06562278]
[0.06638757]
[0.85260193]
[0.06724544]
[0.95890007]
[0.06818085]
[0.96155295]
[0.06917486]
[0.96126548]
[0.07020446]
[0.9609505 ]]
```

4 Discussion

4.1 Why perceptron can't classify XOR data but MLP can do ?

4.1.1 Perceptron

Let us try to use perceptron to classify XOR data.

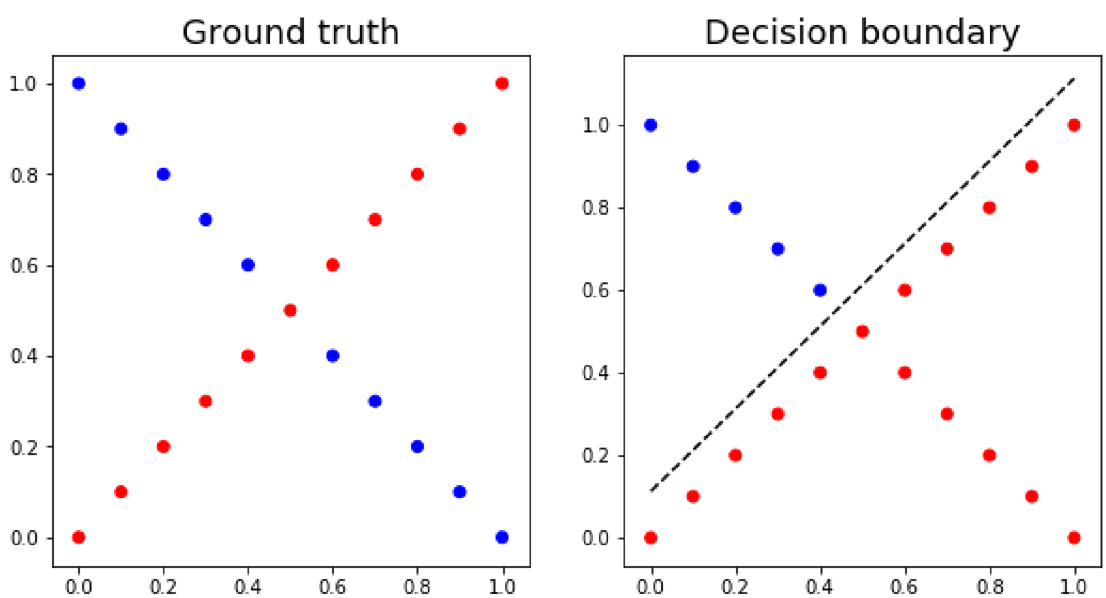
And then find one perceptron only has one decision boundary.

Bacause of that, XOR data can't be classified by one perceptron.

```
In [11]: def draw_decision_boundary(w):
             x = np.linspace(0,1,2)
             y = - (w[0] * x + w[2]) / w[1]
             plt.plot(x, y, '--', c='black')

         def show_data_with_boundary(xs, ys, ts, w):
             cm = LinearSegmentedColormap.from_list(
                 'mymap', [(1, 0, 0), (0, 0, 1)], N=2)
             n = len(xs)
             plt.figure(figsize=(5*n, 5))
```

```
        for i, x, y, t in zip(range(n), xs, ys, ts):
            y = np.round(y)
            plt.subplot(1,n, i+1)
            plt.title(t, fontsize=18)
            plt.scatter(x[:,0], x[:,1], c=y[:,0], cmap=cm)
            if "boundary" in t or "decision" in t:
                for i in range(w.shape[1]):
                    draw_decision_boundary(w[:, i])
        plt.show()
```

```
In [12]: perceptron = NN([2,1], 10)
         x_XOR, y_XOR = generate_XOR_easy()
         for _ in range(10000):
             for i in range(x_XOR.shape[0]):
                 y = perceptron.forward(x_XOR[i:i+1])
                 loss_XOR = loss(y, y_XOR[i:i+1])
                 perceptron.backward(derivative_loss(y, y_XOR[i:i+1]))
                 perceptron.update()

         y = perceptron.forward(x_XOR)
         show_data_with_boundary(
             [x_XOR, x_XOR], [y_XOR, np.round(y)],
             ['Ground truth', 'Decision boundary'], perceptron.l[0].w
         )
```



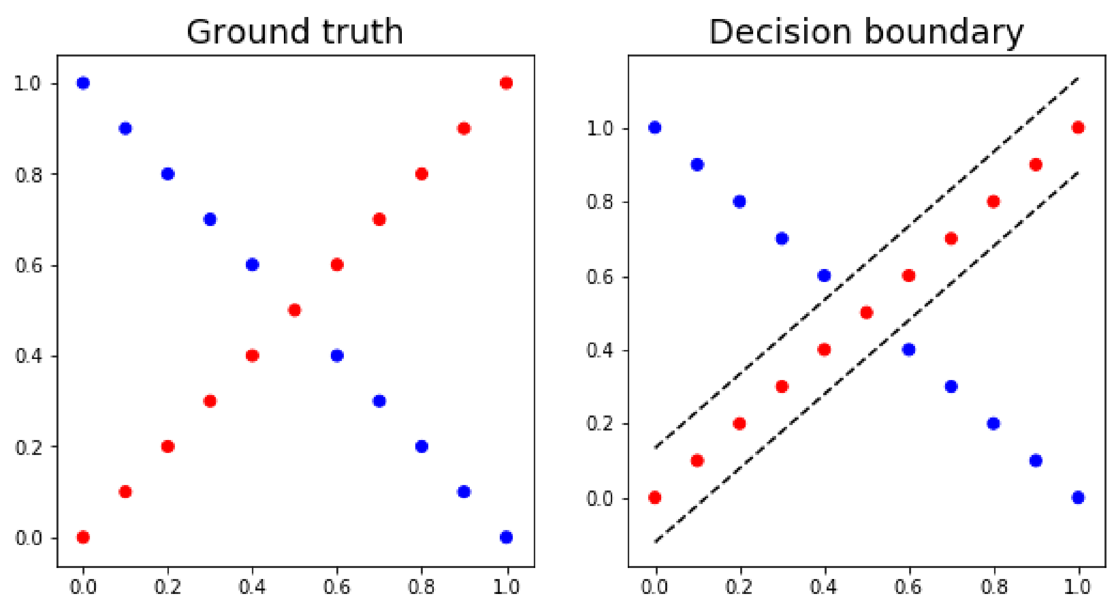
But if we has multi perceptron ?

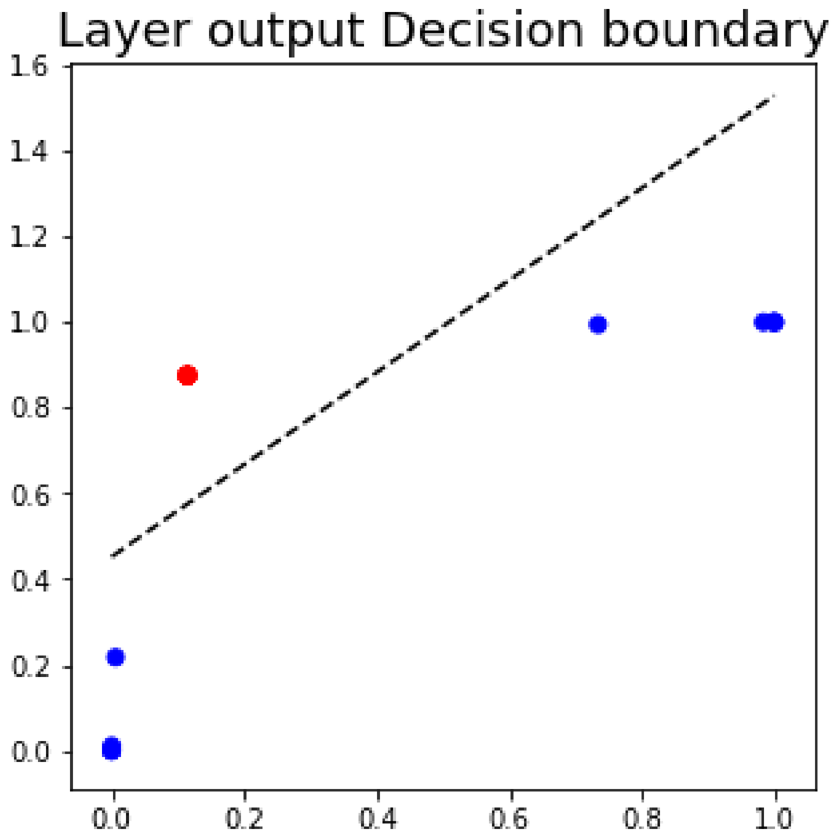
4.1.2 MLP (Mutli Layer Perceptron)

It has two layer, first layer has two perceptron.

XOR data can be classified by two decision boundary.
And input data through first layer will be mapped into new space.
In that space, all data can be classified by one decision boundary. Therefore second layer only has one perceptron.

```
In [13]: MLP = NN([2,2,1], 10)
         x_XOR, y_XOR = generate_XOR_easy()
         for _ in range(10000):
             y = MLP.forward(x_XOR)
             loss_XOR = loss(y, y_XOR)
             MLP.backward(derivative_loss(y, y_XOR))
             MLP.update()

         y = MLP.forward(x_XOR)
         show_data_with_boundary(
             [x_XOR, x_XOR], [y_XOR, np.round(y)],
             ['Ground truth', 'Decision boundary'], MLP.l[0].w
         )
         show_data_with_boundary(
             [MLP.l[0].y], [np.round(y)],
             ['Layer output Decision boundary'], MLP.l[1].w
         )
```




Layer output Decision boundary
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0