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# Explicit solutions for shearing and radial stresses in curved beams

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**Abstract**

In this paper the formulae for the shearing and radial stresses in curved beams are derived analytically based on the solution for a Volterra integral equation of the second kind. These formulae satisfy both the equilibrium equations and the static boundary conditions on the surfaces of the beams. As some applications, the resulting solutions are used to calculate the shearing and radial stresses in a cantilevered curved beam subjected to a concentrated force at its free end. The numerical results are compared with other existing approximate solutions as well as the corresponding solutions based on the theory of elasticity. The calculations show a better agreement between the present solution and the one based on the theory of elasticity. The resulting formulae can be applied to more general cases of curved beams with arbitrary shapes of cross-sections.


*Keywords:* Curved beam; Shearing stress; Radial stress; Integral equation; FEM

## 1. Introduction

So far there exists a widespread discussion on determination of normal stress in curved beams in advanced mechanics of materials while the shearing stress is evaluated by using either the formula for the case of straight beams or other approximate formulae (Oden and Ripperger, 1967; Liu, 1985). Although the derivations of the shearing stress are all resulted from an integral equation, the solution of such an equation for various boundary conditions is still very complicated and thus causes inconveniences for the numerical simulation.

This paper aims to derive the formulae of shearing and radial stresses in the curved beams with arbitrary shapes of cross-sections. The present solution has not appeared in the existing literature yet.

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## 2. Stress resultants and normal stresses in curved beams

Let us consider a curved beam with constant cross-section, as shown in Fig. 1(a). The radius of curvature at any point is denoted by $R$, and the geometry of the beam defined by establishing two orthogonal coordinate systems: one fixed coordinate system $(x, \bar{y}, z)$, with its origin located on a convenient cross-section, and the other curvilinear system $(s, y, z)$ in which $s$ is the arc length measured along the geometric axis, $y$ is a radial coordinate directed toward the center of curvature of this axis, and $z$ is directed normal to the plane of the beam. The components of the external forces per unit arc length in the $y$- and $z$-directions are denoted by $p_y(s)$ and $p_z(s)$, respective, and the twisting of any cross-section is assumed to be zero or negligible (Oden and Ripperger, 1967). Suppose that the stress resultants $N_s$, $V_y$, $V_z$, $M_y$ and $M_z$ are in the positive direction (see also Fig. 1(a)). The cross-section of the beam is shown in Fig. 1(b), where $h_1$, $h_2$ are the distance from centroid axis $z$ of the cross-section to the upper and lower edges, respectively, and $G$ is the centroid of the cross-section.

Navier's assumption is still adopted in the present study. According to this assumption, the formula for normal stress in curved beams can be derived (Oden and Ripperger, 1967)

$$\sigma_s = \frac{N_s}{A} - \frac{M_z}{RA} + \frac{M_zJ_y - M_yJ_{yz}}{J_yJ_z - J_{yz}^2}\frac{y}{1-\frac{y}{R}} + \frac{M_yJ_z - M_zJ_{yz}}{J_yJ_z - J_{yz}^2}\frac{z}{1-\frac{y}{R}}, \tag{1}$$

where $A$ is the total area of the cross-section, and $J_y$, $J_{yz}$, $J_z$ are defined by the following integrals respectively

$$J_y = \int_A \frac{z^2}{1-\frac{y}{R}}\mathrm{d}A, \quad J_{yz} = \int_A \frac{yz}{1-\frac{y}{R}}\mathrm{d}A, \quad J_z = \int_A \frac{y^2}{1-\frac{y}{R}}\mathrm{d}A. \tag{2}$$

## 3. Shearing and radial stresses in curved beams

Let us now examine the equilibrium of a slice of the element of the sectional area $A'$, as shown in Fig. 2(a). For simplicity, the dimension $b(y)$ of $A'$ is assumed to be parallel to the $z$-axis. The normal stress, $\sigma_s$, on $A'$ compose a normal force as follows

$$F = \int_{A'} \sigma_s\,\mathrm{d}A = \left(\frac{N_s}{A} - \frac{M_z}{AR}\right)A' + \frac{M_zJ_y - M_yJ_{yz}}{J_yJ_z - J_{yz}^2}\int_{A'}\frac{y}{1-\frac{y}{R}}\mathrm{d}A + \frac{M_yJ_z - M_zJ_{yz}}{J_yJ_z - J_{yz}^2}\int_{A'}\frac{z}{1-\frac{y}{R}}\mathrm{d}A. \tag{3}$$

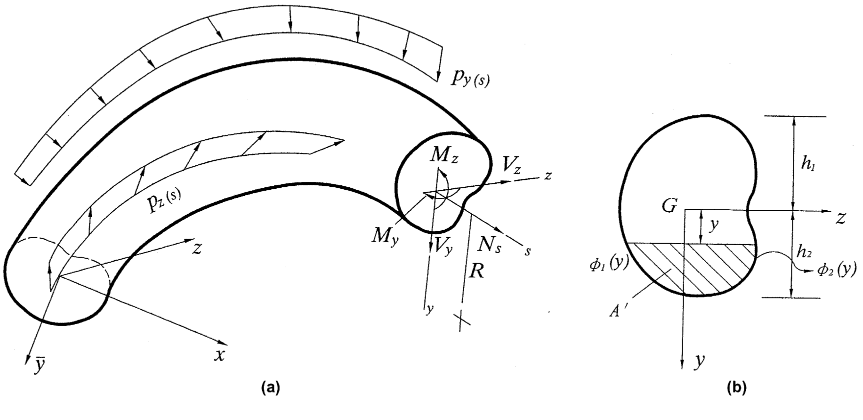


Fig. 1. Bending of a curved beam with arbitrary cross-sectional shapes.

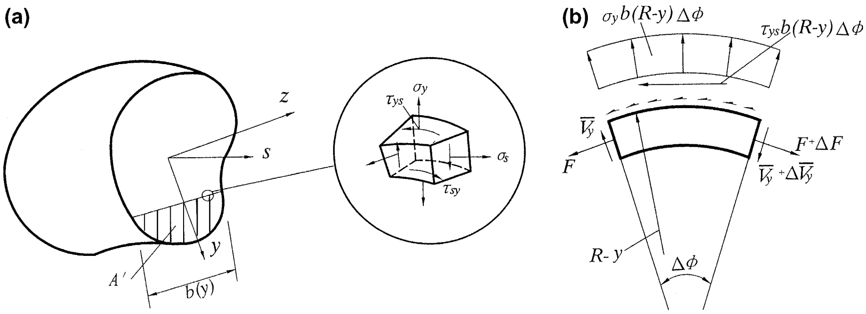


Fig. 2. (a) Stresses on a curved beam element. (b) Forces acting on a portion of the element of cross-sectional area $A'$.

It is assumed that the values of shearing stress at every point of any line paralleled to $z$-axis on the cross-section are all the same, and they are only a function of $y$. Similarly, the shearing stress, $\tau_{ys}(=\tau_{sy})$, on $A'$ compose a force $\overline{V}_y$ parallel to the cross-section below

$$\overline{V}_y = \int_{A'} \tau_{ys}\, \mathrm{d}A. \tag{4}$$

Based on the analysis of force balances for a portion of the element of the cross-section $A'$, as shown in Fig. 2(b), the shearing and radial stresses can be expressed as (Oden and Ripperger, 1967)

$$\tau_{ys} = \frac{1}{b(y)\left(1-\frac{y}{R}\right)}\left(\frac{\partial F}{\partial s} - \frac{1}{R}\overline{V}_y\right), \tag{5}$$

$$\sigma_y = \frac{1}{b(y)\left(1-\frac{y}{R}\right)}\left(\frac{F}{R} + \frac{\partial \overline{V}_y}{\partial s}\right). \tag{6}$$

It can be observed that Eq. (5) is an integral equation with respect to the variable $\tau_{ys}$. Substituting Eq. (3) and Eq. (4) into Eq. (5), and using the following equilibrium equation for a typical curved beam (Oden and Ripperger, 1967)

$$\frac{\mathrm{d}N_s}{\mathrm{d}s} = \frac{V_y}{R}, \quad \frac{\mathrm{d}V_z}{\mathrm{d}s} = -p_z, \quad \frac{\mathrm{d}V_y}{\mathrm{d}s} = -p_y - \frac{N_s}{R}, \quad \frac{\mathrm{d}M_y}{\mathrm{d}s} = V_z, \quad \frac{\mathrm{d}M_z}{\mathrm{d}s} = V_y. \tag{7}$$

Eq. (5) changes into a specific form as

$$\tau_{ys} = \frac{R}{b(y)(R-y)}\left[\frac{J_y V_y - J_{yz} V_z}{H}\int_y^{h_2} \frac{b(y)y}{1-\frac{y}{R}}\mathrm{d}y + \frac{J_z V_z - J_{yz} V_y}{H}\int_y^{h_2} \frac{\int_{\varphi_1(y)}^{\varphi_2(y)} z\,\mathrm{d}z}{1-\frac{y}{R}}\mathrm{d}y\right]$$
$$- \frac{1}{b(y)(R-y)}\int_y^{h_2} \tau_{ys} b(y)\,\mathrm{d}y \quad (h_1 \leqslant y \leqslant h_2), \tag{8}$$

where $H = J_y J_z - J_{yz}^2$. $\phi_1(y)$ and $\phi_2(y)$ are the integral upper and lower limits, as shown in Fig. 1(b). It is clear that the above equation is a Volterra integral equation of the second kind, and can be rewritten as

$$\tau_{ys}(y) - \lambda \int_{h_2}^{y} K(y,t)\tau_{ys}(t)\,\mathrm{d}t = f(y), \tag{9}$$

where

$$\lambda = 1, \tag{10}$$

$$f(y)=\frac{1}{b(y)(1-\frac{y}{R})}\left[\frac{J_yV_y-J_{yz}V_z}{H}\int_y^{h_2}\frac{b(y)y}{1-\frac{y}{R}}\mathrm{d}y+\frac{J_zV_z-J_{yz}V_y}{H}\int_y^{h_2}\frac{\int_{\varphi_1(y)}^{\varphi_2(y)}z\,\mathrm{d}z}{1-\frac{y}{R}}\mathrm{d}y\right]. \tag{11}$$

The kernel of the equation is

$$k(y,t)=\frac{b(t)}{b(y)(R-y)} \tag{12}$$

and the iterated kernels of the equation are given as follows:

$$k_2(y,t)=\int_t^y\frac{b(u)}{b(y)(R-y)}\frac{b(t)}{b(u)(R-u)}\mathrm{d}u=\frac{b(t)}{b(y)(R-y)}\ln\left(\frac{R-t}{R-y}\right),$$

$$k_3(y,t)=\int_t^y\frac{b(u)}{b(y)(R-y)}\frac{b(t)}{b(u)(R-u)}\ln\left(\frac{R-t}{R-u}\right)\mathrm{d}u=\frac{b(t)}{2b(y)(R-y)}\ln^2\left(\frac{R-t}{R-y}\right),$$

$$\ldots$$

The general form, $k_m(y,t)$ $(m=1,2,\ldots)$ can be expressed as

$$k_m(y,t)=\frac{1}{(m-1)!}\frac{b(t)}{b(y)(R-y)}\ln^{(m-1)}\left(\frac{R-t}{R-y}\right). \tag{13}$$

The solution kernel of the integral equation is thus written as

$$\Gamma(y,t;\lambda)=\frac{b(t)}{b(y)(R-y)}\sum_{m=1}^{\infty}\frac{\ln^{(m-1)}\left(\frac{R-t}{R-y}\right)}{(m-1)!}=\frac{b(t)}{b(y)(R-y)}\mathrm{e}^{\ln\left(\frac{R-t}{R-y}\right)}=\frac{b(t)(R-t)}{b(y)(R-y)^2}\quad(t\leqslant y). \tag{14}$$

Applying Eq. (14), Eq. (8) can be expressed in the following form:

$$\begin{aligned}\tau_{ys}&=f(y)+\lambda\int_{h_2}^{y}\Gamma(y,t;\lambda)f(t)\,\mathrm{d}t\\&=\frac{R}{b(y)(R-y)}\left[\frac{J_yV_y-J_{yz}V_z}{H}\int_y^{h_2}\frac{b(y)y}{1-\frac{y}{R}}\mathrm{d}y+\frac{J_zV_z-J_{yz}V_y}{H}\int_y^{h_2}\frac{\int_{\varphi_1(y)}^{\varphi_2(y)}z\,\mathrm{d}z}{1-\frac{y}{R}}\mathrm{d}y\right]\\&\quad+\frac{R}{b(y)(R-y)^2}\int_{h_2}^{y}\left[\frac{J_yV_y-J_{yz}V_z}{H}\int_t^{h_2}\frac{b(t)t}{1-\frac{t}{R}}\mathrm{d}t+\frac{J_zV_z-J_{yz}V_y}{H}\int_t^{h_2}\frac{\int_{\varphi_1(t)}^{\varphi_2(t)}z\,\mathrm{d}z}{1-\frac{t}{R}}\mathrm{d}t\right]\mathrm{d}t\quad(h_1\leqslant y\leqslant h_2).\end{aligned} \tag{15}$$

Substituting Eq. (15) into Eq. (4) yields the expression for $\overline{V}_y$, using the result and applying Eq. (3) and Eq. (7), the radial stress in Eq. (6) has its final expression below

$$\begin{aligned}\sigma_y&=\frac{1}{b(y)(R-y)}\left[\left(\frac{N_s}{A}-\frac{M_z}{RA}\right)\int_y^{h_2}b(y)\mathrm{d}y+\frac{(M_zJ_y-M_yJ_{yz})}{H}\int_y^{h_2}\frac{Rb(y)y}{R-y}\mathrm{d}y+\frac{(M_yJ_z-M_zJ_{yz})}{H}\right.\\&\quad\left.\times\int_y^{h_2}\frac{R(\int_{\varphi_1(y)}^{\varphi_2(y)}z\,\mathrm{d}z)}{R-y}\mathrm{d}y\right]+\frac{R^3(p_y+\frac{N_s}{R})}{Hb(y)(R-y)}\int_y^{h_2}\frac{1}{R-y}\left[-J_y\int_y^{h_2}\frac{b(y)y}{R-y}\mathrm{d}y+J_{yz}\int_y^{h_2}\frac{\int_{\varphi_1(y)}^{\varphi_2(y)}z\,\mathrm{d}z}{R-y}\mathrm{d}y\right]\mathrm{d}y\\&\quad+\frac{R^3p_z}{Hb(y)(R-y)}\int_y^{h_2}\frac{1}{R-y}\left[J_{yz}\int_y^{h_2}\frac{b(y)y}{R-y}\mathrm{d}y-J_z\int_y^{h_2}\frac{\int_{\varphi_1(y)}^{\varphi_2(y)}z\,\mathrm{d}z}{R-y}\mathrm{d}y\right]\mathrm{d}y+\frac{R^3(p_y+\frac{N_s}{R})}{Hb(y)(R-y)}\end{aligned}$$

$$\times \int_y^{h_2} \left\{ \frac{1}{(R-y)^2} \int_{h_2}^{y} \left[ -J_z \int_t^{h_2} \frac{b(t)t}{R-t} \mathrm{d}t + J_{yz} \int_t^{h_2} \frac{\int_{\varphi_1(t)}^{\varphi_2(t)} z\,\mathrm{d}z}{R-t} \mathrm{d}t \right] \mathrm{d}t \right\} \mathrm{d}y + \frac{R^3 p_z}{Hb(y)(R-y)}$$

$$\times \int_y^{h_2} \left\{ \frac{1}{(R-y)^2} \int_{h_2}^{y} \left[ J_{yz} \int_t^{h_2} \frac{b(t)t}{R-t} \mathrm{d}t - J_z \int_t^{h_2} \frac{\int_{\varphi_1(t)}^{\varphi_2(t)} z\,\mathrm{d}z}{R-t} \mathrm{d}t \right] \mathrm{d}t \right\} \mathrm{d}y \quad (h_1 \leqslant y \leqslant h_2). \tag{16}$$

The resulting Eqs. (15) and (16) are the formulae for the shearing and radial stresses in curved beams, respectively. For the case of zero distributed loading, i.e., $p_y = 0$ and $p_z = 0$, it can be easily proven that the shearing and radial stresses expressed in Eqs. (15) and (16) satisfy the boundary condition for surfaces of the beam.

## 4. Numerical examples and analysis

### 4.1. The case for trapezoid cross-section

First, let us consider a circular beam with trapezoid cross-section, as shown in Fig. 3. A concentrated force, $P$, acts at its free end, and the other end is assumed to be fixed. For $R = 9\,\mathrm{cm}$, section constants for the cross-section, shown in Fig. 3, can be determined by Eq. (2) in the following

$$J_y = 11.59\ \mathrm{cm}^4, \quad J_{yz} = -5.394\ \mathrm{cm}^4, \quad J_z = 48.608\ \mathrm{cm}^4.$$

According to the model by Oden and Ripperger (1967), the formulae for shearing and radial stresses over a specific section $A - A(\theta = 30°)$ of the beam reduce to

$$\tau_{ys} = \left(\frac{V_y}{A}\right) \frac{AR}{(0.222y + 2.722)(R-y)} \left( \frac{J_y \overline{Q}_z - J_{yz} \overline{Q}_y}{J_z} - \frac{A'}{AR} \right),$$

$$\sigma_y = \left(\frac{V_y}{A}\right) \frac{AR}{\sqrt{3}(0.222y + 2.722)(R-y)} \left[ \frac{A'}{AR} - \frac{J_y \overline{Q}_z - J_{yz} \overline{Q}_y}{H} \right],$$

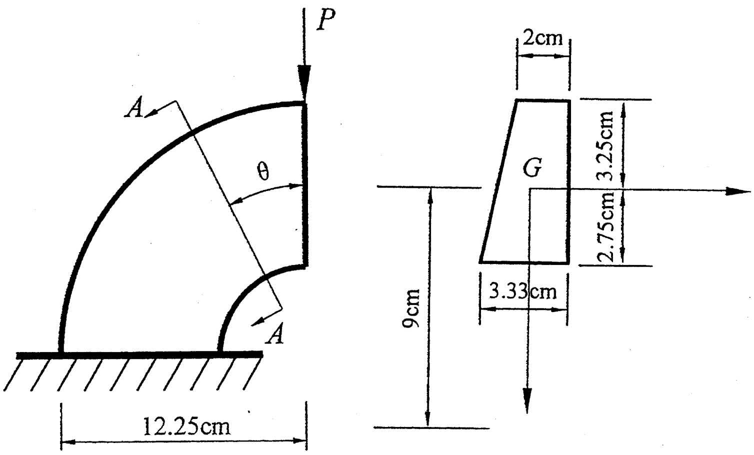


Fig. 3. A curved beam with trapezoid section.

in which

$$A' = \int_y^{h_2} b(y)\mathrm{d}y = \int_y^{h_2} (0.222y + 2.722)\,\mathrm{d}y, \quad \overline{Q}_y = R\int_y^{h_2} \frac{1}{R-y}\left(\int_{-0.222y-1.361}^{1.361} z\,\mathrm{d}z\right)\mathrm{d}y,$$

$$\overline{Q}_z = R\int_y^{h_2} \frac{b(y)y}{R-y}\mathrm{d}y = R\int_y^{h_2} \frac{(0.222y^2 + 2.722y)}{R-y}\mathrm{d}y.$$

In contrast, the present formulae for the case can be obtained by letting $V_z$, $M_y$, $p_y$ and $p_z$ in Eqs. (15) and (16) be zeros, and their lengthy expressions are omitted due to limitation of space.

### 4.2. The case for rectangular cross-section

Secondly, the cross-sectional shape of the beam is assumed to be rectangular. The width and height are denoted by $b$ and $h$, respectively. For the purpose of comparisons, several existing solutions together with the present solution are all listed in the following.

#### 4.2.1. Plane elasticity solution

In the theory of elasticity, the plane problem is solved in polar coordinates, and the formulae for shearing and radial stresses over a specific section $A-A(\theta = 45°)$ of the beam become (Timoshenko and Goodier, 1951)

$$\tau_{ys} = \left(\frac{V_y}{bh}\right)\frac{h}{D}\left[-\frac{R_1^2R_2^2}{r^3} + \frac{R_1^2+R_2^2}{r} - r\right], \quad \sigma_y = \left(\frac{V_y}{bh}\right)\frac{h}{D}\left[\frac{R_1^2R_2^2}{r^3} + \frac{R_1^2+R_2^2}{r} + r\right],$$

where the polar coordinate $r = R - y$, and $R_1 = R - h/2$, $R_2 = R + h/2$, $V_y = P/\sqrt{2}$, $D = R_1^2 - R_2^2 + (R_1^2 + R_2^2)\ln(R_2/R_1)$.

#### 4.2.2. Formulae in this paper

In this paper, the formula for the shearing stress reduces to

$$\tau_{ys} = \left(\frac{V_y}{bh}\right)\frac{R^2bh}{J_z(R-y)}\left[\left(y-\frac{h}{2}\right) + R\ln\left(\frac{R-y}{R-\frac{h}{2}}\right)\right]$$
$$+\left(\frac{V_y}{bh}\right)\frac{R^2bh}{J_z(R-y)^2}\left[\frac{y^2}{2} - \left(R+\frac{h}{2}\right)y + Ry\ln\left(\frac{R-y}{R-\frac{h}{2}}\right) - R^2\ln\left(\frac{R-y}{R-\frac{h}{2}}\right) + \frac{Rh}{2} + \frac{h^2}{8}\right].$$

For the case that $\theta = 45°$, $N_s = -P/\sqrt{2} = -V_y$, $M_z = -PR/\sqrt{2} = -RV_y$, $p_y = p_z = 0$. Substituting these parameters into Eq. (10) gives

$$\sigma_y = \left(\frac{V_y}{bh}\right)\frac{R^2bh}{J_z(R-y)}\left[\left(\frac{h}{4} - \frac{y}{2}\right) + \left(\frac{h^2}{8} - \frac{R^2}{2}\right)\left(\frac{1}{R-\frac{h}{2}} - \frac{1}{R-y}\right) - R\ln\left(\frac{R-y}{R-\frac{h}{2}}\right)\right].$$

#### 4.2.3. Formulae given by Oden and Reppierger

The formulae for the shearing and radial stresses are given by Oden and Ripperger (1967) as

$$\tau_{ys} = \left(\frac{V_y}{bh}\right)\frac{R^2bh}{J_z(R-y)}\left[\left(y-\frac{h}{2}\right) + R\ln\left(\frac{R-y}{R-\frac{h}{2}}\right)\right] + \left(\frac{V_y}{bh}\right)\frac{1}{(R-y)}\left(y-\frac{h}{2}\right),$$

$$\sigma_y = \left(\frac{V_y}{bh}\right)\frac{R^2bh}{8J_z(R-y)}\left[(h^2-R^2)\left(\frac{1}{R-\frac{h}{2}} - \frac{1}{R-y}\right) + \left(y-\frac{h}{2}\right) + 2R\ln\left(\frac{R-y}{R-\frac{h}{2}}\right)\right].$$

#### 4.2.4. Formulae given by Liu

Liu (1985) introduced directly the shearing stress formula $\tau_{ys} = V_y S_z^*/I_z b$ for the straight beams into the integral in Eq. (4). It follows that the corresponding formulae become

$$\tau_{ys} = \left(\frac{V_y}{bh}\right)\frac{h}{(R-y)}\left\{\frac{bR^3}{J_z}\left[\ln\frac{2(R-y)}{2R-h} - \left(\frac{h}{2R} - \frac{y}{R}\right)\right] - \left(\frac{1}{2} - \frac{3y}{2h} + \frac{2y^3}{h^3}\right)\right\},$$
$$\sigma_y = \left(\frac{V_y}{bh}\right)\frac{h}{(R-y)}\left\{\frac{bR^3}{J_z}\left[-\ln\frac{2(R-y)}{2R-h} + \left(\frac{h}{2R} - \frac{y}{R}\right)\right] + \left(\frac{1}{2} - \frac{3y}{2h} + \frac{2y^3}{h^3}\right)\right\}.$$

#### 4.2.5. Formula for the case of the straight beams

It is well known the formula of the shearing stress in a straight beam can be written as

$$\tau_{ys} = \frac{V_y S_z^*}{I_z b} = \left(\frac{V_y}{bh}\right)\left[\frac{3}{2}\left(1 - 4\frac{y^2}{h^2}\right)\right],$$

where $I_z = bh^3/12$. When the straight beams are subjected to bending, the radial stresses are regarded as bearing stresses, which are generally negligible. Hence there exists no specific formula for $\sigma_y$.

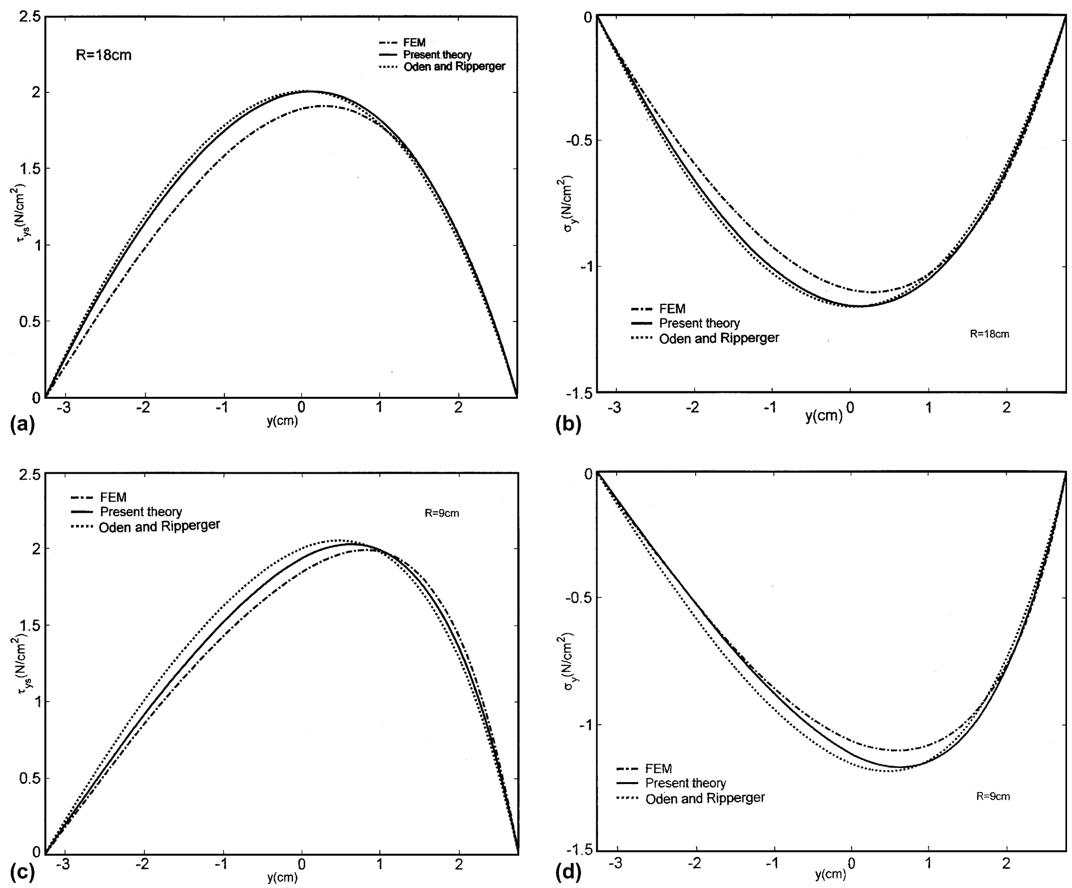


Fig. 4. Shearing and radial stress distributions on $y$-axis across the trapezoid beam at $\theta = 30°$ for various radii of curvature.

All numerical results are displayed in Figs. 4(a)–(d) and 5(a)–(c). The nondimensional parameter related to the shearing stress, as shown in Fig. 5, is denoted by $\beta_i = \tau_{ys}A/V_y$, where $i = 1, 2, 3, 4, 5$ correspond to the above-mentioned cases for the plane elasticity solution, the formula in this paper, the formula given by Oden and Reppierger, the formula given by Liu and the formula for the case of the straight beams, respectively.

Fig. 5(a)–(c) present the shearing stress distributions over the rectangular section ($\theta = 45°$) of the beam, which are obtained from various formulae of the beam for a rectangular cross-section with $R/h = 1.5$, 1 and 0.75. The above results clearly indicate that a larger curvature of the beam will result in a bigger difference between the numerical results (Oden and Ripperger, 1967; Liu, 1985) and the results by the theory of elasticity. However, the comparisons show that the present results agree well with the ones of the theory of elasticity and a three-dimensional finite element analysis by ANSYS program.

In view of Figs. 4 and 5, when the value of $R/h$ ($h = h_1 + h_2$) decreases, the point of maximum shearing stress for the beam is shifted from a certain point above the centroid of the cross-section to another point below that, and the shearing and radial stresses satisfy the mechanical boundary conditions at the top and bottom surfaces of the beams. The present results are compared with those from FEM. A good agreement can be observed.

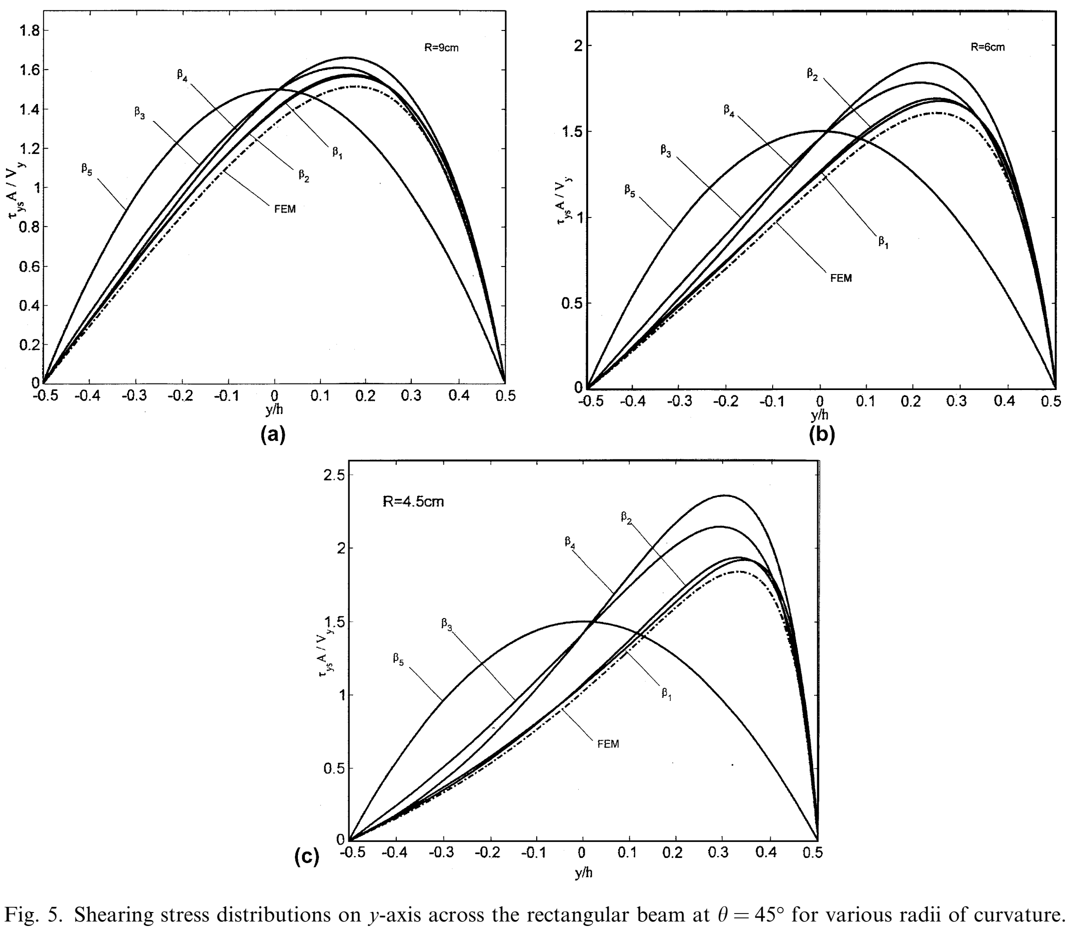


Fig. 5. Shearing stress distributions on $y$-axis across the rectangular beam at $\theta = 45°$ for various radii of curvature.

## 5. Conclusions

In present study the formulae for the shearing and radial stresses in curved beams are derived analytically based on the solution for a Volterra integral equation of the second kind. The numerical results indicate that the present solution is in good agreement with the solution of theory of elasticity, and its use can greatly enhance exactness and improve the efficiency in computation as compared with other approximate solutions based on certain additional assumptions. The resulting formulae can be applied to more general cases of curved beams with arbitrary shapes of cross-sections.

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