Do you believe in Real numbers?

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Motivation

Communication Theory
CHARLES M GOLDIE & RICHARD G.E. PINCH
London Mathematical Society Student Texts 20
Algorithmic information theory
G.J. Chaitin
INFORMATION RANDOMNESS & INCOMPLETENESS
PAPERS ON ALGORITHMIC INFORMATION THEORY
G J Chaitin
World Scientific
TEXTS AND MONOGRAPHS IN COMPUTER SCIENCE
AN INTRODUCTION TO KOLMOGOROV COMPLEXITY AND ITS APPLICATIONS
Ming Li
Paul Vitányi
Springer-Verlag
GÖDEL, ESCHER, BACH:
an Eternal Golden Braid
DOUGLAS R. HOFSTADTER
REVERSE MATHEMATICS
PROOFS FROM THE INSIDE OUT
JOHN STILLWELL
CAMBRIDGE INTRODUCTIONS TO PHILOSOPHY
An Introduction to Non-Classical Logic
From If to Is
GRAHAM PRIEST
Second Edition
CAMBRIDGE

Philosophical Hazards

Side effects of this talk may include

- Indifference
- Mild interest
- Five stages of grief
- Intuitionism

Today's menu

1. The arithmetic definition of real numbers
2. Computable numbers
3. Algorithmically random numbers

Peano Axioms (1889)

1. 0 is a natural number
2. If n is a natural number, its successor $S(n)$ is a natural number
3. For all n, $S(n) \neq 0$
4. For all m, n, $S(m) \neq S(n) \Rightarrow m \neq n$

We call $\mathbb{N} = \{0, S(0), S(S(0)), \ldots\}$ the set of natural numbers.

Arithmetic on Peano numbers

Addition and multiplication are defined recursively

For all $m,\ n \in \mathbb{N}$

- $m + 0 = m$, and $m + S(n) = S(m + n)$
- $m * 0 = 0$, and $m * S(n) = m * n + m$

$+$ and $*$ are associative and commutative.
$*$ is distributive over $+$.

Integers

$$\mathbb{Z} = \{(m, n) \mid m, n \in \mathbb{N}\},$$

where (m, n) is interpreted as $m - n$.

Relations on $\mathbb{Z}$:

- $(m_1, n_1) = (m_2, n_2) \ \Leftrightarrow m_1 + n_2 = m_2 + n_1$
- $(m_1, n_1) < (m_2, n_2) \ \Leftrightarrow m_1 + n_2 < m_2 + n_1$

Arithmetic on Integers

- $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$
- $(m_1, n_1) * (m_2, n_2) = (m_1 * m_2 + n_1 * n_2, m_1 * n_2 + m_2 * n_1)$
- $-(m, n) = (n, m)$ (Additive inverse)

$(\mathbb{Z}, +, *)$ is a *ring*.

Rationals

$$\mathbb{Q} = \{(p, q) \mid p, q \in \mathbb{Z}, q > 0\},$$

where (p, q) is interpreted as $\frac{p}{q}$.

Relations on $\mathbb{Q}$

- $(p_1, q_1) = (p_2, q_2) \Leftrightarrow p_1 * q_2 = p_2 * q_1$
- $(p_1, q_1) < (p_2, q_2) \Leftrightarrow p_1 * q_2 < p_2 * q_1$

Arithmetic on Rationals

- $(p_1, q_1) + (p_2, q_2) = (p_1 * q_2 + p_2 * q_1, q_1 * q_2)$
- $(p_1, q_1) * (p_2, q_2) = (p_1 * p_2, q_1 * q_2)$
- $(p, q)^{-1} = (q, p)$ (Multiplicative inverse)

$(\mathbb{Q}, +, *)$ is a *field*.

Dedekind cuts

No relation with the Snyder cut

A real number is a set $L \subset \mathbb{Q}$ with an upper bound and such that if $s \in L$ and $t < s$, then $t \in L$, e.g.

$$\sqrt{2} = \{q \in \mathbb{Q} \mid q \leq 0 \vee q^2 \leq 2\}$$

Relations on $\mathbb{R}$

- $L_1 = L_2$ (set equality)
- $L_1 < L_2 \Leftrightarrow L_1 \subset L_2$

Arithmetic operations are left as an exercise.

Computable numbers (Turing 1936)

A real number x is *computable* if its decimal expansion can be computed to an arbitrary precision using a Turing machine.

Equivalently,

- There exists a computable function that, given $\varepsilon > 0$, produces a rational q such that $|x - q| < \varepsilon$
- The characteristic function of x, taken as a Dedekind cut is computable

Computable numbers

Computable numbers contain

- rational numbers (trivially)
- algebraic numbers (by using Newton's method)
- e, π
- computable functions (sin, cos, exp, ...) of computable numbers

However, computable numbers are *countable*!

Kolmogorov complexity

aka Program size complexity, algorithmic entropy

The Kolmogorov complexity $K_A(s)$ of a sequence $s = (s_1, \ldots, s_n)$ in a given language A is the length of the shortest program in A that computes s.

"0101010101010101" $\longrightarrow$ `"01" * 8`
"0010100111010101" $\longrightarrow$ `"0010100111010101"`

Algorithmically random numbers (Martin-Löf 1966)

A real number is algorithmically random if the Kolmogorov complexity of its truncated (binary) expansion is "maximal" ($\sim$ diverges)

Properties of ARNs

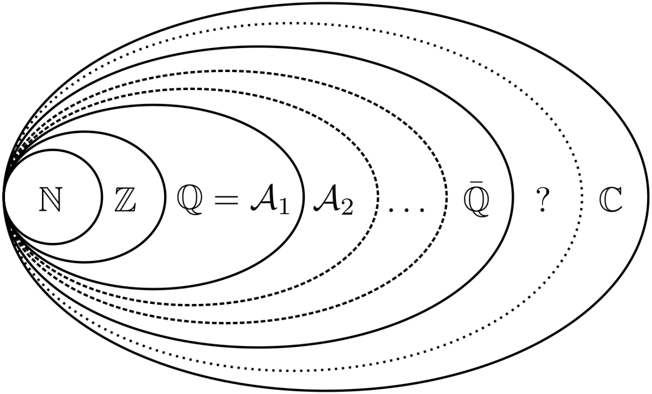
Algorithmically Random Numbers are

- not computable (and vice versa)
- incompressible ($\sim$ entropy)
- indistinguishable from random noise by any statistical test

Closing thoughts

Several interesting questions

- Do non computable real numbers actually exist?
- If not, what does it say about the use of real numbers in applied maths?
- Is there such a thing as "fundamental randomness", and is it ARNs?


$\mathbb{N}$
$\mathbb{Z}$
$\mathbb{Q} = \mathcal{A}_1$
$\mathcal{A}_2$
$\dots$
$\bar{\mathbb{Q}}$
?
$\mathbb{C}$

That's all folks

Slides and references can be found at
`https://github.com/csimal/are-real-numbers-real`