1 Original equations

Scalar field Lagrangian, energy-momentum tensor and field equation:

$$
\begin{aligned}
L &= (\partial^\rho\psi)(\partial_\rho\psi) - V(\psi), && (1)\\
T_{\alpha\beta} &= (\partial_\alpha\psi)(\partial_\beta\psi) - \frac{1}{2}g_{\alpha\beta}\left((\partial^\rho\psi)(\partial_\rho\psi) - V(\psi)\right), && (2)\\
\nabla^\rho\partial_\rho\psi &= -\frac{1}{2}\frac{\partial V}{\partial\psi}. && (3)
\end{aligned}
$$

Original metric:

$$
ds^2 = A\,dt^2 - B\,dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4}
$$

In this parameterization, the shift vector is zero and the lapse function is $\sqrt{A}$. Let

$$
\Phi = \frac{\partial\psi}{\partial r}, \qquad \Pi = \sqrt{\frac{B}{A}}\frac{\partial\psi}{\partial t}. \tag{5}
$$

Kodama vector as a time evolution vector field:

$$
\frac{\partial}{\partial\tau} = K^\alpha\partial_\alpha = \frac{1}{\sqrt{AB}}\frac{\partial}{\partial t}. \tag{6}
$$

Let the new radial coordinate be the same as the original: $r(\tau,\rho) = \rho$. Then

$$
dt = \frac{1}{\sqrt{AB}}d\tau + \frac{\partial t}{\partial\rho}d\rho = \frac{1}{\sqrt{AB}}(d\tau + C d\rho). \qquad dr = d\rho. \tag{7}
$$

The metric in the new coordinates:

$$
ds^2 = \frac{1}{B}d\tau^2 + 2\frac{C}{B}d\tau d\rho - \frac{B^2 - C^2}{B}d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{8}
$$

Using $x^i = (\rho,\theta,\phi)$, the definition of lapse and shift is $ds^2 = (N\,d\tau)^2 + g_{ij}(dx^i + N^i d\tau)(dx^j + N^j d\tau)$. From (8), their values are

$$
N = \sqrt{\frac{B}{B^2 - C^2}}, \qquad N^i = \left(-\frac{C}{B^2 - C^2},\ 0,\ 0\right). \tag{9}
$$

The fields:

$$
\Phi = \frac{\partial\psi}{\partial\rho} - C\frac{\partial\psi}{\partial\tau}, \qquad \Pi = B\frac{\partial\psi}{\partial\tau}. \tag{10}
$$

Kodama vector and its four-divergence in the $\tau - \rho$ system:

$$
\begin{aligned}
K^\alpha &= \delta^\alpha_0, && (11)\\
\nabla_\rho K^\rho &= \partial_\rho K^\rho + \Gamma^\rho_{\sigma\rho}K^\sigma = \Gamma^\rho_{0\rho} = 0. && (12)
\end{aligned}
$$

From (2) and (8-10), we get

$$
\begin{aligned}
T^0_{\ 0} &= \frac{B(\Phi^2+\Pi^2+BV)+2C\Phi\Pi}{2B^2}, && (13)\\
T^1_{\ 0} &= -\frac{\Phi\Pi}{B^2}. && (14)
\end{aligned}
$$

Evolution equations [1]:

$$
\begin{aligned}
\frac{\partial\psi}{\partial\tau} &= \frac{\Pi}{B}, && (15)\\
\frac{\partial\Phi}{\partial\tau} &= \frac{B\frac{\partial}{\partial\rho}\Pi - C\frac{\partial}{\partial\rho}\Phi}{B^2-C^2} + \frac{B(B-1)\Pi - C(B+1)\Phi}{\rho(B^2-C^2)} \\
&\quad + \frac{8\pi\rho(C\Phi - B\Pi)BV + BC\frac{\partial V}{\partial\psi}}{2(B^2-C^2)}, && (16)\\
\frac{\partial\Pi}{\partial\tau} &= \frac{B\frac{\partial}{\partial\rho}\Phi - C\frac{\partial}{\partial\rho}\Pi}{B^2-C^2} + \frac{B(B+1)\Phi - C(B-1)\Pi}{\rho(B^2-C^2)} \\
&\quad + \frac{8\pi\rho(C\Pi - B\Phi)BV - B^2\frac{\partial V}{\partial\psi}}{2(B^2-C^2)} && (17)\\
\frac{\partial B}{\partial\tau} &= 8\pi\rho\Phi\Pi, && (18)\\
\frac{\partial C}{\partial\tau} &= -4\pi\rho(\Phi^2+\Pi^2). && (19)
\end{aligned}
$$

By expanding the equations in series about $\rho = 0$, we can get the parity relations by assuming that the equations are nonsingular. With a precision of order ρ^4, we get the following results:

$$
\begin{aligned}
B(-\rho) &= B(\rho), \qquad B(0) = 1, && (20)\\
C(-\rho) &= -C(\rho), && (21)\\
\Phi(-\rho) &= -\Phi(\rho), && (22)\\
\Pi(-\rho) &= \Pi(\rho). && (23)
\end{aligned}
$$

The initial state can be determined by solving the following equation:

$$
\frac{\partial B}{\partial\rho} = 4\pi\rho\left(B(\Phi^2+\Pi^2+BV)+2C\Phi\Pi\right) - \frac{B(B-1)}{\rho}. \qquad (24)
$$

For simplicity, we can choose

$$
\Phi(\rho) = 0. \qquad (25)
$$

2 Equations for the numerical simulation

Substitutions to avoid singularities:

$$\begin{aligned}
\Phi(\tau,\rho) &= \rho\phi(\tau,\rho), && (26)\\
B(\tau,\rho) &= 1+\rho^2\beta(\tau,\rho), && (27)\\
C(\tau,\rho) &= \rho\gamma(\tau,\rho). && (28)
\end{aligned}$$

Evolution equations:

$$\begin{aligned}
\frac{\partial\phi}{\partial\tau} &= \frac{-\rho\gamma\frac{\partial}{\partial\rho}\phi + B\frac{1}{\rho}\frac{\partial\Pi}{\partial\rho}}{B^2-C^2} + \frac{B\beta\Pi - \gamma(B+2)\phi}{B^2-C^2}\\
&\quad + \frac{8\pi(C\Phi - B\Pi)BV + BC\frac{1}{\rho}\frac{\partial V}{\partial\psi}}{2(B^2-C^2)}, && (29)\\
\frac{\partial\Pi}{\partial\tau} &= \frac{B\rho\frac{\partial}{\partial\rho}\phi - \rho\gamma\frac{\partial\Pi}{\partial\rho}}{B^2-C^2} + \frac{B(B+2)\phi - \rho^2\beta\gamma\Pi}{B^2-C^2}\\
&\quad + \frac{8\pi\rho(C\Pi - B\Phi)BV - B^2\frac{\partial V}{\partial\psi}}{2(B^2-C^2)}, && (30)\\
\frac{\partial\beta}{\partial\tau} &= 8\pi\phi\Pi, && (31)\\
\frac{\partial\gamma}{\partial\tau} &= -4\pi(\Phi^2+\Pi^2). && (32)
\end{aligned}$$

Ordinary differential equation for the calculation of the initial state:

$$\frac{\partial\beta}{\partial\rho} = \frac{4\pi B(\Phi^2+\Pi^2+BV) - 3\beta}{\rho} + 8\pi\gamma\Phi\Pi - \rho\beta^2. \qquad (33)$$

Initial conditions (see Figure 1):

$$\begin{aligned}
\phi(0,\rho) &= \begin{cases} c\exp\left(\frac{d}{(\rho-a)^2-b^2}\right) & \text{if } a-b<\rho<a+b,\\ 0 & \text{otherwise,}\end{cases} && (34)\\
\Pi(0,\rho) &= 0, && (35)\\
\beta(0,0) &= 0, && (36)\\
\gamma(0,\rho) &= 0. && (37)
\end{aligned}$$

References

[1] I. Rácz, Class. Quant. Grav. **23** (2006) 115-124, gr-qc/0511052

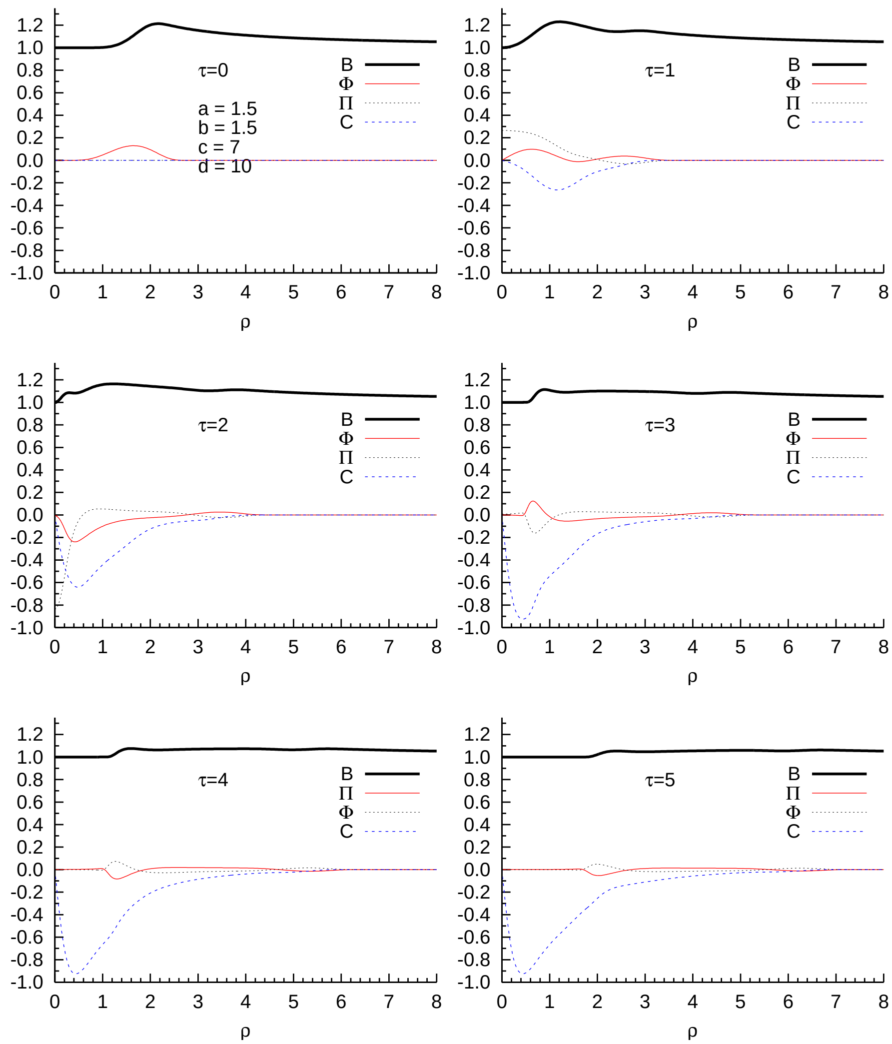

Figure 1: Initial state and time evolution in the massless/interactionless case.

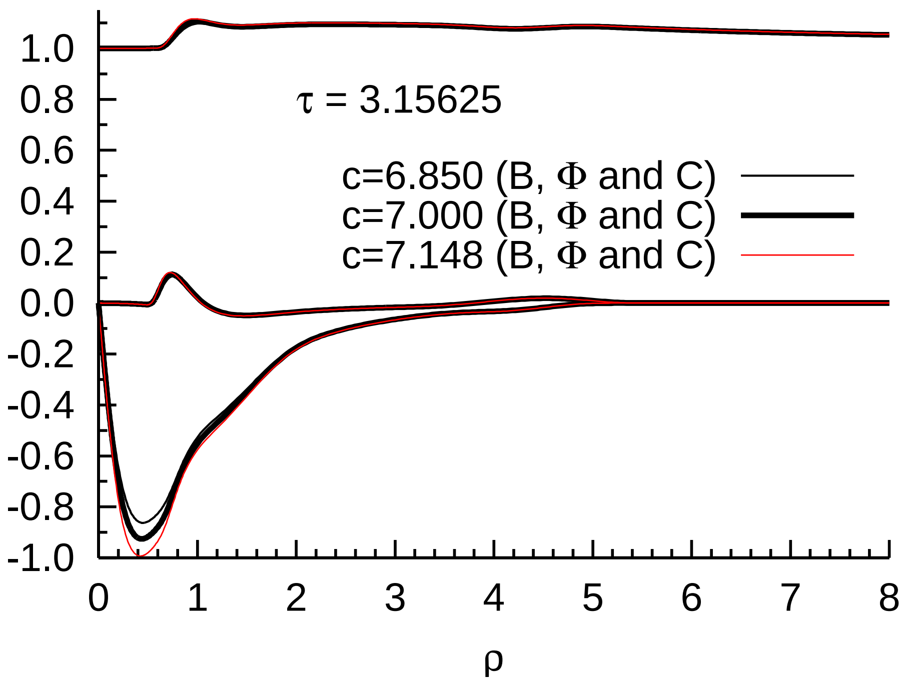

Figure 2: Near the instability. The simulation crashes for $c \geq c_{\rm critical} \approx 7.148$ at $\tau \approx 3.16$. (With $C_F = \Delta\tau/\Delta\rho = 1/64$. With larger values of the Courant factor, the simulation crashes earlier and $c_{\rm critical}$ is smaller.)