APPENDIX

Proof of Theorem 1

***Proof*:** Let $\vec{W} = \mathbf{A}\vec{X} = [W_1, ..., W_m]$, since $\vec{X} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_{\vec{X}})$ then we have $\vec{W} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_{\vec{W}})$, where $\mathbf{\Sigma}_{\vec{W}} = \mathbf{A}\mathbf{\Sigma}_{\vec{X}}\mathbf{A}^\top$. Note that $\mathbf{\Sigma}_{\vec{X}} = \sigma^2\mathbf{I}$, and $\mathbf{I}$ denotes the identity matrix. Let $\vec{Z} = ReLU(\vec{W}) = \max(\vec{W}, \mathbf{0}) = [Z_1, ..., Z_m]$, we first discuss the relationship between $\vec{W}$ and $\vec{Z}$.

Obviously, the variance of W_i can be expressed as $var(W_i) = (\eta_i\sigma)^2, i = 1, ..., m$, η_i is related with $\mathbf{A}$, then the probability density function of W_i, denoted by $f_{W_i}(x)$, is expressed as

$$f_{W_i}(x) = \frac{1}{\sqrt{2\pi}\eta_i\sigma} e^{-\frac{x^2}{2(\eta_i\sigma)^2}}.$$

According to $Z_i = \max(W_i, 0)$, we have the mean and the variance of Z_i as below:

$$\begin{aligned}
&E(Z_i) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\eta_i\sigma} e^{-\frac{x^2}{2(\eta_i\sigma)^2}} \max(x, 0)dx \\
&= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\eta_i\sigma} e^{-\frac{x^2}{2(\eta_i\sigma)^2}} xdx = \frac{\eta_i\sigma}{\sqrt{2\pi}}; \\
&E(Z_i^2) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\eta_i\sigma} e^{-\frac{x^2}{2(\eta_i\sigma)^2}} \max(x, 0)^2dx \\
&= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\eta_i\sigma} e^{-\frac{x^2}{2(\eta_i\sigma)^2}} x^2dx = \frac{(\eta_i\sigma)^2}{2}; \\
&var(Z_i) = E(Z_i^2) - E(Z_i)^2 = \frac{\pi - 1}{2\pi}(\eta_i\sigma)^2.
\end{aligned}$$

Our purpose is to compute the covariance matrix of $\vec{Z}$, that is $\mathbf{\Sigma}_{\vec{Z}}$. After computing $var(Z_i)$, which is on the diagonal position of $\mathbf{\Sigma}_{\vec{Z}}$, we now calculate $cov(Z_i, Z_j), i \neq j$, by

$$cov(Z_i, Z_j) = E(Z_iZ_j) - E(Z_i)E(Z_j).$$

Assume that the joint probability density function of W_i and W_j is written as

$$f_{W_i,W_j}(x, y) = \frac{1}{2\pi\eta_i\eta_j\sigma^2\sqrt{1 - \rho_{ij}^2}} p(x, y, \rho_{ij}, \eta_i, \eta_j),$$

where $p(x,y,\rho_{ij},\eta_i,\eta_j) = e^{-\frac{1}{2(1-\rho_{ij}^2)}\left(\frac{x^2}{(\eta_i\sigma)^2}-\frac{2\rho_{ij}xy}{\eta_i\eta_j\sigma^2}+\frac{y^2}{(\eta_j\sigma)^2}\right)}$, and ρ_{ij} is the correlation coefficient between W_i and W_j. Then,

$$E(Z_iZ_j) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{W_i,W_j}(x,y)\max(x,0)\max(y,0)dxdy$$
$$= \int_0^{\infty}\int_0^{\infty} f_{W_i,W_j}(x,y)xydxdy$$
$$= \int_0^{\infty}\int_0^{\infty} \frac{1}{2\pi\eta_i\eta_j\sigma^2\sqrt{1-\rho_{ij}^2}}p(x,y,\rho_{ij},\eta_i,\eta_j)xydxdy.$$

Furthermore, define $u = \frac{x}{\eta_i\sigma\sqrt{1-\rho_{ij}^2}}, v = \frac{y}{\eta_j\sigma\sqrt{1-\rho_{ij}^2}}$, the computation of $E(Z_iZ_j)$ is transformed to be

$$E(Z_iZ_j) = M_{ij}\sigma^2,$$

where $M_{ij} = \frac{\left(\sqrt{1-\rho_{ij}^2}\right)^3\eta_i\eta_j}{2\pi}\int_0^{\infty}\int_0^{\infty} e^{-\frac{1}{2}(u^2-2\rho_{ij}uv+v^2)}uvdudv.$

Hence,

$$cov(Z_i,Z_j) = E(Z_iZ_j) - E(Z_i)E(Z_j)$$
$$= M_{ij}\sigma^2 - \frac{\eta_i\eta_j\sigma^2}{2\pi} = (M_{ij} - \frac{\eta_i\eta_j}{2\pi})\sigma^2.$$

According to the expressions of $var(Z_i)$ and $cov(Z_i,Z_j)$, it is clear to see that $\mathbf{\Sigma}_{\vec{Z}}$ can be formulated as

$$\mathbf{\Sigma}_{\vec{Z}} = \sigma^2\hat{\mathbf{\Sigma}}_{\vec{Z}},$$

where $\hat{\mathbf{\Sigma}}_{\vec{Z}}$ is only determined by the matrix $\mathbf{A}$.

Therefore, the covariance matrix of $\vec{Y} = \vec{B}\vec{Z}$, *i.e.* $\mathbf{\Sigma}_{\vec{Y}}$ is written as

$$\mathbf{\Sigma}_{\vec{Y}} = \mathbf{B}\mathbf{\Sigma}_{\vec{Z}}\mathbf{B}^\top = \sigma^2\mathbf{B}\hat{\mathbf{\Sigma}}_{\vec{Z}}\mathbf{B}^\top.$$

Due to $E[\|\vec{Y}-E[\vec{Y}]\|_2^2] = tr(\mathbf{\Sigma}_{\vec{Y}}) = \sigma^2 tr(\mathbf{B}\hat{\mathbf{\Sigma}}_{\vec{Z}}\mathbf{B}^\top)$ ($tr(\cdot)$ denotes the trace of a matrix) and $E[\|\vec{X}-E[\vec{X}]\|_2^2] = tr(\mathbf{\Sigma}_{\vec{X}}) = n\sigma^2$, then we have

$$E[\|\vec{Y}-E[\vec{Y}]\|_2^2] = \alpha E[\|\vec{X}-E[\vec{X}]\|_2^2],$$

where $\alpha = \frac{tr(\mathbf{B}\hat{\mathbf{\Sigma}}_{\vec{Z}}\mathbf{B}^\top)}{n}$. That means $E[\|\vec{Y}-E[\vec{Y}]\|_2^2]$ and $E[\|\vec{X}-E[\vec{X}]\|_2^2]$ are linearly related. ∎