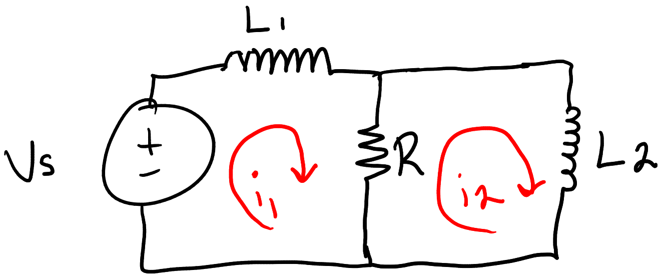
Lecture 4

Continuing Second Order Circuits

Method 2: Operator Method



Given

$R = 1\,\Omega$

$L_1 = 1\,H$

$L_2 = 2H$

$V_L = L\frac{di}{dt}$

i_1) $-V_s + V_{L_1} + R(i_1 - i_2) = 0$

$$\Rightarrow L_1\frac{di_1}{dt} + R(i_1 - i_2) = V_s$$

i_2) $R(i_2 - i_1) + L_2\frac{di_2}{dt} = 0$

Substitute Values

$$\frac{di_1}{dt} + i_1 - i_2 = V_s \quad (1)$$

$$i_2 - i_1 + 2\frac{di_2}{dt} = 0 \quad (2)$$

* differential operator $\Rightarrow \frac{d}{dt} = s$

$$\Rightarrow s i_1 + i_1 - i_2 = V_s \Rightarrow (s+1)i_1 - i_2 = V_s$$

$$\Rightarrow i_2 - i_1 + 2s i_2 = 0 \Rightarrow -i_1 + (2s+1)i_2 = 0$$

* Cramer's Rule for i_2

For i_1: $\begin{vmatrix} V_s & -1 \\ 0 & 2s+1 \end{vmatrix}$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\frac{\begin{array}{cc} i_1 & i_2 \end{array} \\ \begin{vmatrix} s+1 & V_s \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} s+1 & -1 \\ -1 & 2s+1 \end{vmatrix}} = \frac{0 - (-V_s)}{[2s^2+3s+1] - (1)}$$

$$\Rightarrow \frac{V_s}{2s^2+3s} = i_2 \quad \text{or} \quad V_s = i_2(2s^2+3s)$$

* Sub back in $\frac{d}{dt}$

Divide by 2

$$2\frac{d^2 i_2}{dt^2} + 3\frac{di_2}{dt} = V_s$$

$$\Rightarrow \frac{d^2 i_2}{dt^2} + \frac{3}{2}\frac{di_2}{dt} = \frac{1}{2}V_s$$

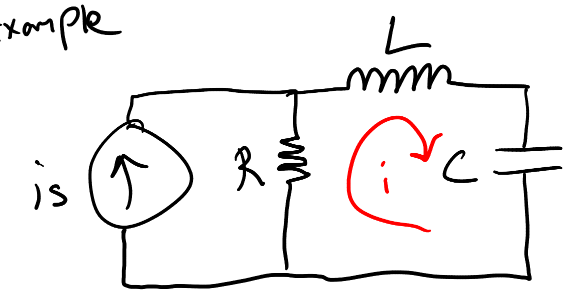
$$2\alpha = \frac{3}{2} \qquad \omega_o^2 = 0$$

$$\alpha = \frac{3}{4} \qquad \omega_o = 0$$

* Steps for the operator method

1) Identify the variable for which the solution is desired
2) Write 2 differential equations in terms of x_1 and x_2
3) Use the operator $s = \frac{d}{dt}$ and $\frac{1}{s} = \int dt$ to obtain 2 algebraic equations
4) Use Cramer's Rule to solve for desired variable
5) Rearrange equation and convert back to derivative form.

* Example



Given

$R = 1\,\Omega$

$L = 2\,H$

$C = 0.5\,F$

Find the 2^{nd} order diff. eq. using the direct method in terms of i

Mesh eq.)

$$R(i - i_s) + L\frac{di}{dt} + V_c = 0$$

$$\Rightarrow i - i_s + 2\frac{di}{dt} + V_c = 0$$

$$\Rightarrow V_c = -i + i_s - 2\frac{di}{dt}$$

$$\frac{dV_c}{dt} = -\frac{di}{dt} + \frac{di_s}{dt} - 2\frac{d^2 i}{dt^2}$$

Remember

$$i_c = C\frac{dv}{dt}$$

Plug in

$$i = -\frac{1}{2}\frac{di}{dt} + \frac{1}{2}\frac{di_s}{dt} - \frac{d^2i}{dt^2}$$

* Multiply $\frac{dv}{dt}$ by $C = \frac{1}{2} = 0.5F$

Rearrange

$$\boxed{\frac{d^2i}{dt^2} + 0.5\frac{di}{dt} + i = 0.5\frac{di_s}{dt} \text{ Amp}}$$

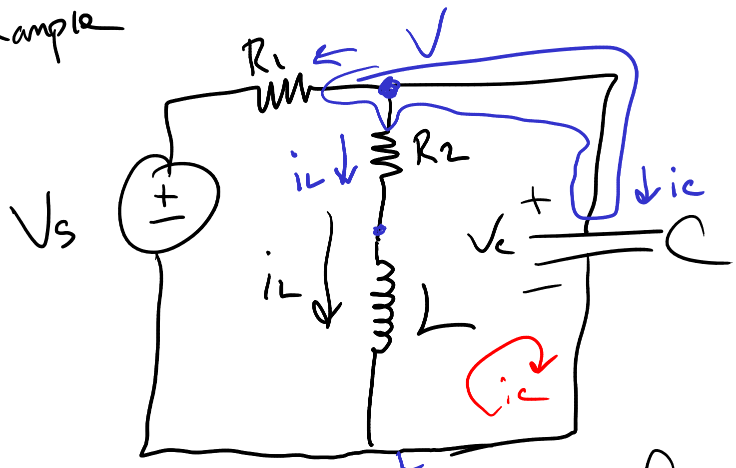
$2\alpha = \frac{1}{2}$

$\therefore \alpha = 0.25$

$\omega_o^2 = 1$

$\omega_o = 1 \frac{rad}{s}$

* Example



Given

$R_1 = 1\,k\Omega$

$R_2 = 1\,\Omega$

$L = 1\,mH$

$C = 1\,mF$

Find the differential equation for the voltage V_c using the operator method

$X_1 = i_L$

$X_2 = V_c$

At Node V

$$\frac{1}{R_1}(V - V_s) + i_L + C\frac{dv}{dt} = 0 \leftarrow$$

Current through inductor

$$V_c = i_L R_2 + L\frac{di_L}{dt} \Rightarrow i_L R_2 + L\frac{di_L}{dt} - V_c = 0$$

$$\frac{d}{dt} \to s$$

$$(1)\,i_L + \left(\frac{1}{R_1} + sC\right)V = \frac{V_s}{R_1} \quad (1)$$

$\times R_1$

Replace $\frac{d}{dt} \to s$

$$i_L R_2 + sLi_L - V = 0$$

$$(R_2 + sL)\,i_L + (-1)V = 0 \quad (2)$$

$$R_1 i_L + (1+sR_1C)V = V_s \quad (3)$$

*Cramer's Rule

$$\frac{\begin{vmatrix} R_1 & V_s \\ R_2+sL & 0 \end{vmatrix}}{\begin{vmatrix} R_1 & 1+sR_1C \\ R_2+sL & -1 \end{vmatrix}} = \frac{0-(R_2+sL)V_s}{-R_1-(1+sR_1C)(R_2+sL)}$$

$$= \frac{-(R_2+sL)V_s}{-R_1-\left[R_2+sL+sR_1R_2C+s^2R_1LC\right]}$$

$$= \frac{-(R_2+sL)V_s}{\left[-R_1LC\right]s^2-\left[L+R_1R_2C\right]s-\left[R_1+R_2\right]}$$

$10^{-3}+\cancel{10^{3}\times 10^{-3}}$

Plug in values

$$\Rightarrow \frac{-(1+10^{-3}s)V_s}{-10^{-3}s^2-\left[10^{-3}+1\right]s-\left[10^{3}+1\right]}$$

Divide by 10^{-3} and factor out -1 from denominator

$$= \frac{(s+1000)V_s}{s^2+1001s+1001\times10^3} = V_c$$

*Rearrange and convert back

$$V_c(s^2+1001s+1001\times10^3) = (s+1000)V_s$$

$$\Rightarrow \boxed{\frac{d^2V_c}{dt^2}+1001\frac{dV_c}{dt}+1001\times10^3V_c = \frac{dV_s}{dt}+1000V_s} \text{ Volts}$$

* Solution of 2^{nd} order differential equation

Equation is of the form

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 = f(t)$$

Where a_2, a_1, a_0 are known constants and $f(t)$ is the forcing function.

* The complete response $x(t)$ is given by

$$x(t) = \underbrace{x_n}_{\text{Natural Response}} + \underbrace{x_f}_{\text{Forced Response}}$$

The natural response $\longrightarrow$ when $f(t) = 0$

The forced response $\longrightarrow$ when $f(t) \neq 0$

* The natural response will satisfy the equation

$$a_2 \frac{d^2x_n}{dt^2} + a_1 \frac{dx_n}{dt} + a_0 x_n = 0$$

Where $x_n = Ae^{st}$

$$\frac{dx_n}{dt} = sAe^{st} \implies \frac{d^2x_n}{dt} = s^2Ae^{st}$$

$$a_2 s^2 Ae^{st} + a_1 sAe^{st} + a_0 Ae^{st} = 0$$

Add x_n back and factor

$$\implies (a_2 s^2 + a_1 s + a_0) x_n = 0$$

* Since $x_n = 0$ is a trivial solution, we require

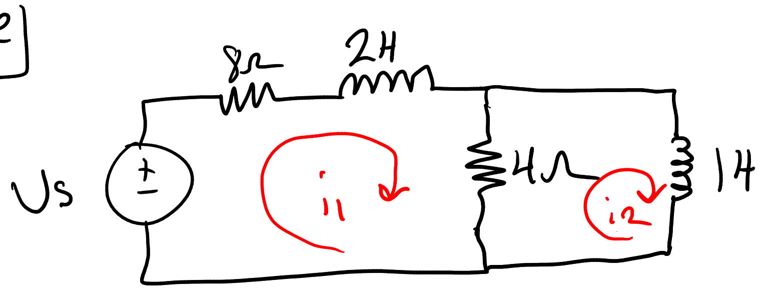
$$\underbrace{a_2 s^2 + a_1 s + a_0}_{\text{Characteristic Equation}} = 0 \qquad s^n = \frac{d^n}{dt^n} \text{ holds}$$

$$s_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2a_0}}{2a_2} \qquad s_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2a_0}}{2a_2}$$

* When there are two roots,

$$X_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

*Example)



Find the natural response of the current i_2

Mesh i_1)

$$-V_s + 8i_1 + 2\frac{di_1}{dt} + 4(i_1 - i_2) = 0$$

$$\Rightarrow 12 i_1 + 2\frac{di_1}{dt} - 4 i_2 = V_s \quad (1)$$

Mesh i_2)

$$4(i_2 - i_1) + \frac{di_2}{dt} = 0$$

$$\Rightarrow -4i_1 + 4i_2 + \frac{di_2}{dt} = 0 \quad (2)$$

use $s = \frac{d}{dt}$

$$(12 + 2s) i_1 - 4 i_2 = V_s$$

$$-4 i_1 + (4 + s) i_2 = 0$$

$$i_2 = \frac{\begin{vmatrix} 12+2s & V_s \\ -4 & 0 \end{vmatrix}}{\begin{vmatrix} 12+2s & -4 \\ -4 & 4+s \end{vmatrix}} = \frac{0 - (-4V_s)}{48 + 20s + 2s^2 - 16}$$

Divide by 2

$$\Rightarrow \frac{4V_s}{2s^2 + 20s + 32} = \frac{2V_s}{s^2 + 10s + 16} = i_2$$

$$\Rightarrow \underbrace{(s^2 + 10s + 16)}_{\text{Characteristic Equation}} i_2 = 2V_s$$

Roots are

$s_1 = -2$

$s_2 = -8$

$$\boxed{X_n = A_1 e^{-2t} + A_2 e^{-8t}}$$

We will find A_1 and A_2 Next time