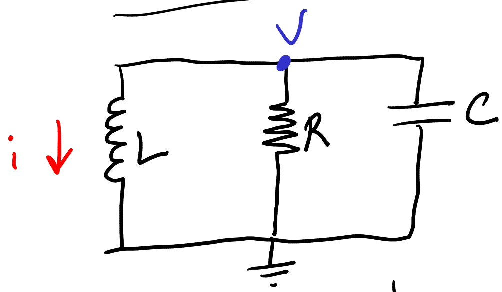
Lecture 5

Natural Response of the Unforced Parallel RLC Circuit



* Circuit will have a second order differential equation

$$\frac{d^2}{dt^2}x(t) + 2\alpha\frac{d}{dt}x(t) + \omega_0^2 x(t) = \cancel{f(t)} \rightarrow 0$$

No source

* We will choose the output to be $v(t)$

$$\frac{d^2}{dt^2}v(t) + 2\alpha\frac{d}{dt}v(t) + \omega_0^2 v(t) = 0$$

KCL at V

$$i + i_R + i_C = 0$$

$$\frac{1}{L}\int_0^t v\,d\tau + \underbrace{i(0)}_{\text{constant}} + \frac{v}{R} + C\frac{dv}{dt} = 0$$

Remember

$$v_L = L\frac{di_L}{dt}$$

or

$$i = \int_0^t \frac{v_L}{L}\,d\tau + i(0)$$

$$i_C = C\frac{dv}{dt}$$

Taking the derivative and rearrange

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$

* Divide by C

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \qquad \frac{d}{dt} \rightarrow s$$

$$\Rightarrow \left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right)v = 0$$

Characteristic Equation $\Leftrightarrow$ C.E.

$$2\alpha = \frac{1}{RC} \qquad \omega_0^2 = \frac{1}{LC}$$

$$\therefore \alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

* The two roots of the C.E. are

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \quad \text{and} \quad s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

* When $s_1 \neq s_2$, the solution is

$$v_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

* The roots can be written as

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad \text{and} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

* We define the damped resonant frequency as

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$\frac{\partial}{\partial t}$ partial

$\frac{d}{dt}$ timed Derivative

* The roots of the C.E. assume possible conditions:

1) $\alpha^2 > \omega_0^2 \longrightarrow$ Two real, distinct roots $\longrightarrow$ overdamped
2) $\alpha^2 = \omega_0^2 \longrightarrow$ Two real, equal roots $\longrightarrow$ critically damped
3) $\alpha^2 < \omega_0^2 \longrightarrow$ Two complex roots $\longrightarrow$ underdamped

* Considering an overdamped RLC circuit at $t=0$

$$\Rightarrow \quad V_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$\boxed{V(0) = A_1 + A_2}$$

We need another equation to solve for A_1 and A_2
Let's solve for the derivative

$A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}$

KCL from earlier

$$\frac{1}{L}\int_0^t v\,d\tau + i(0) + \frac{v}{R} + C\frac{dv}{dt} \leftarrow t=0$$

$$\Rightarrow i(0) + \frac{V(0)}{R} + C\frac{dv(0)}{dt} = 0$$

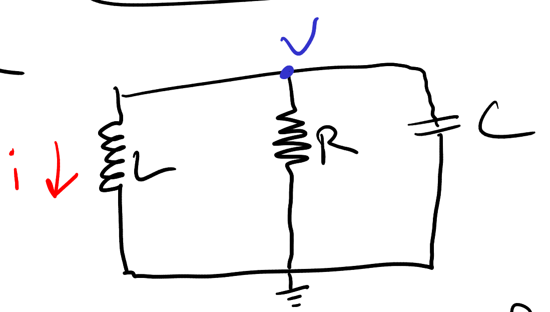
$$\Rightarrow \frac{dv(0)}{dt} = -\frac{V(0)}{RC} - \frac{i(0)}{C}$$

* Also

$$\frac{dv(0)}{dt} = s_1 A_1 + s_2 A_2$$

$$\Rightarrow \boxed{s_1 A_1 + s_2 A_2 = -\frac{V(0)}{RC} - \frac{i(0)}{C}}$$

* Example



Given
$R = \frac{2}{3}\Omega$, $L = 1\,H$
$C = \frac{1}{2}F$, $V(0) = 10V$, $i(0) = 2A$

Find the natural response of $v(t)$

$$C.E. = s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

or $s^2+3s+2=0 \Rightarrow (s+1)(s+2)=0$

$S_1=-1 \quad S_2=-2$ $\quad \alpha=\frac{1}{2RC}$ and $\omega_0^2=\frac{1}{LC}$

Circuit is overdamped!!!

$\alpha=\frac{3}{2} \qquad \omega_0^2=2$

$\alpha^2=\frac{9}{4}$

$$V_n=A_1e^{-t}+A_2e^{-2t}$$

* To find A_1 and A_2

$V_n(0)=A_1+A_2 \Rightarrow A_1+A_2=10$ $\qquad A_1=10-A_2$

$S_1A_1+S_2A_2=\frac{-V(0)}{RC}-\frac{i(0)}{C}$

$\Rightarrow -A_1-2A_2=\frac{-10}{1/3}-\frac{2}{1/2}$

$\Rightarrow -A_1-2A_2=-34$

$-10+A_2-2A_2=-34$

$-A_2=-24 \Rightarrow A_2=24$ and $A_1=10-24$

$A_1=-14$

$$\boxed{V_n=-14e^{-t}+24e^{-2t}\ V}$$

LTSpice Time

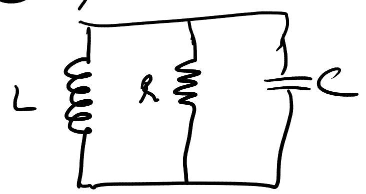
* Critically damped Parallel RLC

- Happens when $\alpha^2=\omega_0^2$, therefore $S_1=S_2$
- Solution will be of the form,

$$V_n=e^{S_1t}(A_1t+A_2)$$

* Given:

$L=1H,\ R=1\Omega,\ C=\frac{1}{4}F,\ V(0)=5V,\ i(0)=-6A$



$C.E.=s^2+4s+4$

$S_1=S_2=-2$

Therefore, $V_n=e^{-2t}\overbrace{(A_1t+A_2)}^{\text{second}}$

At $t=0$

$$v(0) = \cancel{e^{-2(0)}}_{1}\left(A_1\cancel{(0)}_{0} + A_2\right)$$

$$v(0) = A_2 = 5V$$

* To find A_1, we will find the derivative at $t=0$

$$\frac{dv}{dt} = A_1 \underbrace{e^{-2t}}_{1} - \underbrace{2A_1 t e^{-2t}}_{0} - 2A_2 \underbrace{e^{-2t}}_{1}$$

$$\frac{dv(0)}{dt} = A_1 - 2A_2$$

$$\frac{dv(0)}{dt} = -\frac{v(0)}{RC} - \frac{i(0)}{C}$$ ← from Earlier

* Therefore

$$A_1 - 2A_2 = \frac{-5}{1/4} - \frac{-6}{1/4}$$

$$A_1 - 10 = -20 - (-24)$$

$$A_1 = 14$$

$$\boxed{V_n = e^{-2t}(14t + 5)\ V}$$

* Undamped Parallel RLC

When $\alpha^2 < \omega_0^2$ or when $L < 4R^2C$

* Roots will be complex conjugates

$$S_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$ where $j = \sqrt{-1}$

* We call $\sqrt{\omega_0^2 - \alpha^2}$ the damped resonant frequency ω_d

$$\Rightarrow S_{1,2} = -\alpha \pm j\omega_d$$

* The natural response is,

$$V_n = e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

* Euler's Identity

$$e^{\pm j\omega_d t} = \cos(\omega_d t) \pm j\sin(\omega_d t)$$

$$V_n = e^{-\alpha t}\left[\underbrace{(A_1+A_2)}_{B_1}\cos(\omega_d t) + \underbrace{j(A_1-A_2)}_{B_2}\sin(\omega_d t)\right]$$

* Because A_1 and A_2 remain arbitrary we replace them with

$$V_n = e^{-\alpha t}\left(B_1\cos(\omega_d t) + B_2\sin(\omega_d t)\right)$$

Where A_1 and A_2 must be complex conjugates so that B_1 and B_2 are real numbers

* The underdamped response is oscillatory with a decaying magnitude

At $t=0$, $V_n(0) = B_1$ * Take derivative to find B_2

$$\frac{dV_n}{dt} = \underbrace{e^{-\alpha t}}_{1}\left[(\omega_d B_2 - \alpha B_1)\cos(\omega_d t) - \underbrace{(\omega_d B_1 + \alpha B_2)\sin(\omega_d t)}_{0}\right]$$

$$\frac{dV_n(0)}{dt} = \omega_d B_2 - \alpha B_1$$

* From earlier

$$\frac{dV(0)}{dt} = \frac{-V(0)}{RC} - \frac{i(0)}{C}$$

$$\Rightarrow \omega_d B_2 - \alpha B_1 = \frac{-V(0)}{RC} - \frac{i(0)}{C}$$

$$\Rightarrow \boxed{\omega_d B_2 = \alpha B_1 - \frac{V(0)}{RC} - \frac{i(0)}{C}}$$

* Example

Given

$R = \frac{25}{3}\Omega$, $L = 0.1H$, $C = 1mF$, $V(0) = 10V$, $i(0) = -0.6A$

$$\alpha = \frac{1}{2RC} = 60 \qquad \omega_0^2 = \frac{1}{LC} = 10^4$$

$\alpha^2 < \omega_0^2$ so underdamped circuit

* Find ω_d

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^4 - 3.6\times10^3} = 80\ \frac{rad}{s}$$

$$\therefore \quad S_1 = -60 + j80$$

$$S_2 = -60 - j80$$

$$V_n = B_1 e^{-60t} \cos(80t) + B_2 e^{-60t} \sin(80t)$$

$$B_1 = v(0) = 10V$$

$$\omega_d B_2 = \alpha B_1 - \frac{v(0)}{RC} - \frac{i(0)}{C}$$

$$B_2 = 0$$

$$\boxed{V_n = 10e^{-60t} \cos(80t) \ V}$$

HW Problems to be graded

Q1

Q2

Q8