Multi-channel Weighted Nuclear Norm Minimization for Real Color Image Denoising

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1. Proof of Theorem 1.

Proof. 1. Firstly, we proof that the sequence $\{\mathbf{A}_k\}$ generated by Algorithm 1 is upper bounded. Let $\mathbf{X}_{k+1}+\rho_k^{-1}\mathbf{A}_k=\mathbf{U}_k\mathbf{\Sigma}_k\mathbf{V}_k^\top$ be its SVD in the $(k+1)$-th iteration. According to Corollary 1 of [?], we can have the SVD of $\mathbf{Z}_{k+1}$ as $\mathbf{Z}_{k+1}=\mathbf{U}_k\hat{\mathbf{\Sigma}}_k\mathbf{V}_k^\top=\mathbf{U}_k\mathcal{S}_{\frac{\boldsymbol{w}}{\rho_k}}(\mathbf{\Sigma}_k)\mathbf{V}_k^\top$. Then we have

$$
\begin{aligned}
\|\mathbf{A}_{k+1}\|_F &= \|\mathbf{A}_k+\rho_k(\mathbf{X}_{k+1}-\mathbf{Z}_{k+1})\|_F && (1)\\
&= \rho_k\|\rho_k^{-1}\mathbf{A}_k+\mathbf{X}_{k+1}-\mathbf{Z}_{k+1}\|_F && (2)\\
&= \rho_k\|\mathbf{U}_k\mathbf{\Sigma}_k\mathbf{V}_k^\top-\mathbf{U}_k\mathcal{S}_{\frac{\boldsymbol{w}}{\rho_k}}(\mathbf{\Sigma}_k)\mathbf{V}_k^\top\|_F && (3)\\
&= \rho_k\|\mathbf{\Sigma}_k-\mathcal{S}_{\frac{\boldsymbol{w}}{\rho_k}}(\mathbf{\Sigma}_k)\|_F && (4)\\
&= \rho_k\sqrt{\sum_i(\mathbf{\Sigma}_k^{ii}-\mathcal{S}_{\frac{w_i}{\rho_k}}(\mathbf{\Sigma}_k^{ii}))^2} && (5)\\
&\le \rho_k\sqrt{\sum_i(\frac{w_i}{\rho_k})^2}=\sqrt{\sum_i w_i^2}. && (6)
\end{aligned}
$$

The inequality above can be proofed as follows: given the diagonal matrix $\mathbf{\Sigma}_k$, we define $\mathbf{\Sigma}_k^{ii}$ as the i-th element of $\mathbf{\Sigma}_k^{ii}$. If $\mathbf{\Sigma}_k^{ii}\ge\frac{w_i}{\rho_k}$, we have $\mathcal{S}_{\frac{w_i}{\rho_k}}(\mathbf{\Sigma}_k^{ii})=\mathbf{\Sigma}_k^{ii}-\frac{w_i}{\rho_k}\ge 0$. If $\mathbf{\Sigma}_k^{ii}<\frac{w_i}{\rho_k}$, we have $\mathcal{S}_{\frac{w_i}{\rho_k}}(\mathbf{\Sigma}_k^{ii})=0<\mathbf{\Sigma}_k^{ii}+\frac{w_i}{\rho_k}$. After all, we have $|\mathbf{\Sigma}_k^{ii}-\mathcal{S}_{\frac{w_i}{\rho_k}}(\mathbf{\Sigma}_k^{ii})|\le\frac{w_i}{\rho_k}$ and hence the inequality holds true. Hence, the sequence $\{\mathbf{A}_k\}$ is upper bounded.

2. Secondly, we proof that the sequence of Lagrangian function $\{\mathcal{L}(\mathbf{X}_{k+1},\mathbf{Z}_{k+1},\mathbf{A}_k,\rho_k)\}$ is also upper bounded. Since the global optimal solution of $\mathbf{X}$ and $\mathbf{Z}$ in corresponding subproblems, we always have $\mathcal{L}(\mathbf{X}_{k+1},\mathbf{Z}_{k+1},\mathbf{A}_k,\rho_k)\le\mathcal{L}(\mathbf{X}_k,\mathbf{Z}_k,\mathbf{A}_k,\rho_k)$. Based on the updating rule that $\mathbf{A}_{k+1}=\mathbf{A}_k+\rho_k(\mathbf{X}_{k+1}-\mathbf{Z}_{k+1})$, we have $\mathcal{L}(\mathbf{X}_{k+1},\mathbf{Z}_{k+1},\mathbf{A}_{k+1},\rho_{k+1})=\mathcal{L}(\mathbf{X}_{k+1},\mathbf{Z}_{k+1},\mathbf{A}_k,\rho_k)+\langle\mathbf{A}_{k+1}-\mathbf{A}_k,\mathbf{X}_{k+1}-\mathbf{Z}_{k+1}\rangle+\frac{\rho_{k+1}-\rho_k}{2}\|\mathbf{X}_{k+1}-\mathbf{Z}_{k+1}\|_F^2=\mathcal{L}(\mathbf{X}_{k+1},\mathbf{Z}_{k+1},\mathbf{A}_k,\rho_k)+\frac{\rho_{k+1}+\rho_k}{2\rho_k^2}\|\mathbf{A}_{k+1}-\mathbf{A}_k\|_F^2$. Since the sequence $\{\|\mathbf{A}_k\}$ is upper bounded, the sequence $\{\|\mathbf{A}_{k+1}-\mathbf{A}_k\|_F\}$ is also upper bounded. Denote by a the upper bound of $\{\|\mathbf{A}_{k+1}-\mathbf{A}_k\|_F\}$, we have $\mathcal{L}(\mathbf{X}_{k+1},\mathbf{Z}_{k+1},\mathbf{A}_{k+1},\rho_{k+1})\le\mathcal{L}(\mathbf{X}_1,\mathbf{Z}_1,\mathbf{A}_0,\rho_0)+a\sum_{k=0}^{\infty}\frac{\rho_{k+1}+\rho_k}{2\rho_k^2}=\mathcal{L}(\mathbf{X}_1,\mathbf{Z}_1,\mathbf{A}_0,\rho_0)+a\sum_{k=0}^{\infty}\frac{\mu+1}{2\mu^k\rho_0}\le\mathcal{L}(\mathbf{X}_1,\mathbf{Z}_1,\mathbf{A}_0,\rho_0)+\frac{a}{\rho_0}\sum_{k=0}^{\infty}\frac{1}{\mu^{k-1}}$. The last inequality holds since $\mu+1<2\mu$. Since $\sum_{k=0}^{\infty}\frac{1}{\mu^{k-1}}<\infty$, the sequence of Lagrangian function $\mathcal{L}(\mathbf{X}_{k+1},\mathbf{Z}_{k+1},\mathbf{A}_{k+1},\rho_{k+1})$ is upper bound.

3. Thirdly, we proof that the sequences of $\{\mathbf{X}_k\}$ and $\{\mathbf{Z}_k\}$ are upper bounded. Since $\|\mathbf{W}(\mathbf{Y}-\mathbf{X})\|_F^2+\|\mathbf{Z}\|_{\boldsymbol{w},*}=\mathcal{L}(\mathbf{X}_k,\mathbf{Z}_k,\mathbf{A}_{k-1},\rho_{k-1})-\langle\mathbf{A}_k,\mathbf{X}_k-\mathbf{Z}_k\rangle-\frac{\rho_k}{2}\|\mathbf{X}_k-\mathbf{Z}_k\|_F^2=\mathcal{L}(\mathbf{X}_k,\mathbf{Z}_k,\mathbf{A}_{k-1},\rho_{k-1})+\frac{1}{2\rho_k}(\|\mathbf{A}_{k-1}\|_F^2-\|\mathbf{A}_k\|_F^2)$. Thus $\{\mathbf{W}(\mathbf{Y}-\mathbf{X}_k)\}$ and $\{\mathbf{Z}_k\}$ are upper bounded, and hence the sequence $\{\mathbf{X}_k\}$ is bounded by Cauchy-Schwarz inequality and triangle inequality. We can obtain that $\lim_{k\to\infty}\|\mathbf{X}_{k+1}-\mathbf{Z}_{k+1}\|_F=\lim_{k\to\infty}\rho_k^{-1}\|\mathbf{A}_{k+1}-\mathbf{A}_k\|_F=0$ and the equation (1) is proofed.

4. Then we can proof that $\lim_{k\to\infty}\|\mathbf{X}_{k+1}-\mathbf{X}_k\|_F=\lim_{k\to\infty}\|(\mathbf{W}^\top\mathbf{W}+\frac{\rho_k}{2}\mathbf{I})^{-1}(\mathbf{W}^\top\mathbf{W}\mathbf{Y}-\mathbf{W}^\top\mathbf{W}\mathbf{Z}_k-\frac{1}{2}\mathbf{A}_k)-\rho_k^{-1}(\mathbf{A}_k-\mathbf{A}_{k-1})\|_F\le\lim_{k\to\infty}\|(\mathbf{W}^\top\mathbf{W}+\frac{\rho_k}{2}\mathbf{I})^{-1}(\mathbf{W}^\top\mathbf{W}\mathbf{Y}-\mathbf{W}^\top\mathbf{W}\mathbf{Z}_k-\frac{1}{2}\mathbf{A}_k)\|_F+\rho_k^{-1}\|\mathbf{A}_k-\mathbf{A}_{k-1}\|_F=0$ and hence (2) is proofed.

5. Then (3) can be proofed by checking that $\lim_{k\to\infty}\|\mathbf{Z}_{k+1}-\mathbf{Z}_k\|_F=\lim_{k\to\infty}\|\mathbf{X}_k+\rho_k^{-1}\mathbf{A}_{k-1}-\mathbf{Z}_k+\mathbf{X}_{k+1}-\mathbf{X}_k+\rho_k^{-1}\mathbf{A}_{k-1}+\rho_k^{-1}\mathbf{A}_k-\rho_k^{-1}\mathbf{A}_{k+1}\|_F\le\lim_{k\to\infty}\|\mathbf{\Sigma}_{k-1}-\mathcal{S}_{\boldsymbol{w}/\rho_{k-1}}(\mathbf{\Sigma}_{k-1})\|_F+\|\mathbf{X}_{k+1}-\mathbf{X}_k\|_F+\rho_k^{-1}\|\mathbf{A}_{k-1}+\mathbf{A}_{k+1}-\mathbf{A}_k\|_F=0$, where $\mathbf{U}_{k-1}\mathbf{\Sigma}_{k-1}\mathbf{V}_{k-1}^\top$ is the SVD of the matrix $\mathbf{X}_k+\rho_{k-1}\mathbf{A}_{k-1}$. $\square$