Supplementary File to "Multi-channel Weighted Nuclear Norm Minimization for Real Color Image Denoising"

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1. Proof of Theorem 1.

Theorem 1. *Assume the weights in $\boldsymbol{w}$ are in a non-descending order, the sequence $\{\mathbf{X}_k\}$, $\{\mathbf{Z}_k\}$, and $\{\mathbf{A}_k\}$ generated in Algorithm 1 satisfy:*

$$(1) \lim_{k\rightarrow\infty} \|\mathbf{X}_{k+1} - \mathbf{Z}_{k+1}\|_F = 0; \tag{1}$$

$$(2) \lim_{k\rightarrow\infty} \|\mathbf{X}_{k+1} - \mathbf{X}_{k}\|_F = 0; \tag{2}$$

$$(3) \lim_{k\rightarrow\infty} \|\mathbf{Z}_{k+1} - \mathbf{Z}_{k}\|_F = 0. \tag{3}$$

Proof. 1. Firstly, we proof that the sequence $\{\mathbf{A}_k\}$ generated by Algorithm 1 is upper bounded. Let $\mathbf{X}_{k+1} + \rho_k^{-1}\mathbf{A}_k = \mathbf{U}_k\boldsymbol{\Sigma}_k\mathbf{V}_k^\top$ be its SVD in the $(k+1)$-th iteration. According to Corollary 1 of [1], we can have the SVD of $\mathbf{Z}_{k+1}$ as $\mathbf{Z}_{k+1} = \mathbf{U}_k\hat{\boldsymbol{\Sigma}}_k\mathbf{V}_k^\top = \mathbf{U}_k\mathcal{S}_{\frac{\boldsymbol{w}}{\rho_k}}(\boldsymbol{\Sigma}_k)\mathbf{V}_k^\top$. Then we have

$$\begin{aligned}
\|\mathbf{A}_{k+1}\|_F &= \|\mathbf{A}_k + \rho_k(\mathbf{X}_{k+1} - \mathbf{Z}_{k+1})\|_F && (4)\\
&= \rho_k\|\rho_k^{-1}\mathbf{A}_k + \mathbf{X}_{k+1} - \mathbf{Z}_{k+1}\|_F && (5)\\
&= \rho_k\|\mathbf{U}_k\boldsymbol{\Sigma}_k\mathbf{V}_k^\top - \mathbf{U}_k\mathcal{S}_{\frac{\boldsymbol{w}}{\rho_k}}(\boldsymbol{\Sigma}_k)\mathbf{V}_k^\top\|_F && (6)\\
&= \rho_k\|\boldsymbol{\Sigma}_k - \mathcal{S}_{\frac{\boldsymbol{w}}{\rho_k}}(\boldsymbol{\Sigma}_k)\|_F && (7)\\
&= \rho_k\sqrt{\sum_i(\boldsymbol{\Sigma}_k^{ii} - \mathcal{S}_{\frac{w_i}{\rho_k}}(\boldsymbol{\Sigma}_k^{ii}))^2} && (8)\\
&\leq \rho_k\sqrt{\sum_i(\frac{w_i}{\rho_k})^2} = \sqrt{\sum_i w_i^2}. && (9)
\end{aligned}$$

The inequality above can be proofed as follows: given the diagonal matrix $\boldsymbol{\Sigma}_k$, we define $\boldsymbol{\Sigma}_k^{ii}$ as the i-th element of $\boldsymbol{\Sigma}_k^{ii}$. If $\boldsymbol{\Sigma}_k^{ii} \geq \frac{w_i}{\rho_k}$, we have $\mathcal{S}_{\frac{w_i}{\rho_k}}(\boldsymbol{\Sigma}_k^{ii}) = \boldsymbol{\Sigma}_k^{ii} - \frac{w_i}{\rho_k} \geq 0$. If $\boldsymbol{\Sigma}_k^{ii} < \frac{w_i}{\rho_k}$, we have $\mathcal{S}_{\frac{w_i}{\rho_k}}(\boldsymbol{\Sigma}_k^{ii}) = 0 < \boldsymbol{\Sigma}_k^{ii} + \frac{w_i}{\rho_k}$. After all, we have $|\boldsymbol{\Sigma}_k^{ii} - \mathcal{S}_{\frac{w_i}{\rho_k}}(\boldsymbol{\Sigma}_k^{ii})| \leq \frac{w_i}{\rho_k}$ and hence the inequality holds true. Hence, the sequence $\{\mathbf{A}_k\}$ is upper bounded.

2. Secondly, we proof that the sequence of Lagrangian function $\{\mathcal{L}(\mathbf{X}_{k+1}, \mathbf{Z}_{k+1}, \mathbf{A}_k, \rho_k)\}$ is also upper bounded. Since the global optimal solution of $\mathbf{X}$ and $\mathbf{Z}$ in corresponding subproblems, we always have $\mathcal{L}(\mathbf{X}_{k+1}, \mathbf{Z}_{k+1}, \mathbf{A}_k, \rho_k) \leq \mathcal{L}(\mathbf{X}_k, \mathbf{Z}_k, \mathbf{A}_k, \rho_k)$. Based on the updating rule that $\mathbf{A}_{k+1} = \mathbf{A}_k + \rho_k(\mathbf{X}_{k+1} - \mathbf{Z}_{k+1})$, we have $\mathcal{L}(\mathbf{X}_{k+1}, \mathbf{Z}_{k+1}, \mathbf{A}_{k+1}, \rho_{k+1}) = \mathcal{L}(\mathbf{X}_{k+1}, \mathbf{Z}_{k+1}, \mathbf{A}_k, \rho_k) + \langle\mathbf{A}_{k+1} - \mathbf{A}_k, \mathbf{X}_{k+1} - \mathbf{Z}_{k+1}\rangle + \frac{\rho_{k+1}-\rho_k}{2}\|\mathbf{X}_{k+1} - \mathbf{Z}_{k+1}\|_F^2 = \mathcal{L}(\mathbf{X}_{k+1}, \mathbf{Z}_{k+1}, \mathbf{A}_k, \rho_k) + \frac{\rho_{k+1}+\rho_k}{2\rho_k^2}\|\mathbf{A}_{k+1} - \mathbf{A}_k\|_F^2$. Since the sequence $\{\|\mathbf{A}_k\}$ is upper bounded, the sequence $\{\|\mathbf{A}_{k+1} - \mathbf{A}_k\|_F\}$ is also upper bounded. Denote by a the upper bound of $\{\|\mathbf{A}_{k+1} - \mathbf{A}_k\|_F\}$, we have $\mathcal{L}(\mathbf{X}_{k+1}, \mathbf{Z}_{k+1}, \mathbf{A}_{k+1}, \rho_{k+1}) \leq \mathcal{L}(\mathbf{X}_1, \mathbf{Z}_1, \mathbf{A}_0, \rho_0) + a\sum_{k=0}^{\infty}\frac{\rho_{k+1}+\rho_k}{2\rho_k^2} = \mathcal{L}(\mathbf{X}_1, \mathbf{Z}_1, \mathbf{A}_0, \rho_0) + a\sum_{k=0}^{\infty}\frac{\mu+1}{2\mu^k\rho_0} \leq \mathcal{L}(\mathbf{X}_1, \mathbf{Z}_1, \mathbf{A}_0, \rho_0) + \frac{a}{\rho_0}\sum_{k=0}^{\infty}\frac{1}{\mu^{k-1}}$. The last inequality holds since $\mu + 1 < 2\mu$. Since $\sum_{k=0}^{\infty}\frac{1}{\mu^{k-1}} < \infty$, the sequence of Lagrangian function $\mathcal{L}(\mathbf{X}_{k+1}, \mathbf{Z}_{k+1}, \mathbf{A}_{k+1}, \rho_{k+1})$ is upper bound.

3. Thirdly, we proof that the sequences of $\{\mathbf{X}_k\}$ and $\{\mathbf{Z}_k\}$ are upper bounded. Since $\|\mathbf{W}(\mathbf{Y} - \mathbf{X})\|_F^2 + \|\mathbf{Z}\|_{\boldsymbol{w},*} = \mathcal{L}(\mathbf{X}_k, \mathbf{Z}_k, \mathbf{A}_{k-1}, \rho_{k-1}) - \langle\mathbf{A}_k, \mathbf{X}_k - \mathbf{Z}_k\rangle - \frac{\rho_k}{2}\|\mathbf{X}_k - \mathbf{Z}_k\|_F^2 = \mathcal{L}(\mathbf{X}_k, \mathbf{Z}_k, \mathbf{A}_{k-1}, \rho_{k-1}) + \frac{1}{2\rho_k}(\|\mathbf{A}_{k-1}\|_F^2 - \|\mathbf{A}_k\|_F^2)$. Thus $\{\mathbf{W}(\mathbf{Y} - \mathbf{X}_k)\}$ and $\{\mathbf{Z}_k\}$ are upper bounded, and hence the sequence $\{\mathbf{X}_k\}$ is bounded by Cauchy-Schwarz inequality and triangle inequality. We can obtain that $\lim_{k\rightarrow\infty}\|\mathbf{X}_{k+1} - \mathbf{Z}_{k+1}\|_F = \lim_{k\rightarrow\infty}\rho_k^{-1}\|\mathbf{A}_{k+1} - \mathbf{A}_k\|_F = 0$ and the equation (1) is proofed.

4. Then we can proof that $\lim_{k\rightarrow\infty}\|\mathbf{X}_{k+1} - \mathbf{X}_k\|_F = \lim_{k\rightarrow\infty}\|(\mathbf{W}^\top\mathbf{W} + \frac{\rho_k}{2}\mathbf{I})^{-1}(\mathbf{W}^\top\mathbf{W}\mathbf{Y} - \mathbf{W}^\top\mathbf{W}\mathbf{Z}_k - \frac{1}{2}\mathbf{A}_k) - \rho_k^{-1}(\mathbf{A}_k - \mathbf{A}_{k-1})\|_F \leq \lim_{k\rightarrow\infty}\|(\mathbf{W}^\top\mathbf{W} + \frac{\rho_k}{2}\mathbf{I})^{-1}(\mathbf{W}^\top\mathbf{W}\mathbf{Y} - \mathbf{W}^\top\mathbf{W}\mathbf{Z}_k - \frac{1}{2}\mathbf{A}_k)\|_F + \rho_k^{-1}\|\mathbf{A}_k - \mathbf{A}_{k-1}\|_F = 0$ and hence (2) is proofed.

5. Then (3) can be proofed by checking that $\lim_{k\rightarrow\infty}\|\mathbf{Z}_{k+1} - \mathbf{Z}_k\|_F = \lim_{k\rightarrow\infty}\|\mathbf{X}_k + \rho_k^{-1}\mathbf{A}_{k-1} - \mathbf{Z}_k + \mathbf{X}_{k+1} - \mathbf{X}_k + \rho_k^{-1}\mathbf{A}_{k-1} + \rho_k^{-1}\mathbf{A}_k - \rho_k^{-1}\mathbf{A}_{k+1}\|_F \leq \lim_{k\rightarrow\infty}\|\boldsymbol{\Sigma}_{k-1} - \mathcal{S}_{\boldsymbol{w}/\rho_{k-1}}(\boldsymbol{\Sigma}_{k-1})\|_F + \|\mathbf{X}_{k+1} - \mathbf{X}_k\|_F + \rho_k^{-1}\|\mathbf{A}_{k-1} + \mathbf{A}_{k+1} - \mathbf{A}_k\|_F = 0$, where $\mathbf{U}_{k-1}\boldsymbol{\Sigma}_{k-1}\mathbf{V}_{k-1}^\top$ is the SVD of the matrix $\mathbf{X}_k + \rho_{k-1}\mathbf{A}_{k-1}$. □

Fig. 2 shows a scene denoised by the compared methods.

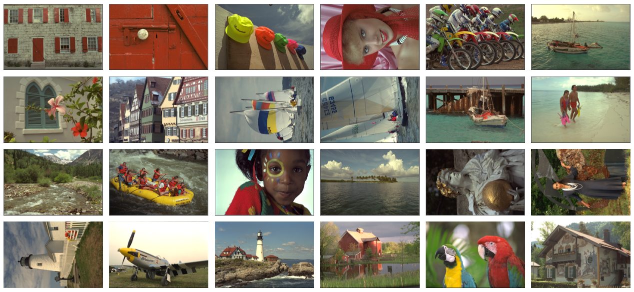

Figure 1. The 24 high quality color images from the Kodak PhotoCD Dataset.

We can see that the methods of CBM3D and NC would remain some noise on the recovered images. The methods of MLP, TNRD, and "WNNM0", which process separately the channels of color images, would over-smooth the images and generate false colors or artifacts. The method "WNNM1", which process jointly the channels of color images, would not generate false colors, but still over-smooth the image. The "WNNM2", which is the WNNM model solved by ADMM algorithm, would remain some noise on the image. By employing the proposed MC-WNNM model, our method preserves the structures (e.g., textures in windows and grass) better across the R, G, B channels and generate less artifacts than other denoising methods, leading to visually pleasant outputs.

References

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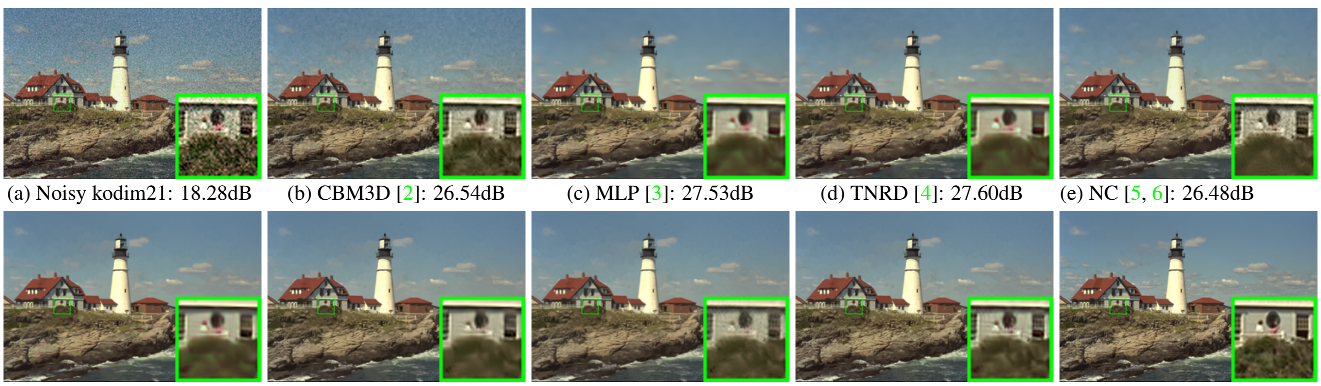

(a) Noisy kodim21: 18.28dB (b) CBM3D [2]: 26.54dB (c) MLP [3]: 27.53dB (d) TNRD [4]: 27.60dB (e) NC [5, 6]: 26.48dB

(f) WNNM0: 27.80dB (g) WNNM1: 27.75dB (h) WNNM2: 27.12dB (i) MC-WNNM: **28.34**dB (j) Clean kodim21

Figure 2. Denoised images of different methods on the image "kodim21" degraded by AWGN with different standard derivations of $\sigma_r = 40, \sigma_g = 20, \sigma_b = 30$ on R, G, B channels, respectively. The images are better to be zoomed in on screen.