SOLUTIONS OF ORTHOGONAL PROCRUSTES PROBLEMS UNDER PARTIALLY KNOWN PRIOR

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Abstract

The orthogonal Procrustes problem aims to find an orthogonal matrix which transforms one given matrix into another by minimizing their Frobenius matrix norm. This problem can be applied in applications such as permutation theory, machine learning, and camera calibration, *etc*. In real cases, the permutation matrix may have been partially known, the dictionaries can be partially learned from external data, and the calibration of camera should be done under some fixed priors. This prior information makes the original orthogonal Procrustes problem more difficult. In this paper, we consider the solution of this problem under partially known priors, which includes the original orthogonal Procrustes problem as a special case with no such prior.

Key words: orthogonal Procrustes problem, pratially known priors

1. Introduction

The classical orthogonal Procrustes problem has been applied in psychometrics, multidimensional scaling, factor analysis, machine learning, computer vision, optical imaging, and robotics.

2. Definition of the Probelm and Solution

Let $\mathbf{A}, \mathbf{B} \in \mathcal{R}^{n \times m}$ be two given data matrices. Define $\mathbf{X} \in \mathcal{R}^{n \times p}$ and $\mathbf{P} \in \mathcal{R}^{n \times q}$ where $p + q = n$. $\mathbf{X}$ is the partially known prior which could be used to guide the solutions of $\mathbf{P}$. We formulate the orthogonal Procrustes problem with partially known priors as:

$$\hat{\mathbf{P}} = \arg\min_{\mathbf{D}} \|\mathbf{B} - [\mathbf{X}\ \mathbf{P}]\mathbf{A}\|_F^2 \quad s.t. \quad \mathbf{P}^\top \mathbf{P} = \mathbf{I}_{q \times q}, \mathbf{X}^\top \mathbf{P} = \mathbf{0}_{p \times q}. \tag{1}$$

In fact, as have been proofed, if the matrix $\mathbf{BA}^\top$ has no zero singular value, then the solution of $\hat{\mathbf{P}} = \mathbf{UV}^\top$ is unique and we do not need any preceding results.

We crop the matrix $\mathbf{A}$ into two parts: $\mathbf{A}_X \in \mathcal{R}^{p \times m}$ and $\mathbf{A}_P \in \mathcal{R}^{q \times m}$ to interact with $\mathbf{X}$ and $\mathbf{P}$, respectively. Then we have

$$\begin{aligned} &\|\mathbf{B} - [\mathbf{X}\ \mathbf{P}]\mathbf{A}\|_F^2 = \|\mathbf{B} - [\mathbf{X}\ \mathbf{P}][\mathbf{A}_X^\top\ \mathbf{A}_P^\top]^\top\|_F^2 = \|\mathbf{B} - [\mathbf{X}\ \mathbf{P}][\mathbf{A}_X^\top\ \mathbf{A}_P^\top]^\top\|_F^2 \\ &= \|\mathbf{B} - \mathbf{XA}_X^\top - \mathbf{PA}_P^\top]\|_F^2 = \|\mathbf{B} - \mathbf{XA}_X^\top - \mathbf{PA}_P^\top]\|_F^2 \end{aligned} \tag{2}$$

The $\mathbf{B} - \mathbf{XA}_X^\top$ is a known data matrix and we replace it with $\mathbf{B}^* = \mathbf{B} - \mathbf{XA}_X^\top$. In the following Results 1, we remove the notation $*$ and use $\mathbf{B}$ as the finally known data matrix.

Results 1: Let $\mathbf{A}, \mathbf{B} \in \mathcal{R}^{n \times m}$ be two given data matrices, given partially known prior of $\mathbf{X}^\top \mathbf{X} = \mathbf{I}_{p \times p}$, the solution of the following problem

$$\hat{\mathbf{P}} = \arg\min_{\mathbf{P}} \|\mathbf{B} - \mathbf{PA}\|_F^2 \quad s.t. \quad \mathbf{P}^\top \mathbf{P} = \mathbf{I}_{q \times q}, \mathbf{X}^\top \mathbf{P} = \mathbf{0}_{p \times q} \tag{3}$$

is $\hat{\mathbf{P}} = \mathbf{UV}^\top$, where $\mathbf{U}$ and $\mathbf{V}$ are the orthogonal matrices obtained by perfroming economy (aka. reduced) SVD:

$$(\mathbf{I}_{n \times n} - \mathbf{XX}^\top)\mathbf{BA}^\top = \mathbf{U\Sigma V}^\top \tag{4}$$

proof:

$$\hat{\mathbf{P}} = \arg\min_{\mathbf{P}} \|\mathbf{B} - \mathbf{PA}\|_F^2 \tag{5}$$

FIGURE 1.
Projection of item discrimination vectors onto V_{θ_T} hyperplance for a six item three-dimensional approximate sample structure.

2.1. Unidimensionality from the Weak LI Conditional Covariance Perspective

2.2. Foundational Issues Facilitated by Infinite Test Length Unidimensional MLI1 Modeling

2.3. Interpreting Conditional Covariances Geometrically to Assess Latent Multidimensional Structure

2.4. NIRT-Based Statistical Procedures, Emphasizing Conditional Covariances

3. Test Fairness

3.1. Multidimensional Model for DIF (MMD)

3.2. Model-Based Parameterization of the amount of DIF in Various Settings

3.3. MMD- Inspired DIF Statistical Procedures

FIGURE 2.
Comparison of Θ_F and Θ_R distribution with $\Theta_F|X_V = k$ and $\Theta_R|X_V = k$ distributions.

3.4. Implementation of DIF/DBF Procedures

FIGURE 3.
Item discrimination vectors of a 22 item validity sector.

4. Formative Assessment Skills Diagnosis: A New Test Paradigm

4.1. A Brief Survey of Psychometric Skills Diagnostic Models

4.2. The Unified Model and Generalizations Making it Useful

4.3. Application of the Unified Model to PSAT Data

4.4. Skills Diagnosis: The New Paradigm?

5. Dimensionality, Equity, and Diagnostic Software

6. Concluding Remarks

FIGURE 4.
Panel index versus bundle DBF $\hat{\beta}$/item.

FIGURE 5.
North Carolina End-of-Grade Math Skills Test Subscores.

FIGURE 6.
PSAT Score Report *Plus* Skills Mastery Reporting.

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