

SOLUTIONS OF ORTHOGONAL PROCRUSTES PROBLEMS UNDER PARTIALLY KNOWN PRIOR

JUN XU

DEPARTMENT OF COMPUTING, THE HONG KONG POLYTECHNIC UNIVERSITY

This article is based on the Presidential Address William Stout gave on June 23, 2002 at the 67th Annual Meeting of the Psychometric Society held in Chapel Hill, North Carolina. —Editor

I wish to especially thank Sarah Hartz and Louis Roussos for their suggestions that helped shape this paper. I wish to thank all my former Ph.D. students: Without their contributions, the content of this paper would have been vastly different and much less interesting!

Requests for reprints should be sent to William Stout, Department of Statistics, University of Illinois, 725 S. Wright Street, Champaign IL 61820. E-Mail: stout@stat.uiuc.edu

Dedication: I want to dedicate this paper to my wife, Barbara Meihoefer, who was lost to illness in this year of my presidency. For, in addition to all the wonderful things she meant to me personally and the enormous support she gave concerning my career, she truly enjoyed and greatly appreciated my psychometric colleagues and indeed found psychometrics an important and fascinating intellectual endeavor, in particular finding the skills diagnosis area exciting and important: She often took time from her career as a business manager and entrepreneur to attend psychometric meetings with me and to discuss research projects with my colleagues and me. She would have enjoyed this paper.—William Stout

Abstract

The orthogonal Procrustes problem aims to find an orthogonal matrix which transforms one given matrix into another by minimizing their Frobenius matrix norm. This problem can be applied in applications such as permutation theory, machine learning, and camera calibration, *etc.* In real cases, the permutation matrix may have been partially known, the dictionaries can be partially learned from external data, and the calibration of camera should be done under some fixed priors. This prior information makes the original orthogonal Procrustes problem more difficult. In this paper, we consider the solution of this problem under partially known priors, which includes the original orthogonal Procrustes problem as a special case with no such prior.

Key words: orthogonal Procrustes problem, partially known priors

1. Introduction

The famous orthogonal Procrustes problem can be used in many research area.

2. Definition of the Problem and Solution

Let $\mathbf{A}, \mathbf{B} \in \mathcal{R}^{n \times m}$ be two given data matrices. Define $\mathbf{X} \in \mathcal{R}^{n \times p}$ and $\mathbf{P} \in \mathcal{R}^{n \times q}$ where $p + q = n$. \mathbf{X} is the partially known prior which could be used to guide the solutions of \mathbf{P} . We formulate the orthogonal Procrustes problem with partially known priors as:

$$\hat{\mathbf{P}} = \arg \min_{\mathbf{D}} \|\mathbf{B} - [\mathbf{X} \mathbf{P}] \mathbf{A}\|_F^2 \quad s.t. \quad \mathbf{P}^\top \mathbf{P} = \mathbf{I}_{q \times q}, \mathbf{X}^\top \mathbf{P} = \mathbf{0}_{p \times q}. \quad (1)$$

In fact, as have been proofed, if the matrix $\mathbf{B} \mathbf{A}^\top$ has no zero singular value, then the solution of $\hat{\mathbf{P}} = \mathbf{U} \mathbf{V}^\top$ is unique and we do not need any preceding results.

We crop the matrix \mathbf{A} into two parts: $\mathbf{A}_X \in \mathcal{R}^{p \times m}$ and $\mathbf{A}_P \in \mathcal{R}^{q \times m}$ to interact with \mathbf{X} and \mathbf{P} , respectively. Then we have

$$\begin{aligned} \|\mathbf{B} - [\mathbf{X} \mathbf{P}] \mathbf{A}\|_F^2 &= \|\mathbf{B} - [\mathbf{X} \mathbf{P}] [\mathbf{A}_X^\top \mathbf{A}_P^\top]^\top\|_F^2 = \|\mathbf{B} - [\mathbf{X} \mathbf{P}] [\mathbf{A}_X^\top \mathbf{A}_P^\top]^\top\|_F^2 \\ &= \|\mathbf{B} - \mathbf{X} \mathbf{A}_X^\top - \mathbf{P} \mathbf{A}_P^\top\|_F^2 = \|\mathbf{B} - \mathbf{X} \mathbf{A}_X^\top - \mathbf{P} \mathbf{A}_P^\top\|_F^2 \end{aligned} \quad (2)$$

The $\mathbf{B} - \mathbf{X} \mathbf{A}_X^\top$ is a known data matrix and we replace it with $\mathbf{B}^* = \mathbf{B} - \mathbf{X} \mathbf{A}_X^\top$. In the following Results 1, we remove the notation $*$ and use \mathbf{B} as the finally known data matrix.

Results 1: The solution of

$$\hat{\mathbf{P}} = \arg \min_{\mathbf{D}} \|\mathbf{B} - \mathbf{P} \mathbf{A}\|_F^2 \quad s.t. \quad \mathbf{P}^\top \mathbf{P} = \mathbf{I}_{q \times q}, \mathbf{X}^\top \mathbf{P} = \mathbf{0}_{p \times q} \quad (3)$$

is $\hat{\mathbf{P}} = \mathbf{U} \mathbf{V}^\top$, where \mathbf{U} and \mathbf{V} are the orthogonal matrices obtained by performing economy (aka. reduced) SVD:

$$(\mathbf{I}_{n \times n} - \mathbf{X} \mathbf{X}^\top) \mathbf{B} \mathbf{A}^\top = \mathbf{U} \Sigma \mathbf{V}^\top \quad (4)$$

proof:

2.1. *Unidimensionality from the Weak LI Conditional Covariance Perspective*

2.2. *Foundational Issues Facilitated by Infinite Test Length Unidimensional MLII Modeling*

2.3. *Interpreting Conditional Covariances Geometrically
to Assess Latent Multidimensional Structure*

2.4. *NIRT-Based Statistical Procedures, Emphasizing Conditional Covariances*

FIGURE 1.

Projection of item discrimination vectors onto V_{θ_T} hyperplane for a six item three-dimensional approximate sample structure.

3. Test Fairness

3.1. Multidimensional Model for DIF (MMD)

3.2. Model-Based Parameterization of the amount of DIF in Various Settings

3.3. MMD- Inspired DIF Statistical Procedures

FIGURE 2.

Comparison of Θ_F and Θ_R distribution with $\Theta_F|X_V = k$ and $\Theta_R|X_V = k$ distributions.

3.4. Implementation of DIF/DBF Procedures

FIGURE 3.

Item discrimination vectors of a 22 item validity sector.

FIGURE 4.

Panel index versus bundle DBF $\hat{\beta}$ /item.

4. Formative Assessment Skills Diagnosis: A New Test Paradigm

4.1. A Brief Survey of Psychometric Skills Diagnostic Models

4.2. The Unified Model and Generalizations Making it Useful

4.3. Application of the Unified Model to PSAT Data

4.4. Skills Diagnosis: The New Paradigm?

5. Dimensionality, Equity, and Diagnostic Software

6. Concluding Remarks

FIGURE 5.
North Carolina End-of-Grade Math Skills Test Subscores.

FIGURE 6.
PSAT Score Report *Plus* Skills Mastery Reporting.

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