

# SOLUTIONS OF ORTHOGONAL PROCRUSTES PROBLEMS UNDER PARTIALLY KNOWN PRIOR

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## Abstract

The orthogonal Procrustes problem aims to find an orthogonal matrix which transforms one given matrix into another by minimizing their Frobenius matrix norm. This problem can be applied in applications such as permutation theory, machine learning, and camera calibration, *etc.* In real cases, the permutation matrix may have been partially known, the dictionaries can be partially learned from external data, and the calibration of camera should be done under some fixed priors. This prior information makes the original orthogonal Procrustes problem more difficult. In this paper, we consider the solution of this problem under partially known priors, which includes the original orthogonal Procrustes problem as a special case with no such prior.

Key words: orthogonal Procrustes problem, partially known priors

## 1. Introduction

The classical orthogonal Procrustes problem has been applied in psychometrics, multidimensional scaling, factor analysis, machine learning, computer vision, optical imaging, and robotics.

## 2. Definition of the Problem and Solution

Let  $\mathbf{A}, \mathbf{B} \in \mathcal{R}^{n \times m}$  be two given data matrices. Define  $\mathbf{X} \in \mathcal{R}^{n \times p}$  and  $\mathbf{P} \in \mathcal{R}^{n \times q}$  where  $p + q = n$ .  $\mathbf{X}$  is the partially known prior which could be used to guide the solutions of  $\mathbf{P}$ . We formulate the orthogonal Procrustes problem with partially known priors as:

$$\hat{\mathbf{P}} = \arg \min_{\mathbf{D}} \|\mathbf{B} - [\mathbf{X} \mathbf{P}] \mathbf{A}\|_F^2 \quad s.t. \quad \mathbf{P}^\top \mathbf{P} = \mathbf{I}_{q \times q}, \mathbf{X}^\top \mathbf{P} = \mathbf{0}_{p \times q}. \quad (1)$$

In fact, as have been proofed, if the matrix  $\mathbf{B} \mathbf{A}^\top$  has no zero singular value, then the solution of  $\hat{\mathbf{P}} = \mathbf{U} \mathbf{V}^\top$  is unique and we do not need any preceding results.

We crop the matrix  $\mathbf{A}$  into two parts:  $\mathbf{A}_X \in \mathcal{R}^{p \times m}$  and  $\mathbf{A}_P \in \mathcal{R}^{q \times m}$  to interact with  $\mathbf{X}$  and  $\mathbf{P}$ , respectively. Then we have

$$\begin{aligned} \|\mathbf{B} - [\mathbf{X} \mathbf{P}] \mathbf{A}\|_F^2 &= \|\mathbf{B} - [\mathbf{X} \mathbf{P}] [\mathbf{A}_X^\top \mathbf{A}_P^\top]^\top\|_F^2 = \|\mathbf{B} - [\mathbf{X} \mathbf{P}] [\mathbf{A}_X^\top \mathbf{A}_P^\top]^\top\|_F^2 \\ &= \|\mathbf{B} - \mathbf{X} \mathbf{A}_X^\top - \mathbf{P} \mathbf{A}_P^\top\|_F^2 = \|\mathbf{B} - \mathbf{X} \mathbf{A}_X^\top - \mathbf{P} \mathbf{A}_P^\top\|_F^2 \end{aligned} \quad (2)$$

The  $\mathbf{B} - \mathbf{X} \mathbf{A}_X^\top$  is a known data matrix and we replace it with  $\mathbf{B}^* = \mathbf{B} - \mathbf{X} \mathbf{A}_X^\top$ . In the following Results 1, we remove the notation  $*$  and use  $\mathbf{B}$  as the finally known data matrix.

**Results 1:** Let  $\mathbf{A}, \mathbf{B} \in \mathcal{R}^{n \times m}$  be two given data matrices, given partially known prior of  $\mathbf{X}^\top \mathbf{X} = \mathbf{I}_{p \times p}$ , the solution of the following problem

$$\hat{\mathbf{P}} = \arg \min_{\mathbf{P}} \|\mathbf{B} - \mathbf{P} \mathbf{A}\|_F^2 \quad s.t. \quad \mathbf{P}^\top \mathbf{P} = \mathbf{I}_{q \times q}, \mathbf{X}^\top \mathbf{P} = \mathbf{0}_{p \times q} \quad (3)$$

is  $\hat{\mathbf{P}} = \mathbf{U} \mathbf{V}^\top$ , where  $\mathbf{U}$  and  $\mathbf{V}$  are the orthogonal matrices obtained by performing economy (aka. reduced) SVD:

$$(\mathbf{I}_{n \times n} - \mathbf{X} \mathbf{X}^\top) \mathbf{B} \mathbf{A}^\top = \mathbf{U} \Sigma \mathbf{V}^\top \quad (4)$$

*proof:* Since  $\mathbf{P}^\top \mathbf{P} = \mathbf{I}_{q \times q}$ , we have

$$\hat{\mathbf{P}} = \arg \min_{\mathbf{P}} \|\mathbf{B} - \mathbf{P} \mathbf{A}\|_F^2 = \arg \min_{\mathbf{P}} \|\mathbf{B}\|_F^2 + \|\mathbf{P} \mathbf{A}\|_F^2 - 2\text{Tr}(\mathbf{B}^\top \mathbf{P} \mathbf{A}) = \arg \max_{\mathbf{P}} \text{Tr}(\mathbf{A} \mathbf{B}^\top \mathbf{P}). \quad (5)$$

We can use Lagrange multiplier method and define the Lagrange function as:

$$\mathcal{L} = \text{Tr}(\mathbf{A} \mathbf{B}^\top \mathbf{P}) - \text{Tr}(\Gamma_1 (\mathbf{P}^\top \mathbf{P} - \mathbf{I}_{q \times q})) - \text{Tr}(\Gamma_2 (\mathbf{P}^\top \mathbf{X})), \quad (6)$$

where  $\Gamma$  is the augmented Lagrange multiplier. Take the derivative of  $\mathcal{L}$  with respect to  $\mathbf{P}$  and set it to 0, we can get

$$\frac{\partial \mathcal{L}}{\partial \mathbf{P}} = \mathbf{B} \mathbf{A}^\top - \mathbf{P} (\Gamma_1 + \Gamma_1^\top) - \mathbf{X} \Gamma_2^\top \quad (7)$$

Note that if the partially known prior were not present, the solution is clearly the solution of the original orthogonal Procrustes problem, i.e.,  $\hat{\mathbf{P}} = \mathbf{U} \mathbf{V}^\top$ , where  $\mathbf{U}$  and  $\mathbf{V}$  are the orthogonal matrices

obtained by performing economy (aka. reduced) SVD:  $\mathbf{BA}^\top = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ . The difference between the solutions of the original orthogonal Procrustes problem and its partially known prior version quantify the effect on the residual of requiring  $\mathbf{P}$  to be orthogonal to the external prior  $\mathbf{P}^\top \mathbf{X}$ .

In this section, we examine the sensitivity of the solution to perturbation in the data. To measure this sensitivity, we give the relative residuals and the Fro-norm condition numbers of the solutions. The condition number of the matrix  $\mathbf{A}$  is defined as  $k_F(\mathbf{A}) = \frac{\sigma_1}{\sigma_r}$ , where  $r = \text{rank}(\mathbf{A})$ .

### *2.1. Unidimensionality from the Weak LI Conditional Covariance Perspective*

### *2.2. Foundational Issues Facilitated by Infinite Test Length Unidimensional MLII Modeling*

### *2.3. Interpreting Conditional Covariances Geometrically to Assess Latent Multidimensional Structure*

### *2.4. NIRT-Based Statistical Procedures, Emphasizing Conditional Covariances*

FIGURE 1.

Projection of item discrimination vectors onto  $V_{\theta_T}$  hyperplane for a six item three-dimensional approximate sample structure.

## **3. Test Fairness**

### *3.1. Multidimensional Model for DIF (MMD)*

### *3.2. Model-Based Parameterization of the amount of DIF in Various Settings*

### *3.3. MMD- Inspired DIF Statistical Procedures*

FIGURE 2.

Comparison of  $\Theta_F$  and  $\Theta_R$  distribution with  $\Theta_F|X_V = k$  and  $\Theta_R|X_V = k$  distributions.

### *3.4. Implementation of DIF/DBF Procedures*

FIGURE 3.

Item discrimination vectors of a 22 item validity sector.

FIGURE 4.  
Panel index versus bundle DBF  $\hat{\beta}/\text{item}$ .

FIGURE 5.  
North Carolina End-of-Grade Math Skills Test Subscores.

#### **4. Formative Assessment Skills Diagnosis: A New Test Paradigm**

*4.1. A Brief Survey of Psychometric Skills Diagnostic Models*

*4.2. The Unified Model and Generalizations Making it Useful*

*4.3. Application of the Unified Model to PSAT Data*

*4.4. Skills Diagnosis: The New Paradigm?*

#### **5. Dimensionality, Equity, and Diagnostic Software**

#### **6. Concluding Remarks**

FIGURE 6.  
PSAT Score Report *Plus* Skills Mastery Reporting.

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