to be orthogonal to the external prior $\mathbf{P}^{\top}\mathbf{X}$.

\section{Uniqueness of Solution $\mathbf{\hat{P}}$}

Now we discuss the uniqueness of the solution $\mathbf{\hat{P}}$. Our discussion is according to the rankness of the $\mathbf{\Sigma}$ generated in the SVD. We first give a lemma describing the rank of $\mathbf{X}\mathbf{X}^{\top}$ where $\mathbf{X}\in\mathbb{R}^{n\times p}$ is the partially known orthogonal matrix.

\emph{Lemma 1}: Let $\mathbf{X}\in\mathbb{R}^{n\times p}$ be orthogonal matrix with $\mathbf{X}^{\top}\mathbf{X}=\mathbf{I}\_{p\times p}$, then $\text{rank}(\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top})\ge n-p$.

\emph{Proof}: We firstly proof that $\text{rank}(\mathbf{X}\mathbf{X}^{\top})=p$. The upper bound of the $\text{rank}(\mathbf{X}\mathbf{X}^{\top})$ can be determined by $\text{rank}(\mathbf{X}\mathbf{X}^{\top})\le\min\{\text{rank}(\mathbf{X}),\text{rank}(\mathbf{X}^{\top})\}=p$. The lower bound of the $\text{rank}(\mathbf{X}\mathbf{X}^{\top})$ can be determined by Sylvester's inequality as $\text{rank}(\mathbf{X}\mathbf{X}^{\top})\ge\text{rank}(\mathbf{X})+\text{rank}(\mathbf{X}^{\top})-p=2p-p=p$. Hence, we have $\text{rank}(\mathbf{X}\mathbf{X}^{\top})=p$. Then, $\text{rank}(\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top})\ge\text{rank}(\mathbf{I}\_{n\times n})-\text{rank}(\mathbf{X}\mathbf{X}^{\top})\ge n-p$.

$\hfill\blacksquare$

The rank of $\mathbf{\Sigma}$ largely depends on the rank of $\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top}$, $\mathbf{B}$ and $\mathbf{A}$. Note that the rank of $\mathbf{B}$ and $\mathbf{A}$ are not larger than $m$ and $q$, respectively. The rank of $\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top}$ is at least $n-q=p$ (since rank($\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top}$) $\ge$ $\text{rank}(\mathbf{I}\_{n\times n}) - \text{rank}(\mathbf{X}\mathbf{X}^{\top})$ $\ge$ $n-p=q$). From above observations, we can see that the rank of $\mathbf{\Sigma}$ can be equal to or less than $q$.

\textbf{Results 2}: If $\text{rank}(\mathbf{\Sigma})=q$ (we call this non-degenerative case), $\mathbf{\Sigma}$ may have distinct or multile non-zero singular values. In the SVD of $(\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top})\mathbf{B}\mathbf{A}^{\top}

=

\mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\top}$, the singular vectors in $\mathbf{U}$ and $\mathbf{V}$

can be determined up to orientation. Hence, we can reformulate the SVD as

\begin{equation}

(\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top})\mathbf{B}\mathbf{A}^{\top}

=

\mathbf{U}^{\*}\mathbf{K}\_{u}\mathbf{\Sigma}\mathbf{K}\_{v}(\mathbf{V}^{\*})^{\top},

\end{equation}

where $\mathbf{U}^{\*}\in \mathbb{R}^{n\times q}$ and $\mathbf{V}^{\*}\in \mathbb{R}^{q\times q}$ are arbitrarily orientated singular vectors of $\mathbf{U}$ and $\mathbf{V}$, respectively. $\mathbf{\Sigma}\in \mathbb{R}^{q\times q}$ are diagonal matrix with singular values are arranged in weak descending order along the diagonal, i.e., $\mathbf{\Sigma}\_{11}\ge\mathbf{\Sigma}\_{22}\ge...\ge\mathbf{\Sigma}\_{qq}\ge0$. The $\mathbf{K}\_{u}$ and $\mathbf{K}\_{v}$ are diagnonal matrices with $+1$ or $-1$ as diagonal elements in arbitrary distribution. If we fix $\mathbf{K}\_{u}$, then $\mathbf{K}\_{v}$ is uniquely determined to meet the requirement that the diagonal elements of $\mathbf{\Sigma}$ should be nonnegative. And the orientations of the singular vectors of $\mathbf{U}^{\*}$ is fixed, then the $\mathbf{U}=\mathbf{U}^{\*}\mathbf{K}\_{u}$ is determined, so does the orientations of the singular vectors of $\mathbf{V}^{\*}$ and $\mathbf{V}^{\top}=\mathbf{K}\_{v}(\mathbf{V}^{\*})^{\top}$. In this case, the solution of $\mathbf{\hat{P}}=\mathbf{U}\mathbf{V}^{\top}=\mathbf{U}^{\*}\mathbf{K}\_{u}\mathbf{K}\_{v}(\mathbf{V}^{\*})^{\top}$ is unique. The case that the $\mathbf{\Sigma}$ have multiple singular values also has unique solution of $\mathbf{\hat{P}}$, which can be discussed in a similar way.

If $0\le\text{rank}(\mathbf{\Sigma})=r< q$, there is $q-r$ (at least one) zero singular values and we call this case the degenerative case. The previous discussion on non-degerative case still can be applied to the singular vectors related to the nonzero singular values, and this part is still unique. However, the singular vectors related to the zero singular values can be in arbitrary orientations as long as they satisfy the orthogonal conditions that $\mathbf{U}^{\top}\mathbf{U}=\mathbf{V}^{\top}\mathbf{V}=\mathbf{V}\mathbf{V}^{\top}=\mathbf{I}\_{q\times q}$. Note that $\mathbf{U}\in \mathbb{R}^{n\times q}$, so $\mathbf{U}\mathbf{U}^{\top}$ no longer equals to the identity matrix of order $n$. From Equ. (12), we can get

\begin{equation}

\mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\top}\mathbf{P}^{\top}

=

\mathbf{P}\mathbf{V}\mathbf{\Sigma}\mathbf{U}^{\top}

\end{equation}

Right multiplying each side by $\mathbf{P}\mathbf{V}$ and then left multiplying each side by $\mathbf{U}^{\top}$, we can get

\begin{equation}

\mathbf{\Sigma}

=

\mathbf{U}^{\top}\mathbf{P}\mathbf{V}\mathbf{\Sigma}\mathbf{U}^{\top}\mathbf{P}\mathbf{V}

\end{equation}

Hence, we can define a diagonal matrix $\mathbf{D}=\mathbf{U}^{\top}\mathbf{P}\mathbf{V}\in\mathbb{R}^{q\times q}$, the diagonal elements of which are

\begin{displaymath}

\mathbf{D}\_{ii}= \left\{ \begin{array}{ll}

1 & \textrm{if $1\le i\le r$};\\

\pm 1 & \textrm{if $r< i \le q$}.\\

\end{array} \right.

\end{displaymath}

Thus, we obtain that $\mathbf{P}=\mathbf{U}\mathbf{D}\mathbf{V}^{\top}$, where $\mathbf{D}$ is defined above. Hence, once we get the solution of $\mathbf{\hat{P}}=\mathbf{U}\mathbf{V}^{\top}$ in problem (3), the final solution for $\mathbf{P}$ when $\text{rank}(\mathbf{\Sigma})<q$ is not unique since the matrix $\mathbf{D}$ is not uniquely determined. In fact, since the number of $\mathbf{D}$ with different diagonal combinations is $2^{q-r}$, the number of solutions for $\mathbf{P}$ is $2^{q-r}$ given fixed $\mathbf{U}$ and $\mathbf{V}$.

%Since the solution of $\mathbf{\hat{P}}$ is not unique when $\text{rank}(\mathbf{\Sigma})<q$, we define the set of solutions for in a formal manner and discuss its properties. The solution set can be defined as:

%\begin{equation}

%\mathcal{S}=\{\mathbf{S}\in\mathbb{R}^{n\times q}: \mathbf{S}^{\top}\mathbf{S}=\mathbf{I}\_{q\times q}, \mathbf{X}^{\top}\mathbf{S}=\mathbf{0}\_{p\times q}, \|\mathbf{B}-\mathbf{S}\mathbf{A}\|\_{F}^{2}=\min\_{\mathbf{P}}\|\mathbf{B}-\mathbf{P}\mathbf{A}\|\_{F}^{2}\}

%\end{equation}

%\section{Sensitivity of $\mathbf{\hat{P}}$ to Data Perturbations}

%In this section, we examine the sensitivity of the solution to perturbation in the data. To measure this sensitivity, we give the relative residuals and the Fro-norm condition numbers of the solutions. The condition number of the matrix $\mathbf{A}$ is defined as $k\_{F}(\mathbf{A})=\frac{\sigma\_{1}}{\sigma\_{r}}$, where $r=\text{rank}(\mathbf{A})$.

%\begin{figure}[h]

%\centerline{\includegraphics[width=254pt]{/figure07.eps}}

%\caption{North Carolina End-of-Grade Math Skills Test Subscores.}

%\end{figure}

\section{Concluding Remarks}

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%\vfill\eject

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