Note that if the partially known columns were not existed, which means $\mathbf{X}=\oldemptyset$, the solution is clearly reduced to the solution of the original orthogonal Procrustes problem, i.e., $\mathbf{\hat{P}} = \mathbf{U}\mathbf{V}^{\top}$, where $\mathbf{U}$ and $\mathbf{V}$ are the orthogonal matrices obtained by performing economy (a.k.a. reduced) SVD:

$\mathbf{B}\mathbf{A}^{\top} = \mathbf{U}\mathbf{\mathbf{\Sigma}}\mathbf{V}^{\top}$.

The difference between the two solutions could reflect the effect on the residual difference of requiring $\mathbf{P}$ to be orthogonal to the additional constraints, i.e., $\mathbf{X}^{\top}\mathbf{P}=\mathbf{0}\_{p\times q}$.

\section{Uniqueness of Solution $\mathbf{\hat{P}}$}

Now we discuss the uniqueness of the solution $\mathbf{\hat{P}}$. Our discussion is according to the rankness of the $\mathbf{\Sigma}$ generated in the SVD. We first give a lemma describing the rank of $\mathbf{X}\mathbf{X}^{\top}$ where $\mathbf{X}\in\mathbb{R}^{n\times p}$ is the orthogonal matrix consisting of partially known columns.

\emph{Lemma 1}: Let $\mathbf{X}\in\mathbb{R}^{n\times p}$ be orthogonal matrix with $\mathbf{X}^{\top}\mathbf{X}=\mathbf{I}\_{p\times p}$, then $\text{rank}(\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top})\ge n-p$.

\emph{Proof}: We firstly proof that $\text{rank}(\mathbf{X}\mathbf{X}^{\top})=p$. The upper bound of $\text{rank}(\mathbf{X}\mathbf{X}^{\top})$ can be determined by $\text{rank}(\mathbf{X}\mathbf{X}^{\top})\le\min\{\text{rank}(\mathbf{X}),\text{rank}(\mathbf{X}^{\top})\}=p$. The lower bound of $\text{rank}(\mathbf{X}\mathbf{X}^{\top})$ can be determined by Sylvester's inequality as $\text{rank}(\mathbf{X}\mathbf{X}^{\top})\ge\text{rank}(\mathbf{X})+\text{rank}(\mathbf{X}^{\top})-p=2p-p=p$. Hence, we have $\text{rank}(\mathbf{X}\mathbf{X}^{\top})=p$. Then, $\text{rank}(\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top})\ge\text{rank}(\mathbf{I}\_{n\times n})-\text{rank}(\mathbf{X}\mathbf{X}^{\top})\ge n-p$.

$\hfill\blacksquare$

The rank of $\mathbf{\Sigma}$ depends on the rank of $\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top}$, $\mathbf{B}$ and $\mathbf{A}$. Note that the rank of $\mathbf{B}$ and $\mathbf{A}$ are not less than or equal to $m$ and $\min\{q,m\}$, respectively. The rank of $\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top}$ is at least $n-p=q$. From above observations, we can see that the rank of $\mathbf{\Sigma}$ can be equal to or less than $\min\{q,m\}$.

\textbf{Results 2}: If $(\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top})\mathbf{B}\mathbf{A}^{\top}\in\mathbb{R}^{n\times q}$ is nonsingular, then $\text{rank}(\mathbf{\Sigma})=q$. $\mathbf{\Sigma}$ may have distinct or multiple non-zero singular values. In the SVD of $(\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top})\mathbf{B}\mathbf{A}^{\top}

=

\mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\top}$, the singular vectors in $\mathbf{U}$ and $\mathbf{V}$

can be determined up to orientation. Hence, we can reformulate the SVD as

\begin{equation}

(\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top})\mathbf{B}\mathbf{A}^{\top}

=

\mathbf{U}^{\*}\mathbf{K}\_{u}\mathbf{\Sigma}\mathbf{K}\_{v}(\mathbf{V}^{\*})^{\top},

\end{equation}

where $\mathbf{U}^{\*}\in \mathbb{R}^{n\times q}$ and $\mathbf{V}^{\*}\in \mathbb{R}^{q\times q}$ are arbitrarily orientated singular vectors of $\mathbf{U}$ and $\mathbf{V}$, respectively. $\mathbf{\Sigma}\in \mathbb{R}^{q\times q}$ are diagonal matrix with singular values are arranged in weak descending order along the diagonal, i.e., $\mathbf{\Sigma}\_{11}\ge\mathbf{\Sigma}\_{22}\ge...\ge\mathbf{\Sigma}\_{qq}\ge0$. The $\mathbf{K}\_{u}$ and $\mathbf{K}\_{v}$ are diagonal matrices with $+1$ or $-1$ as diagonal elements in arbitrary distribution. If we fix $\mathbf{K}\_{u}$, then $\mathbf{K}\_{v}$ is uniquely determined to meet the requirement that the diagonal elements of $\mathbf{\Sigma}$ should be nonnegative. And the orientations of the singular vectors of $\mathbf{U}^{\*}$ is fixed, then the $\mathbf{U}=\mathbf{U}^{\*}\mathbf{K}\_{u}$ is determined, so does the orientations of the singular vectors of $\mathbf{V}^{\*}$ and $\mathbf{V}^{\top}=\mathbf{K}\_{v}(\mathbf{V}^{\*})^{\top}$. In this case, the solution of $\mathbf{\hat{P}}=\mathbf{U}\mathbf{V}^{\top}=\mathbf{U}^{\*}\mathbf{K}\_{u}\mathbf{K}\_{v}(\mathbf{V}^{\*})^{\top}$ is unique. The case that the $\mathbf{\Sigma}$ have multiple singular values also has unique solution of $\mathbf{\hat{P}}$, which can be discussed in a similar way.

If $(\mathbf{I}\_{n\times n}-\mathbf{X}\mathbf{X}^{\top})\mathbf{B}\mathbf{A}^{\top}$ is singular, then $0\le r=\text{rank}(\mathbf{\Sigma})< q$, and there is $q-r$ (at least one) zero singular values. The previous discussion on nonsingular case still can be applied to the singular vectors corresponding to the nonzero singular values, and the production of these singular vectors in $\mathbf{U}$ and $\mathbf{V}$ is still unique. However, the singular vectors corresponding to the zero singular values could be in arbitrary orientations as long as they satisfy the orthogonal conditions that $\mathbf{U}^{\top}\mathbf{U}=\mathbf{V}^{\top}\mathbf{V}=\mathbf{V}\mathbf{V}^{\top}=\mathbf{I}\_{q\times q}$. Note that $\mathbf{U}\in \mathbb{R}^{n\times q}$, so $\mathbf{U}\mathbf{U}^{\top}$ no longer equals to the identity matrix of order $n\times n$. From Eq. (\ref{e13}), we can get

\begin{equation}

\mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\top}\mathbf{P}^{\top}

=

\mathbf{P}\mathbf{V}\mathbf{\Sigma}\mathbf{U}^{\top}

\end{equation}

Right multiplying both sides by $\mathbf{P}\mathbf{V}$ and then left multiplying each side by $\mathbf{U}^{\top}$, we can get that

\begin{equation}

\mathbf{\Sigma}

=

\mathbf{U}^{\top}\mathbf{P}\mathbf{V}\mathbf{\Sigma}\mathbf{U}^{\top}\mathbf{P}\mathbf{V}

\end{equation}

Hence, we can define a diagonal matrix $\mathbf{D}=\mathbf{U}^{\top}\mathbf{P}\mathbf{V}\in\mathbb{R}^{q\times q}$, the diagonal elements of which are

\begin{displaymath}

\mathbf{D}\_{ii}= \left\{ \begin{array}{ll}

1 & \textrm{if $1\le i\le r$};\\

\pm 1 & \textrm{if $r< i \le q$}.\\

\end{array} \right.

\end{displaymath}

Thus, we obtain that $\mathbf{P}=\mathbf{U}\mathbf{D}\mathbf{V}^{\top}$, where $\mathbf{D}$ is defined above. That is to say, when $\text{rank}(\mathbf{\Sigma})<q$ , once we get the solution of $\mathbf{\hat{P}}=\mathbf{U}\mathbf{V}^{\top}$ in problem (\ref{e4}), the final solutions for problem (\ref{e4}) are not unique and determined up to the specific matrix $\mathbf{D}$. In fact, since the number of $\mathbf{D}$ with different diagonal combinations is $2^{q-r}$, the number of solutions $\mathbf{P}$ for problem (\ref{e4}) is $2^{q-r}$ given fixed $\mathbf{U}$ and $\mathbf{V}$.

%Since the solution of $\mathbf{\hat{P}}$ is not unique when $\text{rank}(\mathbf{\Sigma})<q$, we define the set of solutions for in a formal manner and discuss its properties. The solution set can be defined as:

%\begin{equation}

%\mathcal{S}=\{\mathbf{S}\in\mathbb{R}^{n\times q}: \mathbf{S}^{\top}\mathbf{S}=\mathbf{I}\_{q\times q}, \mathbf{X}^{\top}\mathbf{S}=\mathbf{0}\_{p\times q}, \|\mathbf{B}-\mathbf{S}\mathbf{A}\|\_{F}^{2}=\min\_{\mathbf{P}}\|\mathbf{B}-\mathbf{P}\mathbf{A}\|\_{F}^{2}\}

%\end{equation}

%\section{Sensitivity of $\mathbf{\hat{P}}$ to Data Perturbations}

%In this section, we examine the sensitivity of the solution to perturbation in the data. To measure this sensitivity, we give the relative residuals and the Fro-norm condition numbers of the solutions. The condition number of the matrix $\mathbf{A}$ is defined as $k\_{F}(\mathbf{A})=\frac{\sigma\_{1}}{\sigma\_{r}}$, where $r=\text{rank}(\mathbf{A})$.

%\begin{figure}[h]

%\centerline{\includegraphics[width=254pt]{/figure07.eps}}

%\caption{North Carolina End-of-Grade Math Skills Test Subscores.}

%\end{figure}

\section{Concluding Remarks}

In this paper, we studied the classical orthogonal Procrustes problem and gave the solution of a generalized version of the original problem with partially known columns, which included the original orthogonal Procrustes problem as a special case (i.e., when there is no prior known columns at all). We studied the sufficiency and necessary conditions of the solution for the generalized problem and discuss the number of solutions under the nonsingular and singular cases. As the potential future work, we are highly motivated to study how the partially known columns will influence the solutions of the two-sided \cite{schonemann1968two} or weighted \cite{Lissitz1976,Koschat1991} orthogonal Procrustes problems.

\linespacing{0.5}

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