Rank Minimization for Sylvester Equation

Abstract

Sylvester equation is widely used in many problems in system and automatic control. In this paper, we prove that the well-known Roth's similarity theorem (Roth (1952)) is a special case of a rank minimization theorem for Sylvester equation.

Keywords: Rank Minimization, Sylvester Equation

1. Introduction

The Sylvester matrix equation is widely used in system and automatic control community Dehghan (2011); Hajarian (2016); Wu (2008); Hu (2006); Roth (1952). In Dehghan (2011), Dehghan and Hajarian propose two algorithms for finding Hermitian reflexive and skew-Hermitian solutions of Sylvester matrix equations. In Hajarian (2016) Hajarian proposes a gradient based iterative method to solve the Sylvester matrix equation. In Wu (2008), Wu *et al.* solves the Sylvester matrix equatio via Kronecker map. In Hu (2006), Hu and Cheng proposed a polynomial solution to the Sylvester matrix equation. Denote by $\boldsymbol{A} \in \mathbb{F}^{m\times m}$, $\boldsymbol{B} \in \mathbb{F}^{n\times n}$, and $\boldsymbol{C} \in \mathbb{F}^{m\times n}$ three given matrices over some field $\mathbb{F}$. Denote by $\mathrm{GL}(n, \mathbb{F}) = \{\boldsymbol{M} \in \mathbb{F}^{n\times n} | \det \boldsymbol{M} \neq 0\}$ the general linear group of degree n over the field $\mathbb{F}$. The following theorem is the well-known Roth's similarity theorem, which provides a sufficient and necessary condition for guaranting the consistency of the Sylvester equation (1.1) in terms of an equivalence between two associated matrices.

Theorem 1.1 (Roth (1952)). *The matrix equation*

$$\boldsymbol{AX} - \boldsymbol{XB} = \boldsymbol{C} \tag{1.1}$$

is solvable w.r.t $\boldsymbol{X} \in \mathbb{F}^{m\times n}$ *if and only if there exists a matrix* $\boldsymbol{P} \in GL(m+n, \mathbb{F})$ *such that*

$$\boldsymbol{P}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}\boldsymbol{P}$$

2. Main Results

We now generalize the Roth's similarity theorem to a general result based on rank minimization. The purpose of this note is to prove the following theorem.

Theorem 2.1. *Given $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ defined above, denote by we have*

$$\min\{\operatorname{rank}(\boldsymbol{AX} - \boldsymbol{XB} - \boldsymbol{C}) | \boldsymbol{X} \in \mathbb{F}^{m\times n}\}$$

$$= \min\left\{\operatorname{rank}\left[\boldsymbol{P}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} - \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}\boldsymbol{P}\right] | \boldsymbol{P} \in \mathrm{GL}(m+n, \mathbb{F})\right\}.$$

Proof. Denote by

$$\alpha(\boldsymbol{X}) = \boldsymbol{AX} - \boldsymbol{XB} - \boldsymbol{C}, \tag{2.1}$$

and

$$\beta(\boldsymbol{P}) = \boldsymbol{P}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} - \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}\boldsymbol{P}. \tag{2.2}$$

Define

$$R_\alpha = \min \operatorname{rank}\{\alpha(\boldsymbol{X}) | \boldsymbol{X} \in \mathbb{F}^{m\times n}\} \tag{2.3}$$

and

$$R_\beta = \min \operatorname{rank}\{\beta(\boldsymbol{P}) | \boldsymbol{P} \in \mathrm{GL}(m+n, \mathbb{F}).\} \tag{2.4}$$

From Tian (2002); TianMT (2000), we can obtain that

$$R_\alpha = \operatorname{rank}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} - \operatorname{rank}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}. \tag{2.5}$$

For $\boldsymbol{P} \in \mathrm{GL}(m+n, \mathbb{F})$, we can obtain that

$$\begin{aligned} \operatorname{rank}\beta(\boldsymbol{P}) &\geq \operatorname{rank}\boldsymbol{P}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} - \operatorname{rank}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}\boldsymbol{P} \\ &= \operatorname{rank}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} - \operatorname{rank}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} \\ &= R_\alpha. \end{aligned} \tag{2.6}$$

Hence, we have that $R_\beta \geq R_\alpha$. On the other hand, we denote by

$$\boldsymbol{P_X} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{X} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}, \tag{2.7}$$

then we have

$$\beta(\boldsymbol{P_X}) = \begin{pmatrix} \mathbf{0} & -\alpha(\boldsymbol{X}) \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{2.8}$$

Therefore,

$$R_\beta \leq \min\{\mathrm{rank}\beta(\boldsymbol{P_X}) | \boldsymbol{X} \in \mathbb{F}^{m\times n}\} = \min\{\mathrm{rank}\alpha(\boldsymbol{X})\} = R_\alpha. \tag{2.9}$$

This completes the proof. □

Reference

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