Rank Minimization for Sylvester Equation

Abstract

Sylvester equation is widely used in many problems in system and automatic control. In this paper, we prove that the well known Roth's similarity theorem (Roth (1952)) is a special case of a rank minimization theorem for Sylvester equation.

Keywords: Rank Minimization, Sylvester Equation

1. Introduction

The Sylvester matrix equation is widely used in system and automatic control community Dehghan (2011); Hajarian (2016); Wu (2008); Hu (2006); Roth (1952). In Dehghan (2011), Dehghan and Hajarian propose two algorithms for finding Hermitian reflexive and skew-Hermitian solutions of Sylvester matrix equations. In Hajarian (2016) Hajarian proposes a gradient based iterative method to solve the Sylvester matrix equation. In Wu (2008), Wu *et al.* solves the Sylvester matrix equatio via Kronecker map. In Hu (2006), Hu and Cheng proposed a polynomial solution to the Sylvester matrix equation. Denote by $\boldsymbol{A} \in \mathbb{F}^{m\times m}$, $\boldsymbol{B} \in \mathbb{F}^{n\times n}$, and $\boldsymbol{C} \in \mathbb{F}^{m\times n}$ three given matrices over some field $\mathbb{F}$. Denote by $\mathrm{GL}(n, \mathbb{F}) = \{\boldsymbol{M} \in \mathbb{F}^{n\times n} | \det\boldsymbol{M} \neq 0\}$ the general linear group of degree n over the field $\mathbb{F}$. The following theorem is the well-known Roth's similarity theorem, which provides a sufficient and necessary condition for guaranting the consistency of the Sylvester equation (1.1) in terms of an equivalence between two associated matrices.

Theorem 1.1 (Roth (1952)). *The matrix equation*

$$\boldsymbol{AX} - \boldsymbol{XB} = \boldsymbol{C} \tag{1.1}$$

is solvable w.r.t $\boldsymbol{X} \in \mathbb{F}^{m\times n}$ *if and only if there exists a matrix* $\boldsymbol{P} \in GL(m+n, \mathbb{F})$ *such that*

$$\boldsymbol{P}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}\boldsymbol{P}$$

2. Main Results

We now generalize the Roth's similarity theorem to a general result based on rank minimization. The purpose of this note is to prove the following theorem.

Theorem 2.1. *Given $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ defined above, denote by we have*

$$\min\{\operatorname{rank}(\boldsymbol{AX} - \boldsymbol{XB} - \boldsymbol{C}) | \boldsymbol{X} \in \mathbb{F}^{m \times n}\}$$

$$= \min\left\{\operatorname{rank}\left[\boldsymbol{P}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} - \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}\boldsymbol{P}\right] | \boldsymbol{P} \in \mathrm{GL}(m+n, \mathbb{F})\right\}.$$

Proof. Denote by

$$\alpha(\boldsymbol{X}) = \boldsymbol{AX} - \boldsymbol{XB} - \boldsymbol{C}, \tag{2.1}$$

and

$$\beta(\boldsymbol{P}) = \boldsymbol{P}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} - \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}\boldsymbol{P}. \tag{2.2}$$

Define

$$R_\alpha = \min \operatorname{rank}\{\alpha(\boldsymbol{X}) | \boldsymbol{X} \in \mathbb{F}^{m \times n}\} \tag{2.3}$$

and

$$R_\beta = \min \operatorname{rank}\{\beta(\boldsymbol{P}) | \boldsymbol{P} \in \mathrm{GL}(m+n, \mathbb{F}).\} \tag{2.4}$$

From Tian (2002); TianMT (2000), we can obtain that

$$R_\alpha = \operatorname{rank}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} - \operatorname{rank}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}. \tag{2.5}$$

For $\boldsymbol{P} \in \mathrm{GL}(m+n, \mathbb{F})$, we can obtain that

$$\begin{aligned} \operatorname{rank}\beta(\boldsymbol{P}) &\geq \operatorname{rank}\boldsymbol{P}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} - \operatorname{rank}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}\boldsymbol{P} \\ &= \operatorname{rank}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} - \operatorname{rank}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} \\ &= R_\alpha. \end{aligned} \tag{2.6}$$

Hence, we have that $R_\beta \geq R_\alpha$. On the other hand, we denote by

$$\boldsymbol{P_X} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{X} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}, \tag{2.7}$$

then we have

$$\beta(\boldsymbol{P_X}) = \begin{pmatrix} \mathbf{0} & -\alpha(\boldsymbol{X}) \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{2.8}$$

Therefore,

$$R_\beta \leq \min\{\text{rank}\beta(\boldsymbol{P_X})|\boldsymbol{X} \in \mathbb{F}^{m\times n}\} = \min\{\text{rank}\alpha(\boldsymbol{X})\} = R_\alpha. \tag{2.9}$$

This completes the proof. □

Reference

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