Welcome to CS166!

Course Staff

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Living in Exciting Times

- The course staff will do its best understand your circumstances and accommodate as best we can.
- We expect that you'll similarly be understanding and accommodating as we figure out the most effective way to run this course.
- Please keep in touch with us to let us know how we're doing – we want this to be a good experience for everyone involved.
- And most importantly – ***please take care of yourself!*** The university has many resources available; take advantage of them if you need them. [1][2][3][4][5][6][7].

On Zoom Lectures

- We plan to hold these lectures every Tuesday and Thursday from 3:00PM – 4:20PM, Pacific time.
- To respect your privacy, *we do not plan to record these lectures* – please feel free to speak freely and ask questions!
  - Use the “Raise Hand” button under “Participants” to get our attention.
- These lectures are open to anyone on campus, regardless of whether they’re enrolled in CS166.
- I plan to record a parallel set of lecture videos that will go up on Canvas after each lecture.

Why Study Data Structures?

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- ***Explore where theory meets practice.***
  - Some of the data structures we'll cover are used extensively in practice. Many were invented in Silicon Valley!
- ***Challenge your intuition for the limits of efficiency.***
  - You'd be amazed how many times we'll take a problem you're sure you know how to solve and then see how to solve it faster.
- ***See the beauty of theoretical computer science.***
  - We'll cover some amazingly clever theoretical techniques in the course of this class. You'll love them.
- ***Equip yourself to solve complex problems.***
  - Powerful data structures make excellent building blocks for solving seemingly difficult problems.

The Course Website

http://cs166.stanford.edu

Course Requirements

- We plan on having five ***problem sets***.
  - Problem sets may be completed individually or in a pair.
  - They're a mix of written problems and C/C++ coding exercises.
  - You'll submit one copy of the problem set regardless of how many people worked on it.
  - Need to find a partner? Use Piazza, call into office hours, or send us an email.
- We plan of having six ***individual assessments***.
  - Similar to problem sets, except that they must be completed individually.
  - Course staff can answer clarifying questions, but otherwise it's up to you to work out how to solve them.
- We plan to have a final ***research project***.
  - We'll hammer out details in the next couple of weeks. Expect to work in a group, do a deep dive into a topic, and get lots of support from us.

Prerequisites

- ***CS161*** (Design and Analysis of Algorithms)
  - And, transitively, CS103, CS106B, and CS109.
  - We'll assume familiarity with asymptotic notation, correctness proofs, algorithmic strategies (e.g. divide-and-conquer, dynamic programming), classical algorithms, recurrence relations, universal hashing, techniques for manipulating random variables, etc.
- ***CS107*** (Computer Organization and Systems)
  - We'll assume comfort working in C and C++ from the command-line, designing and testing nontrivial programs, and manipulating bitwise representations of data. You should have some knowledge of the memory hierarchy.

Individual Assessment 0

- Individual Assessment 0 goes out today. It's due next Tuesday at 2:30PM Pacific time.
- This is mostly designed as a refresher of topics from the prerequisite courses CS103, CS107, CS109, and CS161.
- If you're mostly comfortable with these problems and are just "working through some rust," then you're probably in the right place!

Grading

- CS166 is graded on an S/NC basis this quarter. All assignments will be graded on an S/NC basis as well.
- To earn a passing grade in CS166, you must earn a passing grade on
  - all five problem sets,
  - five of the six individual assessments, and
  - the research project.
- Everyone has four free ***resubmits*** for the problem sets and individual assessments. If you earn an NC on a problem set or individual assessment, we'll contact you with what you need to patch up, and you can then resubmit, timetable TBA.
- Think of these as "mega late days." You can use them to extend deadlines or entirely redo things if they don't pan out well the first time.

Let’s Get Started!

Range Minimum Queries

The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

  Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[i + 1], …, A[j – 1], A[j]?

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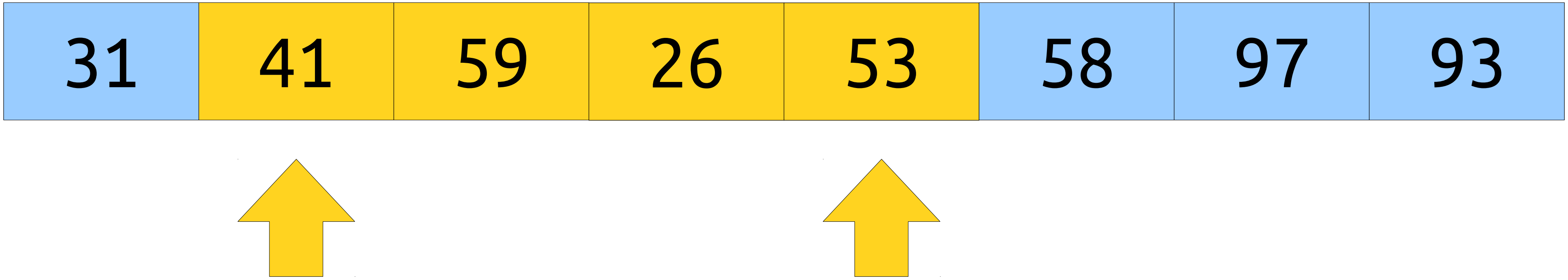

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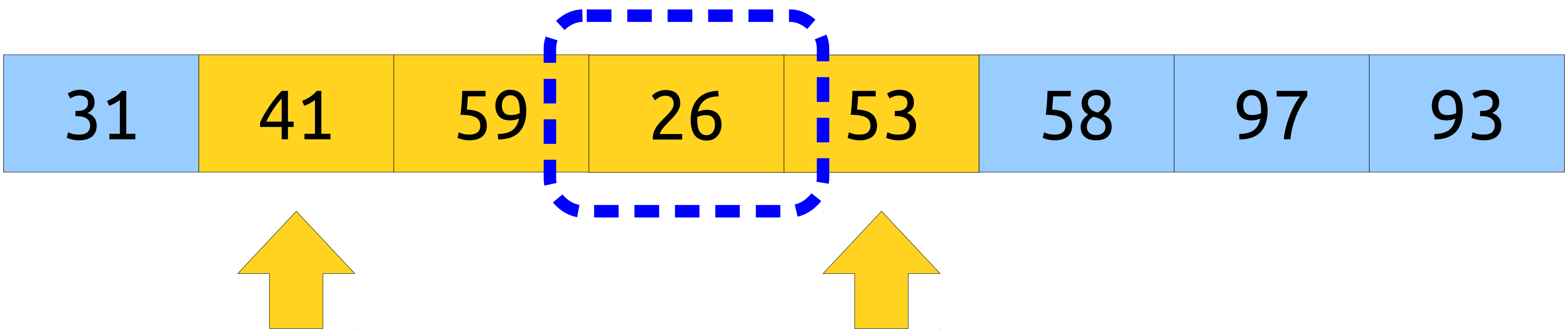

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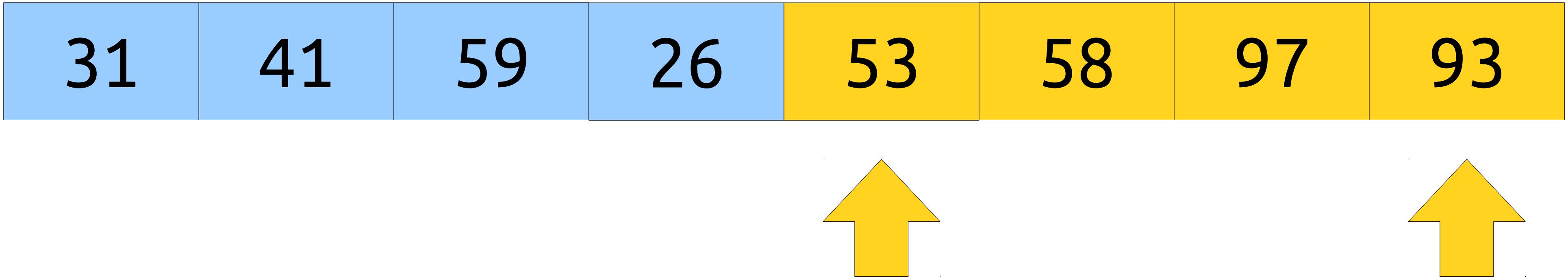

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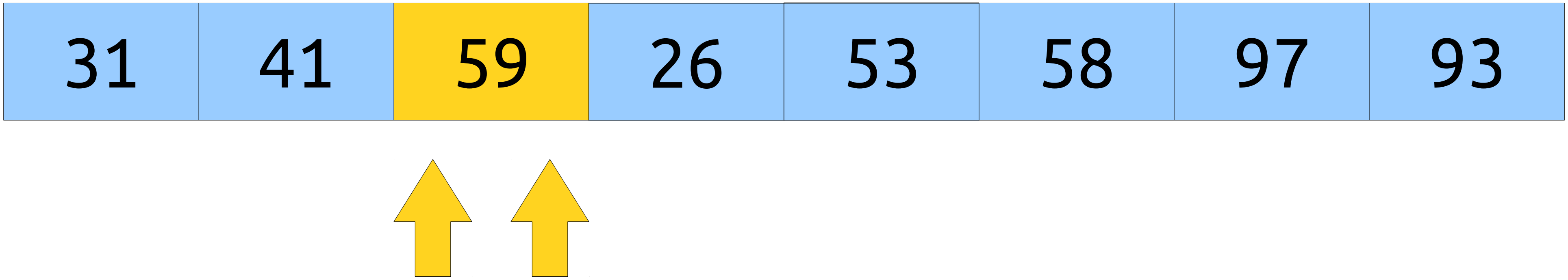

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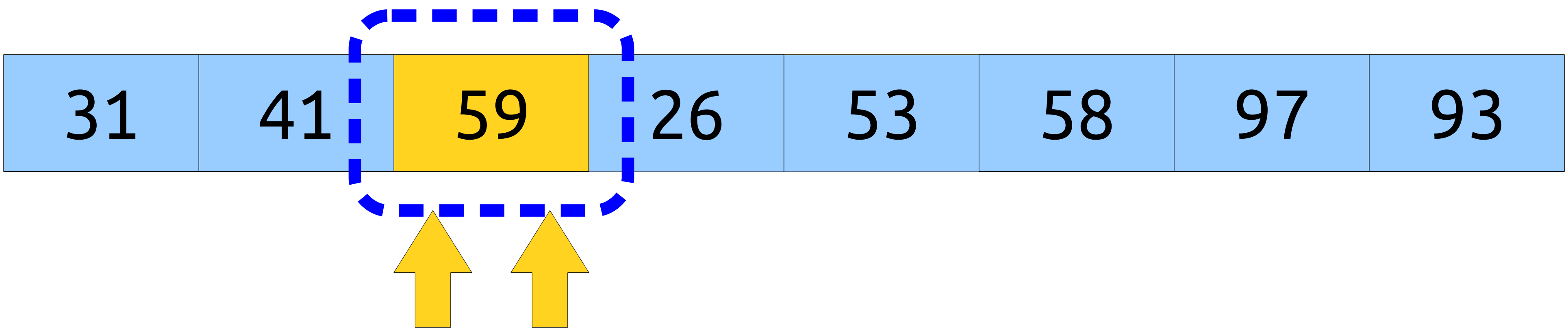

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- Notation: We'll denote a range minimum query in array A between indices i and j as $\mathbf{RMQ_A}(\boldsymbol{i, j})$.

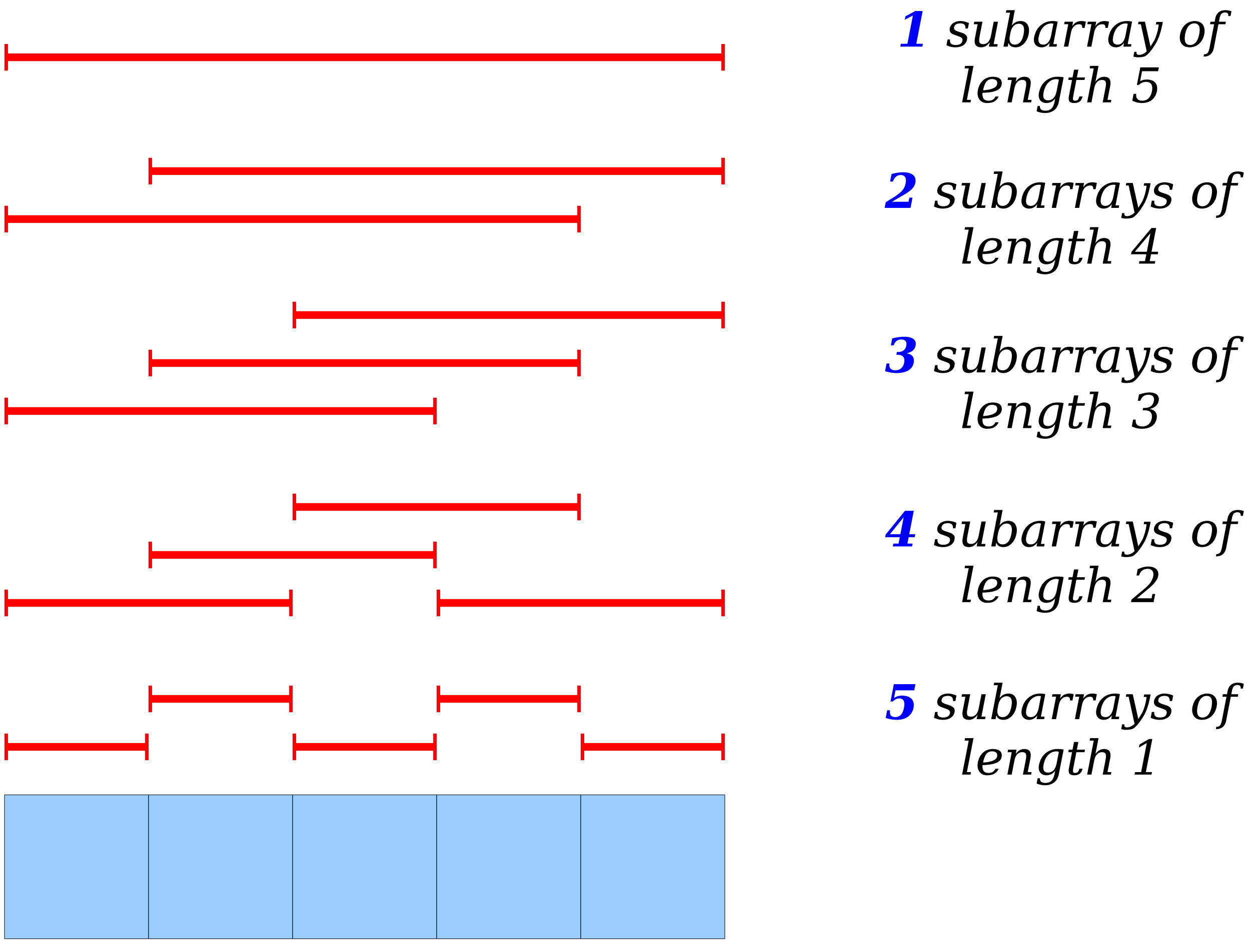
- For simplicity, let's assume 0-indexing.

A Trivial Solution

- There's a simple O(n)-time algorithm for evaluating $\mathrm{RMQ}_A(i, j)$: just iterate across the elements between i and j, inclusive, and take the minimum!
- So… why is this problem at all algorithmically interesting?
- Suppose that the array A is fixed in advance and you're told that we're going to make multiple queries on it.
- Can we do better than the naïve algorithm?

An Observation

- In an array of length n, there are only $\Theta(n^2)$ distinct possible queries.
- Why?



A Different Approach

- There are only $\Theta(n^2)$ possible RMQs in an array of length n.
- If we precompute all of them, we can answer RMQ in time O(1) per query.

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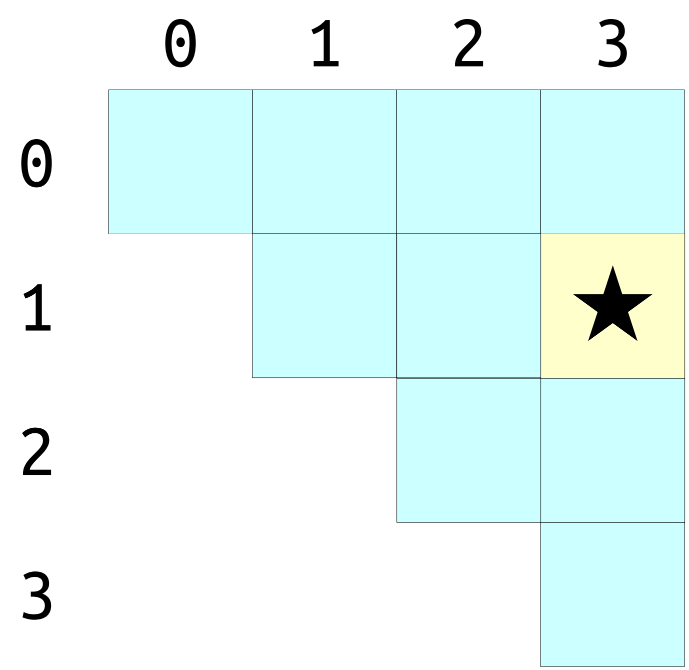

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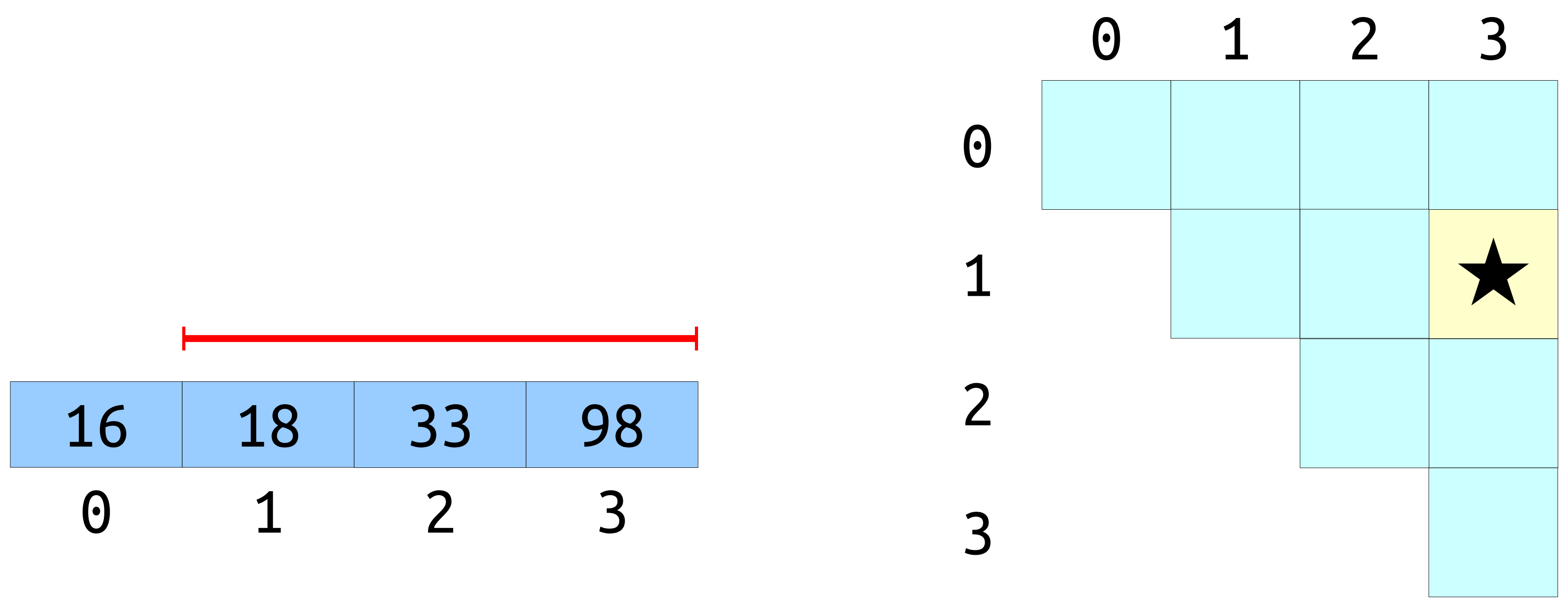

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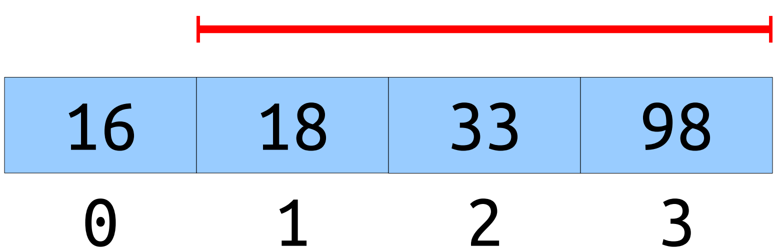

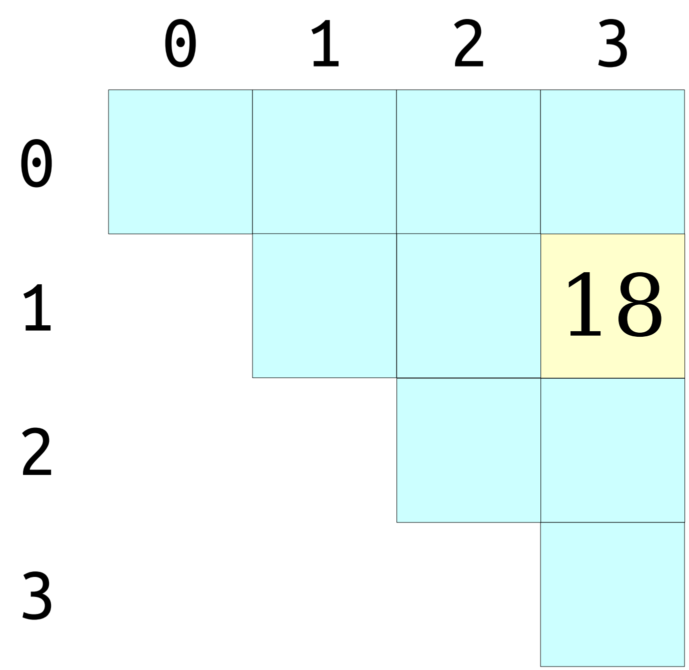

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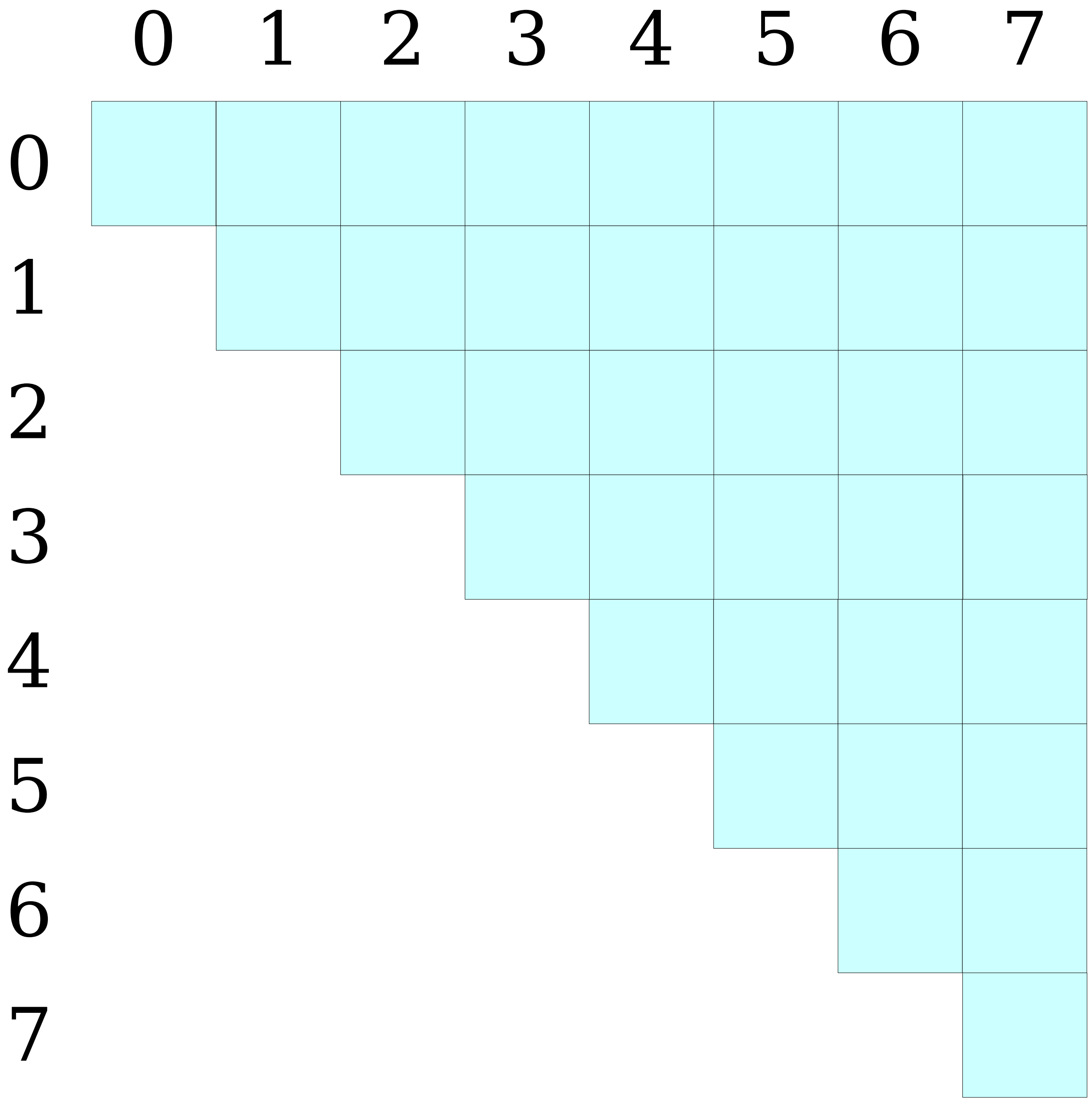
Building the Table

- One simple approach: for each entry in the table, iterate over the range in question and find the minimum value.
- How efficient is this?
    - Number of entries: $\Theta(n^2)$.
    - Time to evaluate each entry: $O(n)$.
    - Time required: $O(n^3)$.
- The runtime is $O(n^3)$ using this approach. Is it also $\Theta(n^3)$?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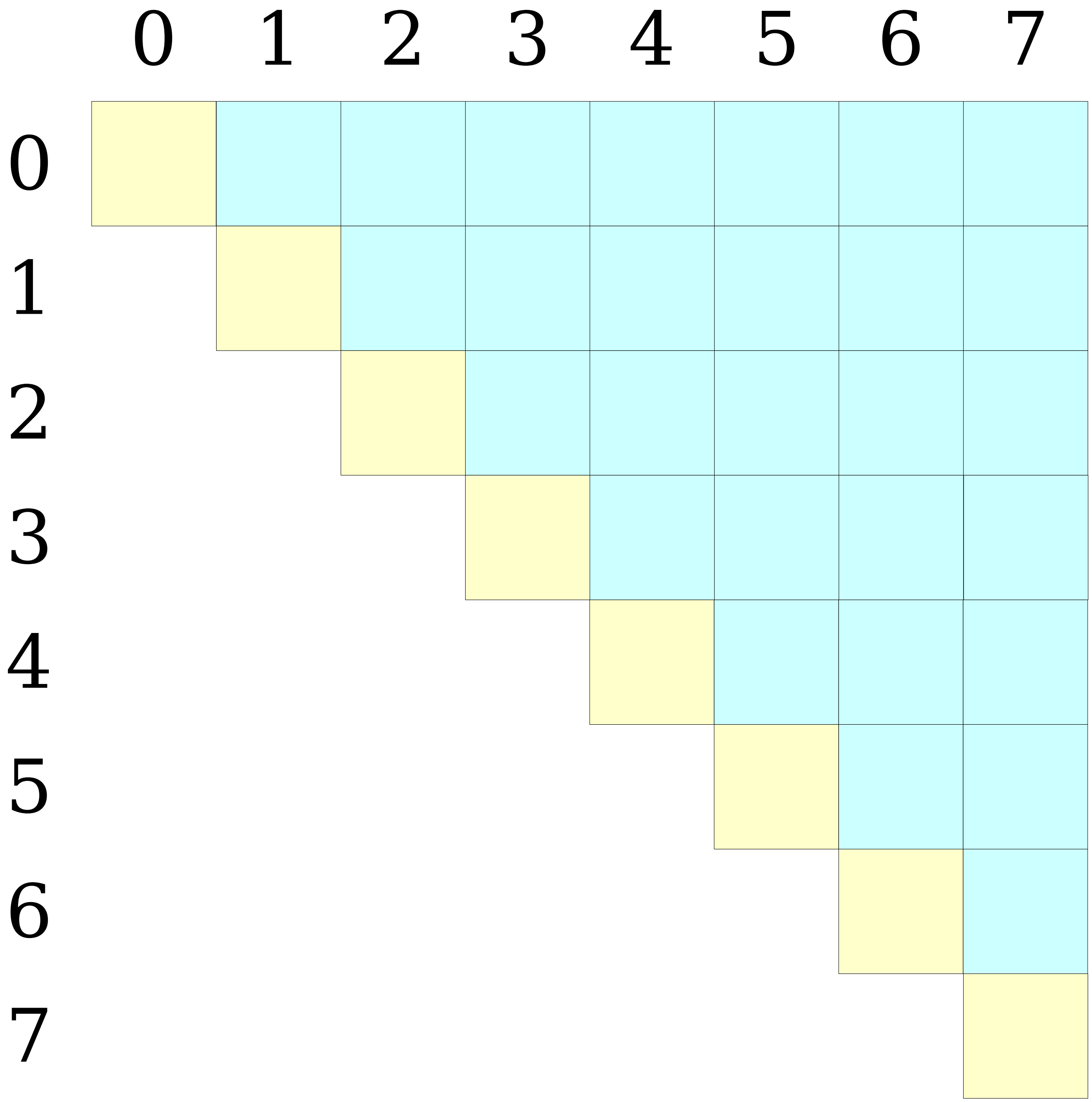
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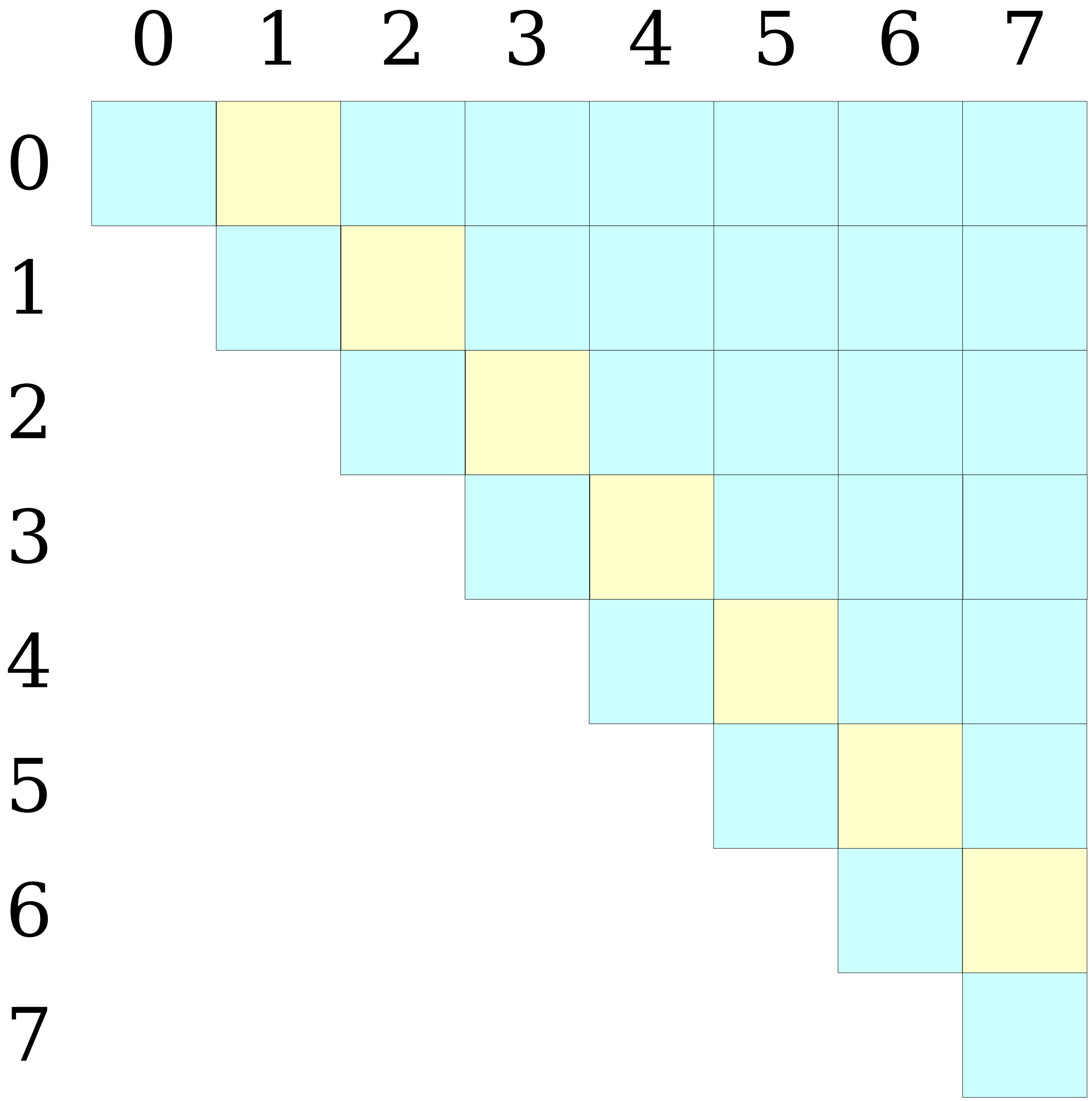
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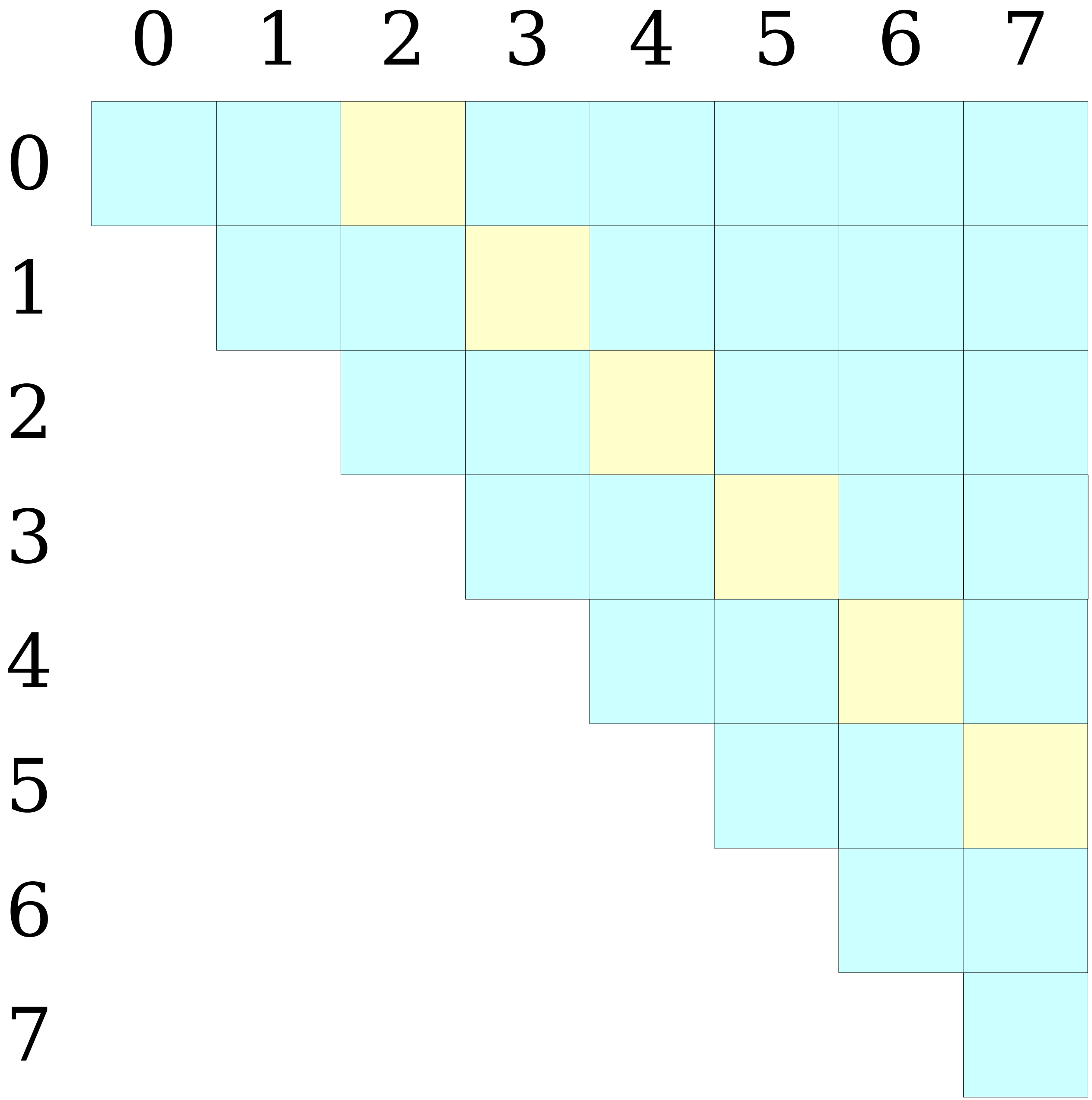
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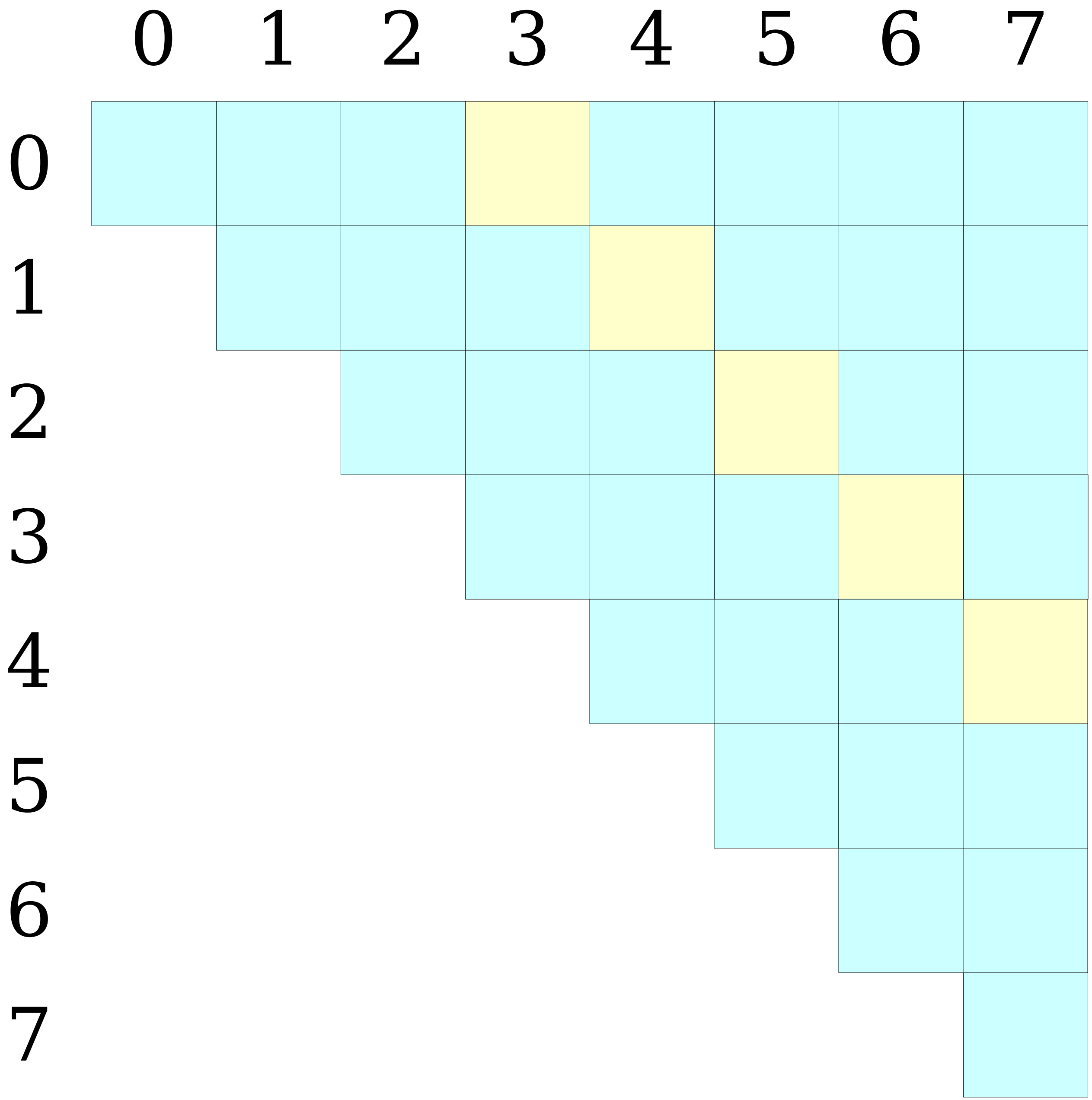
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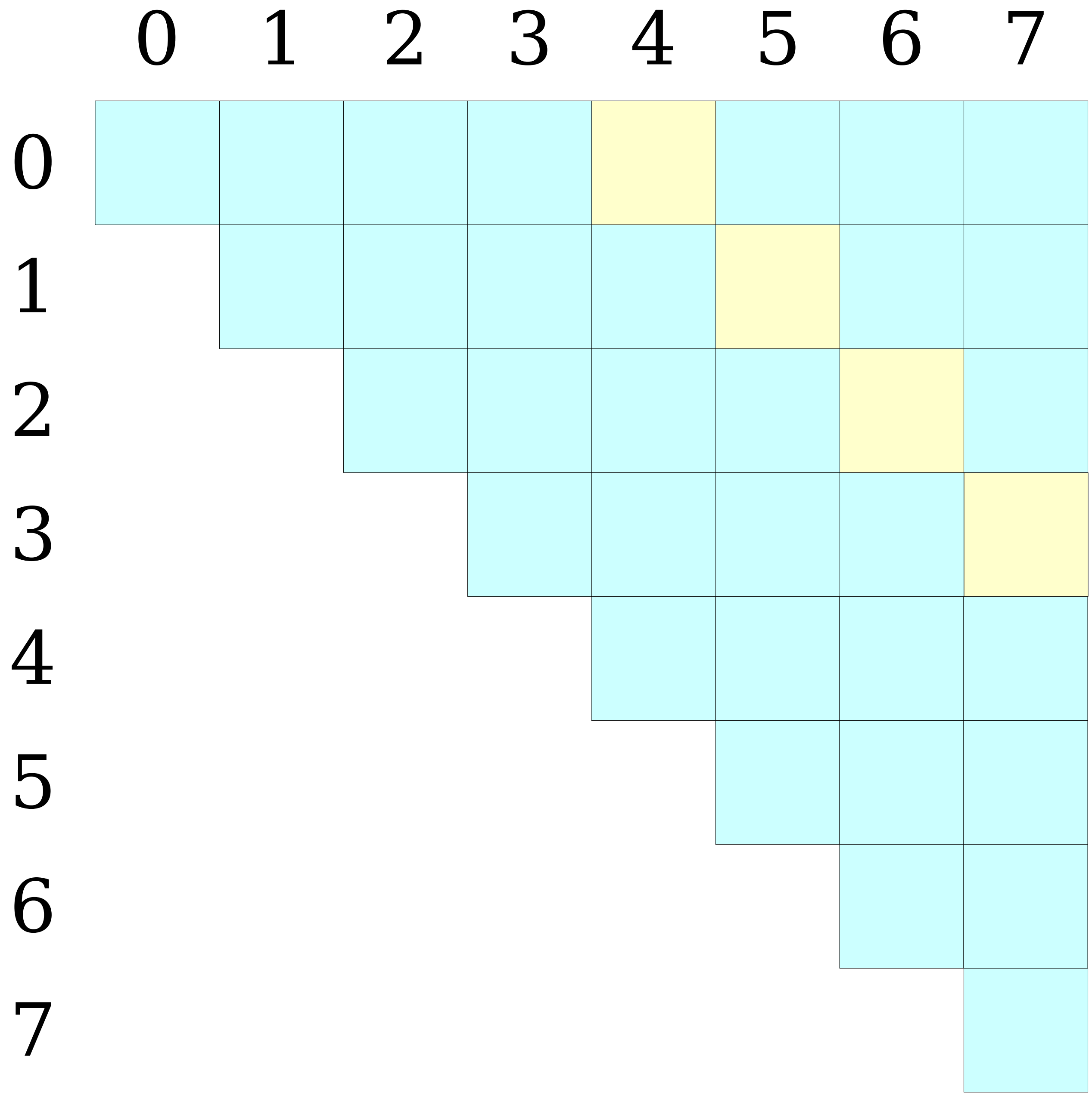
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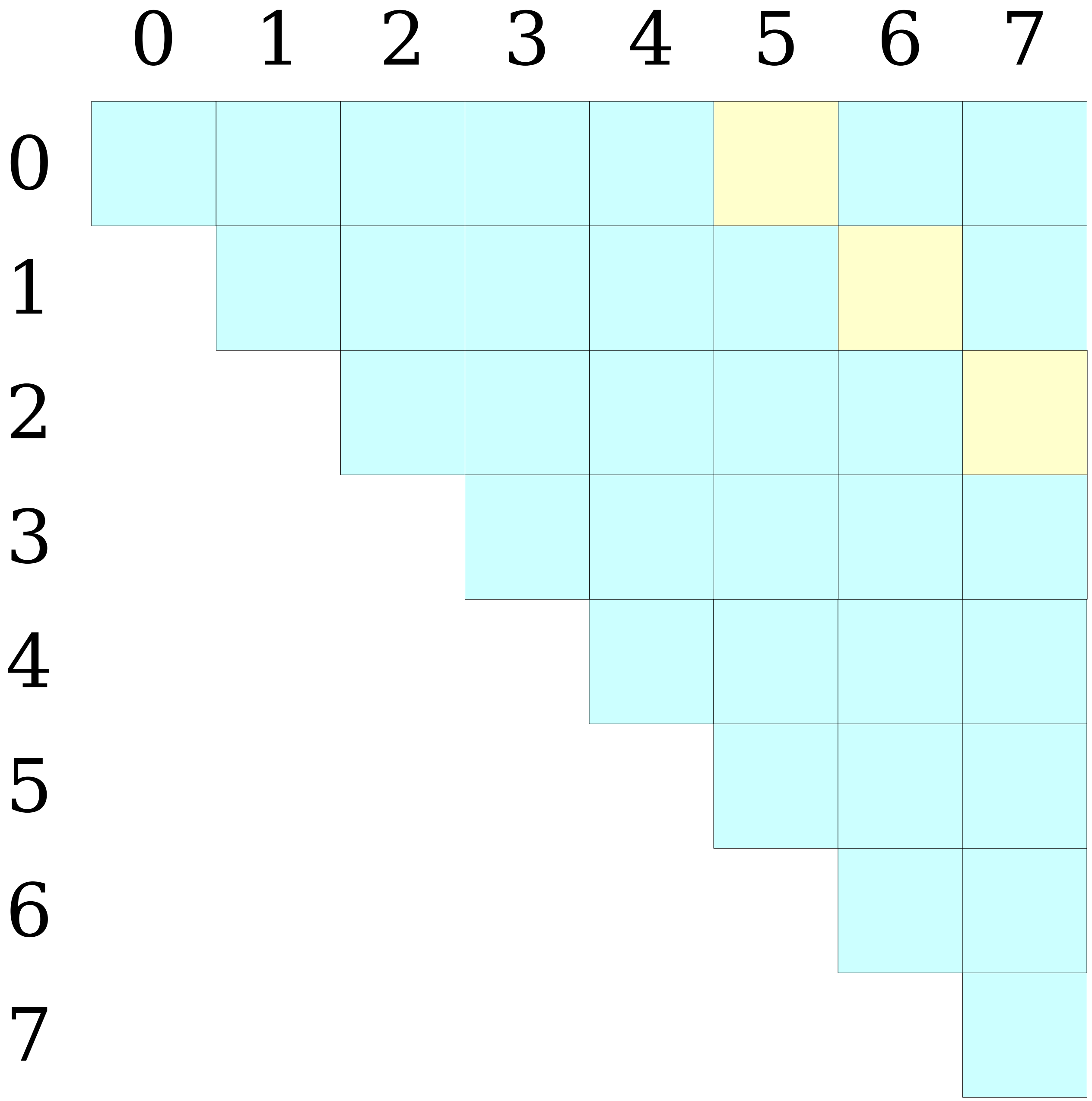
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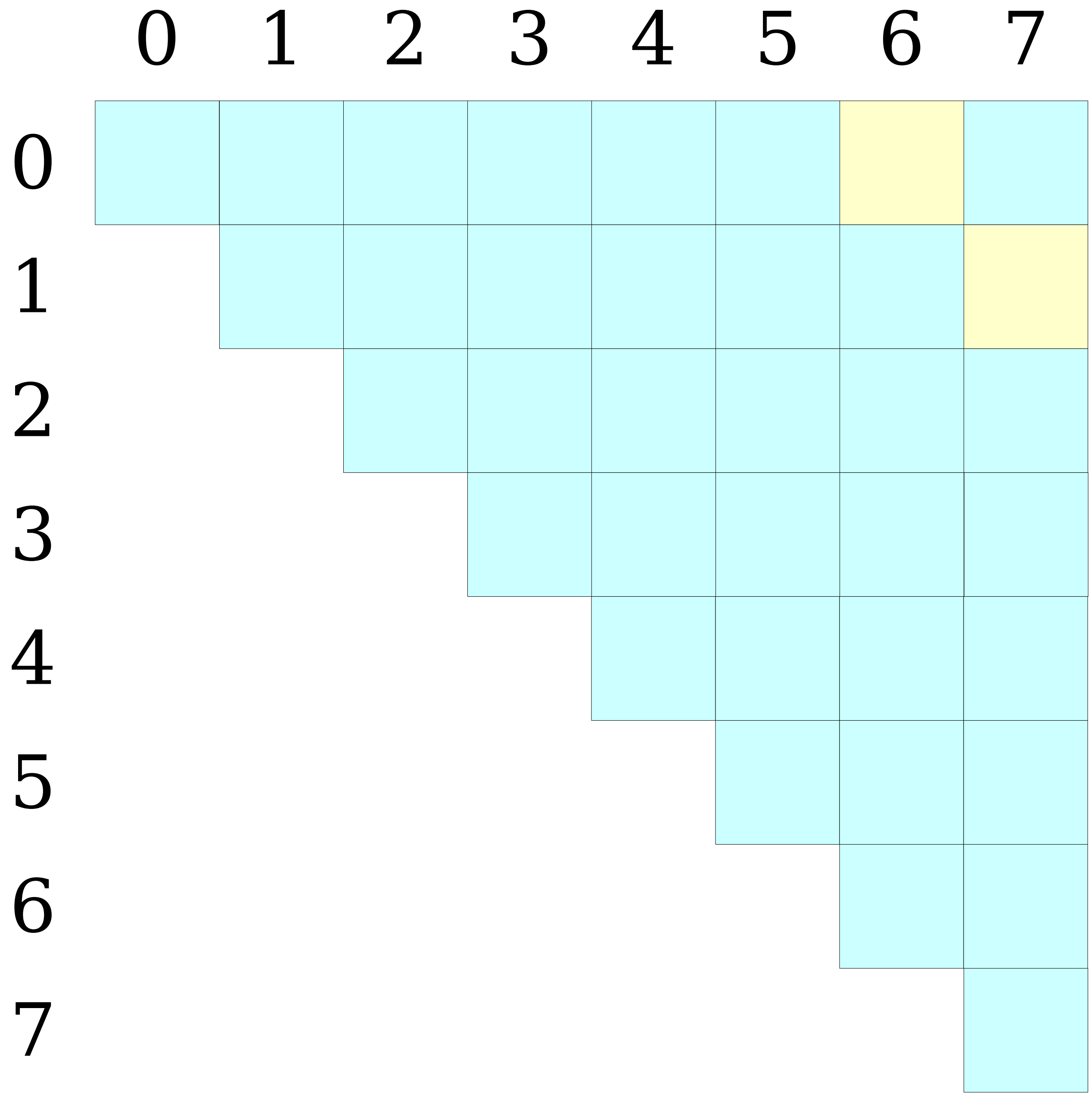
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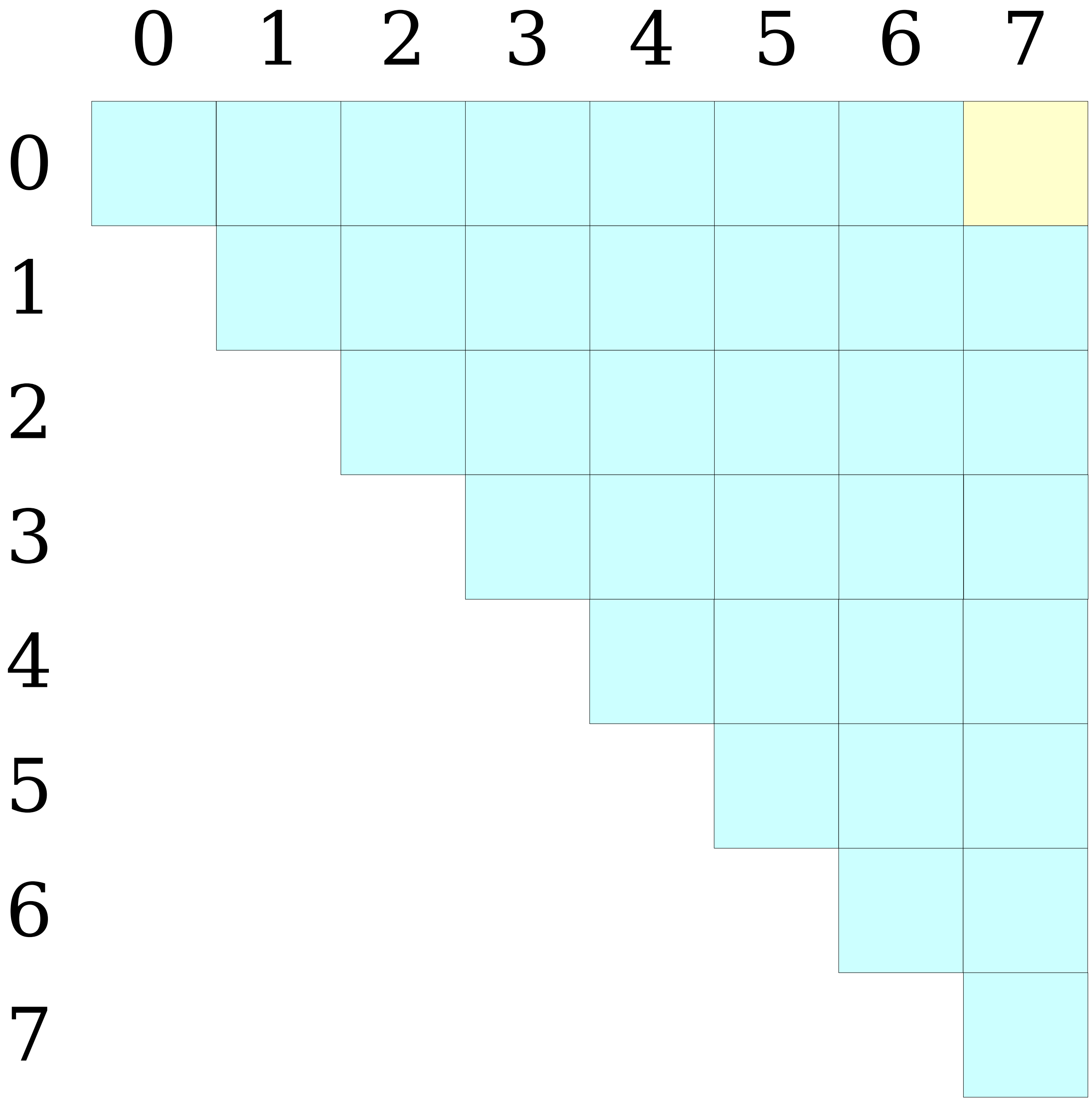
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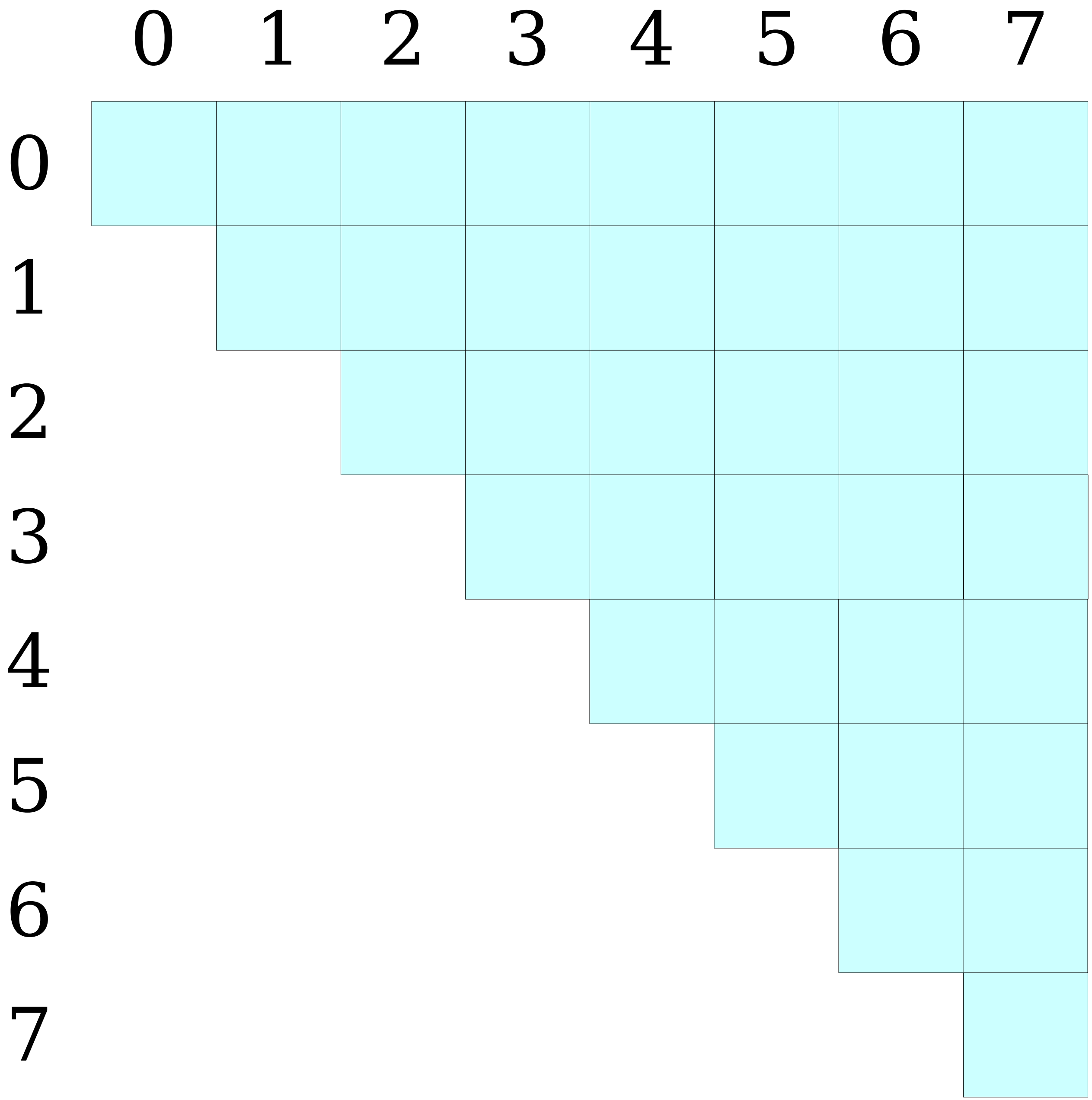
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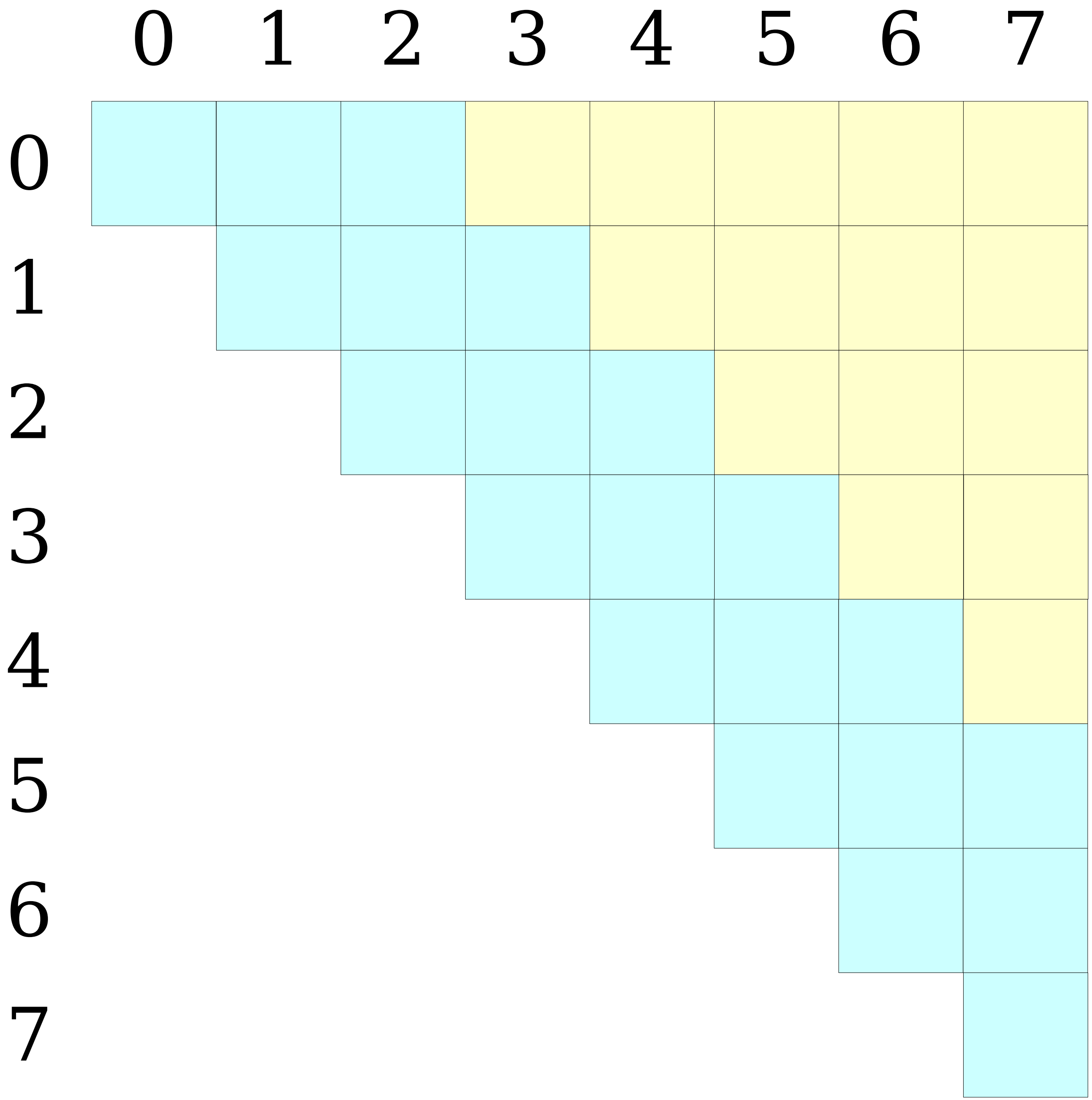
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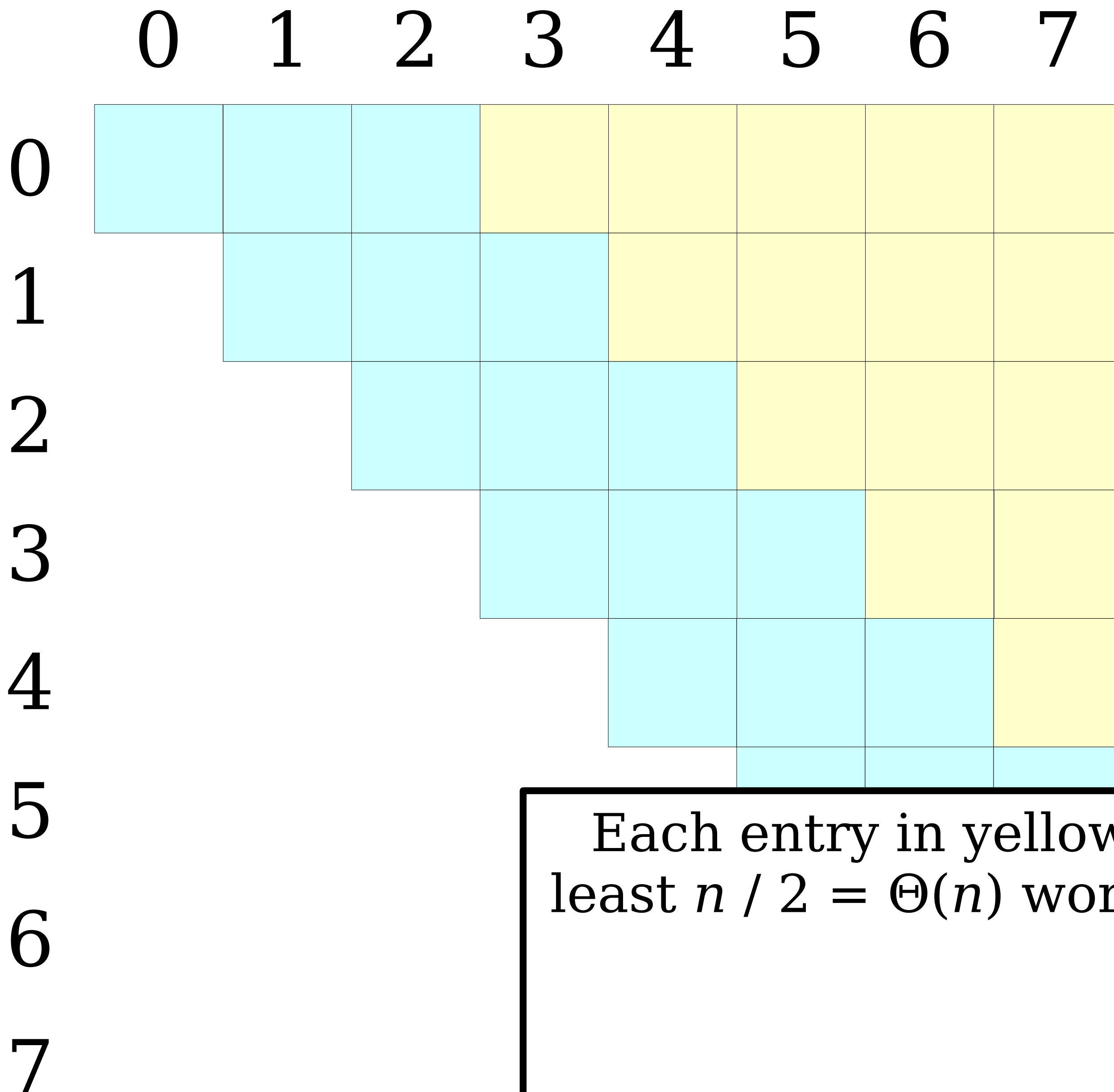

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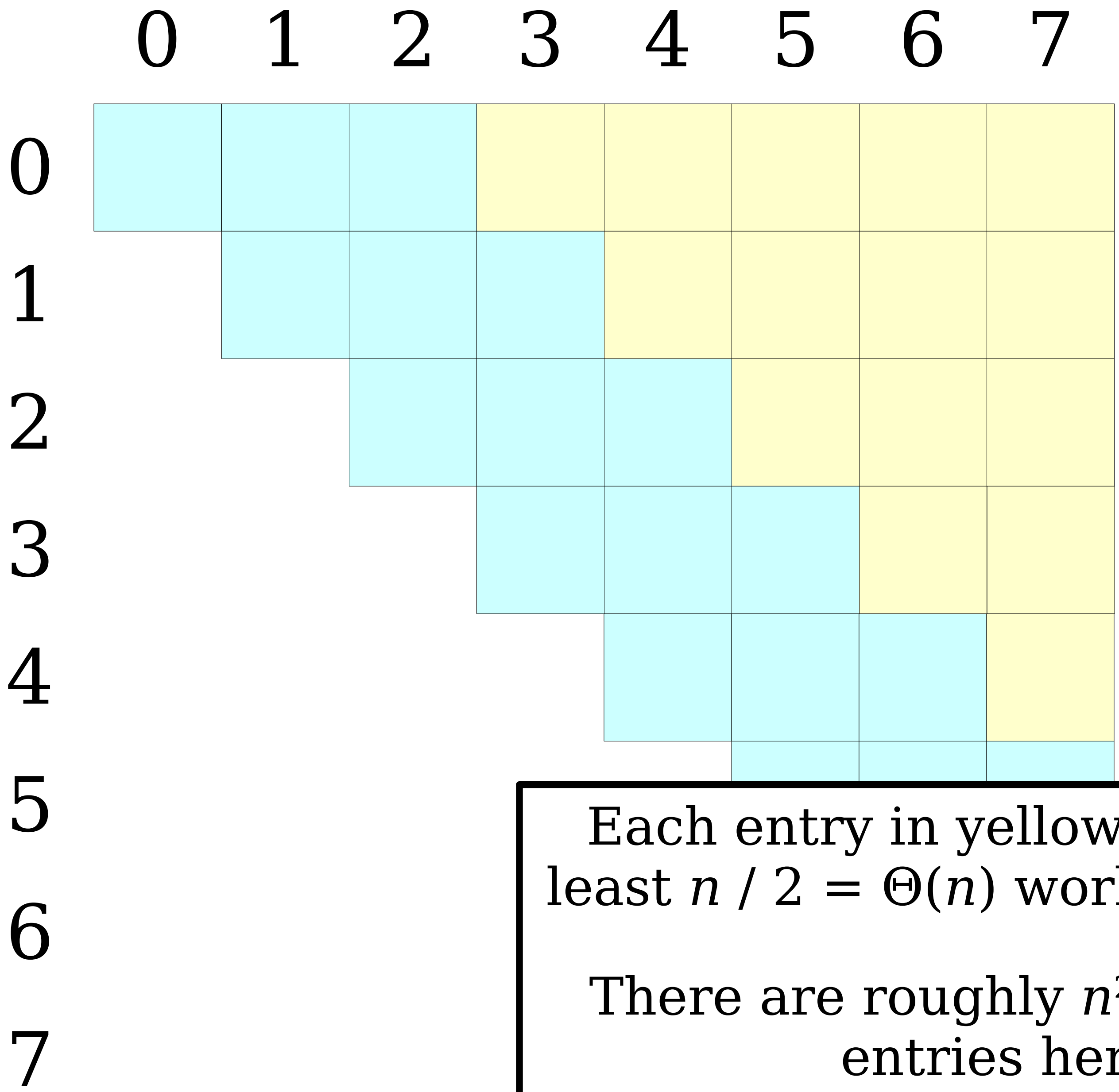

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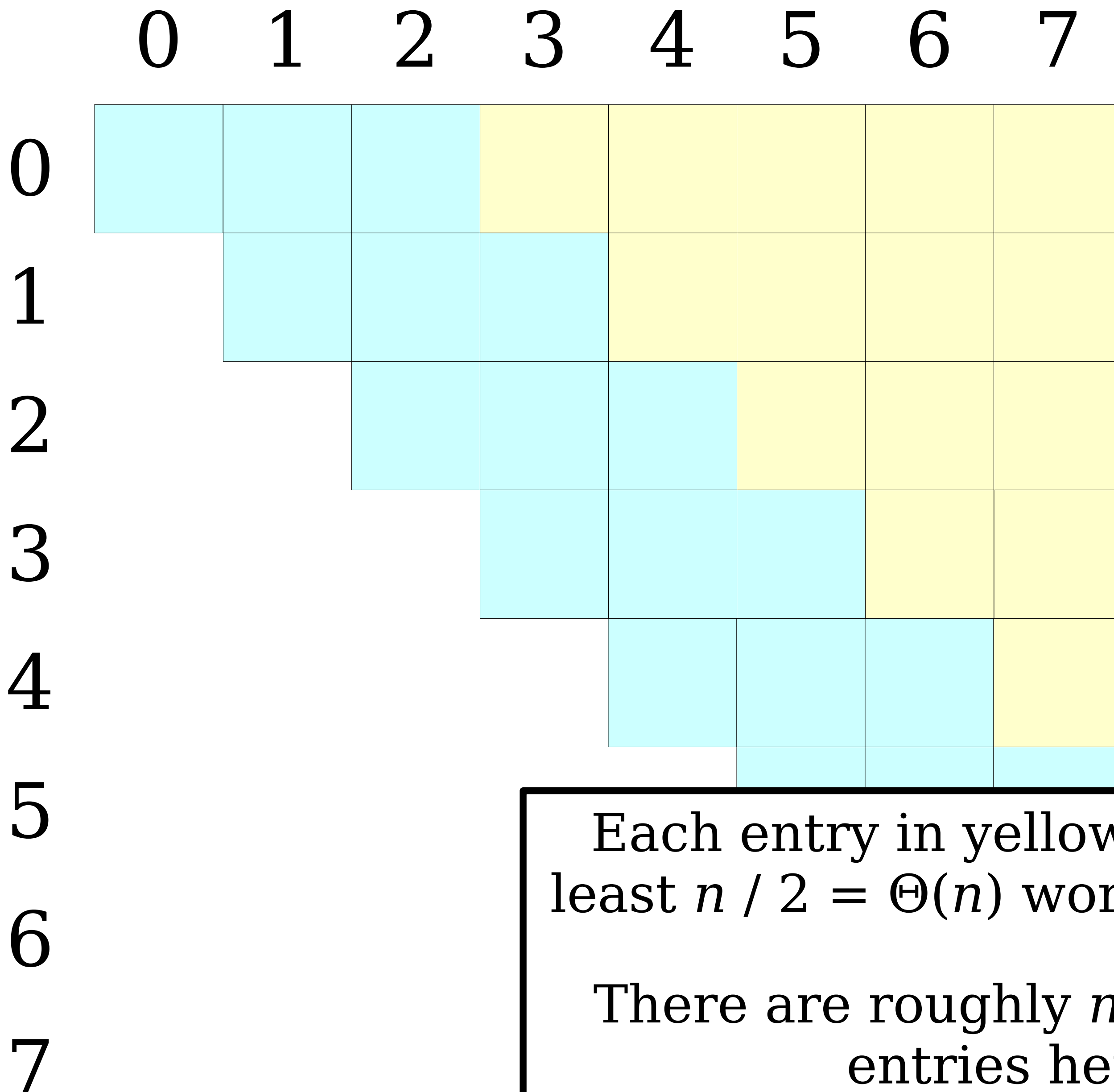
There are roughly $n^2 / 8 = \Theta(n^2)$ entries here.



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Total work required: $\Theta(n^3)$

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

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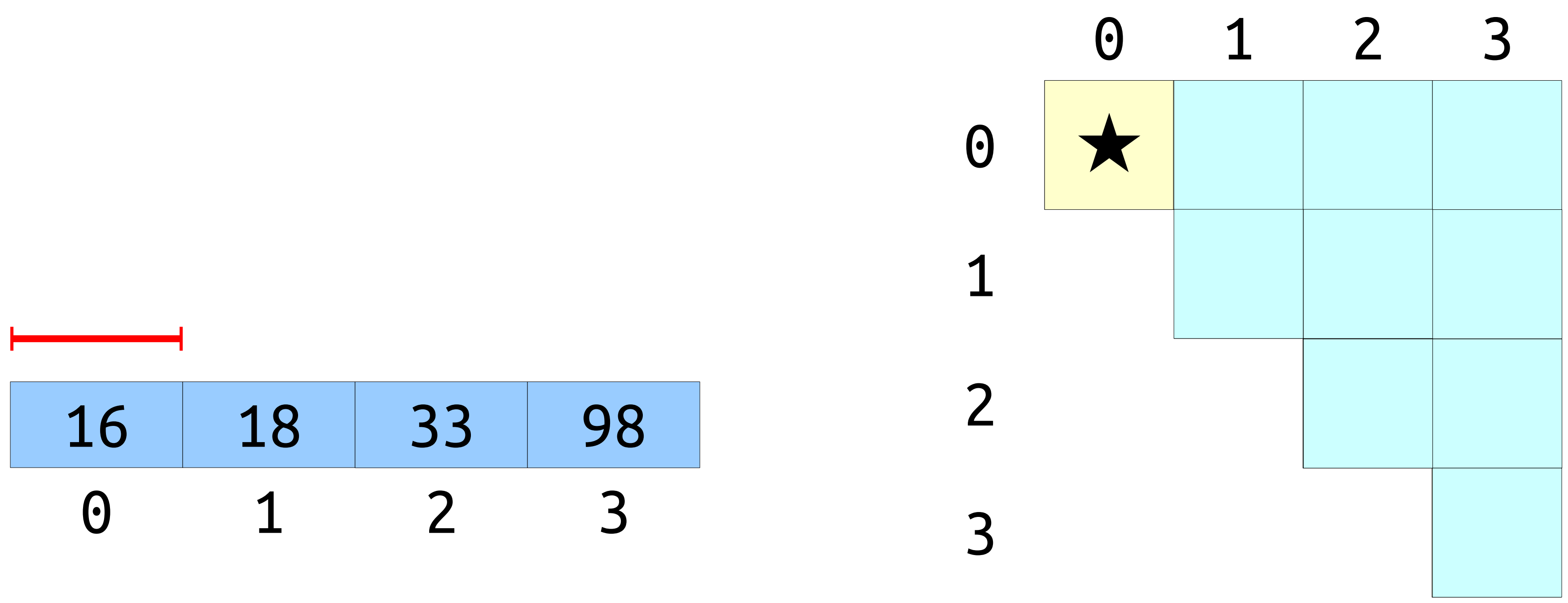

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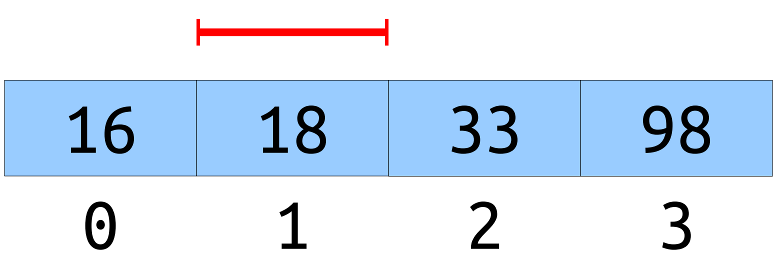

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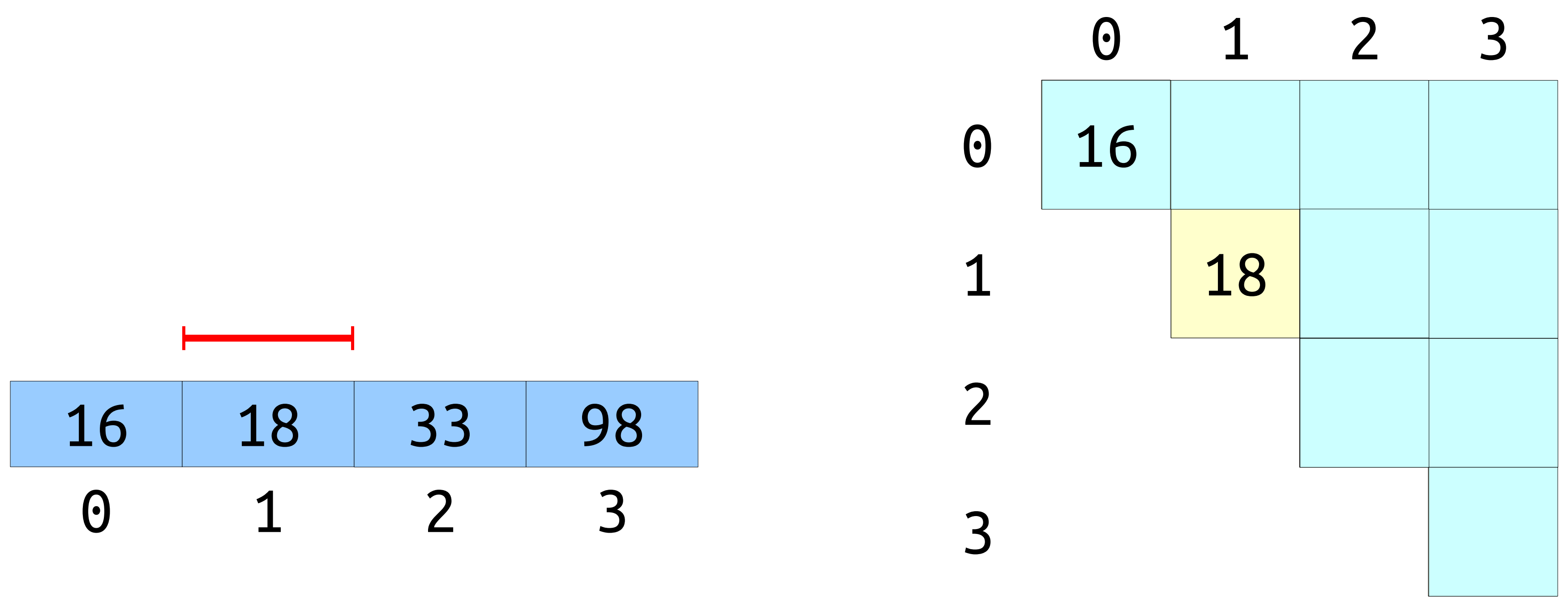

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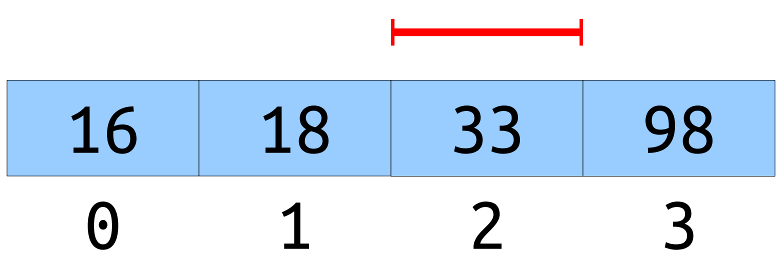

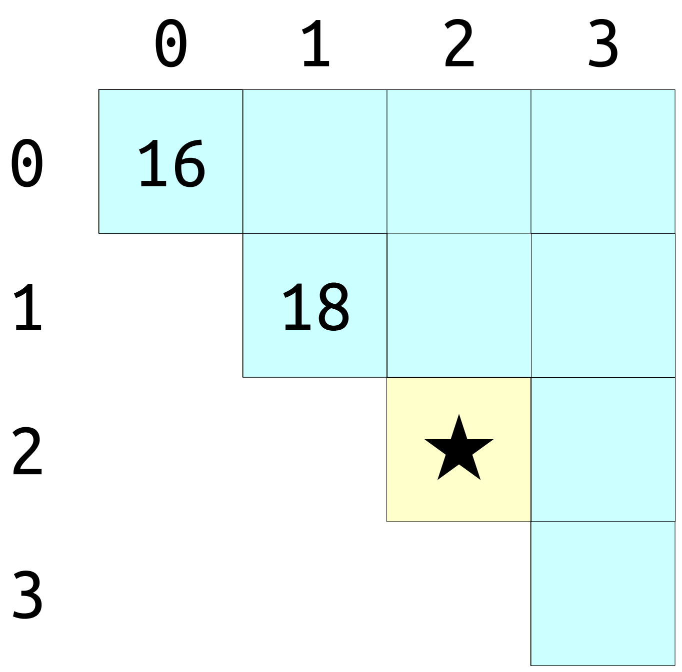

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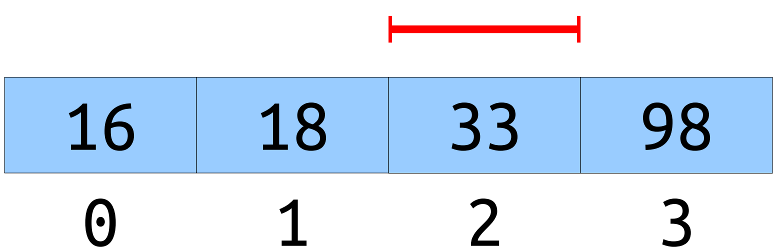

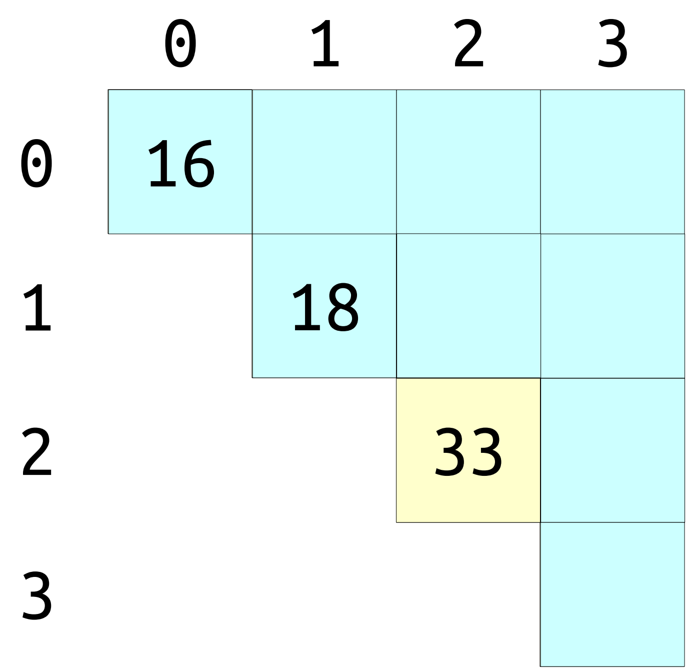

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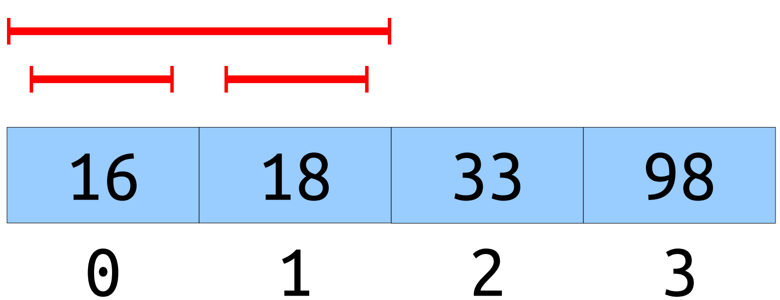

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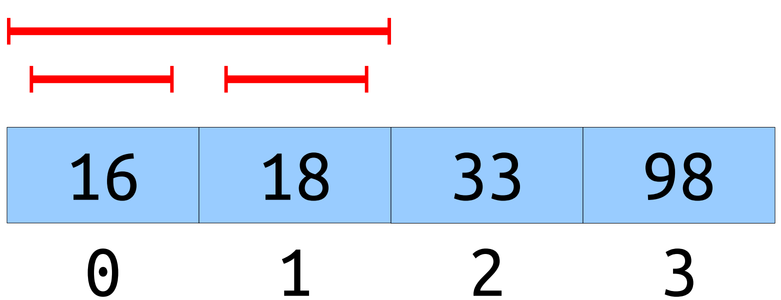

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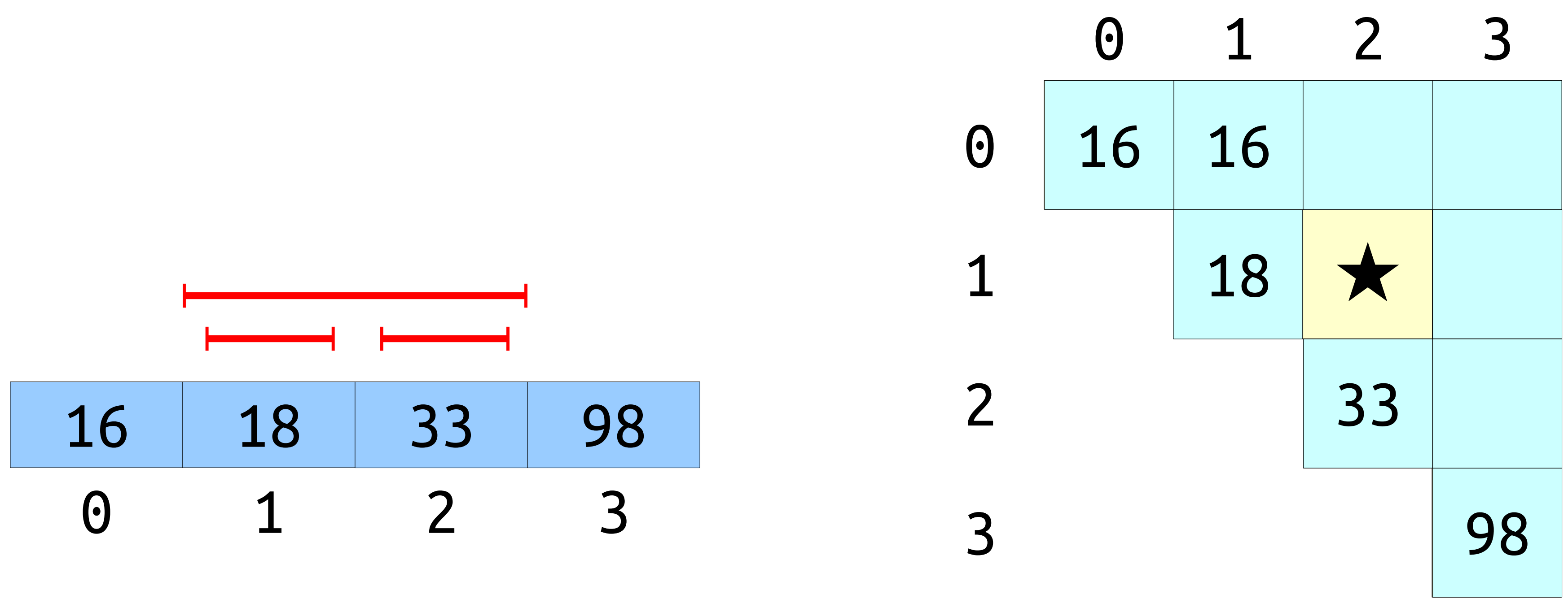

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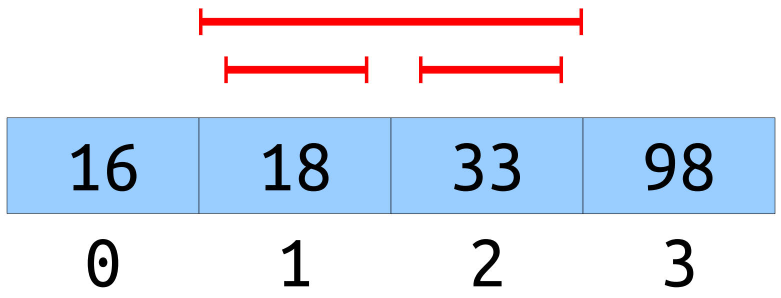

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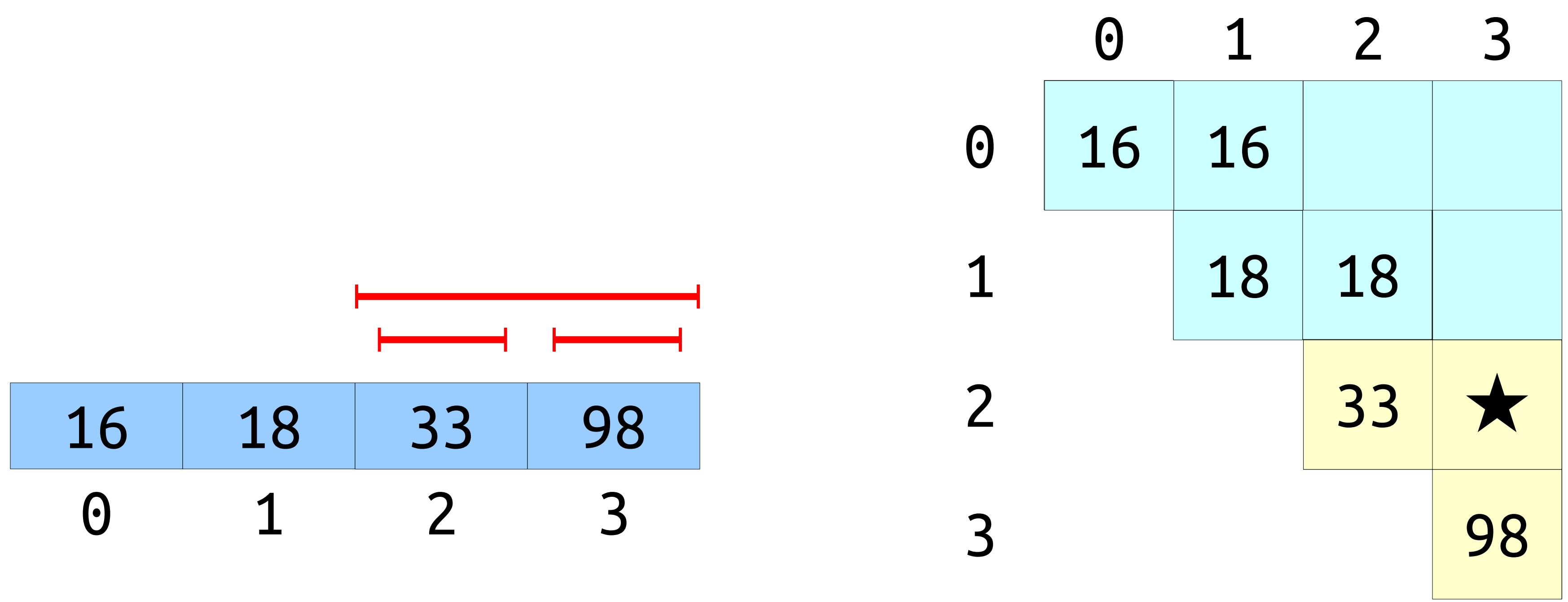

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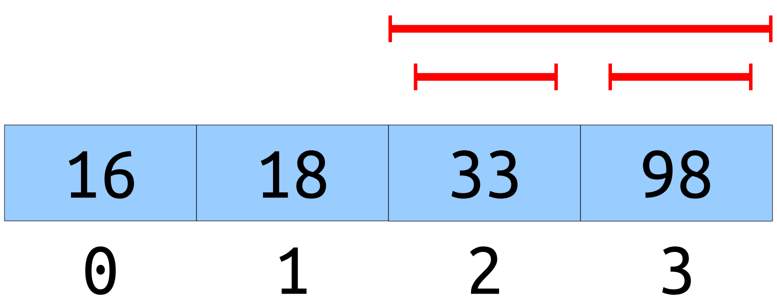

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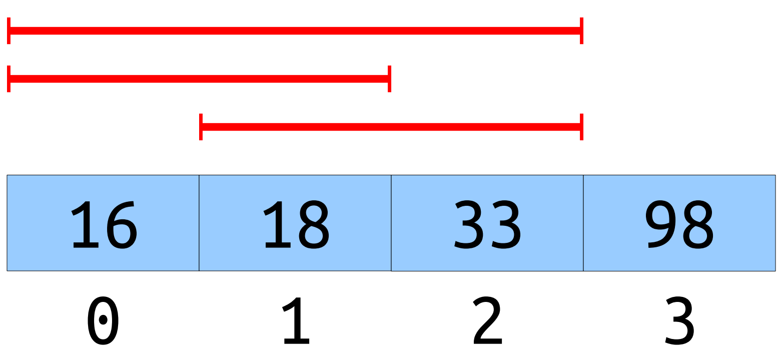

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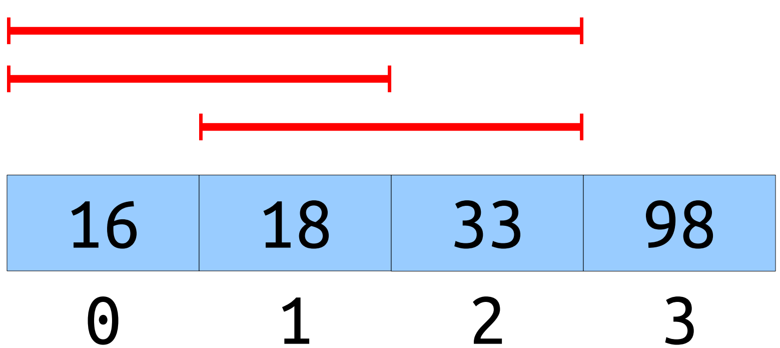

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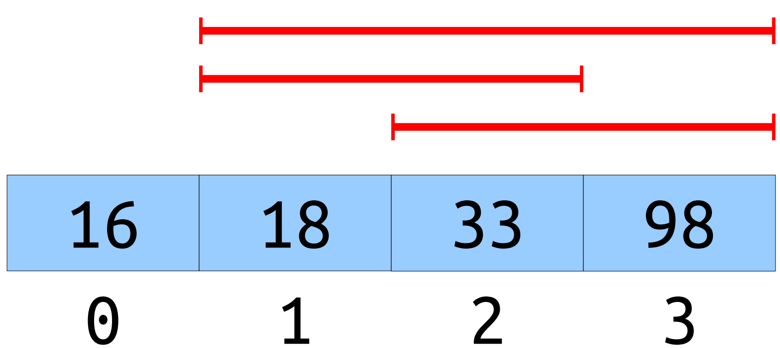

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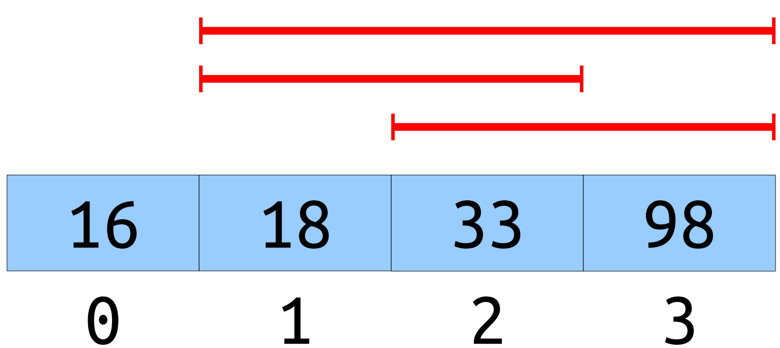

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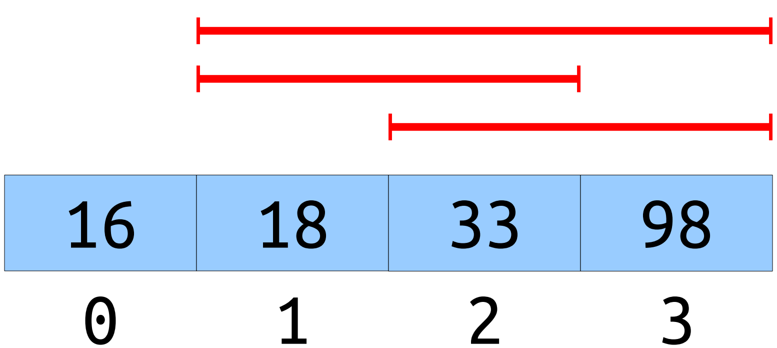

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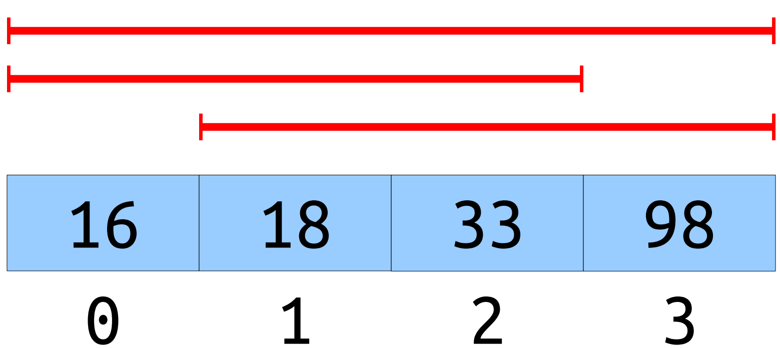

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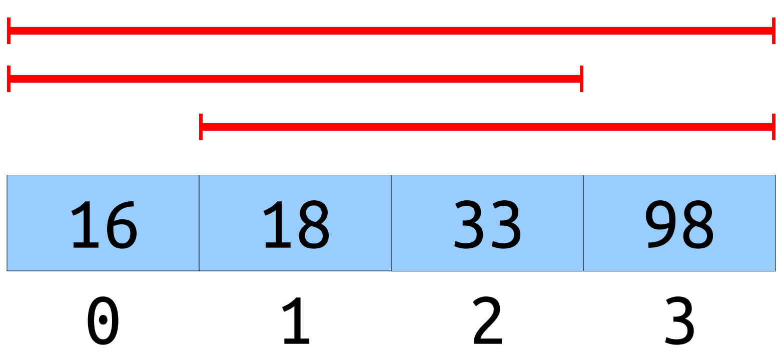

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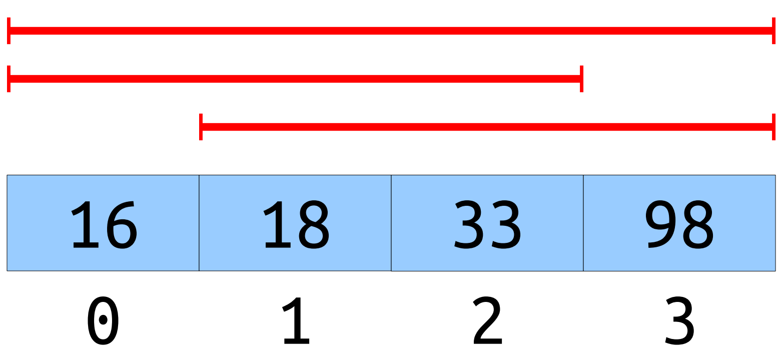

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Some Notation

- We'll say that an RMQ data structure has time complexity **$\langle p(n), q(n) \rangle$** if
  - preprocessing takes time at most $p(n)$ and
  - queries take time at most $q(n)$.
- We now have two RMQ data structures:
  - $\langle O(1), O(n) \rangle$ with no preprocessing.
  - $\langle O(n^2), O(1) \rangle$ with full preprocessing.
- These are two extremes on a curve of tradeoffs: no preprocessing versus full preprocessing.
- ***Question:*** *Is there a "golden mean" between these extremes?*

Another Approach: ***Block Decomposition***

31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97

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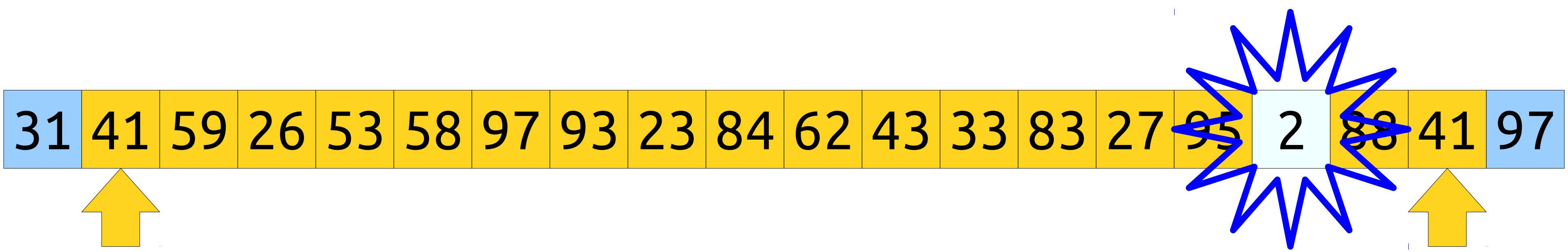
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- Split the input into O(n / b) blocks of some "block size" b.

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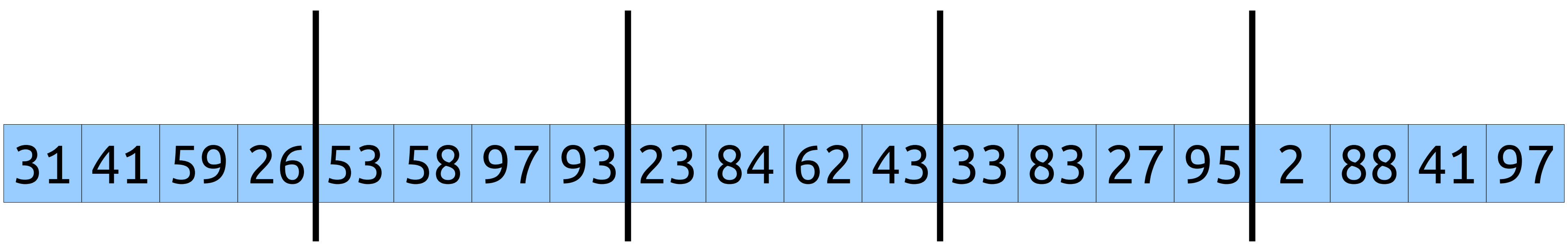

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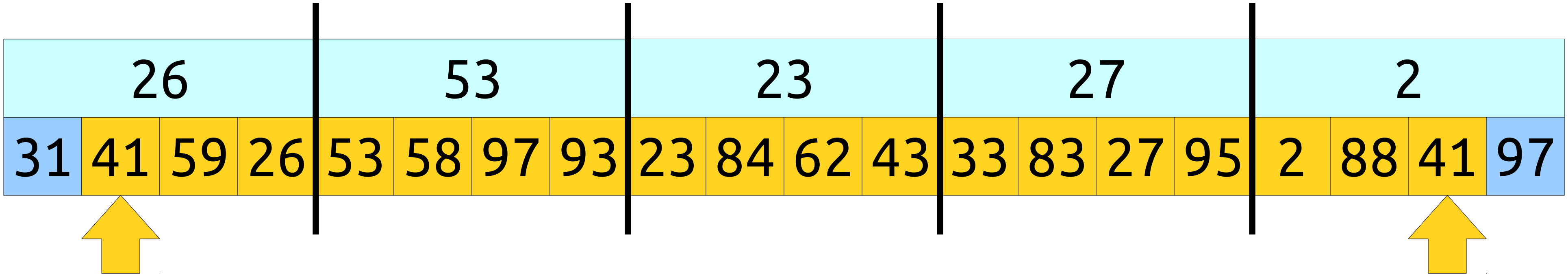

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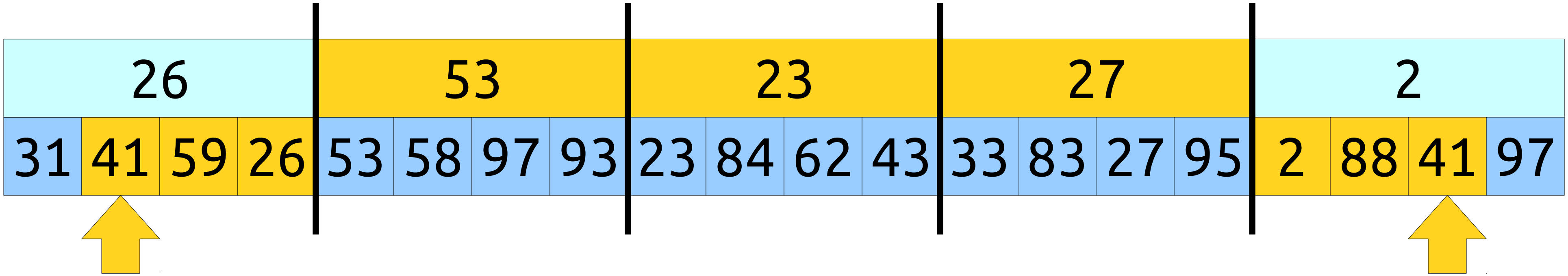

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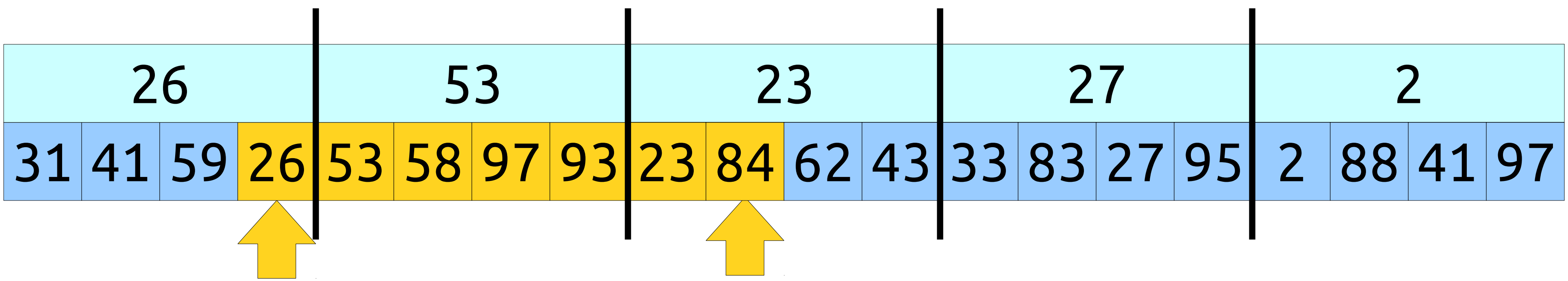

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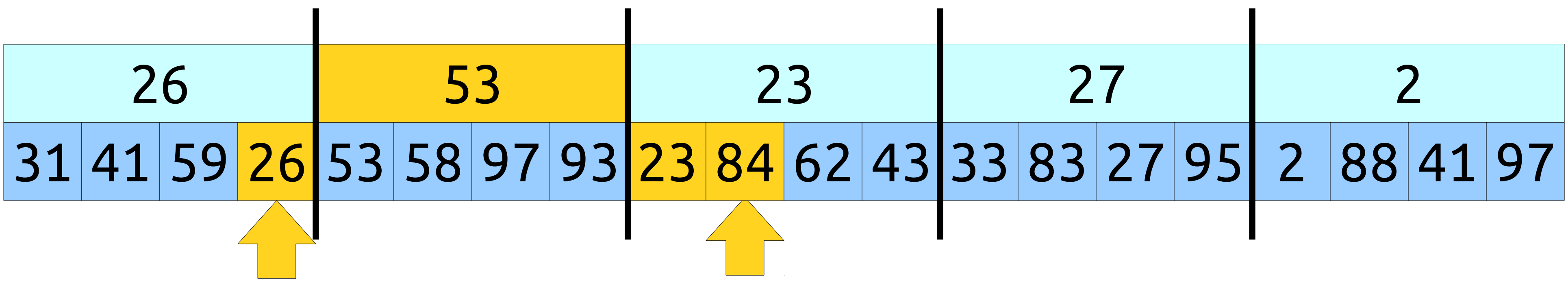

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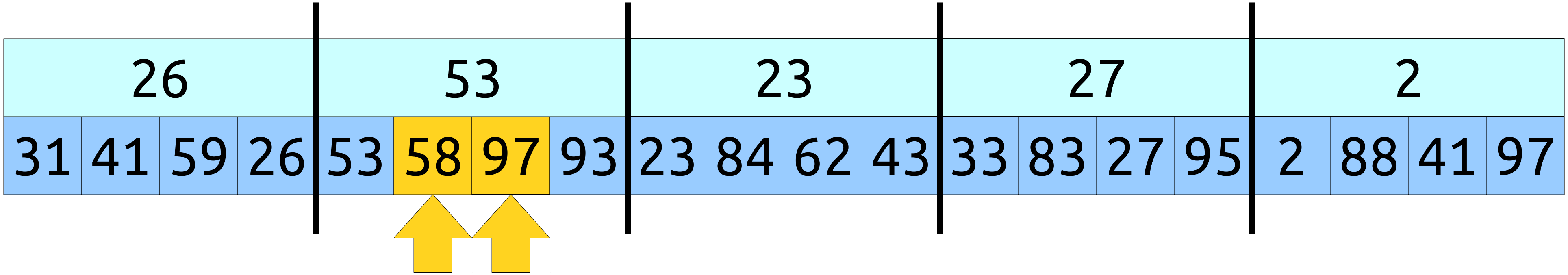
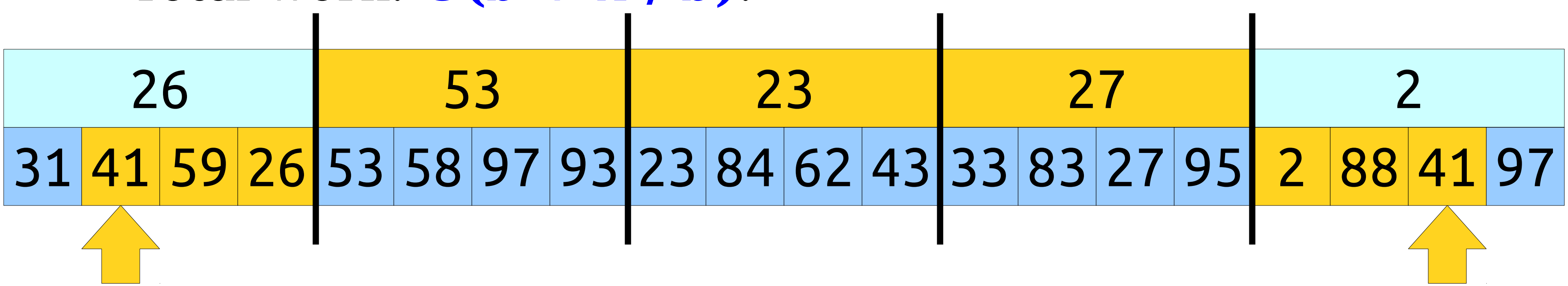


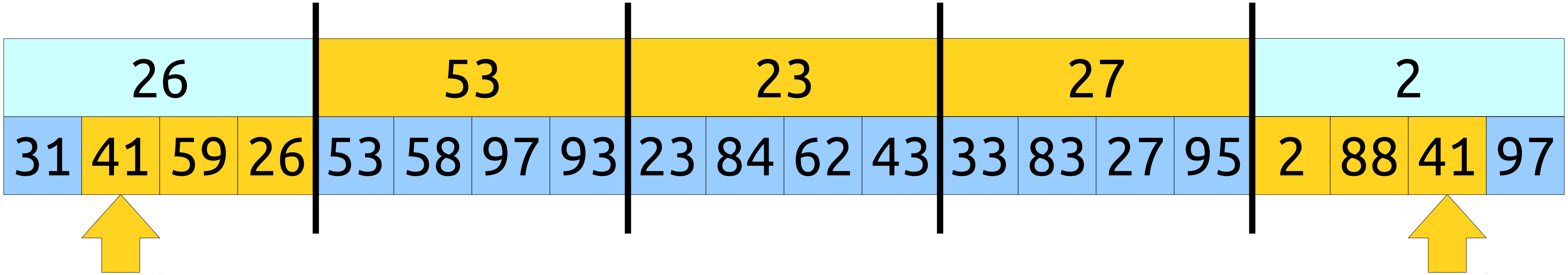
Analyzing the Approach

- Let's analyze this approach in terms of n and b.
- Preprocessing time:
  - O(b) work on O(n / b) blocks to find minima.
  - Total work: **O(n)**.
- Time to evaluate RMQ$_A$(i, j):
  - O(1) work to find block indices (divide by block size).
  - O(b) work to scan inside i and j's blocks.
  - O(n / b) work looking at block minima between i and j.
  - Total work: **O(b + n / b)**.



Intuiting O($\boldsymbol{b}$ + $\boldsymbol{n / b}$)

- As b increases:
  - The $\boldsymbol{b}$ term rises (more elements to scan within each block).
  - The $\boldsymbol{n / b}$ term drops (fewer blocks to look at).
- As b decreases:
  - The $\boldsymbol{b}$ term drops (fewer elements to scan within a block).
  - The $\boldsymbol{n / b}$ term rises (more blocks to look at).
- Is there an optimal choice of b given these constraints?



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Summary of Approaches

- Three solutions so far:
  - Full preprocessing: $\langle O(n^2),\ O(1)\rangle$.
  - Block partition: $\langle O(n),\ O(n^{1/2})\rangle$.
  - No preprocessing: $\langle O(1),\ O(n)\rangle$.
- Modest preprocessing yields modest performance increases.
- ***Question:*** Can we do better?

A Second Approach: ***Sparse Tables***

An Intuition

- The $\langle O(n^2), O(1) \rangle$ solution gives fast queries because every range we might look up has already been precomputed.
- This solution is slow overall because we have to compute the minimum of every possible range.
- ***Question:*** Can we still get constant-time queries without preprocessing all possible ranges?

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26	26	26	26	26
1		41	41	26	26	26	26	26
2			59	26	26	26	26	26
3				26	26	26	26	26
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26	26	26	26	26
1		41	41	26	26	26	26	26
2			59	26	26	26	26	26
3				26	26	26	26	26
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

0	1	2	3	4	5	6	7
31	41	59	26	53	58	97	93

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

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31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
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7								93

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	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
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0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
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6							97	93
7								93

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0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
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0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
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	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
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0	31	31	31	26				
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	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
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	0	1	2	3	4	5	6	7
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1		41	41	26	26			
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3				26	26	26	26	
4					53	53	53	53
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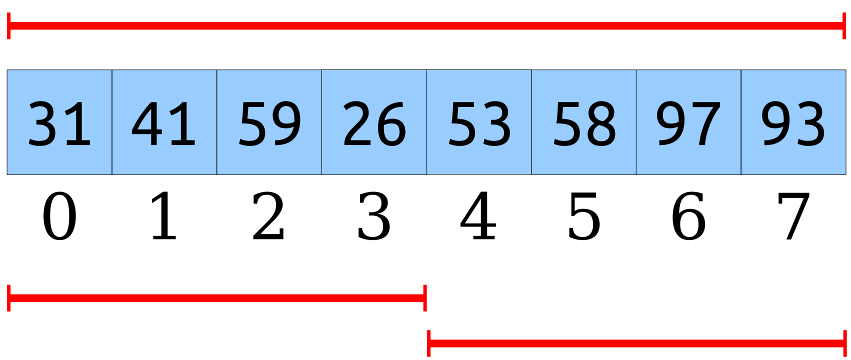
An Observation



	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
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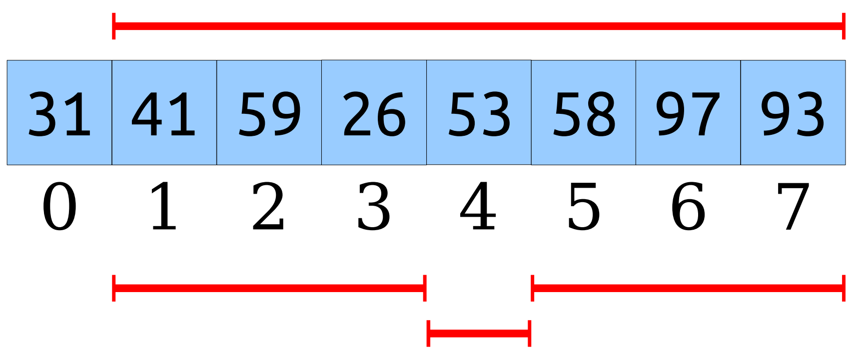

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				★
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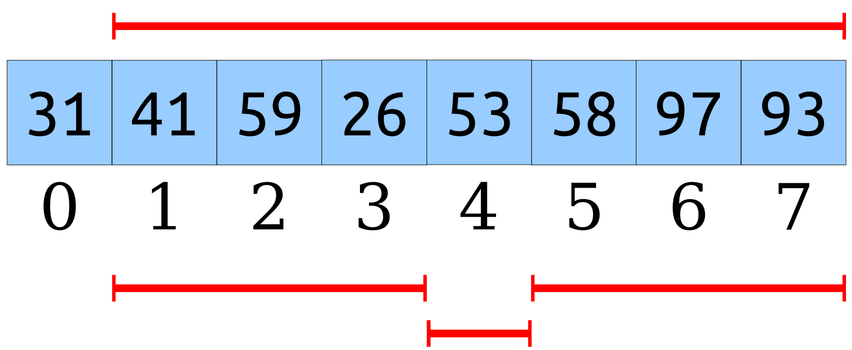
An Observation



	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				★
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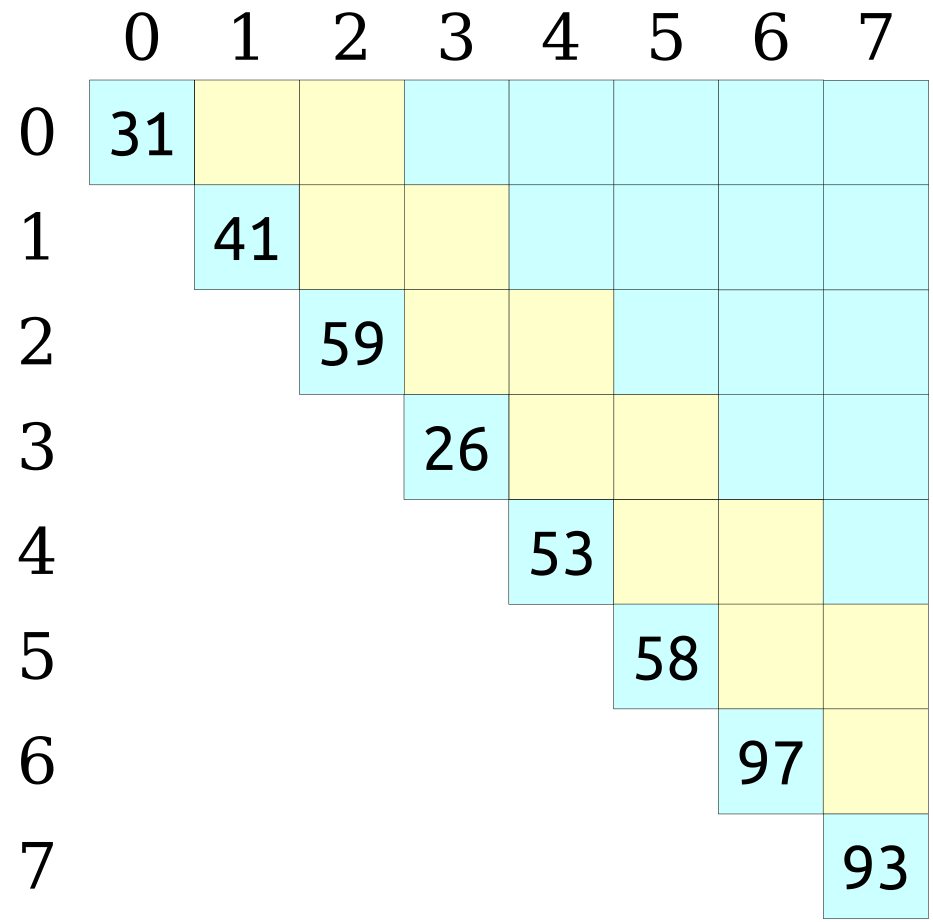

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	0	1	2	3	4	5	6	7
0	31							
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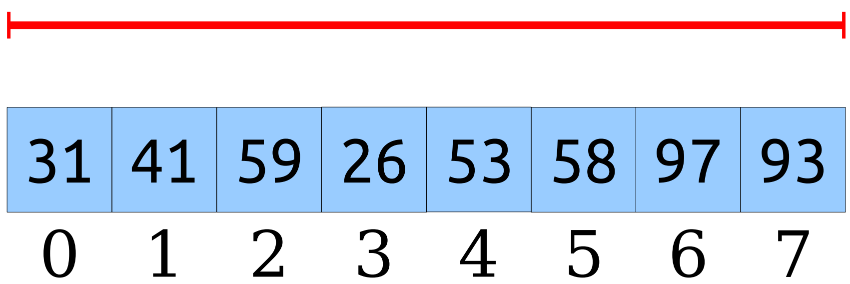
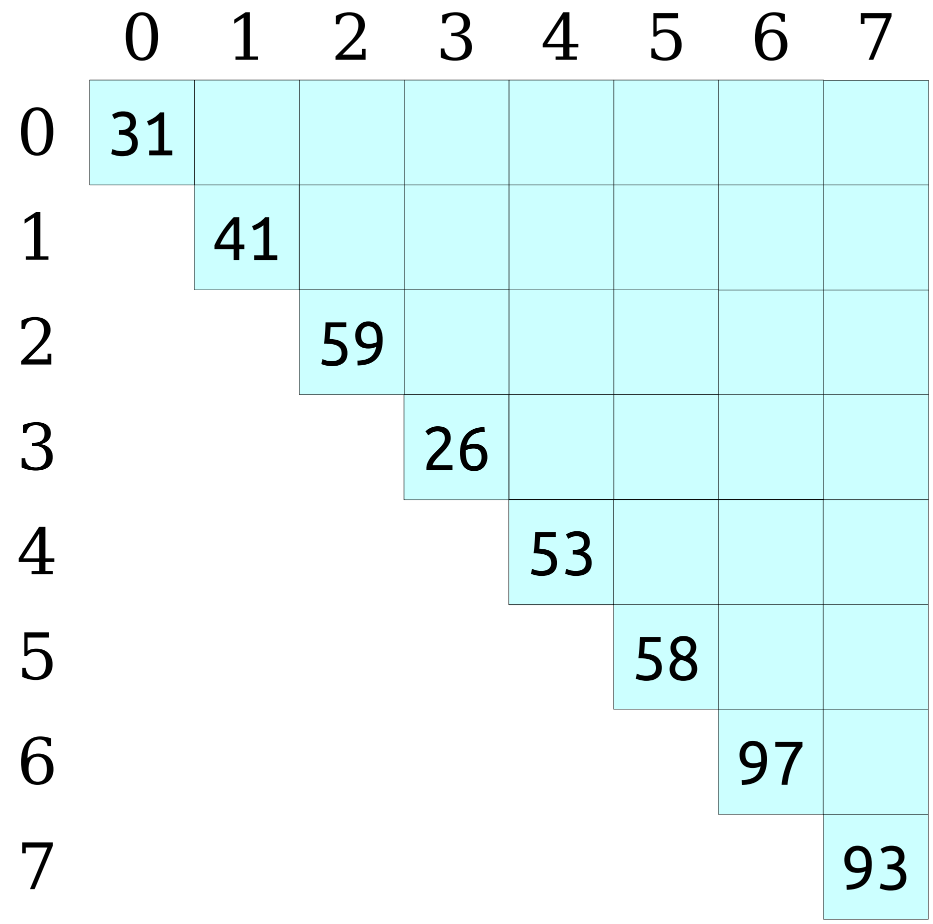
An Observation





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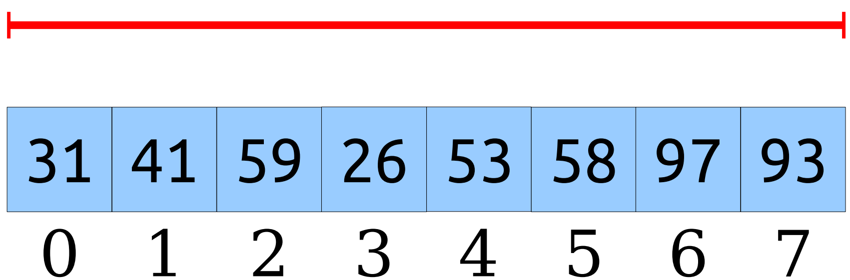

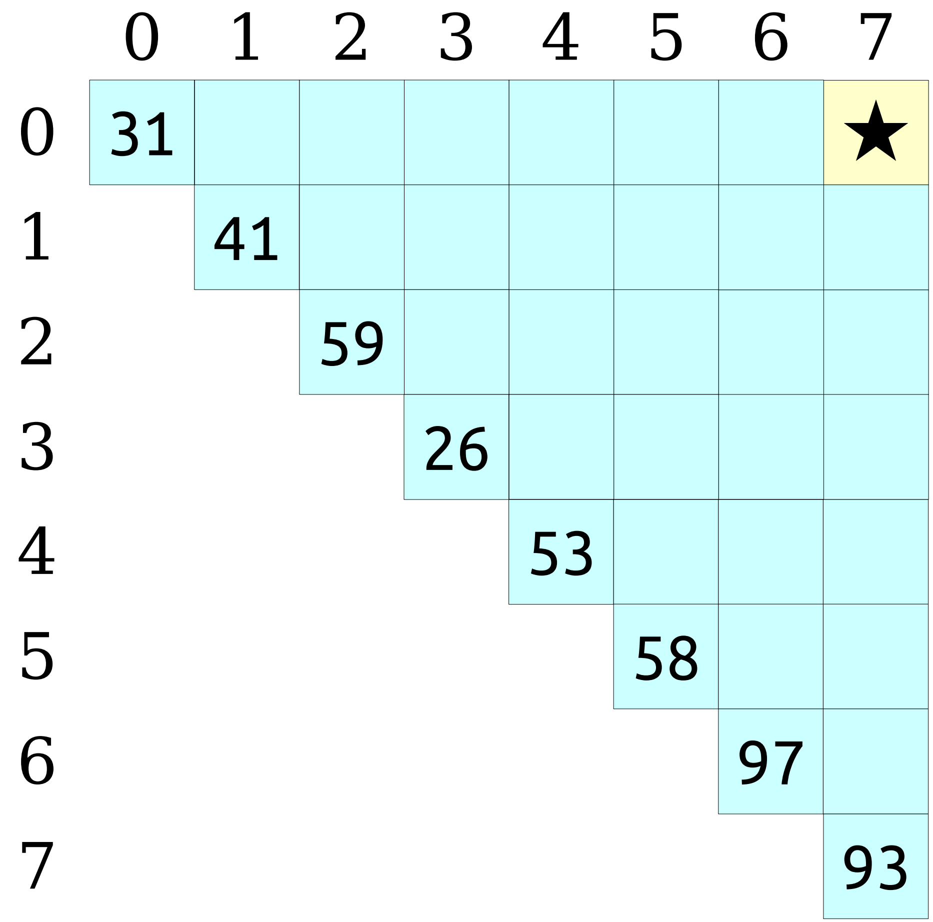

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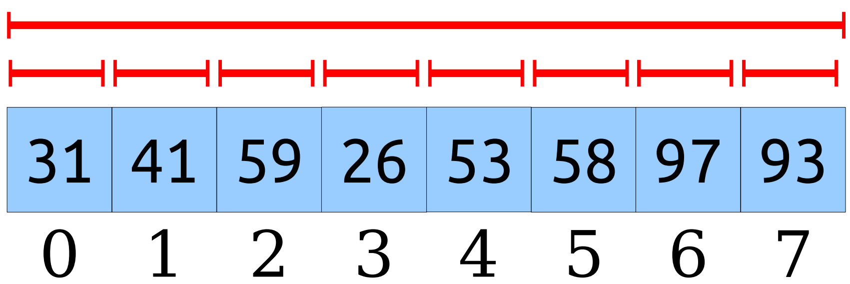
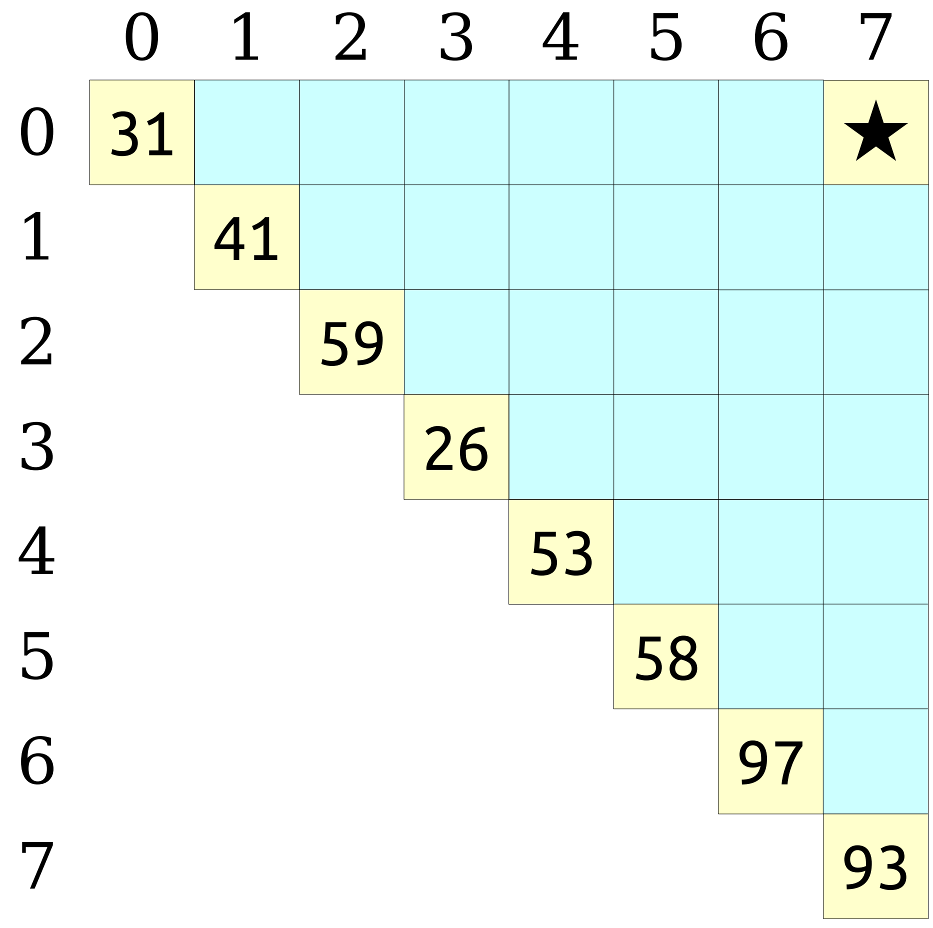
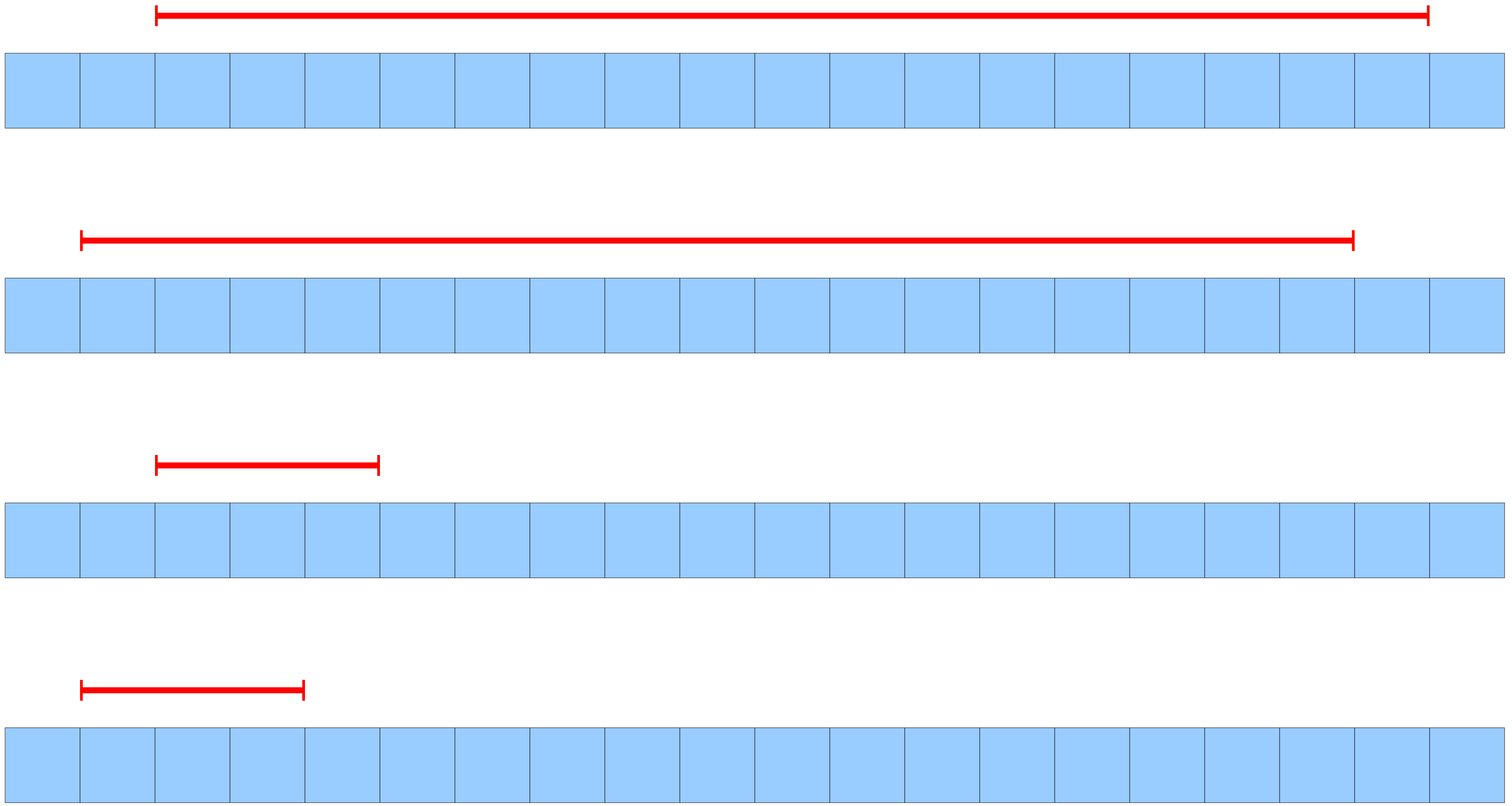




The Intuition

- It's still possible to answer any query in time O(1) without precomputing RMQ over all ranges.
- If we precompute the answers over too many ranges, the preprocessing time will be too large.
- If we precompute the answers over too few ranges, the query time won't be O(1).
- ***Goal:*** Precompute RMQ over a set of ranges such that
  - There are $o(n^2)$ total ranges, but
  - there are enough ranges to support O(1) query times.

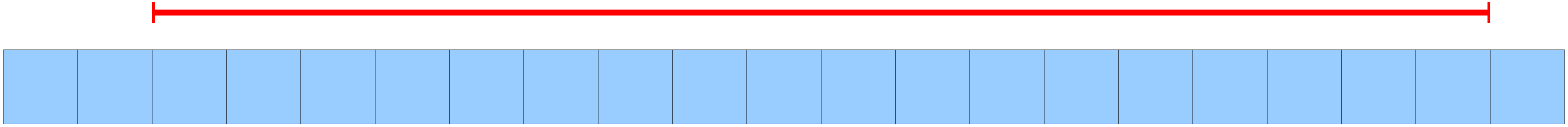
Some Observations



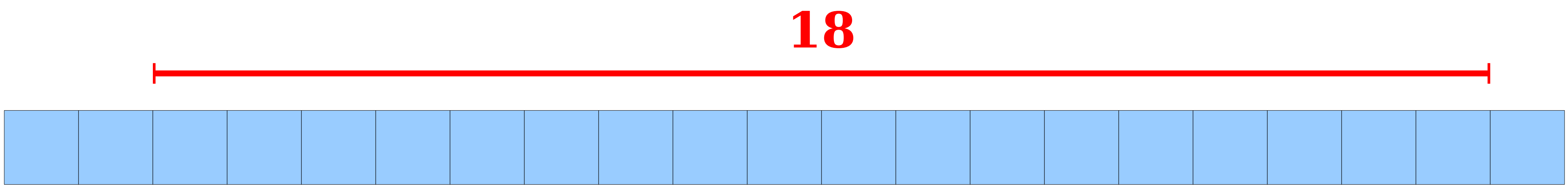
The Approach

- For each index i, compute RMQ for ranges starting at i of size 1, 2, 4, 8, 16, …, 2^k as long as they fit in the array.
  - Gives both large and small ranges starting at any point in the array.
  - Only $O(\log n)$ ranges computed for each array element.
  - Total number of ranges: $O(n \log n)$.
- ***Claim:*** Any range in the array can be formed as the union of two of these ranges.

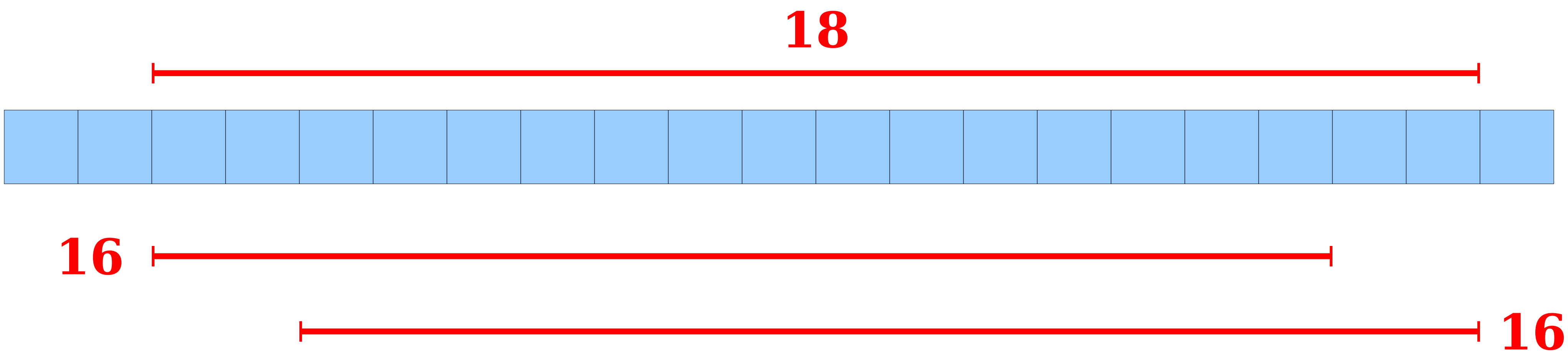
Creating Ranges



Creating Ranges



Creating Ranges



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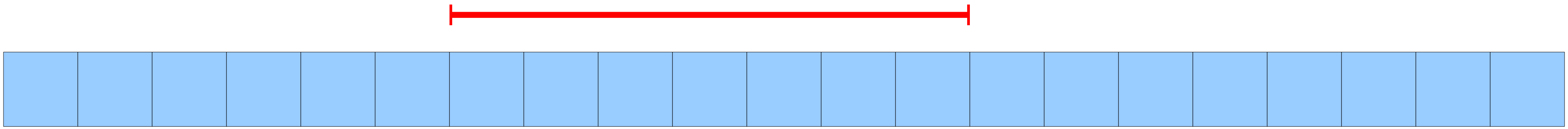

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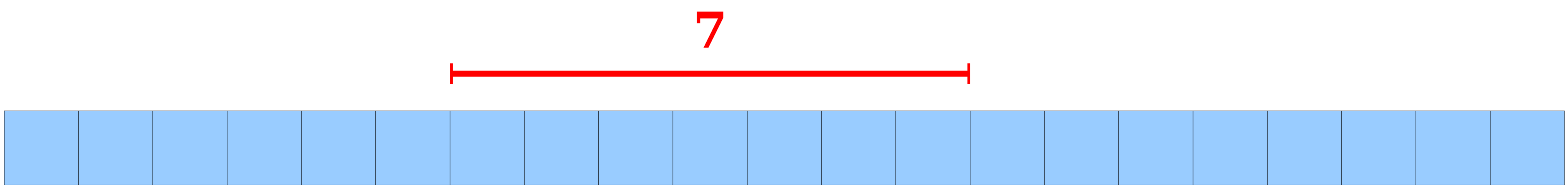

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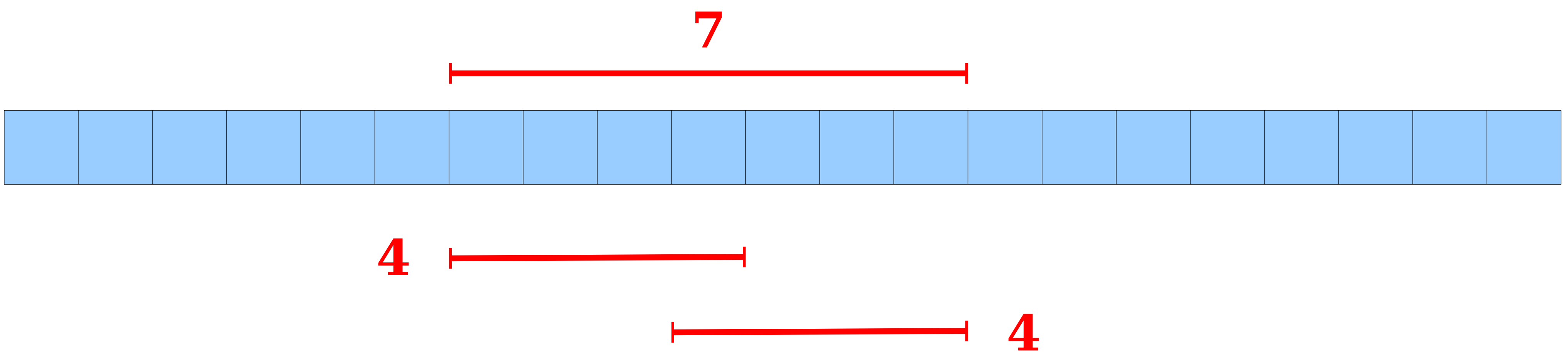

Doing a Query

- To answer $\mathrm{RMQ}_A(i, j)$:
  - Find the largest k such that $2^k \le j - i + 1$.
    - With the right preprocessing, this can be done in time O(1); you'll figure out how in an upcoming assignment. 😃
  - The range $[i, j]$ can be formed as the overlap of the ranges $[i, i + 2^k - 1]$ and $[j - 2^k + 1, j]$.
  - Each range can be looked up in time O(1).
  - Total time: **O(1)**.

Precomputing the Ranges

- There are O(*n* log *n*) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(*n* log *n*).

	2^0	2^1	2^2	2^3
0				
1				
2				
3				
4				
5				
6				
7				

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

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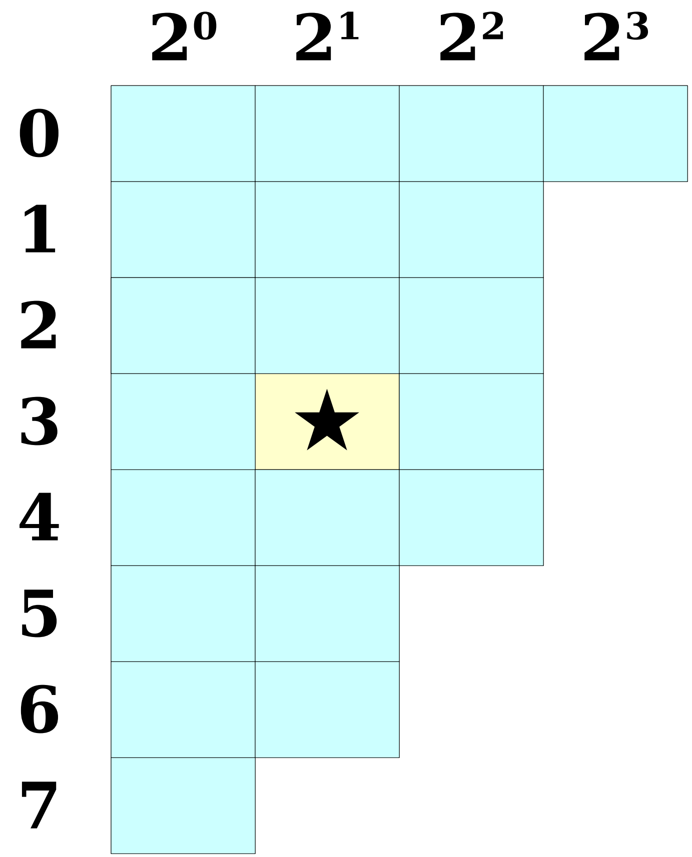

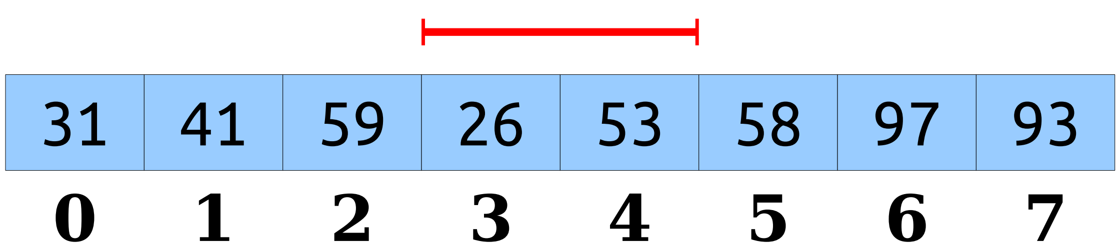

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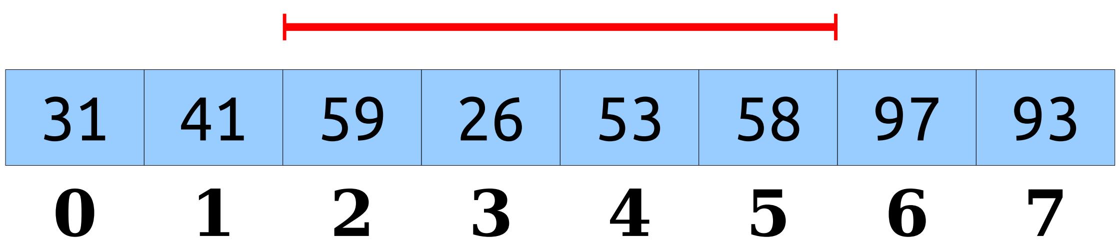

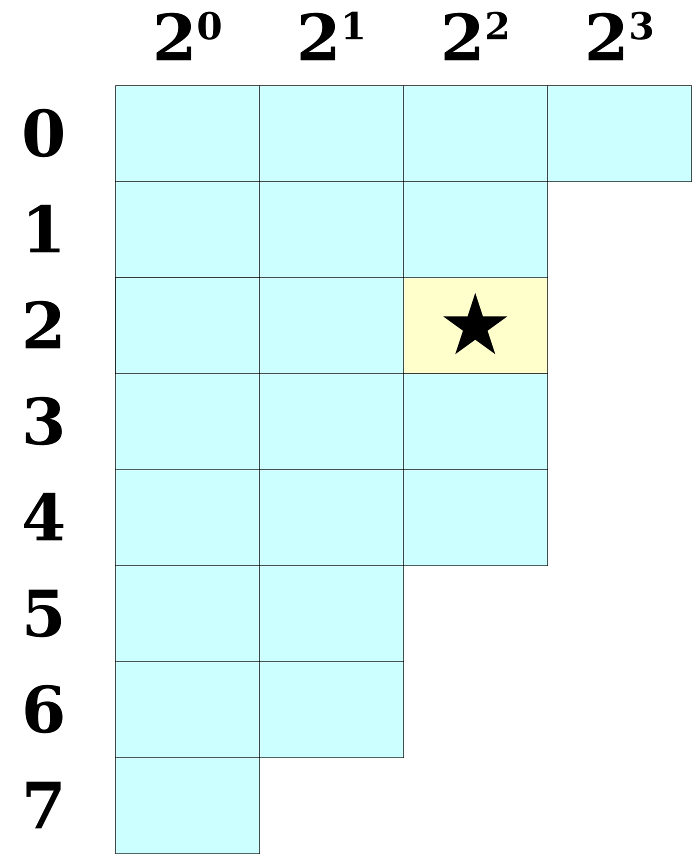

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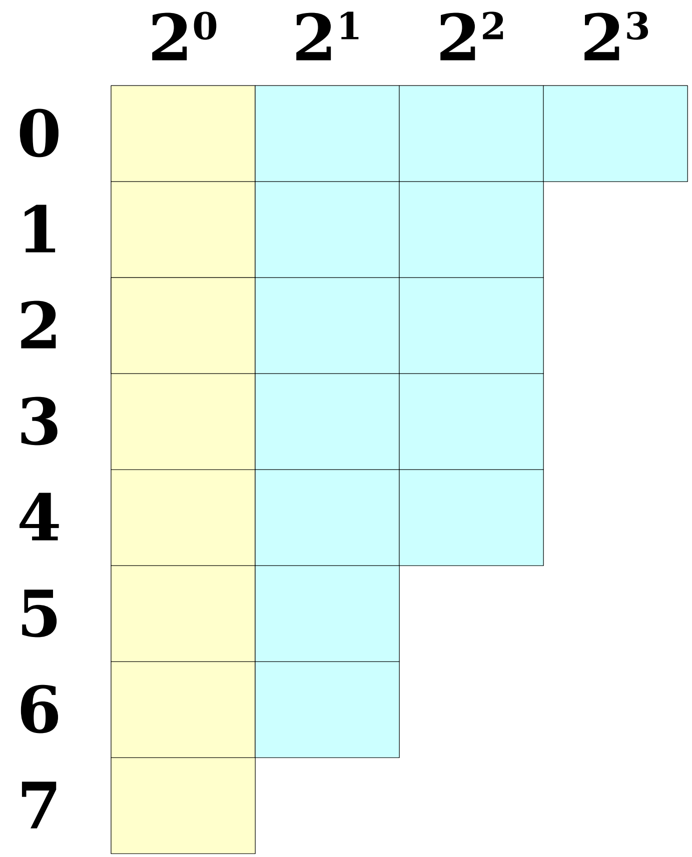

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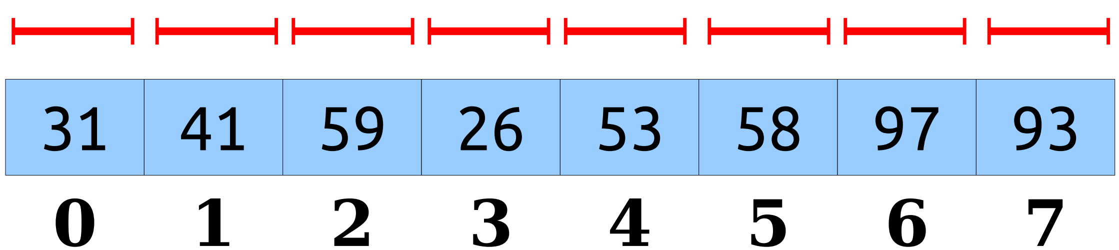

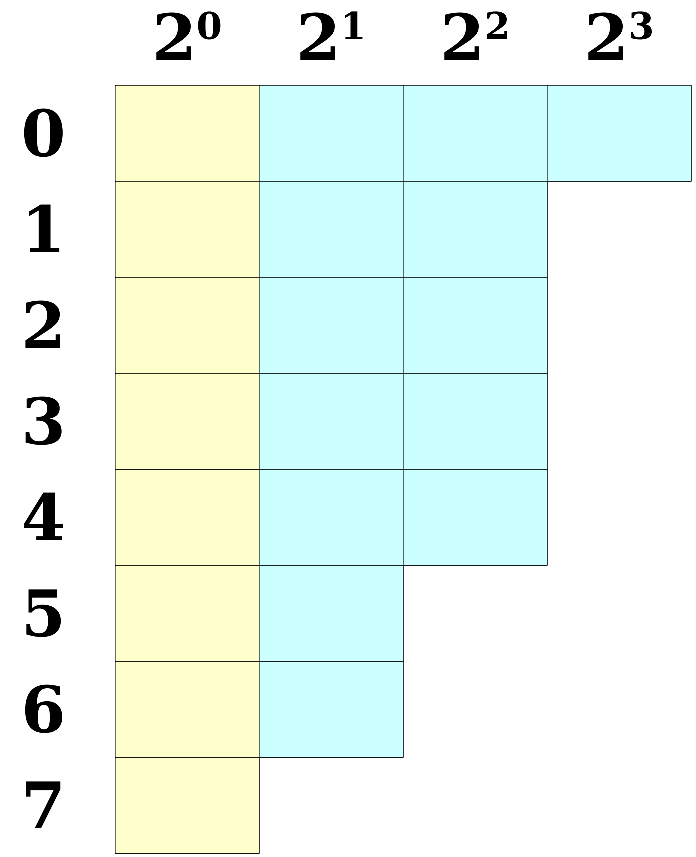

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0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

	2^0	2^1	2^2	2^3
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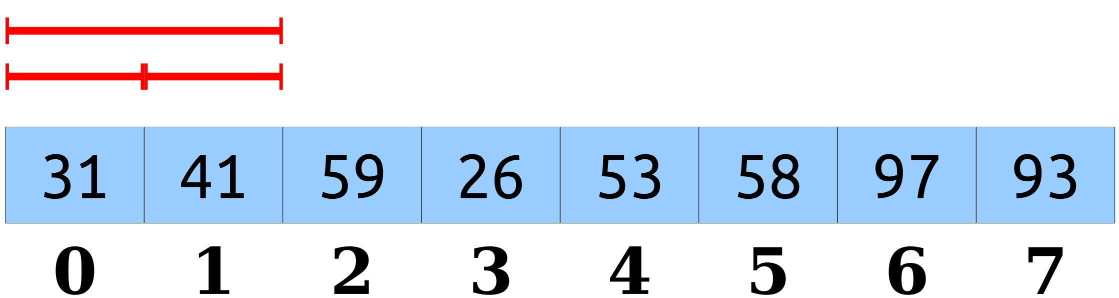

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Precomputing the Ranges

- There are O(*n* log *n*) ranges to precompute.
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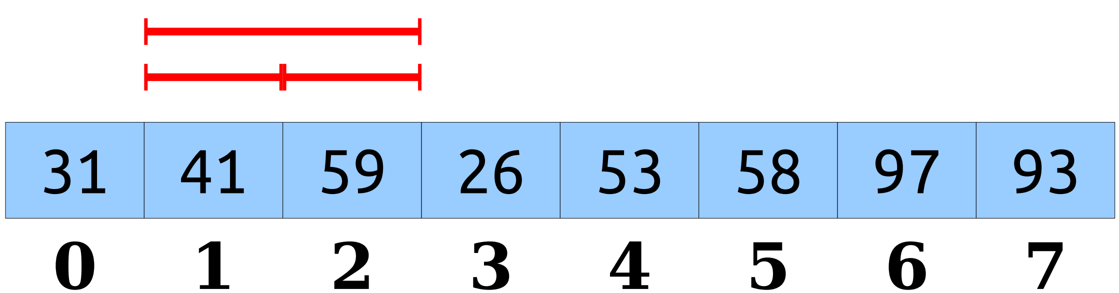

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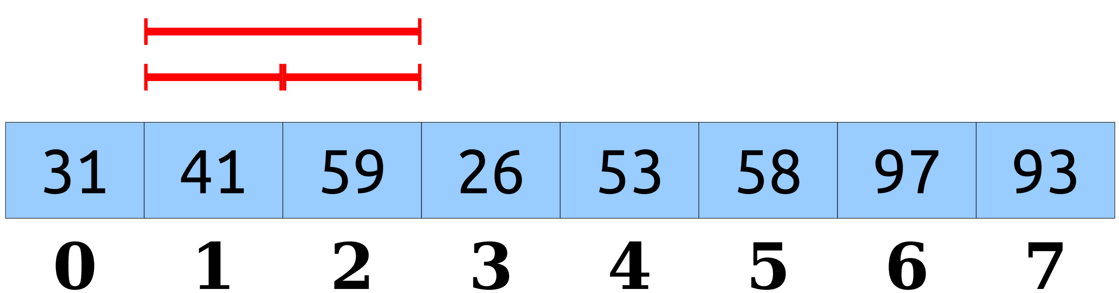

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Sparse Tables

- This data structure is called a ***sparse table***.
- It gives an **⟨O(*n* log *n*), O(1)⟩** solution to RMQ.
- This is asymptotically better than precomputing all possible ranges!

The Story So Far

- We now have the following solutions for RMQ:
  - Precompute all: $\langle O(n^2), \quad O(1)\rangle$.
  - Sparse table: $\langle O(n \log n), \quad O(1)\rangle$.
  - Blocking: $\langle O(n), \quad O(n^{1/2})\rangle$.
  - Precompute none: $\langle O(1), \quad O(n)\rangle$.
- ***Can we do better?***

A Third Approach: ***Hybrid Strategies***

Blocking Revisited

<table>
<tr><td colspan="3">31</td><td colspan="3">26</td><td colspan="3">23</td><td colspan="3">62</td><td colspan="3">27</td></tr>
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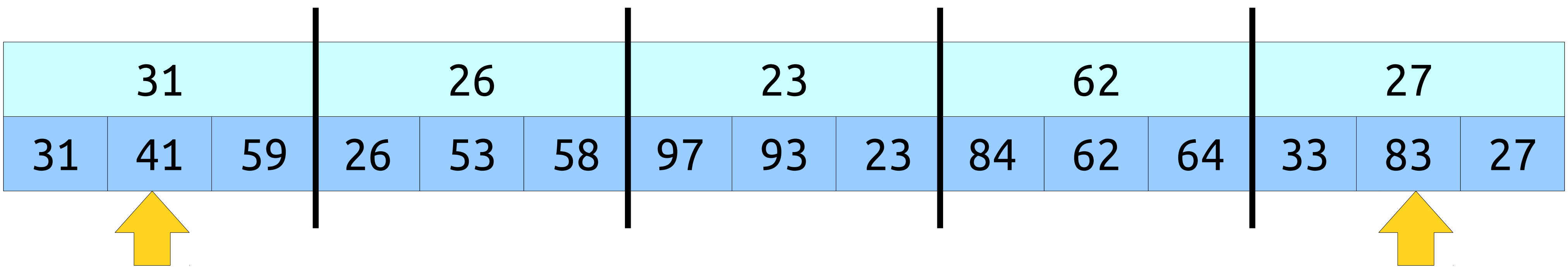
</table>

Blocking Revisited



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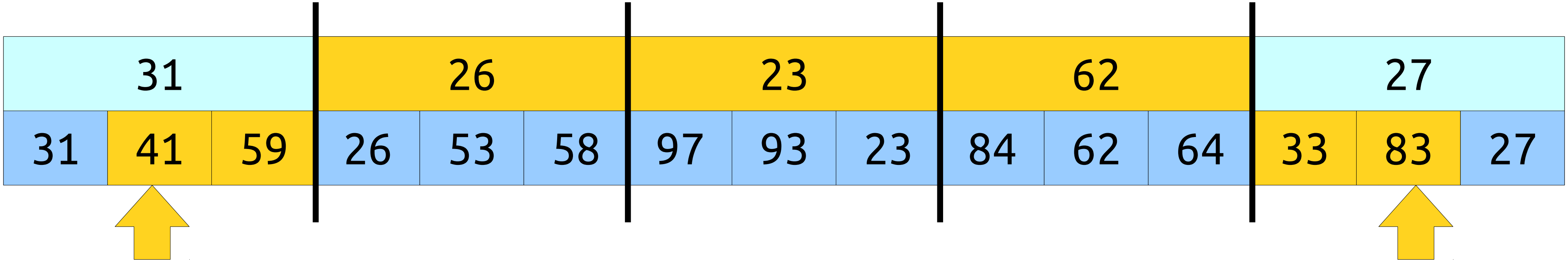

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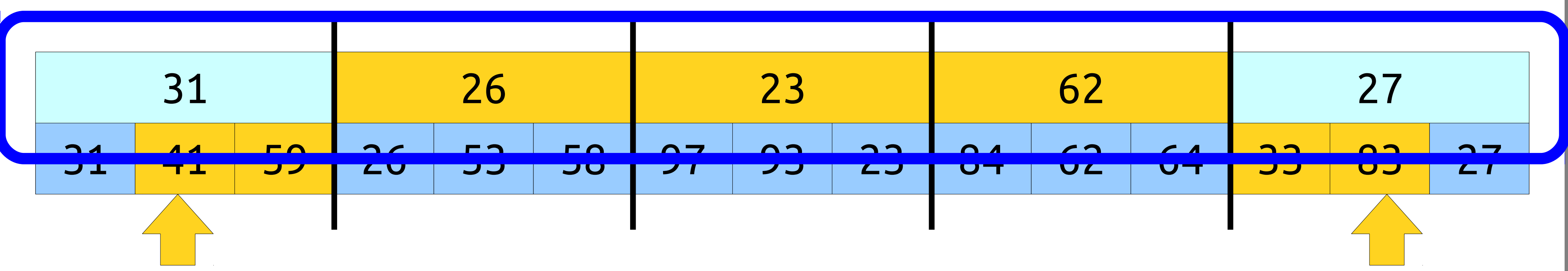
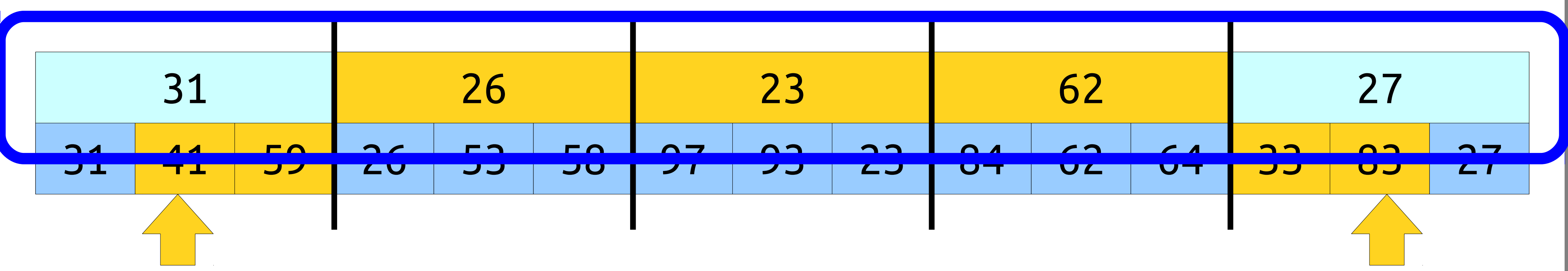


Blocking Revisited

This is just RMQ on the block minima!



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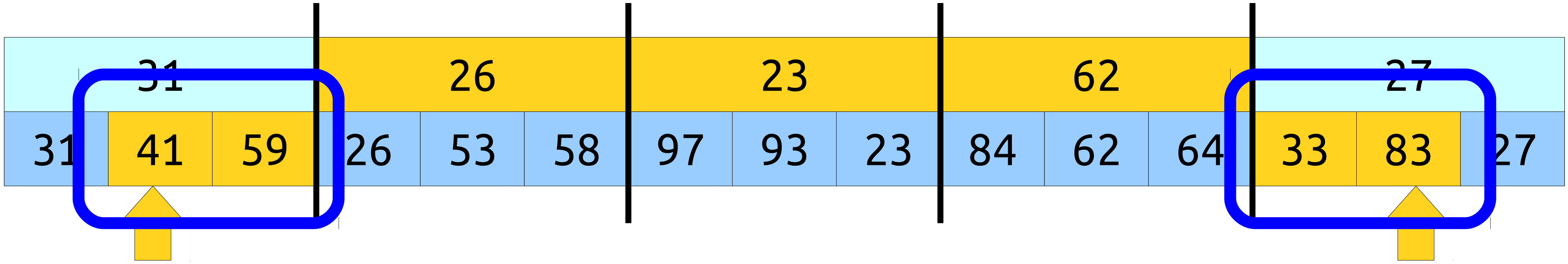

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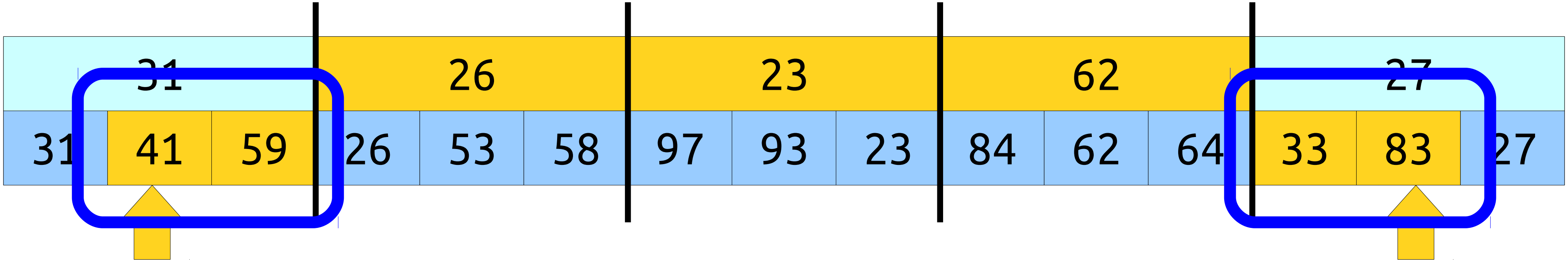
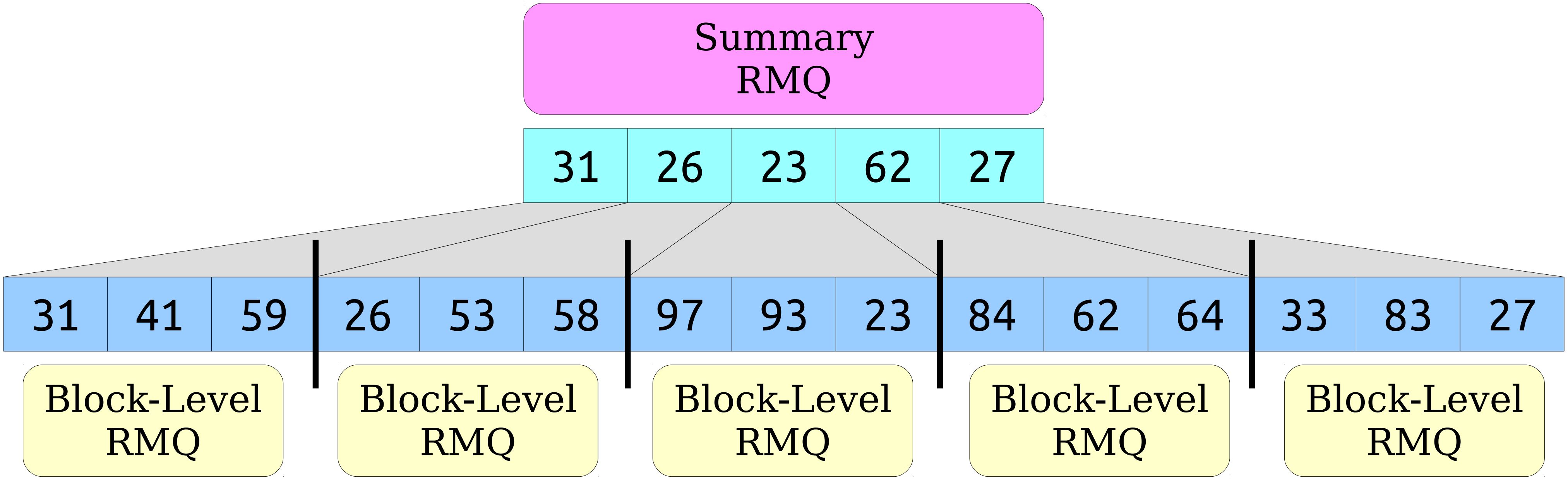


This is just RMQ inside the blocks!

The Framework

- Split the input into blocks of size b.
- Form an array of the block minima.
- Construct a “summary” RMQ structure over the block minima.
- Construct “block” RMQ structures for each block.
- Aggregate the results together.



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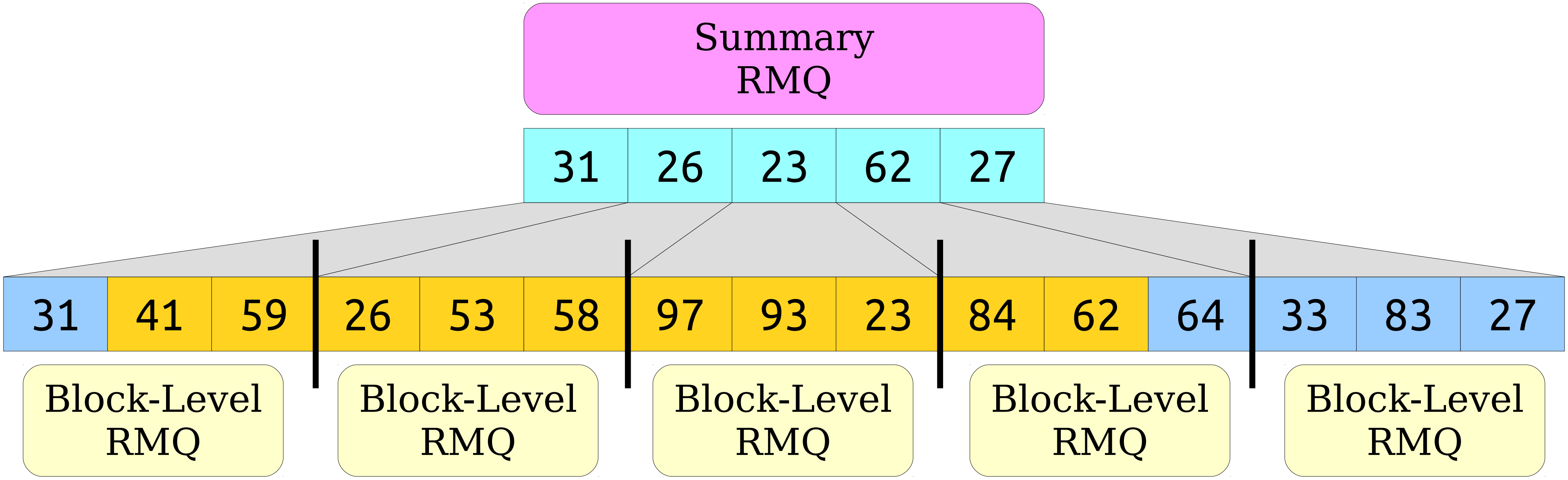

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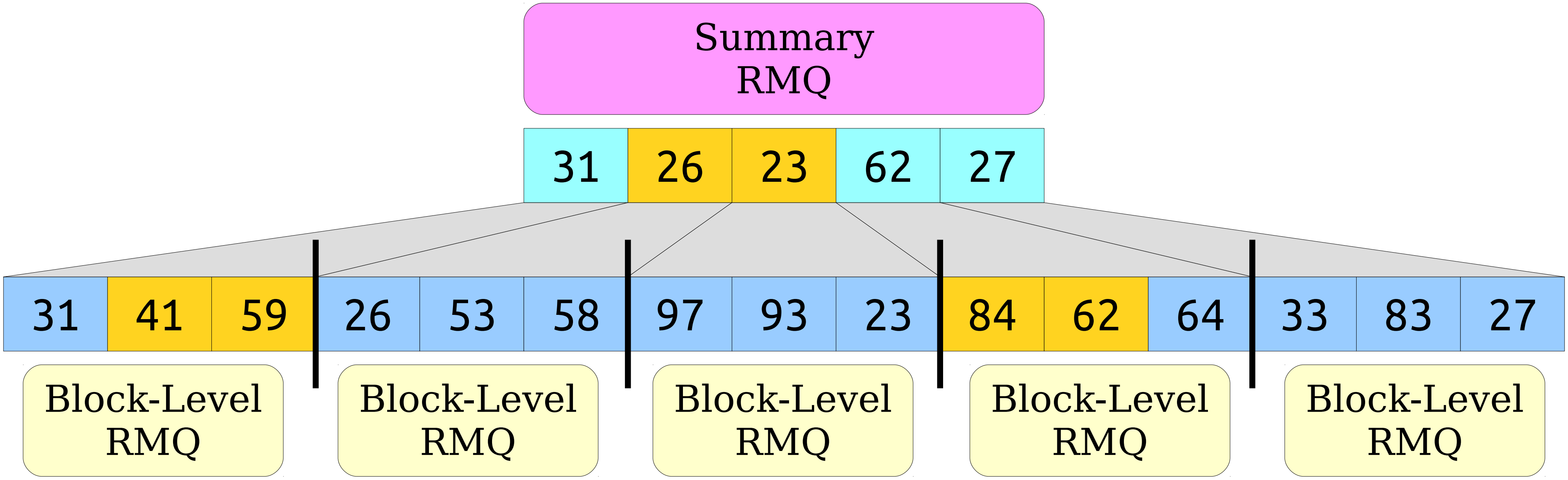

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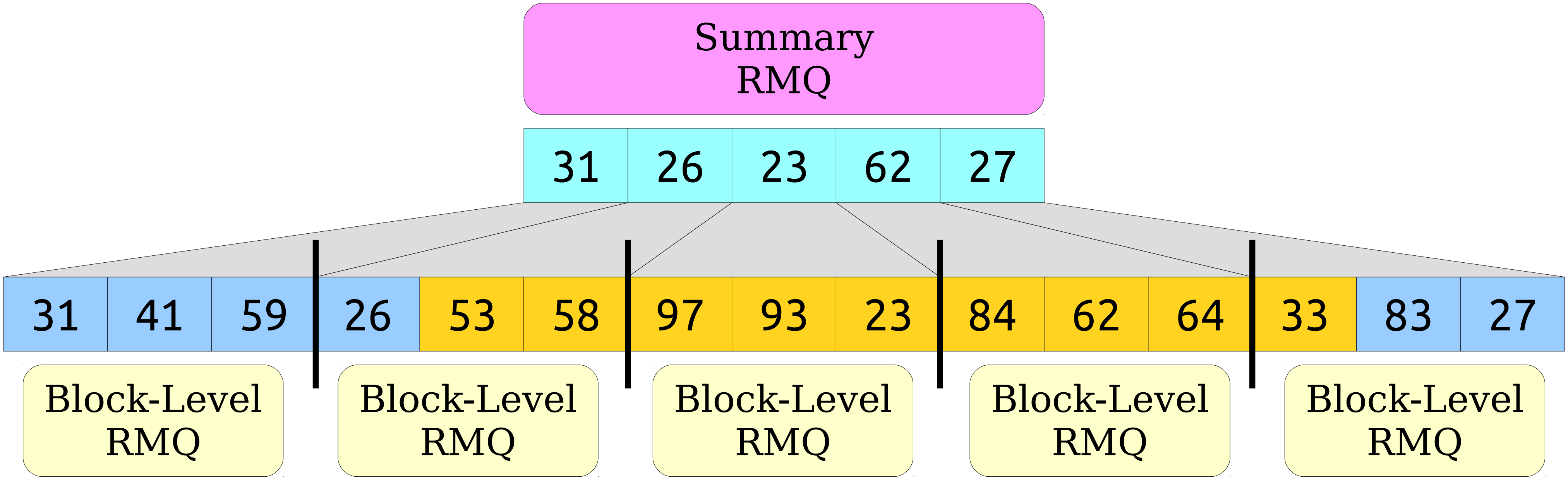

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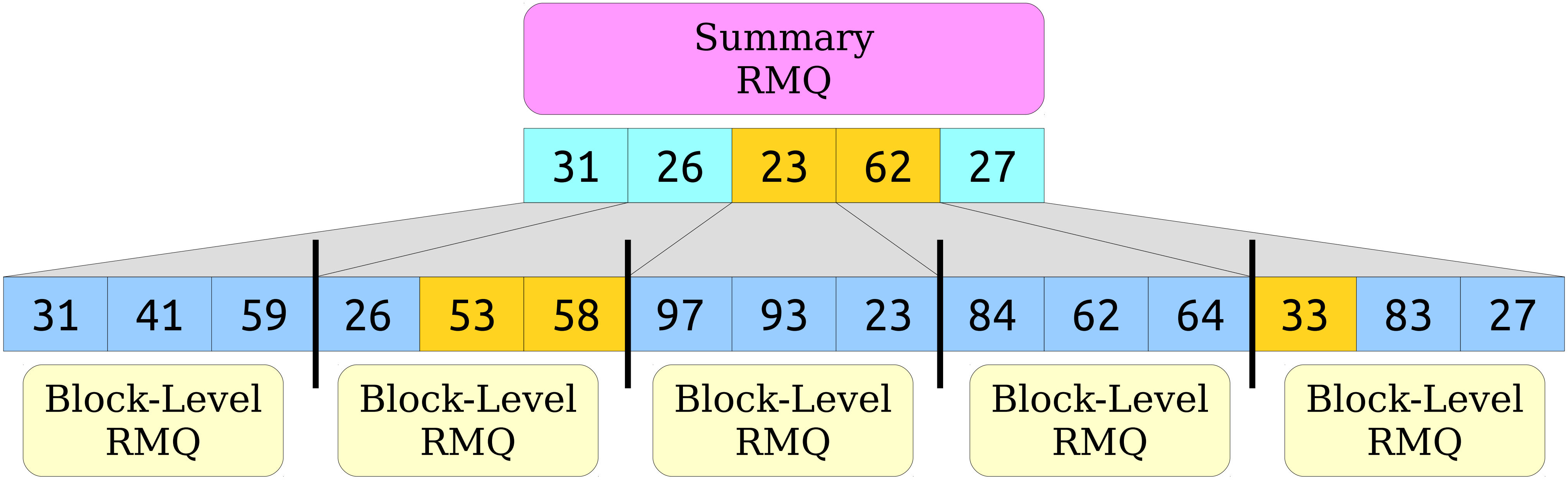
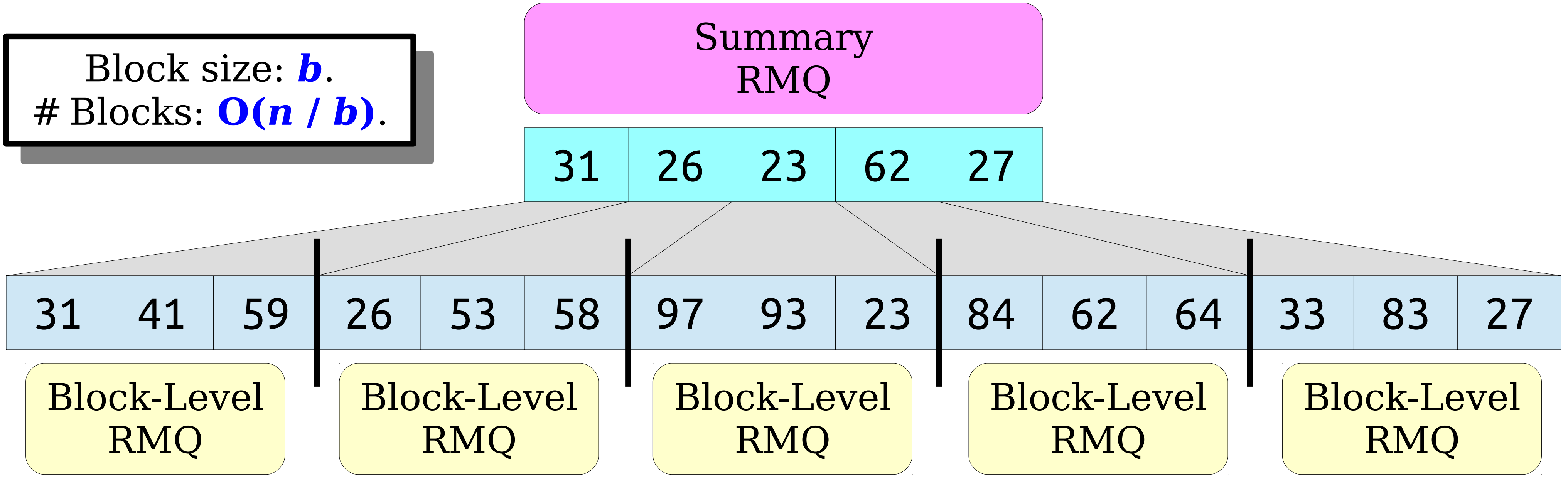


Analyzing Efficiency

- Suppose we use a $\langle p_1(n), q_1(n)\rangle$-time RMQ for the summary RMQ and a $\langle p_2(n), q_2(n)\rangle$-time RMQ for each block, with block size b.
- What is the preprocessing time for this hybrid structure?
    - $\mathbf{O(n)}$ time to compute the minima of each block.
    - $\mathbf{O(p_1(n / b))}$ time to construct RMQ on the minima.
    - $\mathbf{O((n / b)\ p_2(b))}$ time to construct the block RMQs
- Total construction time is $\mathbf{O(n + p_1(n / b) + (n / b)\ p_2(b))}$.



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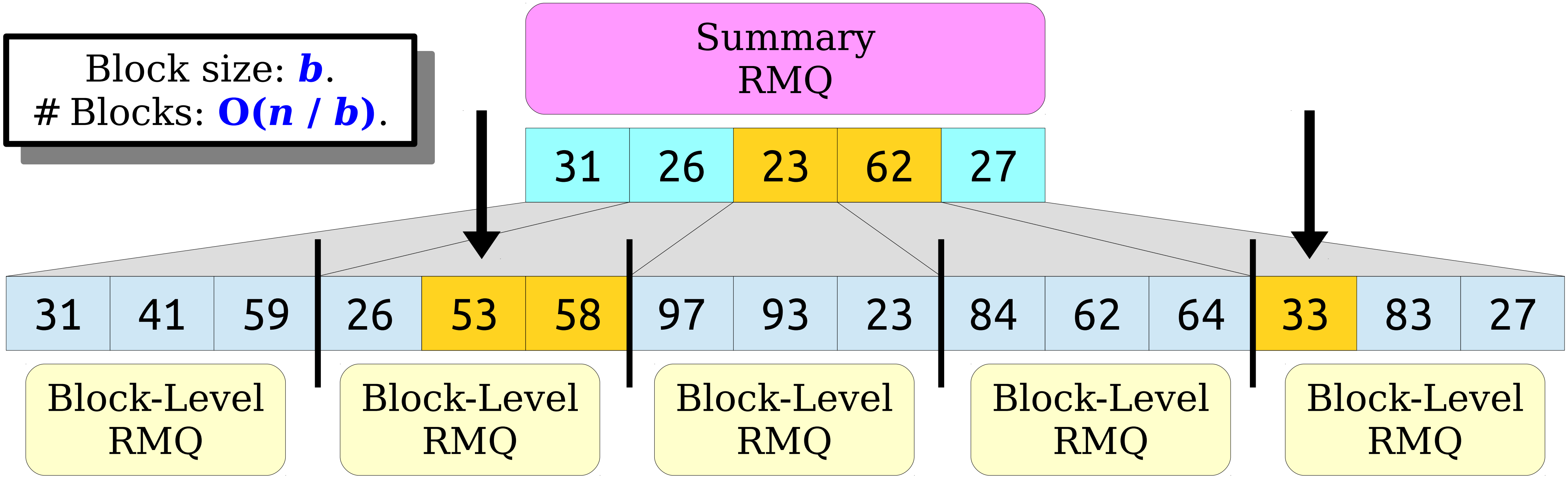
- What is the query time for this hybrid structure?
  - $\mathbf{O(q_1(n / b))}$ time to query the summary RMQ.
  - $\mathbf{O(q_2(b))}$ time to query the block RMQs.
- Total query time: $\mathbf{O(q_1(n / b) + q_2(b))}$.



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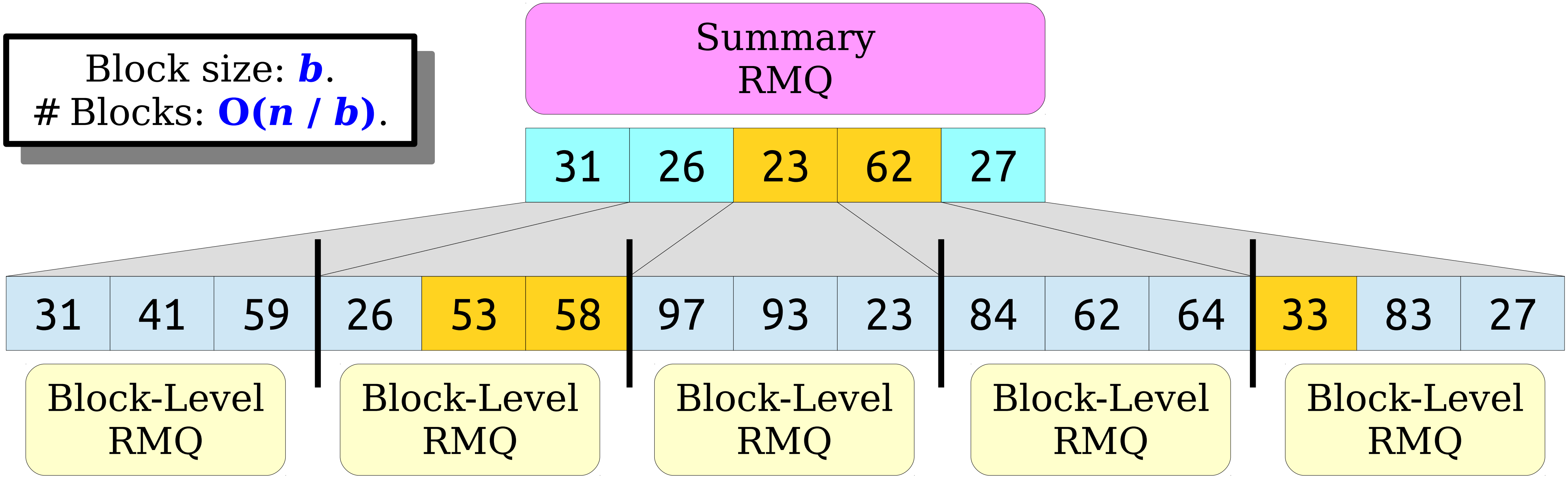
- Hybrid preprocessing time:

$$\mathbf{O(n + p_1(n / b) + (n / b)p_2(b))}$$

- Hybrid query time:

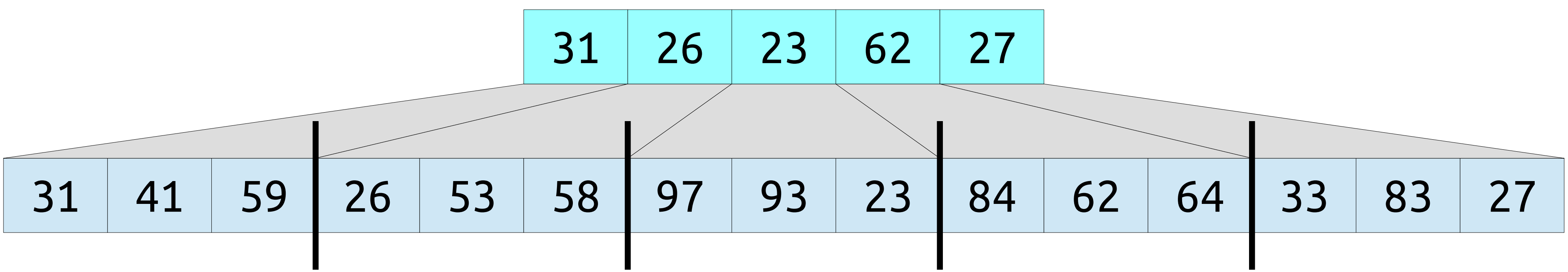
$$\mathbf{O(q_1(n / b) + q_2(b))}$$



A Sanity Check

- The ⟨O(n), O($n^{1/2}$)⟩ block-based structure from earlier uses this framework with the ⟨O(1), O(n)⟩ no-preprocessing RMQ structure and $b = n^{1/2}$.

Do no further preprocessing than just computing the block minima.



Don't do anything fancy per block. Just do linear scans over each of them.

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- Looks good so far!

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$b = n^{1/2}$

An Observation

- We can use any data structures we'd like for the summary and block RMQs.
- Suppose we use an $\langle O(n \log n), O(1) \rangle$ sparse table for the summary RMQ.
- If the block size is b, the time to construct a sparse table over the (n / b) blocks is $\mathbf{O((n / b) \log (n / b))}$.
- ***Cute trick:*** If $\mathbf{b = \Theta(\log n)}$, the time to construct a sparse table over the minima is

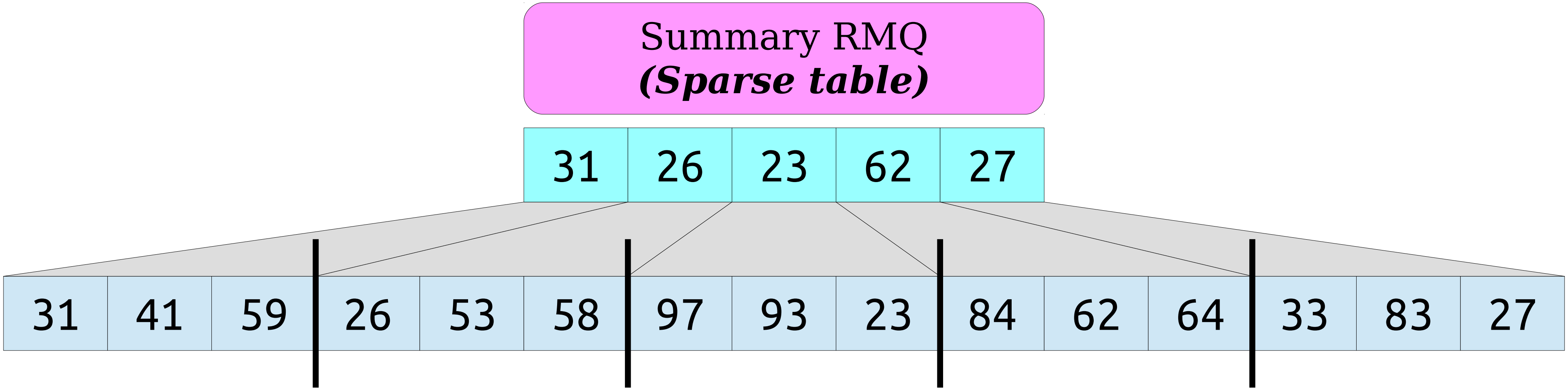
$$O((n / \log n) \log(n / \log n))$$

$$= O((n / \log n) \log n) \qquad \textit{(O is an upper bound)}$$

$$= \mathbf{O(n)}. \qquad \textit{(logs cancel out)}$$

One Possible Hybrid

- Set the block size to log *n*.
- Use a sparse table for the summary RMQ.
- Use the “no preprocessing” structure for each block.



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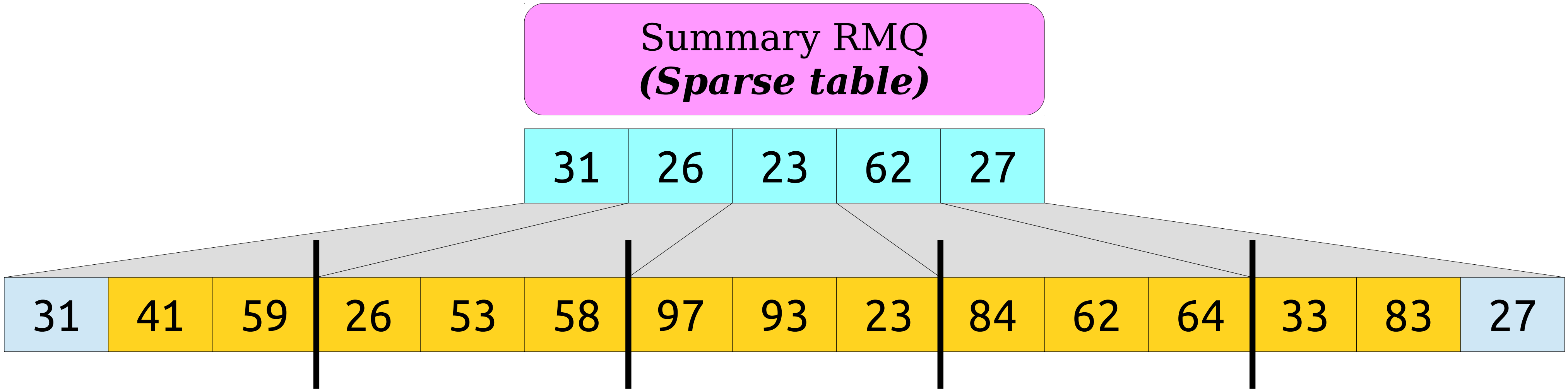

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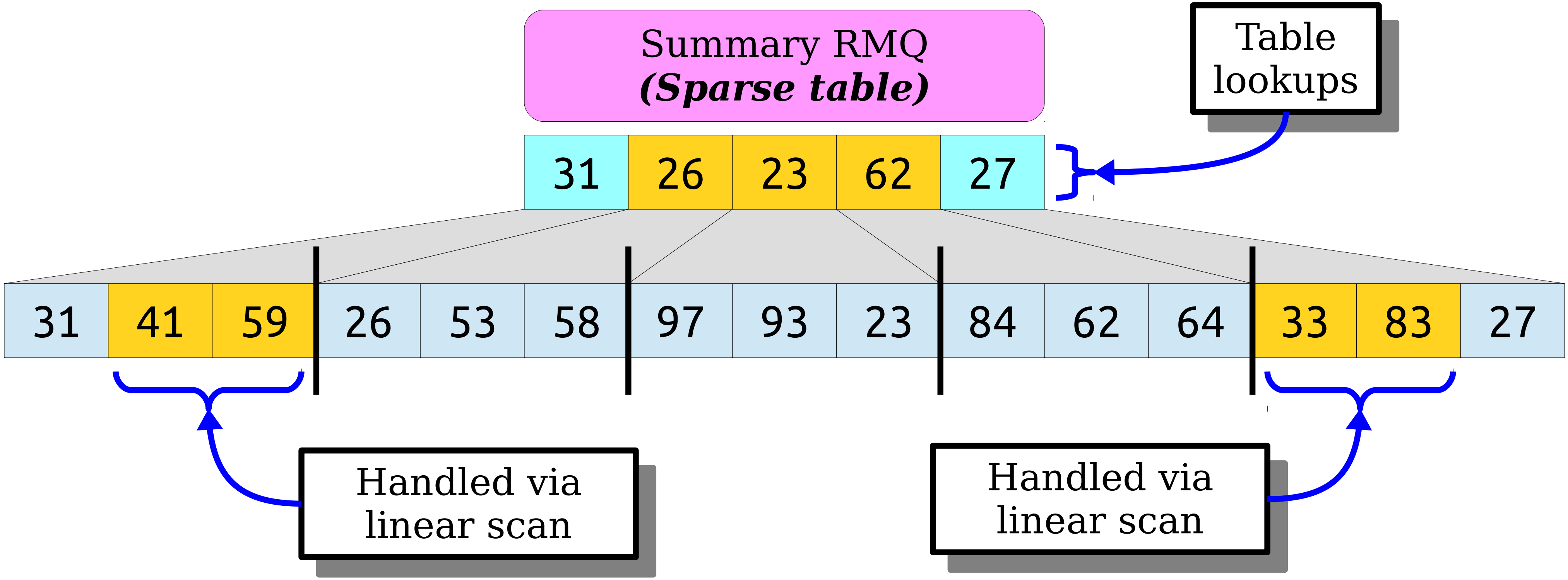

One Possible Hybrid

- Set the block size to log n.
- Use a sparse table for the summary RMQ.
- Use the “no preprocessing” structure for each block.

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = \log n$

One Possible Hybrid

- Set the block size to log n.
- Use a sparse table for the summary RMQ.
- Use the “no preprocessing” structure for each block.
- Preprocessing time:

$$O(n + p_1(n / b) + (n / b)\, p_2(b))$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

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- Set the block size to log n.
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- Preprocessing time:

$$\begin{aligned} &\mathrm{O}(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = \ &\mathrm{O}(n + n + n / \log n) \end{aligned}$$

For Reference

$p_1(n) = \mathrm{O}(n \log n)$
$q_1(n) = \mathrm{O}(1)$

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\begin{aligned}
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= \;&O(n + n + n / \log n) \\
= \;&\mathbf{O(n)}
\end{aligned}
$$

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- Query time:

$$O(q_1(n / b) + q_2(b))$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

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- Preprocessing time:

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\ p_2(b)) \\ = {} & O(n + n + n / \log n) \\ = {} & \mathbf{O(\boldsymbol{n})} \end{aligned}$$

- Query time:

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ = {} & O(1 + \log n) \end{aligned}$$

For Reference

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= {} & \mathbf{O(n)}
\end{aligned}
$$

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$$
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= {} & O(1 + \log n) \\
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- Query time:

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ = {} & O(1 + \log n) \\ = {} & \mathbf{O(\log n)} \end{aligned}$$

- An **⟨O(n), O(log n)⟩** solution!

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

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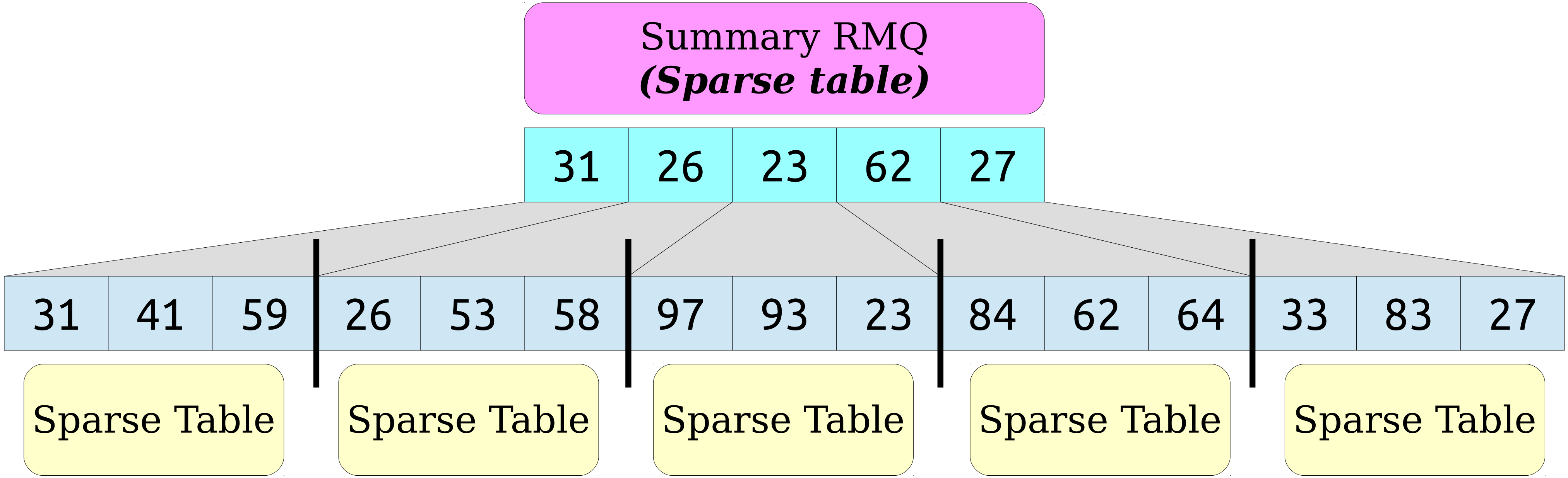
$q_2(n) = O(n)$

$b = \log n$

Another Hybrid

- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the summary and block RMQ structures with a block size of log n.



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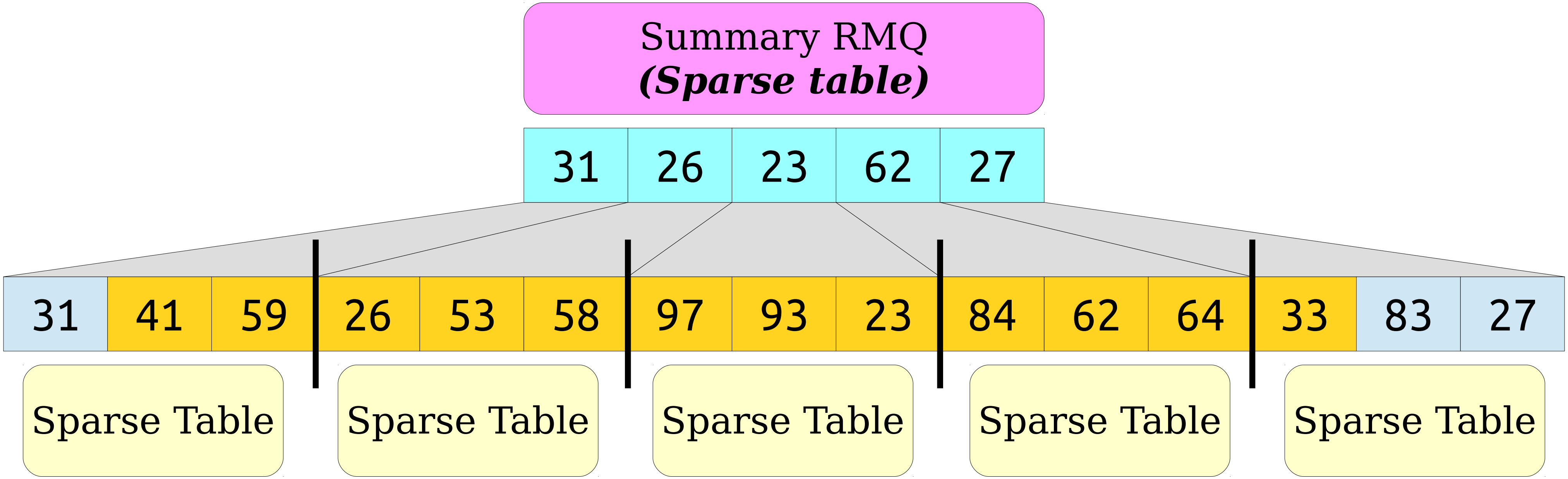

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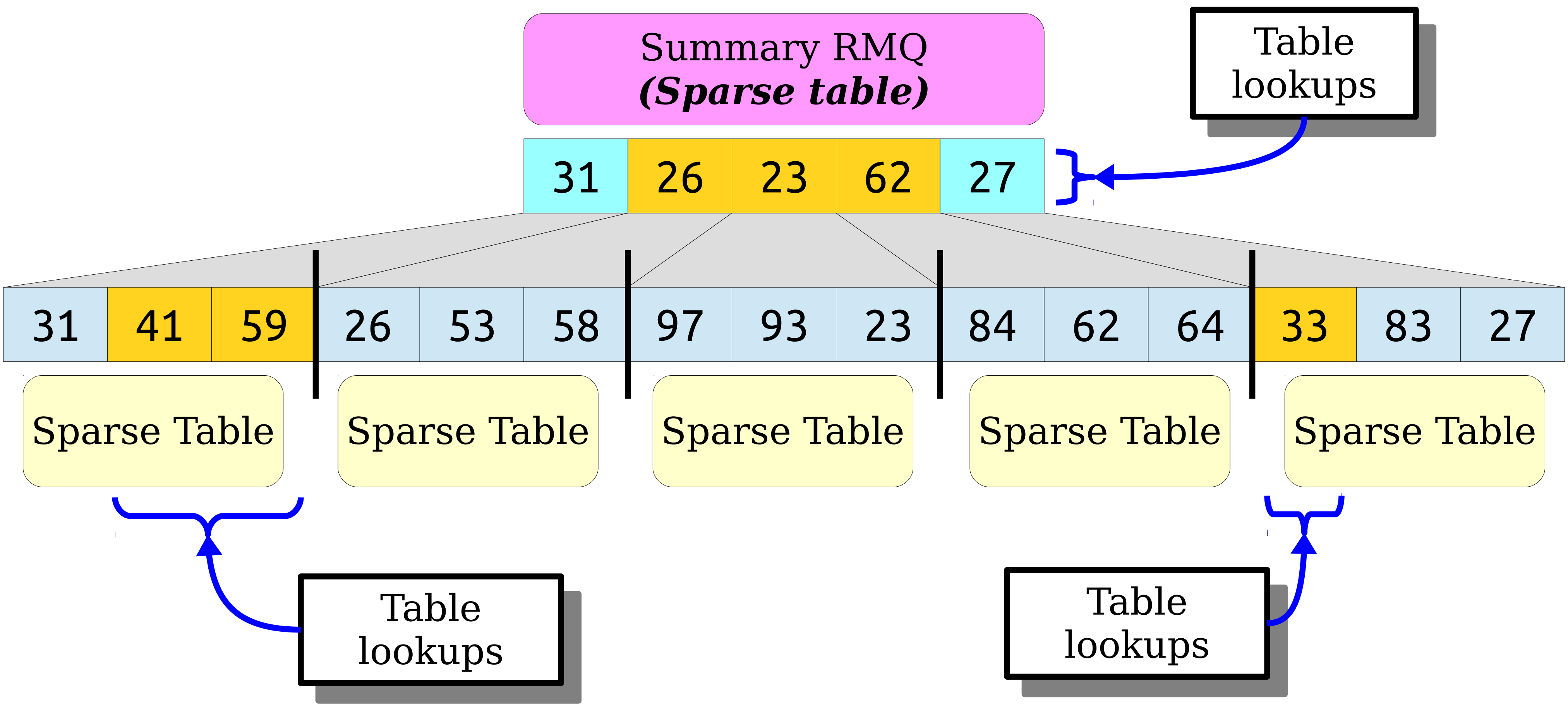

Another Hybrid

- Let's suppose we use the $\langle O(n \log n), O(1)\rangle$ sparse table for both the summary and block RMQ structures with a block size of $\log n$.

For Reference

$p_1(n) = O(n \log n)$
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$q_2(n) = O(1)$

$b = \log n$

Another Hybrid

- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the summary and block RMQ structures with a block size of log n.
- The preprocessing time is

$$O(n + p_1(n / b) + (n / b)\, p_2(b))$$

For Reference

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$q_2(n) = O(1)$

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\begin{aligned}
& O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
= \; & O(n + n + (n / \log n)\, b \log b)
\end{aligned}
$$

For Reference

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= {} & O(n + n + (n / \log n)\, b \log b) \\
= {} & O(n + (n / \log n) \log n \log \log n)
\end{aligned}
$$

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= {} & O(n + n + (n / \log n)\, b \log b) \\
= {} & O(n + (n / \log n) \log n \log \log n) \\
= {} & \mathbf{O(n \log \log n)}
\end{aligned}
$$

For Reference

$p_1(n) = O(n \log n)$
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$p_2(n) = O(n \log n)$
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$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = {} & O(n + n + (n / \log n)\, b \log b) \\ = {} & O(n + (n / \log n) \log n \log \log n) \\ = {} & \mathbf{O(n \log \log n)} \end{aligned}$$

- The query time is

$$O(q_1(n / b) + q_2(b))$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n \log n)$
$q_2(n) = O(1)$

$b = \log n$

Another Hybrid

- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the summary and block RMQ structures with a block size of log n.
- The preprocessing time is

$$
\begin{aligned}
&\mathrm{O}(n + p_1(n / b) + (n / b)\, p_2(b)) \\
=\ &\mathrm{O}(n + n + (n / \log n)\, b \log b) \\
=\ &\mathrm{O}(n + (n / \log n) \log n \log \log n) \\
=\ &\mathbf{O(n \log \log n)}
\end{aligned}
$$

- The query time is

$$
\begin{aligned}
&\mathrm{O}(q_1(n / b) + q_2(b)) \\
=\ &\mathbf{O(1)}
\end{aligned}
$$

For Reference

$p_1(n) = \mathrm{O}(n \log n)$
$q_1(n) = \mathrm{O}(1)$

$p_2(n) = \mathrm{O}(n \log n)$
$q_2(n) = \mathrm{O}(1)$

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\begin{aligned}
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=\ & O(n + n + (n / \log n)\, b \log b) \\
=\ & O(n + (n / \log n) \log n \log \log n) \\
=\ & \mathbf{O(n \log \log n)}
\end{aligned}
$$

- The query time is

$$
\begin{aligned}
& O(q_1(n / b) + q_2(b)) \\
=\ & \mathbf{O(1)}
\end{aligned}
$$

- We have an **⟨O(n log log n), O(1)⟩** solution to RMQ!

For Reference

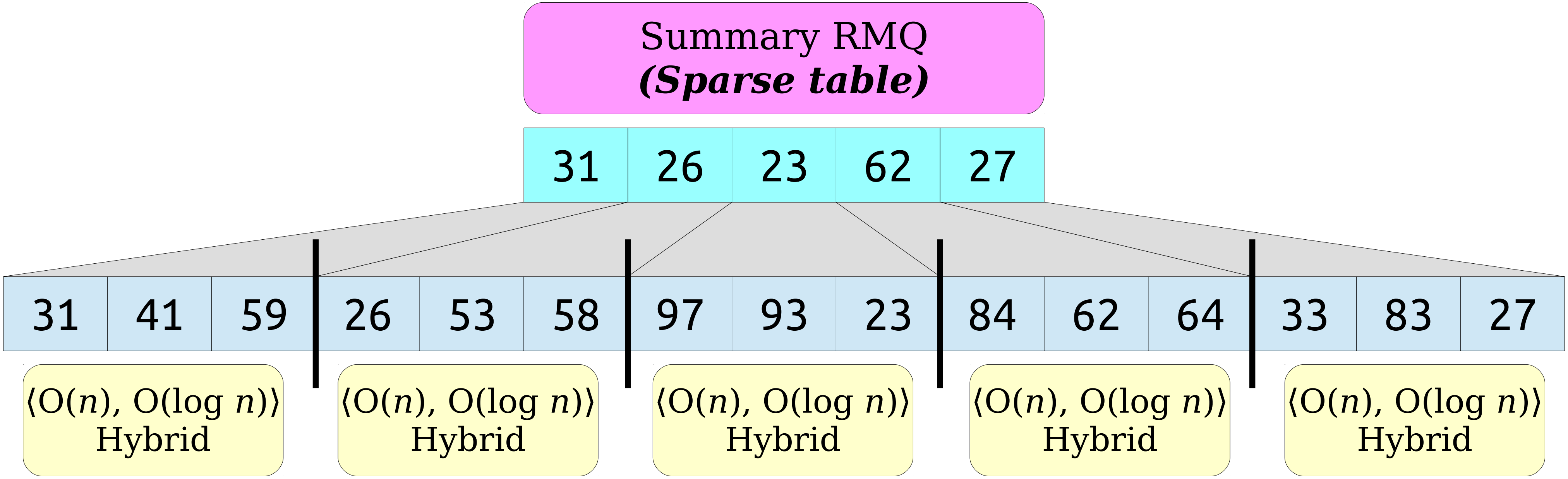
$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n \log n)$
$q_2(n) = O(1)$

$b = \log n$

One Last Hybrid

- Suppose we use a sparse table for the summary RMQ and the ⟨O(n), O(log n)⟩ solution for the block RMQs. Let's choose b = log n.



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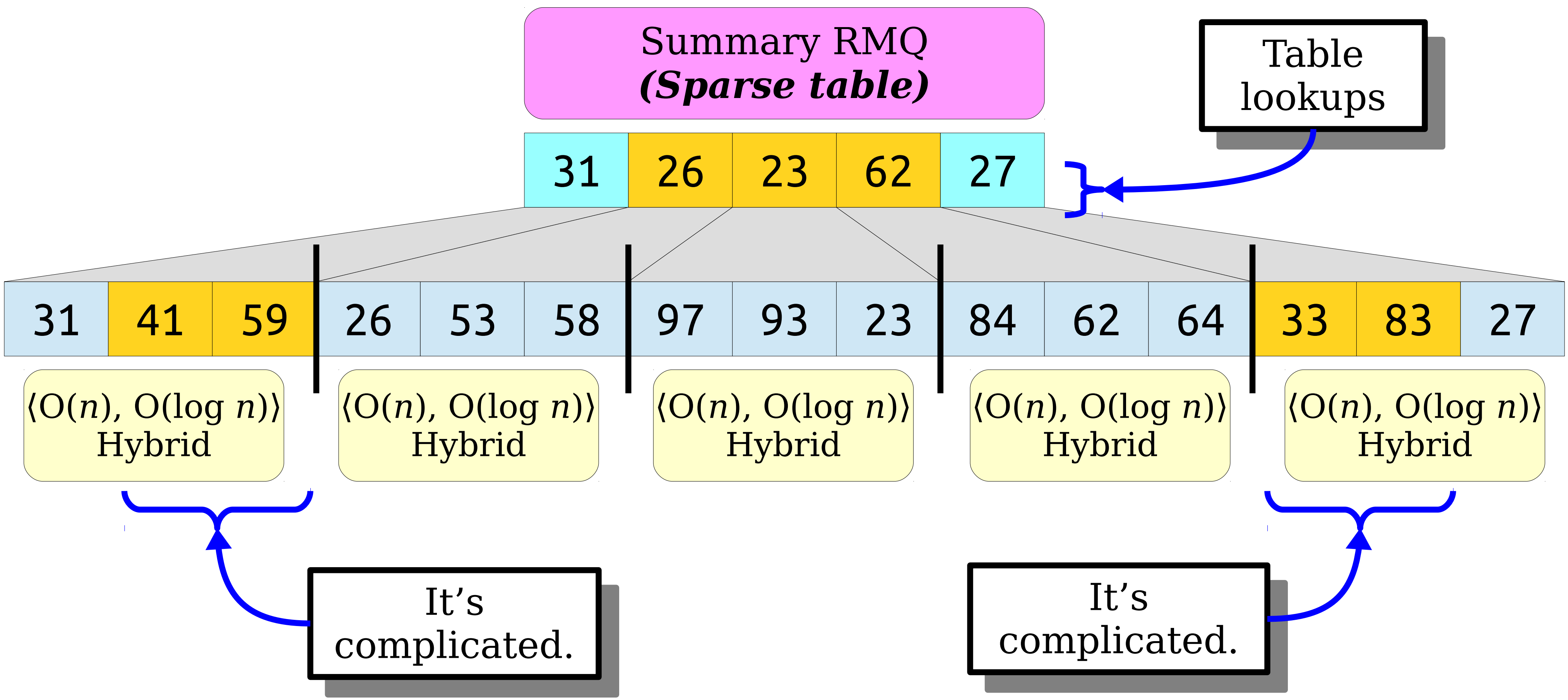
- Suppose we use a sparse table for the summary RMQ and the ⟨O(n), O(log n)⟩ solution for the block RMQs. Let's choose b = log n.



One Last Hybrid

- Suppose we use a sparse table for the summary RMQ and the ⟨O(n), O(log n)⟩ solution for the block RMQs. Let's choose $b = \log n$.

For Reference

$p_1(n) = O(n \log n)$
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- Suppose we use a sparse table for the summary RMQ and the $\langle O(n), O(\log n) \rangle$ solution for the block RMQs. Let's choose $b = \log n$.
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For Reference

$p_1(n) = O(n \log n)$
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- Suppose we use a sparse table for the summary RMQ and the ⟨O(n), O($\log n$)⟩ solution for the block RMQs. Let's choose $b = \log n$.
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$$
\begin{aligned}
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= {} & O(n + n + (n / \log n)\, b) \\
= {} & O(n + n + (n / \log n) \log n)
\end{aligned}
$$

For Reference

$p_1(n) = O(n \log n)$
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$$
\begin{aligned}
&\quad O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
&= O(n + n + (n / \log n)\, b) \\
&= O(n + n + (n / \log n) \log n) \\
&= \mathbf{O(n)}
\end{aligned}
$$

For Reference

$p_1(n) = O(n \log n)$
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- The query time is

$$O(q_1(n / b) + q_2(b))$$

For Reference

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- The query time is

$$\begin{aligned} &O(q_1(n / b) + q_2(b)) \\ = {} &O(1 + \log \log n) \end{aligned}$$

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$p_1(n) = O(n \log n)$
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&= O(n + n + (n / \log n) \log n) \\
&= \mathbf{O(n)}
\end{aligned}
$$

- The query time is

$$
\begin{aligned}
&\quad O(q_1(n / b) + q_2(b)) \\
&= O(1 + \log \log n) \\
&= \mathbf{O(\log \log n)}
\end{aligned}
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- We have an **⟨O(n), O(log log n)⟩** solution to RMQ!

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

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$q_2(n) = O(\log n)$

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Where We Stand

- We've seen a bunch of RMQ structures today:
  - No preprocessing: $\langle O(1), O(n) \rangle$
  - Full preprocessing: $\langle O(n^2), O(1) \rangle$
  - Block partition: $\langle O(n), O(n^{1/2}) \rangle$
  - Sparse table: $\langle O(n \log n), O(1) \rangle$
  - Hybrid 1: $\langle O(n), O(\log n) \rangle$
  - Hybrid 2: $\langle O(n \log \log n), O(1) \rangle$
  - Hybrid 3: $\langle O(n), O(\log \log n) \rangle$

Where We Stand

We've seen a bunch of RMQ structures today:

- No preprocessing: $\langle O(1), O(n) \rangle$
- Full preprocessing: $\langle O(n^2), O(1) \rangle$
- Block partition: $\langle O(n), O(n^{1/2}) \rangle$
- Sparse table: $\langle O(n \log n), O(1) \rangle$
- Hybrid 1: $\langle O(n), O(\log n) \rangle$
- Hybrid 2: $\langle O(n \log \log n), O(1) \rangle$
- Hybrid 3: $\langle O(n), O(\log \log n) \rangle$

Where We Stand

We've seen a bunch of RMQ structures today:

- No preprocessing: ⟨O(1), O(n)⟩
- Full preprocessing: ⟨O(n^2), O(1)⟩
- Block partition: ⟨O(n), O($n^{1/2}$)⟩
- Sparse table: ⟨O(n log n), O(1)⟩
- Hybrid 1: ⟨O(n), O(log n)⟩
- Hybrid 2: ⟨O(n log log n), O(1)⟩
- Hybrid 3: ⟨O(n), O(log log n)⟩

Is there an $\langle O(n), O(1) \rangle$ solution to RMQ?

Yes!

Next Time

- ***Cartesian Trees***
  - A data structure closely related to RMQ.
- ***The Method of Four Russians***
  - A technique for shaving off log factors.
- ***The Fischer-Heun Structure***
  - A clever, asymptotically optimal RMQ structure.