CS 538: Complexity of Computation Fall 2022

Homework 2: Collection of Problems in Complexity Theory

Instructor: Karthik C. S. Grader: Surya Teja Gavva

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2.1 Oblivious Turing Machines

Solve Chapter 1, Exercise 9 in theory.cs.princeton.edu/complexity/book.pdf.

2.2 NP, CoNP, and Cook Reductions

Read and digest the proof of Theorems 2.28 and 2.35 in www.wisdom.weizmann.ac.il/~oded/CC/r1.pdf.

2.3 Complement Complexity Class

For any complexity class $\mathcal{C}$, define Co-$\mathcal{C} := \{\{0,1\}^* \setminus S : S \in \mathcal{C}\}$. Prove that $\mathcal{C} \subseteq$ Co-$\mathcal{C}$ if and only if $\mathcal{C} =$ Co-$\mathcal{C}$.

2.4 Immerman–Szelepcsényi theorem

Read and digest the proof of Theorem 5.14 in www.wisdom.weizmann.ac.il/~oded/CC/r2.pdf.

2.5 Square Root Module for Primality Testing

Solve Chapter 6, Exercise 6.16 in www.wisdom.weizmann.ac.il/~oded/CC/r3.pdf.

2.6 Interactive Proof for Matrix Inversion

Consider the following language: set of all invertible matrices over $\mathbb{F}_2$. Design a 2 round* sublinear communication† IP protocol for this language with completeness 1 and soundness 1/2, where the verifier can run only in linear time of the input matrix size.

*In a 2-round protocol, the Verifier sends one message and the Prover sends one message.

†Given an $n \times n$ matrix, the input size is n^2 bits, and the communication is said to be sublinear, if both the Verifier and Prover send $o(n^2)$ bits.

2.7 Hadamard PCP

Construct a Probabilistically Checkable Proof System for 3-SAT on n variables and $O(n)$ clauses with the following parameters: completeness:1, soundness: 0.99, query complexity: $O(1)$, and randomness complexity: $O(n)$.

2.8 Limits of Gap Amplification

In class (on November 17, 2022), we saw the proof of the following lemma:

Lemma 1. *Let $\lambda < d$, and $|\Sigma|$ be arbitrary constants. There exists a constant $\beta := \beta(\lambda, d, |\Sigma|)$ such that for every t and every d-regular constraint graph G over alphabet Σ with self-loops and $\lambda(G) < \lambda$, $\mathsf{UNSAT}(G^t) \geq \beta\sqrt{t} \cdot \min(\mathsf{UNSAT}(G), 1/t)$ (where $\lambda(G)$ denotes the second largest eignvalue of G and $\mathsf{UNSAT}(G)$ denotes fraction of constraints of G that every assignment leaves unsatisfied).*

We would like to understand the limitations of the above gap amplification procedure. Consider the following construction of expanders:

Theorem 2. *For infinitely many integers d, there exist infinitely many n and a d-regular graph on n vertices G with the following properties:*

1. *G has girth $\frac{2}{3}\log_d n$.*
2. *$\lambda(G) = 2\sqrt{d-1}$.*
3. *every two-partition of G is violated by at least a $1/2 - 2/\sqrt{d-1}$ fraction of edges.*

Using the above construction, show that for every pair of constants d and t there exists an integer n and a d-regular constraint expander G with self-loops on n vertices over alphabet $\{0, 1\}$ such that $\mathsf{UNSAT}(G) \leq 1/2 + O(1/\sqrt{d})$, but $\mathsf{UNSAT}(G^t) = 1/2$.

2.9 Inapproximability of Clique from weak PCP Theorem

Let Q_d denote the d-regular Boolean Hypercube graph on vertex set $\{0, 1\}^d$. We can equipartition the edge set into $E_1, \ldots, E_d$, where E_i contain all the edges of the form $(\mathbf{x}, \mathbf{x} \oplus \mathbf{e}_i)$, where $\mathbf{x} \in \{0, 1\}^d$ and $\mathbf{e}_i$ is the point with 1 in the i^{th} coordinate and 0 everywhere else. We say that a 2-CSP is defined over Q_d, to mean that every vertex in Q_d is a variable of the 2-CSP and for every edge $e := (\mathbf{x}, \mathbf{x} \oplus \mathbf{e}_i)$ in Q_d we have a two arity constraint over the pair of variables $(\mathbf{x}, \mathbf{x} \oplus \mathbf{e}_i)$. Therefore the equipartition of the edges also naturally implies an equipartition of the set of constraints into $\Pi_1, \ldots, \Pi_d$.

Design a polynomial time (Karp) reduction from 2-CSP instance φ over $Q_{\log n}$ and constant alphabet size to an instance (G_φ, k) of the Clique problem with the following guarantees for some absolute constant $\varepsilon > 0$:

Completeness: If there is an assignment to the vertices in $Q_{\log n}$ such that all the constraints in φ are satisfied then there is a clique of size k in G_φ.

Soundness: Suppose that for every assignment σ to the vertices in $Q_{\log n}$ there exists $i \in [\log n]$ such that σ satisfies at most 99% of the constraints in Π_i then there is no clique of size $(1-\varepsilon) \cdot k$ in G_φ.

2.10 Exact Set Cover

Prove the NP-hardness of the Exact Cover Problem defined as follows. For every $\varepsilon > 0$, given as input an integer k and a collection of subsets $S_1, \ldots, S_n$ of universe $[n]$, distinguish between the following cases:

Completeness: There exists mutually disjoint k sets in the collection, say $S_{i_1}, \ldots, S_{i_k}$, such that $\bigcup_{j\in[k]} S_{i_j} = [n]$.

Soundness: For every k mutually disjoint sets in the collection, say $S_{i_1}, \ldots, S_{i_k}$, we have $\left|\bigcup_{j\in[k]} S_{i_j}\right| \leq \varepsilon \cdot n$.

2.11 2-to-2 Games Theorem

Watch this video by Subhash Khot on the recent progress on the Unique Games Conjecture: `www.youtube.com/watch?v=Q3bu4Kkr5cQ`.

2.12 Distributed PCP[‡]

Prove that assuming the Strong Exponential Time Hypothesis, for every $\varepsilon > 0$, no algorithm running in time $n^{2-\varepsilon}$, given as input an integer s and two sets of vectors $A, B \subseteq \mathbb{R}^d$, where $|A| = |B| = n$, and $d = \text{polylog } n$, can distinguish between the following cases:

Completeness: There exists $(a, b) \in A \times B$ such that $\langle a, b\rangle \geq s$.

Soundness: For every $(a, b) \in A \times B$ we have $\langle a, b\rangle \leq s/2$.

2.13 A Strong Assumption and Weak Result for Vertex Cover

Prove that assuming the Strong Exponential Time Hypothesis, for every $\varepsilon > 0$, no algorithm running in time $n^{2-\varepsilon}$, given as input a graph on n vertices and $n^{1+o(1)}$ edges and an integer k, can distinguish between the following cases:

[‡]Hint: Use the inapproximability of k-MaxCover problem that we showed in class (on December 8, 2022) but with $k = 2$.

Completeness: There exists a vertex cover of size k.

Soundness: Every vertex cover is of size at least $1.5 \cdot k$.