Problem 4.1. The data set accompanying this homework gives daily returns for three stocks: TSLA, AAPL, and IBM.

(a) Calculate the historical (regressed, no intercept) beta, for each of these assets as of Dec 31, 2014. In each case, calculate the appropriate t-statistic on the coefficient to test the null hypothesis $\beta = 0$ and state whether you reject the null hypothesis.

(b) Compute the holdings vector $\boldsymbol{h} \in \mathbb{R}^3$ for the unique portfolio of TSLA, AAPL, and IBM which is dollar-neutral (ie. self-financing) and which has holding of 100 dollars in AAPL and zero exposure to beta as of Dec 31, 2014. In other words $\boldsymbol{h} \cdot \boldsymbol{\beta} = 0$ where $\boldsymbol{\beta}$ is the vector of betas computed in part (a).

(c) Compute the daily returns of the portfolio from (b) over the period Jan 1, 2015 to Dec 31, 2015. Assume that each day, the portfolio is rebalanced back to the initial holdings vector $\boldsymbol{h} \in \mathbb{R}^3$. Plot the cumulative sum of the log returns.

(d) Compute the realized correlation of the returns in part (c) to the market's return, using the S&P 500 as a proxy for the market portfolio. Construct a statistical test of the null hypothesis that the correlation is zero. Is the realized correlation significantly different from zero at the 95% level?

Problem 4.2. Suppose you are a fund-of-funds manager with investments in n different hedge funds for some $n \geq 2$. Let r_i denote the annualized return of the i-th fund. Suppose that

$$r_i = \beta r_M + \epsilon_i, \quad \mathrm{var}(\epsilon_i) = \sigma_i^2$$

where r_M denotes the return of the market portfolio (approximated by the S&P 500 in the US) with variance σ_M^2. Suppose that ϵ_i and ϵ_j are independent random variables if $i \neq j$, and that ϵ_i is independent from r_M for all $i = 1, \ldots, n$. Suppose that your fund-of-funds has invested $h_i > 0$ dollars in the i-th hedge fund, so their profit/loss is

$$\pi = h'r = \sum_i h_i r_i.$$

Throughout the following, assume $h = (1/n, 1/n, \ldots, 1/n) \in \mathbb{R}^n$ for simplicity, ie. the fund-of-funds has one unit of capital evenly distributed across its constituents.

(a) Calculate $\mathbb{E}[h'r]$ and $\mathbb{V}[h'r]$. Note that $\mathbb{V}[h'r]$ can be expressed as $\mathbb{V}(h'r) = f(\beta, \sigma_M^2) + g(\sigma_1^2, \ldots, \sigma_n^2)$; find functions $f()$ and $g()$ explicitly.

(b) Take $\beta = 0.5$ and $\sigma_M = 0.2$. Assume that each constituent fund has an annualized volatility target of 10% and all $\sigma_i \approx 0.03$. The "fraction of variance explained by the market" for the fund-of-funds is defined to be $f/(f + g)$. Numerically compute and plot this fraction as a function of n for $n = 2 \ldots 30$.

(c) Take the same assumptions as (b). Further assume that each ϵ_i has a Sharpe ratio of 1.5, so that $\mathbb{E}[\epsilon_i] = 1.5 \cdot \sigma_i$, and the market's expected annual return is $\mathbb{E}[r_M] = 0.07$. The fund-of-funds charges a fee of 0.01 on capital. Numerically compute and plot the Sharpe ratio, $\mathbb{E}[h'r - 0.01]/\sqrt{\mathbb{V}[h'r]}$ as a function of n for $n = 2 \ldots 30$. How does this change if the Sharpe ratio of ϵ_i is 2.0 rather than 1.5?

(d) If the fund-of-funds could simply invest in a single fund with the same properties as the others except that this fund has $\beta = 0$ and $\sigma_i = 0.1$, would that be better or worse, in terms of Sharpe ratio, than the above scenario?