

Convex Optimization and Constraints

- What kinds of constraints can be handled easily in portfolio optimization?
- Is there always a unique global optimum?
- In order to progress, we need a general framework to address these kinds of questions.
- That is the framework of convex optimization.
- Most of what we'll say here can be found in Boyd and Vandenberghe (2009), but that's 700 pages!
- We're going to try to condense it down a bit.

Convex sets

- We begin with a little math review that we are going to need for what follows.
- The mathematics of convex sets and convex functions is extremely important in finance.
- Most of the optimization problems we'll encounter are convex, and convex optimization problems enjoy many favorable properties that are not shared by their non-convex counterparts.
- Each non-convex optimization problem has a dual that is convex.
- Moreover, model-free no-arbitrage relationships guarantee that many derivative pricing functions are convex.
- We do not assume previous knowledge of convex spaces or functions, so start with the basic definitions.

- A $n \times n$ symmetric real matrix M is said to be *positive definite*, a condition denoted by

$$M \succ 0$$

if

$$\langle x, Mx \rangle > 0 \text{ for all } x \in \mathbb{R}^n \setminus \{0\}$$

- The same matrix is said to be *positive semidefinite*, denoted $M \succeq 0$, if

$$\langle x, Mx \rangle \geq 0 \text{ for all } x \in \mathbb{R}^n$$

Furthermore we write $A \prec B$ if $A - B \prec 0$, etc.

- The line segment between x_1 and x_2 in a vector space V is all points $x = \theta x_1 + (1 - \theta)x_2$ with $0 \leq \theta \leq 1$.
- A set C is *convex* if it contains the line segment between any two points in the set, i.e.

$$x_1, x_2 \in C \Rightarrow \theta x_1 + (1 - \theta)x_2 \in C \quad \forall \theta \in [0, 1]$$

- A *convex combination* of x_1, \dots, x_k is any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k$$

with $\theta_1 + \dots + \theta_k = 1, \theta_i \geq 0$.

- The *convex hull* of a set S , denoted $\text{conv}(S)$, is the set of all convex combinations of points in S .

The convex hull is also the smallest convex set that contains S , or the intersection of all convex sets containing S .

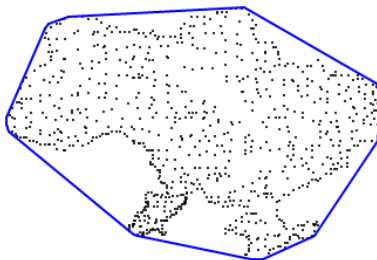


Figure: When S is a bounded subset of the plane, the convex hull may be visualized as the shape formed by a rubber band stretched around S .

Definition 2.1

A *norm* on a vector space V is a function $\| \cdot \|$ from $V \rightarrow \mathbb{R}$ that satisfies: $\|x\| \geq 0$ with equality iff $x = 0$, $\|tx\| = |t| \|x\|$ for $t \in \mathbb{R}$, and

$$\|x + y\| \leq \|x\| + \|y\|. \quad (2.1)$$

- Eq. (2.1) is known as the *triangle inequality*.

- The most familiar examples of norms are $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$ for $p > 0$.
- For $p = 2$ the associated distance $d(x, y) = \|x - y\|_2$ is the usual Euclidean distance, but any norm defines a distance function, or *metric*, $d(x, y) = \|x - y\|$.
- An *open ball* is defined to be a set of the form

$$B(x; \epsilon) := \{y : d(x, y) < \epsilon\} \quad \text{where } \epsilon > 0.$$

- The associated closed ball is denoted $\overline{B}(x; \epsilon)$.
- Both $B(x; \epsilon)$ and $\overline{B}(x; \epsilon)$ are examples of convex sets, for any norm.

- To establish convexity of a set C , one can always try to apply the definition directly.

A faster way might be to show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ellipsoids, etc.) by operations that preserve convexity, such as

- intersection
- affine functions
- perspective function
- linear-fractional functions

- Some more examples of convex sets are provided by familiar sets from matrix theory. S^n , the space of $n \times n$ symmetric, real matrices, is convex. S_+^n denotes the subspace of S^n with non-negative eigenvalues, or equivalently, the subspace

$$\{A \in S^n : v^T A v \geq 0 \ \forall v\}.$$

- S_{++}^n is defined similarly, but for *strictly positive* matrices, i.e. those satisfying $v^T A v > 0$ for any nonzero v .
- We can easily show that S_{++}^n is also convex:

$$v^T [\theta A + (1 - \theta) B] v = \theta v^T A v + (1 - \theta) v^T B v \geq 0 \text{ if } A, B \in S_{++}^n$$

- If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is affine, meaning that

$$f(x) = Ax + b \text{ with } A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

then the image (or inverse image) of a convex set under f is convex.

- Examples include scaling, translation, projection and the solution set of a linear matrix inequality

$$\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\} \text{ with } A_i, B \in S^p.$$

- The perspective function $P : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is defined by

$$P(x, t) = x/t, \quad \text{dom}P = \{(x, t) \mid t > 0\}$$

- It is a fact that images and inverse images of convex sets under perspective are convex.

- A linear-fractional function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined as

$$f(x) = (Ax + b)/(c^T x + d),$$

with

$$\text{dom}(f) = \{x \mid c^T x + d > 0\}.$$

- Images and inverse images of convex sets under linear-fractional functions are convex.

Convex functions

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *convex* if $\text{dom} f$ is a convex set and

$$f(\theta u + (1 - \theta)v) \leq \theta f(u) + (1 - \theta)f(v) \quad (2.2)$$

for all $u, v \in \text{dom} f$ and $0 < \theta < 1$.

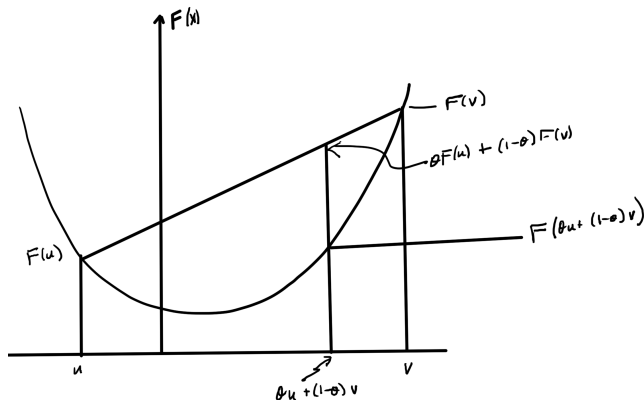


Figure: Illustrating the definition of a convex function.

- Function f is *concave* if $-f$ is convex.
- A function f is *strictly convex* if the inequality in (2.2) is strict.

Examples of convex functions on \mathbb{R} include:

- 1 affine,
- 2 exponential e^{ax} for any $a \in \mathbb{R}$,
- 3 powers x^α on \mathbb{R}_{++} for $\alpha \geq 1$ or $\alpha \leq 0$,
- 4 powers of absolute value $|x|^p$ on \mathbb{R} , for $p \geq 1$, and
- 5 negative entropy $x \log x$ on \mathbb{R}_{++} .

- On \mathbb{R}^n affine functions are convex and concave; all norms on any normed vector space are convex.

Examples of convex functions on $\mathbb{R}^{m \times n}$ include

- ① the affine trace function $f(X) = \text{tr}(A^T X) + b$, and
- ② the spectral (maximum singular value) norm

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = \left(\lambda_{\max}(X^T X) \right)^{1/2}.$$

A key property which facilitates the proof of many theorems is the following:

Theorem 2.2

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if and only if the function $g : \mathbb{R} \rightarrow \mathbb{R}$,

$$g(t) = f(x + tv) \quad \text{with} \quad \text{dom}(g) = \{t \mid x + tv \in \text{dom} f\}$$

is convex (in t) for any $x \in \text{dom} f$ and $v \in \mathbb{R}^n$.

Hence one can check convexity of f by checking convexity of functions of one variable.

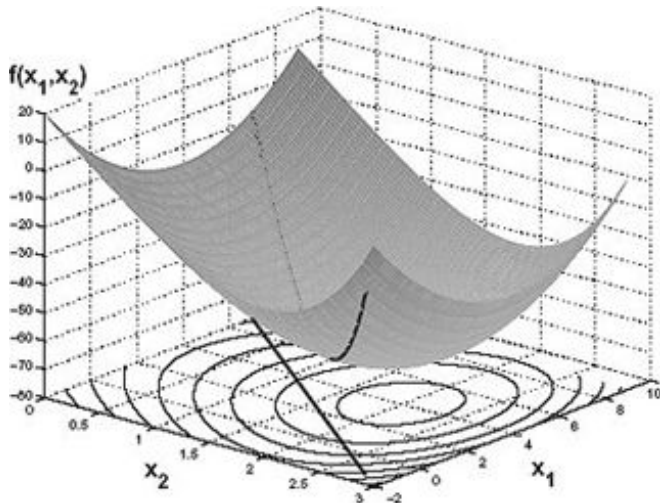


Figure: Two-dimensional convex functions and their 1d slices.

- A function f is *differentiable* if $\text{dom} f$ is open and the gradient exists at each $x \in \text{dom} f$.

The following fact is known as the *first-order condition* for convexity.

Theorem 2.3

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable with convex domain. Then f is convex if and only if

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \text{ for all } x, y \in \text{dom} f. \quad (2.3)$$

In other words the first-order approximation of f is global underestimator.

For $n = 2$, (2.3) says that the surface $z = f(x, y)$ never passes below any of its tangent planes.

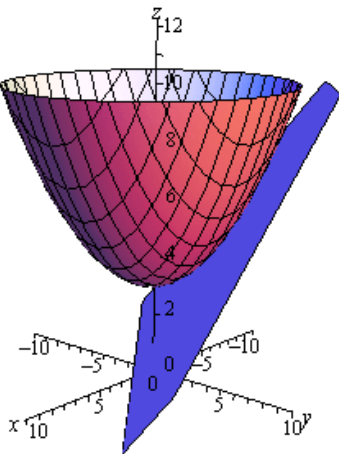


Figure: The *first-order condition* for convexity says the surface $z = f(x, y)$ never passes below any of its tangent planes.

- The proof of Theorem 2.3 is easy given other facts presented above; for example, it suffices to check convexity on all one-dimensional subsets of the domain, and on such subsets, (2.3) simplifies.

- There are also *second-order conditions* for checking convexity of differentiable functions.

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *twice differentiable* if $\text{dom} f$ is open and the Hessian $\nabla^2 f(x) \in S^n$ exists at each $x \in \text{dom} f$.

Theorem 2.4

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be twice differentiable with convex domain. Then f is a convex function if and only if

$$\nabla^2 f(x) \succeq 0 \text{ for all } x \in \text{dom} f.$$

If $\nabla^2 f(x) \succ 0$ for all $x \in \text{dom} f$, then f is strictly convex.

- The proof, which we omit, proceeds by relating the second derivative $g''(t)$ of the 1d function $g(t) = f(x + tv)$ to $\langle v, \nabla^2 f(x)v \rangle$ and using related results from ordinary calculus.

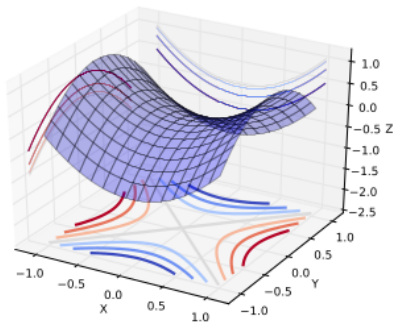


Figure: If $\nabla^2 f(x) \succ 0$ fails to be true for both f and $-f$, then there is probably some point which has both positive and negative curvature directions. Such a point is called a *saddle point*.

- Let's see some examples of multivariate functions for which we can verify convexity using the second-order condition.
- For our first example, consider a quadratic function:

$$f(x) = (1/2)x^T Px + q^T x + r \text{ (with } P \in S^n)$$

- Then one calculates $\nabla f(x) = Px + q$, $\nabla^2 f(x) = P$.
- The function $f(x)$ is convex if $P \succeq 0$.

- A second example is the familiar least-squares objective:
 $f(x) = \|Ax - b\|^2$, which has

$$\nabla f(x) = 2A^T(Ax - b), \quad \nabla^2 f(x) = 2A^T A$$

- This is convex for any A .
- The quadratic-over-linear function $f(x, y) = x^2/y$ is convex for $y > 0$ (just compute the Hessian).

- The log-sum-exp function

$$f(x) = \log \sum_k \exp x_k$$

is convex.

- One can compute the Hessian easily enough:

$$\nabla^2 f(x) = \frac{1}{1'z} \text{diag}(z) - \frac{1}{(1'z)^2} z z^T \quad \text{for } z_k = \exp x_k$$

but there are easier ways of verifying convexity in this case, such as using the composition theorem which we shall discuss later.

- Along similar lines, the geometric mean $f(x) = (\prod_k x_k)^{1/n}$ on R_{++}^n is concave.

- Define the α -*sublevel* set of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$C_\alpha = \{x \in \text{dom} f \mid f(x) \leq \alpha\}$$

- Fact: sublevel sets of convex functions are convex.

- Define the *epigraph* of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$\text{epi} f = \{(x, t) \in \mathbb{R}^{n+1} \mid x \in \text{dom} f, f(x) \leq t\}$$

- A function f is convex if and only if $\text{epi} f$ is a convex set.

This is another way of checking convexity.

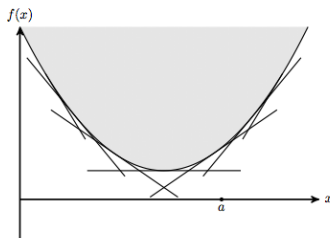


Figure: The shaded region is the epigraph.

- If f is convex, then

$$f(\mathbb{E}z) \leq \mathbb{E}f(z)$$

for any random variable z .

- This is known as *Jensen's inequality*.
- The basic inequality (2.2) that defines convexity is a special case with a discrete distribution.

In practice a great way to prove convexity of f is to show that f is obtained from simple convex functions by operations that preserve convexity:

- nonnegative weighted sum
- composition with affine function
- pointwise maximum and supremum
- composition (sometimes; see Theorem 2.6 below)
- minimization
- perspective

- Examples include the log barrier for linear inequalities

$$f(x) = - \sum \log(b_i - a_i^T x),$$

with

$$\text{dom} f = \{x \mid a_i^T x < b_i, i = 1, \dots, m\}$$

and any norm or affine function: $f(x) = \|Ax + b\|$.

- If f_1, \dots, f_m are convex, then

$$f(x) = \max\{f_1(x), \dots, f_m(x)\}$$

is convex.

- For example, a piecewise-linear function

$$f(x) = \max_{i=1, \dots, m} (a_i^T x + b_i)$$

is convex.

Theorem 2.5

Similarly, if $f(x, y)$ is convex in y for each $x \in A$, then

$$g(y) := \sup_{x \in A} f(x, y) \quad (2.4)$$

is a convex function of y .

Proof. In terms of epigraphs, the pointwise supremum of functions corresponds to the intersection of epigraphs: in terms of (2.4) we have

$$\text{epig} = \bigcap_{x \in A} \text{epif}(x, \cdot).$$

Thus, the result follows from the fact that the intersection of a family of convex sets is convex. \square

From theorem 2.5, it follows immediately that several more interesting classes of functions are, in fact, convex.

- 1 Given any set C , the distance function to the farthest point in C is a convex function:

$$f(x) = \sup_{y \in C} \|x - y\|$$

- 2 the maximum eigenvalue of symmetric matrix: for $X \in S^n$,

$$\lambda_{\max}(X) = \sup_{\|y\|_2=1} y^T X y.$$

and then $\lambda_{\max}(X)$ is a convex function of X .

The *extended-value extension* \tilde{f} of f is defined as:

$$\tilde{f}(x) = f(x) \text{ if } x \in \text{dom} f, \text{ and } \tilde{f}(x) = \infty \text{ if } x \notin \text{dom} f.$$

Theorem 2.6

Consider the composition of $g : \mathbb{R}^n \rightarrow \mathbb{R}$ and $h : \mathbb{R} \rightarrow \mathbb{R}$:

$$f(x) = h(g(x))$$

where h is convex. Then f is convex if either: (a) g is convex and \tilde{h} nondecreasing, or (b) g is concave and \tilde{h} nonincreasing.

- *Proof (Sketch).* For $n = 1$ and differentiable g, h :

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x).$$

- This needs to be ≥ 0 .
- So for example, if h is convex then the first term is nonnegative, and if g is convex and $h' \geq 0$ at all points in its domain then the second term is nonnegative too.

- Applications of Theorem 2.6: $\exp g(x)$ is convex if g is convex.
- Also $1/g(x)$ is convex if g is concave and positive.

- If $f(x, y)$ is convex in (x, y) and C is a convex set, then

$$g(x) = \inf_{y \in C} f(x, y)$$

is convex.

- Therefore, the distance to a set $d(x, S) = \inf_{y \in S} \|x - y\|$ is convex if S is convex.

Definition 2.7

The *convex conjugate* of a function f is

$$f^*(y) = \sup_{x \in \text{dom} f} \left(y^T x - f(x) \right). \quad (2.5)$$

It is also known as the Legendre-Fenchel transformation or Fenchel transformation (after Adrien-Marie Legendre and Werner Fenchel).

- This is another way of generating convex functions: f^* is convex (even if f is not).
- Also, this conjugation operation is fundamental to the construction of the *dual problem*, which we will discuss later.
- The dual is a very important practical tool for actually solving optimization problems numerically.

The standard form optimization problem

- The *standard form optimization problem* is defined to be:

$$\begin{aligned} & \text{minimize } f_0(x) \\ & \text{subject to } f_i(x) \leq 0, \quad i = 1, \dots, m \\ & \quad \quad \quad h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

- The optimal value will be denoted

$$p^* = \inf \{ f_0(x) \mid f_i(x) \leq 0, \quad h_j(x) = 0 \ (\forall i, j) \}$$

- By convention, $p^* = \infty$ if the problem is infeasible (no x satisfies the constraints), and $p^* = -\infty$ if the problem is unbounded below.
- A constraint $f_i, i > 0$ is said to be *active at* x_0 if $f_i(x_0) = 0$.

- A point x is said to be *feasible* if $x \in \text{dom}f_0$ and it satisfies the constraints.
- The *feasible set* is the set of all feasible x .
- A feasible x is said to be *optimal* if $f_0(x) = p^*$.
- We will use X_{opt} to denote the set of optimal points.
- A point z is *locally optimal* if it's optimal for the original problem with added constraint that $\|x - z\| \leq R$ for some $R > 0$.

- The standard form optimization problem has an implicit constraint

$$x \in \mathcal{D} = \bigcap_{i=0}^m \text{dom}(f_i) \cap \bigcap_{j=1}^p \text{dom}(h_j)$$

- This \mathcal{D} is called the domain of the problem.
- A problem is said to be *unconstrained* if it has no explicit constraints ($m = p = 0$).
- For example, minimization of

$$f_0(x) = - \sum_i \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints
 $a_i^T x < b_i$.

- The standard form *convex* optimization problem is the special case of the above in which f_0, f_1, \dots, f_m are convex and equality constraints are affine.
- In other words, the p equality constraints can be written as $Ax = b$ where $b \in \mathbb{R}^p$.
- We formalize these properties in the next definition.

Definition 2.8

The *standard form convex optimization problem* is

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

where all f_i are convex.

- Consider any of the functions $f_i(x)$, $i = 1, \dots, m$ in a convex optimization problem.
- The domain of a convex function is convex, so $\text{dom}(f_i)$ is a convex set.
- Furthermore, the inverse image of a half-space under a convex function is also a convex set, so

$$\{x \in \text{dom}(f_i) : f_i(x) \leq 0\}$$

is a convex set.

- The intersection of convex sets is convex, so the entire feasible set of a convex optimization problem is convex, as is the domain of the problem

$$\mathcal{D} = \bigcap_{i=1}^m \text{dom}(f_i).$$

Theorem 2.9

Any locally optimal point of a convex problem is (globally) optimal.

- *Proof.* Suppose x is locally optimal, which means $\exists R > 0$ such that

$$f_0(x') \geq f_0(x) \text{ for all feasible } x' \text{ with } d(x, x') \leq R \quad (2.6)$$

- Suppose also that $\exists y$ with $f_0(y) < f_0(x)$, hence $d(y, x) > R$ or else (2.6) would be a contradiction.
- Define

$$z = \theta y + (1 - \theta)x \text{ with } \theta := \frac{R}{2\|x - y\|} < \frac{1}{2}$$

- Note z is a convex combination of two feasible points, hence feasible itself.
- Note that $d(z, x) = \|z - x\| = \theta \|y - x\| = R/2$, so $z \in B(x; R)$.

- Finally,

$$f_0(z) \leq \theta f_0(y) + (1 - \theta)f_0(x) < f_0(x) \quad \text{since } f_0(y) < f_0(x)$$

which contradicts our assumption that x is locally optimal, so y with $f_0(y) < f_0(x)$ cannot exist.

- This completes the proof of Theorem 2.9.

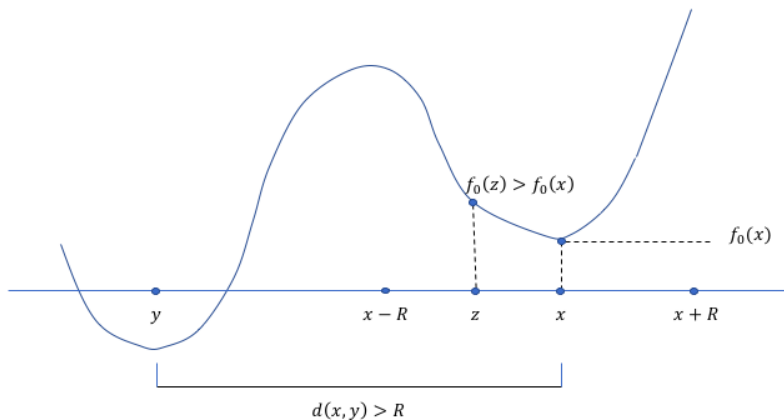


Figure: See proof of Theorem 2.9.

Theorem 2.10

If f is differentiable and convex, then any point $x_0 \in \text{dom} f$ satisfying $\nabla f(x_0) = 0$ is a global minimum of $f(x)$.

Proof. By the first-order condition for convexity,

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \forall y.$$

At $x = x_0$, $\nabla f(x_0) = 0$ and hence this reduces to $f(y) \geq f(x_0) \quad \forall y$.

□

- Two problems are (informally) *equivalent* if the solution of one is readily obtained from the solution of the other, and vice-versa.
- Convex problems can be equivalent to non-convex ones.
- There are some common transformations that preserve convexity, and which sometimes transform the problem into one that's easier to deal with.
- The standard-form convex problem in Definition 2.8 is equivalent to the following one:

$$\begin{aligned} &\text{minimize (over } z) : f_0(Fz + x_0) \\ &\text{subject to} : f_i(Fz + x_0) \leq 0, \quad i = 1, \dots, m \end{aligned}$$

where F and x_0 are such that

$$Ax = b \iff x = Fz + x_0 \text{ for some } z.$$

- Similarly, there is a technique known as introducing *slack variables* for linear inequalities.
- The problem

$$\text{minimize } f_0(x) \quad \text{subject to } a_i^T x \leq b_i, \quad i = 1, \dots, m$$

is equivalent to

$$\begin{aligned} &\text{minimize (over } x, s) : f_0(x) \\ &\quad \text{subject to : } a_i^T x + s_i = b_i, \quad \text{and } s_i \geq 0, \\ &\quad \quad \quad i = 1, \dots, m, \end{aligned}$$

The Lagrangian and the dual problem

- For a general optimization problem in standard form (not necessarily convex), define the *Lagrangian*
 $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$, with $\text{dom} L = \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p$ as follows:

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{j=1}^p \nu_j h_j(x).$$

- The elements of λ, ν are called *Lagrange multipliers*.

- Define the *Lagrange dual* function: $g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow R$, as

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu)$$

- Note that g is concave, because $-g$ is convex.
- To see this,

$$-g(\lambda, \nu) = \sup_{x \in \mathcal{D}} \{-f_0(x) - \lambda \cdot f(x) - \nu \cdot h(x)\}$$

- The right-hand side is convex by Theorem 2.5.
- Note also that g can be $-\infty$ for some λ, ν .

Theorem 2.11 (lower bound property)

If $\lambda \succeq 0$, then $g(\lambda, \nu) \leq p^*$.

Proof. For any feasible \tilde{x} , one has

$$f_0(\tilde{x}) \geq L(\tilde{x}, \lambda, \nu) \geq \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = g(\lambda, \nu)$$

where we used that $\lambda \succeq 0$. Minimizing over all feasible \tilde{x} gives $p^* \geq g(\lambda, \nu)$. \square

- The Lagrange dual is closely related to the conjugate function which we introduced above in (2.5).
- Consider the problem

$$\begin{aligned} & \text{minimize } f_0(x) \\ & \text{subject to } Ax \preceq b, Cx = d \end{aligned}$$

- The Lagrange dual function is

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \text{dom} f_0} \left(f_0(x) + (A^T \lambda + C^T \nu)^T x - b^T \lambda - d^T \nu \right) \\ &= -f_0^*(-A^T \lambda - C^T \nu) - b^T \lambda - d^T \nu \end{aligned} \quad (2.7)$$

- Hence it simplifies the computation of the dual function if the conjugate of f_0 is known.

- Recall that by Theorem 2.11, if $\lambda \succeq 0$, then $g(\lambda, \nu) \leq p^*$.
- This provides all sorts of lower bounds for p^* , but some are better than others.
- What if we maximized $g(\lambda, \nu)$ over the space of all $\lambda \succeq 0$?
- Indeed, the *Lagrange dual problem* is the following:

$$\text{maximize } g(\lambda, \nu) \text{ subject to } \lambda \succeq 0. \quad (2.8)$$

- If we could solve this problem, we would know the best lower bound on p^* that could be obtained by applying the lower bound property of the dual function.
- Eq. (2.8) is generally a convex optimization problem, even if the original one wasn't!
- The optimal value of the dual problem is typically denoted d^* .
- The pair λ, ν are said to be *dual feasible* if $\lambda \succeq 0$, and $(\lambda, \nu) \in \text{dom}(g)$.

- As an example, consider the standard form LP and its dual:

$$\text{minimize } c^T x \quad \text{subject to } Ax = b, \ x \succeq 0$$

- The Lagrangian is

$$\begin{aligned} L(x, \lambda, \nu) &= c^T x + \nu^T (Ax - b) - \lambda^T x \\ &= -b^T \nu + (c + A^T \nu - \lambda)^T x \end{aligned}$$

- L is affine in x , hence

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = \begin{cases} -b^T \nu & c + A^T \nu - \lambda = 0 \\ -\infty & \text{otherwise} \end{cases}$$

- Hence the dual problem is

$$\text{maximize } -b^T \nu \quad \text{subject to } A^T \nu + c \succeq 0$$

Strong duality and the duality gap

- The statement $d^* \leq p^*$ always holds, and is known as *weak duality*.
- The weak duality bound is sometimes useful for finding nontrivial lower bounds for difficult problems.
- The equality $d^* = p^*$ does not usually hold, but when it holds, it is known as *strong duality*.
- Strong duality usually does hold for convex problems, under conditions that we'll make precise very soon.
- The difference $p^* - d^*$ is known as the *duality gap*, and is always positive.

- More generally, dual feasible points allow us to bound how suboptimal a given feasible point is, without knowing the exact value of p^* .
- Indeed, if x is primal feasible and (λ, ν) is dual feasible, then

$$f_0(x) - p^* \leq f_0(x) - g(\lambda, \nu).$$

- We refer to the gap between primal and dual objectives, $f_0(x) - g(\lambda, \nu)$, as the duality gap associated with the primal feasible point $x, (\lambda, \nu)$.
- A primal-dual feasible pair $x, (\lambda, \nu)$ localizes the optimal value of the primal (and dual) problems to an interval:

$$p^*, d^* \in [g(\lambda, \nu), f_0(x)].$$

Slater's condition

- Any condition that guarantees strong duality in convex problems is called a *constraint qualification*.
- The most well-known one is *Slater's condition*, for which we need a few definitions.
- The *affine hull* $\text{aff}(S)$ of a set S in a vector space is the smallest affine set containing S , or equivalently, the intersection of all affine sets containing S .
- The *relative interior* of a set S , defined as:

$$\text{relint}(S) := \{x \in S : \exists \epsilon > 0, B(x, \epsilon) \cap \text{aff}(S) \subseteq S\},$$

- If this definition seems abstract, just think of the set

$$\{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}.$$

Its interior (with respect to \mathbb{R}^3) is empty, but its relative interior is an open unit disk in the xy -plane.

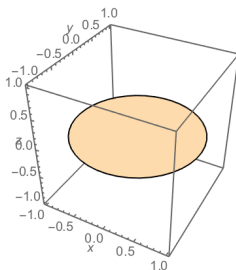


Figure: The shaded region is the relative interior of the set $\{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}$.

Theorem 2.12 (Slater's condition)

Strong duality holds for a convex problem if

$\exists x \in \text{relint}(\mathcal{D})$ such that: $f_i(x) < 0$ for $i = 1, \dots, m$, and $Ax = b$.
(2.9)

Under these conditions, the dual optimum is attained if $p^ > -\infty$.*

- When (2.9) holds, the problem is said to be *strictly feasible*, so a simple mnemonic for Slater's condition is: "strictly feasible implies strongly dual."
- The theorem can be sharpened in various ways.
- For example, affine inequalities do not need to hold with strict inequality.
- There exist non-convex problems with strong duality, it just isn't guaranteed in general.

Karush-Kuhn-Tucker (KKT) conditions

- We'll now begin some considerations which will lead us up to the important *Karush-Kuhn-Tucker (KKT) conditions*, which are practically what one uses to actually solve many optimization problems.

- Assume strong duality holds, x^* is primal optimal, (λ^*, ν^*) is dual optimal.
- Then

$$f_0(x^*) = g(\lambda^*, \nu^*) \quad \text{by strong duality}$$

and

$$\begin{aligned} g(\lambda^*, \nu^*) &= \inf_x L(x, \lambda^*, \nu^*) \\ &\leq L(x^*, \lambda^*, \nu^*) \\ &= f_0(x^*) + \sum_i \lambda_i^* f_i(x^*) + \sum_j \nu_j^* h_j(x^*) \end{aligned}$$

where the inequality is because, for any function $\phi(x)$ and for any single point x^* , one has $\inf_x \phi(x) \leq \phi(x^*)$.

- Now, because x^* is feasible, all $h_j(x^*) = 0$.
- So we have

$$f_0(x^*) = g(\lambda^*, \nu^*) \leq f_0(x^*) + \sum_i \lambda_i^* f_i(x^*)$$

- By dual feasibility, $\lambda_i^* \geq 0$ for all i .
- By primal feasibility, $f_i(x^*) \leq 0$ for all i , so the product satisfies

$$\lambda_i^* f_i(x^*) \leq 0$$

- Hence the sum also satisfies $\sum_i \lambda_i^* f_i(x^*) \leq 0$.

- So we have

$$f_0(x^*) \leq f_0(x^*) + \underbrace{\sum_i \lambda_i^* f_i(x^*)}_{\text{negative or zero}}$$

- Subtracting $f_0(x^*)$ from both side means that the “negative or zero” bit is, in fact, zero and the inequality is, in fact, an equality.

- To summarize: if strong duality holds, x^* is primal optimal, (λ^*, ν^*) is dual optimal, then

$$\lambda_i^* f_i(x^*) = 0 \quad \text{for all } i = 1, \dots, m.$$

- This gives us a way of seeing which constraints are active at the solution.
- If some $\lambda_i^* > 0$ then $f_i(x^*)$ must be zero, meaning the constraint is *active*, or alternatively if $f_i(x^*) < 0$ (meaning the constraint isn't active) then $\lambda_i^* = 0$.
- So any active constraints will have nonzero Lagrange multipliers in the solution to the dual problem.
- Any non-active constraints will have their corresponding Lagrange multipliers in the dual-optimal solution equal to zero.

- Since the above chain of inequalities is an equality, we have shown that

$$\inf_x L(x, \lambda^*, \nu^*) = L(x^*, \lambda^*, \nu^*)$$

- In other words, x^* minimizes $L(x, \lambda^*, \nu^*)$.
- We have thus constructed an *unconstrained* optimization problem with the same optimal point x^* as the original, constrained problem.
- One could of course argue that the constraints are still there, but buried in the fact that one must solve the dual problem to find λ^*, ν^* .

Definition 2.13

A *primal-dual pair* is a pair

$$\zeta = \{x, (\lambda, \nu)\} \in \mathcal{D} \times \text{dom}(g).$$

In a problem with differentiable f_i, h_i , the primal-dual pair ζ is said to satisfy the *KKT conditions* if the following four things hold:

- ❶ primal constraints: $f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p$
- ❷ dual constraints: $\lambda \succeq 0$
- ❸ complementary slackness: $\lambda_i f_i(x) = 0, i = 1, \dots, m$
- ❹ first-order condition:

$$\nabla f_0(x) + \sum_i \lambda_i \nabla f_i(x) + \sum_j \nu_j \nabla h_j(x) = 0.$$

- From our discussion above, if strong duality holds and x, λ, ν are optimal, then they must satisfy the KKT conditions.
- This is the case for convex and nonconvex problems alike.

- Furthermore, if $\tilde{x}, \tilde{\lambda}, \tilde{\nu}$ satisfy KKT for a convex problem, then they are optimal.
- To see this, note that the first two KKT conditions state that \tilde{x} is primal feasible.
- Since $\tilde{\lambda} \succeq 0$, it follows that $L(x, \tilde{\lambda}, \tilde{\nu})$ is convex in x ; the last KKT condition states that its gradient with respect to x vanishes at $x = \tilde{x}$, so it follows that \tilde{x} minimizes $L(x, \tilde{\lambda}, \tilde{\nu})$ by the first-order condition.
- Furthermore,

$$f_0(\tilde{x}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu}) = g(\tilde{\lambda}, \tilde{\nu}).$$

- In summary, for any convex optimization problem with differentiable objective and constraint functions, any points that satisfy the KKT conditions are primal and dual optimal, and have zero duality gap.

- If Slater's condition is satisfied in a convex problem, then x is optimal if and only if there exist λ, ν that satisfy the KKT conditions with the given x .
- This can be viewed as a generalization of the optimality condition $\nabla f_0(x) = 0$ for unconstrained problems.

- A point x_0 is said to be *regular* if the gradients of the active constraints at x_0 are linearly independent.
- For problems without strong duality, the KKT conditions are necessary for *local* optimality under a mild assumption: If x^* is locally optimal and x^* is a regular point then there exists λ^*, ν^* such that the KKT conditions hold.

- An important example in which the KKT conditions can be written down, and solved, explicitly is *equality constrained convex quadratic minimization*:

$$\begin{aligned} &\text{minimize } (1/2)x^T Px + q^T x + r \\ &\text{subject to } Ax = b \end{aligned}$$

where $P \in S_+^n$.

- The KKT conditions for this problem are

$$Ax^* = b, \quad Px^* + q + A^T \nu^* = 0$$

which we can write equivalently as a single matrix equation,

$$\begin{pmatrix} P & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x^* \\ \nu^* \end{pmatrix} = \begin{pmatrix} -q \\ b \end{pmatrix} \quad (2.10)$$

- Solving this set of $m + n$ equations in the $m + n$ variables (x^*, ν^*) gives the optimal primal and dual variables.
- Assuming the relevant inverses exist, one could apply the block inversion formula to get an explicit formula for x^* that doesn't depend on ν^* (exercise).

- Now consider a version of the above in which we have linear inequality constraints.
- For simplicity, we'll drop the linear term in f_0 .
- So the “primal” or original problem is, for $P \in S_{++}^n$,

$$\text{minimize } x^T P x \quad \text{subject to } Ax \preceq b.$$

- The dual function is

$$g(\lambda) = \inf_x \left(x^T P x + \lambda^T (Ax - b) \right) = -\frac{1}{4} \lambda^T A P^{-1} A^T \lambda - b^T \lambda.$$

- The dual problem is then maximizing a quadratic-plus-linear function $g(\lambda)$ over the space $\lambda \succeq 0$, which seems simpler.
- Slater's condition specifies that $p^* = d^*$ if $A\tilde{x} \prec b$ for some \tilde{x} , but in fact more is true: for this class of problems, $p^* = d^*$ in any case.

Sensitivity Analysis

- We want to be able to analyze how sensitive the optimal value is to changes in the constraints.
- To this end, define the *perturbed problem* as

$$\text{minimize } f_0(x) \text{ subject to } f_i(x) \leq u_i, h_j(x) = v_j$$

for some vectors $u \in \mathbb{R}^m, v \in \mathbb{R}^p$.

- In other words, the perturbed problem is just like the original, but with potentially non-zero parameters u_i and v_j on the right-hand sides, where formerly one had zeros.
- When u_i is positive it means that we have relaxed the i -th inequality constraint; when u_i is negative, it means that we have tightened the constraint.
- Define $p^*(u, v)$ as the optimal value of the perturbed problem, noting that $p^*(0, 0)$ equals what we called p^* before.

- Suppose now that $p^*(u, v)$ is differentiable at $u = 0, v = 0$.
- Then, provided strong duality holds, the optimal dual variables λ^*, ν^* are related to the gradient of $p^*(\cdot, \cdot)$ at $u = 0, v = 0$:

$$\lambda_i^* = -\frac{\partial p^*}{\partial u_i}(0, 0), \quad \nu_j^* = -\frac{\partial p^*}{\partial v_j}(0, 0). \quad (2.11)$$

- Thus, assuming the conditions above (differentiability and strong duality), the optimal Lagrange multipliers are exactly the local sensitivities of the optimal value with respect to constraint perturbations. Tightening the i -th inequality constraint a small amount (i.e., taking u_i small and negative) yields an increase in p^* of approximately $-\lambda_i^* u_i$.
- In the same way, loosening the i -th constraint a small amount (i.e., taking u_i small and positive) yields a decrease in p^* of approximately $-\lambda_i^* u_i$.

- There is an economic interpretation of (2.11).
- We consider (for simplicity) a convex problem with no equality constraints, which satisfies Slater's condition.
- The variable x determines how a firm operates, and the objective f_0 is -1 times the profit.
- Each constraint $f_i(x) \leq 0$ represents a limit on some resource such as labor, steel, or warehouse space.
- Then $-p^*(u)$ tells us how much more or less profit could be made if more, or less, of each resource were made available to the firm.

- According to (2.11), λ_i^* tells us approximately how much more profit the firm could make, for a small increase in availability of resource i .
- It follows that λ_i^* would be the natural or equilibrium price for resource i , if it were possible for the firm to buy or sell it.
- If the firm could buy resource i , at a price that is less than λ_i^* , then it should do so because it could then operate in a way that increases its profit more than the cost of buying the resource.
- For this reason λ_i^* is called the *shadow price*.

Computational Frameworks

- There exist a large number of computational tools for performing numerical optimization.
- Since convex optimization problems are particularly well-behaved – any local minimum is a global one – it stands to reason that many general-purpose solvers do give reasonable results when applied, in particular, to convex problems.

- Nevertheless, as our reference for convex optimization is Boyd and Vandenberghe (2009), it is especially noteworthy that the authors of this reference have, with their collaborators in Diamond and Boyd (2016), also overseen the development of a set of python programs known as CVXPY.
- The purpose of CVXPY is to allow the user to specify convex optimization problems in familiar mathematical notation, before passing the problem (in the correct form!) to an unseen underlying lower-level optimizer.
- It automatically transforms the problem into standard form, calls a solver, and unpacks the results.

- The default solver called by CVXPY under the hood is ECOS (Embedded Conic Solver), see Domahidi, Chu, and Boyd (2013) for more details.
- Disciplined convex programming (DCP) is a system for constructing mathematical expressions with known curvature from a given library of base functions.
- Another feature of CVXPY is that it is capable of using DCP to check that the specified optimization problems, as written by the user, are indeed convex problems.
- Lest the unsuspecting user write a problem which, in spite of their best intentions, turned out not to be a convex one, and hence nullifying the usual guarantees about local minima.

- We show a basic example involving constrained regression.


```
from cvxpy import *
import numpy
import matplotlib.pyplot as plt

# Problem data.
n = 15
m = 10
numpy.random.seed(1)
A = numpy.random.randn(n, m)
b = numpy.random.randn(n, 1)

# gamma must be positive due to DCP rules.
gamma = Parameter(sign="positive")

# Construct the problem.
x = Variable(m)
obj = Minimize(sum_squares(A*x - b) + gamma*norm(x, 1))
prob = Problem(obj)
```

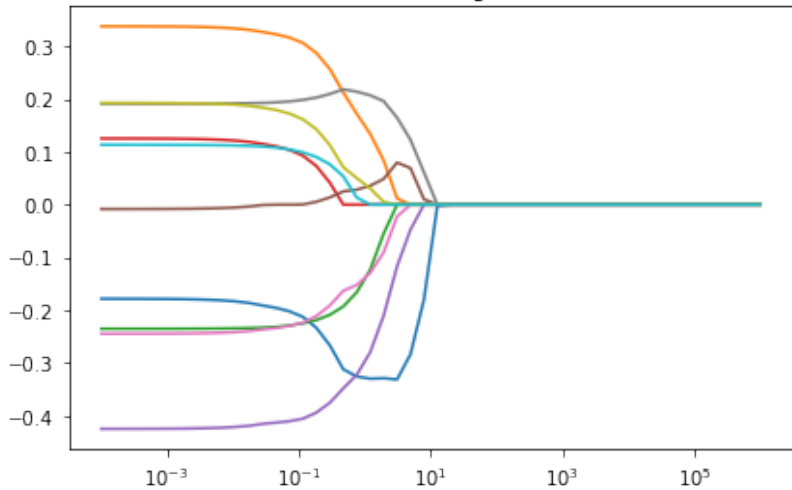
```

# Construct a trade-off curve of  $\|Ax-b\|^2$  vs.  $\|x\|_1$ 
x_values = []
gamma_vals = numpy.logspace(-4, 6)
for val in gamma_vals:
    gamma.value = val
    prob.solve()
# Use expr.value to get the numerical value
x_values.append(x.value)

for i in range(m):
    plt.plot(gamma_vals, [xi[i,0] for xi in x_values])
plt.xscale('log')
plt.title('Entries of  $x$  vs  $\gamma$ ')
plt.tight_layout()
plt.show()

```

Entries of x vs gamma



Problem 2.14

Prove theorem 2.2, theorem 2.3, and theorem 2.4.

Problem 2.15

Referring to (2.10), find the explicit solution for x^ by applying the block matrix inversion formula. Which matrices must you assume are invertible for this formula to apply? If P is positive semi-definite but has a non-trivial null space, what can you say about existence/uniqueness of solutions to the original equality-constrained optimization problem that lead to (2.10)?*

Problem 2.16

Referring to the sub-section on Sensitivity Analysis, suppose that $p^(u, v)$ is differentiable at $u = 0, v = 0$. Then, provided strong duality holds, show that*

$$\lambda_i^* = -\frac{\partial p^*}{\partial u_i}(0, 0), \quad \nu_j^* = -\frac{\partial p^*}{\partial v_j}(0, 0).$$

In other words show (2.11).





Boyd, Stephen and Lieven Vandenberghe (2009). *Convex optimization*. Cambridge university press.



Diamond, Steven and Stephen Boyd (2016). "CVXPY: A Python-Embedded Modeling Language for Convex Optimization". In: *Journal of Machine Learning Research* 17.83, pp. 1–5.



Domahidi, A., E. Chu, and S. Boyd (2013). "ECOS: An SOCP solver for embedded systems". In: *European Control Conference (ECC)*, pp. 3071–3076.