Problem 1.13

<u>If</u>:

If $u()$ is concave, by Jensen's ineq. we have

$$\mathbb{E}(u(w+\tilde{z})) \le u(\mathbb{E}(w+\tilde{z}))$$

$$= u(w + \mathbb{E}(\tilde{z}))$$

$$= u(w) \quad (w \text{ is const}, \mathbb{E}(\tilde{z}) = 0)$$

<u>Only If</u>:

Assume for $\forall w$, $\forall \tilde{z}$ with $\mathbb{E}(\tilde{z}) = 0$ we have

$$\mathbb{E}(u(w+\tilde{z})) \le u(\mathbb{E}(w+\tilde{z}))$$

then for $\forall a, b \in \mathbb{R}$ and for $\lambda \in [0,1]$, let $w = \lambda a + (1-\lambda) b$ and $\tilde{z} = \begin{cases} a-w & \mathbb{P}=\lambda \\ b-w & \mathbb{P}=1-\lambda \end{cases}$

then we have

$$u(\lambda a + (1-\lambda) b) = u(w + \mathbb{E}(\tilde{z}))$$

$$\geq \mathbb{E}(u(w+\tilde{z}))$$

$$= \lambda u(w+a-w) + (1-\lambda) u(w+b-w)$$

$$= \lambda u(a) + (1-\lambda) u(b)$$

so by definition u is concave!

QED

Problem 1.14

The Arrow-Pratt approximation is

$$\Pi \approx \frac{1}{2}\sigma^2 A(w) = \frac{1}{2}k\sigma^2$$

given $u(w) = -\frac{1}{k}\exp(-kw)$, then we want to prove ~~that~~

$$\mathbb{E}[u(w+\tilde{z})] = u(w-\Pi)$$

$$LHS = \mathbb{E}[u(\tilde{w})] = \mathbb{E}\left[-\frac{1}{k}e^{-k\tilde{w}}\right]$$

$$= -\frac{1}{k}\mathbb{E}\left[e^{-k\tilde{w}}\right] \leftarrow \text{log normal}$$

$$= -\frac{1}{k}\exp\left(-k\mu + \frac{k^2\sigma^2}{2}\right)$$

$$RHS = u(w-\Pi) = -\frac{1}{k}\exp\left(-kw + \frac{k^2\sigma^2}{2}\right)$$

$$= -\frac{1}{k}\exp\left(-k\mu + \frac{k^2\sigma^2}{2}\right)$$

$$= LHS$$

QED

Problem 1.15

(a) $\Rightarrow$ (b)

v is more risk-averse than u

$\Rightarrow \Pi_v \geq \Pi_u$

$\Rightarrow \frac{1}{2}\sigma^2 A_v(w) \geq \frac{1}{2}\sigma^2 A_u(w)$

$\Rightarrow A_v(w) \geq A_u(w)$

(b) $\Rightarrow$ (c)

$$A_v(w) = -\frac{v''(w)}{v'(w)}, \quad A_u(w) = -\frac{u''(w)}{u'(w)}$$

$$A_v(w) \geq A_u(w) \Rightarrow -\frac{v''(w)}{v'(w)} \geq -\frac{u''(w)}{u'(w)}$$

$$\Rightarrow \frac{v''(w)}{v'(w)} \leq \frac{u''(w)}{u'(w)}$$

$$\Rightarrow u'(w)\,v''(w) \le v'(w)\,u''(w)$$

$$\Rightarrow v''(w) \le \frac{v'(w)}{u'(w)} \cdot u''(w)$$

assume $v''(w) = \phi''(u)u'(w) + \phi'(u)u''(w)$

then we have

$$\begin{cases} \phi''(w) \le 0 \quad (\text{since } u'(w) > 0) \\ \phi'(w) = \dfrac{v'(w)}{u'(w)} \ge 0 \end{cases}$$

so $\phi(w)$ is concave function. According to chain rule it's easy to show that

$$v(w) = \phi(u(w)) + h(u(w))$$

where $h(w)$ is affine and doesn't affect the concavity of ϕ.

(c) $\Rightarrow$ (a)

$$v(w - \Pi_v) = \mathbb{E}\left(v(w + \tilde{z})\right)$$

$$= \mathbb{E}\left(\phi(u(w + \tilde{z}))\right)$$

$$\le \phi\left[\mathbb{E}(u(w+\tilde{z}))\right]$$

$$= \phi[u(w-\Pi u)]$$

$$= v(w-\Pi v)$$

Since v is increasing we conclude

$$\Pi v \ge \Pi u$$

Thus we can say (a), (b), (c) are equivalent.

Problem 1.16

$$v(x) = a + bu(x)$$

$$\Rightarrow \begin{cases} v'(x) = bu'(x) \\ v''(x) = bu''(x) \end{cases}$$

$$\Rightarrow A_v(x) = -\frac{v''(x)}{v'(x)}$$

$$= -\frac{bu''(x)}{bu'(x)}$$

$$= -u''(x)/u'(x)$$

$$= A_u(x)$$

they have the same risk-aversion!

QED