**Problem 1.** *Prove Lemma 11.1.*

**Problem 2.** *Consider optimally trading a single stock over $T = 30$ days. Each period is one day, and you can trade once per day. The stock's daily return volatility is $\sigma$. Suppose your forecast is 50 basis points for the first period, and decays exponentially with half-life 5 days. This means that*

$$\alpha_t := \mathbb{E}[r_{t,t+1}] = 50 \times 10^{-4} \times 2^{-t/5}. \tag{0.1}$$

*Let $c(\delta)$ be the cost, in dollars, of trading $\delta$ dollars of this stock. For selling, $\delta < 0$. Following Almgren, we assume that*

$$c(\delta) = PX\left(\frac{\gamma\sigma}{2}\frac{X}{V}\left(\frac{\Theta}{V}\right)^{1/4} + \operatorname{sign}(X)\,\eta\sigma\left|\frac{X}{V}\right|^{\beta}\right), \quad X = \delta/P$$

*where $P$ is the current price in dollars, $X$ is the signed trade size in shares, $V$ is the daily volume in shares, $\Theta$ is the total number of shares outstanding, and finally $\gamma = 0.314$ and $\eta = 0.142$ and $\beta = 0.6$ are constants fit to market data. For concreteness, suppose the asset we are trading has*

$$P = \$40,\ V = 2 \times 10^6,\ \Theta = 2 \times 10^8, \quad \sigma = 0.02.$$

*For a trading path $\boldsymbol{x} = (x_0, x_1, \ldots, x_T)$ where $x_t$ denotes dollar holdings of the stock at time $t$, define the profit (also in dollars) as*

$$\pi(\boldsymbol{x}) = \sum_{t=1}^{t}[x_t r_{t,t+1} - c(x_t - x_{t-1})]$$

*This is a random variable due to the presence of $r_{t,t+1}$ which you can assume is Gaussian with mean $\alpha_t := \mathbb{E}[r_{t,t+1}]$ and variance $\mathbb{V}[r_{t,t+1}] = \sigma^2$. In this problem, always assume $x_0 = 0$ is fixed.*

(a) *Find the sequence of positions $x_1, x_2, \ldots, x_T$ that maximizes*

$$u(x_1, \ldots, x_T) = \sum_{t=1}^{T}\left[x_t\alpha_t - \frac{\kappa}{2}\sigma^2 x_t^2 - c(x_t - x_{t-1})\right] \tag{0.2}$$

*with risk-aversion $\kappa = 10^{-7}$. Set tolerance so that your algorithm does not terminate unless each $x_t \in \mathbb{R}$ is within a distance of one dollar to the true optimal path. Plot the optimal path $\boldsymbol{x}^* := (x_0 = 0, x_1^*, \ldots, x_T^*)$ and also report its values in a table. Also report the computation time.*

*Submit your code and a clear explanation of the algorithm you used, why your chose it over other possible algorithms, and how you know that it converges. For example, if you used a method that requires convexity, explain why the function you are optimizing is convex.*

(b) *Use the program you wrote in part (a) to plot expected profit of the optimal path, $\mathbb{E}[\pi(\boldsymbol{x}^*)]$ and* ex ante *Sharpe ratio of the optimal path, defined as*

$$\textit{Sharpe}(\boldsymbol{x}^*) = \sqrt{252}\ \frac{\mathbb{E}[\pi(\boldsymbol{x}^*)]}{\sqrt{\mathbb{V}[\pi(\boldsymbol{x}^*)]}}$$

*as a function of $\kappa$, as a function of the half-life (which was taken to be 5 in equation (0.1) above), as a function of the initial strength (taken to be 50 in equation (0.1)), and as a function of $\sigma$. So you need to do eight plots in all: profit and Sharpe ratio, each as a function of one of four parameters (holding the others fixed). Choose appropriate intervals around the parameter values in part (a). Note that $\kappa$ cannot be negative in reasonable models.*