Notes for M.A. Armstrong's *Groups and Symmetry*

Christian Stigen Larsen

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1 Symmetries of the Tetrahedron

Symmetry group Captures the rules of how symmetries combine for a given object.

Order of operations In the *product*[*] xyz, do z first, then y and finally x. If order doesn't matter in G, it's commutative (or ***abelian***). Remember to label geometric vertices.

2 Axioms

Group Set G with *multiplication* (addition, rotation, etc.) satisfying

- ***associativity***, i.e. $(xy)z = x(yz)$
- ***identity element*** $e \in G$ such that $xe = x = ex$
- ***inverse*** $e \in G$ such that $x^{-1}x = e = xx^{-1}$

Properties common to all groups

- The identify element of a group is unique.
- The inverse of each element of a group is unique.

3 Numbers

Addition of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

- Identity is zero
- $-x$ is the inverse

Multiplication

- For $\mathbb{Q} - \{0\}$, $\mathbb{Q}^{\mathbf{pos}}$, $\mathbb{R} - \{0\}$, $\mathbb{R}^{\mathbf{pos}}$, $\{+1, -1\}$, $\mathbb{C} - \{0\}$, $\mathcal{C}^{\dagger}$, $\{\pm 1, \pm i\}$: $e = 1$ and $x^{-1} = 1/x$.

$\mathbb{Z}$ **under addition modulus** n $e = 0$, $x^{-1} = n - x$ for $x \neq 0$, finite *abelian* group and denoted $\mathbb{Z}_n$.

$\mathbb{Z}$ **under multiplication modulus** n Requires n to be prime.

4 Dihedral Groups

When $n \geqslant 3$ we can manufacture a plate whish has n equal sides. These are the non-commutative *dihedral rotational symmetry groups* D_n. E.g. $D_3 = \{e, r, r^2, s, rs, r^2s\}$. $x^m x^n = x^{m+n}$ and $(x^m)^n = x^{mn}$ provided we interpret $x^0 = e$. For any multiplication table, each element in G appears only once in every given column or row.

$r^n = e$, $s^2 = e$, $sr = r^{n-1}s$, $r^{n-1} = r^{-1}$, etc.

Each element is of form r^a, $r^a s$ where $0 \leqslant a \leqslant n-1$.

For $k = a +_n b$, $r^a r^b = r^k$ and $r^a(r^b s) = r^k s$. For $l = a +_n (n-b)$, $(r^a s) r^b = r^l s$ and $(r^a s)(r^b s) = r^l$ — thus r and s ***generate*** D_n.

The ***order*** $|G|$ is the number of elements in the group. If $x^n = e$, then the *element* x has *finite* order n when n is the smallest such n.

5 Subgroups and Generators

A ***subgroup*** of G is a subset of G which itself forms a group under the multiplication of G. For H to be a

[*] Rotations, flips, multiplications, additions, etc. Same order as functional composition.

[†] Complex numbers of modulus 1.

subgroup of G, $H < G$:

- $xy \in G$ for any $x, y \in H$
- $e_H \in G$
- For any $x \in H$, $x^{-1} \in G$
- Associativity in G implies the same for H.

Subgroup generated by x**, or** $\langle x \rangle$ For an element x in G, the set of all x^n is a subgroup of G (remember $x^0 = e$). Finite order m means $x^0 = e, x^1, \ldots, x^{m-1}$. So order of $x \in G$ is precisely the order of $\langle x \rangle$. If $\langle x \rangle = G$, i.e., generates all of G, then G is a ***cyclic group***.

Subgroup generated by X If $X < G$[‡] and, for example, r,s,r^2,sr (called *words* of X).

Theorems

- A non-empty subset H of a group G is a subgroup of G if and only if xy^{-1} belongs to H whenever x and y belong to H.
- The intersection of two subgroups of a group is itself a subgroup.
- Every subgroup of $\mathbb{Z}$ is cyclic. Every subgroup of a cyclic group is cyclic.

6 Permutations

A *permutation* is a bijection[§] from a set X to itself (e.g., replace all 3s with 1s). The collecticon of *all* permutations of X forms a group S_x under composition of functions (who each perform one specific permutation). When X consists of the first n positive integers, we get the ***symmetric group*** S_n of degree n and order $n!$. S_3 is not abelian

$(a_1 a_2 \ldots a_k)$ is called a ***cyclic permutation***, sending a_1 to a_2, ..., a_k to a_1. Its length is k and a cyclic permutation of length k is called a ***k-cycle***. A 2-cycle is called a ***transposition***. Every element of S_n can be written as many such ***disjoint***, meaning no integer is moved by more than one of them. Therefore they are *commutative*.

[‡] X is a subgroup of G.

[§] A one-to-one mapping between the elements of two sets, meaning you can always go backwards as well.

A few tricks

- Each *element* of S_n can be written as a product of cyclic permutations, and any cyclic permutation can be written as a product of transpositions: $(a_1 a_2 \ldots a_k) = (a_1 a_k) \ldots (a_1 a_3)(a_1)(a_2)$. Therefore, each *element* of S_n can be written as a product of transpositions.
- $(ab) = (1a)(1b)(1a)$
- $(1k) = (k-1, k)\ldots(34)(23)(12)(23)(34)\ldots(k-1, k)$

Theorems

- The transpositions in S_n together generate S_n.
- The transpositions $(12), (13), \ldots, (1n)$ together generate S_n.
- The transpositions $(12), (23), \ldots, (n-1, n)$ together generate S_n.
- The transposition (12) and the n-cycle $(12 \ldots n)$ together generate S_n.

Any *element* α of S_n can be written as a product of *transpositions* in many different ways. But the number of transpositions is always even or always odd. If α *can* be written as the product of an even number of transpositions, then its sign must be $+1$; for odd, it is -1. Therefore, by the first trick above, a *cyclic permutation* is even precisely when its length is odd.

Theorems

- The even permutations in S_n form a subgroup of order $n!/2$ called the ***alternating group*** A_n of degree n.
- For $n \geqslant 3$ the 3-cycles generate A_n.

7 Isomorphisms

If two multiplication tables have corresponding elements and products, they are ***isomorphic***.

Theorem Two groups G and G' are ***isomorphic*** if there is a bijection ϕ from G to G' which satisfies $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$. The function ϕ is called an ***isomorphism*** between G and G'. This is written $G \cong G'$.