Selected Notes on Infinite Series

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Geometric series

$\sum_{n=0}^{\infty} a_n$ where $|a_{n+1} = ra_n$ for all $n \geq 0$ is called a *geometric series*, i.e. each term is a multiple r of the previous.

Theorem If $|r| < 1$, then the series converges with the sum $\frac{a_0}{1-r}$. If $|r| \geq 1$, it diverges.

Theorem If $\lim_{n\to\infty} a_n \neq 0$ or the limit does not exist, then $\sum a_n$ diverges.

Theorem The harmonic series $\sum \frac{1}{n}$ diverges.

The integral test

Theorem If $f(n) = a_n$ for all n, then $\sum a_n$ and $\int f(x)\,\mathrm{d}x$ either both converge or diverge.

The comparison test

Suppose $\sum a_n$ and $\sum b_n$ are positive-term series. Then

1. $\sum a_n$ converges if $\sum b_n$ converges *and* $a_n \leq b_n$ for all n.
2. $\sum a_n$ diverges if $\sum b_n$ diverges *and* $a_n \geq b_n$ for all n.

The limit comparison test

Suppose $\sum a_n$ and $\sum b_n$ are positive-term series. If $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists and $0 < L < +\infty$, then either *both converge* or *both diverge*.

Alternating series

Theorem If $a_n > a_{n+1} > 0$ for all n and $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Theorem If the series $\sum |a_n|$ converges (*absolute convergence*), then so does the series $\sum a_n$.

The ratio test

Theorem If $\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ exists or is infinite, then the $\sum a_n$ of non-zero terms *converges absolutely* if $\rho < 1$, diverges if $\rho > 1$ and is inconclusive if $\rho = 1$.

The root test

If $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists or is infinite, then the infinite series $\sum a_n$ *converges absolutely* if $\rho < 1$, diverges if $\rho > 1$. The test is inconclusive for $\rho = 1$.

Power series

Theorem If $\sum a_n x^n$ is a power series, then either

1. The series converges absolutely for all x, or
2. The series converges only when $x = 0$, or
3. There exists a number $R > 0$ (*radius of convergence*) such that $\sum a_n x^n$ converges absolutely if $|x| < R$ and diverges if $|x| > R$.

After finding the open interval $\langle x_0 - R, x_0 + R\rangle$, make sure to test each end point for convergence or divergence.