MAT200 — Mathematical Methods 2

Compulsory Assignment 1

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Problem 1 (i)

The radicand must be zero or greater, or

$$\begin{aligned} 0 &\leqslant x^2 - 6x + \frac{1}{4}y^2 \,\Big|\, \cdot -4, +y^2 \\ y^2 &\geqslant 4x(6-x) \end{aligned}$$

The right-hand side is zero for $x = 0$ and $x = 6$, and negative for $x < 0$ and $x > 6$. Thus, the inequality is true for $0 < x < 6$.

For plotting $\mathcal{D}(f)$, we note that

$$y_- \leqslant -2\sqrt{x(6-x)} \wedge y_+ \geqslant 2\sqrt{x(6-x)} \text{ for } 0 < x < 6 \tag{1}$$

In fact, taken together, they form an ellipse—see figure 1 on page 8.

Problem 1 (ii)

$C = 0$ is given by (1) above,

$$\begin{aligned} f(x,y) = \sqrt{x^2 - 6x + \frac{1}{4}y^2} &= 0 \\ x^2 - 6x + \frac{1}{4}y^2 &= 0 \\ y &= \pm 2\sqrt{x(6-x)} \text{ for } 0 \leqslant x \leqslant 6 \end{aligned}$$

$C = 4$ is given by

$$\begin{aligned} f(x,y) = \sqrt{x^2 - 6x + \frac{1}{4}y^2} &= 4 \\ \frac{1}{4}y^2 &= 16 - x^2 + 6x \\ y &= \pm 2\sqrt{(x+2)(8-x)} \end{aligned}$$

The plot of these two level-curves (or *contour curves*) is given in figure 1 on page 8. Note that we could have used scaling and shifting, which is much easier, but I felt like doing it this way for fun.

Problem 1 (iii)

Any level-curve is defined by $f(x, y) = C$. For the one corresponding to $(5, 6)$, we could insert $x_0 = 5$ and $y_0 = 6$ and find C, but we don't need to: When we perform implicit differentiation of the resulting expression—to find the slope in $(5, 6)$, or any point, in fact—we see that $C' = 0$. Therefore we go straight to differentiation.

$$\begin{aligned}
C &= x^2 - 6x + \frac{1}{4}y^2 \\
0 &= 2x - 6 + \frac{1}{2}yy' \\
y' &= \frac{12 - 4x}{y} \\
y'(5, 6) &= -\frac{4}{3}
\end{aligned}$$

Now that we have the slope, we can find the tangent line going through this point by back-calculating the y-value for $x = 0$ to find ℓ.

$$\begin{aligned}
\ell_{(x_0, y_0)} &= y_0 - x_0 y'(x_0, y_0) + y'(x_0, y_0)x \\
\ell_{(5,6)} &= 6 + \frac{4 \cdot 5}{3} - \frac{4}{3}x \\
\ell_{(5,6)} &= -\frac{4}{3}x + \frac{38}{3}
\end{aligned}$$

This is plotted in figure 2 on page 9.

Problem 2 (i)

$$f(x,y,z) = e^z + e^{2y}\arctan\left(\frac{y}{x}\right)$$

$$\frac{\partial f}{\partial x} = e^{2y}\frac{1}{\frac{x^2}{y^2}+1}\left(\frac{-y}{x^2}\right) = \frac{-ye^{2y}}{x^2+y^2}$$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(uv) = u'v + uv' \text{ where } u = e^{2y} \text{ and } v = \arctan\frac{y}{x} \\
&= 2e^{2y}\arctan\left(\frac{y}{x}\right) + \frac{e^{2y}}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = 2e^{2y}\arctan\left(\frac{y}{x}\right) + \frac{xe^{2y}}{x^2+y^2} \\
\frac{\partial f}{\partial z} &= e^z
\end{aligned}$$

Problem 2 (ii)

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{-ye^{2y}}{x^2+y^2}\right) = \frac{2xye^{2y}}{(x^2+y^2)^2}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(2e^{2y}\arctan\left(\frac{y}{x}\right) + \frac{xe^{2y}}{x^2+y^2}\right) \\
&= \frac{-2ye^{2y}}{x^2+y^2} + \frac{e^{2y}(x^2+y^2) - xe^{2y}\cdot 2x}{(x^2+y^2)^2} = -e^{2y}\frac{2y(x^2+y^2) - (x^2+y^2) + 2x^2)}{(x^2+y^2)^2} \\
&= -\frac{e^{2y}}{(x^2+y^2)^2}\left(2yx^2 + 2y^3 - x^2 - y^2 + 2x^2\right) \\
&= -\frac{e^{2y}}{(x^2+y^2)^2}\left(y^2(2y-1) + x^2(2y+1)\right)
\end{aligned}$$

Problem 2 (iii)

$$g(s,t) = f(s, st, s+t) = e^{s+t} + e^{2st} \arctan t$$

$$\frac{\partial g}{\partial s} = e^{s+t} + 2te^{2st} \arctan t$$

$$\begin{aligned}\frac{\partial g}{\partial t} &= e^{s+t} + (uv)' \text{ where } u = e^{2st} \text{ and } v = \arctan t \\ &= e^{s+t} + u'v + u + v' \\ &= e^{s+t} + 2se^{2st} \arctan t + \frac{e^{2st}}{1+t^2}\end{aligned}$$

Problem 3 (i)

To find the tangent plane, we must find $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$. We start by organizing F, setting $g = xy - \frac{\pi}{4}$ and getting $g'_x = y$ and $g'_y = x$.

$$\begin{aligned}F &= (2y-2x)\sin^2(xy - \frac{\pi}{4}) = 2y\sin^2(xy - \frac{\pi}{4}) - 2x\sin^2(xy - \frac{\pi}{4}) \\ &= 2y\sin^2 g - 2x\sin^2 g\end{aligned}$$

$$\begin{aligned}F_x &= 0 + 2y^2 2\sin g\cos g - 2\sin^2 g - 2xy2\sin g\cos g \\ &= 2y^2\sin 2g - 2\sin^2 g - 2xy\sin 2g \\ &= 2y(y-x)\sin 2g - 2\sin^2 g\end{aligned}$$

$$\begin{aligned}F_y &= 2\sin^2 g + 2y2x\sin g\cos g - 0 - 2x2x\sin g\cos g \\ &= 2\sin^2 g + x\sin 2g(2y-2x)\end{aligned}$$

Inserting the point $(0,2)$ gives

$$\begin{aligned}g(0,2) &= -\frac{\pi}{4} \\ F_x(0,2) &= 8\sin -\frac{\pi}{2} - 2\sin^2 -\frac{\pi}{4} = -8 - 1 = -9 \\ F_y(0,2) &= 1 \\ F(0,2) &= 2(2-0)\sin^2(0 - \frac{\pi}{4}) = 4\sin^2(-\frac{\pi}{4}) = 2\end{aligned}$$

The tangent plane T is given by

$$\begin{aligned} T - F(a,b) &= F_x(a,b)(x-a) + F_y(a,b)(y-b) \\ T + 2 &= xF_x(0,2) + F_y(0,2)(y-2) \end{aligned}$$

Plugging in the above values, we get the tangent plane equation.

$$\begin{aligned} T - 2 &= xF_x(0,2) + F_y(0,2)(y-2) \\ &= -9x + y - 2 \\ T &= y - 9x \end{aligned}$$

Problem 3 (ii)

Using the tangent equations as the component slopes of F at a point (x_0, y_0), the linear approximation is given by

$$L(x,y) = F(x_0,y_0) + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)$$

We'll choose the point $(x_0, y_0) = (-1, 0)$ to approximate $F(-1.05, 0.02)$.

$$\begin{aligned} F(x,y) \approx L_{(-1,0)}(x,y) &= F(-1,0) + F_x(-1,0)(x+1) + F_y(-1,0)(y) \\ &= 1 - x - 1 + 3y = 3y - x \\ L_{(-1,0)}(-1.05, 0.02) &= 3 \cdot 0.02 + 1.05 = 1.11 \end{aligned}$$

For comparison, the correct value $F(-1.05, 0.02) \approx 1.11493$, so the approximation seems reasonable. Of course, we could have used the tangent plane equation from the previous problem directly. At $(-1, 0)$, it is

$$\begin{aligned} T_{(-1,0)}(x,y) &= F_x(-1,0)(x+1) + F_y(-1,0)(y-0) + F(-1,0) \\ &= -x - 1 + 3y + 1 = 3y - x \end{aligned}$$

The two methods are exactly the same.

Problem 3 (iii)

We could simply use $f'(x) = -\frac{F_x}{F_y}$, as we have done in 3 (iv), but I wanted to do it the hard way as well. First we need some helper equations.

$$\begin{aligned}
u &= xy - \frac{\pi}{4} \\
\sin^2 u &= \frac{1 - \cos 2u}{2} \iff \cos 2u = 1 - 2\sin^2 u \\
2 \sin u \cos u &= \sin 2u \\
u' &= xy' + y \\
\sin^2 u' &= 2 \sin u \cos u \cdot u' = \sin 2u(xy' + y) = xy' \sin 2u + y \sin 2u
\end{aligned}$$

Simplifying $F(x, y) - 1 = 0$,

$$\begin{aligned}
2(y - x)\frac{1 - \cos 2u}{2} &= 1 \\
(y - x)(1 - \cos 2u) &= 1
\end{aligned}$$

Multiplying out

$$\begin{aligned}
(y - x)(1 - \cos 2u) &= 1 \\
y - y \cos 2u - x + x \cos 2u &= 1
\end{aligned}$$

Differentiating both sides using above helpers,

$$\begin{aligned}
y' - y' \cos 2u + y \sin 2u \cdot 2u' - 1 + \cos 2u - x \sin 2u \cdot 2u' &= 0 \\
y' - y' \cos 2u + 2y \sin (2u)(y + xy') - 1 + \cos 2u - 2x \sin (2u)(y + xy') &= 0 \\
y' - y' \cos 2u + 2y^2 \sin^2 u + 2xyy' \sin 2u - 1 + \cos 2u + 2xy \sin 2u + 2x^2 y' \sin 2u &= 0
\end{aligned}$$

Grouping y'-terms on the left, rest on the right,

$$y' \left(1 - \cos 2u + 2xy \sin 2u - 2x^2 \sin 2u\right) = -2y^2 \sin 2u + 2xy \sin 2u + 1 - \cos 2u$$

Dividing and reorganizing using $\cos 2u = 1 - 2\sin^2 u$,

$$y' = \frac{-2y^2 \sin 2u + 2xy \sin 2u + 1 - \cos 2u}{1 - \cos 2u + 2xy \sin 2u - 2x^2 \sin 2u} = \frac{-2y(y - x) \sin 2u + 1 - 1 + 2\sin^2 u}{1 - 1 + 2\sin^2 u + x \sin 2u(2y - 2x)}$$

Finally, we get the *exact* same result as in 3 (iv).

$$y' = \frac{-2y(y - x) \sin 2u + 2\sin^2 u}{2\sin^2 u + x \sin 2u(2y - 2x)} = \underline{\underline{-\frac{y(y - x) \sin 2u - \sin^2 u}{x(y - x) \sin 2u + \sin^2 u}}}$$

Problem 3 (iv)

First we need to find $y(x)$.

$$\begin{aligned} F(0,y) &= 1 \\ 2y\sin^2 -\frac{\pi}{4} &= 1 \\ y &= 1 \text{ when } F(0,y) = 1 \end{aligned}$$

We thus have $x = 0$ and $y = 1$. This gives $2u = -\frac{\pi}{2}$, $\sin 2u = -1$, $\cos 2u = 0$. To find $f'(x)$, we could have used the result in 3 (iii), but instead we'll do

$$f'(x) = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}$$

This is true because F_x and F_y are not altered by adding a constant -1, because $\frac{\partial}{\partial x}(-1) = 0$ and the same for $\frac{\partial}{\partial y}$. Therefore we get

$$f'(x) = \frac{-2y(y-x)\sin(2g) + 2\sin^2(g)}{2\sin^2(g) + x\sin(2g)(2y-2x)} = -\frac{y(y-x)\sin 2u - \sin^2 u}{x(y-x)\sin 2u + \sin^2 u}$$

Inserting $x = 0$ and $y = y(0) = 1$ we get

$$f'(0) = -\frac{1 \cdot -1 - \frac{1}{2}}{\frac{1}{2}} = \frac{6}{2} = 3$$

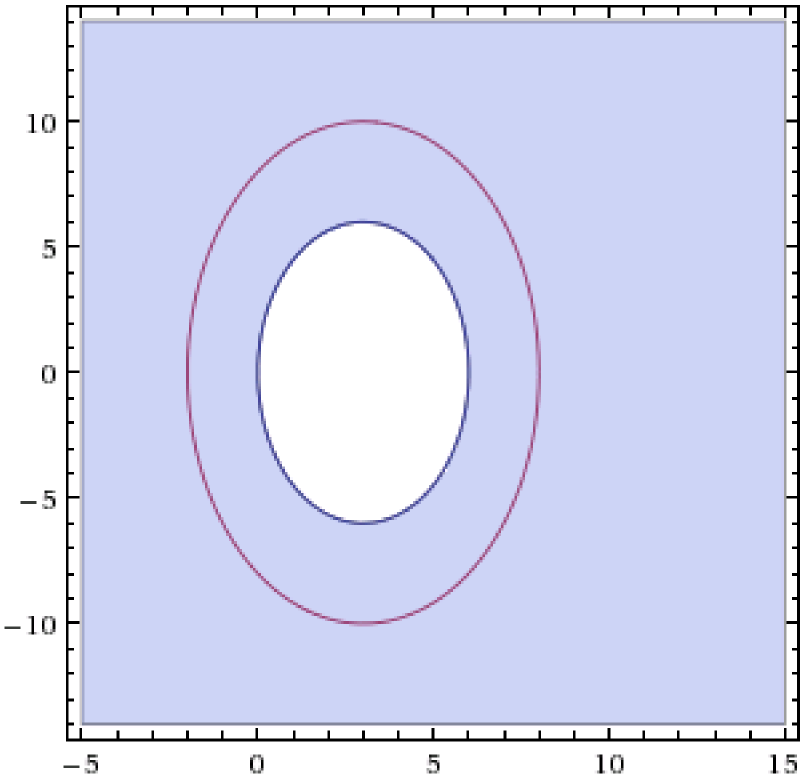

Figure 1: Problem 1 (i) and (ii): Light blue area is $\mathcal{D}(f)$, dark blue line is $C = 0$ and purple line is $C = 4$.

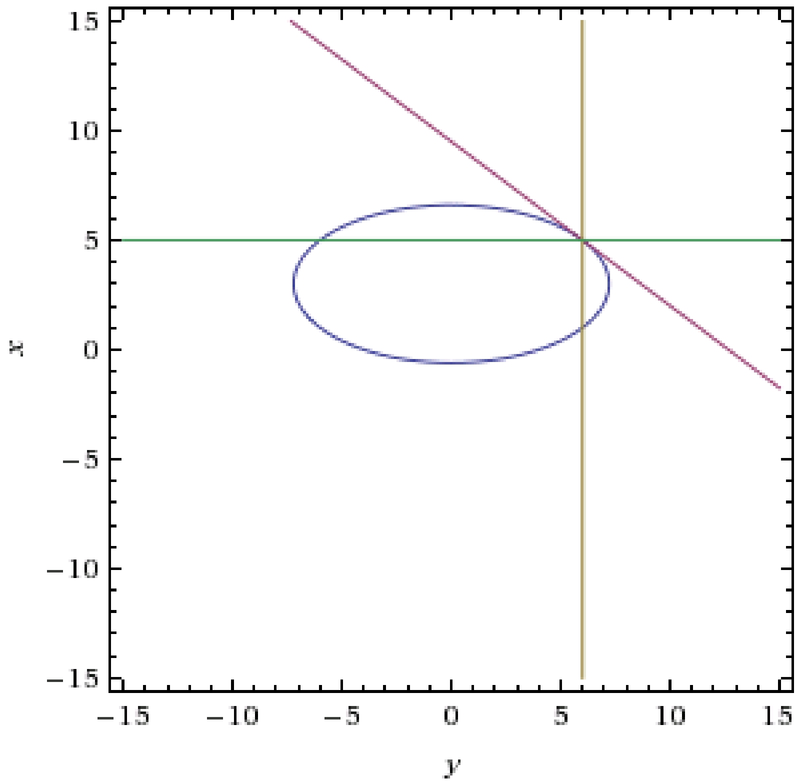

Figure 2: Plot of tangent line $\ell_{(5,6)}$ in problem 1 (iii).