MAT200 — Mathematical Methods 2

Compulsory Assignment 3

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Problem 1 (i)

By Gauss-Jordan elimination,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 4 \\ 4 & 3 & 1 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

giving the solution $x = 1$, $y = 2$ and $z = -1$.

Problem 1 (ii)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 3 & 5 & 9 \\ 3 & 1 & -1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & -2 & -4 & -6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We can't go further, but we see from the first row that $x = z$, and by the second that $y + 2z = 3$ or $y = 3 - 2z$. Thus either z or x can be freely chosen.

Problem 1 (iii)

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 2 \\ 1 & 0 & 0 & -1 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 0 & 2 & -2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 2 & -2 & 0 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right]$$

The last row gives $0 = -5$, meaning *the system is inconsistent*.

Problem 2

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & a \\ 1 & a & 1 \end{bmatrix}$$

We'll start by looking at $\det(A) = 0$,

$$\begin{aligned} \det(A) = (-6 - a^2) + 5a = -a^2 + 5a - 6 \\ -a^2 + 5a - 6 = 0 \\ a^2 - 5a + 6 = 0 \\ a \in \{2, 3\} \end{aligned}$$

The system has unique solutions for $a \notin \{2, 3\}$. But we're after infinite solutions, so let's look at $a = 2$.

$$\begin{aligned} x + 5y &= 6 \\ -6y + 2z &= p \\ x + 2y + z &= p + 4 \end{aligned}$$

Using Gauss-Jordan,

$$\left[\begin{array}{ccc|c} 1 & 5 & 0 & 6 \\ 0 & -6 & 2 & p \\ 1 & 2 & 1 & p+4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 5 & 0 & 6 \\ 1 & 2 & 1 & p+4 \\ 0 & -6 & 2 & p \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 5 & 0 & 6 \\ 0 & -3 & 1 & p-2 \\ 0 & -6 & 2 & p \end{array}\right]$$
$$\to \left[\begin{array}{ccc|c} 1 & 0 & 5/3 & 6 + \frac{5(p-2)}{3} \\ 0 & -3 & 1 & p-2 \\ 0 & 0 & 4 & 3p-4 \end{array}\right]$$

Not completed.

Problem 3 (i)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 12 \\ 5 & 13 \end{bmatrix}$$

Problem 3 (ii)

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -2 & 6 \\ 4 & -2 \end{bmatrix}$$

Problem 3 (iii)

$$\begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ -1 & 1 & 3 \end{bmatrix}$$

Problem 4 (i)

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 1 \cdot (6 - 0) - 0 + 0 = 6$$

Problem 4 (ii)

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 4 & 5 \end{vmatrix} = 1 \cdot -7 - 2 \cdot 7 + 3 \cdot -7 = (-3 + 3) \cdot 7 = 0$$

Problem 4 (iii)

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 0 - 1 \cdot 0 - 0 = 0$$

Problem 4 (iv)

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 0 - 1 \cdot -2 + 2 \cdot 1 = 4$$

Problem 4 (v)

$$\begin{vmatrix} 1 & 3 & 6 & 1 \\ 0 & 4 & 5 & 0 \\ 0 & 5 & 4 & 0 \\ 2 & 6 & 3 & 1 \end{vmatrix} = 1 \begin{vmatrix} 4 & 5 & 0 \\ 5 & 4 & 0 \\ 6 & 3 & 1 \end{vmatrix} + 0 + 0 - 2 \begin{vmatrix} 3 & 6 & 1 \\ 4 & 5 & 0 \\ 5 & 4 & 0 \end{vmatrix}$$
$$= (4 \cdot 4 - 5 \cdot 5) - 2 \cdot (3 \cdot 5 - 6 \cdot 0 + 1(16 - 25)) = 9$$

Problem 5 (i)

We construct a matrix A and know that if $\det(A) \neq 0$, the given vectors are linearly independent.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$
$$\det(A) = 0 - 3 + 1 = -2$$

As the determinant $\neq 0$, the vectors are linearly independent.

Problem 5 (ii)

$$\begin{vmatrix} 1 & 3 & 2 \\ 2 & 2 & 2 \\ 3 & 1 & 2 \end{vmatrix} = 2 + 6 - 8 = 0$$

The vectors are linearly independent.

Problem 5 (iii)

$$\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} = 7 - 6 = 1$$

Thus, u_1 and u_2 are linearly independent.

$$\begin{vmatrix} 2 & 15 \\ 7 & 9 \end{vmatrix} = 18 - 7 \cdot 15 \neq 0$$

Thus, u_2 and u_3 are also linearly independent. That means all vectors are linearly independent, by transitivity.

Problem 5 (iv)

Taking $a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 = 0$ in matrix form,

$$a \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 4 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 6 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Gauss-Jordan,

$$\left[\begin{array}{ccc|c} a & 2b & 0 & 0 \\ 3a & 0 & 6c & 0 \\ 2a & 4b & 0 & 0 \\ 5a & b & 9c & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} a & 2b & 0 & 0 \\ 0 & -6b & 6c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -9b & 9c & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} a & 2b & 0 & 0 \\ 0 & -6b & 6c & 0 \\ 0 & -9b & 9c & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} a & 2b & 0 & 0 \\ 0 & -18b & 18c & 0 \\ 0 & -18b & 18c & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} a & 2b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We now have $a = -2b$, or $b = -a/2$ inserting into $-b + c = 0$ gives $c = -a/2$ as well. Thus, the vectors cannot be linearly independent.