Exercise set 1

Problem 1:

Consider the experiment of flipping an unbiased coin 5 independent times. Let H denote the outcome "head" and T the outcome "tail".

a) Compute the probability of HHTHT and THHHT.

b) Modify the R-code from the number of heads example in the lectures to simulate the probability of three heads occurring in the 5 trials.
Can you also calculate this probability exactly? (Hint: Binomial distribution.)

c) Redo point b) for a biased coin for which the probability of heads is 0.6. (Hint: See the help file for the `sample` function in R and set the `prob`-option.)

Problem 2:

For each of the following, determine the constant c such that $f(x)$ satisfies the condition of being the probability mass function (pmf) for a discrete random variable X, and then depict each $f(x)$ as a bar graph:

a) $f(x) = x/c, \quad x = 1, 2, 3, 4.$

b) $f(x) = c(x+1)^2, \quad x = 0, 1, 2, 3.$

Problem 3:

For the probability distributions in problem 2 find $E(X)$ and $E(g(X))$ with $g(x) = x^3$.

Problem 4:

A continuous random variable X has probability density function (pdf)

$$f(x) = \begin{cases} 4x(1-x^2) & \text{,for } 0 \leq x \leq 1 \\ 0 & \text{,otherwise.} \end{cases}$$

a) Calculate $P(X < 0.5)$, $\mathrm{E}(X)$, $\mathrm{Var}(X)$ and $\mathrm{SD}(X)$.

b) Find the cumulative distribution function (cdf), $F(x)$, and use this to calculate $P(X < 0.3)$ and $P(X > 0.7)$.

c) Make a skecth of $f(x)$ and $F(x)$ in R.

Problem 5:

Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} k(1-x^2) & \text{for } -1 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.

a) Determine k and make a sketch of $f(x)$ in R.

b) Calculate $P(X \leq 0.5)$.

c) Let $Y = 1 + x^2$. Calculate $\mathrm{E}(Y)$.

Problem 6:

The gas price on different days at two gas stations in Sandnes has been registered. Let X be the gas price at one of the stations and let Y be the gas price on the other station on a random day. Registrations of X and Y on 10 randomly chosen days are shown in the table below.

dag i	1	2	3	4	5	6	7	8	9	10
x_i	13.89	13.39	12.20	14.35	14.10	13.39	13.96	14.15	13.69	12.57
y_i	13.99	13.39	12.65	14.25	13.99	13.09	13.66	14.25	13.36	12.57

a) Plot the gas prices at the two stations against each other on a scatter plot using R.

b) Calculate the empirical correlation between the gas prices and comment the result. (Hint: Use the `cor`-function in R.)

Problem 7:

In a blind test one wants to find out whether people can distinguish between the taste of Pepsi and Coca Cola. 10 persons are each given 5 samples to test drink. One of these samples contains Pepsi, the other four Coca Cola. The test people are not told which are which and they are asked to try to pick out the Pepsi sample. Let X be the number of correct identifications.

a) What kind of a distribution does X have, if they in fact taste the same? Find $P(X \geq 3)$, $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

b) Let us now imagine that the test goes on and on (with more than 10 persons if necessary) and let N be the number of persons needed for the first correct identification. What kind of distribution does N have and what is $E(N)$?

Problem 8:

Let X be exponentially distributed with density function $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Make a sketch of this density (e.g. for $\lambda = 1$) and compute the expectation and the variance of X.

Problem 9:

Let X be a random variable that has the beta-distribution given by

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \leq x \leq 1, \alpha > 0, \beta > 0$$

where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

and for integers $\Gamma(\alpha) = (\alpha - 1)!$

a) Write up the formula for $f(x)$ and make a sketch of $f(x)$ for each of the cases (i) $\alpha = \beta = 1$, (ii) $\alpha = 2, \beta = 1$ and (iii) $\alpha = 1, \beta = 2$.

b) Use R to draw samples of different sizes (e.g. size 10, 100, 1000 and 10000) from each of the beta-distributions specified in a). Calculate the average of each of these samples and compare to the expectation which for the beta distribution is given by the formula $E(X) = \alpha/(\alpha + \beta)$.

Problem 10:

The height of Norwegian men in their twenties follow a normal distribution with mean 180 cm and standard deviation 6.5 cm.

Use the R functions `pnorm` and `qnorm` to calculate the following probabilities.

a) The probability that at man has a height of less than 175.
b) The probability that at man has a height of more than 190.
c) The probability that at man has a height between 170 to 180.
d) The height which is such that only 1% of the men have a higher height.

Problem 11:

Three investments give returns X, Y and Z characterized by $E(X) = E(Y) = E(Z) = 0.4$, $\text{Var}(X) = \text{Var}(Y) = \text{Var}(Z) = 1.0$. Moreover, there are the following correlations between these returns: $\rho(X,Y) = 0$, $\rho(X,Z) = 0.5$ and $\rho(Y,Z) = -0.5$:

a) Consider the returns from the following four investments: (i) $2X$, (ii) $X+Y$, (iii) $X+Z$ and (iv) $Y+Z$. Compute the expectation and variance for each of these cases.
Which investment would you prefer if variance is taken as a measure of risk?