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Lecture 9
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# Notes 9.1: Mathematical derivation of low-pass filtered spike trains

In order to make the signal continuous, we low-pass filter the spike train. A low-pass filter only allows low frequencies. A dirac delta signal is very short and rapidly varying which tells you that it contains high frequencies (In fact it contains all frequencies in equal amounts as one sees by taking the Fourier transform). After low-pass filtering, we 'blur' the signal, so its duration increases, and frequency decreases. Figure 1 shows an example.

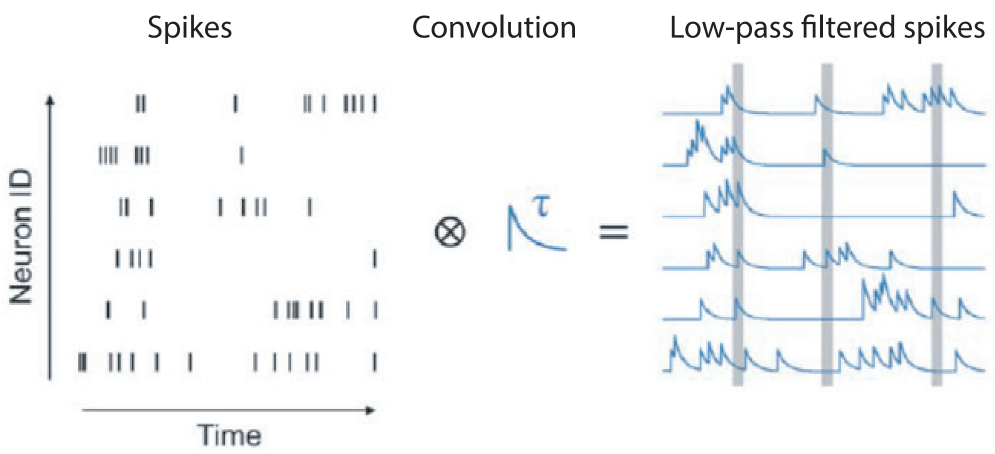


Figure 1: Low-pass filtering spike trains. Figure taken from Jurjut et al in PloS ONE Feb 2011.

The input signal is the spike train $Y(t)$ or $X(t)$, and after it passes through the low-pass filter it becomes $o_1(t)$, and respectively $r_1(t)$.

We are given the input spike trains

$$Y(t) = \sum_i \delta(t - t^i_{post}) \tag{1}$$

$$X(t) = \sum_i \delta(t - t^i_{pre}) \tag{2}$$

And the time constants of the low-pass filters $\tau_-$ and $\tau_+$, and we want to write down the expressions for the output of the filter

$$o_1(t) = \int_0^\infty Y(t-s)\exp\left(-\frac{s}{\tau_-}\right)ds \tag{3}$$

$$r_1(t) = \int_0^\infty X(t-s)\exp\left(-\frac{s}{\tau_+}\right)ds \tag{4}$$

There are two ways of how you can get to the final solution.

## A. Using the definition of the convolution operation $\star$

Low-pass filtering means taking the convolution of the spike train with an exponential decaying function (Figure 1). A convolution is defined as

$$f(t) \star g(t) = \int_0^t f(t-s)g(s)ds \tag{5}$$

If $f(t) = Y(t)$ and $g(t) = \exp(-\frac{s}{\tau_-})$ then

$$o_1(t) = f(t) \star g(t) = \int_0^\infty Y(t-s) \exp\Big(-\frac{s}{\tau_-}\Big)ds \tag{6}$$

$$\tag{7}$$

Note that in the definition of convolution one integrates from $-\infty$ to $\infty$, but since the spikes start at $t = 0$, we only integrate on the positive side of the interval. Applying the same derivation for $f(t) = X(t)$ and $g(t) = \exp(-\frac{s}{\tau_+})$

$$r_1(t) = f(t) \star g(t) = \int_0^\infty X(t-s) \exp\Big(-\frac{s}{\tau_+}\Big)ds \tag{8}$$

$$\tag{9}$$

## B. Solving the Ordinary Differential Equation of a low-pass filter

The input signal is the spike train $Y(t)$ or $X(t)$, and after it passes through the low-pass filter it becomes $o_1(t)$, and respectively $r_1(t)$. The differential equation that describes the temporal evolution of a low-pass filtered signal is

$$\dot{o_1} = \frac{-o_1}{\tau_-} + Y(t) \tag{10}$$

$$\dot{r_1} = \frac{-r_1}{\tau_+} + X(t) \tag{11}$$

The solutions of the two Ordinary Differential Equations (ODE) are

$$o_1(t) = \int_0^\infty Y(t-s) \exp\Big(-\frac{s}{\tau_-}\Big)ds \tag{12}$$

$$r_1(t) = \int_0^\infty X(t-s) \exp\Big(-\frac{s}{\tau_+}\Big)ds \tag{13}$$

Reminder about Laplace transformations

$$\mathcal{L}(f(t)) = \widetilde{f}(s) = \int_0^\infty f(t)\exp(-st)dt \tag{14}$$

Results we will need

- $\mathcal{L}(\exp(wt)) = \frac{1}{s-w}$
- $\mathcal{L}(\frac{df}{dt}) = s\widetilde{f}(s) - f(0)$
- $\mathcal{L}(\delta(t-t_0)) = \exp(-st_0)$
- Relation to convolution: $\mathcal{L}(f_1(t) \star f_2(t)) = \widetilde{f_1}(s)\widetilde{f_2}(s)$ or $f_1(t) \star f_2(t) = \mathcal{L}^{-1}(\widetilde{f_1}(s)\widetilde{f_2}(s))$

$$\dot{o(t)}_1 + \frac{o(t)_1}{\tau_-} = Y(t) = \sum_i \delta(t - t^i) \tag{15}$$

Doing a Laplace transformation gives us

$$\mathcal{L}(\dot{o_1}(t)) + \mathcal{L}\left(\frac{\dot{o_1(t)}}{\tau_-}\right) = \mathcal{L}(\sum_i \delta(t - t^i)) \tag{16}$$

$$s\widetilde{o}(s) - o(0) + \frac{\widetilde{o}(s)}{\tau_-} = \sum_i \exp(-st_i) \tag{17}$$

We set the initial condition $o(t = 0) = 0$, so

$$\widetilde{o}(s)\left(s + \frac{1}{\tau_-}\right) = \widetilde{o}(s)\left(\frac{s\tau_- + 1}{\tau_-}\right) = \sum_i \exp(-st_i) \tag{18}$$

We now solve for $\widetilde{o}(s)$

$$\widetilde{o}(s) = \frac{\tau_-}{s\tau_- + 1}\sum_i \exp(-st_i) = \frac{1}{s + \frac{1}{\tau_-}}\sum_i \exp(-st_i) \tag{19}$$

Now apply the inverse Laplacian transformation

$$o(t) = \mathcal{L}^{-1}\left(\frac{1}{s + \frac{1}{\tau_-}}\sum_i \exp(-st_i)\right)$$

Using the relation to convolution

$$o(t) = \mathcal{L}^{-1}\left(\frac{1}{s + \frac{1}{\tau_-}}\right) \star \mathcal{L}^{-1}\left(\sum_i \exp(-st_i)\right) \tag{20}$$

$$= \exp\left(\frac{-t}{\tau_-}\right) \star (\sum_i \delta(t - t^i)) \tag{21}$$

$$= \exp\left(\frac{-t}{\tau_-}\right) \star Y(t) \tag{22}$$

$$= \int_0^\infty Y(t - s)\exp\left(-\frac{s}{\tau_-}\right)ds \tag{23}$$