Boolean Differential Equations

Bernd Steinbach
Institute of Computer Science
Freiberg University of Mining and Technology
Freiberg, Germany
Email: steinb@informatik.tu-freiberg.de

Christian Posthoff
Department of Computing and Information Technology
The University of the West Indies
Trinidad & Tobago
Email: christian@posthoff.de

***Abstract*— The expressiveness of Boolean Algebras is significantly extended by the *Boolean Differential Calculus* (BDC). The additionally defined differentials of Boolean variables, differentials and further differential operators of Boolean functions as well as several derivative operations of Boolean functions allow to model changes of function values together with changes of the values of variables and many other properties of Boolean functions.**

A Boolean equation equals two given Boolean functions. Its solution is a set of Boolean vectors. We introduce in this paper *Boolean Differential Equations* (BDE). A BDE is an equation that includes derivative operations and differential operators of an unknown Boolean function. We show in this paper that, completely different from a Boolean equation, the solution of a Boolean differential equation is a set of Boolean functions. Hence, Boolean differential equations allow to describe and handle sets of Boolean functions.

For an easier understanding we repeat the definition of the derivative operations and explain the essentials of Boolean differential equations using an example of the most restricted BDE. For a special class of Boolean differential equations we introduce the theoretical background for its solution and give a very simple solution algorithm. Furthermore we show how more general classes can be solved.

I. Introduction

The *Boolean Differential Calculus* (BDC) defines differentials of Boolean variables, differentials and further differential operators applied to Boolean functions as well as several derivative operations of Boolean functions. There are many theorems which describe relationships between them [1], [2], [4], and [5].

The BDC extends Boolean algebra [2]. While the Boolean algebra focuses on values of Boolean functions, the BDC allows the evaluation of *changes* of the function values. Such changes can be investigated for certain pairs of function values as well as with regard to whole subspaces. Due to the same basic data structures, BDC can be applied to any task described by Boolean functions and equations together with the Boolean algebra. Widely used is the BDC in analysis, synthesis, and testing of digital circuits [2], [4], and [5].

All these differentials of Boolean variables, differential operations and derivative operations of Boolean functions are Boolean functions with special properties. In this paper, we show that a *Boolean Differential Equation* (BDE) describes a set of Boolean functions and give an algorithm for solving a certain class of Boolean differential equations [2]. The necessary theory and the solution algorithm for the given class of Boolean differential equations were developed by the first author in his Ph.D. Thesis [3] (written in German).

The remaining part of this paper is structured as follows. In Section II definitions of all derivative operations together with short explanations are given. This restricted knowledge of the Boolean Differential Calculus contributes to a better understanding of the rest of the paper. In Section III essential aspects of Boolean Differential Equations are explained. To that end we consider a special BDE where a single derivative operation appears. Next we describe in Section IV both the theoretical background and an algorithm to solve an important class of Boolean differential equations. In Section V we give some remarks how more general Boolean differential equations can be solved.

II. Derivative Operations

Derivative operations of the BDE are calculated for a given Boolean function with regard to a single variable or a subset of variables. These selected variables and the type of operation describe the studied direction of change. In this section we restrict ourselves to the definition of the derivative operations. For more details about the derivative operations, their relationships and applications we refer to [2] and [4].

There are three groups of derivative operations of a Boolean function $f(\mathbf{x})$. All of them include a derivative that evaluates certain changes of function values, a minimum that evaluates function values 1 for all patterns $\mathbf{x} = \mathbf{c}_0$ of certain subsets, and a maximum that evaluates the existence of function values 1 for certain subsets of $\mathbf{x} = \mathbf{c}_0$. We describe a single variable by an italic letter like x_i and a set of variables by a bold letter like $\mathbf{x}_0$.

The first group of derivative operations explores the change of the function value with regard to the change of a single variable x_i. Hence, the subsets, evaluated by simple derivative operations, include two function values which are reached by changing x_i.

Definition 1: Let $f(\mathbf{x}) = f(x_i, \mathbf{x}_1)$ be a logic function of n variables. Then

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = f(x_i = 0, \mathbf{x}_1) \oplus f(x_i = 1, \mathbf{x}_1) \tag{1}$$

is the (simple) derivative,

$$\min_{x_i} f(\mathbf{x}) = f(x_i = 0, \mathbf{x}_1) \wedge f(x_i = 1, \mathbf{x}_1) \tag{2}$$

the (simple) minimum and

$$\max_{x_i} f(\mathbf{x}) = f(x_i = 0, \mathbf{x}_1) \vee f(x_i = 1, \mathbf{x}_1) \tag{3}$$

the (simple) maximum of the logic function $f(\mathbf{x})$ with regard to the variable x_i.

The second group of derivative operations explores the change of a function with regard to the set of variables $\mathbf{x}_0$. Hence, the subsets, evaluated by *vectorial* derivative operations, include again two function values which are reached by changing all variables of the set $\mathbf{x}_0$ at the same point of time.

Definition 2: Let $f(\mathbf{x}) = f(\mathbf{x}_0, \mathbf{x}_1)$ be a logic function of n variables. Then

$$\frac{\partial f(\mathbf{x}_0, \mathbf{x}_1)}{\partial \mathbf{x}_0} = f(\mathbf{x}_0, \mathbf{x}_1) \oplus f(\overline{\mathbf{x}}_0, \mathbf{x}_1) \tag{4}$$

is the vectorial derivative,

$$\min_{\mathbf{x}_0} f(\mathbf{x}_0, \mathbf{x}_1) = f(\mathbf{x}_0, \mathbf{x}_1) \wedge f(\overline{\mathbf{x}}_0, \mathbf{x}_1) \tag{5}$$

the vectorial minimum, and

$$\max_{\mathbf{x}_0} f(\mathbf{x}_0, \mathbf{x}_1) = f(\mathbf{x}_0, \mathbf{x}_1) \vee f(\overline{\mathbf{x}}_0, \mathbf{x}_1) \tag{6}$$

the vectorial maximum of the logic function $f(\mathbf{x}_0, \mathbf{x}_1)$ with regard to the variables of $\mathbf{x}_0$.

The result of each derivative operation of a Boolean function is again a Boolean function. Hence, derivative operations can be executed iteratively for the result of a previous derivative operation. The third group of derivative operations uses this possibility for all variables of the set of variables $\mathbf{x}_0$. Hence, the subsets, evaluated by m-fold derivative operations, include $2^{|\mathbf{x}_0|}$ function values.

Definition 3: Let $f(\mathbf{x}) = f(\mathbf{x}_0, \mathbf{x}_1)$ be a logic function of n variables, and let $\mathbf{x}_0 = (x_1, x_2, \ldots, x_m)$. Then

$$\frac{\partial^m f(\mathbf{x}_0, \mathbf{x}_1)}{\partial x_1 \partial x_2 \ldots \partial x_m} = \frac{\partial}{\partial x_m}(\ldots(\frac{\partial}{\partial x_2}(\frac{\partial f(\mathbf{x}_0, \mathbf{x}_1)}{\partial x_1}))\ldots) \tag{7}$$

is the m-fold derivative,

$$\min_{\mathbf{x}_0}{}^m f(\mathbf{x}_0, \mathbf{x}_1) = \min_{x_m}(\ldots(\min_{x_2}(\min_{x_1} f(\mathbf{x}_0, \mathbf{x}_1)))\ldots) \tag{8}$$

the m-fold minimum,

$$\max_{\mathbf{x}_0}{}^m f(\mathbf{x}_0, \mathbf{x}_1) = \max_{x_m}(\ldots(\max_{x_2}(\max_{x_1} f(\mathbf{x}_0, \mathbf{x}_1)))\ldots) \tag{9}$$

the m-fold maximum and

$$\Delta_{\mathbf{x}_0} f(\mathbf{x}_0, \mathbf{x}_1) = \min_{\mathbf{x}_0}{}^m f(\mathbf{x}_0, \mathbf{x}_1) \oplus \max_{\mathbf{x}_0}{}^m f(\mathbf{x}_0, \mathbf{x}_1) \tag{10}$$

the Δ - operation of the function $f(\mathbf{x}_0, \mathbf{x}_1)$ with regard to the set of variables $\mathbf{x}_0$.

III. Essence of a Boolean Differential Equation Explored by Examples

Let us take the Boolean function

$$f(\mathbf{x}) = f(x_1, x_2, x_3) = x_1 \vee x_2\, x_3 \quad . \tag{11}$$

Using the definition (4) we get the vectorial derivative with regard to (x_1, x_3) as follows:

$$\begin{aligned}
\frac{\partial f(x_1, x_2, x_3)}{\partial (x_1, x_3)} &= f(x_1, x_2, x_3) \oplus f(\overline{x}_1, x_2, \overline{x}_3) \\
&= (x_1 \vee x_2\, x_3) \oplus (\overline{x}_1 \vee x_2\, \overline{x}_3) \\
&= (x_1 \vee \overline{x}_1\, x_2\, x_3) \oplus (\overline{x}_1 \vee x_1\, x_2\, \overline{x}_3) \\
&= (x_1 \oplus \overline{x}_1\, x_2\, x_3) \oplus (\overline{x}_1 \oplus x_1\, x_2\, \overline{x}_3) \\
&= 1 \oplus \overline{x}_1\, x_2\, x_3 \oplus x_1\, x_2\, \overline{x}_3 \\
&= 1 \oplus x_2\, (\overline{x}_1\, x_3 \oplus x_1\, \overline{x}_3) \\
&= 1 \oplus x_2\, (x_1 \oplus x_3) \\
&= \overline{x}_2 \vee \overline{(x_1 \oplus x_3)} \\
&= \overline{x}_2 \vee (\overline{x}_1 \oplus x_3) \quad .
\end{aligned} \tag{12}$$

Hence, the result of this vectorial derivative is the Boolean function

$$g(x_1, x_2, x_3) = \overline{x}_2 \vee (\overline{x}_1 \oplus x_3) \tag{13}$$

and we have the Boolean differential equation:

$$\frac{\partial f(x_1, x_2, x_3)}{\partial (x_1, x_3)} = g(x_1, x_2, x_3) \quad . \tag{14}$$

The function $g(x_1, x_2, x_3)$ (13) is uniquely defined by the given function $f(x_1, x_2, x_3)$ (11), the definition of the vectorial derivative (4) and the direction of change described by the taken subset of variables (x_1, x_3).

All calculation steps of (12) can also be executed in the reverse direction. Hence, the function $f(x_1, x_2, x_3)$ (11) is a solution of the Boolean differential equation (14) where the function $g(x_1, x_2, x_3)$ is defined by (13).

Now the question arises whether the function $f(x_1, x_2, x_3)$ is uniquely defined by the function $g(x_1, x_2, x_3)$ (13) and Boolean differential equation (14). The answer to this question is *NO*. There are 15 other Boolean functions $f_i(x_1, x_2, x_3)$ which solve the BDE (14) for the function $g(x_1, x_2, x_3)$ (13). All 16 solution functions of the BDE (14) for the the function

$g(x_1, x_2, x_3)$ (13) are:

$$\begin{aligned}
f_0(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \ , && (15)\\
f_1(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus \overline{x}_2\,(\overline{x}_1 \oplus x_3) \ , && (16)\\
f_2(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus \overline{x}_2\,(x_1 \oplus x_3) \ , && (17)\\
f_3(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus \overline{x}_2 \ , && (18)\\
f_4(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus x_2\,(\overline{x}_1 \oplus x_3) \ , && (19)\\
f_5(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus (\overline{x}_1 \oplus x_3) \ , && (20)\\
f_6(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus (x_1 \oplus x_2 \oplus x_3) \ , && (21)\\
f_7(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus (\overline{x}_2 \vee (\overline{x}_1 \oplus x_3)) \ , && (22)\\
f_8(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus x_2\,(x_1 \oplus x_3) \ , && (23)\\
=& (x_1\,\overline{x}_2\,\overline{x}_3 \oplus x_1\,\overline{x}_2\,x_3 \oplus \\
& x_1\,x_2\,x_3) \oplus (x_1\,x_2\,\overline{x}_3 \oplus \overline{x}_1\,x_2\,x_3) \\
=& x_1 \vee x_2\,x_3 \ , && (24)\\
f_9(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus (\overline{x}_1 \oplus x_2 \oplus x_3) \ , && (25)\\
f_{10}(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus (x_1 \oplus x_3) \ , && (26)\\
f_{11}(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus (\overline{x}_2 \vee (x_1 \oplus x_3)) \ , && (27)\\
f_{12}(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus x_2 && (28)\\
f_{13}(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus (x_2 \vee (\overline{x}_1 \oplus x_3)) \ , && (29)\\
f_{14}(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus (x_2 \vee (x_1 \oplus x_3)) \ , && (30)\\
f_{15}(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \oplus 1 \ . && (31)
\end{aligned}$$

It can be verified directly (by inserting these 16 functions into the BDE) (14) that the calculated vectorial derivatives are equal to the function $g(x_1, x_2, x_3)$ (13). In the same way it can be checked that no other function of three variables (x_1, x_2, x_3) solves the BDE (14) for the function $g(x_1, x_2, x_3)$ (13).

The enumeration of these 16 solution function shows that all solution function have a common basic structure

$$f_i(x_1,x_2,x_3) = g_0(x_1,x_2,x_3) \oplus h_i(x_1,x_2,x_3) \qquad (32)$$

with

$$\begin{aligned}
g_0(x_1,x_2,x_3) =& x_1\,(\overline{x}_2 \vee x_3) \\
=& x_1\,\overline{x}_2 \vee x_1\,x_3 \\
=& x_1\,\overline{x}_2 \vee x_1\,x_3 \vee 0 \\
=& x_1\,\overline{x}_2 \vee x_1\,x_1\,x_3 \vee x_1\,\overline{x}_1\,\overline{x}_3 \\
=& x_1 \wedge (\overline{x}_2 \vee x_1\,x_3 \vee \overline{x}_1\,\overline{x}_3) \\
=& x_1 \wedge (\overline{x}_2 \vee (\overline{x}_1 \oplus x_3)) \\
=& x_1 \wedge g(x_1,x_2,x_3) \ . && (33)
\end{aligned}$$

The variable x_1 is selected from the set (x_1, x_3) used to define the direction of the vectorial derivative. The other variable x_3 can be chosen in (33) to specify the function $g_0(x_1, x_2, x_3) = x_3\,(\overline{x}_2 \vee x_1)$. This function $g_0(x_1, x_2, x_3)$ can be used to create together with the same 16 functions $h_i(x_1, x_2, x_3)$ exactly the same set of 16 solution function of the BDE (14) for the function $g(x_1, x_2, x_3)$ (13).

The transformation for the solution function $f_8(x_1, x_2, x_3)$ (24) confirms that the used initial function $f(x_1, x_2, x_3) = f_8(x_1, x_2, x_3)$ (11) is an element of the solution set.

All functions $h_i(x_1, x_2, x_3)$ hold the BDE

$$\frac{\partial h_i(x_1,x_2,x_3)}{\partial(x_1,x_3)} = 0 \ . \qquad (34)$$

This property can be checked easily. Either the functions $h_i(x_1, x_2, x_3)$ does not depend on (x_1, x_3), than we have e.g.

$$\begin{aligned}
\frac{\partial h_3(x_1,x_2,x_3)}{\partial(x_1,x_3)} =& \frac{\partial(\overline{x}_2)}{\partial(x_1,x_3)} \\
=& \overline{x}_2 \oplus x_2 = 0 \ , && (35)
\end{aligned}$$

or the variables (x_1, x_3) appear in the function $h_i(x_1, x_2, x_3)$ connected by an $\oplus$-operation, than we have e.g. $h_i(x_1, x_2, x_3)$

$$\begin{aligned}
\frac{\partial h_1(x_1,x_2,x_3)}{\partial(x_1,x_3)} =& \frac{\partial(\overline{x}_2\,(\overline{x}_1 \oplus x_3))}{\partial(x_1,x_3)} \\
=& \overline{x}_2\,(\overline{x}_1 \oplus x_3) \oplus \overline{x}_2\,(x_1 \oplus \overline{x}_3) \\
=& \overline{x}_2\,\overline{(x_1 \oplus x_3)} \oplus \overline{x}_2\,\overline{(x_1 \oplus x_3)} = 0 \ . && (36)
\end{aligned}$$

There is one remaining question concerning the BDE of a single derivative operation like (14): Are there solution functions $f(x_1, x_2, x_3)$ for each function $g(x_1, x_2, x_3)$? The answer to this question is *NO* too. It can be verified, for instance, by checking all 256 functions $f(x_1, x_2, x_3)$ that no function $f(x_1, x_2, x_3)$ exists as solution of the BDE (14) where the function $g(x_1, x_2, x_3) = x_1\,x_2\,x_3$. The reason for that come from the definion (4) of the vectorial derivative.

Due to

$$g(\mathbf{x}_0, \mathbf{x}_1) = \frac{\partial f(\mathbf{x}_0, \mathbf{x}_1)}{\partial \mathbf{x}_0} = f(\mathbf{x}_0, \mathbf{x}_1) \oplus f(\overline{\mathbf{x}}_0, \mathbf{x}_1) \qquad (37)$$

we get

$$g(\overline{\mathbf{x}}_0, \mathbf{x}_1) = \frac{\partial f(\overline{\mathbf{x}}_0, \mathbf{x}_1)}{\partial \mathbf{x}_0} = f(\overline{\mathbf{x}}_0, \mathbf{x}_1) \oplus f(\mathbf{x}_0, \mathbf{x}_1) \qquad (38)$$

and consequently

$$g(\mathbf{x}_0, \mathbf{x}_1) = g(\overline{\mathbf{x}}_0, \mathbf{x}_1) \qquad (39)$$

which can be expressed by

$$\begin{aligned}
g(\mathbf{x}_0, \mathbf{x}_1) =& g(\overline{\mathbf{x}}_0, \mathbf{x}_1) \\
g(\mathbf{x}_0, \mathbf{x}_1) \oplus g(\overline{\mathbf{x}}_0, \mathbf{x}_1) =& g(\overline{\mathbf{x}}_0, \mathbf{x}_1) \oplus g(\overline{\mathbf{x}}_0, \mathbf{x}_1) \\
g(\mathbf{x}_0, \mathbf{x}_1) \oplus g(\overline{\mathbf{x}}_0, \mathbf{x}_1) =& 0 \\
\frac{\partial g(\mathbf{x}_0, \mathbf{x}_1)}{\partial \mathbf{x}_0} =& 0 \ . && (40)
\end{aligned}$$

Hence, the function $g(x_1, x_2, x_3)$ in the BDE (14) must hold the condition (40) in order to find solution functions $f(x_1, x_2, x_3)$. For that reason (40) is called *integrability condition* for the vectorial derivatives of Boolean functions.

We learn form from this example:

1) A Boolean differential equation (14) includes the unknown function $f(x_1, x_2, x_3)$.
2) There are solutions of a BDE like (14) only if the right-hand function $g(x_1, x_2, x_3)$ holds a special integrability condition.
3) As shown in (32), the general solution of an inhomogeneous BDE is built using a single special solution of the

inhomogeneous BDE and the set of all solutions of the associated homogeneous BDE. The associated homogeneous BDE is developed by replacing the righthand side of an inhomogeneous BDE by 0.

4) *Generally, the solution of a Boolean differential equation is a set of Boolean functions*. This is a significant difference to Boolean equations. The solution of a Boolean equation is a set of Boolean vectors.

A Boolean differential equation like (14) where the left hand side consists of a single derivative operation of an unknown Boolean function $f(\mathbf{x})$ and the right hand side is specified by an expression of a known function $g(\mathbf{x})$ can be specified for each derivative operation. For all of them both the integrability condition and a formula of the type (32) to calculate all solutions is known. However, these solution formulas require the solution set of the associated homogeneous BDE.

IV. Boolean Differential Equations of Simple and Vectorial Derivatives

More generally, we explore as next Boolean Differential Equations which depend on the function $f(\mathbf{x})$ as well as their simple and vectorial derivatives. Such a BDE has the following general structure.

$$D_l\left(f(\mathbf{x}), \frac{\partial f(\mathbf{x})}{\partial x_1}, \frac{\partial f(\mathbf{x})}{\partial x_2}, \ldots, \frac{\partial f(\mathbf{x}_0, \mathbf{x}_1)}{\partial \mathbf{x}_0} \ldots, \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right) = D_r\left(f(\mathbf{x}), \frac{\partial f(\mathbf{x})}{\partial x_1}, \frac{\partial f(\mathbf{x})}{\partial x_2}, \ldots, \frac{\partial f(\mathbf{x}_0, \mathbf{x}_1)}{\partial \mathbf{x}_0} \ldots, \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right) \tag{41}$$

All derivatives in (41) are Boolean functions. Hence, applying the rules of the Boolean Algebra allows to transform the general BDE (41) into an equivalent homogeneous BDE

$$D\left(f(\mathbf{x}), \frac{\partial f(\mathbf{x})}{\partial x_1}, \frac{\partial f(\mathbf{x})}{\partial x_2}, \ldots, \frac{\partial f(\mathbf{x}_0, \mathbf{x}_1)}{\partial \mathbf{x}_0} \ldots, \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right) = 0 \ . \tag{42}$$

The solution of a Boolean differential equation (42) is again a set of functions $f(\mathbf{x})$. It is not necessary to verify whether there is a solution of the BDE (42), because the empty set is a possible solution, too.

The following definition supports both the understanding and the practical solution algorithms of Boolean differential equations (42).

Definition 4: Let be $g(\mathbf{x})$ a solution function of (42). Then

1)
$$\left[g(\mathbf{x}), \frac{\partial g(\mathbf{x})}{\partial x_1}, \frac{\partial g(\mathbf{x})}{\partial x_2}, ..., \frac{\partial g(\mathbf{x})}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{c}} \tag{43}$$
is a local solution for $\mathbf{x} = \mathbf{c}$,

2)
$$\mathrm{D}(u_0, u_1, ..., u_{2^n-1}) = 0 \tag{44}$$
is the Boolean equation, associated to the Boolean differential equation (42), and has the set of local solutions SLS.

3)
$$\nabla g(\mathbf{x}) = \left(g(\mathbf{x}), \frac{\partial g(\mathbf{x})}{\partial x_1}, \frac{\partial g(\mathbf{x})}{\partial x_2}, ..., \frac{\partial g(\mathbf{x})}{\partial \mathbf{x}}\right) . \tag{45}$$

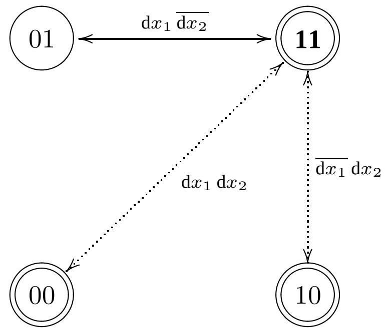

Fig. 1. Graph of the function $g(x_1, x_2) = x_1 \vee \overline{x}_2$: labels in the nodes indicates the values of x_1 and x_2, bold labels indicate the selected reference node, the simply circled node indicates the function value $g(x_1 = 0, x_2 = 1) = 0$, the doubly circled nodes indicate function values $g(x_1, x_2) = 1$, $\mathrm{d}x_i$ is the differential of a variable x_i, normal arrows indicate the change of function values between the nodes, and dotted arrows indicate that the function values do not change between the nodes.

The local solution (43) is the key to solve the BDE (42). The function values of $g(\mathbf{x})$ and of all simple and vectorial derivatives of $g(\mathbf{x})$ for a single point $\mathbf{x} = \mathbf{c}$ specify the function $g(\mathbf{x})$ completely. The following simple example helps to understand the reason for that.

Example 1: Figure 1 represents the function $g(x_1, x_2) = x_1 \vee \overline{x}_2$ and emphasizes

1) the function value
$$g(x_1 = 1, x_2 = 1) = 1 \ , \tag{46}$$

2) indicated by the direction of change $\mathrm{d}x_1\,\overline{\mathrm{d}x_2}$
$$\left.\frac{\partial g(\mathbf{x})}{\partial x_1}\right|_{(x_1,x_2)=(11)} = 1 \ , \tag{47}$$

3) indicated by the direction of change $\overline{\mathrm{d}x_1}\,\mathrm{d}x_2$
$$\left.\frac{\partial g(\mathbf{x})}{\partial x_2}\right|_{(x_1,x_2)=(11)} = 0 \ , \tag{48}$$

4) and indicated by the direction of change $\mathrm{d}x_1\,\mathrm{d}x_2$
$$\left.\frac{\partial g(\mathbf{x})}{\partial (x_1, x_2)}\right|_{(x_1,x_2)=(11)} = 0 \ . \tag{49}$$

¿From the known function value 1 (46) for $(x_1, x_2) = (11)$ and the known change of the function value in the direction of x_1 starting from $(x_1, x_2) = (11)$ (47) follows that $g(x_1 = 0, x_2 = 1) = 0$. ¿From the known function value 1 (46) for $(x_1, x_2) = (11)$ and the knowledge that the function value in the direction of x_2 starting from $(x_1, x_2) = (11)$ is unchanged (48) follows that $g(x_1 = 1, x_2 = 0) = 1$. ¿From the known function value 1 (46) for $(x_1, x_2) = (11)$ and the knowledge that the function value in the direction of (x_1, x_2) starting from $(x_1, x_2) = (11)$ is unchanged (49) follows that $g(x_1 = 0, x_2 = 0) = 1$. Hence, the function $g(x_1, x_2) = x_1 \vee \overline{x}_2$ is reconstructed based on the information of (46), (47), (48), and (49), only.

We learned from Example 1 that the values of a solution function $g(\mathbf{x})$ and all its simple and vectorial derivatives in

TABLE I

RELATIONSHIP BETWEEN THE INDEX OF A MODEL VARIABLE u_i, ITS BINARY CODE AND THE ASSOCIATED ELEMENT OF THE BDE (42)

Index	Binary Code	u_i	Associated Element
0	$(0\dots00)$	u_0	$f(\mathbf{x})$
1	$(0\dots01)$	u_1	$\frac{\partial f(\mathbf{x})}{\partial x_1}$
2	$(0\dots10)$	u_2	$\frac{\partial f(\mathbf{x})}{\partial x_2}$
3	$(0\dots11)$	u_3	$\frac{\partial f(\mathbf{x})}{\partial (x_1,x_2)}$
:	:	:	:
2^n-1	$(1\dots11)$	u_{2^n-1}	$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}$

a single selected point of the Boolean space describe this solution function completely. Hence, these elements carry the information about all solutions. In each expression of a BDE (42), both the function $f(\mathbf{x})$ and and all its simple and vectorial derivatives can only appear negated or not negated. Therefore we can model these elements by Boolean variables u_i. Table I shows that a value 1 in the binary code of the index i of u_i indicates the variables used in the associated derivative. Using the relationships of Table I, the BDE (42) is mapped into the associated Boolean equation (44). The solution of this associated Boolean equation (44) is a set of Boolean vectors $\mathbf{u}$ called *set of local solutions SLS*.

As explored in Example 1, a local solution describes potential solution functions with regard to each of the 2^n points of the Boolean space B^n. The main Theorem describes a restriction for the global solution functions.

Theorem 1: If the logic function $f(\mathbf{x})$ is a solution function of the Boolean differential equation (42), then all logic functions

$$f(x_1, x_2, ..., x_n) = f(x_1 \oplus c_1, x_2 \oplus c_2, ..., x_n \oplus c_n) \quad (50)$$

for $\mathbf{c} = (c_1, \dots, c_n) \in B^n$ are solution functions of (42) too.

Proof: The expression $x_i \oplus c_i$ can be expressed by:

$$x_i \oplus c_i = \begin{cases} x_i & : \quad c_i = 0 \\ \overline{x}_i & : \quad c_i = 1 \end{cases} \quad (51)$$

1) Due to the Shannon decomposition

$$f(\mathbf{x}_0, x_i, \mathbf{x}_1) = \overline{x}_i\, f(\mathbf{x}_0, 0, \mathbf{x}_1) \oplus x_i\, f(\mathbf{x}_0, 1, \mathbf{x}_1), \quad (52)$$
$$f(\mathbf{x}_0, \overline{x}_i, \mathbf{x}_1) = \overline{x}_i\, f(\mathbf{x}_0, 1, \mathbf{x}_1) \oplus x_i\, f(\mathbf{x}_0, 0, \mathbf{x}_1), \quad (53)$$

it follows that

$$f(\mathbf{x}_0, x_i, \mathbf{x}_1) \mid_{(\mathbf{x}_0,x_i,\mathbf{x}_1)=(\mathbf{c}_0,c_i,\mathbf{c}_1)} = f(\mathbf{x}_0, \overline{x}_i, \mathbf{x}_1) \mid_{(\mathbf{x}_0,x_i,\mathbf{x}_1)=(\mathbf{c}_0,\overline{c}_i,\mathbf{c}_1)}, \quad (54)$$

$$\frac{\partial f(\mathbf{x}_0, x_i, \mathbf{x}_1)}{\partial \mathbf{x}_0} \mid_{(\mathbf{x}_0,x_i,\mathbf{x}_1)=(\mathbf{c}_0,c_i,\mathbf{c}_1)} = \frac{\partial f(\mathbf{x}_0, \overline{x}_i, \mathbf{x}_1)}{\partial \mathbf{x}_0} \mid_{(\mathbf{x}_0,x_i,\mathbf{x}_1)=(\mathbf{c}_0,\overline{c}_i,\mathbf{c}_1)}, \quad (55)$$

$$\begin{aligned} \frac{\partial f(\mathbf{x}_0, x_i, \mathbf{x}_1)}{\partial (\mathbf{x}_0, x_i)} \mid_{(\mathbf{x}_0,x_i,\mathbf{x}_1)=(\mathbf{c}_0,\mathbf{c}_0,\mathbf{c}_1)} &= \\ f(\mathbf{x}_0 = \mathbf{c}_0, x_i = c_i, \mathbf{x}_1 = \mathbf{c}_1) &\oplus \\ f(\mathbf{x}_0 = \mathbf{c}_0, \overline{x}_i = \overline{c}_i, \mathbf{x}_1 = \mathbf{c}_1) &= \\ f(\mathbf{x}_0 = \mathbf{c}_0, \overline{x}_i = \overline{c}_i, \mathbf{x}_1 = \mathbf{c}_1) &\oplus \\ f(\mathbf{x}_0 = \mathbf{c}_0, x_i = c_i, \mathbf{x}_1 = \mathbf{c}_1) &= \\ \frac{\partial f(\mathbf{x}_0, \overline{x}_i, \mathbf{x}_1)}{\partial (\mathbf{x}_0, x_i)} \mid_{(\mathbf{x}_0,x_i,\mathbf{x}_1)=(\mathbf{c}_0,\overline{c}_i,\mathbf{c}_1)}&, \end{aligned} \quad (56)$$

and, therefore (50) is true for $\mathbf{c} = (\mathbf{c}_0, c_i, \mathbf{c}_1) = (\mathbf{0}, 1, \mathbf{0})$.

2) All 2^n vectors $\mathbf{c}$ can be created by the antivalence $\oplus$ of vectors with $|\mathbf{c}| = 1$. The theorem is completely proven when 1. is applied 2^n times for different solution functions in each iteration.

■

It can be proven that (50) defines an equivalence relation. Hence,

1) (50) divides all logic functions $f : B^n \to B$ into disjoint classes, and
2) the solution of the Boolean differential equation (42) will only consist of k complete equivalence classes of functions defined by (50), where $k \geq 0$

Based on Theorem 1 we can specify the solution process for the BDE (42). This solution process consists of four steps:

1) Create the associated Boolean equation (44) of the BDE (42).
2) Solve the Boolean equation (44): as solution we get the set of all local solutions $SLS(\mathbf{u})$. Each local solution is *necessary but not sufficient* for a global solution function of the BDE (42). The sufficient condition for a global solution function of the BDE (42) is that a local solution exists for each point of the Boolean space B^n:

$$\forall \mathbf{c} \in B^n \quad \nabla f(\mathbf{x}) \mid_{\mathbf{x}=\mathbf{c}} \in SLS(\mathbf{u}) \ . \quad (57)$$

Some of the local solutions for a global solution function of the BDE (42) can be equal to each other.
3) The set $SLS(\mathbf{u})$ expresses each local solution by the function value u_0 in one point of B^n and the values of changed $u_i, 0 < i \leq 2^n$, in all the other points of B^n. In order to solve the Boolean differential equation (42), it is easier to manipulate function values instead of values of the derivatives. The vector $\mathbf{v}$ of values of the function contains the same information as the vector $\mathbf{u}$ and can be calculated by (58). The function `d2v` (*derivative to*

TABLE II

INDEX PAIRS FOR THE EXCHANGE OF FUNCTION VALUES

$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$0 \Leftrightarrow 1$	$0 \Leftrightarrow 2$	$0 \Leftrightarrow 4$	$0 \Leftrightarrow 8$	$0 \Leftrightarrow 16$
$2 \Leftrightarrow 3$	$1 \Leftrightarrow 3$	$1 \Leftrightarrow 5$	$1 \Leftrightarrow 9$	$1 \Leftrightarrow 17$
$4 \Leftrightarrow 5$	$4 \Leftrightarrow 6$	$2 \Leftrightarrow 6$	$2 \Leftrightarrow 10$	$2 \Leftrightarrow 18$
$6 \Leftrightarrow 7$	$5 \Leftrightarrow 7$	$3 \Leftrightarrow 7$	$3 \Leftrightarrow 11$	$3 \Leftrightarrow 19$
$8 \Leftrightarrow 9$	$8 \Leftrightarrow 10$	$8 \Leftrightarrow 12$	$4 \Leftrightarrow 12$	$4 \Leftrightarrow 20$
$10 \Leftrightarrow 11$	$9 \Leftrightarrow 11$	$9 \Leftrightarrow 13$	$5 \Leftrightarrow 13$	$5 \Leftrightarrow 21$
$12 \Leftrightarrow 13$	$12 \Leftrightarrow 14$	$10 \Leftrightarrow 14$	$6 \Leftrightarrow 14$	$6 \Leftrightarrow 22$
$14 \Leftrightarrow 15$	$13 \Leftrightarrow 15$	$11 \Leftrightarrow 15$	$7 \Leftrightarrow 15$	$7 \Leftrightarrow 23$
$16 \Leftrightarrow 17$	$16 \Leftrightarrow 18$	$16 \Leftrightarrow 20$	$16 \Leftrightarrow 24$	$8 \Leftrightarrow 24$
$18 \Leftrightarrow 19$	$17 \Leftrightarrow 19$	$17 \Leftrightarrow 21$	$17 \Leftrightarrow 25$	$9 \Leftrightarrow 25$
$20 \Leftrightarrow 21$	$20 \Leftrightarrow 22$	$18 \Leftrightarrow 22$	$18 \Leftrightarrow 26$	$10 \Leftrightarrow 26$
$22 \Leftrightarrow 23$	$21 \Leftrightarrow 23$	$19 \Leftrightarrow 23$	$19 \Leftrightarrow 27$	$11 \Leftrightarrow 27$
$24 \Leftrightarrow 25$	$24 \Leftrightarrow 26$	$24 \Leftrightarrow 28$	$20 \Leftrightarrow 28$	$12 \Leftrightarrow 28$
$26 \Leftrightarrow 27$	$25 \Leftrightarrow 27$	$25 \Leftrightarrow 29$	$21 \Leftrightarrow 29$	$13 \Leftrightarrow 29$
$28 \Leftrightarrow 29$	$28 \Leftrightarrow 30$	$26 \Leftrightarrow 30$	$22 \Leftrightarrow 30$	$14 \Leftrightarrow 30$
$30 \Leftrightarrow 31$	$29 \Leftrightarrow 31$	$27 \Leftrightarrow 31$	$23 \Leftrightarrow 31$	$15 \Leftrightarrow 31$

value) transforms the set $SLS(\mathbf{u})$ into the set $SLS'(\mathbf{v})$:

$$\begin{aligned} v_0 &= u_0, \\ v_i &= u_0 \oplus u_i, \quad \text{with} \quad i = 1, 2, ..., 2^n - 1. \end{aligned} \tag{58}$$

4) Due to (50) the exchange of x_i and $\overline{x}_i$ does not change the set of solution functions. Hence, the intersection of the given set $SLS'(\mathbf{v})$ with the exchanged set $SLST(\mathbf{v})$ for all variables x_i separates the global solution functions from local solutions which are not sufficient. This change can be implemented by exchanging pairs of function values v_i (59) in the set $SLS'(\mathbf{v})$:

$$\begin{aligned} v_{(m+2\,k\cdot 2^{i-1})} &\Longleftrightarrow v_{(m+(2\,k+1)\cdot 2^{i-1})}, \\ \text{with} \quad i &= 1, 2, ..., n, \\ m &= 0, 1, ..., 2^{i-1} - 1, \\ k &= 0, 1, ..., 2^{n-i} - 1. \end{aligned} \tag{59}$$

The function $\mathtt{epv}$ (*exchange pairs of values*) exchanges function values of $SLS'(\mathbf{v})$ using formula (59) with respect to a given index i and returns the set $SLST(\mathbf{v})$. Table II lists the index pairs defined by (59) to solve a Boolean differential equation (42) of up to 5 variables. The value of i indicates which variable x_i of the desired solution function must change. The set of global solution functions of the bde (42) is described by the solution vectors $(\mathbf{v})$ as follows:

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) = &\overline{x}_1\, \overline{x}_2\, \ldots\, \overline{x}_n \wedge v_0 \vee \\ &x_1\, \overline{x}_2\, \ldots\, \overline{x}_n \wedge v_1 \vee \\ &\ldots \vee \\ &x_1\, x_2\, \ldots\, x_n \wedge v_{2^n-1} \end{aligned} \tag{60}$$

Using both the function $\mathtt{d2v}$ which transforms the variables u_i associated with the derivatives into the variables u_i associated with function values, and the function $\mathtt{epv}$ which exchanges function values of $SLS'(\mathbf{v})$ using formula (59) with respect to a given index i the algorithm 1 finds all solution functions of a given BDE (42). Due to (50) the solution functions are classes.

Algorithm 1 Separation of Function Classes

Require: BDE (42) with function $f(\mathbf{x})$ containing n variables

Ensure: set of Boolean vectors $\mathbf{v} = (v_0, v_1, \ldots, v_{2^n-1})$ that describe substituted in (60) the set of all solution functions of the BDE (42)

1: $SLS(\mathbf{u}) \leftarrow$ solution of BE (44) associated with BDE (42)
2: $SLS'(\mathbf{v}) \leftarrow \mathtt{d2v}(SLS(\mathbf{u}))$
3: **for** $i \leftarrow 1$ to n **do**
4: $\quad SLST(\mathbf{v}) \leftarrow \mathtt{epv}(SLS'(\mathbf{v}), i)$
5: $\quad SLS'(\mathbf{v}) \leftarrow SLS'(\mathbf{v}) \cap SLST(\mathbf{v})$
6: **end for**

Example 2: The complete solution process will be shown for the Boolean differential equation

$$\frac{\partial f(x_1, x_2)}{\partial x_1} \oplus \frac{\partial f(x_1, x_2)}{\partial x_2} \oplus \frac{\partial f(x_1, x_2)}{\partial (x_1, x_2)} = 1 \ , \tag{61}$$

which describes all bent functions of two variables.

Using the references of Table I we get the associated Boolean equation (62) for the BDE to solve (61).

$$u_1 \oplus u_2 \oplus u_3 = 1 \tag{62}$$

In step 1 of the algorithm 1, the associated Boolean equation is solved with the result $SLS(\mathbf{u})$ (63) as solution set.

$$SLS(\mathbf{u}) = \begin{array}{ccc} u_1 & u_2 & u_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \tag{63}$$

In step 2 of the algorithm 1, the function $\mathtt{d2v}(SLS(\mathbf{u}))$ maps the solution set (63) into $SLS'(\mathbf{v})$ (64) using (58).

$$SLS'(\mathbf{v}) = \begin{array}{cccc} v_0 & v_1 & v_2 & v_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array} \tag{64}$$

Because $f(x_1, x_2)$ depends on two variables, step 4 and 5 in the loop from row 3 until row 6 of algorithm 1 must be executed for $i = 1$ and $i = 2$.

The exchange of function values of $SLS'(\mathbf{v})$ (64) with regard to (59) and the index $i = 1$ creates $SLST(\mathbf{v})$ (65). Therefor the function $\mathtt{epv}(SLS'(\mathbf{v}), i = 1)$ exchanges the columns v_0 with v_1 and v_2 with v_3. The indices for these exchanges are enumerated in column $i = 1$ of Table II.

$$SLST(\mathbf{v}) \quad = \quad \begin{array}{cccc} v_0 & v_1 & v_2 & v_3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array} \tag{65}$$

The result of the intersection of $SLS'(\mathbf{v})$ (64) and the transformed set of local sulutions $SLST(\mathbf{v})$ (65) has in this special case again the result $SLS'(\mathbf{v})$ (64).

Next the steps 4 and 5 must be executed for $i = 2$.

The exchange of function values of $SLS'(\mathbf{v})$ (64) with regard to (59) and the index $i = 2$ creates $SLST(\mathbf{v})$ (66). Therefor the function $\mathtt{epv}(SLS'(\mathbf{v}), i = 2)$ exchanges the columns v_0 with v_2 and v_1 with v_3. The indices for these exchanges are enumerated in column $i = 2$ of Table II.

$$SLST(\mathbf{v}) \quad = \quad \begin{array}{cccc} v_0 & v_1 & v_2 & v_3 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array} \tag{66}$$

The result of the intersection of $SLS'(\mathbf{v})$ (64) and the transformed set of local solutions $SLST(\mathbf{v})$ (66) in step 5 for the second iteration in algorithm 1 has in this special case again the result $SLS'(\mathbf{v})$ (64). Hence, the set of 8 vectors of function values (64) is the final result of algorithm 1 and describes due to (60) the following set of 8 solution functions $f(x_1, x_2)$.

$$f_1(x_1, x_2) = x_1 \wedge \overline{x}_2 \tag{67}$$
$$f_2(x_1, x_2) = \overline{x}_1 \wedge x_2 \tag{68}$$
$$f_3(x_1, x_2) = x_1 \wedge x_2 \tag{69}$$
$$f_4(x_1, x_2) = x_1 \vee x_2 \tag{70}$$
$$f_5(x_1, x_2) = \overline{x}_1 \vee x_2 \tag{71}$$
$$f_6(x_1, x_2) = x_1 \vee \overline{x}_2 \tag{72}$$
$$f_7(x_1, x_2) = \overline{x}_1 \vee \overline{x}_2 \tag{73}$$
$$f_8(x_1, x_2) = \overline{x}_1 \wedge \overline{x}_2 \tag{74}$$

These 8 functions are the 8 nonlinear functions of two Boolean variables. The other 8 functions of B^2 are linear functions which are specified by $f_l(x_1, x_2) = c_0 \oplus c_1\, x_1 \oplus c_2\, x_2$. Hence, the calculated nonlinear functions are bent functions.

V. More General Boolean Differential Equations

The BDE explored in Section IV are restricted to simple and vectorial derivatives. Here we study how more general Boolean differential equations can be solved.

First, we extend the BDE (42) to a BDE in which each *de*rivative *o*peration ($DEO_i(f(\mathbf{x}))$) of the function $f(\mathbf{x})$ as defined in Section II can appear. A homogeneous version of such an extended BDE is (75), where $DEO_i(f(\mathbf{x}))$ can be each simple derivative operation, each vectorial derivative operation as well as each m-fold derivative operation of the unknown function $f(\mathbf{x})$.

$$D(f(\mathbf{x}), DEO_1(f(\mathbf{x})), \ldots, DEO_k(f(\mathbf{x}))) = 0. \tag{75}$$

It is not necessary to create a new solution algorithm for the BDE (75) because there are theorems of the Boolean differential calculus which allow to express each derivative operation of $f(\mathbf{x})$ using the function $f(\mathbf{x})$ and their simple and vectorial derivatives. Hence, each BDE (75) can be transformed into the BDE (42) and then solved using algorithm 1.

The transformation relation for the simple minimum is

$$\min_{x_i} f(\mathbf{x}) = f(\mathbf{x}) \wedge \overline{\frac{\partial f(\mathbf{x})}{\partial x_i}} \ , \tag{76}$$

and for the simple maximum it is

$$\max_{x_i} f(\mathbf{x}) = f(\mathbf{x}) \vee \frac{\partial f(\mathbf{x})}{\partial x_i} \ . \tag{77}$$

Similarly, the transformation relations are for the vectorial minimum

$$\min_{\mathbf{x}_0} f(\mathbf{x}_0, \mathbf{x}_1) = f(\mathbf{x}_0, \mathbf{x}_1) \wedge \overline{\frac{\partial f(\mathbf{x}_0, \mathbf{x}_1)}{\partial \mathbf{x}_0}} \ , \tag{78}$$

and for the vectorial maximum

$$\max_{\mathbf{x}_0} f(\mathbf{x}_0, \mathbf{x}_1) = f(\mathbf{x}_0, \mathbf{x}_1) \vee \frac{\partial f(\mathbf{x}_0, \mathbf{x}_1)}{\partial \mathbf{x}_0} \ . \tag{79}$$

In order to have short formulas, we give the transformation relations for the m-fold derivative operations with regard to the set of variables $(x_1, x_2) \in \mathbf{x}$. It is known, that all these formulas can be generalized to each subset of variables.

$$\min_{(x_1,x_2)}{}^2 f(\mathbf{x}) = f(\mathbf{x}) \wedge \overline{\frac{\partial f(\mathbf{x})}{\partial x_1}} \wedge \overline{\frac{\partial f(\mathbf{x})}{\partial x_2}} \wedge \overline{\frac{\partial f(\mathbf{x})}{\partial (x_1, x_2)}} \tag{80}$$
$$\max_{(x_1,x_2)}{}^2 f(\mathbf{x}) = f(\mathbf{x}) \vee \frac{\partial f(\mathbf{x})}{\partial x_1} \vee \frac{\partial f(\mathbf{x})}{\partial x_2} \vee \frac{\partial f(\mathbf{x})}{\partial (x_1, x_2)} \tag{81}$$
$$\frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} = \frac{\partial f(\mathbf{x})}{\partial x_1} \oplus \frac{\partial f(\mathbf{x})}{\partial x_2} \oplus \frac{\partial f(\mathbf{x})}{\partial (x_1, x_2)} \tag{82}$$
$$\Delta_{(x_1,x_2)} f(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_1} \vee \frac{\partial f(\mathbf{x})}{\partial x_2} \vee \frac{\partial f(\mathbf{x})}{\partial (x_1, x_2)} \tag{83}$$

Secondly, we extend the BDE (75) by the logic variables x_i.

$$D(f(\mathbf{x}), DEO_1(f(\mathbf{x})), \ldots, DEO_k(f(\mathbf{x})), x_1, \ldots, x_n) = 0. \tag{84}$$

Using BDE (84) not only classes of functions but each set of Boolean functions can be described. In [3] algorithm 1 was extended to algorithm *Separation of Functions* that solves the more general BDE (84).

Thirdly, we extend BDE (84) to BDE (85) in which additionally each *di*fferential *o*peration ($DIO_i(f(\mathbf{x}))$) of the function $f(\mathbf{x})$ and each differential $\mathrm{d}x_i$ of a Boolean variable x_i can appear. Definitions of these differential operations are given in [2], [3], [4], and [5].

$$D(f(\mathbf{x}), DEO_1(f(\mathbf{x})), \ldots, DEO_k(f(\mathbf{x})), x_1, \ldots, x_n, \\ DIO_1(f(\mathbf{x})), \ldots, DIO_l(f(\mathbf{x})), \mathrm{d}x_1, \ldots, \mathrm{d}x_n) = 0. \tag{85}$$

All differential operations can be expressed by derivative operations and differentials of the variables. The relationships for these transformations are given in [3] and [5]. The application of these transformation simplifies BDE (85) to BDE (86).

$$D(f(\mathbf{x}), DEO_1(f(\mathbf{x})), \ldots, DEO_k(f(\mathbf{x})), \\ x_1, \ldots, x_n, \mathrm{d}x_1, \ldots, \mathrm{d}x_n) = 0. \tag{86}$$

The algorithm *Separation of Functions* of [3] allows to solve the BDE (86), too.

VI. Conclusion

This paper introduces into the theory of Boolean differential equations (BDE). The most important feature of a BDE is that the solution of each BDE is a set of Boolean functions. Hence, a BDE allow to describe a set of Boolean functions. We give an algorithm that allows to solve important classes of BDE. These classes of BDEs have special classes of Boolean functions as solution. The given solution algorithm can easily be implemented using XBOOLE [2].

More general classes BDEs are introduced which allow to specify each set of Boolean function. In order to restrict the size of this paper we refer to [3] where an algorithm to solve these most general BDEs is given.

Knowing the operation of the Boolean differential calculus, it is easy to specify a BDE that describes a set of Boolean functions with certain properties. However, it is a challenge for the future to find Boolean differential equations for function classes characterized by properties which are not directly covered by the operations of the Boolean differential calculus.

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