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Lecture 4
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# Solution 4.1: Basic Model Circuit

Make use of the laws given on the exercise sheet.

- First we note that $R_{Cl}$ is set to infinity:

$$I_{Cl} = \lim_{R_{Cl}\to\infty} \frac{V}{R_{Cl}} = 0.$$

  Kirchhoff's current law then implies that

$$I_{Na} = -I_K.$$

  Insert Ohm's law to obtain

$$\frac{V_m - E_{Na}}{R_{Na}} = -\frac{V_m - E_K}{R_K}.$$

  Now we solve for $R_K$ and insert the physical quantities

$$R_K = -\frac{R_{Na} \cdot (V_m - E_K)}{(V_m - E_{Na})} = 0.5\ \text{M}\Omega.$$

- Similarly we calculate

$$\begin{aligned} I_{Na} + I_K + I_{Cl} &= 0 && \text{(Kirchhoff's Law)} \\ \frac{V_m - E_{Na}}{R_{Na}} + \frac{V_m - E_K}{R_K} + \frac{V_m - E_{Cl}}{R_{Cl}} &= 0 && \text{(Ohm's Law)} \\ V_m = \frac{E_{Na}/R_{Na} + E_K/R_K + E_{Cl}/R_{Cl}}{1/R_{Na} + 1/R_K + 1/R_{Cl}} & && \text{(Solving for } V_m\text{)} \\ V_m &= -51.4\ \text{mV} && \text{(inserting)} \end{aligned}$$

- And finally

$$\begin{aligned} I_{Na} &= -I_K - I_{Cl} && \text{(Kirchhoff's Law)} \\ I_{Na} = -\frac{V_m - E_K}{R_K} - \frac{V_m - E_{Cl}}{R_{Cl}} & && \text{(Ohm's Law)} \\ I_{Na} &= -5.7\ \text{nA} && \text{(inserting)} \end{aligned}$$

  Sodium is more concentrated outside the cell, potassium inside. Therefore, when the membrane potential is between the two reversal potentials ($E_K$, $E_{Na}$) as is the case here, positively charged $Na^+$ ions flow inward.

  The resistance we find to be

$$R_{Na} = \frac{V_m - E_{Na}}{I_{Na}} = 18.4\ \text{M}\Omega$$

# Solution 4.2: The Membrane Capacitance

1. The necessary charge $Q$ to cause a membrane potential $V_m$ across the membrane capacitance $C_m$ is

$$Q = C_m \cdot V_m = 1\text{nF} \cdot 70\text{mV} = 7 \cdot 10^{-11}\text{C}.$$

   In terms of monovalent ions carrying one elementary charge this is

$$N_{ion} = \frac{7 \cdot 10^{-11}\text{C}}{1.6 \cdot 10^{-19}\text{C/Ion}} = 4.4 \cdot 10^8\ \text{Ions}$$

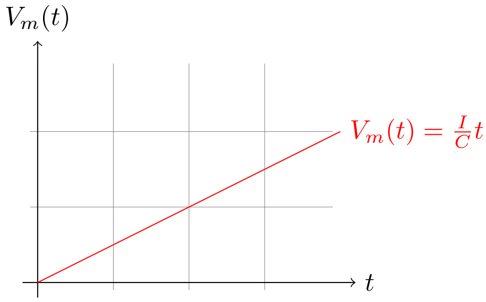
2. The membrane capacitance determines how much current is required to charge an ideal membrane at a given rate:

$$\begin{aligned} I &= C_m \cdot \frac{dV}{dt} \\ &= 1\text{nF} \cdot 1\frac{\text{mV}}{\text{ms}} = 1\text{nA} \end{aligned}$$

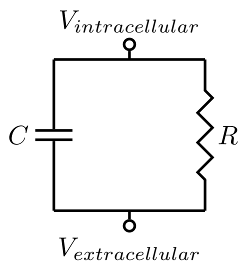
Note: the formula $C \cdot \frac{dV}{dt} = I$ will be used a lot in the exercised about currents in the neuron.

3. Without leakage $V_m(t)$ depends linearly on $t$ because

$$V_m = \frac{Q}{C} = \frac{I \cdot t}{C}$$




4. The resistor and capacitor are in parallel (the leak current through the resistor slowly decreases the potential across the capacitor).




5. We know that as $t \to \infty$ the membrane potential tends to $-38.3$ mV. This we insert into the given equation

$$-38.3\text{mV} - (-45\text{mV}) = \lim_{t\to\infty} \Delta V_m(t) = \lim_{t\to\infty} R \cdot I_{in}\left(1 - e^{-t/\tau}\right) = R \cdot I_{in} = R \cdot 9.2 \text{ nA}.$$

Solving for $R$ gives

$$R = \frac{-38.3\text{mV} - (-45\text{mV})}{9.2\text{nA}} = 728 \text{ k}\Omega.$$

We use the relation $\tau = R \cdot C$ to obtain

$$C = \frac{\tau}{R} = \frac{0.124\text{s}}{728 \text{ k}\Omega} = 170 \text{ nF}$$

## Solution 4.3: Cable Equation

1. The injection site is at $x = 0$

$$v(0) = \frac{i_e R_\lambda}{2} e^{|0|/\lambda} = \frac{i_e R_\lambda}{2}$$

Divide by $i_e$ to get

$$\text{Input resistance} = \frac{v(0)}{i_e} = \frac{R_\lambda}{2}$$

2. The action potential will propagate if the potential at the second node rises above the $-50\text{mV}$ threshold.

$$\frac{i_e R_\lambda}{2} = v(0) = V - V_{\text{rest}} = 30\text{mV} - (-70\text{mV}) = 100\text{mV}$$

$$v(2\lambda) = 100\text{mV} \cdot e^{-|2\lambda|/\lambda} = 13.53\text{mV}$$

$$V|_{x=2\lambda} = V_{\text{rest}} + v(2\lambda) = -70\text{mV} + 13.53\text{mV} = -56.47\text{mV}$$

   which is below the threshold; the action potential is not propagated.

3. Changing the radius will change $\lambda$ to $\lambda'$, where

$$\lambda' = \sqrt{\frac{2a \cdot r_m}{2 \cdot r_L}} = \sqrt{2}\lambda$$

   so that we get

$$v(2\lambda) = 100\text{mV} \cdot e^{-|2\lambda|/(\sqrt{2}\lambda)} = 24.31\text{mV}$$

$$V|_{x=2\lambda} = V_{\text{rest}} + v(2\lambda) = -70\text{mV} + 24.31\text{mV} = -45.69\text{mV}$$

   so that we cross the threshold this time and the action potential is propagated.

4. Note that the additional capacitance only affects the dynamical part of the solution (*i.e.* how long it takes to reach the steady state), not the steady state itself. Therefore we only need to take into account the additional resistance here, which will change $\lambda$:

$$\lambda'' = \sqrt{\frac{a \cdot 4r_m}{2 \cdot r_L}} = 2\lambda$$

   so that we get

$$v(2\lambda) = 100\text{mV} \cdot e^{-|2\lambda|/(2\lambda)} = 36.79\text{mV}$$

$$V|_{x=2\lambda} = V_{\text{rest}} + v(2\lambda) = -70\text{mV} + 36.79\text{mV} = -33.21\text{mV}.$$

   We cross the threshold again and the action potential is propagated.

## Solution 4.4: Circuit Elements

Resistor – Ion Channels
Capacitance – Lipid bilayer
Battery – Ionic concentration gradient