National Taiwan University - Calculus 2 for Class 01-09

Worksheet 1 : Volumes

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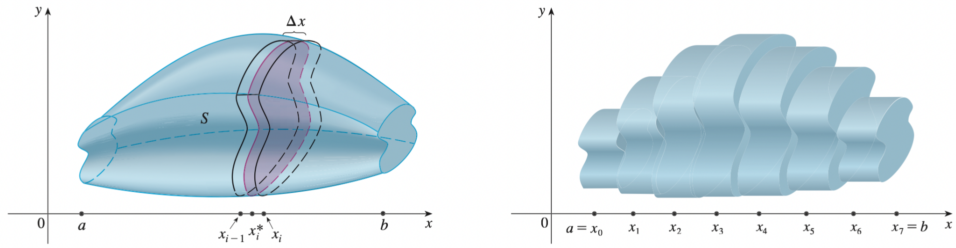
Definition of Volume.

For a *cylinder* (of an arbitrary base), its volume equals to $V = Ah$ where A is its base area and h is its height.



But how do we compute the volume of a general solid S? To do this, we place S in a coordinate system. Suppose that the x coordinates of S is contained in the interval $[a, b]$ (see the left figure). Then we cut S by a family of parallel planes, $P_{x_i} : x = x_i$, where $a = x_1 < x_2 < ... < x_n = b$ are constants (see the right figure). Thus we slice S into many small slabs which are parts of S between two adjacent planes P_{x_i} and $P_{x_{i+1}}$, $i = 1, 2, \ldots, n-1$. Each slab can be approximated by a cylinder with height $x_{i+1} - x_i = \Delta x_i$ and base $S \cap P_{x_i}$. If we know the area of the cross-section $S \cap P_x$ for all $x \in [a, b]$ which is denoted by $A(x)$, then the approximating cylinders have volumes $A(x_i)\Delta x_i$, $i = 1, 2, \ldots, n-1$. Adding volumes of these cylinders, we can approximate the volume of S by

$$\sum_{i=1}^{n-1} A(x_i)\Delta x_i$$

which is a finite Riemann sum of $A(x)$ over the interval $[a, b]$. When Δx_i tends to 0, the slices become thinner and thus the approximation becomes better. Hence, we will *define the volume of S as the limit of these Riemann sums :*

$$V(S) = \int_a^b A(x)\, dx$$

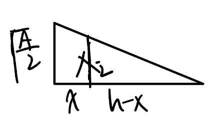
In conclusion, the volume of S is the *definite integral of the cross-sectional area sliced by a family of parallel planes.*

Exercise 1.

(a) Follow the steps and find the volume of the pyramid whose base is a square with area A and whose height is h. (Hint: Textbook p 454 [1])

- Place the pyramid in a coordinate system. To simplify the computation, we can choose the origin to be 底面中心 and the x-axis to be central axis.

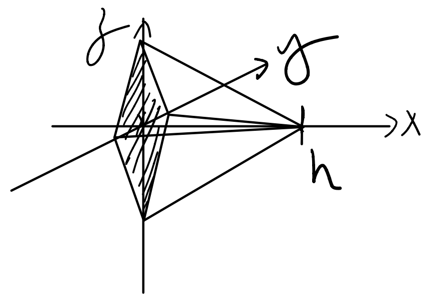
- Let P_{x_0} be the plane $x = x_0$. The cross section $S \cap P_{x_0}$ is a square (shape) with area $A\left(\frac{h-x_0}{h}\right)^2$.



$\Rightarrow \frac{h-x}{h} = \frac{\square}{\frac{\sqrt{A}}{2}} \Rightarrow x_i = \frac{\sqrt{A}}{2} \times \left(\frac{h-x}{h}\right)$

- Write the volume of S as a definite integral of $A(x) =$ area of $S \cap P_x$, and compute the volume.

$V(x) = \int_0^h A(x)\,dx = \int_0^h \left[\left(\frac{\sqrt{A}}{2}\left(\frac{h-x}{h}\right)\right) \times 2\right]^2 \times \frac{1}{2}\, dx$



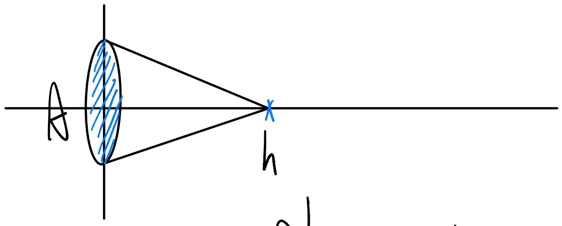
$\frac{Ah}{3}$

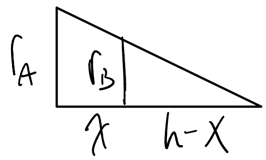
$A\int_0^h \left(\frac{h-x}{h}\right)^2 dx = \frac{-A}{h^2}\int_0^h (x-h)^2\, d(x-h)$

$= \frac{-A}{h^2}\left(\frac{(x-h)^3}{3}\Big|_0^h\right) = \frac{A}{h^2} \times \frac{h^3}{3} = \frac{Ah}{3}$ #

[1] Geogebra demo : www.geogebra.org/m/AkkphsNF

(b) Imitate the above procedure and find the volume of a circular cone whose base is a disc of area A and whose height is h.





$$\frac{h-x}{h} = \frac{r_B}{r_A}$$

$$r_B = r_A \times \left(\frac{h-x}{h}\right)$$

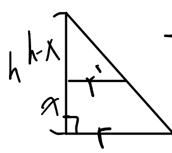
$$(r_B)^2\pi = (r_A)^2 \times \left(\frac{h-x}{h}\right)^2 \times \pi$$

$$V(x) = \int_0^h A(x)\,dx$$

$$= \int_0^h A\left(\frac{h-x}{h}\right)^2 dx = \frac{A}{h^2}\int (x-h)^2\,d(x-h) = \frac{A}{h^2} \times \frac{(x-h)^3}{3}\Big|_0^h = -\frac{-h^3 \times A}{3h^2} = \frac{Ah}{3}$$

(c) Actually, a pyramid or a circular cone are just special kinds of 'cones'. Suppose that B is a region on a plane P and O is a point not on P. A cone with base B and apex O consists all points on line segments that join the apex O to a point of B. Note that the base B could be a square, a disc, or any irregular shape. If the area of base is A and the distance from the apex O to the plane P (which is also known as the height of the cone) is h, can you derive the volume of the cone in terms of A and h?

1° let 底面面積 A



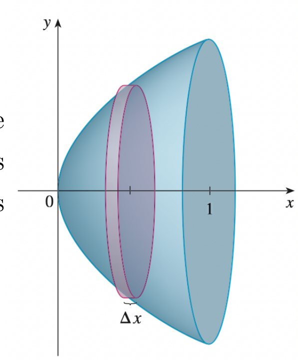
$$\frac{h-x}{h} = \sqrt{\frac{A(x)}{A}} \Rightarrow A(x) = \left(\frac{h-x}{h}\right)^2 A$$

2° $$V(x) = \int_0^h A(x)\,dx = \int_0^h \frac{(h-x)^2}{h^2} A\,dx$$

$$= \frac{A}{h^2}\int_0^h (x-h)^2\,d(x-h) = \frac{A}{h^2} \times \frac{(x-h)^3}{3}\Big|_0^h = \frac{Ah}{3}$$

Volumes of Solids of Revolution

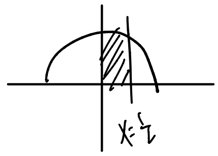
If we revolve a plane region about a line, we obtain a **solid of revolution**. To compute the volume of a solid of revolution, we often cut the solid with planes perpendicular to the axis of revolution. Then the cross sections are just disks or annular rings. Integrating the cross section areas, we can easily obtain the volume.



Exercise 2.

(a) Let D be the region under the circle $y = \sqrt{1-x^2}$, above the x axis, between lines $x = 0$ and $x = \frac{1}{2}$. Rotate D about the x-axis and we obtain a solid S. Find the volume of S.

- Let P_{x_0} be the plane $x = x_0$, $0 \le x_0 \le 1/2$. The cross section $S \cap P_{x_0}$ is a disk with radius $\sqrt{1-(x_0)^2}$ and area $(1-(x_0)^2)\pi$.
- Write the volume of S as a definite integral and compute the volume.



- S is part of a unit ball. Can you compute the volume of a ball with radius r? the volume of a cap of a ball $\{(x,y,z)\,|\,x^2+y^2+z^2 \le R^2, x \ge r\}$ where $0 < r < R$?

(b) Let D be the disk $\{(x,y)\,|\,x^2+(y-2)^2 \le 1\}$. Rotate D about the x-axis and we obtain a donuts shape solid S. Find the volume of S.

- Let P_{x_0} be the plane $x = x_0$, $-1 \le x_0 \le 1$. The cross section $S \cap P_{x_0}$ is a annular ring with outer radius 2, inner radius 1 and area π.
- Write the volume of S as a definite integral and compute the volume. [2]

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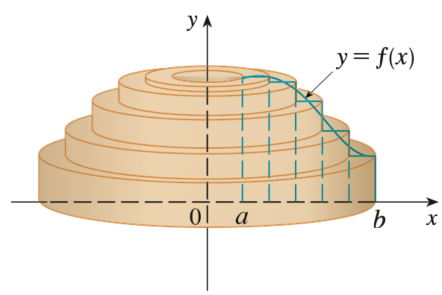
Volumes by Cylindrical Shells

There is another way to compute the volume of a solid of revolution which is called the *Method of Cylindrical Shells*. Consider a solid S obtained by rotating about the y-axis the plane region D bounded by $y = f(x) \ge 0$, $y = 0$, $x = a$, and $x = b$, where $0 \le a < b$.



To compute its volume, we first divide D by lines $x = x_i$, where $a = x_0 < x_1 < \cdots < x_n = b$. Then the part of D between $x = x_{i-1}$ and $x = x_i$ is approximated by a rectangle $[x_{i-1}, x_i] \times [0, f(\bar{x}_i)]$ where $\bar{x}_i = \dfrac{x_{i-1}+x_i}{2}$. Rotating the rectangle about the y-axis, we obtain a thin cylindrical shell with outer radius x_i, inner radius x_{i-1}, and height $f(\bar{x}_i)$. Hence this cylindrical shell has volume $\pi(x_i^2 - x_{i-1}^2)f(\bar{x}_i)$ which is

$$2\pi\bar{x}_i f(\bar{x}_i)\Delta x_i \text{ where } \Delta x_i = x_i - x_{i-1}.$$

By adding up volumes of these cylindrical shells and taking limit, we obtain the volume of S : [a]

$$V(S) = \int_a^b 2\pi x f(x)\, dx.$$

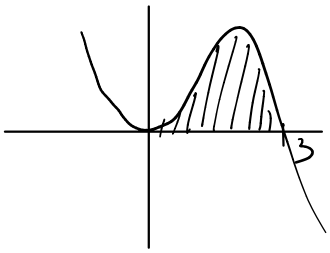
[a] Geogebra Demo : https://www.geogebra.org/m/ggfye7dj

[2] Geogebra Demo : https://www.geogebra.org/m/GjrsJvU9

Exercise 3.

Let D be the region under $y = f(x) = 3x^2 - x^3$, above the x-axis, from $x = 0$ to $x = 3$.

(a) Find the volume of the solid obtained by rotating D about the y-axis.



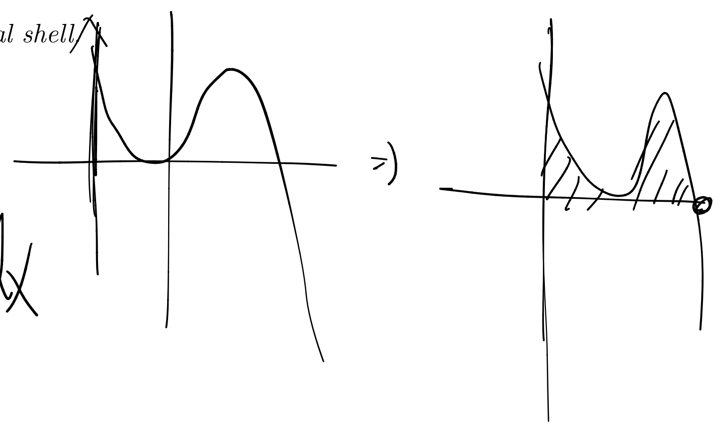
$V = \int_0^3 2\pi x f(x)\,dx$, $f(x) = 3x^2 - x^3$

$= 2\pi \int_0^3 x(3x^2 - x^3)\,dx$

$= \frac{243\pi}{10}$ #

(b) Find the volume of the solid obtained by rotating D about the line $x = -2$.

`Hint.` *Change the outer and inner radii of a cylindrical shell.*



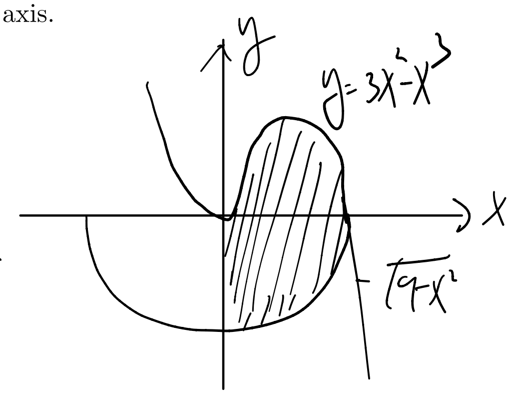
let $F(x) = f(x-2)$

$V = \int_2^5 2\pi x\left(3(x-2)^2 - (x-2)^3\right)dx$

$= 71.3\pi$ #

(c) Let $\tilde{D}$ be the region under $y = f(x) = 3x^2 - x^3$, above $y = g(x) = -\sqrt{9 - x^2}$, from $x = 0$ to $x = 3$. Find the volume of the solid obtained by rotating $\tilde{D}$ about the y axis.

`Hint.` *Change the height of a cylindrical shell.*



$V = \int_0^3 2\pi x(3x^2 - x^3)\,dx + \frac{\cancel{4}^{\,2}}{\cancel{3}}\pi \times \cancel{3^3}^{\,3^2} \times \frac{1}{\cancel{2}}$

$= \frac{243\pi}{10} + 18\pi = 42.3\pi$ #