Calculus 1 Quiz 1 (2.1-3.4)

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10/04/2023

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In the following, you shall NOT use L'Hospital's rule to compute the limit.

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1. (30%) Find the limits or show that it doesn't exist.

+5 (a) (10%) $\lim_{x\to\infty} \frac{[x]}{x}$, where $[x]$ is the greatest integer function.

+6 (b) (10%) $\lim_{x\to 0^-} \frac{x}{\sqrt{1-\cos(3x)}}$.

+10 (c) (10%) $\lim_{x\to -\infty} \tan^{-1}\left(\frac{2x^3 - x^{\frac{1}{3}}}{x^2+1}\right)$.

$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}}$

(a) 1° $x \le [x] < x+1 \Rightarrow \frac{x}{x} \le \frac{[x]}{x} < \frac{x+1}{x}$ +2

2° $\lim_{x\to\infty} \frac{x}{x} \le \lim_{x\to\infty} \frac{[x]}{x} < \lim_{x\to\infty} \frac{x+1}{x}$

By squeeze +1, $\lim_{x\to\infty} \frac{[x]}{x} = 1$ +2 ※

(b) $\lim_{x\to 0^-} \frac{x}{\sqrt{1-\cos 3x}} = \lim_{x\to 0^-} \frac{x}{\sqrt{2}\sin\frac{3x}{2}} = \lim_{x\to 0^-} \frac{1}{\sqrt{2}} \times \frac{\frac{3x}{2}}{\sin\frac{3x}{2}} \times \frac{2}{3} = \cancel{\frac{\sqrt{2}}{3}}$ ※ +6

(c) 1° $\lim_{x\to -\infty} \frac{2x^3 - x^{\frac{1}{3}}}{x^2+1} = \lim_{t\to\infty} \frac{-2t^3 + t^{\frac{1}{3}}}{t^2+1} = -\infty$

$\therefore \lim_{x\to -\infty} \tan^{-1}\left(\frac{2x^3 - x^{\frac{1}{3}}}{x^2+1}\right) = \frac{-\pi}{2}$ ※

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2. (20%)

+10 (a) (10%) Evaluate $\lim_{x \to 1} \frac{x^{100} - 1}{x - 1}$.

+7 (b) (10%) Show that the derivative of even function is odd.

(a) $\frac{x^{100}-1}{x-1} = \frac{(x-1)(1+x+x^2+x^3+\cdots+x^{99})}{(x-1)}$

$\therefore \lim_{x\to1} \frac{x^{100}-1}{x-1} = \lim_{x\to1}(1+x+x^2+x^3+\cdots+x^{99}) = 100$

(b) 1° let a even function $f(x) \Rightarrow f(-x) = f(x)$

2° $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$, $f'(-x) = \lim_{\Delta x\to0} \frac{f(-x-\Delta x) - f(-x)}{-\Delta x} = \lim_{\Delta x\to 0} - \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$= -f'(x)$

3° Therefore, the derivative of even is odd.

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3. (30%)

+15 (a) (15%) Compute $f'(x)$ and $f''(x)$ where $f(x) = e^{e^x}$.

+13 (b) (15%) $\frac{d}{dx}\sin(x + \tan(x + \cos(x)))$.

(a) $f(x) = e^{e^x}$

1° $f'(x) = e^{e^x} \cdot \ln e \times (e^x)' = e^{e^x} \times e^x = e^{e^x+x}$

2° $f''(x) = (e^{e^x} \cdot e^x)' = e^{e^x} \times e^x \times e^x + e^{e^x} \times e^x = e^{e^x+x}(e^x+1)$

(b) $\frac{d}{dx} \sin(x + \tan(x+\cos x))$

$= \cos(x+\tan(x+\cos x)) \times (1+\sec^2(x+\cos x)) \times (1-\sin x)$

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4. (20%+5%) Consider the function

+10

$$f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

+9

+5

Compute $f'(x)$ for both $x \neq 0$ (10%) and $x = 0$ (10%).
Also, is f' continuous (extra credit 5%)?

(a) 1° $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ (compute $f'(x)$ when $x=0$)

$= \lim_{x\to 0} \frac{x^3\sin\left(\frac{1}{x^2}\right) - 0}{x} = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x^2}\right)$

2° $-1 \le \sin\left(\frac{1}{x^2}\right) \le 1$

$\Rightarrow -x^2 \le x^2\sin\left(\frac{1}{x^2}\right) \le x^2 \Rightarrow \lim_{x\to 0}(-x^2) \le \lim_{x\to 0} x^2\sin\left(\frac{1}{x^2}\right) \le \lim_{x\to 0} x^2$

By squeeze, $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x^2}\right) = 0$

$\therefore$ $f'(x)$ when $x=0$ is 0

(b) Compute $f'(x)$ when $x \neq 0$:

$f'(x) = 3x^2\sin\left(\frac{1}{x^2}\right) + x^3\cos\left(\frac{1}{x^2}\right)(-2x^{-3}) = \left(3x^2\sin\left(\frac{1}{x^2}\right) - 2\cos\left(\frac{1}{x^2}\right)\right)$

(c) 1° $\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(3x^2\sin\left(\frac{1}{x^2}\right) - 2\cos\left(\frac{1}{x^2}\right)\right)$

$\because \lim_{x\to 0} 3x^2\sin\left(\frac{1}{x^2}\right) = 0$ but $\lim_{x\to 0} 2\cos\left(\frac{1}{x^2}\right) = \text{DNE} \Rightarrow \lim_{x\to 0} f'(x)$ DNE

$\therefore$ f' is not continuous at $x=0$.