Calculus I (05) Date : 2022/10

Exercise 4 : Linearization, Extreme Values and Mean Value Theorem (3.10~4.2)

1. (a) Use the linearization of $f(x) = x^{-2x}$ at $x = 1$ to approximate $(0.96)^{-1.92}$.
   (b) Use an appropriate linearization to approximate a solution to the equation $x^{-2x} = 0.95$.

2. Find the absolute maximum and absolute minimum values of f on the given interval I.
   (a) $f(x) = \dfrac{e^x}{1+x^2}$, $I = [0, 3]$.
   (b) $f(x) = x^a(1-x)^b$, $I = [0, 1]$, where a and b are positive numbers.

3. Suppose that $0 \le a < b$. Using the Mean Value Theorem, prove that
$$\frac{1}{1+b^2} < \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} < \frac{1}{1+a^2}$$

4. Prove the identity $\sin^{-1}\left(\dfrac{x-1}{x+1}\right) = 2\tan^{-1}\sqrt{x} - \dfrac{\pi}{2}$.

5. (a) Find the linearization of $f(x) = \sin^{-1} x$ at $x = 0.5$. Denote the linearization by $L(x)$.
   (b) Use linear approximation to estimate $\sin^{-1}(0.49)$.
   (c) Let $g(x) = \sin^{-1} x - L(x)$. Use the Mean Value Theorem twice to estimate $|g(0.49) - g(0.5)|$ and get an upper bound for the quantity.

6. (a) Show that the equation $3x + 2\cos x + 5 = 0$ has exactly one real root.
   (b) Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

7. Let f be continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b) = 0$. Show that there exists $c \in (a, b)$ such that $f'(c) = f(c)$.

8. Suppose that $a_0, a_1, \cdots, a_n$ are real numbers satisfying
$$a_0 + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$$
   Show that the equation
$$a_0 + a_1x + \cdots + a_nx^n = 0$$
   has at least one real root in $[0, 1]$.

9. (a) Show that $e^x \ge 1 + x$ for $x \ge 0$.
   (b) Deduce that $e^x \ge 1 + x + \dfrac{1}{2}x^2$ for $x \ge 0$.
   (c) Prove that for $x \ge 0$ and any positive integer n, $e^x \ge 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!}$.

10. (a) Suppose that f and g are differentiable on an open interval containing $[a, b]$ and $f(a) > g(a)$, $f(b) > g(b)$. Show that if the equation $f(x) = g(x)$ has exactly one solution on $[a, b]$, then at the solution $x_0 \in [a, b]$, $f(x)$ and $g(x)$ have the same tangent line.
    (b) For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution?

11. (Optional) Show that if f is a differentiable function that satisfies
$$\frac{f(x+n) - f(x)}{n} = f'(x)$$
    for all real numbers x and all positive integers n, then f is a linear function.

12. (Optional) Suppose that f is a differentiable function. If $f'(a) > 0$ and $f'(b) < 0$, show that there exists $c \in (a, b)$ such that $f'(c) = 0$. (Note that f' may not be continuous.)

Answers:

1. (a) 1.08
   (b) 1.025

2. (a) Max $= f(3) = \frac{e^3}{10}$, Min $= f(0) = 1$
   (b) Max $= f\left(\frac{a}{a+b}\right) = \frac{a^a b^b}{(a+b)^{a+b}}$, Min $= f(0) = f(1) = 0$

3. Hint: Apply the Mean Value Theorem to $f(x) = \tan^{-1} x$ on $[a, b]$

4. Hint: Let $f(x) = \sin^{-1}\left(\frac{x-1}{x+1}\right) - 2\tan^{-1}\sqrt{x}$. Show that $f'(x) = 0$.

5. (a) $L(x) = \frac{\pi}{6} + \frac{2}{\sqrt{3}}(x - \frac{1}{2})$
   (b) $\frac{\pi}{6} - \frac{2}{\sqrt{3}}(0.01)$
   (c) $|g(0.49) - g(0.5)| \leq \frac{4}{10000\sqrt{27}}$

6. (a) Use the Intermediate Value Theorem for the existence of the root and then apply the Mean Value Theorem to prove the uniqueness.
   (b) Hint: proof by contradiction.

7. Hint: Apply the Mean Value Theorem to $g(x) = e^{-x} f(x)$

8. Hint: Apply the Mean Value Theorem to $f(x) = a_0 x + \frac{a_1}{2}x^2 + \cdots + \frac{a_n}{n+1}x^{n+1}$ on $[0, 1]$

9. (a) Show that $f(x) = e^x - (1 + x)$ is increasing for $x \geq 0$ and $f(0) = 0$
   (b) Use (a) to show that $f(x) = e^x - (1 + x + \frac{x^2}{2})$ is increasing for $x \geq 0$.
   (c) Use induction.

10. (a) Hint: Consider $h(x) = f(x) - g(x)$. Show that $h(x) \geq 0$ for all $x \in [a, b]$.
    (b) $k = 2\sqrt{e}$

11. Hint: Show that $f'(x) = 0$

12. Hint: f has maximum value on $[a, b]$ occurring at an interior point c in (a, b)