National Taiwan University - Calculus 3 for Class 01-09

Worksheet 2 : Introduction to Polar Coordinates

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Reference : Stewart §10.3

In this worksheet, we introduce a new coordinate system to describe points on a plane.

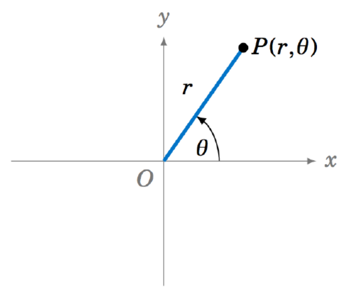
Introduction of Polar Coordinates

(1) **Definition.**

In the polar coordinate system, a point P is described by a pair of numbers (r, θ) for which

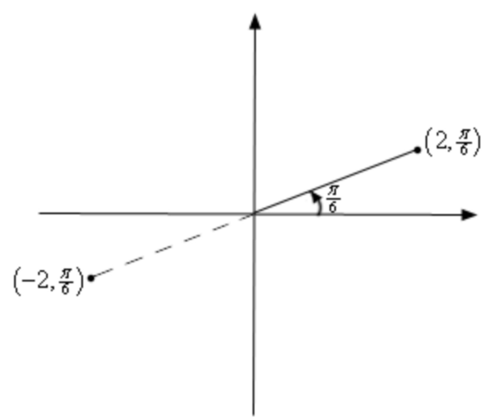
- $r =$ the distance from the origin (the pole) to the point P,
- $\theta =$ the (oriented) angle between segment OP and the positive x-axis.

r and θ are, respectively, called the radical and angular coordinates of the point P.



(2) **Remarks on negetive r or θ.**

- If $\theta < 0$, then the angle is taken *clockwise* from the positive x-axis.
- By convention, if $r < 0$, then the point (r, θ) refers to the point by reflecting $(-r, \theta)$ about the origin.



(3) **Warning.**

For this reason, unlike Cartesian/rectangular coordinates, the polar coordinates of a given point are far from being unique : there are many ways to represent the same point in polar coordinates! For example, in polar coordinates, the pairs

$$(-2, \frac{\pi}{6}),\ (2, \frac{7\pi}{6}),\ (2, -\frac{5\pi}{6})$$

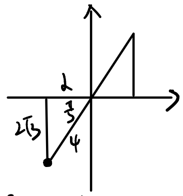
represent the same point!

Exercise 1. Convert each of the following points into the given coordinate system.

(a) Convert $(-4, \frac{\pi}{3})$ into Cartesian coordinates.

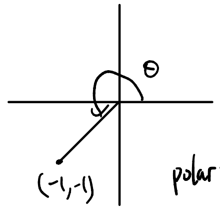
(b) Convert $(-1, -1)$ into polar coordinates with $r > 0$ and $0 \le \theta < 2\pi$.

(a)



$(-4, \frac{\pi}{3}) \Rightarrow (-2, -2\sqrt{3})$

(b)



polar: $(\sqrt{2}, \frac{5\pi}{4})$

In general, we can convert between Cartesian and polar coordinates easily by the following pair of equations (the proof of which will be left as an exercise to readers).

Theorem. Let (x, y) and (r, θ) be, respectively, the Cartesian and polar coordinates of the same point. Then

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \end{cases} \text{ and conversely, } \begin{cases} r^2 &= x^2 + y^2 \\ \tan(\theta) &= \dfrac{y}{x} \text{ if } x \neq 0 \end{cases}.$$

Example. Using the above theorem, we can convert the Cartesian equation of a curve $f(x, y) = 0$ into its 'polar' counterpart $f(r\cos\theta, r\sin\theta) = 0$ (or vice versa). For example, the horizontal line $y = 1$ would have a 'polar equation' $r\cos\theta = 1$. Hence, $r = \sec\theta$ is part of the horizontal line $y = 1$ for which $\cos\theta \neq 0$ (equivalently, $\theta \neq \frac{\pi}{2} \pm k\pi$).

Exercise 2.

(a) Fix a non-zero real number a, sketch the curve $x^2 + y^2 = 2ax$.

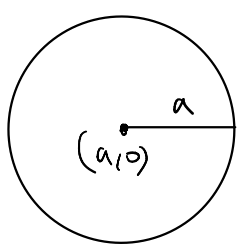
(b) Convert the Cartesian equation of the above curve into a polar equation of the form $r = f(\theta)$.

(a)

$x^2 - 2ax + y^2 = 0$

$x^2 - 2ax + a^2 + y^2 = a^2$

$(x-a)^2 + y^2 = a^2$



(b)

let $x = r\cos\theta$, $y = r\sin\theta$

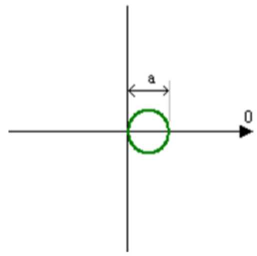
$x^2 + y^2 = 2ax \Rightarrow r^2 = 2ar\cos\theta$

$\Rightarrow r = 2a\cos\theta$

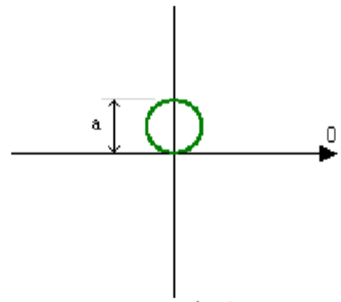
Polar Curves. A polar curve $r = f(\theta)$ describes the dependence of the length of r on the polar angle θ. In this worksheet, we will introduce four kinds of standard curves.

(1) **Circles that pass the origin.**

(a) $r = a\cos\theta$, $(0 \le \theta < \pi)$ (b) $r = a\sin\theta$, $(0 \le \theta < \pi)$.



r = a cos θ



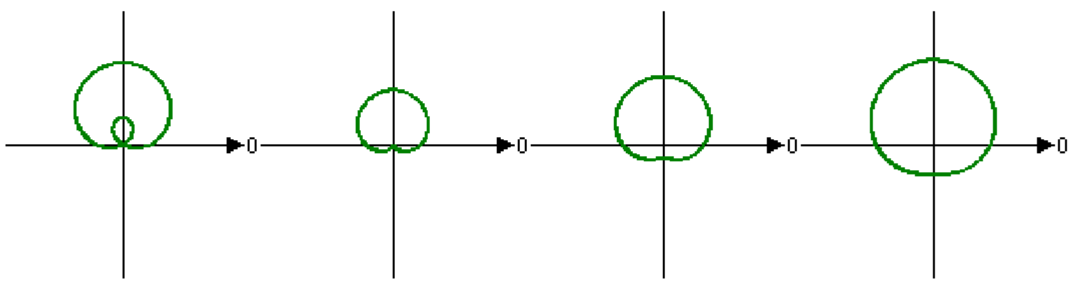
r = a sin θ

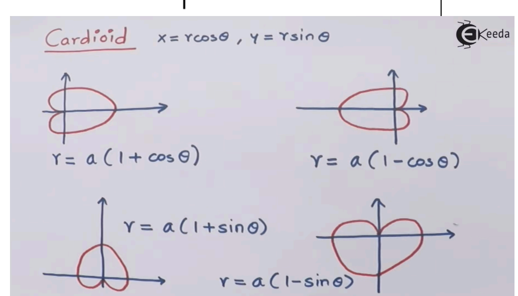
(2) **Cardioids.** A cardioid is given by a polar equation of the form

$$r = a \pm b\sin\theta \text{ or } r = a \pm \cos\theta \text{ with } a, b > 0.$$

$\frac{a}{b} < 1$ $\frac{a}{b} = 1$ $1 < \frac{a}{b} < 2$ $\frac{a}{b} \ge 2$

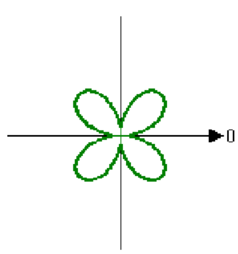
r = a + b sin θ



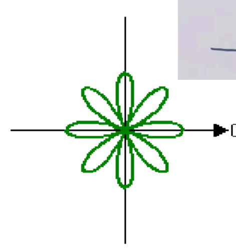


(3) **Rose curves.** A rose curve is given by a polar equation of the form $r = a\sin n\theta$ or $r = a\cos n\theta$ where $a \neq 0$ and n is an integer > 1.

If n is an even integer, then the rose will have 2n petals.

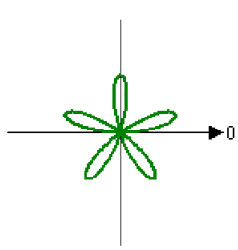


r = a sin 2θ

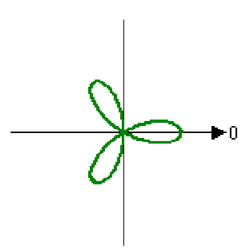


r = a cos 4θ

If n is an odd integer, then the rose will have n petals.



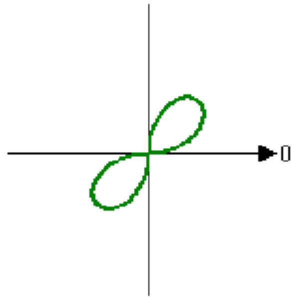
r = a sin 5θ



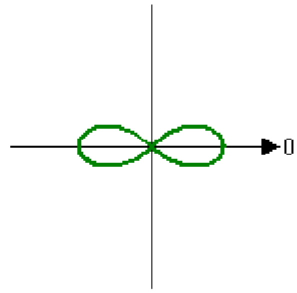
r = a cos 3θ

(4) **Lemniscates** This is given by a polar equation of the form.

$$r^2 = a^2 \sin 2\theta \text{ or } r^2 = a^2 \cos 2\theta \text{ where } a \neq 0.$$



$r^2 = a^2 \sin 2\theta$



$r^2 = a^2 \cos 2\theta$

Exercise 3.

(a) Sketch the following polar curves.

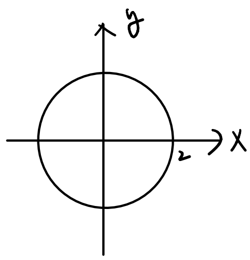
(a) $r = 2$ (b) $\theta = \dfrac{\pi}{4}$ (c) $r = \theta$, $\theta > 0$ (spirals) .

(b) Given a polar curve $r = f(\theta)$. Fixed a number ϕ, describe the curve $r = f(\theta + \phi)$.

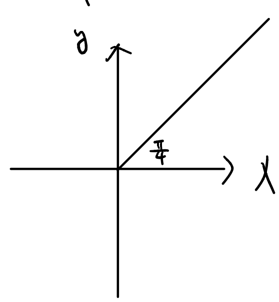
(c) Sketch the polar curve $r = 1 + 2\cos\theta$ and $r = \cos\left(3\theta + \dfrac{\pi}{3}\right)$

(a)

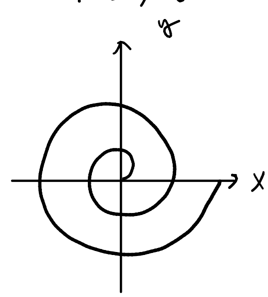
(I) $r = 2$



(II) $\theta = \frac{\pi}{4}$



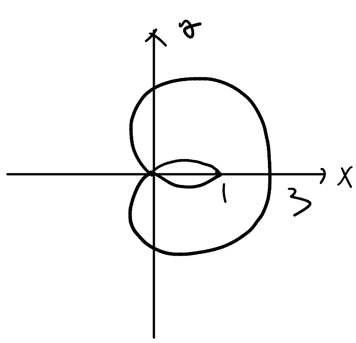
(III) $r = \theta$, $\theta > 0$

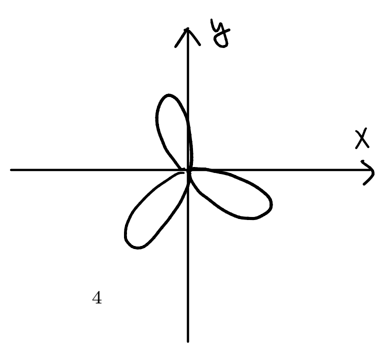


(b) let $\alpha = \theta + \phi \Rightarrow \theta = \alpha - \phi$

$\therefore f(\theta) = f(\alpha - \phi) \Rightarrow$ decrease the angle of every point on the given curve by ϕ.

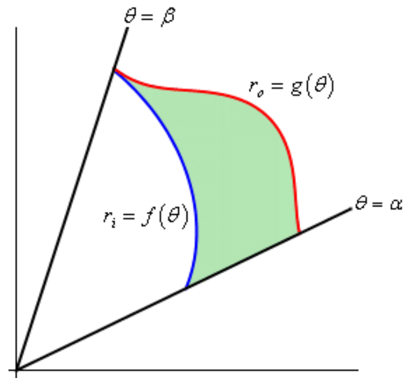
(c) (I) $r = 1 + 2\cos\theta \Rightarrow \frac{1}{2} < 1$ (II) $r = \cos\left(3\theta + \frac{\pi}{3}\right)$





Polar Regions. The region enclosed by the polar curves $r = f(\theta)$, $r = g(\theta)$ and the straight lines $\theta = \alpha$, $\theta = \beta$ can be described, in set notation, as

$$\{(r, \theta) \,:\, f(\theta) \le r \le g(\theta) \text{ and } \alpha \le \theta \le \beta\}.$$



Example. By a 'polar rectangle', we refer to a set of the form

$$D = D(a, b, \alpha, \beta) = \{(r, \theta),\; a < r < b \text{ and } \alpha < \theta < \beta\}.$$

This is given by the following region.

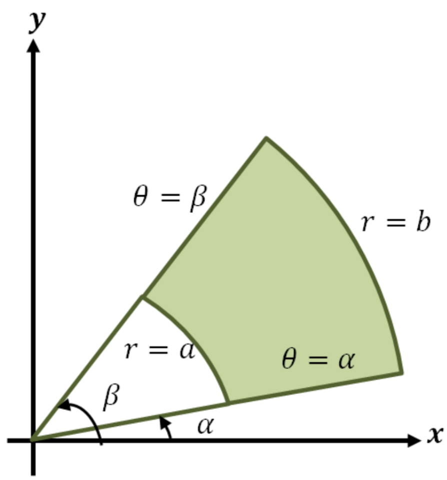
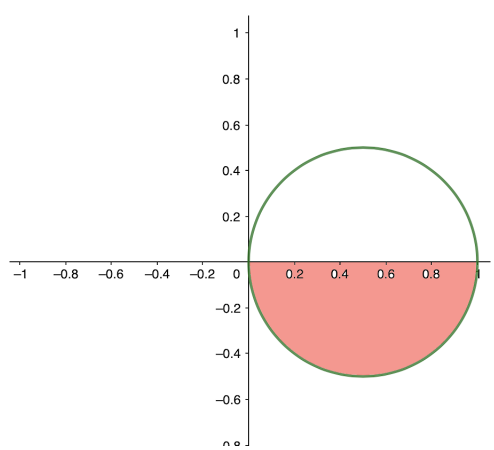
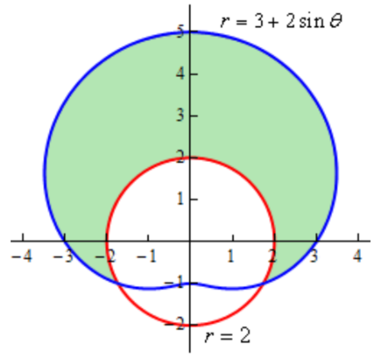


Figure. Polar rectangle $D(a, b, \alpha, \beta)$.

Example. The set $S = \{(r, \theta) \,:\, \cos\theta \le r \le 0 \text{ and } \frac{\pi}{2} \le \theta \le \pi\}$ represents the lower half disk enclosed by the circle $r = \cos\theta$. (See Figure below)



Exercise 4. Describe the following region in polar coordinates as a set.



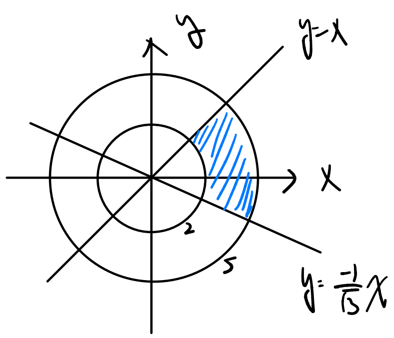
$S = \left\{ (r,\theta) : 2 \le r \le 3 + 2\sin\theta \,,\ \frac{-\pi}{6} \le \theta \le \frac{7\pi}{6} \right\}$

Exercise 5. Sketch the following regions. Hence describe the regions as sets in polar coordinates.

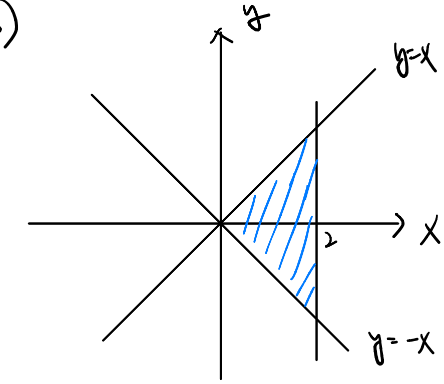
(a) $\{(x, y) \in \mathbb{R}^2 : 2 \le x^2 + y^2 \le 5 \text{ and } -\frac{1}{\sqrt{3}}x \le y \le x\}$ in Cartesian coordinates.

(b) The triangular region $\{(x, y) \in \mathbb{R}^2 : -x \le y \le x,\ 0 \le x \le 2\}$ in Cartesian coordinates.

(a)



(b)



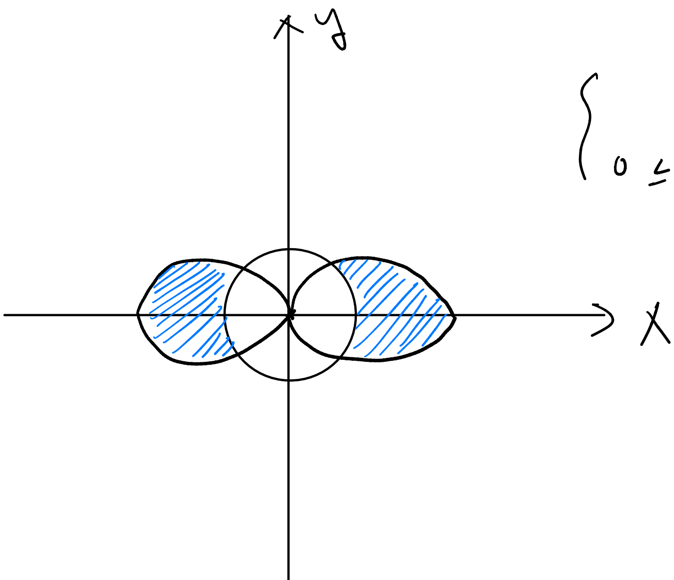
$0 \le r\cos\theta \le 2$

$0 \le r \le \frac{2}{\cos\theta}$

$S = \left\{ (r,\theta) : \sqrt{2} \le r \le \sqrt{5},\ \frac{-\pi}{6} \le \theta \le \frac{\pi}{4} \right\}$

$S = \left\{ (r,\theta) : 0 \le r \le \frac{2}{\cos\theta},\ \frac{-\pi}{4} \le \theta \le \frac{\pi}{4} \right\}$

Exercise 6. Sketch the region described by the set $\{(r, \theta) : 1 \le r \le 2\sqrt{\cos(2\theta)}\}$.



$\begin{cases} r \ge 1 \\ 0 \le r^2 \le 4\cos 2\theta \end{cases} \Rightarrow 1 \le r^2 \le 4\cos 2\theta$