National Taiwan University - Calculus 2 for Class 01-09

Worksheet 2 : Applications of Integration

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Arc Length.
The concepts of the method of Riemann Sum and integration are both directed at measuring total accumulation. The length of a curve can be described as the total length of a combination of short curves. Any *smooth* curve that is short enough would look like a line segment, and the length would be approximately $\sqrt{(dx)^2+(dy)^2}$. Thus we can get an arc length formula if we integrate the expression.

The Arc Length Formula for Graphs: Consider the graph $y=f(x)$. If f' is continuous on the closed interval $[a,b]$, then the length of $y=f(x),\ a\le x\le b$, is

$$L=\int_a^b\sqrt{1+[f'(x)]^2}\,dx=\int_a^b\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx.$$

A further application of the arc length formula lies in the arc length function defined by

$$s(x)=\int_a^x\sqrt{1+[f'(t)]^2}\,dt.$$

The length function is useful in finding out the exact point you would arrive at after travelling for a fixed distance.

Exercise 1. Find the exact length of the curve

$$y=\frac{x^3}{3}+\frac{1}{4x},\ 1\le x\le 2.$$

1° $L=\int_1^2\sqrt{1+(f'(x))^2}\,dx$

$=\int_1^2\sqrt{1+(x^2-\frac14x^{-2})^2}\,dx$

2° $\int\sqrt{1+(x^2-\frac14x^{-2})^2}\,dx$

$=\int\sqrt{1+x^4+\frac{1}{16}x^{-4}-2\cdot x^2\cdot\frac14x^{-2}}\,dx$

$=\int\sqrt{\frac12+(x^2)^2+(\frac{1}{4x^2})^2}\,dx$

$=\int(x^2+\frac{1}{4x^2})\,dx=\frac{x^3}{3}-\frac{1}{4x}+C$ ※

3° $L=\frac{x^3}{3}-\frac{1}{4x}\Big|_1^2$

$=\frac83-\frac18-(\frac13-\frac14)$

$=\frac73+\frac18=\frac{59}{24}$ ※

Exercise 2. Find the exact length of the curve

$$y = e^x,\ 0 \le x \le 1.$$

1° $L = \int_0^1 \sqrt{1+(e^x)^2}\, dx$

2° $\int \sqrt{1+e^{2x}}\, dx$, let $\sqrt{1+e^{2x}} = u \Rightarrow e^{2x} = u^2 - 1 \Rightarrow 2x = \ln(u^2-1) \Rightarrow x = \frac{1}{2}\ln(u^2-1)$

$\Rightarrow dx = \frac{1}{2} \times \frac{2u}{u^2-1} = \frac{u}{u^2-1} du$

$= \int u \times \frac{u}{u^2-1} du = \int \frac{u^2-1+1}{u^2-1} du = \int 1 du + \int \frac{1}{u^2-1} du$

$\Rightarrow \frac{1}{u^2-1} = \frac{-\frac{1}{2}}{u+1} + \frac{\frac{1}{2}}{u-1}$

$= u + \int \left(\frac{-\frac{1}{2}}{u+1} + \frac{\frac{1}{2}}{u-1} \right) du$

$= u + \frac{1}{2}\left(-\ln|u+1| + \ln|u-1| \right)$

3° $L = u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| \Big|_{\sqrt{2}}^{\sqrt{1+e^2}}$

$= \sqrt{1+e^2} + \frac{1}{2}\ln\left|\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right| - \left(\sqrt{2} + \frac{1}{2}\ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right)$

Exercise 3. Consider the curve

$$y = \frac{2}{3}\sqrt{x^3},\ 1 \le x \le 4.$$

(a) Find the exact length.

(b) Find the length function $s(x)$. Does it have an inverse?

(c) Find an interval for x where the exact length of the corresponding curve is equal to 1.

(a) $y = \frac{2}{3}x^{\frac{3}{2}} \Rightarrow y' = x^{\frac{1}{2}}$

$Len = \int_1^4 \sqrt{1+x}\, dx$

$\int \sqrt{1+x}\, d(x) = \int \sqrt{1+x}\, d(1+x)$

$= \frac{2}{3}(1+x)^{\frac{3}{2}} + C$

$Len = \frac{2}{3}(1+x)^{\frac{3}{2}} \Big|_1^4$

$= \frac{2}{3}\left(\sqrt{125}^3 - \sqrt{2}^3\right)$

(b) $s(x) = \frac{2}{3}(1+x)^{\frac{3}{2}} \Big|_1^x$

$y = \frac{2}{3}(1+x)^{\frac{3}{2}}$

$\left(\frac{3y}{2}\right)^{\frac{2}{3}} - 1 = x \Rightarrow y^{-1} = \left(\frac{3x}{2}\right)^{\frac{2}{3}} - 1$

(c) $\int_1^x \sqrt{1+x} = 1$

$\Rightarrow \frac{2}{3}(1+x)^{\frac{3}{2}} \Big|_1^x = 1$

$\Rightarrow \frac{2}{3}\left((1+x)^{\frac{3}{2}} - \sqrt{8}\right) = 1 \Rightarrow x = \left(\frac{3}{2} + \sqrt{8}\right)^{\frac{2}{3}} - 1$

$\therefore \left[1, \left(\frac{3}{2} + \sqrt{8}\right)^{\frac{2}{3}} - 1\right]$

Average Value and Center of Mass.

Finding the average value of a function is a natural topic in many real world scenarios. It is important to remember that a clear domain must be given before an average can be computed.

The average value of the function f over the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\ dx.$$

The center of mass of an object with a given density function is the average position of all point masses. We treat the whole object as a system and we use the *moment* of the system about the origin to find the center of mass. For a pipe-like object that covers the interval $[a, b]$ of a given density $\rho(x)$, the mass equals

$$m = \int_a^b \rho(x)\, dx$$

and the position of its center of mass equals

$$\overline{x} = \frac{1}{m}\int_a^b x\rho(x)\ dx.$$

$$\overline{X} = \frac{\int_a^b x f(x)\, dx}{m}$$

Exercise 4.

(a) Find the average value of $f(x) = 25 - x^2$ on the interval $[0, 2]$.

(b) Find all values of c in the interval $[0, 2]$ such that $f(c)$ is equal to the average value.

(a) 所求 $= \frac{1}{2}\int_0^2 (-x^2+25)\, dx$

$= \frac{1}{2}\left(\left(\frac{-x^3}{3}+25x\right)\Big|_0^2\right) = \frac{1}{2}\left(\frac{-8}{3}+50\right) = \frac{-4}{3}+25$ ##

(b) $25 - x^2 = 25 - \frac{4}{3}$

$\Rightarrow x = \frac{2}{\sqrt{3}}$ or $\cancel{-\frac{2}{\sqrt{3}}}$ (X) $\Rightarrow c = x = \frac{2\sqrt{3}}{3}$ ##

Exercise 5. Find the mass and the center of mass of a steel pipe with density function $\rho(x) = \dfrac{1}{1+x^2}$ over the interval $[0, \sqrt{3}]$.

$$\bar{x} = \frac{1}{m}\int_0^{\sqrt{3}} x\, f(x)\, dx$$

$$m = \int_0^{\sqrt{3}} f(x)\,dx$$

$$1^\circ \int_0^{\sqrt{3}} \frac{1}{x^2+1}\,dx = \tan^{-1}(x)\Big|_0^{\sqrt{3}} = \frac{\pi}{3}$$

$$2^\circ \quad \bar{x} = \frac{3}{\pi}\int_0^{\sqrt{3}} \frac{x}{1+x^2}\,dx$$

$$= \frac{3}{2\pi}\int_1^4 \frac{1}{1+x^2}\,d(1+x^2) = \frac{3}{2\pi}\ln(1+x^2)\Big|_1^4 = \frac{3\ln 2}{\pi}$$

Exercise 6. **(Optional)** Prove the Mean Value Theorem for Integrals : if f is continuous on $[a, b]$, then there exists c in $[a, b]$ such that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c).$$

Interpret this result geometrically.

let $F(x) = \int_a^x f(t)\,dt$ for some $x \in [a,b]$, F is continuous and differentiable on (a,b)

By MVT, there exist c in (a,b) s.t.

$$F'(c) = \frac{F(b)-F(a)}{b-a} = \int_a^b f(t)dt - \int_a^a f(t)dt$$

where $F'(c) = f(c)$

$$\therefore \quad f(c) = F'(c) = \frac{F(b)-F(a)}{b-a} = \int_a^b f(x)\,dx \Rightarrow f(c) = \int_a^b f(x)\,dx$$