11 □ SEQUENCES, SERIES, AND POWER SERIES

11.1 Sequences

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.

(b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.

(c) The terms a_n become large as n becomes large. In fact, we can make a_n as large as we like by taking n sufficiently large.

2. (a) From Definition 1, a convergent sequence is a sequence for which $\lim\limits_{n\to\infty} a_n$ exists. Examples: $\{1/n\}$, $\{1/2^n\}$

(b) A divergent sequence is a sequence for which $\lim\limits_{n\to\infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$

3. $a_n = n^3 - 1$, so the sequence is $\{1^3-1, 2^3-1, 3^3-1, 4^3-1, 5^3-1, \ldots\} = \{0, 7, 26, 63, 124, \ldots\}$.

4. $a_n = \dfrac{1}{3^n+1}$, so the sequence is $\left\{\dfrac{1}{3^1+1}, \dfrac{1}{3^2+1}, \dfrac{1}{3^3+1}, \dfrac{1}{3^4+1}, \dfrac{1}{3^5+1}, \ldots\right\} = \left\{\dfrac{1}{4}, \dfrac{1}{10}, \dfrac{1}{28}, \dfrac{1}{82}, \dfrac{1}{244}, \ldots\right\}$.

5. $\{2^n+n\}_{n=2}^{\infty}$, so the sequence is $\{2^2+2, 2^3+3, 2^4+4, 2^5+5, 2^6+6, \ldots\} = \{6, 11, 20, 37, 70, \ldots\}$.

6. $\left\{\dfrac{n^2-1}{n^2+1}\right\}_{n=3}^{\infty}$, so the sequence is

$$\left\{\frac{3^2-1}{3^2+1}, \frac{4^2-1}{4^2+1}, \frac{5^2-1}{5^2+1}, \frac{6^2-1}{6^2+1}, \frac{7^2-1}{7^2+1}, \ldots\right\} = \left\{\frac{8}{10}, \frac{15}{17}, \frac{24}{26}, \frac{35}{37}, \frac{48}{50}, \ldots\right\}.$$

7. $a_n = \dfrac{(-1)^{n-1}}{n^2}$, so the sequence is

$$\left\{\frac{(-1)^{1-1}}{1^2}, \frac{(-1)^{2-1}}{2^2}, \frac{(-1)^{3-1}}{3^2}, \frac{(-1)^{4-1}}{4^2}, \frac{(-1)^{5-1}}{5^2}, \ldots\right\} = \left\{1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots\right\}.$$

8. $a_n = \dfrac{(-1)^n}{4^n}$, so the sequence is $\left\{\dfrac{(-1)^1}{4^1}, \dfrac{(-1)^2}{4^2}, \dfrac{(-1)^3}{4^3}, \dfrac{(-1)^4}{4^4}, \dfrac{(-1)^5}{4^5}, \ldots\right\} = \left\{-\dfrac{1}{4}, \dfrac{1}{16}, -\dfrac{1}{64}, \dfrac{1}{256}, -\dfrac{1}{1024}, \ldots\right\}$.

9. $a_n = \cos n\pi$, so the sequence is $\{\cos\pi, \cos 2\pi, \cos 3\pi, \cos 4\pi, \cos 5\pi, \ldots\} = \{-1, 1, -1, 1, -1, \ldots\}$.

10. $a_n = 1 + (-1)^n$, so the sequence is $\{1-1, 1+1, 1-1, 1+1, 1-1, \ldots\} = \{0, 2, 0, 2, 0, \ldots\}$.

11. $a_n = \dfrac{(-2)^n}{(n+1)!}$, so the sequence is

$$\left\{\frac{(-2)^1}{2!}, \frac{(-2)^2}{3!}, \frac{(-2)^3}{4!}, \frac{(-2)^4}{5!}, \frac{(-2)^5}{6!}, \ldots\right\} = \left\{-\frac{2}{2}, \frac{4}{6}, -\frac{8}{24}, \frac{16}{120}, -\frac{32}{720}, \ldots\right\} = \left\{-1, \frac{2}{3}, -\frac{1}{3}, \frac{2}{15}, -\frac{2}{45}, \ldots\right\}.$$

12. $a_n = \dfrac{2n+1}{n!+1}$, so the sequence is $\left\{\dfrac{2+1}{1+1}, \dfrac{4+1}{2+1}, \dfrac{6+1}{6+1}, \dfrac{8+1}{24+1}, \dfrac{10+1}{120+1}, \ldots\right\} = \left\{\dfrac{3}{2}, \dfrac{5}{3}, \dfrac{7}{7}, \dfrac{9}{25}, \dfrac{11}{121}, \ldots\right\}$.

13. $a_1 = 1, a_{n+1} = 2a_n + 1$. $a_2 = 2a_1 + 1 = 2 \cdot 1 + 1 = 3$. $a_3 = 2a_2 + 1 = 2 \cdot 3 + 1 = 7$. $a_4 = 2a_3 + 1 = 2 \cdot 7 + 1 = 15$. $a_5 = 2a_4 + 1 = 2 \cdot 15 + 1 = 31$. The sequence is $\{1, 3, 7, 15, 31, \ldots\}$.

14. $a_1 = 6, a_{n+1} = \dfrac{a_n}{n}$. $a_2 = \dfrac{a_1}{1} = \dfrac{6}{1} = 6$. $a_3 = \dfrac{a_2}{2} = \dfrac{6}{2} = 3$. $a_4 = \dfrac{a_3}{3} = \dfrac{3}{3} = 1$. $a_5 = \dfrac{a_4}{4} = \dfrac{1}{4}$.

The sequence is $\left\{6, 6, 3, 1, \frac{1}{4}, \ldots\right\}$.

15. $a_1 = 2, a_{n+1} = \dfrac{a_n}{1 + a_n}$. $a_2 = \dfrac{a_1}{1 + a_1} = \dfrac{2}{1 + 2} = \dfrac{2}{3}$. $a_3 = \dfrac{a_2}{1 + a_2} = \dfrac{2/3}{1 + 2/3} = \dfrac{2}{5}$. $a_4 = \dfrac{a_3}{1 + a_3} = \dfrac{2/5}{1 + 2/5} = \dfrac{2}{7}$.

$a_5 = \dfrac{a_4}{1 + a_4} = \dfrac{2/7}{1 + 2/7} = \dfrac{2}{9}$. The sequence is $\left\{2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \ldots\right\}$.

16. $a_1 = 2, a_2 = 1, a_{n+1} = a_n - a_{n-1}$. Each term is defined in term of the two preceding terms.

$a_3 = a_2 - a_1 = 1 - 2 = -1$. $a_4 = a_3 - a_2 = -1 - 1 = -2$. $a_5 = a_4 - a_3 = -2 - (-1) = -1$.

The sequence is $\{2, 1, -1, -2, -1, \ldots\}$.

17. $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots\right\}$. The denominator is two times the number of the term, n, so $a_n = \dfrac{1}{2n}$.

18. $\left\{4, -1, \frac{1}{4}, -\frac{1}{16}, \frac{1}{64}, \ldots\right\}$. The first term is 4 and each term is $-\frac{1}{4}$ times the preceding one, so $a_n = 4\left(-\frac{1}{4}\right)^{n-1}$.

19. $\left\{-3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \ldots\right\}$. The first term is -3 and each term is $-\frac{2}{3}$ times the preceding one, so $a_n = -3\left(-\frac{2}{3}\right)^{n-1}$.

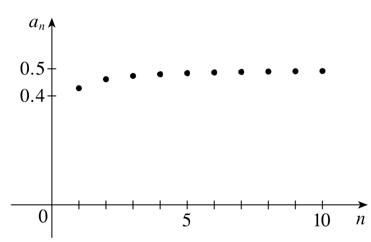
20. $\{5, 8, 11, 14, 17, \ldots\}$. Each term is larger than the preceding term by 3, so $a_n = a_1 + d(n-1) = 5 + 3(n-1) = 3n + 2$.

21. $\left\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \ldots\right\}$. The numerator of the *n*th term is n^2 and its denominator is $n + 1$. Including the alternating signs, we get $a_n = (-1)^{n+1} \dfrac{n^2}{n+1}$.

22. $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$. Two possibilities are $a_n = \sin \dfrac{n\pi}{2}$ and $a_n = \cos \dfrac{(n-1)\pi}{2}$.

23.

n	$a_n = \dfrac{3n}{1+6n}$
1	0.4286
2	0.4615
3	0.4737
4	0.4800
5	0.4839
6	0.4865
7	0.4884
8	0.4898
9	0.4909
10	0.4918

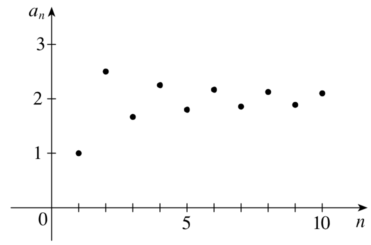


It appears that $\lim\limits_{n\to\infty} a_n = 0.5$.

$$\lim_{n\to\infty} \frac{3n}{1+6n} = \lim_{n\to\infty} \frac{(3n)/n}{(1+6n)/n} = \lim_{n\to\infty} \frac{3}{1/n + 6} = \frac{3}{6} = \frac{1}{2}$$

24.

n	$a_n = 2 + \dfrac{(-1)^n}{n}$
1	1.0000
2	2.5000
3	1.6667
4	2.2500
5	1.8000
6	2.1667
7	1.8571
8	2.1250
9	1.8889
10	2.1000

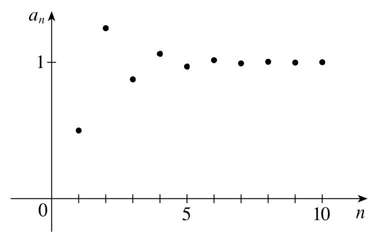


It appears that $\lim\limits_{n\to\infty} a_n = 2$.

$\lim\limits_{n\to\infty}\left(2 + \dfrac{(-1)^n}{n}\right) = \lim\limits_{n\to\infty} 2 + \lim\limits_{n\to\infty}\dfrac{(-1)^n}{n} = 2 + 0 = 2$ since $\lim\limits_{n\to\infty}\dfrac{1}{n} = 0$ and by Theorem 6, $\lim\limits_{n\to\infty}\dfrac{(-1)^n}{n} = 0$.

25.

n	$a_n = 1 + \left(-\frac{1}{2}\right)^n$
1	0.5000
2	1.2500
3	0.8750
4	1.0625
5	0.9688
6	1.0156
7	0.9922
8	1.0039
9	0.9980
10	1.0010

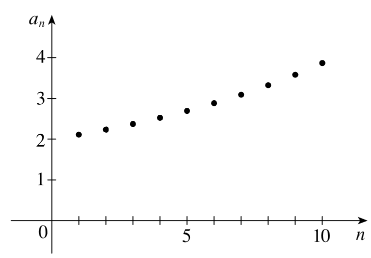


It appears that $\lim\limits_{n\to\infty} a_n = 1$.

$\lim\limits_{n\to\infty}\left(1 + \left(-\frac{1}{2}\right)^n\right) = \lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty}\left(-\frac{1}{2}\right)^n = 1 + 0 = 1$ since $\lim\limits_{n\to\infty}\left(-\frac{1}{2}\right)^n = 0$ by (9).

26.

n	$a_n = 1 + \dfrac{10^n}{9^n}$
1	2.1111
2	2.2346
3	2.3717
4	2.5242
5	2.6935
6	2.8817
7	3.0908
8	3.3231
9	3.5812
10	3.8680



It appears that the sequence does not have a limit.

$\lim\limits_{n\to\infty}\dfrac{10^n}{9^n} = \lim\limits_{n\to\infty}\left(\dfrac{10}{9}\right)^n$, which diverges by (9) since $\frac{10}{9} > 1$.

27. $a_n = \dfrac{5}{n+2} = \dfrac{5/n}{(n+2)/n} = \dfrac{5/n}{1+2/n}$, so $a_n \to \dfrac{0}{1+0} = 0$ as $n\to\infty$. Converges

28. $a_n = 5\sqrt{n+2}$, so $a_n \to \infty$ as $n\to\infty$ since $\lim\limits_{n\to\infty}\sqrt{n+2} = \infty$. Diverges

29. $a_n = \dfrac{4n^2 - 3n}{2n^2 + 1} = \dfrac{(4n^2 - 3n)/n^2}{(2n^2 + 1)/n^2} = \dfrac{4 - 3/n}{2 + 1/n^2}$, so $a_n \to \dfrac{4 - 0}{2 + 0} = 2$ as $n \to \infty$. Converges

30. $a_n = \dfrac{4n^2 - 3n}{2n + 1} = \dfrac{(4n^2 - 3n)/n}{(2n + 1)/n} = \dfrac{4n - 3}{2 + 1/n}$, so $a_n \to \infty$ as $n \to \infty$ since $\lim\limits_{n\to\infty}(4n - 3) = \infty$ and $\lim\limits_{n\to\infty}\left(2 + \dfrac{1}{n}\right) = 2$.

Diverges

31. $a_n = \dfrac{n^4}{n^3 - 2n} = \dfrac{n^4/n^3}{(n^3 - 2n)/n^3} = \dfrac{n}{1 - 2/n^2}$, so $a_n \to \infty$ as $n \to \infty$ since $\lim\limits_{n\to\infty} n = \infty$ and

$\lim\limits_{n\to\infty}\left(1 - \dfrac{2}{n^2}\right) = 1 - 0 = 1$. Diverges

32. $a_n = 2 + (0.86)^n \to 2 + 0 = 2$ as $n \to \infty$ since $\lim\limits_{n\to\infty}(0.86)^n = 0$ by (9) with $r = 0.86$. Converges

33. $a_n = 3^n 7^{-n} = \dfrac{3^n}{7^n} = \left(\dfrac{3}{7}\right)^n$, so $\lim\limits_{n\to\infty} a_n = 0$ by (9) with $r = \dfrac{3}{7}$. Converges

34. $a_n = \dfrac{3\sqrt{n}}{\sqrt{n} + 2} = \dfrac{3\sqrt{n}/\sqrt{n}}{(\sqrt{n} + 2)/\sqrt{n}} = \dfrac{3}{1 + 2/\sqrt{n}} \to \dfrac{3}{1 + 0} = 3$ as $n \to \infty$. Converges

35. Because the natural exponential function is continuous at 0, Theorem 7 enables us to write

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} e^{-1/\sqrt{n}} = e^{\lim_{n\to\infty}(-1/\sqrt{n})} = e^0 = 1$. Converges

36. $a_n = \dfrac{4^n}{1 + 9^n} = \dfrac{4^n/9^n}{(1 + 9^n)/9^n} = \dfrac{(4/9)^n}{(1/9)^n + 1} \to \dfrac{0}{0 + 1} = 0$ as $n \to \infty$ since $\lim\limits_{n\to\infty}\left(\dfrac{4}{9}\right)^n = 0$ and

$\lim\limits_{n\to\infty}\left(\dfrac{1}{9}\right)^n = 0$ by (9). Converges

37. $a_n = \sqrt{\dfrac{1 + 4n^2}{1 + n^2}} = \sqrt{\dfrac{(1 + 4n^2)/n^2}{(1 + n^2)/n^2}} = \sqrt{\dfrac{(1/n^2) + 4}{(1/n^2) + 1}} \to \sqrt{4} = 2$ as $n \to \infty$ since $\lim\limits_{n\to\infty}(1/n^2) = 0$. Converges

38. $a_n = \cos\left(\dfrac{n\pi}{n + 1}\right) = \cos\left(\dfrac{n\pi/n}{(n + 1)/n}\right) = \cos\left(\dfrac{\pi}{1 + 1/n}\right)$, so $a_n \to \cos\pi = -1$ as $n \to \infty$ since $\lim\limits_{n\to\infty} 1/n = 0$.

Converges

39. $a_n = \dfrac{n^2}{\sqrt{n^3 + 4n}} = \dfrac{n^2/\sqrt{n^3}}{\sqrt{n^3 + 4n}/\sqrt{n^3}} = \dfrac{\sqrt{n}}{\sqrt{1 + 4/n^2}}$, so $a_n \to \infty$ as $n \to \infty$ since $\lim\limits_{n\to\infty}\sqrt{n} = \infty$ and

$\lim\limits_{n\to\infty}\sqrt{1 + 4/n^2} = 1$. Diverges

40. If $b_n = \dfrac{2n}{n + 2}$, then $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\dfrac{(2n)/n}{(n + 2)/n} = \lim\limits_{n\to\infty}\dfrac{2}{1 + 2/n} = \dfrac{2}{1} = 2$. Since the natural exponential function is

continuous at 2, by Theorem 7, $\lim\limits_{n\to\infty} e^{2n/(n+2)} = e^{\lim_{n\to\infty} b_n} = e^2$. Converges

41. $\lim\limits_{n\to\infty}|a_n| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)^n}{2\sqrt{n}}\right| = \dfrac{1}{2}\lim\limits_{n\to\infty}\dfrac{1}{n^{1/2}} = \dfrac{1}{2}(0) = 0$, so $\lim\limits_{n\to\infty} a_n = 0$ by (6). Converges

42. $\lim\limits_{n\to\infty} \dfrac{n}{n+\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{n/n}{(n+\sqrt{n})/n} = \lim\limits_{n\to\infty} \dfrac{1}{1+1/\sqrt{n}} = \dfrac{1}{1+0} = 1.$ Thus, $a_n = \dfrac{(-1)^{n+1}n}{n+\sqrt{n}}$ has odd-numbered terms that approach 1 and even-numbered terms that approach -1 as $n \to \infty$, and hence, the sequence $\{a_n\}$ is divergent.

43. $a_n = \dfrac{(2n-1)!}{(2n+1)!} = \dfrac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \dfrac{1}{(2n+1)(2n)} \to 0$ as $n \to \infty$. Converges

44. $a_n = \dfrac{\ln n}{\ln(2n)} = \dfrac{\ln n}{\ln 2 + \ln n} = \dfrac{(\ln n)/\ln n}{(\ln 2 + \ln n)/\ln n} = \dfrac{1}{\frac{\ln 2}{\ln n} + 1} \to \dfrac{1}{0+1} = 1$ as $n \to \infty$. Converges

45. $a_n = \sin n$. This sequence diverges since the terms don't approach any particular real number as $n \to \infty$. The terms take on values between -1 and 1. Diverges

46. $a_n = \dfrac{\tan^{-1} n}{n}$. $\lim\limits_{n\to\infty} \tan^{-1} n = \lim\limits_{x\to\infty} \tan^{-1} x = \dfrac{\pi}{2}$ by (4), so $\lim\limits_{n\to\infty} a_n = 0$. Converges

47. $a_n = n^2 e^{-n} = \dfrac{n^2}{e^n}$. Since $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{2x}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{2}{e^x} = 0$, it follows from Theorem 4 that $\lim\limits_{n\to\infty} a_n = 0$. Converges

48. $a_n = \ln(n+1) - \ln n = \ln\left(\dfrac{n+1}{n}\right) = \ln\left(1 + \dfrac{1}{n}\right) \to \ln(1) = 0$ as $n \to \infty$ because ln is continuous. Converges

49. $0 \le \dfrac{\cos^2 n}{2^n} \le \dfrac{1}{2^n}$ [since $0 \le \cos^2 n \le 1$], so since $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$, $\left\{\dfrac{\cos^2 n}{2^n}\right\}$ converges to 0 by the Squeeze Theorem.

50. $a_n = \sqrt[n]{2^{1+3n}} = (2^{1+3n})^{1/n} = (2^1 2^{3n})^{1/n} = 2^{1/n} 2^3 = 8 \cdot 2^{1/n}$, so

$\lim\limits_{n\to\infty} a_n = 8 \lim\limits_{n\to\infty} 2^{1/n} = 8 \cdot 2^{\lim_{n\to\infty}(1/n)} = 8 \cdot 2^0 = 8$ by Theorem 7, since the function $f(x) = 2^x$ is continuous at 0. Converges

51. $a_n = n\sin(1/n) = \dfrac{\sin(1/n)}{1/n}$. Since $\lim\limits_{x\to\infty} \dfrac{\sin(1/x)}{1/x} = \lim\limits_{t\to 0^+} \dfrac{\sin t}{t}$ [where $t = 1/x$] $= 1$, it follows from Theorem 4 that $\{a_n\}$ converges to 1.

52. $a_n = 2^{-n}\cos n\pi$. $0 \le \left|\dfrac{\cos n\pi}{2^n}\right| \le \dfrac{1}{2^n} = \left(\dfrac{1}{2}\right)^n$, so $\lim\limits_{n\to\infty} |a_n| = 0$ by (9), and $\lim\limits_{n\to\infty} a_n = 0$ by (6). Converges

53. $y = \left(1 + \dfrac{2}{x}\right)^x \Rightarrow \ln y = x \ln\left(1 + \dfrac{2}{x}\right)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+2/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\left(\dfrac{1}{1+2/x}\right)\left(-\dfrac{2}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{2}{1+2/x} = 2 \Rightarrow$$

$\lim\limits_{x\to\infty} \left(1 + \dfrac{2}{x}\right)^x = \lim\limits_{x\to\infty} e^{\ln y} = e^2$, so by Theorem 4, $\lim\limits_{n\to\infty} \left(1 + \dfrac{2}{n}\right)^n = e^2$. Converges

54. $y = x^{1/x} \Rightarrow \ln y = \dfrac{1}{x}\ln x$, so $\lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = \lim\limits_{x\to\infty} \dfrac{1}{x} = 0 \Rightarrow$

$\lim\limits_{x\to\infty} x^{1/x} = \lim\limits_{x\to\infty} e^{\ln y} = e^0 = 1$, so by Theorem 4, $\lim\limits_{n\to\infty} n^{1/n} = 1$. Converges

55. $a_n = \ln(2n^2+1) - \ln(n^2+1) = \ln\left(\dfrac{2n^2+1}{n^2+1}\right) = \ln\left(\dfrac{2+1/n^2}{1+1/n^2}\right) \to \ln 2$ as $n \to \infty$. Converges

56. $\displaystyle\lim_{x\to\infty} \frac{(\ln x)^2}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{2(\ln x)(1/x)}{1} = 2\lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{H}}{=} 2\lim_{x\to\infty} \frac{1/x}{1} = 0$, so by Theorem 4, $\displaystyle\lim_{n\to\infty} \frac{(\ln n)^2}{n} = 0$. Converges

57. $a_n = \arctan(\ln n)$. Let $f(x) = \arctan(\ln x)$. Then $\displaystyle\lim_{x\to\infty} f(x) = \frac{\pi}{2}$ since $\ln x \to \infty$ as $x \to \infty$ and arctan is continuous.

Thus, $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = \frac{\pi}{2}$. Converges

58. $a_n = n - \sqrt{n+1}\,\sqrt{n+3} = n - \sqrt{n^2+4n+3} = \dfrac{n-\sqrt{n^2+4n+3}}{1} \cdot \dfrac{n+\sqrt{n^2+4n+3}}{n+\sqrt{n^2+4n+3}}$

$$= \frac{n^2-(n^2+4n+3)}{n+\sqrt{n^2+4n+3}} = \frac{-4n-3}{n+\sqrt{n^2+4n+3}} = \frac{(-4n-3)/n}{\left(n+\sqrt{n^2+4n+3}\right)/n} = \frac{-4-3/n}{1+\sqrt{1+4/n+3/n^2}},$$

so $\displaystyle\lim_{n\to\infty} a_n = \frac{-4-0}{1+\sqrt{1+0+0}} = \frac{-4}{2} = -2$. Converges

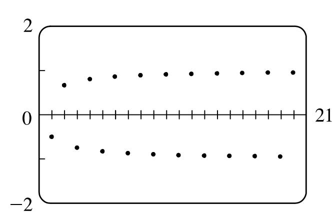
59. $\{0,1,0,0,1,0,0,0,1,\ldots\}$ diverges since the sequence takes on only two values, 0 and 1, and never stays arbitrarily close to either value (or any other value) for n sufficiently large.

60. $\left\{\frac{1}{1},\frac{1}{3},\frac{1}{2},\frac{1}{4},\frac{1}{3},\frac{1}{5},\frac{1}{4},\frac{1}{6},\ldots\right\}$. $a_{2n-1} = \dfrac{1}{n}$ and $a_{2n} = \dfrac{1}{n+2}$ for all positive integers n. $\displaystyle\lim_{n\to\infty} a_n = 0$ since $\displaystyle\lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} \frac{1}{n} = 0$ and $\displaystyle\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{1}{n+2} = 0$. For n sufficiently large, a_n can be made as close to 0 as we like. Converges

61. $a_n = \dfrac{n!}{2^n} = \dfrac{1}{2}\cdot\dfrac{2}{2}\cdot\dfrac{3}{2}\cdot\cdots\cdot\dfrac{(n-1)}{2}\cdot\dfrac{n}{2} \geq \dfrac{1}{2}\cdot\dfrac{n}{2}$ [for $n>1$] $= \dfrac{n}{4} \to \infty$ as $n \to \infty$, so $\{a_n\}$ diverges.

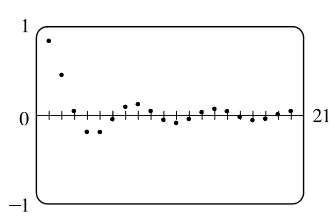
62. $0 < |a_n| = \dfrac{3^n}{n!} = \dfrac{3}{1}\cdot\dfrac{3}{2}\cdot\dfrac{3}{3}\cdot\cdots\cdot\dfrac{3}{(n-1)}\cdot\dfrac{3}{n} \leq \dfrac{3}{1}\cdot\dfrac{3}{2}\cdot\dfrac{3}{n}$ [for $n>2$] $= \dfrac{27}{2n} \to 0$ as $n\to\infty$, so by the Squeeze Theorem and Theorem 6, $\{(-3)^n/n!\}$ converges to 0.

63.



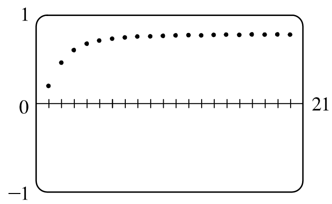
From the graph, it appears that the sequence $\{a_n\} = \left\{(-1)^n \dfrac{n}{n+1}\right\}$ is divergent, since it oscillates between 1 and -1 (approximately). To prove this, suppose that $\{a_n\}$ converges to L. If $b_n = \dfrac{n}{n+1}$, then $\{b_n\}$ converges to 1, and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{1} = L$. But $\dfrac{a_n}{b_n} = (-1)^n$, so $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$ does not exist. This contradiction shows that $\{a_n\}$ diverges.

64.



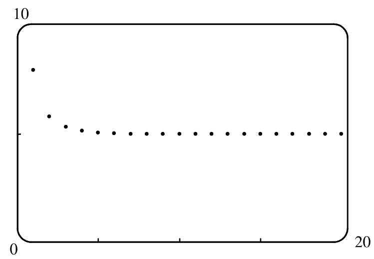
From the graph, it appears that the sequence converges to 0.

$|a_n| = \left|\dfrac{\sin n}{n}\right| = \dfrac{|\sin n|}{|n|} \leq \dfrac{1}{n}$, so $\displaystyle\lim_{n\to\infty} |a_n| = 0$. By (6), it follows that $\displaystyle\lim_{n\to\infty} a_n = 0$.

65. 

From the graph, it appears that the sequence converges to a number between 0.7 and 0.8.

$a_n = \arctan\left(\dfrac{n^2}{n^2+4}\right) = \arctan\left(\dfrac{n^2/n^2}{(n^2+4)/n^2}\right) = \arctan\left(\dfrac{1}{1+4/n^2}\right) \to$ $\arctan 1 = \dfrac{\pi}{4}$ $[\approx 0.785]$ as $n \to \infty$.

66. 

From the graph, it appears that the sequence converges to 5.

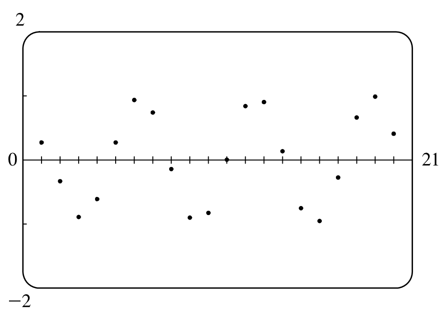
$$5 = \sqrt[n]{5^n} \le \sqrt[n]{3^n + 5^n} \le \sqrt[n]{5^n + 5^n} = \sqrt[n]{2}\,\sqrt[n]{5^n}$$
$$= \sqrt[n]{2} \cdot 5 \to 5 \text{ as } n \to \infty \quad \left[\lim_{n\to\infty} 2^{1/n} = 2^0 = 1\right]$$

Hence, $a_n \to 5$ by the Squeeze Theorem.

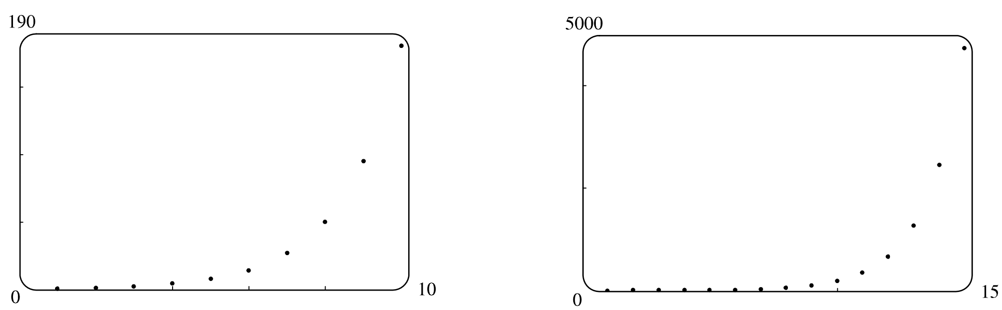
Alternate solution: Let $y = (3^x + 5^x)^{1/x}$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(3^x + 5^x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{3^x \ln 3 + 5^x \ln 5}{3^x + 5^x} = \lim_{x\to\infty} \frac{\left(\frac{3}{5}\right)^x \ln 3 + \ln 5}{\left(\frac{3}{5}\right)^x + 1} = \ln 5,$$

so $\lim\limits_{x\to\infty} y = e^{\ln 5} = 5$, and so $\left\{\sqrt[n]{3^n + 5^n}\right\}$ converges to 5.

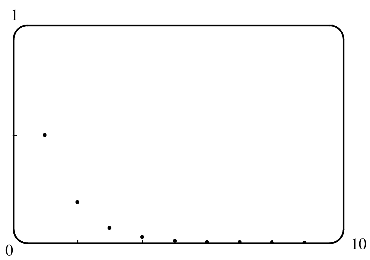
67. 

From the graph, it appears that the sequence $\{a_n\} = \left\{\dfrac{n^2 \cos n}{1 + n^2}\right\}$ is divergent, since it oscillates between 1 and -1 (approximately). To prove this, suppose that $\{a_n\}$ converges to L. If $b_n = \dfrac{n^2}{1+n^2}$, then $\{b_n\}$ converges to 1, and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{1} = L$. But $\dfrac{a_n}{b_n} = \cos n$, so $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$ does not exist. This contradiction shows that $\{a_n\}$ diverges.

68. 

From the graphs, it seems that the sequence diverges. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$. We first prove by induction that $a_n \ge \left(\dfrac{3}{2}\right)^{n-1}$ for all n. This is clearly true for $n = 1$, so let $P(n)$ be the statement that the above is true for n. We must show it is then true for $n+1$. $a_{n+1} = a_n \cdot \dfrac{2n+1}{n+1} \ge \left(\dfrac{3}{2}\right)^{n-1} \cdot \dfrac{2n+1}{n+1}$ (induction hypothesis). But $\dfrac{2n+1}{n+1} \ge \dfrac{3}{2}$

[since $2(2n+1) \geq 3(n+1) \;\Leftrightarrow\; 4n+2 \geq 3n+3 \;\Leftrightarrow\; n \geq 1$], and so we get that $a_{n+1} \geq \left(\frac{3}{2}\right)^{n-1} \cdot \frac{3}{2} = \left(\frac{3}{2}\right)^n$ which is $P(n+1)$. Thus, we have proved our first assertion, so since $\left\{\left(\frac{3}{2}\right)^{n-1}\right\}$ diverges [by (9)], so does the given sequence $\{a_n\}$.

69. 

From the graph, it appears that the sequence approaches 0.

$$0 < a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n} = \frac{1}{2n} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdot \cdots \cdot \frac{2n-1}{2n}$$

$$\leq \frac{1}{2n} \cdot (1) \cdot (1) \cdot \cdots \cdot (1) = \frac{1}{2n} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\left\{\dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n}\right\}$ converges to 0.

70. (a) $a_1 = 1$, $a_{n+1} = 4 - a_n$ for $n \geq 1$. $a_1 = 1$, $a_2 = 4 - a_1 = 4 - 1 = 3$, $a_3 = 4 - a_2 = 4 - 3 = 1$, $a_4 = 4 - a_3 = 4 - 1 = 3$, $a_5 = 4 - a_4 = 4 - 3 = 1$. Since the terms of the sequence alternate between 1 and 3, the sequence is divergent.

(b) $a_1 = 2$, $a_2 = 4 - a_1 = 4 - 2 = 2$, $a_3 = 4 - a_2 = 4 - 2 = 2$. Since all of the terms are 2, $\lim\limits_{n\to\infty} a_n = 2$ and hence, the sequence is convergent.

71. (a) $a_n = 1000(1.06)^n \;\Rightarrow\; a_1 = 1060$, $a_2 = 1123.60$, $a_3 = 1191.02$, $a_4 = 1262.48$, and $a_5 = 1338.23$.

(b) $\lim\limits_{n\to\infty} a_n = 1000 \lim\limits_{n\to\infty} (1.06)^n$, so the sequence diverges by (9) with $r = 1.06 > 1$.

72. (a) Substitute 1 to 6 for n in $I_n = 100\left(\dfrac{1.0025^n - 1}{0.0025} - n\right)$ to get $I_1 = \$0$, $I_2 = \$0.25$, $I_3 = \$0.75$, $I_4 = \$1.50$, $I_5 = \$2.51$, and $I_6 = \$3.76$.

(b) For two years, use $2 \cdot 12 = 24$ for n to get \$70.28.

73. (a) We are given that the initial population is 5000, so $P_0 = 5000$. The number of catfish increases by 8% per month and is decreased by 300 per month, so $P_1 = P_0 + 8\%P_0 - 300 = 1.08P_0 - 300$, $P_2 = 1.08P_1 - 300$, and so on. Thus, $P_n = 1.08P_{n-1} - 300$.

(b) Using the recursive formula with $P_0 = 5000$, we get $P_1 = 5100$, $P_2 = 5208$, $P_3 = 5325$ (rounding any portion of a catfish), $P_4 = 5451$, $P_5 = 5587$, and $P_6 = 5734$, which is the number of catfish in the pond after six months.

74. $a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$ When $a_1 = 11$, the first 40 terms are 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. When $a_1 = 25$, the first 40 terms are 25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4.
The famous Collatz conjecture is that this sequence always reaches 1, regardless of the starting point a_1.

75. If $|r| \geq 1$, then $\{r^n\}$ diverges by (9), so $\{nr^n\}$ diverges also, since $|nr^n| = n\,|r^n| \geq |r^n|$. If $|r| < 1$ then

$$\lim_{x\to\infty} xr^x = \lim_{x\to\infty} \frac{x}{r^{-x}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1}{(-\ln r)\,r^{-x}} = \lim_{x\to\infty} \frac{r^x}{-\ln r} = 0,$$ so $\lim\limits_{n\to\infty} nr^n = 0$, and hence $\{nr^n\}$ converges

whenever $|r| < 1$.

76. (a) Let $\lim\limits_{n\to\infty} a_n = L$. By Definition 2, this means that for every $\varepsilon > 0$ there is an integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$. Thus, $|a_{n+1} - L| < \varepsilon$ whenever $n + 1 > N \iff n > N - 1$. It follows that $\lim\limits_{n\to\infty} a_{n+1} = L$ and so $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$.

(b) If $L = \lim\limits_{n\to\infty} a_n$ then $\lim\limits_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 1/\,(1 + L) \Rightarrow L^2 + L - 1 = 0 \Rightarrow L = \frac{-1+\sqrt{5}}{2}$ (since L has to be nonnegative if it exists).

77. Since $\{a_n\}$ is a decreasing sequence, $a_n > a_{n+1}$ for all $n \geq 1$. Because all of its terms lie between 5 and 8, $\{a_n\}$ is a bounded sequence. By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent; that is, $\{a_n\}$ has a limit L. L must be less than 8 since $\{a_n\}$ is decreasing, so $5 \leq L < 8$.

78. Since $\{a_n\} = \{\cos n\} \approx \{0.54, -0.42, -0.99, -0.65, 0.28, \ldots\}$, the sequence is not monotonic. The sequence is bounded since $-1 \leq \cos n \leq 1$ for all n.

79. $a_n = \dfrac{1}{2n+3}$ is decreasing since $a_{n+1} = \dfrac{1}{2(n+1)+3} = \dfrac{1}{2n+5} < \dfrac{1}{2n+3} = a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$. Note that $a_1 = \frac{1}{5}$.

80. $a_n > a_{n+1} \iff \dfrac{1-n}{2+n} > \dfrac{1-(n+1)}{2+(n+1)} \iff \dfrac{1-n}{2+n} > \dfrac{-n}{n+3} \iff -n^2 - 2n + 3 > -n^2 - 2n \iff 3 > 0$, which is true for all $n \geq 1$, so $\{a_n\}$ is decreasing. Since $a_1 = 0$ and $\lim\limits_{n\to\infty} \dfrac{1-n}{2+n} = \lim\limits_{n\to\infty} \dfrac{1/n - 1}{2/n + 1} = -1$, the sequence is bounded $(-1 < a_n \leq 0)$.

81. The terms of $a_n = n(-1)^n$ alternate in sign, so the sequence is not monotonic. The first five terms are -1, 2, -3, 4, and -5. Since $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} n = \infty$, the sequence is not bounded.

82. Since $\{a_n\} = \left\{2 + \dfrac{(-1)^n}{n}\right\} = \left\{1, 2\frac{1}{2}, 1\frac{2}{3}, \ldots\right\}$, the sequence is not monotonic. The sequence is bounded since $1 \leq a_n \leq \frac{5}{2}$ for all n.

83. $a_n = 3 - 2ne^{-n}$. Let $f(x) = 3 - 2xe^{-x}$. Then $f'(x) = 0 - 2[x(-e^{-x}) + e^{-x}] = 2e^{-x}(x - 1)$, which is positive for $x > 1$, so f is increasing on $(1, \infty)$. It follows that the sequence $\{a_n\} = \{f(n)\}$ is increasing. The sequence is bounded below by $a_1 = 3 - 2e^{-1} \approx 2.26$ and above by 3, so the sequence is bounded.

84. $a_n = n^3 - 3n + 3$. Let $f(x) = x^3 - 3x + 3$. Then $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$, which is positive for $x > 1$, so f is increasing on $(1, \infty)$. It follows that the sequence $\{a_n\} = \{f(n)\}$ is increasing. The sequence is bounded below by $a_1 = 1$, but is not bounded above, so it is not bounded.

85. For $\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\right\}$, $a_1 = 2^{1/2}$, $a_2 = 2^{3/4}$, $a_3 = 2^{7/8}$, ..., so $a_n = 2^{(2^n-1)/2^n} = 2^{1-(1/2^n)}$.

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 2^{1-(1/2^n)} = 2^1 = 2$.

Alternate solution: Let $L = \lim\limits_{n\to\infty} a_n$. (We could show the limit exists by showing that $\{a_n\}$ is bounded and increasing.)

Then L must satisfy $L = \sqrt{2 \cdot L} \quad \Rightarrow \quad L^2 = 2L \quad \Rightarrow \quad L(L-2) = 0$. $L \neq 0$ since the sequence increases, so $L = 2$.

86. (a) Let P_n be the statement that $a_{n+1} \geq a_n$ and $a_n \leq 3$. P_1 is obviously true. We will assume that P_n is true and then show that as a consequence P_{n+1} must also be true. $a_{n+2} \geq a_{n+1} \quad \Leftrightarrow \quad \sqrt{2 + a_{n+1}} \geq \sqrt{2 + a_n} \quad \Leftrightarrow$ $2 + a_{n+1} \geq 2 + a_n \quad \Leftrightarrow \quad a_{n+1} \geq a_n$, which is the induction hypothesis. $a_{n+1} \leq 3 \quad \Leftrightarrow \quad \sqrt{2 + a_n} \leq 3 \quad \Leftrightarrow$ $2 + a_n \leq 9 \quad \Leftrightarrow \quad a_n \leq 7$, which is certainly true because we are assuming that $a_n \leq 3$. So P_n is true for all n, and so $a_1 \leq a_n \leq 3$ (showing that the sequence is bounded), and hence by the Monotonic Sequence Theorem, $\lim\limits_{n\to\infty} a_n$ exists.

(b) If $L = \lim\limits_{n\to\infty} a_n$, then $\lim\limits_{n\to\infty} a_{n+1} = L$ also, so $L = \sqrt{2 + L} \quad \Rightarrow \quad L^2 = 2 + L \quad \Leftrightarrow \quad L^2 - L - 2 = 0 \quad \Leftrightarrow$ $(L+1)(L-2) = 0 \quad \Leftrightarrow \quad L = 2$ [since L can't be negative].

87. $a_1 = 1$, $a_{n+1} = 3 - \dfrac{1}{a_n}$. We show by induction that $\{a_n\}$ is increasing and bounded above by 3. Let P_n be the proposition that $a_{n+1} > a_n$ and $0 < a_n < 3$. Clearly P_1 is true. Assume that P_n is true. Then $a_{n+1} > a_n \quad \Rightarrow \quad \dfrac{1}{a_{n+1}} < \dfrac{1}{a_n} \quad \Rightarrow$ $-\dfrac{1}{a_{n+1}} > -\dfrac{1}{a_n}$. Now $a_{n+2} = 3 - \dfrac{1}{a_{n+1}} > 3 - \dfrac{1}{a_n} = a_{n+1} \quad \Leftrightarrow \quad P_{n+1}$. This proves that $\{a_n\}$ is increasing and bounded above by 3, so $1 = a_1 < a_n < 3$, that is, $\{a_n\}$ is bounded, and hence convergent by the Monotonic Sequence Theorem. If $L = \lim\limits_{n\to\infty} a_n$, then $\lim\limits_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 3 - 1/L \quad \Rightarrow \quad L^2 - 3L + 1 = 0 \quad \Rightarrow \quad L = \frac{3 \pm \sqrt{5}}{2}$. But $L > 1$, so $L = \frac{3 + \sqrt{5}}{2}$.

88. $a_1 = 2$, $a_{n+1} = \dfrac{1}{3 - a_n}$. We use induction. Let P_n be the statement that $0 < a_{n+1} \leq a_n \leq 2$. Clearly P_1 is true, since $a_2 = 1/(3-2) = 1$. Now assume that P_n is true. Then $a_{n+1} \leq a_n \quad \Rightarrow \quad -a_{n+1} \geq -a_n \quad \Rightarrow \quad 3 - a_{n+1} \geq 3 - a_n \quad \Rightarrow$ $a_{n+2} = \dfrac{1}{3 - a_{n+1}} \leq \dfrac{1}{3 - a_n} = a_{n+1}$. Also $a_{n+2} > 0$ [since $3 - a_{n+1}$ is positive] and $a_{n+1} \leq 2$ by the induction hypothesis, so P_{n+1} is true. To find the limit, we use the fact that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} \quad \Rightarrow \quad L = \frac{1}{3 - L} \quad \Rightarrow$ $L^2 - 3L + 1 = 0 \quad \Rightarrow \quad L = \frac{3 \pm \sqrt{5}}{2}$. But $L \leq 2$, so we must have $L = \frac{3 - \sqrt{5}}{2}$.

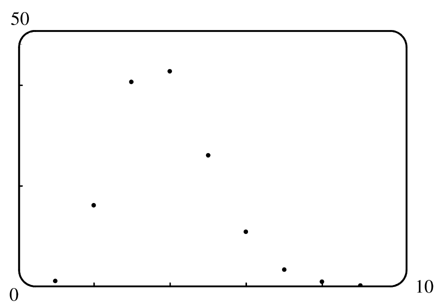
89. (a) Let a_n be the number of rabbit pairs in the nth month. Clearly $a_1 = 1 = a_2$. In the nth month, each pair that is 2 or more months old (that is, a_{n-2} pairs) will produce a new pair to add to the a_{n-1} pairs already present. Thus, $a_n = a_{n-1} + a_{n-2}$, so that $\{a_n\} = \{f_n\}$, the Fibonacci sequence.

(b) $a_n = \dfrac{f_{n+1}}{f_n} \;\Rightarrow\; a_{n-1} = \dfrac{f_n}{f_{n-1}} = \dfrac{f_{n-1} + f_{n-2}}{f_{n-1}} = 1 + \dfrac{f_{n-2}}{f_{n-1}} = 1 + \dfrac{1}{f_{n-1}/f_{n-2}} = 1 + \dfrac{1}{a_{n-2}}$. If $L = \lim\limits_{n\to\infty} a_n$, then $L = \lim\limits_{n\to\infty} a_{n-1}$ and $L = \lim\limits_{n\to\infty} a_{n-2}$, so L must satisfy $L = 1 + \dfrac{1}{L} \;\Rightarrow\; L^2 - L - 1 = 0 \;\Rightarrow\; L = \frac{1+\sqrt{5}}{2}$ [since L must be positive].

90. (a) If f is continuous, then $f(L) = f\left(\lim\limits_{n\to\infty} a_n\right) = \lim\limits_{n\to\infty} f(a_n) = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = L$ by Exercise 76(a).

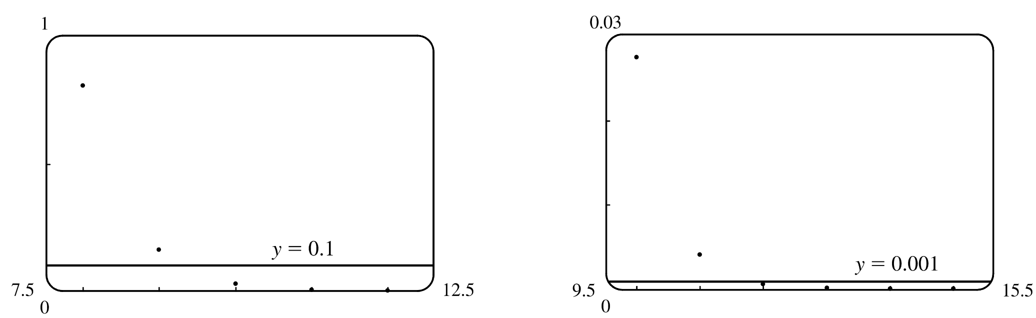
(b) By repeatedly pressing the cosine key on the calculator (that is, taking cosine of the previous answer) until the displayed value stabilizes, we see that $L \approx 0.73909$.

91. (a)



From the graph, it appears that the sequence $\left\{\dfrac{n^5}{n!}\right\}$ converges to 0, that is, $\lim\limits_{n\to\infty} \dfrac{n^5}{n!} = 0$.

(b)



From the first graph, it seems that the smallest possible value of N corresponding to $\varepsilon = 0.1$ is 9, since $n^5/n! < 0.1$ whenever $n \geq 10$, but $9^5/9! > 0.1$. From the second graph, it seems that for $\varepsilon = 0.001$, the smallest possible value for N is 11 since $n^5/n! < 0.001$ whenever $n \geq 12$.

92. Let $\varepsilon > 0$ and let N be any positive integer larger than $\ln(\varepsilon)/\ln|r|$. If $n > N$, then $n > \ln(\varepsilon)/\ln|r| \;\Rightarrow\; n\ln|r| < \ln\varepsilon$ [since $|r| < 1 \Rightarrow \ln|r| < 0$] $\;\Rightarrow\; \ln(|r|^n) < \ln\varepsilon \;\Rightarrow\; |r|^n < \varepsilon \;\Rightarrow\; |r^n - 0| < \varepsilon$, and so by Definition 2, $\lim\limits_{n\to\infty} r^n = 0$.

93. Theorem 6: If $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} -|a_n| = 0$, and since $-|a_n| \leq a_n \leq |a_n|$, we have that $\lim\limits_{n\to\infty} a_n = 0$ by the Squeeze Theorem.

94. Let $L = \lim_{n\to\infty} a_n$ and $f(x) = x^p$, $p > 0$ and $a_n > 0$. Since f is a continuous function,

$$\lim_{n\to\infty} a_n^p = \lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) \text{ [Theorem 7] } = f(L) = \left[\lim_{n\to\infty} a_n\right]^p.$$

95. To Prove: If $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n\to\infty} (a_n b_n) = 0$.

Proof: Since $\{b_n\}$ is bounded, there is a positive number M such that $|b_n| \le M$ and hence, $|a_n|\,|b_n| \le |a_n|\,M$ for all $n \ge 1$. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = 0$, there is an integer N such that $|a_n - 0| < \dfrac{\varepsilon}{M}$ if $n > N$. Then $|a_n b_n - 0| = |a_n b_n| = |a_n|\,|b_n| \le |a_n|\,M = |a_n - 0|\,M < \dfrac{\varepsilon}{M} \cdot M = \varepsilon$ for all $n > N$. Since ε was arbitrary, $\lim_{n\to\infty} (a_n b_n) = 0$.

96. (a) $$\begin{aligned}\frac{b^{n+1} - a^{n+1}}{b - a} &= b^n + b^{n-1}a + b^{n-2}a^2 + b^{n-3}a^3 + \cdots + ba^{n-1} + a^n \\ &< b^n + b^{n-1}b + b^{n-2}b^2 + b^{n-3}b^3 + \cdots + bb^{n-1} + b^n = (n+1)b^n\end{aligned}$$

(b) Since $b - a > 0$, we have $b^{n+1} - a^{n+1} < (n+1)b^n(b-a) \;\Rightarrow\; b^{n+1} - (n+1)b^n(b-a) < a^{n+1} \;\Rightarrow$ $b^n[(n+1)a - nb] < a^{n+1}$.

(c) Substituting in part (b), $(n+1)a - nb = (n+1)\left(1 + \dfrac{1}{n+1}\right) - n\left(1 + \dfrac{1}{n}\right) = 1$, and so

$$b^n = \left(1 + \frac{1}{n}\right)^n < a^{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1}.$$

(d) Substituting in part (b), $(n+1)a - nb = (n+1)\cdot 1 - n\left(1 + \dfrac{1}{2n}\right) = \dfrac{1}{2}$, and so $\left(1 + \dfrac{1}{2n}\right)^n \left(\dfrac{1}{2}\right) < 1 \;\Rightarrow$

$$\left(1 + \frac{1}{2n}\right)^n < 2 \;\Rightarrow\; \left(1 + \frac{1}{2n}\right)^{2n} < 4.$$

(e) $a_n < a_{2n}$ since $\{a_n\}$ is increasing, so $a_n < a_{2n} < 4$.

(f) Since $\{a_n\}$ is increasing and bounded above by 4, $a_1 \le a_n \le 4$, and so $\{a_n\}$ is bounded and monotonic, and hence has a limit by the Monotonic Sequence Theorem.

97. (a) First we show that $a > a_1 > b_1 > b$.

$a_1 - b_1 = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}\left(a - 2\sqrt{ab} + b\right) = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 > 0$ [since $a > b$] $\Rightarrow$ $a_1 > b_1$. Also $a - a_1 = a - \frac{1}{2}(a+b) = \frac{1}{2}(a-b) > 0$ and $b - b_1 = b - \sqrt{ab} = \sqrt{b}\left(\sqrt{b} - \sqrt{a}\right) < 0$, so $a > a_1 > b_1 > b$. In the same way we can show that $a_1 > a_2 > b_2 > b_1$ and so the given assertion is true for $n = 1$. Suppose it is true for $n = k$, that is, $a_k > a_{k+1} > b_{k+1} > b_k$. Then

$$a_{k+2} - b_{k+2} = \tfrac{1}{2}(a_{k+1} + b_{k+1}) - \sqrt{a_{k+1}b_{k+1}} = \tfrac{1}{2}\left(a_{k+1} - 2\sqrt{a_{k+1}b_{k+1}} + b_{k+1}\right) = \tfrac{1}{2}\left(\sqrt{a_{k+1}} - \sqrt{b_{k+1}}\right)^2 > 0,$$

$a_{k+1} - a_{k+2} = a_{k+1} - \frac{1}{2}(a_{k+1} + b_{k+1}) = \frac{1}{2}(a_{k+1} - b_{k+1}) > 0$, and

$$b_{k+1} - b_{k+2} = b_{k+1} - \sqrt{a_{k+1}b_{k+1}} = \sqrt{b_{k+1}}\left(\sqrt{b_{k+1}} - \sqrt{a_{k+1}}\right) < 0 \quad \Rightarrow \quad a_{k+1} > a_{k+2} > b_{k+2} > b_{k+1},$$

so the assertion is true for $n = k + 1$. Thus, it is true for all n by mathematical induction.

(b) From part (a) we have $a > a_n > a_{n+1} > b_{n+1} > b_n > b$, which shows that both sequences, $\{a_n\}$ and $\{b_n\}$, are monotonic and bounded. So they are both convergent by the Monotonic Sequence Theorem.

(c) Let $\lim\limits_{n\to\infty} a_n = \alpha$ and $\lim\limits_{n\to\infty} b_n = \beta$. Then $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{a_n + b_n}{2} \quad \Rightarrow \quad \alpha = \dfrac{\alpha + \beta}{2} \quad \Rightarrow$
$2\alpha = \alpha + \beta \quad \Rightarrow \quad \alpha = \beta$.

98. (a) Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} a_{2n} = L$, there exists N_1 such that $|a_{2n} - L| < \varepsilon$ for $n > N_1$. Since $\lim\limits_{n\to\infty} a_{2n+1} = L$, there exists N_2 such that $|a_{2n+1} - L| < \varepsilon$ for $n > N_2$. Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. If n is even, then $n = 2m$ where $m > N_1$, so $|a_n - L| = |a_{2m} - L| < \varepsilon$. If n is odd, then $n = 2m + 1$, where $m > N_2$, so $|a_n - L| = |a_{2m+1} - L| < \varepsilon$. Therefore $\lim\limits_{n\to\infty} a_n = L$.

(b) $a_1 = 1$, $a_2 = 1 + \frac{1}{1+1} = \frac{3}{2} = 1.5$, $a_3 = 1 + \frac{1}{5/2} = \frac{7}{5} = 1.4$, $a_4 = 1 + \frac{1}{12/5} = \frac{17}{12} = 1.41\overline{6}$,
$a_5 = 1 + \frac{1}{29/12} = \frac{41}{29} \approx 1.413793$, $a_6 = 1 + \frac{1}{70/29} = \frac{99}{70} \approx 1.414286$, $a_7 = 1 + \frac{1}{169/70} = \frac{239}{169} \approx 1.414201$,
$a_8 = 1 + \frac{1}{408/169} = \frac{577}{408} \approx 1.414216$. Notice that $a_1 < a_3 < a_5 < a_7$ and $a_2 > a_4 > a_6 > a_8$. It appears that the odd terms are increasing and the even terms are decreasing. Let's prove that $a_{2n-2} > a_{2n}$ and $a_{2n-1} < a_{2n+1}$ by mathematical induction. Suppose that $a_{2k-2} > a_{2k}$. Then $1 + a_{2k-2} > 1 + a_{2k} \quad \Rightarrow \quad \dfrac{1}{1 + a_{2k-2}} < \dfrac{1}{1 + a_{2k}} \quad \Rightarrow$
$1 + \dfrac{1}{1 + a_{2k-2}} < 1 + \dfrac{1}{1 + a_{2k}} \quad \Rightarrow \quad a_{2k-1} < a_{2k+1} \quad \Rightarrow \quad 1 + a_{2k-1} < 1 + a_{2k+1} \quad \Rightarrow$
$\dfrac{1}{1 + a_{2k-1}} > \dfrac{1}{1 + a_{2k+1}} \quad \Rightarrow \quad 1 + \dfrac{1}{1 + a_{2k-1}} > 1 + \dfrac{1}{1 + a_{2k+1}} \quad \Rightarrow \quad a_{2k} > a_{2k+2}$. We have thus shown, by induction, that the odd terms are increasing and the even terms are decreasing. Also all terms lie between 1 and 2, so both $\{a_n\}$ and $\{b_n\}$ are bounded monotonic sequences and are therefore convergent by the Monotonic Sequence Theorem. Let $\lim\limits_{n\to\infty} a_{2n} = L$. Then $\lim\limits_{n\to\infty} a_{2n+2} = L$ also. We have

$$a_{n+2} = 1 + \frac{1}{1 + 1 + 1/(1 + a_n)} = 1 + \frac{1}{(3 + 2a_n)/(1 + a_n)} = \frac{4 + 3a_n}{3 + 2a_n}$$

so $a_{2n+2} = \dfrac{4 + 3a_{2n}}{3 + 2a_{2n}}$. Taking limits of both sides, we get $L = \dfrac{4 + 3L}{3 + 2L} \quad \Rightarrow \quad 3L + 2L^2 = 4 + 3L \quad \Rightarrow \quad L^2 = 2 \quad \Rightarrow$
$L = \sqrt{2}$ [since $L > 0$]. Thus, $\lim\limits_{n\to\infty} a_{2n} = \sqrt{2}$. Similarly we find that $\lim\limits_{n\to\infty} a_{2n+1} = \sqrt{2}$. So, by part (a), $\lim\limits_{n\to\infty} a_n = \sqrt{2}$.

99. (a) Suppose $\{p_n\}$ converges to p. Then $p_{n+1} = \dfrac{bp_n}{a+p_n} \quad\Rightarrow\quad \lim\limits_{n\to\infty} p_{n+1} = \dfrac{b \lim\limits_{n\to\infty} p_n}{a + \lim\limits_{n\to\infty} p_n} \quad\Rightarrow\quad p = \dfrac{bp}{a+p} \quad\Rightarrow$

$p^2 + ap = bp \quad\Rightarrow\quad p(p+a-b) = 0 \quad\Rightarrow\quad p = 0$ or $p = b - a$.

(b) $p_{n+1} = \dfrac{bp_n}{a+p_n} = \dfrac{\left(\dfrac{b}{a}\right)p_n}{1+\dfrac{p_n}{a}} < \left(\dfrac{b}{a}\right)p_n$ since $1 + \dfrac{p_n}{a} > 1$.

(c) By part (b), $p_1 < \left(\dfrac{b}{a}\right)p_0$, $p_2 < \left(\dfrac{b}{a}\right)p_1 < \left(\dfrac{b}{a}\right)^2 p_0$, $p_3 < \left(\dfrac{b}{a}\right)p_2 < \left(\dfrac{b}{a}\right)^3 p_0$, etc. In general, $p_n < \left(\dfrac{b}{a}\right)^n p_0$,

so $\lim\limits_{n\to\infty} p_n \le \lim\limits_{n\to\infty} \left(\dfrac{b}{a}\right)^n \cdot p_0 = 0$ since $b < a$. $\left[\text{By (9), } \lim\limits_{n\to\infty} r^n = 0 \text{ if } -1 < r < 1. \text{ Here } r = \dfrac{b}{a} \in (0,1)\,.\right]$

(d) Let $a < b$. We first show, by induction, that if $p_0 < b - a$, then $p_n < b - a$ and $p_{n+1} > p_n$.

For $n = 0$, we have $p_1 - p_0 = \dfrac{bp_0}{a+p_0} - p_0 = \dfrac{p_0(b-a-p_0)}{a+p_0} > 0$ since $p_0 < b - a$. So $p_1 > p_0$.

Now we suppose the assertion is true for $n = k$, that is, $p_k < b - a$ and $p_{k+1} > p_k$. Then

$b - a - p_{k+1} = b - a - \dfrac{bp_k}{a+p_k} = \dfrac{a(b-a) + bp_k - ap_k - bp_k}{a+p_k} = \dfrac{a(b-a-p_k)}{a+p_k} > 0$ because $p_k < b - a$. So

$p_{k+1} < b - a$. And $p_{k+2} - p_{k+1} = \dfrac{bp_{k+1}}{a+p_{k+1}} - p_{k+1} = \dfrac{p_{k+1}(b-a-p_{k+1})}{a+p_{k+1}} > 0$ since $p_{k+1} < b - a$. Therefore,

$p_{k+2} > p_{k+1}$. Thus, the assertion is true for $n = k + 1$. It is therefore true for all n by mathematical induction.

A similar proof by induction shows that if $p_0 > b - a$, then $p_n > b - a$ and $\{p_n\}$ is decreasing.

In either case the sequence $\{p_n\}$ is bounded and monotonic, so it is convergent by the Monotonic Sequence Theorem.

It then follows from part (a) that $\lim\limits_{n\to\infty} p_n = b - a$.

DISCOVERY PROJECT Logistic Sequences

1. To write such a program in Maple it is best to calculate all the points first and then graph them. One possible sequence of commands [taking $p_0 = \frac{1}{2}$ and $k = 1.5$ for the difference equation] is

```
t:='t';p(0):=1/2;k:=1.5;
for j from 1 to 20  do p(j):=k*p(j-1)*(1-p(j-1)) od;
plot([seq([t,p(t)] t=0..20)],t=0..20,p=0..0.5,style=point);
```

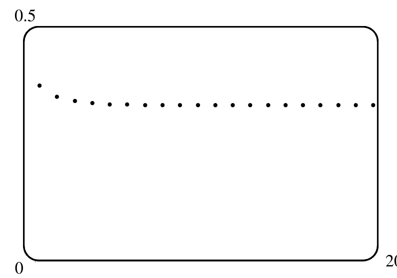
In Mathematica, we can use the following program:

```
p[0]=1/2
k=1.5
p[j_]:=k*p[j-1]*(1-p[j-1])
P=Table[p[t],{t,20}]
ListPlot[P]
```

[continued]

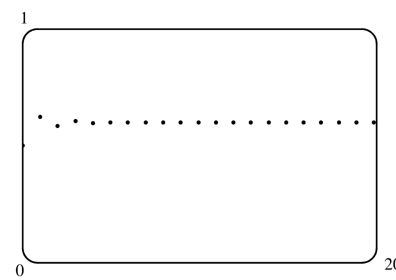
With $p_0 = \frac{1}{2}$ and $k = 1.5$:

n	p_n	n	p_n	n	p_n
0	0.5	7	0.3338465076	14	0.3333373303
1	0.375	8	0.3335895255	15	0.3333353318
2	0.3515625	9	0.3334613309	16	0.3333343326
3	0.3419494629	10	0.3333973076	17	0.3333338329
4	0.3375300416	11	0.3333653143	18	0.3333335831
5	0.3354052689	12	0.3333493223	19	0.3333334582
6	0.3343628617	13	0.3333413274	20	0.3333333958



With $p_0 = \frac{1}{2}$ and $k = 2.5$:

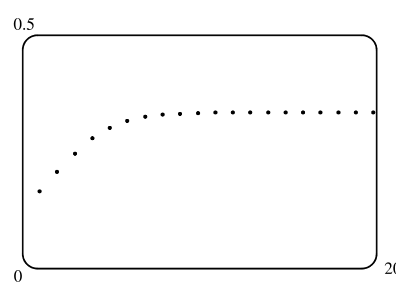
n	p_n	n	p_n	n	p_n
0	0.5	7	0.6004164790	14	0.5999967417
1	0.625	8	0.5997913269	15	0.6000016291
2	0.5859375	9	0.6001042277	16	0.5999991854
3	0.6065368651	10	0.5999478590	17	0.6000004073
4	0.5966247409	11	0.6000260637	18	0.5999997964
5	0.6016591486	12	0.5999869664	19	0.6000001018
6	0.5991635437	13	0.6000065164	20	0.5999999491



Both of these sequences seem to converge (the first to about $\frac{1}{3}$, the second to about 0.60).

With $p_0 = \frac{7}{8}$ and $k = 1.5$:

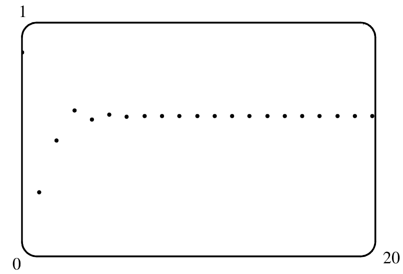
n	p_n	n	p_n	n	p_n
0	0.875	7	0.3239166554	14	0.3332554829
1	0.1640625	8	0.3284919837	15	0.3332943990
2	0.2057189941	9	0.3308775005	16	0.3333138639
3	0.2450980344	10	0.3320963702	17	0.3333235980
4	0.2775374819	11	0.3327125567	18	0.3333284655
5	0.3007656421	12	0.3330223670	19	0.3333308994
6	0.3154585059	13	0.3331777051	20	0.3333321164



[continued]

With $p_0 = \frac{7}{8}$ and $k = 2.5$:

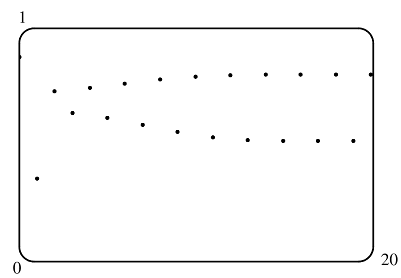
n	p_n	n	p_n	n	p_n
0	0.875	7	0.6016572368	14	0.5999869815
1	0.2734375	8	0.5991645155	15	0.6000065088
2	0.4966735840	9	0.6004159972	16	0.5999967455
3	0.6249723374	10	0.5997915688	17	0.6000016272
4	0.5859547872	11	0.6001041070	18	0.5999991864
5	0.6065294364	12	0.5999479194	19	0.6000004068
6	0.5966286980	13	0.6000260335	20	0.5999997966



The limit of the sequence seems to depend on k, but not on p_0.

2. With $p_0 = \frac{7}{8}$ and $k = 3.2$:

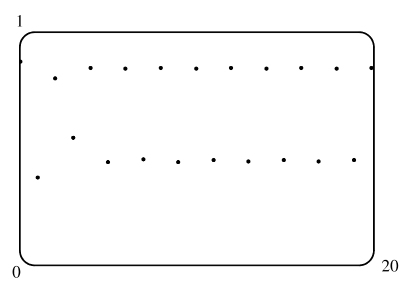
n	p_n	n	p_n	n	p_n
0	0.875	7	0.5830728495	14	0.7990633827
1	0.35	8	0.7779164854	15	0.5137954979
2	0.728	9	0.5528397669	16	0.7993909896
3	0.6336512	10	0.7910654689	17	0.5131681132
4	0.7428395416	11	0.5288988570	18	0.7994451225
5	0.6112926626	12	0.7973275394	19	0.5130643795
6	0.7603646184	13	0.5171082698	20	0.7994538304



It seems that eventually the terms fluctuate between two values (about 0.5 and 0.8 in this case).

3. With $p_0 = \frac{7}{8}$ and $k = 3.42$:

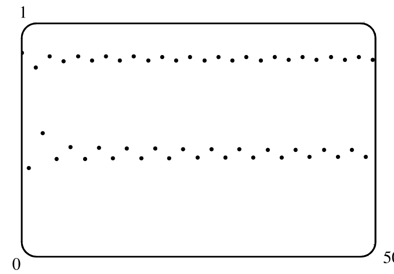
n	p_n	n	p_n	n	p_n
0	0.875	7	0.4523028596	14	0.8442074951
1	0.3740625	8	0.8472194412	15	0.4498025048
2	0.8007579316	9	0.4426802161	16	0.8463823232
3	0.5456427596	10	0.8437633929	17	0.4446659586
4	0.8478752457	11	0.4508474156	18	0.8445284520
5	0.4411212220	12	0.8467373602	19	0.4490464985
6	0.8431438501	13	0.4438243545	20	0.8461207931



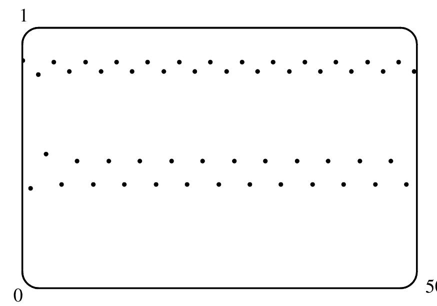
[continued]

With $p_0 = \frac{7}{8}$ and $k = 3.45$:

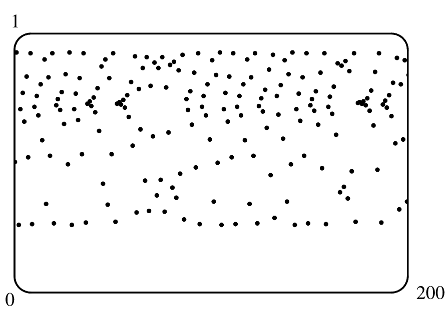
n	p_n	n	p_n	n	p_n
0	0.875	7	0.4670259170	14	0.8403376122
1	0.37734375	8	0.8587488490	15	0.4628875685
2	0.8105962830	9	0.4184824586	16	0.8577482026
3	0.5296783241	10	0.8395743720	17	0.4209559716
4	0.8594612299	11	0.4646778983	18	0.8409445432
5	0.4167173034	12	0.8581956045	19	0.4614610237
6	0.8385707740	13	0.4198508858	20	0.8573758782



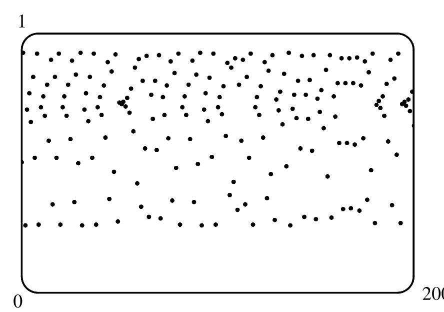
From the preceding graphs, it seems that for k between 3.4 and 3.5, the terms eventually fluctuate between four values. In the graph below, the pattern followed by the terms is 0.395, 0.832, 0.487, 0.869, 0.395, ... Note that even for $k = 3.42$ (as in the first graph), there are four distinct “branches”; even after 1000 terms, the first and third terms in the pattern differ by about 2×10^{-9}, while the first and fifth terms differ by only 2×10^{-10}. With $p_0 = \frac{7}{8}$ and $k = 3.48$:



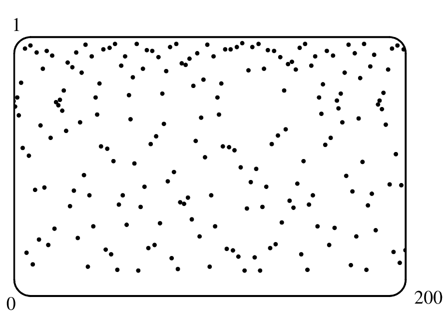
4.



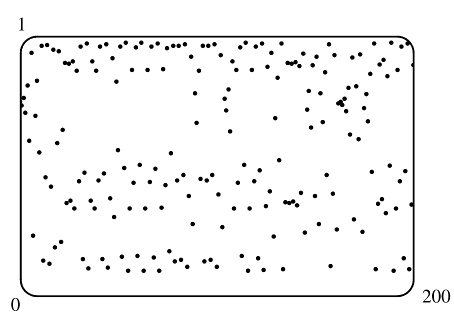
$p_0 = 0.5,\ k = 3.7$



$p_0 = 0.501,\ k = 3.7$

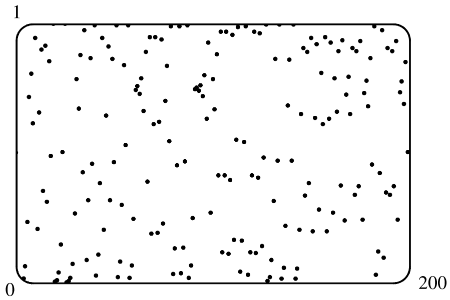


$p_0 = 0.75,\ k = 3.9$



$p_0 = 0.749,\ k = 3.9$

[continued]



$p_0 = 0.5, k = 3.999$

From the graphs, it seems that if p_0 is changed by 0.001, the whole graph changes completely. (Note, however, that this might be partially due to accumulated round-off error in the CAS. These graphs were generated by Maple with 100-digit accuracy, and different degrees of accuracy give different graphs.) There seem to be some some fleeting patterns in these graphs, but on the whole they are certainly very chaotic. As k increases, the graph spreads out vertically, with more extreme values close to 0 or 1.

11.2 Series

1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

2. $\sum\limits_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5.

In other words, it means that $\lim_{n\to\infty} s_n = 5$, where s_n is the nth partial sum, that is, $\sum\limits_{i=1}^{n} a_i$.

3. $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} [2 - 3(0.8)^n] = \lim\limits_{n\to\infty} 2 - 3 \lim\limits_{n\to\infty} (0.8)^n = 2 - 3(0) = 2$

4. $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \dfrac{n^2 - 1}{4n^2 + 1} = \lim\limits_{n\to\infty} \dfrac{(n^2 - 1)/n^2}{(4n^2 + 1)/n^2} = \lim\limits_{n\to\infty} \dfrac{1 - 1/n^2}{4 + 1/n^2} = \dfrac{1 - 0}{4 + 0} = \dfrac{1}{4}$

5. For $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$, $a_n = \dfrac{1}{n^3}$. $s_1 = a_1 = \dfrac{1}{1^3} = 1$, $s_2 = s_1 + a_2 = 1 + \dfrac{1}{2^3} = 1.125$,

$s_3 = s_2 + a_3 \approx 1.1620$, $s_4 = s_3 + a_4 \approx 1.1777$, $s_5 = s_4 + a_5 \approx 1.1857$, $s_6 = s_5 + a_6 \approx 1.1903$,

$s_7 = s_6 + a_7 \approx 1.1932$, $s_8 = s_7 + a_8 \approx 1.1952$. It appears that the series is convergent.

6. For $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt[3]{n}}$, $a_n = \dfrac{1}{\sqrt[3]{n}}$. $s_1 = a_1 = \dfrac{1}{\sqrt[3]{1}} = 1$, $s_2 = s_1 + a_2 = 1 + \dfrac{1}{\sqrt[3]{2}} \approx 1.7937$,

$s_3 = s_2 + a_3 \approx 2.4871$, $s_4 = s_3 + a_4 \approx 3.1170$, $s_5 = s_4 + a_5 \approx 3.7018$, $s_6 = s_5 + a_6 \approx 4.2521$,

$s_7 = s_6 + a_7 \approx 4.7749$, and $s_8 = s_7 + a_8 \approx 5.2749$. It appears that the series is divergent.

7. For $\sum\limits_{n=1}^{\infty} \sin n$, $a_n = \sin n$. $s_1 = a_1 = \sin 1 \approx 0.8415$, $s_2 = s_1 + a_2 \approx 1.7508$,

$s_3 = s_2 + a_3 \approx 1.8919$, $s_4 = s_3 + a_4 \approx 1.1351$, $s_5 = s_4 + a_5 \approx 0.1762$, $s_6 = s_5 + a_6 \approx -0.1033$,

$s_7 = s_6 + a_7 \approx 0.5537$, and $s_8 = s_7 + a_8 \approx 1.5431$. It appears that the series is divergent.

8. For $\sum_{n=1}^{\infty}(-1)^n n$, $a_n=(-1)^n n$. $s_1=a_1=(-1)^1(1)=-1$, $s_2=s_1+a_2=-1+2=1$,

$s_3=s_2+a_3=-2$, $s_4=s_3+a_4=2$, $s_5=s_4+a_5=-3$, $s_6=s_5+a_6=3$,

$s_7=s_6+a_7=-4$, $s_8=s_7+a_8=4$. It appears that the series is divergent.

9. For $\sum_{n=1}^{\infty}\frac{1}{n^4+n^2}$, $a_n=\frac{1}{n^4+n^2}$. $s_1=a_1=\frac{1}{1^4+1^2}=\frac{1}{2}=0.5$, $s_2=s_1+a_2=\frac{1}{2}+\frac{1}{16+4}=0.55$,

$s_3=s_2+a_3\approx 0.5611$, $s_4=s_3+a_4\approx 0.5648$, $s_5=s_4+a_5\approx 0.5663$, $s_6=s_5+a_6\approx 0.5671$,

$s_7=s_6+a_7\approx 0.5675$, and $s_8=s_7+a_8\approx 0.5677$. It appears that the series is convergent.

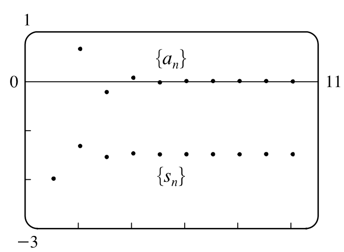
10. For $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}$, $a_n=(-1)^{n-1}\frac{1}{n!}$. $s_1=a_1=\frac{1}{1!}=1$, $s_2=s_1+a_2=1-\frac{1}{2!}=0.5$,

$s_3=s_2+a_3=0.5+\frac{1}{3!}\approx 0.6667$, $s_4=s_3+a_4=0.625$, $s_5=s_4+a_5\approx 0.6333$, $s_6=s_5+a_6\approx 0.6319$,

$s_7=s_6+a_7\approx 0.6321$, and $s_8=s_7+a_8\approx 0.6321$. It appears that the series is convergent.

11.

n	s_n
1	-2
2	-1.33333
3	-1.55556
4	-1.48148
5	-1.50617
6	-1.49794
7	-1.50069
8	-1.49977
9	-1.50008
10	-1.49997

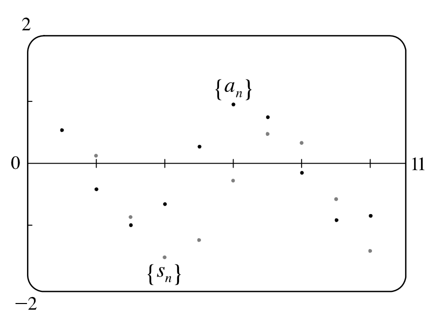


From the graph and the table, it seems that the series converges to -1.5. In fact, it is a geometric series with $a=-2$ and $r=-\frac{1}{3}$, so its sum is $\sum_{n=1}^{\infty}\frac{6}{(-3)^n}=\frac{-2}{1-\left(-\frac{1}{3}\right)}=-1.5$.

Note that the point corresponding to $n=1$ is part of both $\{a_n\}$ and $\{s_n\}$.

12.

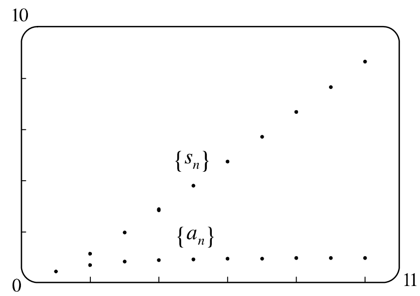
n	s_n
1	0.54030
2	0.12416
3	-0.86584
4	-1.51948
5	-1.23582
6	-0.27565
7	0.47825
8	0.33275
9	-0.57838
10	-1.41745



The series $\sum_{n=1}^{\infty}\cos n$ diverges, since its terms do not approach 0.

13.

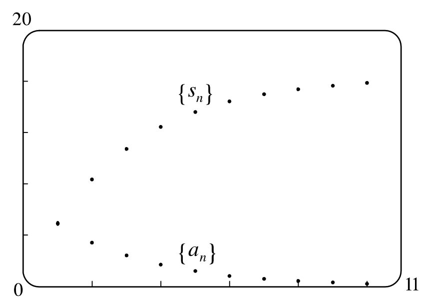
n	s_n
1	0.44721
2	1.15432
3	1.98637
4	2.88080
5	3.80927
6	4.75796
7	5.71948
8	6.68962
9	7.66581
10	8.64639



The series $\sum\limits_{n=1}^{\infty} \dfrac{n}{\sqrt{n^2+4}}$ diverges, since its terms do not approach 0.

14.

n	s_n
1	4.90000
2	8.33000
3	10.73100
4	12.41170
5	13.58819
6	14.41173
7	14.98821
8	15.39175
9	15.67422
10	15.87196



From the graph and the table, we see that the terms are getting smaller and may approach 0, and that the series approaches a value near 16. The series is geometric with $a_1 = 4.9$ and $r = 0.7$, so its sum is $\sum\limits_{n=1}^{\infty} \dfrac{7^{n+1}}{10^n} = \dfrac{4.9}{1-0.7} = \dfrac{4.9}{0.3} = 16.\overline{3}$.

15. (a) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{2n}{3n+1} = \dfrac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (11.1.1).

(b) Since $\lim\limits_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum\limits_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence.

16. (a) Both $\sum\limits_{i=1}^{n} a_i$ and $\sum\limits_{j=1}^{n} a_j$ represent the sum of the first n terms of the sequence $\{a_n\}$, that is, the nth partial sum.

(b) $\sum\limits_{i=1}^{n} a_j = \underbrace{a_j + a_j + \cdots + a_j}_{n \text{ terms}} = na_j$, which, in general, is not the same as $\sum\limits_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

17. For the series $\sum\limits_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n}\right)$,

$$\begin{aligned}
s_n &= \sum_{i=1}^{n}\left(\frac{1}{i+2}-\frac{1}{i}\right) \\
&= \left(\frac{1}{3}-1\right)+\left(\frac{1}{4}-\frac{1}{2}\right)+\left(\frac{1}{5}-\frac{1}{3}\right)+\left(\frac{1}{6}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n-2}\right)+\left(\frac{1}{n+1}-\frac{1}{n-1}\right)+\left(\frac{1}{n+2}-\frac{1}{n}\right) \\
&= -1-\frac{1}{2}+\frac{1}{n+1}+\frac{1}{n+2} \qquad \text{[telescoping series]}
\end{aligned}$$

Thus, $\sum\limits_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n}\right)=\lim\limits_{n\to\infty}s_n=\lim\limits_{n\to\infty}\left(-1-\frac{1}{2}+\frac{1}{n+1}+\frac{1}{n+2}\right)=-1-\frac{1}{2}=-\frac{3}{2}.$ Converges

18. For the series $\sum\limits_{n=4}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$

$$s_n=\sum_{i=4}^{n}\left(\frac{1}{\sqrt{i}}-\frac{1}{\sqrt{i+1}}\right)=\left(\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{5}}\right)+\left(\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{6}}\right)+\cdots+\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)=\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{n+1}}$$

[telescoping series]

Thus, $\sum\limits_{n=4}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)=\lim\limits_{n\to\infty}s_n=\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{n+1}}\right)=\frac{1}{\sqrt{4}}-0=\frac{1}{2}.$ Converges

19. For the series $\sum\limits_{n=1}^{\infty}\frac{3}{n(n+3)}$, $s_n=\sum\limits_{i=1}^{n}\frac{3}{i(i+3)}=\sum\limits_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{i+3}\right)$ [using partial fractions]. The latter sum is

$$\begin{aligned}
&\left(1-\tfrac{1}{4}\right)+\left(\tfrac{1}{2}-\tfrac{1}{5}\right)+\left(\tfrac{1}{3}-\tfrac{1}{6}\right)+\left(\tfrac{1}{4}-\tfrac{1}{7}\right)+\cdots+\left(\tfrac{1}{n-3}-\tfrac{1}{n}\right)+\left(\tfrac{1}{n-2}-\tfrac{1}{n+1}\right)+\left(\tfrac{1}{n-1}-\tfrac{1}{n+2}\right)+\left(\tfrac{1}{n}-\tfrac{1}{n+3}\right) \\
&\qquad = 1+\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{n+1}-\tfrac{1}{n+2}-\tfrac{1}{n+3} \qquad \text{[telescoping series]}
\end{aligned}$$

Thus, $\sum\limits_{n=1}^{\infty}\frac{3}{n(n+3)}=\lim\limits_{n\to\infty}s_n=\lim\limits_{n\to\infty}\left(1+\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{n+1}-\tfrac{1}{n+2}-\tfrac{1}{n+3}\right)=1+\tfrac{1}{2}+\tfrac{1}{3}=\tfrac{11}{6}.$ Converges

20. For the series $\sum\limits_{n=1}^{\infty}\ln\frac{n}{n+1}$,

$$s_n=(\ln 1-\ln 2)+(\ln 2-\ln 3)+(\ln 3-\ln 4)+\cdots+[\ln n-\ln(n+1)]=\ln 1-\ln(n+1)=-\ln(n+1)$$

[telescoping series]

Thus, $\lim\limits_{n\to\infty}s_n=-\infty$, so the series is divergent.

21. For the series $\sum\limits_{n=1}^{\infty}\left(e^{1/n}-e^{1/(n+1)}\right)$,

$$s_n=\sum_{i=1}^{n}\left(e^{1/i}-e^{1/(i+1)}\right)=(e^1-e^{1/2})+(e^{1/2}-e^{1/3})+\cdots+\left(e^{1/n}-e^{1/(n+1)}\right)=e-e^{1/(n+1)}$$

[telescoping series]

Thus, $\sum\limits_{n=1}^{\infty}\left(e^{1/n}-e^{1/(n+1)}\right)=\lim\limits_{n\to\infty}s_n=\lim\limits_{n\to\infty}\left(e-e^{1/(n+1)}\right)=e-e^0=e-1.$ Converges

22. Using partial fractions, the partial sums of the series $\sum_{n=2}^{\infty} \frac{1}{n^3 - n}$ are

$$s_n = \sum_{i=2}^{n} \frac{1}{i(i-1)(i+1)} = \sum_{i=2}^{n} \left(-\frac{1}{i} + \frac{1/2}{i-1} + \frac{1/2}{i+1} \right) = \frac{1}{2} \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{2}{i} + \frac{1}{i+1} \right)$$

$$= \frac{1}{2}\left[\left(\frac{1}{1} - \frac{2}{2} + \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right) + \cdots \right.$$

$$\left. + \left(\frac{1}{n-3} - \frac{2}{n-2} + \frac{1}{n-1} \right) + \left(\frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n} \right) + \left(\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1} \right) \right]$$

Note: In three consecutive expressions in parentheses, the 3rd term in the first expression plus the 2nd term in the second expression plus the 1st term in the third expression sum to 0.

$$= \frac{1}{2}\left(\frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n} - \frac{2}{n} + \frac{1}{n+1} \right) = \frac{1}{4} - \frac{1}{2n} + \frac{1}{2n+2}$$

Thus, $\sum_{n=2}^{\infty} \frac{1}{n^3 - n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{4} - \frac{1}{2n} + \frac{1}{2n+2} \right) = \frac{1}{4}$.

23. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$ is a geometric series with ratio $r = -\frac{4}{3}$. Since $|r| = \frac{4}{3} > 1$, the series diverges.

24. $4 + 3 + \frac{9}{4} + \frac{27}{16} + \cdots$ is a geometric series with ratio $\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, the series converges to $\frac{a}{1-r} = \frac{4}{1 - 3/4} = 16$.

25. $10 - 2 + 0.4 - 0.08 + \cdots$ is a geometric series with ratio $-\frac{2}{10} = -\frac{1}{5}$. Since $|r| = \frac{1}{5} < 1$, the series converges to

$\frac{a}{1-r} = \frac{10}{1 - (-1/5)} = \frac{10}{6/5} = \frac{50}{6} = \frac{25}{3}$.

26. $2 + 0.5 + 0.125 + 0.03125 + \cdots$ is a geometric series with ratio $r = \frac{0.5}{2} = \frac{1}{4}$. Since $|r| = \frac{1}{4} < 1$, the series converges

to $\frac{a}{1-r} = \frac{2}{1 - 1/4} = \frac{2}{3/4} = \frac{8}{3}$.

27. $\sum_{n=1}^{\infty} 12\,(0.73)^{n-1}$ is a geometric series with first term $a = 12$ and ratio $r = 0.73$. Since $|r| = 0.73 < 1$, the series converges

to $\frac{a}{1-r} = \frac{12}{1 - 0.73} = \frac{12}{0.27} = \frac{12(100)}{27} = \frac{400}{9}$.

28. $\sum_{n=1}^{\infty} \frac{5}{\pi^n} = 5 \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \right)^n$. The latter series is geometric with $a = \frac{1}{\pi}$ and ratio $r = \frac{1}{\pi}$. Since $|r| = \frac{1}{\pi} < 1$, it converges to

$\frac{1/\pi}{1 - 1/\pi} = \frac{1}{\pi - 1}$. Thus, the given series converges to $5\left(\frac{1}{\pi - 1} \right) = \frac{5}{\pi - 1}$.

29. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4} \sum_{n=1}^{\infty} \left(-\frac{3}{4} \right)^{n-1}$. The latter series is geometric with $a = 1$ and ratio $r = -\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it

converges to $\frac{1}{1 - (-3/4)} = \frac{4}{7}$. Thus, the given series converges to $\left(\frac{1}{4}\right)\left(\frac{4}{7}\right) = \frac{1}{7}$.

30. $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n} = 3\sum_{n=0}^{\infty} \left(-\tfrac{3}{2}\right)^n$ is a geometric series with ratio $r = -\frac{3}{2}$. Since $|r| = \frac{3}{2} > 1$, the series diverges.

31. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{(e^2)^n}{6^n 6^{-1}} = 6\sum_{n=1}^{\infty} \left(\frac{e^2}{6}\right)^n$ is a geometric series with ratio $r = \dfrac{e^2}{6}$. Since $|r| = \dfrac{e^2}{6}[\approx 1.23] > 1$, the series diverges.

32. $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n} = \sum_{n=1}^{\infty} \frac{6(2^2)^n \cdot 2^{-1}}{3^n} = 3\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ is a geometric series with ratio $r = \dfrac{4}{3}$. Since $|r| = \dfrac{4}{3} > 1$, the series diverges.

33. $\dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{9} + \dfrac{1}{12} + \dfrac{1}{15} + \cdots = \sum\limits_{n=1}^{\infty} \dfrac{1}{3n} = \dfrac{1}{3}\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$. This is a constant multiple of the divergent harmonic series, so it diverges.

34. $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \dfrac{4}{5} + \dfrac{5}{6} + \dfrac{6}{7} + \cdots = \sum\limits_{n=1}^{\infty} \dfrac{n}{n+1}$. This series diverges by the Test for Divergence since

$\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1 \neq 0.$

35. $\dfrac{2}{5} + \dfrac{4}{25} + \dfrac{8}{125} + \dfrac{16}{625} + \dfrac{32}{3125} + \cdots = \sum\limits_{n=1}^{\infty} \left(\dfrac{2}{5}\right)^n$. This series is geometric with $a = \dfrac{2}{5}$ and ratio $r = \dfrac{2}{5}$. Since $|r| = \dfrac{2}{5} < 1$, it converges to $\dfrac{2/5}{1-2/5} = \dfrac{2}{3}$.

36. $\frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{2}{81} + \frac{1}{243} + \frac{2}{729} + \cdots = \left(\frac{1}{3} + \frac{1}{27} + \frac{1}{243} + \cdots\right) + \left(\frac{2}{9} + \frac{2}{81} + \frac{2}{729} + \cdots\right)$, which are both convergent geometric series with sums $\dfrac{1/3}{1-1/9} = \dfrac{3}{8}$ and $\dfrac{2/9}{1-1/9} = \dfrac{1}{4}$, so the original series converges and its sum is $\frac{3}{8} + \frac{1}{4} = \frac{5}{8}$.

37. $\displaystyle\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$ diverges by the Test for Divergence since $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2+n}{1-2n} = \lim_{n\to\infty} \frac{2/n+1}{1/n-2} = -\frac{1}{2} \neq 0.$

38. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{k^2-2k+5}$ diverges by the Test for Divergence since $\displaystyle\lim_{k\to\infty} \frac{k^2}{k^2-2k+5} = \lim_{k\to\infty} \frac{1}{1-2/k+5/k^2} = 1 \neq 0.$

39. $\displaystyle\sum_{n=1}^{\infty} 3^{n+1}4^{-n} = \sum_{n=1}^{\infty} \frac{3^n \cdot 3^1}{4^n} = 3\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$. The latter series is geometric with $a = \dfrac{3}{4}$ and ratio $r = \dfrac{3}{4}$. Since $|r| = \dfrac{3}{4} < 1$, it converges to $\dfrac{3/4}{1-3/4} = 3$. Thus, the given series converges to $3(3) = 9$.

40.
$$\begin{aligned}\sum_{n=1}^{\infty} \left[(-0.2)^n + (0.6)^{n-1}\right] &= \sum_{n=1}^{\infty}(-0.2)^n + \sum_{n=1}^{\infty}(0.6)^{n-1} \qquad \text{[sum of two geometric series]}\\ &= \frac{-0.2}{1-(-0.2)} + \frac{1}{1-0.6} = -\frac{1}{6} + \frac{5}{2} = \frac{7}{3}\end{aligned}$$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4+e^{-n}}$ diverges by the Test for Divergence since $\displaystyle\lim_{n\to\infty} \frac{1}{4+e^{-n}} = \frac{1}{4+0} = \frac{1}{4} \neq 0.$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n}$ diverges by the Test for Divergence since $\displaystyle\lim_{n\to\infty} \frac{2^n+4^n}{e^n} = \lim_{n\to\infty}\left(\frac{2^n}{e^n}+\frac{4^n}{e^n}\right) \geq \lim_{n\to\infty}\left(\frac{4}{e}\right)^n = \infty$

since $\dfrac{4}{e} > 1.$

43. $\displaystyle\sum_{k=1}^{\infty} (\sin 100)^k$ is a geometric series with first term $a = \sin 100$ [≈ -0.506] and ratio $r = \sin 100$. Since $|r| < 1$, the series

converges to $\dfrac{\sin 100}{1-\sin 100} \approx -0.336.$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n}$ diverges by the Test for Divergence since $\displaystyle\lim_{n\to\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n} = \frac{1}{1+0} = 1 \neq 0.$

45. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ diverges by the Test for Divergence since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right) = \ln\left(\lim_{n\to\infty}\frac{n^2+1}{2n^2+1}\right) = \ln\tfrac{1}{2} \neq 0.$$

46. $\displaystyle\sum_{k=0}^{\infty} (\sqrt{2})^{-k} = \sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^k$ is a geometric series with first term $a = \left(\dfrac{1}{\sqrt{2}}\right)^0 = 1$ and ratio $r = \dfrac{1}{\sqrt{2}}$. Since $|r| < 1$, the

series converges to $\dfrac{1}{1-1/\sqrt{2}} = \dfrac{\sqrt{2}}{\sqrt{2}-1} \approx 3.414.$

47. $\displaystyle\sum_{n=1}^{\infty} \arctan n$ diverges by the Test for Divergence since $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan n = \tfrac{\pi}{2} \neq 0.$

48. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right)$ diverges because $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n} = 2\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. (If it converged, then $\displaystyle\frac{1}{2}\cdot 2\sum_{n=1}^{\infty}\frac{1}{n}$ would also converge by Theorem 8(i), but we know from Example 9 that the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.) If the given series converges, then the difference $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right) - \sum_{n=1}^{\infty}\frac{3}{5^n}$ must converge (since $\displaystyle\sum_{n=1}^{\infty}\frac{3}{5^n}$ is a convergent geometric series) and equal $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n}$, but we have just seen that $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n}$ diverges, so the given series must also diverge.

49. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{e^n} = \sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n$ is a geometric series with first term $a = \dfrac{1}{e}$ and ratio $r = \dfrac{1}{e}$. Since $|r| = \dfrac{1}{e} < 1$, the series converges

to $\dfrac{1/e}{1-1/e} = \dfrac{1/e}{1-1/e}\cdot\dfrac{e}{e} = \dfrac{1}{e-1}$. By Example 8, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$. Thus, by Theorem 8(ii),

$$\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right) = \sum_{n=1}^{\infty}\frac{1}{e^n} + \sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \frac{1}{e-1}+1 = \frac{1}{e-1}+\frac{e-1}{e-1} = \frac{e}{e-1}.$$

50. $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ diverges by the Test for Divergence since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{n^2} = \lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty \neq 0.$

51. (a) Many people would guess that $x < 1$, but note that x consists of an infinite number of 9s.

(b) $x = 0.99999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10{,}000} + \cdots = \sum_{n=1}^{\infty} \frac{9}{10^n}$, which is a geometric series with $a_1 = 0.9$ and $r = 0.1$. Its sum is $\frac{0.9}{1-0.1} = \frac{0.9}{0.9} = 1$, that is, $x = 1$.

(c) The number 1 has two decimal representations, $1.00000\ldots$ and $0.99999\ldots$.

(d) Except for 0, all rational numbers that have a terminating decimal representation can be written in more than one way. For example, 0.5 can be written as $0.49999\ldots$ as well as $0.50000\ldots$.

52. $a_1 = 1$, $a_n = (5-n)a_{n-1} \quad\Rightarrow\quad a_2 = (5-2)a_1 = 3(1) = 3$, $a_3 = (5-3)a_2 = 2(3) = 6$, $a_4 = (5-4)a_3 = 1(6) = 6$, $a_5 = (5-5)a_4 = 0$, and all succeeding terms equal 0. Thus, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{4} a_n = 1 + 3 + 6 + 6 = 16$.

53. $0.\overline{8} = \frac{8}{10} + \frac{8}{10^2} + \cdots$ is a geometric series with $a = \frac{8}{10}$ and $r = \frac{1}{10}$. It converges to $\frac{a}{1-r} = \frac{8/10}{1-1/10} = \frac{8}{9}$.

54. $0.\overline{46} = \frac{46}{100} + \frac{46}{100^2} + \cdots$ is a geometric series with $a = \frac{46}{100}$ and $r = \frac{1}{100}$. It converges to $\frac{a}{1-r} = \frac{46/100}{1-1/100} = \frac{46}{99}$.

55. $2.\overline{516} = 2 + \frac{516}{10^3} + \frac{516}{10^6} + \cdots$. Now $\frac{516}{10^3} + \frac{516}{10^6} + \cdots$ is a geometric series with $a = \frac{516}{10^3}$ and $r = \frac{1}{10^3}$. It converges to $\frac{a}{1-r} = \frac{516/10^3}{1-1/10^3} = \frac{516/10^3}{999/10^3} = \frac{516}{999}$. Thus, $2.\overline{516} = 2 + \frac{516}{999} = \frac{2514}{999} = \frac{838}{333}$.

56. $10.1\overline{35} = 10.1 + \frac{35}{10^3} + \frac{35}{10^5} + \cdots$. Now $\frac{35}{10^3} + \frac{35}{10^5} + \cdots$ is a geometric series with $a = \frac{35}{10^3}$ and $r = \frac{1}{10^2}$. It converges to $\frac{a}{1-r} = \frac{35/10^3}{1-1/10^2} = \frac{35/10^3}{99/10^2} = \frac{35}{990}$. Thus, $10.1\overline{35} = 10.1 + \frac{35}{990} = \frac{9999+35}{990} = \frac{10{,}034}{990} = \frac{5017}{495}$.

57. $1.234\overline{567} = 1.234 + \frac{567}{10^6} + \frac{567}{10^9} + \cdots$. Now $\frac{567}{10^6} + \frac{567}{10^9} + \cdots$ is a geometric series with $a = \frac{567}{10^6}$ and $r = \frac{1}{10^3}$. It converges to $\frac{a}{1-r} = \frac{567/10^6}{1-1/10^3} = \frac{567/10^6}{999/10^3} = \frac{567}{999{,}000} = \frac{21}{37{,}000}$. Thus,

$$1.234\overline{567} = 1.234 + \frac{21}{37{,}000} = \frac{1234}{1000} + \frac{21}{37{,}000} = \frac{45{,}658}{37{,}000} + \frac{21}{37{,}000} = \frac{45{,}679}{37{,}000}.$$

58. $5.\overline{71358} = 5 + \frac{71{,}358}{10^5} + \frac{71{,}358}{10^{10}} + \cdots$. Now $\frac{71{,}358}{10^5} + \frac{71{,}358}{10^{10}} + \cdots$ is a geometric series with $a = \frac{71{,}358}{10^5}$ and $r = \frac{1}{10^5}$. It converges to $\frac{a}{1-r} = \frac{71{,}358/10^5}{1-1/10^5} = \frac{71{,}358/10^5}{99{,}999/10^5} = \frac{71{,}358}{99{,}999} = \frac{23{,}786}{33{,}333}$. Thus,

$$5.\overline{71358} = 5 + \frac{23{,}786}{33{,}333} = \frac{166{,}665}{33{,}333} + \frac{23{,}786}{33{,}333} = \frac{190{,}451}{33{,}333}.$$

59. $\sum\limits_{n=1}^{\infty}(-5)^n x^n = \sum\limits_{n=1}^{\infty}(-5x)^n$ is a geometric series with $r = -5x$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|-5x| < 1$ $\Leftrightarrow$ $|x| < \frac{1}{5}$, that is, $-\frac{1}{5} < x < \frac{1}{5}$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{-5x}{1-(-5x)} = \dfrac{-5x}{1+5x}$.

60. $\sum\limits_{n=1}^{\infty}(x+2)^n$ is a geometric series with $r = x+2$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|x+2| < 1$ $\Leftrightarrow$ $-1 < x+2 < 1$ $\Leftrightarrow$ $-3 < x < -1$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{x+2}{1-(x+2)} = \dfrac{x+2}{-x-1}$.

61. $\sum\limits_{n=0}^{\infty}\dfrac{(x-2)^n}{3^n} = \sum\limits_{n=0}^{\infty}\left(\dfrac{x-2}{3}\right)^n$ is a geometric series with $r = \dfrac{x-2}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\left|\dfrac{x-2}{3}\right| < 1$ $\Leftrightarrow$ $-1 < \dfrac{x-2}{3} < 1$ $\Leftrightarrow$ $-3 < x-2 < 3$ $\Leftrightarrow$ $-1 < x < 5$. In that case, the sum of the series is

$$\frac{a}{1-r} = \frac{1}{1-\dfrac{x-2}{3}} = \frac{1}{\dfrac{3-(x-2)}{3}} = \frac{3}{5-x}.$$

62. $\sum\limits_{n=0}^{\infty}(-4)^n(x-5)^n = \sum\limits_{n=0}^{\infty}[-4(x-5)]^n$ is a geometric series with $r = -4(x-5)$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|-4(x-5)| < 1$ $\Leftrightarrow$ $|x-5| < \frac{1}{4}$ $\Leftrightarrow$ $-\frac{1}{4} < x-5 < \frac{1}{4}$ $\Leftrightarrow$ $\frac{19}{4} < x < \frac{21}{4}$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{1}{1-[-4(x-5)]} = \dfrac{1}{4x-19}$.

63. $\sum\limits_{n=0}^{\infty}\dfrac{2^n}{x^n} = \sum\limits_{n=0}^{\infty}\left(\dfrac{2}{x}\right)^n$ is a geometric series with $r = \dfrac{2}{x}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\left|\dfrac{2}{x}\right| < 1$ $\Leftrightarrow$ $2 < |x|$ $\Leftrightarrow$ $x > 2$ or $x < -2$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{1}{1-2/x} = \dfrac{x}{x-2}$.

64. $\sum\limits_{n=0}^{\infty}\dfrac{x^n}{2^n} = \sum\limits_{n=0}^{\infty}\left(\dfrac{x}{2}\right)^n$ is a geometric series with $r = \dfrac{x}{2}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\left|\dfrac{x}{2}\right| < 1$ $\Leftrightarrow$ $-1 < \dfrac{x}{2} < 1$ $\Leftrightarrow$ $-2 < x < 2$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{1}{1-\frac{x}{2}} = \dfrac{2}{2-x}$.

65. $\sum\limits_{n=0}^{\infty}e^{nx} = \sum\limits_{n=0}^{\infty}(e^x)^n$ is a geometric series with $r = e^x$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|e^x| < 1$ $\Leftrightarrow$ $-1 < e^x < 1$ $\Leftrightarrow$ $0 < e^x < 1$ $\Leftrightarrow$ $x < 0$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{1}{1-e^x}$.

66. $\sum\limits_{n=0}^{\infty}\dfrac{\sin^n x}{3^n} = \sum\limits_{n=0}^{\infty}\left(\dfrac{\sin x}{3}\right)^n$ is a geometric series with $r = \dfrac{\sin x}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\left|\dfrac{\sin x}{3}\right| < 1$ $\Leftrightarrow$ $|\sin x| < 3$, which is true for all x. Thus, the sum of the series is $\dfrac{a}{1-r} = \dfrac{1}{1-(\sin x)/3} = \dfrac{3}{3-\sin x}$.

67. After defining f, We use `convert(f,parfrac);` in Maple or `Apart` in Mathematica to find that the general term is $\dfrac{3n^2+3n+1}{(n^2+n)^3} = \dfrac{1}{n^3} - \dfrac{1}{(n+1)^3}$. So the nth partial sum is

$$s_n = \sum_{k=1}^{n}\left(\frac{1}{k^3} - \frac{1}{(k+1)^3}\right) = \left(1 - \frac{1}{2^3}\right) + \left(\frac{1}{2^3} - \frac{1}{3^3}\right) + \cdots + \left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) = 1 - \frac{1}{(n+1)^3}$$

The series converges to $\lim_{n\to\infty} s_n = 1$. This can be confirmed by directly computing the sum using `sum(f,n=1..infinity);` (in Maple) or `Sum[f,{n,1,Infinity}]` (in Mathematica).

68. See Exercise 67 for specific CAS commands.

$\dfrac{1}{n^5-5n^3+4n} = \dfrac{1}{24(n-2)} + \dfrac{1}{24(n+2)} - \dfrac{1}{6(n-1)} - \dfrac{1}{6(n+1)} + \dfrac{1}{4n}$. So the nth partial sum is

$$\begin{aligned} s_n &= \frac{1}{24}\sum_{k=3}^{n}\left(\frac{1}{k-2} - \frac{4}{k-1} + \frac{6}{k} - \frac{4}{k+1} + \frac{1}{k+2}\right) \\ &= \frac{1}{24}\left[\left(\frac{1}{1} - \frac{4}{2} + \frac{6}{3} - \frac{4}{4} + \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-2} - \frac{4}{n-1} + \frac{6}{n} - \frac{4}{n+1} + \frac{1}{n+2}\right)\right] \end{aligned}$$

The terms with denominator 5 or greater cancel, except for a few terms with n in the denominator. So as $n \to \infty$,

$s_n \to \dfrac{1}{24}\left(\dfrac{1}{1} - \dfrac{3}{2} + \dfrac{3}{3} - \dfrac{1}{4}\right) = \dfrac{1}{24}\left(\dfrac{1}{4}\right) = \dfrac{1}{96}$.

69. For $n = 1$, $a_1 = 0$ since $s_1 = 0$. For $n > 1$,

$$a_n = s_n - s_{n-1} = \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1} = \frac{(n-1)n - (n+1)(n-2)}{(n+1)n} = \frac{2}{n(n+1)}$$

Also, $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-1/n}{1+1/n} = 1$.

70. $a_1 = s_1 = 3 - \frac{1}{2} = \frac{5}{2}$. For $n \neq 1$,

$$a_n = s_n - s_{n-1} = \left(3 - n2^{-n}\right) - \left[3 - (n-1)2^{-(n-1)}\right] = -\frac{n}{2^n} + \frac{n-1}{2^{n-1}} \cdot \frac{2}{2} = \frac{2(n-1)}{2^n} - \frac{n}{2^n} = \frac{n-2}{2^n}$$

Also, $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(3 - \frac{n}{2^n}\right) = 3$ because $\displaystyle\lim_{x\to\infty} \frac{x}{2^x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0$.

71. (a) The quantity of the drug in the body after the first tablet is 100 mg. After 8 hours, the body eliminates 75% of the drug, which means 25% remains. Thus, after the second tablet, there is 100 mg plus 25% of the first 100-mg tablet, that is, $100 + 0.25(100) = 125$ mg. After the third tablet, the quantity is $100 + 0.25(125)$ or, equivalently, $100 + 100(0.25) + 100(0.25)^2$. Either expression gives 131.25 mg.

(b) From part (a), we see that $Q_{n+1} = 100 + 0.25\,Q_n$.

(c) $$\begin{aligned} Q_n &= 100 + 100(0.25)^1 + 100(0.25)^2 + \cdots + 100(0.25)^{n-1} \\ &= \sum_{i=1}^{n} 100(0.25)^{i-1} \qquad [\text{geometric with } a = 100 \text{ and } r = 0.25] \end{aligned}$$

The quantity of the antibiotic that remains in the body in the long run is $\displaystyle\lim_{n\to\infty} Q_n = \frac{100}{1-0.25} = \frac{100}{0.75} = 133.\overline{3}$ mg.

72. (a) The concentration of the drug after the first injection is 1.5 mg/L. “Reduced by 90%” is the same as 10% remains, so the concentration after the second injection is $1.5 + 0.10(1.5) = 1.65$ mg/L. The concentration after the third injection is $1.5 + 0.10(1.65)$, or, equivalently, $1.5 + 1.5(0.10) + 1.5(0.10)^2$. Either expression gives us 1.665 mg/L.

(b) $C_n = 1.5 + 1.5(0.10)^1 + 1.5(0.10)^2 + \cdots + 1.5(0.10)^{n-1}$

$= \sum_{i=1}^{n} 1.5(0.10)^{i-1}$ [geometric with $a = 1.5$ and $r = 0.10$].

By (3), $C_n = \dfrac{1.5[1-(0.10)^n]}{1-0.10} = \dfrac{1.5}{0.9}[1-(0.10)^n] = \dfrac{5}{3}[1-(0.10)^n]$ mg/L.

(c) The limiting value of the concentration is $\lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{5}{3}[1-(0.10)^n] = \frac{5}{3}(1-0) = \frac{5}{3}$ mg/L.

73. (a) The quantity of the drug in the body after the first tablet is 150 mg. "Eliminates 95%" is the same as retains 5%, so after the second tablet, there is 150 mg plus 5% of the first 150- mg tablet, that is, $[150 + 150(0.05)]$ mg. After the third tablet, the quantity is $[150 + 150(0.05) + 150(0.05)^2] = 157.875$ mg. After n tablets, the quantity (in mg) is $150 + 150(0.05) + \cdots + 150(0.05)^{n-1}$. We can use Formula 3 to write this as $\dfrac{150(1-0.05^n)}{1-0.05} = \dfrac{3000}{19}(1-0.05^n)$.

(b) The number of milligrams remaining in the body in the long run is $\lim_{n\to\infty}\left[\frac{3000}{19}(1-0.05^n)\right] = \frac{3000}{19}(1-0) \approx 157.895$, only 0.02 mg more than the amount after 3 tablets.

74. (a) The residual concentration just before the second injection is De^{-aT}; before the third, $De^{-aT} + De^{-a2T}$; before the $(n+1)$st, $De^{-aT} + De^{-a2T} + \cdots + De^{-anT}$. This sum is equal to $\dfrac{De^{-aT}\left(1-e^{-anT}\right)}{1-e^{-aT}}$ [Formula 3].

(b) The limiting pre-injection concentration is $\lim_{n\to\infty} \dfrac{De^{-aT}\left(1-e^{-anT}\right)}{1-e^{-aT}} = \dfrac{De^{-aT}(1-0)}{1-e^{-aT}} \cdot \dfrac{e^{aT}}{e^{aT}} = \dfrac{D}{e^{aT}-1}$.

(c) $\dfrac{D}{e^{aT}-1} \ge C \quad\Rightarrow\quad D \ge C\left(e^{aT}-1\right)$, so the minimal dosage is $D = C\left(e^{aT}-1\right)$.

75. (a) The first step in the chain occurs when the local government spends D dollars. The people who receive it spend a fraction c of those D dollars, that is, Dc dollars. Those who receive the Dc dollars spend a fraction c of it, that is, Dc^2 dollars. Continuing in this way, we see that the total spending after n transactions is

$S_n = D + Dc + Dc^2 + \cdots + Dc^{n-1} = \dfrac{D(1-c^n)}{1-c}$ by (3).

(b) $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{D(1-c^n)}{1-c} = \dfrac{D}{1-c}\lim_{n\to\infty}(1-c^n) = \dfrac{D}{1-c}$ $\left[\text{since } 0 < c < 1 \quad\Rightarrow\quad \lim_{n\to\infty} c^n = 0\right]$

$= \dfrac{D}{s}$ [since $c + s = 1$] $= kD$ [since $k = 1/s$]

If $c = 0.8$, then $s = 1 - c = 0.2$ and the multiplier is $k = 1/s = 5$.

76. (a) Initially, the ball falls a distance H, then rebounds a distance rH, falls rH, rebounds r^2H, falls r^2H, etc. The total distance it travels is

$$H + 2rH + 2r^2H + 2r^3H + \cdots = H\left(1 + 2r + 2r^2 + 2r^3 + \cdots\right) = H\left[1 + 2r\left(1 + r + r^2 + \cdots\right)\right]$$
$$= H\left[1 + 2r\left(\frac{1}{1-r}\right)\right] = H\left(\frac{1+r}{1-r}\right) \text{ meters}$$

(b) We know that the ball falls $\frac{1}{2}gt^2$ meters in t seconds, where $g = 9.8\text{ m/s}^2$ is the gravitational acceleration. Thus, the ball falls h meters in $t = \sqrt{2h/g}$ seconds. The total travel time in seconds is

$$\begin{aligned}\sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2H}{g}r} + 2\sqrt{\frac{2H}{g}r^2} + 2\sqrt{\frac{2H}{g}r^3} + \cdots &= \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r} + 2\sqrt{r}^{\,2} + 2\sqrt{r}^{\,3} + \cdots\right] \\ &= \sqrt{\frac{2H}{g}}\left(1 + 2\sqrt{r}\left[1 + \sqrt{r} + \sqrt{r}^{\,2} + \cdots\right]\right) \\ &= \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r}\left(\frac{1}{1-\sqrt{r}}\right)\right] = \sqrt{\frac{2H}{g}}\frac{1+\sqrt{r}}{1-\sqrt{r}}\end{aligned}$$

(c) It will help to make a chart of the time for each descent and each rebound of the ball, together with the velocity just before and just after each bounce. Recall that the time in seconds needed to fall h meters is $\sqrt{2h/g}$. The ball hits the ground with velocity $-g\sqrt{2h/g} = -\sqrt{2hg}$ (taking the upward direction to be positive) and rebounds with velocity $kg\sqrt{2h/g} = k\sqrt{2hg}$, taking time $k\sqrt{2h/g}$ to reach the top of its bounce, where its velocity is 0. At that point, its height is k^2h. All these results follow from the formulas for vertical motion with gravitational acceleration $-g$:

$$\frac{d^2y}{dt^2} = -g \quad\Rightarrow\quad v = \frac{dy}{dt} = v_0 - gt \quad\Rightarrow\quad y = y_0 + v_0t - \tfrac{1}{2}gt^2.$$

number of descent	time of descent	speed before bounce	speed after bounce	time of ascent	peak height
1	$\sqrt{2H/g}$	$\sqrt{2Hg}$	$k\sqrt{2Hg}$	$k\sqrt{2H/g}$	k^2H
2	$\sqrt{2k^2H/g}$	$\sqrt{2k^2Hg}$	$k\sqrt{2k^2Hg}$	$k\sqrt{2k^2H/g}$	k^4H
3	$\sqrt{2k^4H/g}$	$\sqrt{2k^4Hg}$	$k\sqrt{2k^4Hg}$	$k\sqrt{2k^4H/g}$	k^6H
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

The total travel time in seconds is

$$\begin{aligned}\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + \cdots &= \sqrt{\frac{2H}{g}}\left(1 + 2k + 2k^2 + 2k^3 + \cdots\right) \\ &= \sqrt{\frac{2H}{g}}\left[1 + 2k(1 + k + k^2 + \cdots)\right] \\ &= \sqrt{\frac{2H}{g}}\left[1 + 2k\left(\frac{1}{1-k}\right)\right] = \sqrt{\frac{2H}{g}}\frac{1+k}{1-k}\end{aligned}$$

Another method: We could use part (b). At the top of the bounce, the height is $k^2h = rh$, so $\sqrt{r} = k$ and the result follows from part (b).

77. $\sum\limits_{n=2}^{\infty}(1+c)^{-n}$ is a geometric series with $a = (1+c)^{-2}$ and $r = (1+c)^{-1}$, so the series converges when $\left|(1+c)^{-1}\right| < 1 \quad\Leftrightarrow\quad |1+c| > 1 \quad\Leftrightarrow\quad 1+c > 1$ or $1+c < -1 \quad\Leftrightarrow\quad c > 0$ or $c < -2$. We calculate the sum of the series and set it equal to 2: $\dfrac{(1+c)^{-2}}{1-(1+c)^{-1}} = 2 \quad\Leftrightarrow\quad \left(\dfrac{1}{1+c}\right)^2 = 2 - 2\left(\dfrac{1}{1+c}\right) \quad\Leftrightarrow\quad 1 = 2(1+c)^2 - 2(1+c) \quad\Leftrightarrow$

$2c^2 + 2c - 1 = 0 \quad\Leftrightarrow\quad c = \frac{-2\pm\sqrt{12}}{4} = \frac{\pm\sqrt{3}-1}{2}$. However, the negative root is inadmissible because $-2 < \frac{-\sqrt{3}-1}{2} < 0$.

So $c = \frac{\sqrt{3}-1}{2}$.

78. $\sum\limits_{n=0}^{\infty} e^{nc} = \sum\limits_{n=0}^{\infty} (e^c)^n$ is a geometric series with $a = (e^c)^0 = 1$ and $r = e^c$. If $e^c < 1$, it has sum $\dfrac{1}{1-e^c}$, so $\dfrac{1}{1-e^c} = 10 \quad\Rightarrow$

$\frac{1}{10} = 1 - e^c \quad\Rightarrow\quad e^c = \frac{9}{10} \quad\Rightarrow\quad c = \ln\frac{9}{10}$.

79. Assume the harmonic series converges with sum S. Following the outlined method of proof, we have

$$S = \left(1+\frac{1}{2}\right) + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}\right) + \cdots > \left(\frac{1}{2}+\frac{1}{2}\right) + \left(\frac{1}{4}+\frac{1}{4}\right) + \left(\frac{1}{6}+\frac{1}{6}\right) + \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = S.$$

This indicates that $S > S \quad\Leftrightarrow\quad 0 > 0$, which is a contradiction. Thus, the assumption was incorrect and the harmonic series diverges.

80. From the hint, observe that $\dfrac{1}{n-1} + \dfrac{1}{n+1} = \dfrac{(n+1)+(n-1)}{(n-1)(n+1)} = \dfrac{2n}{n^2-1} > \dfrac{2n}{n^2} = \dfrac{2}{n}$ for $n \geq 3$. Now, assuming the harmonic series converges with sum S, we have

$$\begin{aligned} S &= 1 + \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right) + \left(\frac{1}{8}+\frac{1}{9}+\frac{1}{10}\right) + \cdots \\ &= 1 + \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{3}\right) + \left(\frac{1}{5}+\frac{1}{7}+\frac{1}{6}\right) + \left(\frac{1}{8}+\frac{1}{10}+\frac{1}{9}\right) + \cdots \\ &> 1 + \left(\frac{2}{3}+\frac{1}{3}\right) + \left(\frac{2}{6}+\frac{1}{6}\right) + \left(\frac{2}{9}+\frac{1}{9}\right) + \cdots = 1 + \left(1+\frac{1}{2}+\frac{1}{3}+\cdots\right) = 1 + S. \end{aligned}$$

This indicates that $S > 1 + S \quad\Leftrightarrow\quad 0 > 1$, which is a contradiction. Thus, the assumption was incorrect and the harmonic series diverges.

81. From the hint, we'll show that $e^x > 1 + x$ for $x > 0$ by proving that $f(x) = e^x - (1+x) > 0$ for $x > 0$. Since $f(0) = e^0 - 1 = 0$ and $f'(x) = e^x - 1 > 0$ when $x > 0$, f is increasing on $(0, \infty) \quad\Rightarrow\quad$ when $x > 0$, $f(x) > 0 \quad\Rightarrow$ $e^x - (1+x) > 0$ for $x > 0 \quad\Rightarrow\quad e^x > 1 + x$ for $x > 0$. Now,

$$\begin{aligned} e^{1+(1/2)+(1/3)+\cdots+(1/n)} &= e^1 e^{1/2} e^{1/3} \cdots e^{1/n} > (1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right) && \left[\text{since } e^x > 1+x\right] \\ &= \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\cdots\left(\frac{n+1}{n}\right) && \left[\begin{array}{l}\text{each denominator cancels with the} \\ \text{numerator of the preceding fraction}\end{array}\right] \\ &= n + 1 \end{aligned}$$

[continued]

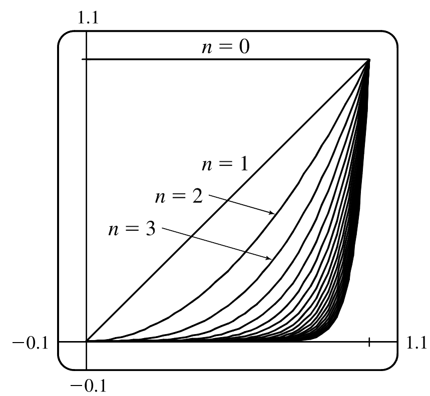
Assuming the harmonic series converges with sum S, we take the limit as $n \to \infty$ to get

$\lim\limits_{n\to\infty} e^{1+(1/2)+(1/3)+\cdots+(1/n)} > \lim\limits_{n\to\infty}(n+1) \quad \Rightarrow$

$e^{\lim\limits_{n\to\infty}[1+(1/2)+(1/3)+\cdots+(1/n)]} > \lim\limits_{n\to\infty}(n+1) \quad [\text{since } e^x \text{ is continuous}] \quad \Rightarrow \quad e^S > \lim\limits_{n\to\infty}(n+1)$. The limit does not exist since $n+1 \to \infty$ as $n \to \infty$. e^S is larger than the value of the limit, so S is not finite valued, which is a contradiction. Thus, the assumption was incorrect and the harmonic series diverges.

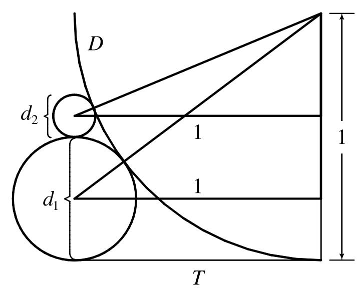
82. The area between $y = x^{n-1}$ and $y = x^n$ for $0 \le x \le 1$ is

$$\int_0^1 (x^{n-1} - x^n)\,dx = \left[\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n} - \frac{1}{n+1}$$
$$= \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)}$$



We can see from the diagram that as $n \to \infty$, the sum of the areas between the successive curves approaches the area of the unit square, that is, 1. So $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+1)} = 1$.

83. Let d_n be the diameter of C_n. We draw lines from the centers of the C_i to the center of D (or C), and using the Pythagorean Theorem, we can write



$1^2 + \left(1 - \tfrac{1}{2}d_1\right)^2 = \left(1 + \tfrac{1}{2}d_1\right)^2 \quad \Leftrightarrow$

$1 = \left(1 + \tfrac{1}{2}d_1\right)^2 - \left(1 - \tfrac{1}{2}d_1\right)^2 = 2d_1$ [difference of squares] $\quad \Rightarrow \quad d_1 = \tfrac{1}{2}$.

Similarly,

$1 = \left(1 + \tfrac{1}{2}d_2\right)^2 - \left(1 - d_1 - \tfrac{1}{2}d_2\right)^2 = 2d_2 + 2d_1 - d_1^2 - d_1 d_2$

$\quad = (2 - d_1)(d_1 + d_2) \quad \Leftrightarrow$

$d_2 = \dfrac{1}{2-d_1} - d_1 = \dfrac{(1-d_1)^2}{2-d_1}$, $1 = \left(1 + \tfrac{1}{2}d_3\right)^2 - \left(1 - d_1 - d_2 - \tfrac{1}{2}d_3\right)^2 \quad \Leftrightarrow \quad d_3 = \dfrac{[1-(d_1+d_2)]^2}{2-(d_1+d_2)}$, and in general,

$d_{n+1} = \dfrac{\left(1 - \sum_{i=1}^n d_i\right)^2}{2 - \sum_{i=1}^n d_i}$. If we actually calculate d_2 and d_3 from the formulas above, we find that they are $\dfrac{1}{6} = \dfrac{1}{2\cdot 3}$ and $\dfrac{1}{12} = \dfrac{1}{3\cdot 4}$ respectively, so we suspect that in general, $d_n = \dfrac{1}{n(n+1)}$. To prove this, we use induction: Assume that for all $k \le n$, $d_k = \dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$. Then $\sum\limits_{i=1}^{n} d_i = 1 - \dfrac{1}{n+1} = \dfrac{n}{n+1}$ [telescoping sum]. Substituting this into our formula for d_{n+1}, we get $d_{n+1} = \dfrac{\left[1 - \dfrac{n}{n+1}\right]^2}{2 - \left(\dfrac{n}{n+1}\right)} = \dfrac{\dfrac{1}{(n+1)^2}}{\dfrac{n+2}{n+1}} = \dfrac{1}{(n+1)(n+2)}$, and the induction is complete.

Now, we observe that the partial sums $\sum_{i=1}^{n} d_i$ of the diameters of the circles approach 1 as $n \to \infty$; that is,

$\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+1)} = 1$, which is what we wanted to prove.

84. $|CD| = b\sin\theta$, $|DE| = |CD|\sin\theta = b\sin^2\theta$, $|EF| = |DE|\sin\theta = b\sin^3\theta$, $\ldots$. Therefore,

$|CD| + |DE| + |EF| + |FG| + \cdots = b\sum\limits_{n=1}^{\infty}\sin^n\theta = b\left(\dfrac{\sin\theta}{1-\sin\theta}\right)$ since this is a geometric series with $r = \sin\theta$

and $|\sin\theta| < 1$ $\left[\text{because } 0 < \theta < \frac{\pi}{2}\right]$.

85. The series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ diverges (geometric series with $r = -1$), so we cannot say that $0 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$.

86. If $\sum\limits_{n=1}^{\infty} a_n$ is convergent, then $\lim\limits_{n\to\infty} a_n = 0$ by Theorem 6, so $\lim\limits_{n\to\infty}\dfrac{1}{a_n} \neq 0$, and so $\sum\limits_{n=1}^{\infty}\dfrac{1}{a_n}$ is divergent by the Test for Divergence.

87. (a) $\sum\limits_{n=1}^{\infty} ca_n = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} ca_i = \lim\limits_{n\to\infty} c\sum\limits_{i=1}^{n} a_i = c\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} a_i = c\sum\limits_{n=1}^{\infty} a_n$, which exists by hypothesis.

(b)
$$\begin{aligned}\sum_{n=1}^{\infty}(a_n - b_n) &= \lim_{n\to\infty}\sum_{i=1}^{n}(a_i - b_i) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\right) = \lim_{n\to\infty}\sum_{i=1}^{n} a_i - \lim_{n\to\infty}\sum_{i=1}^{n} b_i \\ &= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n, \text{ which exists by hypothesis.}\end{aligned}$$

88. If $\sum ca_n$ were convergent, then $\sum(1/c)(ca_n) = \sum a_n$ would be also, by Theorem 8(i). But this is not the case, so $\sum ca_n$ must diverge.

89. Suppose on the contrary that $\sum(a_n + b_n)$ converges. Then $\sum(a_n + b_n)$ and $\sum a_n$ are convergent series. So by Theorem 8(iii), $\sum[(a_n + b_n) - a_n]$ would also be convergent. But $\sum[(a_n + b_n) - a_n] = \sum b_n$, a contradiction, since $\sum b_n$ is given to be divergent.

90. No. For example, take $\sum a_n = \sum n$ and $\sum b_n = \sum(-n)$, which both diverge, yet $\sum(a_n + b_n) = \sum 0$, which converges with sum 0.

91. The partial sums $\{s_n\}$ form an increasing sequence, since $s_n - s_{n-1} = a_n > 0$ for all n. Also, the sequence $\{s_n\}$ is bounded since $s_n \leq 1000$ for all n. So by the Monotonic Sequence Theorem, the sequence of partial sums converges, that is, the series $\sum a_n$ is convergent.

92. (a) RHS $= \dfrac{1}{f_{n-1}f_n} - \dfrac{1}{f_nf_{n+1}} = \dfrac{f_nf_{n+1} - f_nf_{n-1}}{f_n^2f_{n-1}f_{n+1}} = \dfrac{f_{n+1} - f_{n-1}}{f_nf_{n-1}f_{n+1}} = \dfrac{(f_{n-1} + f_n) - f_{n-1}}{f_nf_{n-1}f_{n+1}} = \dfrac{1}{f_{n-1}f_{n+1}} =$ LHS

(b)
$$\begin{aligned}\sum_{n=2}^{\infty}\frac{1}{f_{n-1}f_{n+1}} &= \sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_nf_{n+1}}\right) \quad \text{[from part (a)]} \\ &= \lim_{n\to\infty}\left[\left(\frac{1}{f_1f_2} - \frac{1}{f_2f_3}\right) + \left(\frac{1}{f_2f_3} - \frac{1}{f_3f_4}\right) + \left(\frac{1}{f_3f_4} - \frac{1}{f_4f_5}\right) + \cdots + \left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_nf_{n+1}}\right)\right] \\ &= \lim_{n\to\infty}\left(\frac{1}{f_1f_2} - \frac{1}{f_nf_{n+1}}\right) = \frac{1}{f_1f_2} - 0 = \frac{1}{1\cdot 1} = 1 \quad \text{because } f_n \to \infty \text{ as } n \to \infty.\end{aligned}$$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{f_n}{f_{n-1}f_n} - \frac{f_n}{f_nf_{n+1}}\right)$ [as above]

$$= \sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)$$

$$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1}-\frac{1}{f_3}\right)+\left(\frac{1}{f_2}-\frac{1}{f_4}\right)+\left(\frac{1}{f_3}-\frac{1}{f_5}\right)+\left(\frac{1}{f_4}-\frac{1}{f_6}\right)+\cdots+\left(\frac{1}{f_{n-1}}-\frac{1}{f_{n+1}}\right)\right]$$

$$= \lim_{n\to\infty}\left(\frac{1}{f_1}+\frac{1}{f_2}-\frac{1}{f_n}-\frac{1}{f_{n+1}}\right) = 1+1-0-0 = 2 \quad \text{because } f_n \to \infty \text{ as } n \to \infty.$$

93. (a) At the first step, only the interval $\left(\frac{1}{3},\frac{2}{3}\right)$ (length $\frac{1}{3}$) is removed. At the second step, we remove the intervals $\left(\frac{1}{9},\frac{2}{9}\right)$ and $\left(\frac{7}{9},\frac{8}{9}\right)$, which have a total length of $2\cdot\left(\frac{1}{3}\right)^2$. At the third step, we remove 2^2 intervals, each of length $\left(\frac{1}{3}\right)^3$. In general, at the *n*th step we remove 2^{n-1} intervals, each of length $\left(\frac{1}{3}\right)^n$, for a length of $2^{n-1}\cdot\left(\frac{1}{3}\right)^n = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$. Thus, the total length of all removed intervals is $\sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{1/3}{1-2/3} = 1$ $\left[\text{geometric series with } a=\frac{1}{3} \text{ and } r=\frac{2}{3}\right]$. Notice that at the nth step, the leftmost interval that is removed is $\left(\left(\frac{1}{3}\right)^n, \left(\frac{2}{3}\right)^n\right)$, so we never remove 0, and 0 is in the Cantor set. Also, the rightmost interval removed is $\left(1-\left(\frac{2}{3}\right)^n, 1-\left(\frac{1}{3}\right)^n\right)$, so 1 is never removed. Some other numbers in the Cantor set are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$, $\frac{2}{9}$, $\frac{7}{9}$, and $\frac{8}{9}$.

(b) The area removed at the first step is $\frac{1}{9}$; at the second step, $8\cdot\left(\frac{1}{9}\right)^2$; at the third step, $(8)^2\cdot\left(\frac{1}{9}\right)^3$. In general, the area removed at the nth step is $(8)^{n-1}\left(\frac{1}{9}\right)^n = \frac{1}{9}\left(\frac{8}{9}\right)^{n-1}$, so the total area of all removed squares is

$$\sum_{n=1}^{\infty}\frac{1}{9}\left(\frac{8}{9}\right)^{n-1} = \frac{1/9}{1-8/9} = 1.$$

94. (a)

a_1	1	2	4	1	1	1000
a_2	2	3	1	4	1000	1
a_3	1.5	2.5	2.5	2.5	500.5	500.5
a_4	1.75	2.75	1.75	3.25	750.25	250.75
a_5	1.625	2.625	2.125	2.875	625.375	375.625
a_6	1.6875	2.6875	1.9375	3.0625	687.813	313.188
a_7	1.65625	2.65625	2.03125	2.96875	656.594	344.406
a_8	1.67188	2.67188	1.98438	3.01563	672.203	328.797
a_9	1.66406	2.66406	2.00781	2.99219	664.398	336.602
a_{10}	1.66797	2.66797	1.99609	3.00391	668.301	332.699
a_{11}	1.66602	2.66602	2.00195	2.99805	666.350	334.650
a_{12}	1.66699	2.66699	1.99902	3.00098	667.325	333.675

The limits seem to be $\frac{5}{3}$, $\frac{8}{3}$, 2, 3, 667, and 334. Note that the limits appear to be "weighted" more toward a_2. In general, we guess that the limit is $\dfrac{a_1+2a_2}{3}$.

(b) $a_{n+1} - a_n = \frac{1}{2}(a_n + a_{n-1}) - a_n = -\frac{1}{2}(a_n - a_{n-1}) = -\frac{1}{2}\left[\frac{1}{2}(a_{n-1} + a_{n-2}) - a_{n-1}\right]$

$= -\frac{1}{2}\left[-\frac{1}{2}(a_{n-1} - a_{n-2})\right] = \cdots = \left(-\frac{1}{2}\right)^{n-1}(a_2 - a_1)$

Note that we have used the formula $a_k = \frac{1}{2}(a_{k-1} + a_{k-2})$ a total of $n-1$ times in this calculation, once for each k between 3 and $n+1$. Now we can write

$$\begin{aligned} a_n &= a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) \\ &= a_1 + \sum_{k=1}^{n-1}(a_{k+1} - a_k) = a_1 + \sum_{k=1}^{n-1}\left(-\tfrac{1}{2}\right)^{k-1}(a_2 - a_1) \end{aligned}$$

and so

$$\lim_{n\to\infty} a_n = a_1 + (a_2 - a_1)\sum_{k=1}^{\infty}\left(-\tfrac{1}{2}\right)^{k-1} = a_1 + (a_2 - a_1)\left[\frac{1}{1-(-1/2)}\right] = a_1 + \tfrac{2}{3}(a_2 - a_1) = \frac{a_1 + 2a_2}{3}.$$

95. (a) For $\sum_{n=1}^{\infty}\frac{n}{(n+1)!}$, $s_1 = \frac{1}{1\cdot 2} = \frac{1}{2}$, $s_2 = \frac{1}{2} + \frac{2}{1\cdot 2\cdot 3} = \frac{5}{6}$, $s_3 = \frac{5}{6} + \frac{3}{1\cdot 2\cdot 3\cdot 4} = \frac{23}{24}$,

$s_4 = \frac{23}{24} + \frac{4}{1\cdot 2\cdot 3\cdot 4\cdot 5} = \frac{119}{120}$. The denominators are $(n+1)!$, so a guess would be $s_n = \frac{(n+1)! - 1}{(n+1)!}$.

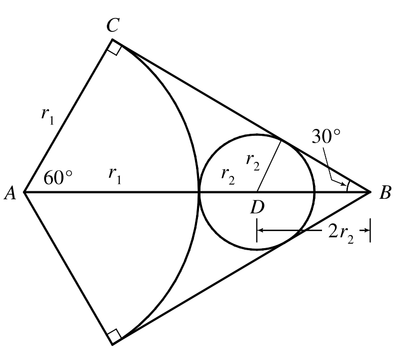
(b) For $n = 1$, $s_1 = \frac{1}{2} = \frac{2! - 1}{2!}$, so the formula holds for $n = 1$. Assume $s_k = \frac{(k+1)! - 1}{(k+1)!}$. Then

$$\begin{aligned} s_{k+1} &= \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+1)!(k+2)} = \frac{(k+2)! - (k+2) + k + 1}{(k+2)!} \\ &= \frac{(k+2)! - 1}{(k+2)!} \end{aligned}$$

Thus, the formula is true for $n = k+1$. So by induction, the guess is correct.

(c) $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{(n+1)! - 1}{(n+1)!} = \lim_{n\to\infty}\left[1 - \frac{1}{(n+1)!}\right] = 1$ and so $\sum_{n=1}^{\infty}\frac{n}{(n+1)!} = 1$.

96.



Let r_1 = radius of the large circle, r_2 = radius of next circle, and so on. From the figure we have $\angle BAC = 60°$ and $\cos 60° = r_1/|AB|$, so $|AB| = 2r_1$ and $|DB| = 2r_2$. Therefore, $2r_1 = r_1 + r_2 + 2r_2 \Rightarrow$ $r_1 = 3r_2$. In general, we have $r_{n+1} = \frac{1}{3}r_n$, so the total area is

$$\begin{aligned} A &= \pi r_1^2 + 3\pi r_2^2 + 3\pi r_3^2 + \cdots = \pi r_1^2 + 3\pi r_2^2\left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots\right) \\ &= \pi r_1^2 + 3\pi r_2^2\cdot\frac{1}{1 - 1/9} = \pi r_1^2 + \tfrac{27}{8}\pi r_2^2 \end{aligned}$$

Since the sides of the triangle have length 1, $|BC| = \frac{1}{2}$ and $\tan 30° = \frac{r_1}{1/2}$. Thus, $r_1 = \frac{\tan 30°}{2} = \frac{1}{2\sqrt{3}} \Rightarrow r_2 = \frac{1}{6\sqrt{3}}$,

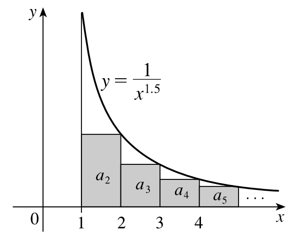
so $A = \pi\left(\frac{1}{2\sqrt{3}}\right)^2 + \frac{27\pi}{8}\left(\frac{1}{6\sqrt{3}}\right)^2 = \frac{\pi}{12} + \frac{\pi}{32} = \frac{11\pi}{96}$. The area of the triangle is $\frac{\sqrt{3}}{4}$, so the circles occupy about 83.1% of the area of the triangle.

11.3 The Integral Test and Estimates of Sums

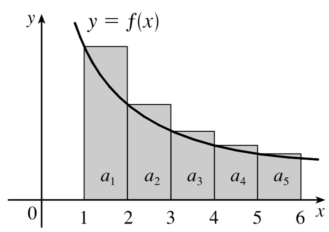
1. The picture shows that $a_2 = \dfrac{1}{2^{1.5}} < \displaystyle\int_1^2 \frac{1}{x^{1.5}}\,dx$,

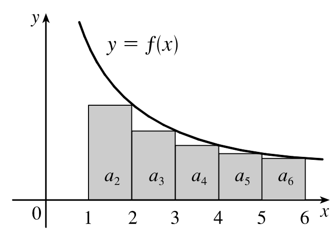
$a_3 = \dfrac{1}{3^{1.5}} < \displaystyle\int_2^3 \frac{1}{x^{1.5}}\,dx$, and so on, so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.5}} < \int_1^{\infty} \frac{1}{x^{1.5}}\,dx$.

The integral converges by (7.8.2) with $p = 1.5 > 1$, so the series converges.



2. From the first figure, we see that $\int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$. From the second figure, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx$. Thus, we have $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$.





3. The function $f(x) = x^{-3}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} x^{-3}\,dx = \lim_{t\to\infty} \int_1^t x^{-3}\,dx = \lim_{t\to\infty} \left[\frac{x^{-2}}{-2}\right]_1^t = \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{2}\right) = \frac{1}{2}.$$

Since this improper integral is convergent, the series $\displaystyle\sum_{n=1}^{\infty} n^{-3}$ is also convergent by the Integral Test.

4. The function $f(x) = x^{-0.3}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} x^{-0.3}\,dx = \lim_{t\to\infty} \int_1^t x^{-0.3}\,dx = \lim_{t\to\infty} \left[\frac{x^{0.7}}{0.7}\right]_1^t = \lim_{t\to\infty} \left(\frac{t^{0.7}}{0.7} - \frac{1}{0.7}\right) = \infty.$$

Since this improper integral is divergent, the series $\displaystyle\sum_{n=1}^{\infty} n^{-0.3}$ is also divergent by the Integral Test.

5. The function $f(x) = \dfrac{2}{5x-1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{2}{5x-1}\,dx = \lim_{t\to\infty} \int_1^t \frac{2}{5x-1}\,dx = \lim_{t\to\infty} \left[\frac{2}{5}\ln(5x-1)\right]_1^t = \lim_{t\to\infty} \left[\frac{2}{5}\ln(5t-1) - \frac{2}{5}\ln 4\right] = \infty.$$

Since this improper integral is divergent, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5n-1}$ is also divergent by the Integral Test.

6. The function $f(x) = \dfrac{1}{(3x-1)^4}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{1}{(3x-1)^4}\,dx = \lim_{t\to\infty} \int_1^t (3x-1)^{-4}\,dx = \lim_{t\to\infty} \left[\frac{1}{(-3)3}(3x-1)^{-3}\right]_1^t = \lim_{t\to\infty} \left[-\frac{1}{9(3t-1)^3} + \frac{1}{9\cdot 2^3}\right] = \frac{1}{72}.$$

Since this improper integral is convergent, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-1)^4}$ is also convergent by the Integral Test.

7. The function $f(x) = \dfrac{x^2}{x^3+1}$ is continuous, positive, and decreasing ($\star$) on $[2, \infty)$, so the Integral Test applies.

$$\int_2^\infty \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty}\int_2^t \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty}\left[\frac{1}{3}\ln\left|x^3+1\right|\right]_2^t = \lim_{t\to\infty}\left[\frac{1}{3}\ln\left(t^3+1\right) - \frac{1}{3}\ln 9\right] = \infty.$$

Since the improper integral is divergent, the series $\sum\limits_{n=2}^{\infty}\dfrac{n^2}{n^3+1}$ is also divergent by the Integral Test.

$$(\star)\text{:}\ \ f'(x) = \frac{(x^3+1)(2x) - x^2(3x^2)}{(x^3+1)^2} = \frac{2x - x^4}{(x^3+1)^2} = -\frac{x(x^3-2)}{(x^3+1)^2} < 0 \text{ for } x \geq 2.$$

8. The function $f(x) = x^2 e^{-x^3}$ is continuous, positive, and decreasing ($\star$) on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty x^2 e^{-x^3}\,dx = \lim_{t\to\infty}\int_1^t x^2 e^{-x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{3}e^{-x^3}\right]_1^t = -\frac{1}{3}\lim_{t\to\infty}\left(e^{-t^3} - e^{-1}\right) = -\frac{1}{3}\left(0 - \frac{1}{e}\right) = \frac{1}{3e}.$$

Since this improper integral is convergent, the series $\sum\limits_{n=1}^{\infty} n^2 e^{-n^3}$ is also convergent by the Integral Test.

$$(\star)\text{:}\ \ f'(x) = x^2 e^{-x^3}(-3x^2) + e^{-x^3}(2x) = xe^{-x^3}(-3x^3+2) = \frac{x(2-3x^3)}{e^{x^3}} < 0 \ \text{ for } x > 1$$

9. The function $f(x) = \dfrac{1}{x(\ln x)^3}$ is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies.

$$\begin{aligned}\int_2^\infty \frac{1}{x(\ln x)^3}\,dx &= \lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{du}{u^3}\quad \left[u = \ln x,\, du = \frac{dx}{x}\right] \quad = \lim_{t\to\infty}\left[-\frac{1}{2}u^{-2}\right]_{\ln 2}^{\ln t}\\ &= \lim_{t\to\infty}\left[-\frac{1}{2(\ln t)^2} + \frac{1}{2(\ln 2)^2}\right] = 0 + \frac{1}{2(\ln 2)^2} = \frac{1}{2(\ln 2)^2}\end{aligned}$$

Since the improper integral is convergent, the series $\sum\limits_{n=2}^{\infty}\dfrac{1}{n(\ln n)^3}$ is also convergent by the Integral Test.

10. The function $f(x) = \dfrac{\tan^{-1}x}{1+x^2}$ is continuous, positive, and decreasing ($\star$) on $[1, \infty)$, so the Integral Test applies.

$$\begin{aligned}\int_1^\infty \frac{\tan^{-1}x}{1+x^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{\tan^{-1}x}{1+x^2}\,dx = \lim_{t\to\infty}\int_{\tan^{-1}1}^{\tan^{-1}t} u\,du \quad \left[u = \tan^{-1}x,\, du = \frac{dx}{1+x^2}\right]\\ &= \lim_{t\to\infty}\left[\frac{1}{2}u^2\right]_{\tan^{-1}1}^{\tan^{-1}t} = \lim_{t\to\infty}\left[\frac{1}{2}(\tan^{-1}t)^2 - \frac{1}{2}(\tan^{-1}1)^2\right] = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \frac{3\pi^2}{32}.\end{aligned}$$

Since the improper integral is convergent, the series $\sum\limits_{n=1}^{\infty}\dfrac{\tan^{-1}n}{1+n^2}$ is also convergent by the Integral Test.

$$(\star)\text{:}\ \ f'(x) = \frac{(1+x^2)\left(\dfrac{1}{1+x^2}\right) - (\tan^{-1}x)(2x)}{(1+x^2)^2} = \frac{1 - 2x\tan^{-1}x}{(1+x^2)^2} < 0 \text{ for } 2x\tan^{-1}x > 1.$$

Both $2x$ and $\tan^{-1}x$ are increasing functions, and when $x = 1$, $2x\tan^{-1}x = 2(1)\tan^{-1}1 = 2\pi/4 > 1$, so it follows that $f'(x) < 0$ when $x \geq 1$.

11. $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{\sqrt{2}}}$ is a p-series with $p = \sqrt{2} > 1$, so it converges by (1).

12. $\sum_{n=3}^{\infty} n^{-0.9999} = \sum_{n=3}^{\infty} \frac{1}{n^{0.9999}}$ is a p-series with $p = 0.9999 \leq 1$, so it diverges by (1). The fact that the series begins with $n = 3$ is irrelevant when determining convergence.

13. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^3}$. This is a p-series with $p = 3 > 1$, so it converges by (1).

14. $\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n+3}$. The function $f(x) = \frac{1}{2x+3}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{1}{2x+3}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{2x+3}\,dx = \lim_{t\to\infty} \left[\tfrac{1}{2}\ln(2x+3)\right]_1^t = \lim_{t\to\infty} \left[\tfrac{1}{2}\ln(2t+3) - \tfrac{1}{2}\ln 5\right] = \infty,$$ so the series $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ diverges.

15. $\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \frac{1}{19} + \cdots = \sum_{n=1}^{\infty} \frac{1}{4n-1}$. The function $f(x) = \frac{1}{4x-1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{1}{4x-1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{4x-1}\,dx = \lim_{t\to\infty} \left[\tfrac{1}{4}\ln(4x-1)\right]_1^t = \lim_{t\to\infty} \left[\tfrac{1}{4}\ln(4t-1) - \tfrac{1}{4}\ln 3\right] = \infty,$$ so the series $\sum_{n=1}^{\infty} \frac{1}{4n-1}$ diverges.

16. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. This is a p-series with $p = \frac{3}{2} > 1$, so it converges by (1).

17. $\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2} = \sum_{n=1}^{\infty} \left(\frac{\sqrt{n}}{n^2} + \frac{4}{n^2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} + \sum_{n=1}^{\infty} \frac{4}{n^2}$. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series with $p = \frac{3}{2} > 1$. $\sum_{n=1}^{\infty} \frac{4}{n^2} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a constant multiple of a convergent p-series with $p = 2 > 1$, so it converges. The sum of two convergent series is convergent, so the original series is convergent.

18. The function $f(x) = \frac{\sqrt{x}}{1+x^{3/2}}$ is continuous and positive on $[1, \infty)$.

$$f'(x) = \frac{(1+x^{3/2})\left(\frac{1}{2}x^{-1/2}\right) - x^{1/2}\left(\frac{3}{2}x^{1/2}\right)}{(1+x^{3/2})^2} = \frac{\frac{1}{2}x^{-1/2} + \frac{1}{2}x - \frac{3}{2}x}{(1+x^{3/2})^2} = \frac{1-2x^{3/2}}{2\sqrt{x}(1+x^{3/2})^2} < 0$$ for $x \geq 1$, so f is decreasing on $[1, \infty)$, and the Integral Test applies.

$$\begin{aligned} \int_1^{\infty} \frac{\sqrt{x}}{1+x^{3/2}}\,dx &= \lim_{t\to\infty} \int_1^t \frac{\sqrt{x}}{1+x^{3/2}}\,dx = \lim_{t\to\infty} \left[\tfrac{2}{3}\ln(1+x^{3/2})\right]_1^t \qquad \left[\begin{array}{c}\text{substitution}\\ \text{with } u = 1+x^{3/2}\end{array}\right] \\ &= \lim_{t\to\infty} \left[\tfrac{2}{3}\ln(1+t^{3/2}) - \tfrac{2}{3}\ln 2\right] = \infty, \end{aligned}$$

so the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^{3/2}}$ diverges.

19. The function $f(x) = \dfrac{1}{x^2+4}$ is continuous, positive, and decreasing on $[1, \infty)$, so we can apply the Integral Test.

$$\int_1^\infty \frac{1}{x^2+4}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+4}\,dx = \lim_{t\to\infty} \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_1^t = \frac{1}{2}\lim_{t\to\infty}\left[\tan^{-1}\left(\frac{t}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right]$$
$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)\right]$$

Therefore, the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+4}$ converges.

20. The function $f(x) = \dfrac{1}{x^2+2x+2}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{1}{x^2+2x+2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(x+1)^2+1}\,dx = \lim_{t\to\infty}\Big[\arctan(x+1)\Big]_1^t$$
$$= \lim_{t\to\infty}[\arctan(t+1) - \arctan 2] = \frac{\pi}{2} - \arctan 2,$$

so the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+2n+2}$ converges.

21. The function $f(x) = \dfrac{x^3}{x^4+4}$ is continuous and positive on $[2, \infty)$, and is also decreasing since

$f'(x) = \dfrac{(x^4+4)(3x^2) - x^3(4x^3)}{(x^4+4)^2} = \dfrac{12x^2 - x^6}{(x^4+4)^2} = \dfrac{x^2(12-x^4)}{(x^4+4)^2} < 0$ for $x > \sqrt[4]{12} \approx 1.86$, so we can use the Integral Test on $[2, \infty)$.

$\displaystyle\int_2^\infty \frac{x^3}{x^4+4}\,dx = \lim_{t\to\infty}\int_2^t \frac{x^3}{x^4+4}\,dx = \lim_{t\to\infty}\left[\tfrac{1}{4}\ln(x^4+4)\right]_2^t = \lim_{t\to\infty}\left[\tfrac{1}{4}\ln(t^4+4) - \tfrac{1}{4}\ln 20\right] = \infty$, so the series $\sum\limits_{n=2}^{\infty}\dfrac{n^3}{n^4+4}$ diverges, and it follows that $\sum\limits_{n=1}^{\infty}\dfrac{n^3}{n^4+4}$ diverges as well.

22. The function $f(x) = \dfrac{3x-4}{x^2-2x} = \dfrac{2}{x} + \dfrac{1}{x-2}$ [by partial fractions] is continuous, positive, and decreasing on $[3, \infty)$ since it is the sum of two such functions, so we can apply the Integral Test.

$\displaystyle\int_3^\infty \frac{3x-4}{x^2-x}\,dx = \lim_{t\to\infty}\int_3^t\left[\frac{2}{x} + \frac{1}{x-2}\right]dx = \lim_{t\to\infty}\Big[2\ln x + \ln(x-2)\Big]_3^t = \lim_{t\to\infty}[2\ln t + \ln(t-2) - 2\ln 3] = \infty.$

The integral is divergent, so the series $\sum\limits_{n=3}^{\infty}\dfrac{3n-4}{n^2-n}$ is divergent.

23. $f(x) = \dfrac{1}{x\ln x}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = -\dfrac{1+\ln x}{x^2(\ln x)^2} < 0$ for $x > 2$, so we can use the Integral Test. $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{t\to\infty}[\ln(\ln x)]_2^t = \lim_{t\to\infty}[\ln(\ln t) - \ln(\ln 2)] = \infty$, so the series $\sum\limits_{n=2}^{\infty}\dfrac{1}{n\ln n}$ diverges.

24. The function $f(x) = \dfrac{\ln x}{x^2}$ is continuous and positive on $[2, \infty)$, and also decreasing since

$f'(x) = \dfrac{x^2(1/x) - (\ln x)(2x)}{(x^2)^2} = \dfrac{x - 2x\ln x}{x^4} = \dfrac{1-2\ln x}{x^3} < 0$ for $x > e^{1/2} \approx 1.65$, so we can use the Integral Test on $[2, \infty)$.

$$\int_2^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_2^t \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left(\left[-\frac{\ln x}{x}\right]_2^t + \int_2^t \frac{1}{x^2}\,dx\right) \qquad \left[\begin{matrix}\text{by parts with}\\ u=\ln x,\ dv=(1/x^2)\,dx\end{matrix}\right]$$

$$= \lim_{t\to\infty}\left(-\frac{\ln t}{t} + \frac{\ln 2}{2} + \left[-\frac{1}{x}\right]_2^t\right) \overset{\text{H}}{=} \lim_{t\to\infty}\left(-\frac{1/t}{1} + \frac{\ln 2}{2} - \frac{1}{t} + \frac{1}{2}\right) = \frac{\ln 2 + 1}{2},$$

so the series $\sum\limits_{n=2}^{\infty} \dfrac{\ln n}{n^2}$ converges.

25. The function $f(x) = xe^{-x} = \dfrac{x}{e^x}$ is continuous and positive on $[1, \infty)$, and also decreasing since

$f'(x) = \dfrac{e^x \cdot 1 - xe^x}{(e^x)^2} = \dfrac{e^x(1-x)}{(e^x)^2} = \dfrac{1-x}{e^x} < 0$ for $x > 1$ [and $f(1) > f(2)$], so we can use the Integral Test on $[1, \infty)$.

$$\int_1^\infty xe^{-x}\,dx = \lim_{t\to\infty}\int_1^t xe^{-x}\,dx = \lim_{t\to\infty}\left(\left[-xe^{-x}\right]_1^t + \int_1^t e^{-x}\,dx\right) \qquad \left[\begin{matrix}\text{by parts with}\\ u=x,\ dv=e^{-x}\,dx\end{matrix}\right]$$

$$= \lim_{t\to\infty}\left(-te^{-t} + e^{-1} + \left[-e^{-x}\right]_1^t\right) = \lim_{t\to\infty}\left(-\frac{t}{e^t} + \frac{1}{e} - \frac{1}{e^t} + \frac{1}{e}\right)$$

$$\overset{\text{H}}{=} \lim_{t\to\infty}\left(-\frac{1}{e^t} + \frac{1}{e} - 0 + \frac{1}{e}\right) = \frac{2}{e},$$

so the series $\sum\limits_{k=1}^{\infty} ke^{-k}$ converges.

26. The function $f(x) = xe^{-x^2} = \dfrac{x}{e^{x^2}}$ is continuous and positive on $[1, \infty)$, and also decreasing since

$f'(x) = \dfrac{e^{x^2} \cdot 1 - xe^{x^2} \cdot 2x}{(e^{x^2})^2} = \dfrac{1 - 2x^2}{e^{x^2}} < 0$ for $x > \sqrt{\frac{1}{2}} \approx 0.7$, so we can use the Integral Test on $[1, \infty)$.

$\displaystyle\int_1^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx = \lim_{t\to\infty}\left[-\tfrac{1}{2}e^{-x^2}\right]_1^t = \lim_{t\to\infty}\left(-\tfrac{1}{2}e^{-t^2} + \tfrac{1}{2}e^{-1}\right) = \frac{1}{2e}$, so the series $\sum\limits_{k=1}^{\infty} ke^{-k^2}$ converges.

27. The function $f(x) = \dfrac{1}{x^2 + x^3} = \dfrac{1}{x^2} - \dfrac{1}{x} + \dfrac{1}{x+1}$ [by partial fractions] is continuous, positive and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty}\int_1^t \left(\frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}\right)dx = \lim_{t\to\infty}\left[-\frac{1}{x} - \ln x + \ln(x+1)\right]_1^t$$

$$= \lim_{t\to\infty}\left[-\frac{1}{t} + \ln\frac{t+1}{t} + 1 - \ln 2\right] = 0 + 0 + 1 - \ln 2$$

The integral converges, so the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2 + n^3}$ converges.

28. The function $f(x) = \dfrac{x}{x^4 + 1}$ is positive, continuous, and decreasing on $[1, \infty)$. [Note that

$f'(x) = \dfrac{x^4 + 1 - 4x^4}{(x^4+1)^2} = \dfrac{1 - 3x^4}{(x^4+1)^2} < 0$ on $[1, \infty)$.] Thus, we can apply the Integral Test.

[continued]

$$\int_1^\infty \frac{x}{x^4+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(2x)}{1+(x^2)^2}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}(x^2)\right]_1^t = \frac{1}{2}\lim_{t\to\infty}\left[\tan^{-1}(t^2)-\tan^{-1}1\right] = \frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\pi}{8}$$

so the series $\sum\limits_{n=1}^{\infty}\dfrac{n}{n^4+1}$ converges.

29. The function $f(x)=\dfrac{\cos\pi x}{\sqrt{x}}$ is neither positive nor decreasing on $[1,\infty)$, so the hypotheses of the Integral Test are not satisfied for the series $\sum\limits_{n=1}^{\infty}\dfrac{\cos\pi n}{\sqrt{n}}$.

30. The function $f(x)=\dfrac{\cos^2 x}{1+x^2}$ is not decreasing on $[1,\infty)$, so the hypotheses of the Integral Test are not satisfied for the series $\sum\limits_{n=1}^{\infty}\dfrac{\cos^2 n}{1+n^2}$.

31. We have already shown (in Exercise 23) that when $p=1$ the series $\sum\limits_{n=2}^{\infty}\dfrac{1}{n(\ln n)^p}$ diverges, so assume that $p\neq 1$.

$f(x)=\dfrac{1}{x(\ln x)^p}$ is continuous and positive on $[2,\infty)$, and $f'(x)=-\dfrac{p+\ln x}{x^2(\ln x)^{p+1}}<0$ if $x>e^{-p}$, so that f is eventually decreasing and we can use the Integral Test.

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^{1-p}}{1-p}\right]_2^t \quad [\text{for } p\neq 1] \;=\; \lim_{t\to\infty}\left[\frac{(\ln t)^{1-p}}{1-p}-\frac{(\ln 2)^{1-p}}{1-p}\right]$$

This limit exists whenever $1-p<0 \;\Leftrightarrow\; p>1$, so the series converges for $p>1$.

32. $f(x)=\dfrac{1}{x\ln x\,[\ln(\ln x)]^p}$ is positive and continuous on $[3,\infty)$. For $p\geq 0$, f clearly decreases on $[3,\infty)$; and for $p<0$, it can be verified that f is ultimately decreasing. Thus, we can apply the Integral Test.

$$I=\int_3^\infty \frac{dx}{x\ln x\,[\ln(\ln x)]^p} = \lim_{t\to\infty}\int_3^t \frac{[\ln(\ln x)]^{-p}}{x\ln x}\,dx = \lim_{t\to\infty}\left[\frac{[\ln(\ln x)]^{-p+1}}{-p+1}\right]_3^t \qquad [\text{for } p\neq 1]$$

$$= \lim_{t\to\infty}\left[\frac{[\ln(\ln t)]^{-p+1}}{-p+1}-\frac{[\ln(\ln 3)]^{-p+1}}{-p+1}\right],$$

which exists whenever $-p+1<0 \;\Leftrightarrow\; p>1$. If $p=1$, then $I=\lim\limits_{t\to\infty}\left[\ln(\ln(\ln x))\right]_3^t=\infty$. Therefore, $\sum\limits_{n=3}^{\infty}\dfrac{1}{n\ln n\,[\ln(\ln n)]^p}$ converges for $p>1$.

33. Clearly the series cannot converge if $p\geq-\frac{1}{2}$, because then $\lim\limits_{n\to\infty}n(1+n^2)^p\neq 0$. So assume $p<-\frac{1}{2}$. Then $f(x)=x(1+x^2)^p$ is continuous, positive, and eventually decreasing on $[1,\infty)$, and we can use the Integral Test.

$$\int_1^\infty x(1+x^2)^p dx = \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{(1+x^2)^{p+1}}{p+1}\right]_1^t = \frac{1}{2(p+1)}\lim_{t\to\infty}\left[(1+t^2)^{p+1}-2^{p+1}\right].$$

This limit exists and is finite $\;\Leftrightarrow\; p+1<0 \;\Leftrightarrow\; p<-1$, so the series $\sum\limits_{n=1}^{\infty}n(1+n^2)^p$ converges whenever $p<-1$.

34. If $p \le 0$, $\lim_{n\to\infty} \frac{\ln n}{n^p} = \infty$ and the series diverges, so assume $p > 0$. $f(x) = \frac{\ln x}{x^p}$ is positive and continuous and $f'(x) < 0$ for $x > e^{1/p}$, so f is eventually decreasing and we can use the Integral Test. Integration by parts gives

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \lim_{t\to\infty}\left[\frac{x^{1-p}\left[(1-p)\ln x - 1\right]}{(1-p)^2}\right]_1^t \text{ (for } p \ne 1) = \frac{1}{(1-p)^2}\left[\lim_{t\to\infty} t^{1-p}\left[(1-p)\ln t - 1\right] + 1\right], \text{ which exists}$$

whenever $1 - p < 0 \quad\Leftrightarrow\quad p > 1$. Thus, $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ converges $\quad\Leftrightarrow\quad p > 1$.

35. Since this is a p-series with $p = x$, $\zeta(x)$ is defined when $x > 1$. Unless specified otherwise, the domain of a function f is the set of real numbers x such that the expression for $f(x)$ makes sense and defines a real number. So, in the case of a series, it's the set of real numbers x such that the series is convergent.

36. (a) $\sum_{n=2}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{1^2}$ [subtract a_1] $= \frac{\pi^2}{6} - 1$

(b) $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=4}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\right) = \frac{\pi^2}{6} - \frac{49}{36}$

(c) $\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{24}$

37. (a) $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4 = \sum_{n=1}^{\infty} \frac{81}{n^4} = 81\sum_{n=1}^{\infty} \frac{1}{n^4} = 81\left(\frac{\pi^4}{90}\right) = \frac{9\pi^4}{10}$

(b) $\sum_{k=5}^{\infty} \frac{1}{(k-2)^4} = \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots = \sum_{k=3}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} - \left(\frac{1}{1^4} + \frac{1}{2^4}\right)$ [subtract a_1 and a_2] $= \frac{\pi^4}{90} - \frac{17}{16}$

38. (a) $f(x) = 1/x^4$ is positive and continuous and $f'(x) = -4/x^5$ is negative for $x > 0$, and so the Integral Test applies.

$\sum_{n=1}^{\infty} \frac{1}{n^4} \approx s_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{10^4} \approx 1.082037.$

$R_{10} \le \int_{10}^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty}\left[\frac{1}{-3x^3}\right]_{10}^t = \lim_{t\to\infty}\left(-\frac{1}{3t^3} + \frac{1}{3\,(10)^3}\right) = \frac{1}{3000}$, so the error is at most $0.000\overline{3}$.

(b) $s_{10} + \int_{11}^{\infty} \frac{1}{x^4}\,dx \le s \le s_{10} + \int_{10}^{\infty} \frac{1}{x^4}\,dx \quad\Rightarrow\quad s_{10} + \frac{1}{3(11)^3} \le s \le s_{10} + \frac{1}{3(10)^3} \quad\Rightarrow$

$1.082037 + 0.000250 = 1.082287 \le s \le 1.082037 + 0.000333 = 1.082370$, so we get $s \approx 1.08233$ with error ≤ 0.00005.

(c) The estimate in part (b) is $s \approx 1.08233$ with error ≤ 0.00005. The exact value given in Exercise 37 is $\pi^4/90 \approx 1.082323$. The difference is less than 0.00001.

(d) $R_n \le \int_n^{\infty} \frac{1}{x^4}\,dx = \frac{1}{3n^3}$. So $R_n < 0.00001 \quad\Rightarrow\quad \frac{1}{3n^3} < \frac{1}{10^5} \quad\Rightarrow\quad 3n^3 > 10^5 \quad\Rightarrow\quad n > \sqrt[3]{(10)^5/3} \approx 32.2$, that is, for $n > 32$.

39. (a) $f(x) = \dfrac{1}{x^2}$ is positive and continuous and $f'(x) = -\dfrac{2}{x^3}$ is negative for $x > 0$, and so the Integral Test applies.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx s_{10} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{10^2} \approx 1.549768.$$

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[\frac{-1}{x}\right]_{10}^{t} = \lim_{t\to\infty}\left(-\frac{1}{t} + \frac{1}{10}\right) = \frac{1}{10}, \text{ so the error is at most } 0.1.$$

(b) $s_{10} + \displaystyle\int_{11}^{\infty} \frac{1}{x^2}\,dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^2}\,dx \quad\Rightarrow\quad s_{10} + \tfrac{1}{11} \leq s \leq s_{10} + \tfrac{1}{10} \quad\Rightarrow$

$1.549768 + 0.090909 = 1.640677 \leq s \leq 1.549768 + 0.1 = 1.649768$, so we get $s \approx 1.64522$ (the average of 1.640677 and 1.649768) with error ≤ 0.005 (the maximum of $1.649768 - 1.64522$ and $1.64522 - 1.640677$, rounded up).

(c) The estimate in part (b) is $s \approx 1.64522$ with error ≤ 0.005. The exact value given in Exercise 36 is $\pi^2/6 \approx 1.644934$. The difference is less than 0.0003.

(d) $R_n \leq \displaystyle\int_n^{\infty} \frac{1}{x^2}\,dx = \frac{1}{n}$. So $R_n < 0.001$ if $\dfrac{1}{n} < \dfrac{1}{1000} \quad\Leftrightarrow\quad n > 1000$.

40. $\displaystyle\sum_{n=1}^{\infty} ne^{-2n}$. $f(x) = xe^{-2x}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies. Using (2),

$$R_n \leq \int_n^{\infty} xe^{-2x}\,dx = \lim_{t\to\infty}\left(\left[-\tfrac{1}{2}xe^{-2x}\right]_n^t + \int_n^t \tfrac{1}{2}e^{-2x}\,dx\right) \qquad \left[\begin{array}{c}\text{using parts with}\\ u = x,\ dv = e^{-2x}\,dx\end{array}\right]$$

$$= \lim_{t\to\infty}\left(\frac{-t}{2e^{2t}} + \frac{n}{2e^{2n}} - \frac{1}{4e^{2t}} + \frac{1}{4e^{2n}}\right) \overset{\text{H}}{=} 0 + \frac{n}{2e^{2n}} - 0 + \frac{1}{4e^{2n}} = \frac{2n+1}{4e^{2n}}$$

To be correct to four decimal places, we want $\dfrac{2n+1}{4e^{2n}} \leq \dfrac{5}{10^5}$. This inequality is true for $n = 6$.

$$s_6 = \sum_{n=1}^{6} \frac{n}{e^{2n}} = \frac{1}{e^2} + \frac{2}{e^4} + \frac{3}{e^6} + \frac{4}{e^8} + \frac{5}{e^{10}} + \frac{6}{e^{12}} \approx 0.1810.$$

41. $\displaystyle\sum_{n=1}^{\infty} (2n+1)^{-6}$. $f(x) = 1/(2x+1)^6$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

Using (2), $R_n \leq \displaystyle\int_n^{\infty} (2x+1)^{-6}\,dx = \lim_{t\to\infty}\left[\frac{-1}{10(2x+1)^5}\right]_n^t = \frac{1}{10(2n+1)^5}$. To be correct to five decimal places,

we want $\dfrac{1}{10(2n+1)^5} \leq \dfrac{5}{10^6} \quad\Leftrightarrow\quad (2n+1)^5 \geq 20{,}000 \quad\Leftrightarrow\quad n \geq \tfrac{1}{2}\left(\sqrt[5]{20{,}000} - 1\right) \approx 3.12$, so use $n = 4$.

$$s_4 = \sum_{n=1}^{4} \frac{1}{(2n+1)^6} = \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} \approx 0.001\,446 \approx 0.00145.$$

42. $f(x) = \dfrac{1}{x(\ln x)^2}$ is positive and continuous and $f'(x) = -\dfrac{\ln x + 2}{x^2(\ln x)^3}$ is negative for $x > 1$, so the Integral Test applies.

Using (2), we need $0.01 > \displaystyle\int_n^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty}\left[\frac{-1}{\ln x}\right]_n^t = \frac{1}{\ln n}$. This is true for $n > e^{100}$, so we would have to add this many terms to find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ to within 0.01, which would be problematic because $e^{100} \approx 2.7 \times 10^{43}$.

43. $\sum\limits_{n=1}^{\infty} n^{-1.001} = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{1.001}}$ is a convergent p-series with $p = 1.001 > 1$. Using (2), we get

$R_n \le \displaystyle\int_n^{\infty} x^{-1.001}\,dx = \lim_{t\to\infty}\left[\frac{x^{-0.001}}{-0.001}\right]_n^t = -1000\lim_{t\to\infty}\left[\frac{1}{x^{0.001}}\right]_n^t = -1000\left(-\frac{1}{n^{0.001}}\right) = \frac{1000}{n^{0.001}}$. We want

$R_n < 0.000\,000\,005 \quad\Leftrightarrow\quad \dfrac{1000}{n^{0.001}} < 5\times 10^{-9} \quad\Leftrightarrow\quad n^{0.001} > \dfrac{1000}{5\times 10^{-9}} \quad\Leftrightarrow$

$n > \left(2\times 10^{11}\right)^{1000} = 2^{1000}\times 10^{11{,}000} \approx 1.07\times 10^{301}\times 10^{11{,}000} = 1.07\times 10^{11{,}301}$.

44. (a) $f(x) = \left(\dfrac{\ln x}{x}\right)^2$ is continuous and positive for $x > 1$, and since $f'(x) = \dfrac{2\ln x\,(1-\ln x)}{x^3} < 0$ for $x > e$, we can apply

the Integral Test. Using a CAS, we get $\displaystyle\int_1^{\infty}\left(\frac{\ln x}{x}\right)^2 dx = 2$, so the series $\sum\limits_{n=1}^{\infty}\left(\dfrac{\ln n}{n}\right)^2$ also converges.

(b) Since the Integral Test applies, the error in $s \approx s_n$ is $R_n \le \displaystyle\int_n^{\infty}\left(\frac{\ln x}{x}\right)^2 dx = \frac{(\ln n)^2 + 2\ln n + 2}{n}$.

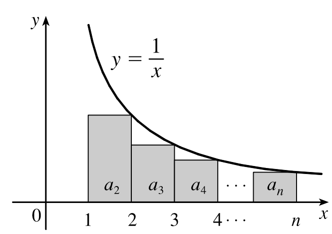
(c) By graphing the functions $y_1 = \dfrac{(\ln x)^2 + 2\ln x + 2}{x}$ and $y_2 = 0.05$, we see that $y_1 < y_2$ for $n \ge 1373$.

(d) Using the CAS to sum the first 1373 terms, we get $s_{1373} \approx 1.94$.

45. (a) From the figure, $a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx$, so with

$f(x) = \dfrac{1}{x}$, $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n} \le \displaystyle\int_1^n \frac{1}{x}\,dx = \ln n$.

Thus, $s_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n} \le 1 + \ln n$.



(b) By part (a), $s_{10^6} \le 1 + \ln 10^6 \approx 14.82 < 15$ and

$s_{10^9} \le 1 + \ln 10^9 \approx 21.72 < 22$.

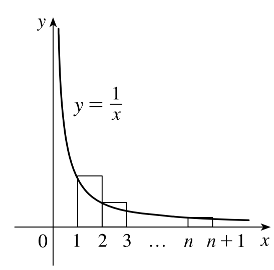
46. (a) The sum of the areas of the n rectangles in the graph to the right is

$1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$. Now $\displaystyle\int_1^{n+1}\frac{dx}{x}$ is less than this sum because

the rectangles extend above the curve $y = 1/x$, so

$\displaystyle\int_1^{n+1}\frac{1}{x}\,dx = \ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, and since

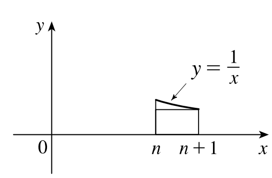
$\ln n < \ln(n+1)$, $0 < 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} - \ln n = t_n$.



(b) The area under $f(x) = 1/x$ between $x = n$ and $x = n+1$ is

$\displaystyle\int_n^{n+1}\frac{dx}{x} = \ln(n+1) - \ln n$, and this is clearly greater than the area of

the inscribed rectangle in the figure to the right $\left[\text{which is } \dfrac{1}{n+1}\right]$, so



$t_n - t_{n+1} = [\ln(n+1) - \ln n] - \dfrac{1}{n+1} > 0$, and so $t_n > t_{n+1}$, so $\{t_n\}$ is a decreasing sequence.

(c) We have shown that $\{t_n\}$ is decreasing and that $t_n > 0$ for all n. Thus, $0 < t_n \le t_1 = 1$, so $\{t_n\}$ is a bounded monotonic sequence, and hence converges by the Monotonic Sequence Theorem.

47. $b^{\ln n} = \left(e^{\ln b}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln b} = n^{\ln b} = \dfrac{1}{n^{-\ln b}}$. $\sum\limits_{n=1}^{\infty} b^{\ln n}$ is a p-series, which converges for all b such that $-\ln b > 1 \quad\Leftrightarrow$

$\ln b < -1 \quad\Leftrightarrow\quad b < e^{-1} \quad\Leftrightarrow\quad b < 1/e$ [with $b > 0$].

48. For the series $\sum\limits_{n=1}^{\infty} \left(\dfrac{c}{n} - \dfrac{1}{n+1}\right)$,

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \left(\frac{c}{i} - \frac{1}{i+1}\right) = \left(\frac{c}{1} - \frac{1}{2}\right) + \left(\frac{c}{2} - \frac{1}{3}\right) + \left(\frac{c}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{c}{n} - \frac{1}{n+1}\right) \\ &= \frac{c}{1} + \frac{c-1}{2} + \frac{c-1}{3} + \frac{c-1}{4} + \cdots + \frac{c-1}{n} - \frac{1}{n+1} = c + (c-1)\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right) - \frac{1}{n+1} \end{aligned}$$

Thus, $\sum\limits_{n=1}^{\infty} \left(\dfrac{c}{n} - \dfrac{1}{n+1}\right) = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left[c + (c-1)\sum\limits_{i=2}^{n} \dfrac{1}{i} - \dfrac{1}{n+1}\right]$. Since a constant multiple of a divergent series is divergent, the last limit exists only if $c - 1 = 0$, so the original series converges only if $c = 1$.

11.4 The Comparison Tests

1. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the discusssion preceding the box titled "The Limit Comparison Test.")

(b) If $a_n < b_n$ for all n, then $\sum a_n$ is convergent. [This is part (i) of the Direct Comparison Test.]

2. (a) If $a_n > b_n$ for all n, then $\sum a_n$ is divergent. [This is part (ii) of the Direct Comparison Test.]

(b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

3. (a) $\dfrac{n}{n^3+5} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$ for all $n \ge 2$. $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^2}$ converges because it is a p-series with $p = 2 > 1$, so $\sum\limits_{n=2}^{\infty} \dfrac{n}{n^3+5}$ converges by part (i) of the Direct Comparison Test.

(b) Use the Limit Comparison Test with $a_n = \dfrac{n}{n^3-5}$ and $b_n = \dfrac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^3-5} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^3}{n^3\left(1 - 5/n^3\right)} = \lim_{n\to\infty} \frac{1}{1 - 5/n^3} = \frac{1}{1-0} = 1 > 0$$

Since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^2}$ is a convergent (partial) p-series $[p = 2 > 1]$, the series $\sum\limits_{n=2}^{\infty} \dfrac{n}{n^3-5}$ also converges.

4. (a) $\dfrac{n^2+n}{n^3-2} > \dfrac{n^2}{n^3-2} > \dfrac{n^2}{n^3} = \dfrac{1}{n}$ for all $n \ge 2$. $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$ diverges because it is a (partial) p-series with $p = 1 \le 1$, so $\sum\limits_{n=2}^{\infty} \dfrac{n^2+n}{n^3-2}$ diverges by part (ii) of the Direct Comparison Test.

(b) Use the Limit Comparison Test with $a_n = \dfrac{n^2 - n}{n^3 + 2}$ and $b_n = \dfrac{1}{n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2 - n}{n^3 + 2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3 - n^2}{n^3 + 2} = \lim_{n\to\infty} \frac{1 - 1/n}{1 + 2/n^3} = \frac{1 - 0}{1 + 0} = 1 > 0$$

Since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$ is a divergent (partial) p-series $[p = 1 \le 1]$, the series $\sum\limits_{n=2}^{\infty} \dfrac{n^2 - n}{n^3 + 2}$ also diverges.

5. An inequality can be used to show that a series converges if its general term can be shown to be less than or equal to the general term of a known convergent series. The only inequality that satisfies this condition is given in part (c) since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is a convergent p-series $[\,p = 2 > 1]$.

6. An inequality can be used to show that a series diverges if its general term can be shown to be greater than or equal to the general term of a known divergent series. The only inequality that satisfies this condition is given in part (c) since $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n} = \dfrac{1}{2} \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ is half of the harmonic series, which is divergent.

7. $\dfrac{1}{n^3 + 8} < \dfrac{1}{n^3}$ for all $n \ge 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3 + 8}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$, which converges because it is a p-series with $p = 3 > 1$.

8. $\dfrac{1}{\sqrt{n} - 1} > \dfrac{1}{\sqrt{n}}$ for all $n \ge 2$, so $\sum\limits_{n=2}^{\infty} \dfrac{1}{\sqrt{n} - 1}$ diverges by direct comparison with $\sum\limits_{n=2}^{\infty} \dfrac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p = \frac{1}{2} \le 1$.

9. $\dfrac{n+1}{n\sqrt{n}} > \dfrac{n}{n\sqrt{n}} = \dfrac{1}{\sqrt{n}}$ for all $n \ge 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n+1}{n\sqrt{n}}$ diverges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p = \frac{1}{2} \le 1$.

10. $\dfrac{n-1}{n^3+1} < \dfrac{n}{n^3+1} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$ for all $n \ge 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n-1}{n^3+1}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

11. $\dfrac{9^n}{3 + 10^n} < \dfrac{9^n}{10^n} = \left(\dfrac{9}{10}\right)^n$ for all $n \ge 1$. $\sum\limits_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ is a convergent geometric series $\left(|r| = \frac{9}{10} < 1\right)$, so $\sum\limits_{n=1}^{\infty} \dfrac{9^n}{3 + 10^n}$ converges by the Direct Comparison Test.

12. $\dfrac{6^n}{5^n - 1} > \dfrac{6^n}{5^n} = \left(\dfrac{6}{5}\right)^n$ for all $n \ge 1$. $\sum\limits_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ is a divergent geometric series $\left(|r| = \frac{6}{5} > 1\right)$, so $\sum\limits_{n=1}^{\infty} \dfrac{6^n}{5^n - 1}$ diverges by the Direct Comparison Test.

13. For $n \ge 2$, $\ln n < n$, so $\dfrac{1}{\ln n} > \dfrac{1}{n}$. Thus, $\sum\limits_{n=2}^{\infty} \dfrac{1}{\ln n}$ diverges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, which diverges because it is a p-series with $p = 1 \le 1$ (the harmonic series).

14. $\dfrac{k\sin^2 k}{1+k^3} \le \dfrac{k}{1+k^3} < \dfrac{k}{k^3} = \dfrac{1}{k^2}$ for all $k \ge 1$, so $\sum\limits_{k=1}^{\infty} \dfrac{k\sin^2 k}{1+k^3}$ converges by direct comparison with $\sum\limits_{k=1}^{\infty} \dfrac{1}{k^2}$, which converges because it is a p-series with $p = 2 > 1$.

15. $\dfrac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}} < \dfrac{\sqrt[3]{k}}{\sqrt{k^3}} = \dfrac{k^{1/3}}{k^{3/2}} = \dfrac{1}{k^{7/6}}$ for all $k \ge 1$, so $\sum\limits_{k=1}^{\infty} \dfrac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$ converges by direct comparison with $\sum\limits_{k=1}^{\infty} \dfrac{1}{k^{7/6}}$, which converges because it is a p-series with $p = \frac{7}{6} > 1$.

16. $\dfrac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} < \dfrac{2k(k^2)}{k(k^2)^2} = \dfrac{2k^3}{k^5} = \dfrac{2}{k^2}$ for all $k \ge 1$, so $\sum\limits_{k=1}^{\infty} \dfrac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$ converges by direct comparison with $2\sum\limits_{k=1}^{\infty} \dfrac{1}{k^2}$, which converges because it is a constant multiple of a p-series with $p = 2 > 1$.

17. $\dfrac{1+\cos n}{e^n} < \dfrac{2}{e^n}$ for all $n \ge 1$. $\sum\limits_{n=1}^{\infty} \dfrac{2}{e^n}$ is a convergent geometric series ($|r| = \frac{1}{e} < 1$), so $\sum\limits_{n=1}^{\infty} \dfrac{1+\cos n}{e^n}$ converges by the Direct Comparison Test.

18. $\dfrac{1}{\sqrt[3]{3n^4+1}} < \dfrac{1}{\sqrt[3]{3n^4}} < \dfrac{1}{\sqrt[3]{n^4}} = \dfrac{1}{n^{4/3}}$ for all $n \ge 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt[3]{3n^4+1}}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{4/3}}$, which converges because it is a p-series with $p = \frac{4}{3} > 1$.

19. $\dfrac{4^{n+1}}{3^n-2} > \dfrac{4 \cdot 4^n}{3^n} = 4\left(\dfrac{4}{3}\right)^n$ for all $n \ge 1$. $\sum\limits_{n=1}^{\infty} 4\left(\dfrac{4}{3}\right)^n = 4\sum\limits_{n=1}^{\infty}\left(\dfrac{4}{3}\right)^n$ is a divergent geometric series $\left(|r| = \frac{4}{3} > 1\right)$, so $\sum\limits_{n=1}^{\infty} \dfrac{4^{n+1}}{3^n-2}$ diverges by the Direct Comparison Test.

20. $\dfrac{1}{n^n} \le \dfrac{1}{n^2}$ for all $n \ge 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^n}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

21. Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt{n^2+1}}$ and $b_n = \dfrac{1}{n}$:

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n}{\sqrt{n^2+1}} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1+(1/n^2)}} = 1 > 0$. Since the harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges, so does $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n^2+1}}$.

22. Use the Limit Comparison Test with $a_n = \dfrac{2}{\sqrt{n}+2}$ and $b_n = \dfrac{1}{\sqrt{n}}$:

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{2\sqrt{n}}{\sqrt{n}+2} = \lim\limits_{n\to\infty} \dfrac{2}{1+2/\sqrt{n}} = 2 > 0$. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ is a divergent p-series $[\,p = \frac{1}{2} \le 1]$, the series $\sum\limits_{n=1}^{\infty} \dfrac{2}{\sqrt{n}+2}$ is also divergent.

23. Use the Limit Comparison Test with $a_n = \dfrac{n+1}{n^3+n}$ and $b_n = \dfrac{1}{n^2}$:

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{(n+1)n^2}{n(n^2+1)} = \lim_{n\to\infty}\frac{n^2+n}{n^2+1} = \lim_{n\to\infty}\frac{1+1/n}{1+1/n^2} = 1 > 0$. Since $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series $[p = 2 > 1]$, the series $\sum\limits_{n=1}^{\infty}\frac{n+1}{n^3+n}$ also converges.

24. Use the Limit Comparison Test with $a_n = \dfrac{n^2+n+1}{n^4+n^2}$ and $b_n = \dfrac{1}{n^2}$:

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{(n^2+n+1)n^2}{n^2(n^2+1)} = \lim_{n\to\infty}\frac{n^2+n+1}{n^2+1} = \lim_{n\to\infty}\frac{1+1/n+1/n^2}{1+1/n^2} = 1 > 0$. Since $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series $[p = 2 > 1]$, the series $\sum\limits_{n=1}^{\infty}\frac{n^2+n+1}{n^4+n^2}$ also converges.

25. Use the Limit Comparison Test with $a_n = \dfrac{\sqrt{1+n}}{2+n}$ and $b_n = \dfrac{1}{\sqrt{n}}$:

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt{1+n}\sqrt{n}}{2+n} = \lim_{n\to\infty}\frac{\sqrt{n^2+n}/\sqrt{n^2}}{(2+n)/n} = \lim_{n\to\infty}\frac{\sqrt{1+1/n}}{2/n+1} = 1 > 0$. Since $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a divergent p-series $[\,p = \frac{1}{2} \le 1]$, the series $\sum\limits_{n=1}^{\infty}\frac{\sqrt{1+n}}{2+n}$ also diverges.

26. Use the Limit Comparison Test with $a_n = \dfrac{n+2}{(n+1)^3}$ and $b_n = \dfrac{1}{n^2}$:

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2(n+2)}{(n+1)^3} = \lim_{n\to\infty}\frac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^3} = 1 > 0$. Since $\sum\limits_{n=3}^{\infty}\frac{1}{n^2}$ is a convergent (partial) p-series $[p = 2 > 1]$, the series $\sum\limits_{n=3}^{\infty}\frac{n+2}{(n+1)^3}$ also converges.

27. Use the Limit Comparison Test with $a_n = \dfrac{5+2n}{(1+n^2)^2}$ and $b_n = \dfrac{1}{n^3}$:

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^3(5+2n)}{(1+n^2)^2} = \lim_{n\to\infty}\frac{5n^3+2n^4}{(1+n^2)^2}\cdot\frac{1/n^4}{1/(n^2)^2} = \lim_{n\to\infty}\frac{\frac{5}{n}+2}{\left(\frac{1}{n^2}+1\right)^2} = 2 > 0$. Since $\sum\limits_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent p-series $[p = 3 > 1]$, the series $\sum\limits_{n=1}^{\infty}\frac{5+2n}{(1+n^2)^2}$ also converges.

28. $\dfrac{n+3^n}{n+2^n} > \dfrac{3^n}{n+2^n} > \dfrac{3^n}{2^n+2^n} = \dfrac{3^n}{2\cdot 2^n} = \dfrac{1}{2}\left(\dfrac{3}{2}\right)^n$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{n+3^n}{n+2^n}$ diverges by direct comparison with $\dfrac{1}{2}\sum\limits_{n=1}^{\infty}\left(\dfrac{3}{2}\right)^n$, which is a constant multiple of a divergent geometric series $[|r| = \frac{3}{2} > 1]$. *Or:* Use the Limit Comparison Test with $a_n = \dfrac{n+3^n}{n+2^n}$ and $b_n = \left(\dfrac{3}{2}\right)^n$.

29. $\dfrac{e^n+1}{ne^n+1} \ge \dfrac{e^n+1}{ne^n+n} = \dfrac{e^n+1}{n(e^n+1)} = \dfrac{1}{n}$ for $n \ge 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{e^n+1}{ne^n+1}$ diverges by direct comparison with the divergent harmonic series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$. *Or:* Use the Limit Comparison Test with $a_n = \dfrac{e^n+1}{ne^n+1}$ and $b_n = \dfrac{1}{n}$.

30. If $a_n = \dfrac{1}{n\sqrt{n^2-1}}$ and $b_n = \dfrac{1}{n^2}$, then

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2}{n\sqrt{n^2-1}} = \lim_{n\to\infty}\frac{n/n}{\sqrt{n^2-1}/n} = \lim_{n\to\infty}\frac{1}{\sqrt{1-1/n^2}} = \frac{1}{1} = 1 > 0$, so $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{n^2-1}}$ converges by the Limit Comparison Test with the convergent series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2}$.

31. $\dfrac{2+\sin n}{n^2} \le \dfrac{2+1}{n^2} = \dfrac{3}{n^2}$, for all $n \ge 1$, so $\displaystyle\sum_{n=1}^{\infty}\frac{2+\sin n}{n^2}$ converges by direct comparison with $3\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$, which converges because it is a constant multiple of a p-series with $p = 2 > 1$.

32. $\dfrac{n^2+\cos^2 n}{n^3} \ge \dfrac{n^2}{n^3} = \dfrac{1}{n}$, for all $n \ge 1$, so $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+\cos^2 n}{n^3}$ diverges by direct comparison with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges because it is a p-series with $p = 1 \le 1$ (the harmonic series).

33. Use the Limit Comparison Test with $a_n = \left(1+\dfrac{1}{n}\right)^2 e^{-n}$ and $b_n = e^{-n}$: $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 1 > 0$. Since $\displaystyle\sum_{n=1}^{\infty}e^{-n} = \sum_{n=1}^{\infty}\frac{1}{e^n}$ is a convergent geometric series $\left[|r| = \frac{1}{e} < 1\right]$, the series $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^2 e^{-n}$ also converges.

34. $\dfrac{e^{1/n}}{n} > \dfrac{1}{n}$ for all $n \ge 1$, so $\displaystyle\sum_{n=1}^{\infty}\frac{e^{1/n}}{n}$ diverges by direct comparison with the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$.

35. Clearly $n! = n(n-1)(n-2)\cdots(3)(2) \ge 2\cdot 2\cdot 2\cdot\cdots\cdot 2\cdot 2 = 2^{n-1}$, so $\dfrac{1}{n!} \le \dfrac{1}{2^{n-1}}$. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$ is a convergent geometric series $\left[|r| = \frac{1}{2} < 1\right]$, so $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$ converges by the Direct Comparison Test.

36. $\dfrac{n!}{n^n} = \dfrac{1\cdot 2\cdot 3\cdot\cdots\cdot(n-1)n}{n\cdot n\cdot n\cdot\cdots\cdot n\cdot n} \le \dfrac{1}{n}\cdot\dfrac{2}{n}\cdot 1\cdot 1\cdot\cdots\cdot 1$ for $n \ge 2$, so since $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n^2}$ converges $[p = 2 > 1]$, $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges also by the Direct Comparison Test.

37. Use the Limit Comparison Test with $a_n = \sin\left(\dfrac{1}{n}\right)$ and $b_n = \dfrac{1}{n}$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin(1/n)}{1/n} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 > 0$. Since $\displaystyle\sum_{n=1}^{\infty}b_n$ is the divergent harmonic series,

$\displaystyle\sum_{n=1}^{\infty}\sin(1/n)$ also diverges. [Note that we could also use l'Hospital's Rule to evaluate the limit:

$\displaystyle\lim_{x\to\infty}\frac{\sin(1/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\cos(1/x)\cdot\left(-1/x^2\right)}{-1/x^2} = \lim_{x\to\infty}\cos\frac{1}{x} = \cos 0 = 1.]$

38. Use the Limit Comparison Test with $a_n = \sin^2\left(\dfrac{1}{n}\right)$ and $b_n = \dfrac{1}{n^2}$:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin^2\left(\dfrac{1}{n}\right)}{\dfrac{1}{n^2}} = \lim_{n\to\infty}\left[\frac{\sin\left(\dfrac{1}{n}\right)}{\dfrac{1}{n}}\right]^2 = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \qquad \left[\text{where } x = \frac{1}{n}\right]$$

[continued]

Now, $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ (see Equation 3.3.5) and the squaring function is continuous at $x = 1$, so $\lim\limits_{x \to 0} \left(\dfrac{\sin x}{x}\right)^2 = 1^2 = 1 > 0$ [Theorem 2.5.8]. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is a convergent p-series $[p = 2 > 1]$, the series $\sum\limits_{n=1}^{\infty} \sin^2\left(\dfrac{1}{n}\right)$ also converges.

39. Use the Limit Comparison Test with $a_n = \dfrac{1}{n} \tan\left(\dfrac{1}{n}\right)$ and $b_n = \dfrac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{1}{n}\tan\left(\dfrac{1}{n}\right)}{\dfrac{1}{n^2}} = \lim_{n\to\infty} n \tan\left(\frac{1}{n}\right) = \lim_{x\to 0} \frac{\tan x}{x} \quad \left[\text{where } x = \frac{1}{n}\right] \quad \overset{\text{H}}{=} \lim_{x\to 0} \frac{\sec^2 x}{1} = \frac{1}{1} = 1 > 0$$

Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is a convergent p-series $[p = 2 > 1]$, the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n} \tan\left(\dfrac{1}{n}\right)$ also converges.

40. Use the Limit Comparison Test with $a_n = \dfrac{1}{n^{1+1/n}}$ and $b_n = \dfrac{1}{n}$. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n}{n^{1+1/n}} = \lim\limits_{n\to\infty} \dfrac{1}{n^{1/n}} = 1$ $\left[\text{since } \lim\limits_{x\to\infty} x^{1/x} = 1 \text{ by l'Hospital's Rule}\right]$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges [harmonic series] $\Rightarrow$ $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{1+1/n}}$ diverges.

41. $\sum\limits_{n=1}^{10} \dfrac{1}{5+n^5} = \dfrac{1}{5+1^5} + \dfrac{1}{5+2^5} + \dfrac{1}{5+3^5} + \cdots + \dfrac{1}{5+10^5} \approx 0.19926$. Now $\dfrac{1}{5+n^5} < \dfrac{1}{n^5}$, so the error is

$$R_{10} \le T_{10} \le \int_{10}^{\infty} \frac{1}{x^5}\,dx = \lim_{t\to\infty} \int_{10}^{t} x^{-5}\,dx = \lim_{t\to\infty} \left[\frac{-1}{4x^4}\right]_{10}^{t} = \lim_{t\to\infty} \left(\frac{-1}{4t^4} + \frac{1}{40{,}000}\right) = \frac{1}{40{,}000} = 0.000\,025.$$

42. $\sum\limits_{n=1}^{10} \dfrac{e^{1/n}}{n^4} = \dfrac{e^{1/1}}{1^4} + \dfrac{e^{1/2}}{2^4} + \dfrac{e^{1/3}}{3^4} + \cdots + \dfrac{e^{1/10}}{10^4} \approx 2.84748$. Now $\dfrac{e^{1/n}}{n^4} \le \dfrac{e}{n^4}$ for $n \ge 1$, so the error is

$$R_{10} \le T_{10} \le \int_{10}^{\infty} \frac{e}{x^4}\,dx = \lim_{t\to\infty} \int_{10}^{t} ex^{-4}\,dx = \lim_{t\to\infty} \left[\frac{-e}{3x^3}\right]_{10}^{t} = \lim_{t\to\infty} \left(\frac{-e}{3t^3} + \frac{e}{3000}\right) = \frac{e}{3000} \approx 0.000\,906.$$

43. $\sum\limits_{n=1}^{10} 5^{-n}\cos^2 n = \dfrac{\cos^2 1}{5} + \dfrac{\cos^2 2}{5^2} + \dfrac{\cos^2 3}{5^3} + \cdots + \dfrac{\cos^2 10}{5^{10}} \approx 0.07393$. Now $\dfrac{\cos^2 n}{5^n} \le \dfrac{1}{5^n}$, so the error is

$$R_{10} \le T_{10} \le \int_{10}^{\infty} \frac{1}{5^x}\,dx = \lim_{t\to\infty} \int_{10}^{t} 5^{-x}\,dx = \lim_{t\to\infty} \left[-\frac{5^{-x}}{\ln 5}\right]_{10}^{t} = \lim_{t\to\infty} \left(-\frac{5^{-t}}{\ln 5} + \frac{5^{-10}}{\ln 5}\right) = \frac{1}{5^{10}\ln 5} < 6.4 \times 10^{-8}.$$

44. $\sum\limits_{n=1}^{10} \dfrac{1}{3^n+4^n} = \dfrac{1}{3^1+4^1} + \dfrac{1}{3^2+4^2} + \dfrac{1}{3^3+4^3} + \cdots + \dfrac{1}{3^{10}+4^{10}} \approx 0.19788$. Now $\dfrac{1}{3^n+4^n} < \dfrac{1}{3^n+3^n} = \dfrac{1}{2\cdot 3^n}$, so the error is

$$\begin{aligned} R_{10} \le T_{10} \le \int_{10}^{\infty} \frac{1}{2\cdot 3^x}\,dx &= \lim_{t\to\infty} \int_{10}^{t} \frac{1}{2}\cdot 3^{-x}\,dx = \lim_{t\to\infty} \left[-\frac{1}{2}\,\frac{3^{-x}}{\ln 3}\right]_{10}^{t} = \lim_{t\to\infty} \left(-\frac{1}{2}\,\frac{3^{-t}}{\ln 3} + \frac{1}{2}\,\frac{3^{-10}}{\ln 3}\right) \\ &= \frac{1}{2\cdot 3^{10}\ln 3} < 7.7 \times 10^{-6}. \end{aligned}$$

45. Since $\dfrac{d_n}{10^n} \le \dfrac{9}{10^n}$ for each n, and since $\sum\limits_{n=1}^{\infty} \dfrac{9}{10^n}$ is a convergent geometric series $\left(|r| = \frac{1}{10} < 1\right)$, $0.d_1d_2d_3\ldots = \sum\limits_{n=1}^{\infty} \dfrac{d_n}{10^n}$ will always converge by the Comparison Test.

46. Clearly, if $p < 0$ then the series diverges, since $\lim_{n\to\infty} \frac{1}{n^p \ln n} = \infty$. If $0 \le p \le 1$, then $n^p \ln n \le n \ln n \Rightarrow$

$\frac{1}{n^p \ln n} \ge \frac{1}{n \ln n}$ and $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges (Exercise 11.3.23), so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ diverges. If $p > 1$, use the Limit Comparison

Test with $a_n = \frac{1}{n^p \ln n}$ and $b_n = \frac{1}{n^p}$. $\sum_{n=2}^{\infty} b_n$ converges, and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$, so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ also converges.

(Or use the Comparison Test, since $n^p \ln n > n^p$ for $n > e$.) In summary, the series converges if and only if $p > 1$.

47. Using Formula 1.5.10, we have $(\ln n)^{\ln \ln n} = e^{(\ln \ln n)(\ln \ln n)} \quad [n > 1] \quad = e^{(\ln \ln n)^2} < e^{\left(\sqrt{\ln n}\right)^2} = e^{\ln n} = n$. So

$\frac{1}{(\ln n)^{\ln \ln n}} > \frac{1}{n}$ for all $n \ge 2$. Thus, the series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln \ln n}}$ diverges by comparison with the partial harmonic series

$\sum_{n=2}^{\infty} \frac{1}{n}$, which diverges.

48. (a) Since $\lim_{n\to\infty} (a_n/b_n) = 0$, there is a number $N > 0$ such that $|a_n/b_n - 0| < 1$ for all $n > N$, and so $a_n < b_n$ since a_n and b_n are positive. Thus, since $\sum b_n$ converges, so does $\sum a_n$ by the Direct Comparison Test.

(b) (i) If $a_n = \frac{\ln n}{n^3}$ and $b_n = \frac{1}{n^2}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$. Now $\sum b_n$ is a convergent p-series $[p = 2 > 1]$, so $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges by part (a).

(ii) If $a_n = 1 - \cos\left(\frac{1}{n^2}\right)$ and $b_n = \frac{1}{n^2}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1 - \cos(1/n^2)}{1/n^2} = \lim_{x\to\infty} \frac{1 - \cos(1/x^2)}{1/x^2} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\sin(1/x^2) \cdot (-2/x^3)}{-2/x^3} = \lim_{x\to\infty} \sin\left(\frac{1}{x^2}\right) = 0.$$

Now $\sum b_n$ is a convergent p-series $[p = 2 > 1]$, so $\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n^2}\right)$ converges by part (a).

49. (a) Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, there is an integer N such that $\frac{a_n}{b_n} > 1$ whenever $n > N$. (Take $M = 1$ in Definition 11.1.3.) Then $a_n > b_n$ whenever $n > N$ and since $\sum b_n$ is divergent, $\sum a_n$ is also divergent by the Comparison Test.

(b) (i) If $a_n = \frac{1}{\ln n}$ and $b_n = \frac{1}{n}$ for $n \ge 2$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty$, so by part (a), $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent.

(ii) If $a_n = \frac{\ln n}{n}$ and $b_n = \frac{1}{n}$, then $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \ln n = \lim_{x\to\infty} \ln x = \infty$, so $\sum_{n=1}^{\infty} a_n$ diverges by part (a).

50. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum b_n$ diverges while $\sum a_n$ converges.

51. $\lim\limits_{n\to\infty} na_n = \lim\limits_{n\to\infty} \dfrac{a_n}{1/n}$, so we apply the Limit Comparison Test with $b_n = \dfrac{1}{n}$. Since $\lim\limits_{n\to\infty} na_n > 0$ we know that either both series converge or both series diverge, and we also know that $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges [p-series with $p = 1$]. Therefore, $\sum a_n$ must be divergent.

52. First we observe that, by l'Hospital's Rule, $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = \lim\limits_{x\to 0} \dfrac{1}{1+x} = 1$. Also, if $\sum a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$ by Theorem 11.2.6. Therefore, $\lim\limits_{n\to\infty} \dfrac{\ln(1+a_n)}{a_n} = \lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1 > 0$. We are given that $\sum a_n$ is convergent and $a_n > 0$. Thus, $\sum \ln(1+a_n)$ is convergent by the Limit Comparison Test.

53. Yes. Since $\sum a_n$ is a convergent series with positive terms, $\lim\limits_{n\to\infty} a_n = 0$ by Theorem 11.2.6, and $\sum b_n = \sum \sin(a_n)$ is a series with positive terms (for large enough n). We have $\lim\limits_{n\to\infty} \dfrac{b_n}{a_n} = \lim\limits_{n\to\infty} \dfrac{\sin(a_n)}{a_n} = 1 > 0$ by Theorem 3.3.5. Thus, $\sum b_n$ is also convergent by the Limit Comparison Test.

54. Since $\sum a_n$ converges, $\lim\limits_{n\to\infty} a_n = 0$, so there exists N such that $|a_n - 0| < 1$ for all $n > N \;\Rightarrow\; 0 \le a_n < 1$ for all $n > N \;\Rightarrow\; 0 \le a_n^2 \le a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

55. (a) False. The series $\sum a_n = \sum \dfrac{1}{n}$ and $\sum b_n = \sum \dfrac{1}{n}$ are both divergent, but $\sum a_n b_n = \sum \dfrac{1}{n} \cdot \dfrac{1}{n} = \sum \dfrac{1}{n^2}$ is convergent.

(b) False. The series $\sum a_n = \sum \dfrac{1}{n^2}$ converges and $\sum b_n = \sum \dfrac{1}{n}$ diverges, but $\sum a_n b_n = \sum \dfrac{1}{n^2} \cdot \dfrac{1}{n} = \sum \dfrac{1}{n^3}$ converges.

(c) True. Since $\sum a_n$ converges, for suffficiently large n, $a_n < 1 \;\Rightarrow\; a_n b_n < b_n$. Since $\sum b_n$ converges, it follows by the Direct Comparison Test that $\sum a_n b_n$ converges.

11.5 Alternating Series and Absolute Convergence

1. (a) An alternating series is a series whose terms are alternately positive and negative.

(b) An alternating series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (-1)^{n-1} b_n$, where $b_n = |a_n|$, converges if $0 < b_{n+1} \le b_n$ for all n and $\lim\limits_{n\to\infty} b_n = 0$. (This is the Alternating Series Test.)

(c) The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n = s - s_n$ and the size of the error is smaller than b_{n+1}; that is, $|R_n| \le b_{n+1}$. (This is the Alternating Series Estimation Theorem.)

2. $\dfrac{2}{3} - \dfrac{2}{5} + \dfrac{2}{7} - \dfrac{2}{9} + \dfrac{2}{11} - \cdots = \sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{2}{2n+1}$. Now $b_n = \dfrac{2}{2n+1} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series converges by the Alternating Series Test.

3. $-\frac{2}{5}+\frac{4}{6}-\frac{6}{7}+\frac{8}{8}-\frac{10}{9}+\cdots=\sum\limits_{n=1}^{\infty}(-1)^n\frac{2n}{n+4}$. Now $\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}\frac{2n}{n+4}=\lim\limits_{n\to\infty}\frac{2}{1+4/n}=\frac{2}{1}\neq 0$. Since $\lim\limits_{n\to\infty}a_n\neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

4. $\frac{1}{\ln 3}-\frac{1}{\ln 4}+\frac{1}{\ln 5}-\frac{1}{\ln 6}+\frac{1}{\ln 7}-\cdots=\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{\ln(n+2)}$. Now $b_n=\frac{1}{\ln(n+2)}>0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty}b_n=0$, so the series converges by the Alternating Series Test.

5. $\sum\limits_{n=1}^{\infty}a_n=\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{3+5n}=\sum\limits_{n=1}^{\infty}(-1)^{n-1}b_n$. Now $b_n=\frac{1}{3+5n}>0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty}b_n=0$, so the series converges by the Alternating Series Test.

6. $\sum\limits_{n=0}^{\infty}a_n=\sum\limits_{n=0}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n+1}}=\sum\limits_{n=0}^{\infty}(-1)^{n+1}b_n$. Now $b_n=\frac{1}{\sqrt{n+1}}>0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty}b_n=0$, so the series converges by the Alternating Series Test.

7. $\sum\limits_{n=1}^{\infty}a_n=\sum\limits_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n+1}=\sum\limits_{n=1}^{\infty}(-1)^nb_n$. Now $\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}\frac{3-1/n}{2+1/n}=\frac{3}{2}\neq 0$. Since $\lim\limits_{n\to\infty}a_n\neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

8. $\sum\limits_{n=1}^{\infty}a_n=\sum\limits_{n=1}^{\infty}(-1)^n\frac{n^2}{n^2+n+1}=\sum\limits_{n=1}^{\infty}(-1)^nb_n$. Now $\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}\frac{n^2}{n^2+n+1}=\lim\limits_{n\to\infty}\frac{1}{1+1/n+1/n^2}=1\neq 0$. Since $\lim\limits_{n\to\infty}a_n\neq 0$, the series diverges by the Test for Divergence.

9. $\sum\limits_{n=1}^{\infty}a_n=\sum\limits_{n=1}^{\infty}(-1)^ne^{-n}=\sum\limits_{n=1}^{\infty}(-1)^nb_n$. Now $b_n=\frac{1}{e^n}>0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty}b_n=0$, so the series converges by the Alternating Series Test.

10. $b_n=\frac{\sqrt{n}}{2n+3}>0$ for $n\geq 1$. $\{b_n\}$ is decreasing for $n\geq 2$ since

$$\left(\frac{\sqrt{x}}{2x+3}\right)'=\frac{(2x+3)\left(\frac{1}{2}x^{-1/2}\right)-x^{1/2}(2)}{(2x+3)^2}=\frac{\frac{1}{2}x^{-1/2}[(2x+3)-4x]}{(2x+3)^2}=\frac{3-2x}{2\sqrt{x}\,(2x+3)^2}<0 \text{ for } x>\tfrac{3}{2}.$$

Also, $\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}\frac{\sqrt{n}/\sqrt{n}}{(2n+3)/\sqrt{n}}=\lim\limits_{n\to\infty}\frac{1}{2\sqrt{n}+3/\sqrt{n}}=0$. Thus, the series $\sum\limits_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{2n+3}$ converges by the Alternating Series Test.

11. $b_n=\frac{n^2}{n^3+4}>0$ for $n\geq 1$. $\{b_n\}$ is decreasing for $n\geq 2$ since

$$\left(\frac{x^2}{x^3+4}\right)'=\frac{(x^3+4)(2x)-x^2(3x^2)}{(x^3+4)^2}=\frac{x(2x^3+8-3x^3)}{(x^3+4)^2}=\frac{x(8-x^3)}{(x^3+4)^2}<0 \text{ for } x>2. \text{ Also,}$$

$\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}\frac{1/n}{1+4/n^3}=0$. Thus, the series $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$ converges by the Alternating Series Test.

12. $b_n = \dfrac{n}{2^n} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 2$ since $\left(\dfrac{x}{2^x}\right)' = \dfrac{2^x - x\,2^x \ln 2}{(2^x)^2} = \dfrac{1 - x\ln 2}{2^x} < 0$ for $x > \dfrac{1}{\ln 2} \approx 1.4$.

Also, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{n}{2^n} \overset{\text{H}}{=} \lim\limits_{n\to\infty} \dfrac{1}{2^n \ln 2} = 0$. Thus, the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{n}{2^n}$ converges by the Alternating Series Test.

13. $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} e^{2/n} = e^0 = 1$, so $\lim\limits_{n\to\infty} (-1)^{n-1} e^{2/n}$ does not exist. Thus, the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$ diverges by the Test for Divergence.

14. $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \arctan n = \frac{\pi}{2}$, so $\lim\limits_{n\to\infty} (-1)^{n-1} \arctan n$ does not exist. Thus, the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \arctan n$ diverges by the Test for Divergence.

15. $a_n = \dfrac{\sin\left(n + \frac{1}{2}\right)\pi}{1 + \sqrt{n}} = \dfrac{(-1)^n}{1 + \sqrt{n}}$. Now $b_n = \dfrac{1}{1 + \sqrt{n}} > 0$ for $n \geq 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=0}^{\infty} \dfrac{\sin\left(n + \frac{1}{2}\right)\pi}{1 + \sqrt{n}}$ converges by the Alternating Series Test.

16. $a_n = \dfrac{n \cos n\pi}{2^n} = (-1)^n \dfrac{n}{2^n} = (-1)^n b_n$. $\{b_n\}$ is decreasing for $n \geq 2$ since

$(x2^{-x})' = x(-2^{-x} \ln 2) + 2^{-x} = 2^{-x}(1 - x\ln 2) < 0$ for $x > \dfrac{1}{\ln 2}$ $[\approx 1.4]$. Also, $\lim\limits_{n\to\infty} b_n = 0$ since

$\lim\limits_{x\to\infty} \dfrac{x}{2^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{2^x \ln 2} = 0$. Thus, the series $\sum\limits_{n=1}^{\infty} \dfrac{n \cos n\pi}{2^n}$ converges by the Alternating Series Test.

17. $\sum\limits_{n=1}^{\infty} (-1)^n \sin\dfrac{\pi}{n}$. $b_n = \sin\dfrac{\pi}{n} > 0$ for $n \geq 2$ and $\sin\dfrac{\pi}{n} \geq \sin\dfrac{\pi}{n+1}$, and $\lim\limits_{n\to\infty} \sin\dfrac{\pi}{n} = \sin 0 = 0$, so the series converges by the Alternating Series Test.

18. $\sum\limits_{n=1}^{\infty} (-1)^n \cos\dfrac{\pi}{n}$. $\lim\limits_{n\to\infty} \cos\dfrac{\pi}{n} = \cos(0) = 1$, so $\lim\limits_{n\to\infty} (-1)^n \cos\dfrac{\pi}{n}$ does not exist and the series diverges by the Test for Divergence.

19. $b_n = \dfrac{n^2}{5^n} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 2$ since

$\left(\dfrac{x^2}{5^x}\right)' = \dfrac{5^x \cdot 2x - x^2\, 5^x \ln 5}{(5^x)^2} = \dfrac{x\,5^x(2 - x\ln 5)}{(5^x)^2} = \dfrac{x(2 - x\ln 5)}{5^x} < 0$ for $x > \dfrac{2}{\ln 5} \approx 1.2$. Also,

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{n^2}{5^n} \overset{\text{H}}{=} \lim\limits_{n\to\infty} \dfrac{2n}{5^n \ln 5} \overset{\text{H}}{=} \lim\limits_{n\to\infty} \dfrac{2}{5^n (\ln 5)^2} = 0$. Thus, the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{n^2}{5^n}$ converges by the Alternating Series Test.

20. $b_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{1} \cdot \dfrac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \dfrac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing and

$\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$ converges by the Alternating Series Test.

21. (a) A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent. If a series is absolutely convergent, then it is convergent.

(b) A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent; that is, if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

(c) Suppose the series of positive terms $\sum_{n=1}^{\infty} b_n$ converges. Then $\sum |(-1)^n\, b_n| = \sum |b_n| = \sum b_n$ also converges, so $\sum_{n=1}^{\infty} (-1)^n\, b_n$ is absolutely convergent (and therefore convergent).

22. $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \left| \dfrac{(-1)^n}{n^4} \right| = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^4}$. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^4}$ is a convergent p-series $[p = 4 > 1]$, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^4}$ is absolutely convergent.

23. $b_n = \dfrac{1}{\sqrt[3]{n^2}} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 1$, and $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt[3]{n^2}} = 0$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{\sqrt[3]{n^2}}$ converges by the Alternating Series Test. Also, observe that $\sum\limits_{n=1}^{\infty} \left| \dfrac{(-1)^{n-1}}{\sqrt[3]{n^2}} \right| = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{2/3}}$ is divergent since it is a p-series with $p = \frac{2}{3} \leq 1$. Thus, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{\sqrt[3]{n^2}}$ is conditionally convergent.

24. Since $\lim\limits_{n\to\infty} \dfrac{n^2}{n^2+1} = \lim\limits_{n\to\infty} \dfrac{1}{1+1/n^2} = \dfrac{1}{1+0} = 1 \neq 0$ and $\lim\limits_{n\to\infty} (-1)^{n+1}$ does not exist, $\lim\limits_{n\to\infty} (-1)^{n+1} \dfrac{n^2}{n^2+1}$ does not exist, so the series $\sum\limits_{n=0}^{\infty} (-1)^{n+1} \dfrac{n^2}{n^2+1}$ diverges by the Test for Divergence.

25. $b_n = \dfrac{1}{5n+1} > 0$ for $n \geq 1$, $\{b_n\}$ is decreasing for $n \geq 1$, and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{5n+1}$ converges by the Alternating Series Test. To determine absolute convergence, choose $a_n = \dfrac{1}{n}$ to get

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{1/n}{1/(5n+1)} = \lim\limits_{n\to\infty} \dfrac{5n+1}{n} = 5 > 0$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{5n+1}$ diverges by the Limit Comparison Test with the harmonic series. Thus, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{5n+1}$ is conditionally convergent.

26. $\sum\limits_{n=1}^{\infty} \dfrac{-n}{n^2+1} = -\sum\limits_{n=1}^{\infty} \dfrac{n}{n^2+1}$. Use the Limit Comparison Test with $a_n = \dfrac{n}{n^2+1}$ and $b_n = \dfrac{1}{n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n/\left(n^2+1\right)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+1/n^2} = \frac{1}{1+0} = 1 > 0$$

Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ is a divergent p-series $[p = 1 \leq 1]$, the series $\sum\limits_{n=1}^{\infty} \dfrac{n}{n^2+1}$ also diverges, and hence, the negative of this series, $\sum\limits_{n=1}^{\infty} \dfrac{-n}{n^2+1}$, diverges.

27. $\sum\limits_{n=1}^{\infty}|a_n| = \sum\limits_{n=1}^{\infty}\left|\dfrac{(-1)^n}{n^2+1}\right| = \sum\limits_{n=1}^{\infty}\dfrac{1}{n^2+1}$. Since $\dfrac{1}{n^2+1} < \dfrac{1}{n^2}$ and $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ is a convergent p-series $[p = 2 > 1]$, the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2+1}$ is convergent by the Direct Comparison Test. Thus, $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^2+1}$ is absolutely convergent.

28. $0 < \left|\dfrac{\sin n}{2^n}\right| < \dfrac{1}{2^n}$ for $n \geq 1$ and $\sum\limits_{n=1}^{\infty}\dfrac{1}{2^n}$ is a convergent geometric series $[r = \frac{1}{2} < 1]$, so $\sum\limits_{n=1}^{\infty}\left|\dfrac{\sin n}{2^n}\right|$ converges by direct comparison and the series $\sum\limits_{n=1}^{\infty}\dfrac{\sin n}{2^n}$ is absolutely convergent.

29. $0 < \left|\dfrac{1+2\sin n}{n^3}\right| < \dfrac{3}{n^3}$ for $n \geq 1$ and $3\sum\limits_{n=1}^{\infty}\dfrac{1}{n^3}$ is a constant times a convergent p-series $[p = 3 > 1]$, so $\sum\limits_{n=1}^{\infty}\left|\dfrac{1+2\sin n}{n^3}\right|$ converges by direct comparison and the series $\sum\limits_{n=1}^{\infty}\dfrac{1+2\sin n}{n^3}$ is absolutely convergent.

30. $b_n = \dfrac{n}{n^2+4} > 0$ for $n \geq 1$, $\{b_n\}$ is decreasing for $n \geq 2$, and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{n^2+4}$ converges by the Alternating Series Test. To determine absolute convergence, choose $a_n = \dfrac{1}{n}$ to get

$\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \lim\limits_{n\to\infty}\dfrac{1/n}{n/(n^2+4)} = \lim\limits_{n\to\infty}\dfrac{n^2+4}{n^2} = \lim\limits_{n\to\infty}\dfrac{1+4/n^2}{1} = 1 > 0$, so $\sum\limits_{n=1}^{\infty}\dfrac{n}{n^2+4}$ diverges by the Limit Comparison Test with the harmonic series. Thus, the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{n^2+4}$ is conditionally convergent.

31. $\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{\ln n}$ converges by the Alternating Series Test since $\lim\limits_{n\to\infty}\dfrac{1}{\ln n} = 0$ and $\left\{\dfrac{1}{\ln n}\right\}$ is decreasing. Now $\ln n < n$, so $\dfrac{1}{\ln n} > \dfrac{1}{n}$, and since $\sum\limits_{n=2}^{\infty}\dfrac{1}{n}$ is the divergent (partial) harmonic series, $\sum\limits_{n=2}^{\infty}\dfrac{1}{\ln n}$ diverges by the Direct Comparison Test. Thus, $\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{\ln n}$ is conditionally convergent.

32. $b_n = \dfrac{n}{\sqrt{n^3+2}} > 0$ for $n \geq 1$, $\{b_n\}$ is decreasing for $n \geq 2$, and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{n}{\sqrt{n^3+2}}$ converges by the Alternating Series Test. To determine absolute convergence, choose $a_n = \dfrac{1}{\sqrt{n}}$ to get

$\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n}}\cdot\dfrac{\sqrt{n^3+2}}{n}\right) = \lim\limits_{n\to\infty}\dfrac{\sqrt{n^3+2}}{\sqrt{n^3}} = \lim\limits_{n\to\infty}\sqrt{1+\dfrac{2}{n^3}} = 1 > 0$, so $\sum\limits_{n=1}^{\infty}\dfrac{n}{\sqrt{n^3+2}}$ diverges by limit comparision with the divergent p-series $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$ $\left[p = \frac{1}{2} \leq 1\right]$. Thus, $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{n}{\sqrt{n^3+2}}$ is conditionally convergent.

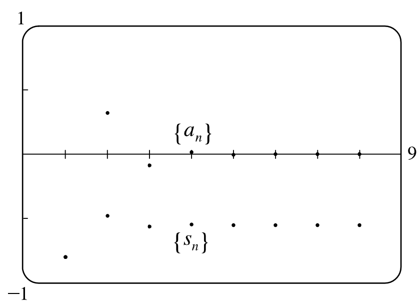
33. $a_n = \dfrac{\cos n\pi}{3n+2} = (-1)^n\dfrac{1}{3n+2} = (-1)^n b_n$. $\{b_n\}$ is decreasing for $n \geq 1$, and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=1}^{\infty}\dfrac{\cos n\pi}{3n+2}$ converges by the Alternating Series Test. To determine absolute convergence, use the Limit Comparison Test with $a_n = \dfrac{1}{n}$:

[continued]

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{1/n}{1/(3n+2)} = \lim\limits_{n\to\infty} \dfrac{3n+2}{n} = \lim\limits_{n\to\infty} \dfrac{3+2/n}{1} = 3 > 0$. Since the harmonic series diverges, so does $\sum\limits_{n=1}^{\infty} \dfrac{1}{3n+2}$. Thus, the series $\sum\limits_{n=1}^{\infty} \dfrac{\cos n\pi}{3n+2}$ is conditionally convergent.

34. The function $f(x) = \dfrac{1}{x\ln x}$ is continuous, positive, and decreasing on $[2, \infty)$.

$\displaystyle\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x\ln x}\,dx = \lim_{t\to\infty} [\ln(\ln x)]_2^t = \lim_{t\to\infty} [(\ln(\ln t) - \ln(\ln 2)] = \infty$, so the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$ diverges by the Integral Test. Now $\{b_n\} = \left\{\dfrac{1}{n\ln n}\right\}$ with $n \geq 2$ is a decreasing sequence of positive terms and $\lim\limits_{n\to\infty} b_n = 0$. Thus, $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$ converges by the Alternating Series Test. It follows that $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$ is conditionally convergent.

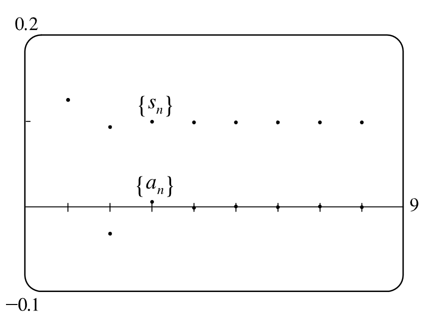
35. 

The graph gives us an estimate for the sum of the series $\sum\limits_{n=1}^{\infty} \dfrac{(-0.8)^n}{n!}$ of -0.55.

$b_8 = \dfrac{(0.8)^n}{8!} \approx 0.000\,004$, so

$$\sum_{n=1}^{\infty} \frac{(-0.8)^n}{n!} \approx s_7 = \sum_{n=1}^{7} \frac{(-0.8)^n}{n!}$$
$$\approx -0.8 + 0.32 - 0.085\overline{3} + 0.0170\overline{6} - 0.002\,731 + 0.000\,364 - 0.000\,042 \approx -0.5507$$

Adding b_8 to s_7 does not change the fourth decimal place of s_7, so the sum of the series, correct to four decimal places, is -0.5507.

36. 

The graph gives us an estimate for the sum of the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{n}{8^n}$ of 0.1.

$b_6 = \dfrac{6}{8^6} \approx 0.000\,023$, so

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{8^n} \approx s_5 = \sum_{n=1}^{5} (-1)^{n-1} \frac{n}{8^n}$$
$$\approx 0.125 - 0.03125 + 0.005\,859 - 0.000\,977 + 0.000\,153 \approx 0.0988$$

Adding b_6 to s_5 does not change the fourth decimal place of s_5, so the sum of the series, correct to four decimal places, is 0.0988.

37. The series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^6}$ satisfies (i) of the Alternating Series Test because $\dfrac{1}{(n+1)^6} < \dfrac{1}{n^6}$ and (ii) $\lim\limits_{n\to\infty} \dfrac{1}{n^6} = 0$, so the series is convergent. Now $b_5 = \dfrac{1}{5^6} = 0.000064 > 0.00005$ and $b_6 = \dfrac{1}{6^6} \approx 0.00002 < 0.00005$, so by the Alternating Series Estimation Theorem, $n = 5$. (That is, since the 6th term is less than the desired error, we need to add the first 5 terms to get the sum to the desired accuracy.)

38. The series $\sum\limits_{n=1}^{\infty} \dfrac{\left(-\frac{1}{3}\right)^n}{n} = \sum\limits_{n=1}^{\infty} (-1)^n \dfrac{1}{n3^n}$ satisfies (i) of the Alternating Series Test because $\dfrac{1}{(n+1)3^{n+1}} < \dfrac{1}{n3^n}$ and (ii) $\lim\limits_{n\to\infty} \dfrac{1}{n3^n} = 0$, so the series is convergent. Now $b_5 = \dfrac{1}{5 \cdot 3^5} \approx 0.0008 > 0.0005$ and $b_6 = \dfrac{1}{6 \cdot 3^6} \approx 0.0002 < 0.0005$, so by the Alternating Series Estimation Theorem, $n = 5$. (That is, since the 6th term is less than the desired error, we need to add the first 5 terms to get the sum to the desired accuracy.)

39. The series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^2 2^n}$ satisfies (i) of the Alternating Series Test because $\dfrac{1}{(n+1)^2 2^{n+1}} < \dfrac{1}{n^2 2^n}$ and (ii) $\lim\limits_{n\to\infty} \dfrac{1}{n^2 2^n} = 0$, so the series is convergent. Now $b_5 = \dfrac{1}{5^2 2^5} = 0.00125 > 0.0005$ and $b_6 = \dfrac{1}{6^2 2^6} \approx 0.0004 < 0.0005$, so by the Alternating Series Estimation Theorem, $n = 5$. (That is, since the 6th term is less than the desired error, we need to add the first 5 terms to get the sum to the desired accuracy.)

40. The series $\sum\limits_{n=1}^{\infty} \left(-\dfrac{1}{n}\right)^n = \sum\limits_{n=1}^{\infty} (-1)^n \dfrac{1}{n^n}$ satisfies (i) of the Alternating Series Test because $\dfrac{1}{(n+1)^{n+1}} < \dfrac{1}{n^n}$ and (ii) $\lim\limits_{n\to\infty} \dfrac{1}{n^n} = 0$, so the series is convergent. Now $b_5 = \dfrac{1}{5^5} = 0.00032 > 0.00005$ and $b_6 = \dfrac{1}{6^6} \approx 0.00002 < 0.00005$, so by the Alternating Series Estimation Theorem, $n = 5$. (That is, since the 6th term is less than the desired error, we need to add the first 5 terms to get the sum to the desired accuracy.)

41. $b_4 = \dfrac{1}{8!} = \dfrac{1}{40{,}320} \approx 0.000\,025$, so

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \approx s_3 = \sum_{n=1}^{3} \frac{(-1)^n}{(2n)!} = -\frac{1}{2} + \frac{1}{24} - \frac{1}{720} \approx -0.459\,722$$

Adding b_4 to s_3 does not change the fourth decimal place of s_3, so by the Alternating Series Estimation Theorem, the sum of the series, correct to four decimal places, is -0.4597.

42. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^6} \approx s_9 = \dfrac{1}{1^6} - \dfrac{1}{2^6} + \dfrac{1}{3^6} - \dfrac{1}{4^6} + \dfrac{1}{5^6} - \dfrac{1}{6^6} + \dfrac{1}{7^6} - \dfrac{1}{8^6} + \dfrac{1}{9^6} \approx 0.985\,552$. Subtracting $b_{10} = 1/10^6$ from s_9 does not change the fourth decimal place of s_9, so by the Alternating Series Estimation Theorem, the sum of the series, correct to four decimal places, is 0.9856.

43. $\sum\limits_{n=1}^{\infty} (-1)^n n e^{-2n} \approx s_5 = -\dfrac{1}{e^2} + \dfrac{2}{e^4} - \dfrac{3}{e^6} + \dfrac{4}{e^8} - \dfrac{5}{e^{10}} \approx -0.105\,025$. Adding $b_6 = 6/e^{12} \approx 0.000\,037$ to s_5 does not change the fourth decimal place of s_5, so by the Alternating Series Estimation Theorem, the sum of the series, correct to four decimal places, is -0.1050.

44. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n4^n} \approx s_6 = \frac{1}{4} - \frac{1}{2\cdot 4^2} + \frac{1}{3\cdot 4^3} - \frac{1}{4\cdot 4^4} + \frac{1}{5\cdot 4^5} - \frac{1}{6\cdot 4^6} \approx 0.223136$. Adding $b_7 = \dfrac{1}{7\cdot 4^7} \approx 0.000\,0087$ to s_6 does not change the fourth decimal place of s_6, so by the Alternating Series Estimation Theorem, the sum of the series, correct to four decimal places, is 0.2231.

45. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{49} - \frac{1}{50} + \frac{1}{51} - \frac{1}{52} + \cdots$. The 50th partial sum of this series is an underestimate, since $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = s_{50} + \left(\frac{1}{51} - \frac{1}{52}\right) + \left(\frac{1}{53} - \frac{1}{54}\right) + \cdots$, and the terms in parentheses are all positive. The result can be seen geometrically in Figure 1.

46. If $p > 0$, $\dfrac{1}{(n+1)^p} \le \dfrac{1}{n^p}$ ($\{1/n^p\}$ is decreasing) and $\displaystyle\lim_{n\to\infty}\frac{1}{n^p} = 0$, so the series converges by the Alternating Series Test. If $p \le 0$, $\displaystyle\lim_{n\to\infty}\frac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. Thus, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}$ converges $\Leftrightarrow$ $p > 0$.

47. Clearly $b_n = \dfrac{1}{n+p}$ is decreasing and eventually positive and $\displaystyle\lim_{n\to\infty} b_n = 0$ for any p. So the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n+p}$ converges (by the Alternating Series Test) for any p for which every b_n is defined, that is, $n + p \ne 0$ for $n \ge 1$, or p is not a negative integer.

48. Let $f(x) = \dfrac{(\ln x)^p}{x}$. Then $f'(x) = \dfrac{(\ln x)^{p-1}(p - \ln x)}{x^2} < 0$ if $x > e^p$ so f is eventually decreasing for every p. Clearly $\displaystyle\lim_{n\to\infty}\frac{(\ln n)^p}{n} = 0$ if $p \le 0$, and if $p > 0$ we can apply l'Hospital's Rule $[\![p+1]\!]$ times to get a limit of 0 as well. So the series $\displaystyle\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(\ln n)^p}{n}$ converges for all p (by the Alternating Series Test).

49. $\sum b_{2n} = \sum 1/(2n)^2$ clearly converges (by direct comparison with the p-series for $p = 2$). So suppose that $\sum(-1)^{n-1}b_n$ converges. Then by Theorem 11.2.8(ii), so does $\sum\left[(-1)^{n-1}b_n + b_n\right] = 2\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right) = 2\sum\dfrac{1}{2n-1}$. But this diverges by direct comparison with the harmonic series, a contradiction. Therefore, $\sum(-1)^{n-1}b_n$ must diverge. The Alternating Series Test does not apply since $\{b_n\}$ is not decreasing.

50. (a) We will prove this by induction. Let $P(n)$ be the proposition that $s_{2n} = h_{2n} - h_n$. $P(1)$ is the statement $s_2 = h_2 - h_1$, which is true since $1 - \frac{1}{2} = \left(1 + \frac{1}{2}\right) - 1$. So suppose that $P(n)$ is true. We will show that $P(n+1)$ must be true as a consequence.

$$\begin{aligned} h_{2n+2} - h_{n+1} &= \left(h_{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}\right) - \left(h_n + \frac{1}{n+1}\right) = (h_{2n} - h_n) + \frac{1}{2n+1} - \frac{1}{2n+2} \\ &= s_{2n} + \frac{1}{2n+1} - \frac{1}{2n+2} = s_{2n+2} \end{aligned}$$

which is $P(n+1)$, and proves that $s_{2n} = h_{2n} - h_n$ for all n.

(b) We know that $h_{2n} - \ln(2n) \to \gamma$ and $h_n - \ln n \to \gamma$ as $n \to \infty$. So

$s_{2n} = h_{2n} - h_n = [h_{2n} - \ln(2n)] - (h_n - \ln n) + [\ln(2n) - \ln n]$, and

$\lim\limits_{n\to\infty} s_{2n} = \gamma - \gamma + \lim\limits_{n\to\infty} [\ln(2n) - \ln n] = \lim\limits_{n\to\infty} (\ln 2 + \ln n - \ln n) = \ln 2$.

51. (a) Since $\sum a_n$ is absolutely convergent, and since $\left|a_n^+\right| \le |a_n|$ and $\left|a_n^-\right| \le |a_n|$ (because a_n^+ and a_n^- each equal either a_n or 0), we conclude by the Direct Comparison Test that both $\sum a_n^+$ and $\sum a_n^-$ must be absolutely convergent. O*r:* Use Theorem 11.2.8.

(b) We will show by contradiction that both $\sum a_n^+$ and $\sum a_n^-$ must diverge. For suppose that $\sum a_n^+$ converged. Then so would $\sum\left(a_n^+ - \frac{1}{2}a_n\right)$ by Theorem 11.2.8. But $\sum\left(a_n^+ - \frac{1}{2}a_n\right) = \sum\left[\frac{1}{2}\left(a_n + |a_n|\right) - \frac{1}{2}a_n\right] = \frac{1}{2}\sum |a_n|$, which diverges because $\sum a_n$ is only conditionally convergent. Hence, $\sum a_n^+$ can't converge. Similarly, neither can $\sum a_n^-$.

52. Let $\sum b_n$ be the rearranged series constructed in the hint. [This series can be constructed by virtue of the result of Exercise 51(b).] This series will have partial sums s_n that oscillate in value back and forth across r. Since $\lim\limits_{n\to\infty} a_n = 0$ (by Theorem 11.2.6), and since the size of the oscillations $|s_n - r|$ is always less than $|a_n|$ because of the way $\sum b_n$ was constructed, we have that $\sum b_n = \lim\limits_{n\to\infty} s_n = r$.

53. Suppose that $\sum a_n$ is conditionally convergent.

(a) $\sum n^2 a_n$ is divergent: Suppose $\sum n^2 a_n$ converges. Then $\lim\limits_{n\to\infty} n^2 a_n = 0$ by Theorem 6 in Section 11.2, so there is an integer $N > 0$ such that $n > N \;\Rightarrow\; n^2 |a_n| < 1$. For $n > N$, we have $|a_n| < \dfrac{1}{n^2}$, so $\sum\limits_{n>N} |a_n|$ converges by comparison with the convergent p-series $\sum\limits_{n>N} \dfrac{1}{n^2}$. In other words, $\sum a_n$ converges absolutely, contradicting the assumption that $\sum a_n$ is conditionally convergent. This contradiction shows that $\sum n^2 a_n$ diverges.

Remark: The same argument shows that $\sum n^p a_n$ diverges for any $p > 1$.

(b) $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{n \ln n}$ is conditionally convergent. It converges by the Alternating Series Test, but does not converge absolutely $\Bigg[$by the Integral Test, since the function $f(x) = \dfrac{1}{x \ln x}$ is continuous, positive, and decreasing on $[2, \infty)$ and $\displaystyle\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{t\to\infty} \int_2^t \frac{dx}{x \ln x} = \lim_{t\to\infty} \Big[\ln(\ln x)\Big]_2^t = \infty\Bigg]$. Setting $a_n = \dfrac{(-1)^n}{n \ln n}$ for $n \ge 2$, we find that $\sum\limits_{n=2}^{\infty} n a_n = \sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{\ln n}$ converges by the Alternating Series Test.

It is easy to find conditionally convergent series $\sum a_n$ such that $\sum n a_n$ diverges. Two examples are $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n}$ and $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{\sqrt{n}}$, both of which converge by the Alternating Series Test and fail to converge absolutely because $\sum |a_n|$ is a p-series with $p \le 1$. In both cases, $\sum n a_n$ diverges by the Test for Divergence.

11.6 The Ratio and Root Tests

1. (a) Since $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 8 > 1$, part (b) of the Ratio Test tells us that the series $\sum a_n$ is divergent.

(b) Since $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 0.8 < 1$, part (a) of the Ratio Test tells us that the series $\sum a_n$ is absolutely convergent (and therefore convergent).

(c) Since $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test fails and the series $\sum a_n$ might converge or it might diverge.

2. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{1}{a_n/a_{n+1}}\right| = \left|\dfrac{1}{2}\right| = \dfrac{1}{2} < 1$ Thus, the series $\sum a_n$ is absolutely convergent (and therefore convergent) by the Ratio Test.

3. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{n+1}{5^{n+1}}\cdot\dfrac{5^n}{n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{1}{5}\cdot\dfrac{n+1}{n}\right| = \dfrac{1}{5}\lim\limits_{n\to\infty}\dfrac{1+1/n}{1} = \dfrac{1}{5}(1) = \dfrac{1}{5} < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{n}{5^n}$ is absolutely convergent by the Ratio Test.

4. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-2)^{n+1}}{(n+1)^2}\cdot\dfrac{n^2}{(-2)^n}\right| = \lim\limits_{n\to\infty}\left|(-2)\dfrac{n^2}{(n+1)^2}\right| = 2\lim\limits_{n\to\infty}\dfrac{1}{(1+1/n)^2} = 2(1) = 2 > 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{(-2)^n}{n^2}$ is divergent by the Ratio Test.

5. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)^n 3^{n+1}}{2^{n+1}(n+1)^3}\cdot\dfrac{2^n n^3}{(-1)^{n-1}3^n}\right| = \lim\limits_{n\to\infty}\left|\left(-\dfrac{3}{2}\right)\dfrac{n^3}{(n+1)^3}\right| = \dfrac{3}{2}\lim\limits_{n\to\infty}\dfrac{1}{(1+1/n)^3} = \dfrac{3}{2}(1) = \dfrac{3}{2} > 1$,

so the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{3^n}{2^n n^3}$ is divergent by the Ratio Test.

6. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-3)^{n+1}}{[2(n+1)+1]!}\cdot\dfrac{(2n+1)!}{(-3)^n}\right| = \lim\limits_{n\to\infty}\left|(-3)\dfrac{1}{(2n+3)(2n+2)}\right| = 3\lim\limits_{n\to\infty}\dfrac{1}{(2n+3)(2n+2)}$

$= 3(0) = 0 < 1$

so the series $\sum\limits_{n=0}^{\infty}\dfrac{(-3)^n}{(2n+1)!}$ is absolutely convergent by the Ratio Test.

7. $\lim\limits_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left|\dfrac{1}{(k+1)!}\cdot\dfrac{k!}{1}\right| = \lim\limits_{k\to\infty}\dfrac{1}{k+1} = 0 < 1$, so the series $\sum\limits_{k=1}^{\infty}\dfrac{1}{k!}$ is absolutely convergent by the Ratio Test.

Since the terms of this series are positive, absolute convergence is the same as convergence.

8. $\lim\limits_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left|\dfrac{(k+1)e^{-(k+1)}}{ke^{-k}}\right| = \lim\limits_{k\to\infty}\left(\dfrac{k+1}{k}\cdot e^{-1}\right) = \dfrac{1}{e}\lim\limits_{k\to\infty}\dfrac{1+1/k}{1} = \dfrac{1}{e}(1) = \dfrac{1}{e} < 1$, so the series $\sum\limits_{k=1}^{\infty}ke^{-k}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

9. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{10^{n+1}}{(n+2)\,4^{2n+3}}\cdot\dfrac{(n+1)\,4^{2n+1}}{10^n}\right] = \lim\limits_{n\to\infty}\left(\dfrac{10}{4^2}\cdot\dfrac{n+1}{n+2}\right) = \dfrac{5}{8} < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{10^n}{(n+1)4^{2n+1}}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

10. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{(n+1)!}{100^{n+1}}\cdot\dfrac{100^n}{n!}\right] = \lim\limits_{n\to\infty}\dfrac{n+1}{100} = \infty$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{n!}{100^n}$ diverges by the Ratio Test.

11. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)\pi^{n+1}}{(-3)^n}\cdot\dfrac{(-3)^{n-1}}{n\pi^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{\pi}{-3}\cdot\dfrac{n+1}{n}\right| = \dfrac{\pi}{3}\lim\limits_{n\to\infty}\dfrac{1+1/n}{1} = \dfrac{\pi}{3}(1) = \dfrac{\pi}{3} > 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{n\pi^n}{(-3)^{n-1}}$ diverges by the Ratio Test. *Or:* Since $\lim\limits_{n\to\infty}|a_n| = \infty$, the series diverges by the Test for Divergence.

12. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)^{10}}{(-10)^{n+2}}\cdot\dfrac{(-10)^{n+1}}{n^{10}}\right| = \lim\limits_{n\to\infty}\left|\dfrac{1}{-10}\left(\dfrac{n+1}{n}\right)^{10}\right| = \dfrac{1}{10}\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{10} = \dfrac{1}{10}(1) = \dfrac{1}{10} < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{n^{10}}{(-10)^{n+1}}$ is absolutely convergent by the Ratio Test.

13. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{\cos[(n+1)\pi/3]}{(n+1)!}\cdot\dfrac{n!}{\cos(n\pi/3)}\right| = \lim\limits_{n\to\infty}\left|\dfrac{\cos[(n+1)\pi/3]}{(n+1)\cos(n\pi/3)}\right| = \lim\limits_{n\to\infty}\dfrac{c}{n+1} = 0 < 1$ (where $0 < c \le 2$ for all positive integers n), so the series $\sum\limits_{n=1}^{\infty}\dfrac{\cos(n\pi/3)}{n!}$ is absolutely convergent by the Ratio Test.

14. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)!}{(n+1)^{n+1}}\cdot\dfrac{n^n}{n!}\right| = \lim\limits_{n\to\infty}\dfrac{(n+1)n^n}{(n+1)^{n+1}} = \lim\limits_{n\to\infty}\dfrac{n^n}{(n+1)^n} = \lim\limits_{n\to\infty}\dfrac{1}{(1+1/n)^n} = \dfrac{1}{e} < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{n!}{n^n}$ is absolutely convergent by the Ratio Test.

15. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)^{100}100^{n+1}}{(n+1)!}\cdot\dfrac{n!}{n^{100}100^n}\right| = \lim\limits_{n\to\infty}\dfrac{100}{n+1}\left(\dfrac{n+1}{n}\right)^{100} = \lim\limits_{n\to\infty}\dfrac{100}{n+1}\left(1+\dfrac{1}{n}\right)^{100}$

$= 0\cdot 1 = 0 < 1$

so the series $\sum\limits_{n=1}^{\infty}\dfrac{n^{100}100^n}{n!}$ is absolutely convergent by the Ratio Test.

16. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{[2(n+1)]!}{[(n+1)!]^2}\cdot\dfrac{(n!)^2}{(2n)!}\right| = \lim\limits_{n\to\infty}\dfrac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim\limits_{n\to\infty}\dfrac{(2+2/n)(2+1/n)}{(1+1/n)(1+1/n)} = \dfrac{2\cdot 2}{1\cdot 1} = 4 > 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{(2n)!}{(n!)^2}$ diverges by the Ratio Test.

17. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)^n(n+1)!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)}\cdot\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(-1)^{n-1}n!}\right| = \lim\limits_{n\to\infty}\dfrac{n+1}{2n+1}$

$= \lim\limits_{n\to\infty}\dfrac{1+1/n}{2+1/n} = \dfrac{1}{2} < 1$,

so the series $1 - \dfrac{2!}{1\cdot 3} + \dfrac{3!}{1\cdot 3\cdot 5} - \dfrac{4!}{1\cdot 3\cdot 5\cdot 7} + \cdots + (-1)^{n-1}\dfrac{n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} + \cdots$ is absolutely convergent by the Ratio Test.

18. $\dfrac{2}{3}+\dfrac{2\cdot 5}{3\cdot 5}+\dfrac{2\cdot 5\cdot 8}{3\cdot 5\cdot 7}+\dfrac{2\cdot 5\cdot 8\cdot 11}{3\cdot 5\cdot 7\cdot 9}+\cdots=\sum\limits_{n=1}^{\infty}\dfrac{2\cdot 5\cdot 8\cdot 11\cdot\cdots\cdot(3n-1)}{3\cdot 5\cdot 7\cdot 9\cdot\cdots\cdot(2n+1)}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|&=\lim_{n\to\infty}\left|\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)(3n+2)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)(2n+3)}\cdot\frac{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}\right|\\&=\lim_{n\to\infty}\frac{3n+2}{2n+3}=\lim_{n\to\infty}\frac{3+2/n}{2+3/n}=\frac{3}{2}>1,\end{aligned}$$

so the given series diverges by the Ratio Test.

19. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left|\dfrac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)(2n+2)}{(n+1)!}\cdot\dfrac{n!}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}\right|=\lim\limits_{n\to\infty}\dfrac{2n+2}{n+1}=\lim\limits_{n\to\infty}\dfrac{2(n+1)}{n+1}=2>1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{n!}$ diverges by the Ratio Test.

20. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left|\dfrac{2^{n+1}\,(n+1)!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)\,(3n+5)}\cdot\dfrac{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}{2^n n!}\right|=\lim\limits_{n\to\infty}\dfrac{2(n+1)}{3n+5}=\dfrac{2}{3}<1$, so the series $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{2^n n!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}$ is absolutely convergent by the Ratio Test.

21. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\dfrac{n^2+1}{2n^2+1}=\lim\limits_{n\to\infty}\dfrac{1+1/n^2}{2+1/n^2}=\dfrac{1}{2}<1$, so the series $\sum\limits_{n=1}^{\infty}\left(\dfrac{n^2+1}{2n^2+1}\right)^n$ is absolutely convergent by the Root Test.

22. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\sqrt[n]{\left|\dfrac{(-2)^n}{n^n}\right|}=\lim\limits_{n\to\infty}\dfrac{2}{n}=0<1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{(-2)^n}{n^n}$ is absolutely convergent by the Root Test.

23. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\sqrt[n]{\left|\dfrac{(-1)^{n-1}}{(\ln n)^n}\right|}=\lim\limits_{n\to\infty}\dfrac{1}{\ln n}=0<1$, so the series $\sum\limits_{n=2}^{\infty}\dfrac{(-1)^{n-1}}{(\ln n)^n}$ is absolutely convergent by the Root Test.

24.
$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{|a_n|}&=\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^{5n}\right|}=\lim_{n\to\infty}\frac{2^5\,n^5}{(n+1)^5}=32\lim_{n\to\infty}\frac{1}{\left(\dfrac{n+1}{n}\right)^5}=32\lim_{n\to\infty}\frac{1}{(1+1/n)^5}\\&=32(1)=32>1,\end{aligned}$$

so the series $\sum\limits_{n=1}^{\infty}\left(\dfrac{-2n}{n+1}\right)^{5n}$ diverges by the Root Test.

25. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\sqrt[n]{\left(1+\dfrac{1}{n}\right)^{n^2}}=\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n=e>1$ [by Equation 3.6.6], so the series $\sum\limits_{n=1}^{\infty}\left(1+\dfrac{1}{n}\right)^{n^2}$ diverges by the Root Test.

26. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\sqrt[n]{|(\arctan n)^n|}=\lim\limits_{n\to\infty}\arctan n=\frac{\pi}{2}>1$, so the series $\sum\limits_{n=0}^{\infty}(\arctan n)^n$ diverges by the Root Test.

27. $\sum\limits_{n=2}^{\infty}a_n=\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n\ln n}{n}=\sum\limits_{n=2}^{\infty}(-1)^n b_n$. Now $b_n=\dfrac{\ln n}{n}>0$ for $n\geq 2$, and $\{b_n\}$ is decreasing for $n\geq 3$ since $\left(\dfrac{\ln x}{x}\right)'=\dfrac{x\cdot\frac{1}{x}-\ln x\cdot 1}{x^2}=\dfrac{1-\ln x}{x^2}<0$ when $\ln x>1$ or $x>e\approx 2.7$. Also, $\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}\dfrac{\ln n}{n}\overset{\text{H}}{=}\lim\limits_{n\to\infty}\dfrac{1/n}{1}=0$, so the series $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n \ln n}{n}$ converges by the Alternating Series Test. To determine absolute convergence, note that

$\left|\dfrac{(-1)^n \ln n}{n}\right| = \dfrac{\ln n}{n} > \dfrac{1}{n}$ for $n \geq 3$, so $\sum\limits_{n=2}^{\infty} \left|\dfrac{(-1)^n \ln n}{n}\right|$ is divergent by direct comparison with $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$, which is divergent.

Hence, $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n \ln n}{n}$ is conditionally convergent.

28. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \sqrt[n]{\left|\left(\dfrac{1-n}{2+3n}\right)^n\right|} = \lim\limits_{n\to\infty} \dfrac{n-1}{3n+2} = \lim\limits_{n\to\infty} \dfrac{1-1/n}{3+2/n} = \dfrac{1}{3} < 1$, so the series $\sum\limits_{n=1}^{\infty} \left(\dfrac{1-n}{2+3n}\right)^n$ is absolutely convergent by the Root Test.

29. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(-9)^{n+1}}{(n+1)10^{n+2}} \cdot \dfrac{n10^{n+1}}{(-9)^n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(-9)n}{10(n+1)}\right| = \dfrac{9}{10} \lim\limits_{n\to\infty} \dfrac{1}{1+1/n} = \dfrac{9}{10}(1) = \dfrac{9}{10} < 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{(-9)^n}{n10^{n+1}}$ is absolutely convergent by the Ratio Test.

30. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(n+1)5^{2n+2}}{10^{n+2}} \cdot \dfrac{10^{n+1}}{n5^{2n}}\right| = \lim\limits_{n\to\infty} \dfrac{5^2(n+1)}{10n} = \dfrac{5}{2} \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right) = \dfrac{5}{2}(1) = \dfrac{5}{2} > 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{n5^{2n}}{10^{n+1}}$ diverges by the Ratio Test. *Or:* Since $\lim\limits_{n\to\infty} a_n = \infty$, the series diverges by the Test for Divergence.

31. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \sqrt[n]{\left|\left(\dfrac{n}{\ln n}\right)^n\right|} = \lim\limits_{n\to\infty} \dfrac{n}{\ln n} = \lim\limits_{x\to\infty} \dfrac{x}{\ln x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{1/x} = \lim\limits_{x\to\infty} x = \infty$, so the series $\sum\limits_{n=2}^{\infty} \left(\dfrac{n}{\ln n}\right)^n$ diverges by the Root Test.

32. $\left|\dfrac{\sin(n\pi/6)}{1+n\sqrt{n}}\right| \leq \dfrac{1}{1+n\sqrt{n}} < \dfrac{1}{n^{3/2}}$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{\sin(n\pi/6)}{1+n\sqrt{n}}$ converges by direct comparison with the convergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ $(p = \frac{3}{2} > 1)$. It follows that the given series is absolutely convergent.

33. $\left|\dfrac{(-1)^n \arctan n}{n^2}\right| < \dfrac{\pi/2}{n^2}$, so since $\sum\limits_{n=1}^{\infty} \dfrac{\pi/2}{n^2} = \dfrac{\pi}{2} \sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges $(p = 2 > 1)$, the given series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n \arctan n}{n^2}$ converges absolutely by the Direct Comparison Test.

34. $\sum\limits_{n=2}^{\infty} a_n = \sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{\sqrt{n}\ln n} = \sum\limits_{n=2}^{\infty} (-1)^n b_n$. Now $b_n = \dfrac{1}{\sqrt{n}\ln n} > 0$ for $n \geq 2$, $\{b_n\}$ is decreasing for $n \geq 2$, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{\sqrt{n}\ln n}$ converges by the Alternating Series Test. Also, observe that $\left|\dfrac{(-1)^n}{\sqrt{n}\ln n}\right| = \dfrac{1}{\sqrt{n}\ln n} > \dfrac{1}{\sqrt{n}\sqrt{n}} = \dfrac{1}{n}$ for $n \geq 2$, so the series $\sum\limits_{n=2}^{\infty} \left|\dfrac{(-1)^n}{\sqrt{n}\ln n}\right|$ is divergent by direct comparison with $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$, which is a divergent (partial) p-series $[p = 1 \leq 1]$. Thus, $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{\sqrt{n}\ln n}$ is conditionally convergent.

35. By the recursive definition, $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{5n+1}{4n+3}\right| = \dfrac{5}{4} > 1$, so the series diverges by the Ratio Test.

36. By the recursive definition, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{2+\cos n}{\sqrt{n}}\right| = 0 < 1$, so the series converges absolutely by the Ratio Test.

37. The series $\sum\limits_{n=1}^{\infty}\dfrac{b_n^n\cos n\pi}{n} = \sum\limits_{n=1}^{\infty}(-1)^n\dfrac{b_n^n}{n}$, where $b_n > 0$ for $n \geq 1$ and $\lim\limits_{n\to\infty} b_n = \dfrac{1}{2}$.

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)^{n+1}b_n^{n+1}}{n+1}\cdot\dfrac{n}{(-1)^n b_n^n}\right| = \lim\limits_{n\to\infty} b_n\dfrac{n}{n+1} = \dfrac{1}{2}(1) = \dfrac{1}{2} < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{b_n^n\cos n\pi}{n}$ is absolutely convergent by the Ratio Test.

38. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)^{n+1}(n+1)!}{(n+1)^{n+1}b_1b_2\cdots b_nb_{n+1}}\cdot\dfrac{n^nb_1b_2\cdots b_n}{(-1)^n\,n!}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)(n+1)n^n}{b_{n+1}(n+1)^{n+1}}\right| = \lim\limits_{n\to\infty}\dfrac{n^n}{b_{n+1}(n+1)^n}$

$$= \lim_{n\to\infty}\frac{1}{b_{n+1}}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{b_{n+1}}\left(\frac{1}{1+1/n}\right)^n = \lim_{n\to\infty}\frac{1}{b_{n+1}(1+1/n)^n} = \frac{1}{\frac{1}{2}e} = \frac{2}{e} < 1$$

so the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n\,n!}{n^nb_1b_2b_3\cdots b_n}$ is absolutely convergent by the Ratio Test.

39. (a) $\lim\limits_{n\to\infty}\left|\dfrac{1/(n+1)^3}{1/n^3}\right| = \lim\limits_{n\to\infty}\dfrac{n^3}{(n+1)^3} = \lim\limits_{n\to\infty}\dfrac{1}{(1+1/n)^3} = 1$. Inconclusive for $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^3}$.

(b) $\lim\limits_{n\to\infty}\left|\dfrac{(n+1)}{2^{n+1}}\cdot\dfrac{2^n}{n}\right| = \lim\limits_{n\to\infty}\dfrac{n+1}{2n} = \lim\limits_{n\to\infty}\left(\dfrac{1}{2}+\dfrac{1}{2n}\right) = \dfrac{1}{2}$. Conclusive (convergent) for $\sum\limits_{n=1}^{\infty}\dfrac{n}{2^n}$.

(c) $\lim\limits_{n\to\infty}\left|\dfrac{(-3)^n}{\sqrt{n+1}}\cdot\dfrac{\sqrt{n}}{(-3)^{n-1}}\right| = 3\lim\limits_{n\to\infty}\sqrt{\dfrac{n}{n+1}} = 3\lim\limits_{n\to\infty}\sqrt{\dfrac{1}{1+1/n}} = 3$. Conclusive (divergent) for $\sum\limits_{n=1}^{\infty}\dfrac{(-3)^{n-1}}{\sqrt{n}}$.

(d) $\lim\limits_{n\to\infty}\left|\dfrac{\sqrt{n+1}}{1+(n+1)^2}\cdot\dfrac{1+n^2}{\sqrt{n}}\right| = \lim\limits_{n\to\infty}\left[\sqrt{1+\dfrac{1}{n}}\cdot\dfrac{1/n^2+1}{1/n^2+(1+1/n)^2}\right] = 1$. Inconclusive for $\sum\limits_{n=1}^{\infty}\dfrac{\sqrt{n}}{1+n^2}$.

40. We use the Ratio Test for the series $\sum\limits_{n=1}^{\infty}\dfrac{(n!)^2}{(kn)!}$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^2/[k(n+1)]!}{(n!)^2/(kn)!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{[k(n+1)]\,[k(n+1)-1]\cdots[kn+1]}\right|$$

Now if $k=1$, then this is equal to $\lim\limits_{n\to\infty}\left|\dfrac{(n+1)^2}{(n+1)}\right| = \infty$, so the series diverges; if $k=2$, the limit is

$\lim\limits_{n\to\infty}\left|\dfrac{(n+1)^2}{(2n+2)(2n+1)}\right| = \dfrac{1}{4} < 1$, so the series converges, and if $k > 2$, then the highest power of n in the denominator is larger than 2, and so the limit is 0, indicating convergence. So the series converges for $k \geq 2$.

41. (a) $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)!}\cdot\dfrac{n!}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x}{n+1}\right| = |x|\lim\limits_{n\to\infty}\dfrac{1}{n+1} = |x|\cdot 0 = 0 < 1$, so by the Ratio Test the series $\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}$ converges for all x.

(b) Since the series of part (a) always converges, we must have $\lim\limits_{n\to\infty}\dfrac{x^n}{n!} = 0$ by Theorem 11.2.6.

42. (a) $R_n = a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots = a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+1}} + \cdots\right)$

$$= a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+3}}\frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}} + \cdots\right)$$

$$= a_{n+1}(1 + r_{n+1} + r_{n+2}r_{n+1} + r_{n+3}r_{n+2}r_{n+1} + \cdots) \quad (\star)$$

$$\le a_{n+1}\left(1 + r_{n+1} + r_{n+1}^2 + r_{n+1}^3 + \cdots\right) \quad \text{[since } \{r_n\} \text{ is decreasing]} \quad = \frac{a_{n+1}}{1 - r_{n+1}}$$

(b) Note that since $\{r_n\}$ is increasing and $r_n \to L$ as $n \to \infty$, we have $r_n < L$ for all n. So, starting with equation ($\star$),

$$R_n = a_{n+1}(1 + r_{n+1} + r_{n+2}r_{n+1} + r_{n+3}r_{n+2}r_{n+1} + \cdots) \le a_{n+1}\left(1 + L + L^2 + L^3 + \cdots\right) = \frac{a_{n+1}}{1 - L}.$$

43. (a) $s_5 = \sum\limits_{n=1}^{5} \frac{1}{n2^n} = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \frac{1}{160} = \frac{661}{960} \approx 0.68854$. Now the ratios

$r_n = \frac{a_{n+1}}{a_n} = \frac{n2^n}{(n+1)2^{n+1}} = \frac{n}{2(n+1)}$ form an increasing sequence, since

$r_{n+1} - r_n = \frac{n+1}{2(n+2)} - \frac{n}{2(n+1)} = \frac{(n+1)^2 - n(n+2)}{2(n+1)(n+2)} = \frac{1}{2(n+1)(n+2)} > 0$. So by Exercise 42(b), the error

in using s_5 is $R_5 \le \frac{a_6}{1 - \lim\limits_{n\to\infty} r_n} = \frac{1/\left(6 \cdot 2^6\right)}{1 - 1/2} = \frac{1}{192} \approx 0.00521$.

(b) The error in using s_n as an approximation to the sum is $R_n = \frac{a_{n+1}}{1 - \frac{1}{2}} = \frac{2}{(n+1)2^{n+1}}$. We want $R_n < 0.00005 \quad \Leftrightarrow$

$\frac{1}{(n+1)2^n} < 0.00005 \quad \Leftrightarrow \quad (n+1)2^n > 20{,}000$. To find such an n we can use trial and error or a graph. We calculate

$(11+1)2^{11} = 24{,}576$, so $s_{11} = \sum\limits_{n=1}^{11} \frac{1}{n2^n} \approx 0.693109$ is within 0.00005 of the actual sum.

44. $s_{10} = \sum\limits_{n=1}^{10} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{10}{1024} \approx 1.988$. The ratios $r_n = \frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} = \frac{1}{2}\left(1 + \frac{1}{n}\right)$ form a

decreasing sequence, and $r_{11} = \frac{11+1}{2(11)} = \frac{12}{22} = \frac{6}{11} < 1$, so by Exercise 42(a), the error in using s_{10} to approximate the sum

of the series $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$ is $R_{10} \le \frac{a_{11}}{1 - r_{11}} = \frac{\frac{11}{2048}}{1 - \frac{6}{11}} = \frac{121}{10{,}240} \approx 0.0118$.

45. (i) Following the hint, we get that $|a_n| < r^n$ for $n \ge N$, and so since the geometric series $\sum_{n=1}^{\infty} r^n$ converges $[0 < r < 1]$, the series $\sum_{n=N}^{\infty} |a_n|$ converges as well by the Direct Comparison Test, and hence so does $\sum_{n=1}^{\infty} |a_n|$, so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then there is an integer N such that $\sqrt[n]{|a_n|} > 1$ for all $n \ge N$, so $|a_n| > 1$ for $n \ge N$. Thus, $\lim\limits_{n\to\infty} a_n \ne 0$, so $\sum_{n=1}^{\infty} a_n$ diverges by the Test for Divergence.

(iii) Consider $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ [diverges] and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ [converges]. For each sum, $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, so the Root Test is inconclusive.

46. (a) $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{[4(n+1)]!\,[1103+26{,}390(n+1)]}{[(n+1)!]^4\,396^{4(n+1)}}\cdot\dfrac{(n!)^4\,396^{4n}}{(4n)!\,(1103+26{,}390n)}\right|$

$$= \lim_{n\to\infty}\frac{(4n+4)(4n+3)(4n+2)(4n+1)(26{,}390n+27{,}493)}{(n+1)^4\,396^4\,(26{,}390n+1103)} = \frac{4^4}{396^4} = \frac{1}{99^4} < 1,$$

so by the Ratio Test, the series $\sum\limits_{n=0}^{\infty}\dfrac{(4n)!\,(1103+26{,}390n)}{(n!)^4\,396^{4n}}$ converges.

(b) $\dfrac{1}{\pi} = \dfrac{2\sqrt{2}}{9801}\sum\limits_{n=0}^{\infty}\dfrac{(4n)!\,(1103+26{,}390n)}{(n!)^4\,396^{4n}}$

With the first term $(n=0)$, $\dfrac{1}{\pi}\approx\dfrac{2\sqrt{2}}{9801}\cdot\dfrac{1103}{1}\quad\Rightarrow\quad\pi\approx 3.141\,592\,73$, so we get 6 correct decimal places of π, which is $3.141\,592\,653\,589\,793\,238$ to 18 decimal places.

With the second term $(n=1)$, $\dfrac{1}{\pi}\approx\dfrac{2\sqrt{2}}{9801}\left(\dfrac{1103}{1}+\dfrac{4!\,(1103+26{,}390)}{396^4}\right)\quad\Rightarrow\quad\pi\approx 3.141\,592\,653\,589\,793\,878$, so we get 15 correct decimal places of π.

11.7 Strategy for Testing Series

1. (a) $\sum\limits_{n=1}^{\infty}\dfrac{1}{5^n} = \sum\limits_{n=1}^{\infty}\left(\dfrac{1}{5}\right)^n$ is a geometric series with ratio $r=\dfrac{1}{5}$. Since $|r|=\dfrac{1}{5}<1$, the series converges.

(b) $\dfrac{1}{5^n+n} < \dfrac{1}{5^n}$ for $n\geq 1$, so $\sum\limits_{n=1}^{\infty}\dfrac{1}{5^n+n}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty}\dfrac{1}{5^n}$, which converges because it is a geometric series with $|r|=\dfrac{1}{5}<1$.

2. (a) $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^{3/2}} = \sum\limits_{n=1}^{\infty}(-1)^n b_n$. Now, $b_n=\dfrac{1}{n^{3/2}}>0$ for $n\geq 1$, $\{b_n\}$ is decreasing for $n\geq 1$, and $\lim\limits_{n\to\infty} b_n = 0$, so the series converges by the Alternating Series Test.

(b) $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{3/2}}$ is a p-series with $p=\dfrac{3}{2}>1$, so it converges.

3. (a) $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{n+1}{3^{n+1}}\cdot\dfrac{3^n}{n}\right| = \lim\limits_{n\to\infty}\dfrac{n+1}{3n} = \lim\limits_{n\to\infty}\left(\dfrac{n}{3n}+\dfrac{1}{3n}\right) = \lim\limits_{n\to\infty}\left(\dfrac{1}{3}+\dfrac{1}{3n}\right) = \dfrac{1}{3}<1,$

so the series $\sum\limits_{n=1}^{\infty}\dfrac{n}{3^n}$ is absolutely convergent (and therefore convergent) by the Ratio Test.

(b) $\lim\limits_{n\to\infty}\dfrac{3^n}{n} \overset{\text{H}}{=} \lim\limits_{n\to\infty}\dfrac{3^n\ln 3}{1}=\infty$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{3^n}{n}$ diverges by the Test for Divergence.

4. (a) $\lim\limits_{n\to\infty}\dfrac{n+1}{n} = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right) = 1\neq 0$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{n+1}{n}$ diverges by the Test for Divergence.

(b) $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|(-1)^n \frac{n+1}{n}\right| = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1 \neq 0$, so the series $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n}$ diverges by the Test for Divergence. $\left[\text{Note that } \lim_{n\to\infty} (-1)^n \frac{n+1}{n} \text{ does not exist.}\right]$

5. (a) Use the Limit Comparison Test with $a_n = \frac{n}{n^2+1}$ and $b_n = \frac{1}{n}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+1/n^2} = 1 > 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series, the series $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$ also diverges.

(b) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n^2+1}\right)^n} = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1/n}{1+1/n^2} = \frac{0}{1} = 0 < 1$, so the series $\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^n$ converges by the Root Test.

6. (a) $\frac{\ln n}{n} > \frac{1}{n}$ for $n \geq 3$, so $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by direct comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges because it is a p-series with $p = 1 \leq 1$.

(b) Let $f(x) = \frac{1}{x \ln x}$. Then f is continuous and positive on $[10, \infty)$, and also decreasing since $f'(x) = -\frac{1+\ln x}{x^2(\ln x)^2} < 0$ for $x > 10$, so we can use the Integral Test. $\int_{10}^{\infty} \frac{1}{x \ln x} dx = \lim_{t\to\infty} \Big[\ln|\ln x|\Big]_{10}^{t} = \lim_{t\to\infty} [\ln(\ln t) - \ln(\ln 10)] = \infty$, so the series $\sum_{n=10}^{\infty} \frac{1}{n \ln n}$ diverges.

7. (a) Since $n! > n^2$ for $n \geq 4$, we have $\frac{1}{n+n!} < \frac{1}{n+n^2} < \frac{1}{n^2}$ for $n \geq 4$. Thus, $\sum_{n=1}^{\infty} \frac{1}{n+n!}$ converges by direct comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

(b) $\frac{1}{n} + \frac{1}{n!} > \frac{1}{n}$ for $n \geq 1$, so $\sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{n!}\right)$ diverges by direct comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges because it is a p-series with $p = 1 \leq 1$.

8. (a) Use the Limit Comparison Test with $a_n = \frac{1}{\sqrt{n^2+1}}$ and $b_n = \frac{1}{n}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{n}{n\sqrt{1+1/n^2}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^2}} = 1 > 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ also diverges.

(b) $\frac{1}{n\sqrt{n^2+1}} < \frac{1}{n\sqrt{n^2}} = \frac{1}{n \cdot n} = \frac{1}{n^2}$ for $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$ converges by direct comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

9. Use the Limit Comparison Test with $a_n = \dfrac{n^2-1}{n^3+1}$ and $b_n = \dfrac{1}{n}$:

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{(n^2-1)n}{n^3+1} = \lim_{n\to\infty}\frac{n^3-n}{n^3+1} = \lim_{n\to\infty}\frac{1-1/n^2}{1+1/n^3} = 1 > 0$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is the divergent harmonic series, the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^2-1}{n^3+1}$ also diverges.

10. $\dfrac{n-1}{n^3+1} < \dfrac{n}{n^3+1} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$ for $n \ge 1$, so $\displaystyle\sum_{n=1}^{\infty}\frac{n-1}{n^3+1}$ converges by direct comparison with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

11. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2-1}{n^3+1} = \sum_{n=1}^{\infty}(-1)^n b_n$. Now $b_n = \dfrac{n^2-1}{n^3+1} > 0$ for $n \ge 2$, $\{b_n\}$ is decreasing for $n \ge 2$, and $\displaystyle\lim_{n\to\infty} b_n = 0$, so the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2-1}{n^3+1}$ converges by the Alternating Series Test. By Exercise 9, $\displaystyle\sum_{n=1}^{\infty}\frac{n^2-1}{n^3+1}$ diverges, so the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2-1}{n^3+1}$ is conditionally convergent.

12. $\displaystyle\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\left|(-1)^n\frac{n^2-1}{n^2+1}\right| = \lim_{n\to\infty}\frac{1-1/n^2}{1+1/n^2} = 1 \ne 0$, so the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2-1}{n^2+1}$ diverges by the Test for Divergence. $\left[\text{Note that } \displaystyle\lim_{n\to\infty}(-1)^n\frac{n^2-1}{n^2+1} \text{ does not exist.}\right]$

13. $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^2} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{2x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{2} = \infty$, so $\displaystyle\lim_{n\to\infty}\frac{e^n}{n^2} = \infty$. Thus, the series $\displaystyle\sum_{n=1}^{\infty}\frac{e^n}{n^2}$ diverges by the Test for Divergence.

14. $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{n^{2n}}{(1+n)^{3n}}} = \lim_{n\to\infty}\frac{n^2}{(1+n)^3} = \lim_{n\to\infty}\frac{1/n}{(1/n+1)^3} = \frac{0}{1} = 0 < 1$, so the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^{2n}}{(1+n)^{3n}}$ converges by the Root Test.

15. Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2,\infty)$, so we can apply the Integral Test.

Since $\displaystyle\int\frac{1}{x\sqrt{\ln x}}\,dx \quad \begin{bmatrix} u = \ln x, \\ du = dx/x \end{bmatrix} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find

$\displaystyle\int_2^{\infty}\frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty}\int_2^t\frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty}\left[2\sqrt{\ln x}\right]_2^t = \lim_{t\to\infty}\left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty$. Since the integral diverges, the given series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$ diverges.

16. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^4}{4^{n+1}}\cdot\frac{4^n}{n^4}\right| = \lim_{n\to\infty}\frac{(n+1)^4}{4n^4} = \frac{1}{4}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^4 = \frac{1}{4}(1) = \frac{1}{4} < 1$, so the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^4}{4^n}$ is absolutely convergent (and therefore convergent) by the Ratio Test.

17. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{\pi^{2n+2}}{(2n+2)!}\cdot\dfrac{(2n)!}{\pi^{2n}}\right| = \lim\limits_{n\to\infty}\dfrac{\pi^2}{(2n+2)(2n+1)} = 0 < 1$, so the series $\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{\pi^{2n}}{(2n)!}$ is absolutely convergent (and therefore convergent) by the Ratio Test.

18. Let $f(x) = x^2e^{-x^3}$. Then f is continuous and positive on $[1,\infty)$, and $f'(x) = \dfrac{x(2-3x^3)}{e^{x^3}} < 0$ for $x \geq 1$, so f is decreasing on $[1,\infty)$ as well, and we can apply the Integral Test. $\int_1^\infty x^2e^{-x^3}\,dx = \lim\limits_{t\to\infty}\left[-\frac{1}{3}e^{-x^3}\right]_1^t = \frac{1}{3e}$, so the integral converges, and hence, the series $\sum\limits_{n=1}^{\infty}n^2e^{-n^3}$ converges.

19. $\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{n^3}+\dfrac{1}{3^n}\right) = \sum\limits_{n=1}^{\infty}\dfrac{1}{n^3}+\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{3}\right)^n$. The first series converges since it is a p-series with $p = 3 > 1$ and the second series converges since it is geometric with $|r| = \frac{1}{3} < 1$. The sum of two convergent series is convergent.

20. $\dfrac{1}{k\sqrt{k^2+1}} < \dfrac{1}{k\sqrt{k^2}} = \dfrac{1}{k^2}$, so $\sum\limits_{k=1}^{\infty}\dfrac{1}{k\sqrt{k^2+1}}$ converges by direct comparison with the convergent p-series $\sum\limits_{k=1}^{\infty}\dfrac{1}{k^2}$ $(p = 2 > 1)$.

21. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{3^{n+1}(n+1)^2}{(n+1)!}\cdot\dfrac{n!}{3^nn^2}\right| = \lim\limits_{n\to\infty}\dfrac{3(n+1)^2}{(n+1)n^2} = 3\lim\limits_{n\to\infty}\dfrac{n+1}{n^2} = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{3^nn^2}{n!}$ converges by the Ratio Test.

22. $\left|\dfrac{\sin 2n}{1+2^n}\right| \leq \dfrac{1}{1+2^n} < \dfrac{1}{2^n} = \left(\dfrac{1}{2}\right)^n$, so the series $\sum\limits_{n=1}^{\infty}\left|\dfrac{\sin 2n}{1+2^n}\right|$ converges by direct comparison with the geometric series $\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{2}\right)^n$ with $|r| = \frac{1}{2} < 1$. Thus, the series $\sum\limits_{n=1}^{\infty}\dfrac{\sin 2n}{1+2^n}$ converges absolutely, implying convergence.

23. $a_k = \dfrac{2^{k-1}3^{k+1}}{k^k} = \dfrac{2^k2^{-1}3^k3^1}{k^k} = \dfrac{3}{2}\left(\dfrac{2\cdot 3}{k}\right)^k$. By the Root Test, $\lim\limits_{k\to\infty}\sqrt[k]{\left(\dfrac{6}{k}\right)^k} = \lim\limits_{k\to\infty}\dfrac{6}{k} = 0 < 1$, so the series $\sum\limits_{k=1}^{\infty}\left(\dfrac{6}{k}\right)^k$ converges. It follows from Theorem 8(i) in Section 11.2 that the given series, $\sum\limits_{k=1}^{\infty}\dfrac{2^{k-1}3^{k+1}}{k^k} = \sum\limits_{k=1}^{\infty}\dfrac{3}{2}\left(\dfrac{6}{k}\right)^k$, also converges.

24. Use the Limit Comparison Test with $a_n = \dfrac{\sqrt{n^4+1}}{n^3+n}$ and $b_n = \dfrac{1}{n}$:

$\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \lim\limits_{n\to\infty}\dfrac{n\sqrt{n^4+1}}{n(n^2+1)} = \lim\limits_{n\to\infty}\dfrac{\sqrt{n^4+1}/n^2}{(n^2+1)/n^2} = \lim\limits_{n\to\infty}\dfrac{\sqrt{1+1/n^4}}{1+1/n^2} = 1 > 0$. Since $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$ is the divergent harmonic series, the series $\sum\limits_{n=1}^{\infty}\dfrac{\sqrt{n^4+1}}{n^3+n}$ also diverges.

25.
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)(3n+2)}\cdot\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}\right| = \lim_{n\to\infty}\frac{2n+1}{3n+2}$$
$$= \lim_{n\to\infty}\frac{2+1/n}{3+2/n} = \frac{2}{3} < 1,$$

so the series $\sum\limits_{n=1}^{\infty}\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}$ converges by the Ratio Test.

26. $b_n = \dfrac{1}{\sqrt{n}-1}$ for $n \geq 2$. $\{b_n\}$ is a decreasing sequence of positive numbers and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^{n-1}}{\sqrt{n}-1}$ converges by the Alternating Series Test.

27. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$. Then $f'(x) = \dfrac{2-\ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\dfrac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim\limits_{n\to\infty} \dfrac{\ln n}{\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{1/n}{1/\left(2\sqrt{n}\right)} = \lim\limits_{n\to\infty} \dfrac{2}{\sqrt{n}} = 0$, so the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{\ln n}{\sqrt{n}}$ converges by the Alternating Series Test.

28. $a_k = \dfrac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)} < \dfrac{\sqrt[3]{k}}{k(\sqrt{k}+1)} < \dfrac{\sqrt[3]{k}}{k\sqrt{k}} = \dfrac{k^{1/3}}{k^{3/2}} = \dfrac{1}{k^{7/6}}$, so the series $\sum\limits_{k=1}^{\infty} \dfrac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)}$ converges by direct comparison with the convergent p-series $\sum\limits_{k=1}^{\infty} \dfrac{1}{k^{7/6}}$ $\left(p = \frac{7}{6} > 1\right)$.

29. $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \left|(-1)^n \cos(1/n^2)\right| = \lim\limits_{n\to\infty} \left|\cos(1/n^2)\right| = \cos 0 = 1$, so the series $\sum\limits_{n=1}^{\infty} (-1)^n \cos(1/n^2)$ diverges by the Test for Divergence.

30. $\lim\limits_{k\to\infty} |a_k| = \lim\limits_{k\to\infty} \left|\dfrac{1}{2+\sin k}\right| = \lim\limits_{k\to\infty} \dfrac{1}{2+\sin k}$, which does not exist (the terms vary between $\frac{1}{3}$ and 1). Thus, the series $\sum\limits_{k=1}^{\infty} \dfrac{1}{2+\sin k}$ diverges by the Test for Divergence.

31. Using the Limit Comparison Test with $a_n = \tan\left(\dfrac{1}{n}\right)$ and $b_n = \dfrac{1}{n}$, we have

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\tan(1/n)}{1/n} = \lim\limits_{x\to\infty} \dfrac{\tan(1/x)}{1/x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{\sec^2(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim\limits_{x\to\infty} \sec^2(1/x) = 1^2 = 1 > 0$. Since $\sum\limits_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum\limits_{n=1}^{\infty} a_n$ is also divergent.

32. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(n \sin \dfrac{1}{n}\right) = \lim\limits_{n\to\infty} \dfrac{\sin(1/n)}{1/n} = \lim\limits_{x\to 0^+} \dfrac{\sin x}{x} = 1 \neq 0$, so the series $\sum\limits_{n=1}^{\infty} n\sin(1/n)$ diverges by the Test for Divergence.

33. $\dfrac{4-\cos n}{\sqrt{n}} \geq \dfrac{4-1}{\sqrt{n}} = \dfrac{3}{n^{1/2}}$ for $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{4-\cos n}{\sqrt{n}}$ diverges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{3}{n^{1/2}}$, which diverges because it is a constant multiple of a p-series with $p = \frac{1}{2} \leq 1$.

34. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{8+(-1)^n n}{n} = \lim\limits_{n\to\infty} \left[\dfrac{8}{n} + (-1)^n\right]$. This limit does not exist since $8/n \to 0$ as $n \to \infty$, but $(-1)^n$ alternates between 1 and -1. Thus, the series $\sum\limits_{n=1}^{\infty} \dfrac{8+(-1)^n n}{n}$ diverges by the Test for Divergence.

35. Use the Ratio Test. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(n+1)!}{e^{(n+1)^2}} \cdot \dfrac{e^{n^2}}{n!}\right| = \lim\limits_{n\to\infty} \dfrac{(n+1)n!\cdot e^{n^2}}{e^{n^2+2n+1}n!} = \lim\limits_{n\to\infty} \dfrac{n+1}{e^{2n+1}} = 0 < 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n!}{e^{n^2}}$ converges.

36. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\left(\dfrac{n^2+2n+2}{5^{n+1}}\cdot\dfrac{5^n}{n^2+1}\right) = \lim\limits_{n\to\infty}\left(\dfrac{1+2/n+2/n^2}{1+1/n^2}\cdot\dfrac{1}{5}\right) = \dfrac{1}{5} < 1$, so $\sum\limits_{n=1}^{\infty}\dfrac{n^2+1}{5^n}$ converges by the Ratio Test.

37. $\displaystyle\int_2^{\infty}\frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x}-\frac{1}{x}\right]_1^t$ [using integration by parts] $\overset{\text{H}}{=} 1$. So $\sum\limits_{n=1}^{\infty}\dfrac{\ln n}{n^2}$ converges by the Integral Test, and since $\dfrac{k\ln k}{(k+1)^3} < \dfrac{k\ln k}{k^3} = \dfrac{\ln k}{k^2}$, the given series $\sum\limits_{k=1}^{\infty}\dfrac{k\ln k}{(k+1)^3}$ converges by the Direct Comparison Test.

38. Since $\left\{\dfrac{1}{n}\right\}$ is a decreasing sequence, $e^{1/n} \le e^{1/1} = e$ for all $n \ge 1$, and $\sum\limits_{n=1}^{\infty}\dfrac{e}{n^2}$ converges ($p = 2 > 1$), so $\sum\limits_{n=1}^{\infty}\dfrac{e^{1/n}}{n^2}$ converges by the Direct Comparison Test. (Or use the Integral Test.)

39. $\sum\limits_{n=1}^{\infty}a_n = \sum\limits_{n=1}^{\infty}(-1)^n\dfrac{1}{\cosh n} = \sum\limits_{n=1}^{\infty}(-1)^n\,b_n$. Now $b_n = \dfrac{1}{\cosh n} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty}b_n = 0$, so the series converges by the Alternating Series Test.

Or: Write $\dfrac{1}{\cosh n} = \dfrac{2}{e^n+e^{-n}} < \dfrac{2}{e^n}$ and $\sum\limits_{n=1}^{\infty}\dfrac{1}{e^n}$ is a convergent geometric series, so $\sum\limits_{n=1}^{\infty}\dfrac{1}{\cosh n}$ is convergent by the Direct Comparison Test. So $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{1}{\cosh n}$ is absolutely convergent and therefore convergent.

40. Let $f(x) = \dfrac{\sqrt{x}}{x+5}$. Then $f(x)$ is continuous and positive on $[1,\infty)$, and since $f'(x) = \dfrac{5-x}{2\sqrt{x}\,(x+5)^2} < 0$ for $x > 5$, $f(x)$ is eventually decreasing, so we can use the Alternating Series Test. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n}}{n+5} = \lim\limits_{n\to\infty}\dfrac{1}{n^{1/2}+5n^{-1/2}} = 0$, so the series $\sum\limits_{j=1}^{\infty}(-1)^j\dfrac{\sqrt{j}}{j+5}$ converges.

41. $\lim\limits_{k\to\infty}a_k = \lim\limits_{k\to\infty}\dfrac{5^k}{3^k+4^k} =$ [divide by 4^k] $\lim\limits_{k\to\infty}\dfrac{(5/4)^k}{(3/4)^k+1} = \infty$ since $\lim\limits_{k\to\infty}\left(\dfrac{3}{4}\right)^k = 0$ and $\lim\limits_{k\to\infty}\left(\dfrac{5}{4}\right)^k = \infty$.

Thus, $\sum\limits_{k=1}^{\infty}\dfrac{5^k}{3^k+4^k}$ diverges by the Test for Divergence.

42.
$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{(n!)^n}{n^{4n}}\right|} = \lim_{n\to\infty}\frac{n!}{n^4} = \lim_{n\to\infty}\left[\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdot\frac{n-3}{n}\cdot(n-4)!\right]$$
$$= \lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)(n-4)!\right] = \infty,$$
so the series $\sum\limits_{n=1}^{\infty}\dfrac{(n!)^n}{n^{4n}}$ diverges by the Root Test.

43. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^{n^2/n} = \lim\limits_{n\to\infty}\dfrac{1}{[(n+1)/n]^n} = \dfrac{1}{\lim\limits_{n\to\infty}(1+1/n)^n} = \dfrac{1}{e} < 1$, so the series $\sum\limits_{n=1}^{\infty}\left(\dfrac{n}{n+1}\right)^{n^2}$ converges by the Root Test.

44. $0 \leq n\cos^2 n \leq n$, so $\dfrac{1}{n + n\cos^2 n} \geq \dfrac{1}{n+n} = \dfrac{1}{2n}$. Thus, $\sum\limits_{n=1}^{\infty} \dfrac{1}{n + n\cos^2 n}$ diverges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n}$, which is a constant multiple of the (divergent) harmonic series.

45. $a_n = \dfrac{1}{n^{1+1/n}} = \dfrac{1}{n \cdot n^{1/n}}$, so let $b_n = \dfrac{1}{n}$ and use the Limit Comparison Test. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{1}{n^{1/n}} = 1 > 0$ [see Exercise 4.4.63], so the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{1+1/n}}$ diverges by comparison with the divergent harmonic series.

46. Note that $(\ln n)^{\ln n} = \left(e^{\ln\ln n}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln\ln n} = n^{\ln\ln n}$ and $\ln\ln n \to \infty$ as $n \to \infty$, so $\ln\ln n > 2$ for sufficiently large n. For these n we have $(\ln n)^{\ln n} > n^2$, so $\dfrac{1}{(\ln n)^{\ln n}} < \dfrac{1}{n^2}$. Since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^2}$ converges $[p = 2 > 1]$, so does $\sum\limits_{n=2}^{\infty} \dfrac{1}{(\ln n)^{\ln n}}$ by the Direct Comparison Test.

47. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} (2^{1/n} - 1) = 1 - 1 = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$ converges by the Root Test.

48. Use the Limit Comparison Test with $a_n = \sqrt[n]{2} - 1$ and $b_n = 1/n$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^{1/n} - 1}{1/n} = \lim_{x\to\infty} \frac{2^{1/x} - 1}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{2^{1/x} \cdot \ln 2 \cdot (-1/x^2)}{-1/x^2} = \lim_{x\to\infty} (2^{1/x} \cdot \ln 2) = 1 \cdot \ln 2 = \ln 2 > 0.$$

So since $\sum\limits_{n=1}^{\infty} b_n$ diverges (harmonic series), so does $\sum\limits_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$.

Alternate solution: $\sqrt[n]{2} - 1 = \dfrac{1}{2^{(n-1)/n} + 2^{(n-2)/n} + 2^{(n-3)/n} + \cdots + 2^{1/n} + 1}$ [rationalize the numerator] $\geq \dfrac{1}{2n}$, and since $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n} = \dfrac{1}{2}\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges (harmonic series), so does $\sum\limits_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$ by the Direct Comparison Test.

11.8 Power Series

1. A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, where x is a variable and the c_n's are constants called the coefficients of the series.

More generally, a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$ is called a power series in $(x-a)$ or a power series centered at a or a power series about a, where a is a constant.

2. (a) Given the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

In most cases, R can be found by using the Ratio Test.

(b) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (a), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers; that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

3. If $a_n = \dfrac{x^n}{n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{n+1}\cdot\dfrac{n}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{nx}{n+1}\right| = \lim\limits_{n\to\infty}\left(\dfrac{1}{1+1/n}\,|x|\right) = |x|$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n}$ converges when $|x| < 1$, so the radius of convergence is $R = 1$. Now we'll check the endpoints, that is, $x = \pm 1$. When $x = 1$, the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$ diverges since it is the harmonic series. When $x = -1$, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}$ converges by the Alternating Series Test. Thus, the interval of convergence is $[-1, 1)$.

4. If $a_n = (-1)^n nx^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{(-1)^n\,nx^n}\right| = \lim_{n\to\infty}\left|(-1)\frac{n+1}{n}x\right| = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)|x|\right] = |x|.$$ By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}(-1)^n nx^n$ converges when $|x| < 1$, so the radius of convergence $R = 1$. Now we'll check the endpoints, that is, $x = \pm 1$. Both series $\sum\limits_{n=1}^{\infty}(-1)^n n(\pm 1)^n = \sum\limits_{n=1}^{\infty}(\mp 1)^n n$ diverge by the Test for Divergence since $\lim\limits_{n\to\infty}|(\mp 1)^n n| = \infty$. Thus, the interval of convergence is $I = (-1, 1)$.

5. If $a_n = \sqrt{n}\,x^n$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{\sqrt{n+1}\,x^{n+1}}{\sqrt{n}\,x^n}\right| = \lim\limits_{n\to\infty}\left|\sqrt{\dfrac{n+1}{n}}\,x\right| = \lim\limits_{n\to\infty}\sqrt{1+\dfrac{1}{n}}\,|x| = |x|$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\sqrt{n}\,x^n$ converges when $|x| < 1$, so $R = 1$. When $x = \pm 1$, both series $\sum\limits_{n=1}^{\infty}\sqrt{n}\,(\pm 1)^n$ diverge by the Test for Divergence since $\lim\limits_{n\to\infty}|\sqrt{n}\,(\pm 1)^n| = \infty$. Thus, the interval of convergence is $(-1, 1)$.

6. If $a_n = \dfrac{(-1)^n x^n}{\sqrt[3]{n}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{\sqrt[3]{n+1}}\cdot\frac{\sqrt[3]{n}}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)x\sqrt[3]{n}}{\sqrt[3]{n+1}}\right| = \lim_{n\to\infty}\sqrt[3]{\frac{1}{1+1/n}}\,|x| = |x|.$$ By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n x^n}{\sqrt[3]{n}}$ converges when $|x| < 1$, so $R = 1$. When $x = 1$, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{\sqrt[3]{n}}$ converges by the Alternating Series Test. When $x = -1$, the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt[3]{n}}$ diverges since it is a p-series $\left(p = \frac{1}{3} \le 1\right)$. Thus, the interval of convergence is $(-1, 1]$.

7. If $a_n = \dfrac{n}{5^n}\,x^n$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)\,x^{n+1}}{5^{n+1}}\cdot\dfrac{5^n}{n\,x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{n+1}{5n}\,x\right| = \lim\limits_{n\to\infty}\left(\dfrac{1}{5}+\dfrac{1}{5n}\right)|x| = \dfrac{|x|}{5}$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{n}{5^n}\,x^n$ converges when $\dfrac{|x|}{5} < 1 \;\Leftrightarrow\; |x| < 5$, so $R = 5$. When $x = \pm 5$, both series

$\sum_{n=1}^{\infty} \frac{n(\pm 5)^n}{5^n} = \sum_{n=1}^{\infty} (\pm 1)^n n$ diverge by the Test for Divergence since $\lim_{n\to\infty} |(\pm 1)^n n| = \infty$. Thus, the interval of convergence is $(-5, 5)$.

8. If $a_n = \frac{5^n}{n} x^n$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{5^{n+1} x^{n+1}}{(n+1)} \cdot \frac{n}{5^n x^n}\right| = \lim_{n\to\infty} \left|\frac{5n}{n+1} x\right| = \lim_{n\to\infty} \left(\frac{5}{1+1/n} |x|\right) = 5|x|$. By the Ratio Test, the series $\sum_{n=2}^{\infty} \frac{5^n}{n} x^n$ converges when $5|x| < 1 \Leftrightarrow |x| < \frac{1}{5}$, so $R = \frac{1}{5}$. When $x = \frac{1}{5}$, the series $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges since it is the (partial) harmonic series. When $x = -\frac{1}{5}$, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n}$ converges by the Alternating Series Test. Thus, the interval of convergence is $\left[-\frac{1}{5}, \frac{1}{5}\right)$.

9. If $a_n = \frac{x^n}{n\,3^n}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)\,3^{n+1}} \cdot \frac{n\,3^n}{x^n}\right| = \lim_{n\to\infty} \left|\frac{n}{3(n+1)} x\right| = \lim_{n\to\infty} \left(\frac{1}{3+3/n} |x|\right) = \frac{|x|}{3}$. By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{n\,3^n}$ converges when $\frac{|x|}{3} < 1 \Leftrightarrow |x| < 3$, so $R = 3$. When $x = 3$, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since it is the harmonic series. When $x = -3$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the Alternating Series Test. Thus, the interval of convergence is $[-3, 3)$.

10. If $a_n = \frac{n}{n+1} x^n$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)\,x^{n+1}}{(n+1)+1} \cdot \frac{n+1}{n\,x^n}\right| = \lim_{n\to\infty} \left|\frac{n^2+2n+1}{n^2+2n} x\right| = \lim_{n\to\infty} \left(\frac{1+2/n+1/n^2}{1+2/n} |x|\right) = |x|.$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{n}{n+1} x^n$ converges when $|x| < 1$, so $R = 1$. When $x = \pm 1$, both series $\sum_{n=1}^{\infty} \frac{n(\pm 1)^n}{n+1}$ diverge by the Test for Divergence since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$ and $\lim_{n\to\infty} \frac{n(-1)^n}{n+1}$ does not exist. Thus, the interval of convergence is $(-1, 1)$.

11. If $a_n = \frac{x^n}{2n-1}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{2n+1} \cdot \frac{2n-1}{x^n}\right| = \lim_{n\to\infty} \left(\frac{2n-1}{2n+1} |x|\right) = \lim_{n\to\infty} \left(\frac{2-1/n}{2+1/n} |x|\right) = |x|$. By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$ converges when $|x| < 1$, so $R = 1$. When $x = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges by direct comparison with $\sum_{n=1}^{\infty} \frac{1}{2n}$ since $\frac{1}{2n-1} > \frac{1}{2n}$ and $\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges since it is a constant multiple of the harmonic series. When $x = -1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ converges by the Alternating Series Test. Thus, the interval of convergence is $[-1, 1)$.

12. If $a_n = \frac{(-1)^n x^n}{n^2}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1} x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-1)^n x^n}\right| = \lim_{n\to\infty} \left|\frac{(-1)xn^2}{(n+1)^2}\right| = \lim_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^2 |x|\right] = 1^2 \cdot |x| = |x|.$$

[continued]

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n x^n}{n^2}$ converges when $|x| < 1$, so $R = 1$. When $x = 1$, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^2}$ converges by the Alternating Series Test. When $x = -1$, the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges since it is a p-series with $p = 2 > 1$. Thus, the interval of convergence is $[-1, 1]$.

13. If $a_n = \dfrac{x^n}{n!}$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{x^n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x}{n+1}\right| = |x| \lim\limits_{n\to\infty} \dfrac{1}{n+1} = |x| \cdot 0 = 0 < 1$ for *all* real x. So, by the Ratio Test, $R = \infty$ and $I = (-\infty, \infty)$.

14. Here the Root Test is easier. If $a_n = n^n x^n$, then $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} n\,|x| = \infty$ if $x \neq 0$, so $R = 0$ and $I = \{0\}$.

15. If $a_n = \dfrac{x^n}{n^4\, 4^n}$, then

$\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{(n+1)^4\, 4^{n+1}} \cdot \dfrac{n^4\, 4^n}{x^n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{n^4}{(n+1)^4} \cdot \dfrac{x}{4}\right| = \lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right)^4 \dfrac{|x|}{4} = 1^4 \cdot \dfrac{|x|}{4} = \dfrac{|x|}{4}$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n^4\, 4^n}$ converges when $\dfrac{|x|}{4} < 1 \quad\Leftrightarrow\quad |x| < 4$, so $R = 4$. When $x = 4$, the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^4}$ converges since it is a p-series ($p = 4 > 1$). When $x = -4$, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^4}$ converges by the Alternating Series Test. Thus, the interval of convergence is $[-4, 4]$.

16. If $a_n = 2^n n^2 x^n$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{2^{n+1}(n+1)^2 x^{n+1}}{2^n n^2 x^n}\right| = \lim\limits_{n\to\infty} 2\left(\dfrac{n+1}{n}\right)^2 |x| = 2\,|x|$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} 2^n n^2 x^n$ converges when $2\,|x| < 1 \quad\Leftrightarrow\quad |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. When $x = \pm\frac{1}{2}$, both series $\sum\limits_{n=1}^{\infty} 2^n n^2 \left(\pm\frac{1}{2}\right)^n = \sum\limits_{n=1}^{\infty} (\pm 1)^n n^2$ diverge by the Test for Divergence since $\lim\limits_{n\to\infty} \left|(\pm 1)^n n^2\right| = \infty$. Thus, the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

17. If $a_n = \dfrac{(-1)^n\, 4^n}{\sqrt{n}}\, x^n$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(-1)^{n+1}\, 4^{n+1}\, x^{n+1}}{\sqrt{n+1}} \cdot \dfrac{\sqrt{n}}{(-1)^n\, 4^n\, x^n}\right| = \lim\limits_{n\to\infty} \sqrt{\dfrac{n}{n+1}} \cdot 4\,|x| = 4\,|x|$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n\, 4^n}{\sqrt{n}}\, x^n$ converges when $4\,|x| < 1 \quad\Leftrightarrow\quad |x| < \frac{1}{4}$, so $R = \frac{1}{4}$. When $x = \frac{1}{4}$, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. When $x = -\frac{1}{4}$, the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ diverges since it is a p-series $\left(p = \frac{1}{2} \leq 1\right)$. Thus, the interval of convergence is $\left(-\frac{1}{4}, \frac{1}{4}\right]$.

18. If $a_n = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n5^n}\, x^n$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(-1)^n x^{n+1}}{(n+1)\, 5^{n+1}} \cdot \dfrac{n5^n}{(-1)^{n-1} x^n}\right| = \lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right) \dfrac{|x|}{5} = 1 \cdot \dfrac{|x|}{5} = \dfrac{|x|}{5}$.

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n5^n}\, x^n$ converges when $\dfrac{|x|}{5} < 1 \quad\Leftrightarrow\quad |x| < 5$, so $R = 5$. When $x = 5$, the series

$\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n}$ converges by the Alternating Series Test. When $x = -5$, the series $\sum\limits_{n=1}^{\infty} \dfrac{-1}{n}$ diverges since it is a constant multiple of the harmonic series. Thus, the interval of convergence is $(-5, 5]$.

19. If $a_n = \dfrac{n}{2^n(n^2+1)}\, x^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{2^{n+1}(n^2+2n+2)} \cdot \frac{2^n(n^2+1)}{n\,x^n}\right| = \lim_{n\to\infty}\frac{n^3+n^2+n+1}{n^3+2n^2+2n}\cdot\frac{|x|}{2}$$
$$= \lim_{n\to\infty}\frac{1+1/n+1/n^2+1/n^3}{1+2/n+2/n^2}\cdot\frac{|x|}{2} = \frac{|x|}{2}$$

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{n}{2^n(n^2+1)}\,x^n$ converges when $\dfrac{|x|}{2} < 1 \;\Leftrightarrow\; |x| < 2$, so $R = 2$. When $x = 2$, the series $\sum\limits_{n=1}^{\infty}\dfrac{n}{n^2+1}$ diverges by the Limit Comparison Test with $b_n = \dfrac{1}{n}$. When $x = -2$, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n n}{n^2+1}$ converges by the Alternating Series Test. Thus, the interval of convergence is $[-2, 2)$.

20. If $a_n = \dfrac{x^{2n}}{n!}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{2n+2}}{(n+1)!}\cdot\dfrac{n!}{x^{2n}}\right| = \lim\limits_{n\to\infty}\dfrac{|x^2|}{n+1} = 0 < 1$ for *all* real x. So, by the Ratio Test, $R = \infty$ and $I = (-\infty, \infty)$.

21. If $a_n = \dfrac{(x-2)^n}{n^2+1}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(x-2)^{n+1}}{(n+1)^2+1}\cdot\dfrac{n^2+1}{(x-2)^n}\right| = |x-2|\lim\limits_{n\to\infty}\dfrac{n^2+1}{(n+1)^2+1} = |x-2|$. By the Ratio Test, the series $\sum\limits_{n=0}^{\infty}\dfrac{(x-2)^n}{n^2+1}$ converges when $|x-2| < 1 \quad [R = 1] \;\Leftrightarrow\; -1 < x-2 < 1 \;\Leftrightarrow\; 1 < x < 3$. When $x = 1$, the series $\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{1}{n^2+1}$ converges by the Alternating Series Test; when $x = 3$, the series $\sum\limits_{n=0}^{\infty}\dfrac{1}{n^2+1}$ converges by direct comparison with the p-series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ $[p = 2 > 1]$. Thus, the interval of convergence is $I = [1, 3]$.

22. If $a_n = \dfrac{(-1)^n}{(2n-1)2^n}(x-1)^n$, then

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)^{n+1}(x-1)^{n+1}}{(2n+1)\,2^{n+1}}\cdot\dfrac{(2n-1)\,2^n}{(-1)^n(x-1)^n}\right| = \lim\limits_{n\to\infty}\dfrac{2n-1}{2n+1}\cdot\dfrac{|x-1|}{2} = \dfrac{|x-1|}{2}$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{(2n-1)\,2^n}(x-1)^n$ converges when $\dfrac{|x-1|}{2} < 1 \;\Leftrightarrow\; |x-1| < 2 \quad [R = 2] \;\Leftrightarrow\; -2 < x-1 < 2 \;\Leftrightarrow\; -1 < x < 3$. When $x = 3$, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{2n-1}$ converges by the Alternating Series Test. When $x = -1$, the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{2n-1}$ diverges by the Limit Comparison Test with $b_n = \dfrac{1}{n}$. Thus, the interval of convergence is $(-1, 3]$.

23. If $a_n = \dfrac{(x+2)^n}{2^n\,\ln n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{(x+2)^{n+1}}{2^{n+1}\ln(n+1)}\cdot\dfrac{2^n\ln n}{(x+2)^n}\right| = \lim\limits_{n\to\infty}\dfrac{\ln n}{\ln(n+1)}\cdot\dfrac{|x+2|}{2} = \dfrac{|x+2|}{2}$ since

$\lim\limits_{n\to\infty}\dfrac{\ln n}{\ln(n+1)} = \lim\limits_{x\to\infty}\dfrac{\ln x}{\ln(x+1)} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{1/x}{1/(x+1)} = \lim\limits_{x\to\infty}\dfrac{x+1}{x} = \lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right) = 1$. By the Ratio Test, the series

$\sum\limits_{n=2}^{\infty} \dfrac{(x+2)^n}{2^n \ln n}$ converges when $\dfrac{|x+2|}{2} < 1 \;\Leftrightarrow\; |x+2| < 2 \;\; [R=2] \;\Leftrightarrow\; -2 < x+2 < 2 \;\Leftrightarrow\; -4 < x < 0$.

When $x=-4$, the series $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{\ln n}$ converges by the Alternating Series Test. When $x=0$, the series $\sum\limits_{n=2}^{\infty} \dfrac{1}{\ln n}$ diverges by the Limit Comparison Test with $b_n = \dfrac{1}{n}$ (or by direct comparison with the harmonic series). Thus, the interval of convergence is $[-4, 0)$.

24. If $a_n = \dfrac{\sqrt{n}}{8^n}(x+6)^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}\,(x+6)^{n+1}}{8^{n+1}} \cdot \frac{8^n}{\sqrt{n}(x+6)^n}\right| = \lim_{n\to\infty}\sqrt{\frac{n+1}{n}} \cdot \frac{|x+6|}{8}$$
$$= \lim_{n\to\infty}\sqrt{1+\frac{1}{n}} \cdot \frac{|x+6|}{8} = \frac{|x+6|}{8}$$

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n}}{8^n}(x+6)^n$ converges when $\dfrac{|x+6|}{8} < 1 \;\Leftrightarrow\; |x+6| < 8 \;\; [R=8] \;\Leftrightarrow$ $-8 < x+6 < 8 \;\Leftrightarrow\; -14 < x < 2$. When $x=2$, the series $\sum\limits_{n=1}^{\infty}\sqrt{n}$ diverges by the Test for Divergence since $\lim\limits_{n\to\infty}|a_n| = \lim\limits_{n\to\infty}\sqrt{n} = \infty > 0$. Similarly, when $x=-14$, the series $\sum\limits_{n=1}^{\infty}(-1)^n\sqrt{n}$ diverges. Thus, the interval of convergence is $(-14, 2)$.

25. If $a_n = \dfrac{(x-2)^n}{n^n}$, then $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \lim\limits_{n\to\infty}\dfrac{|x-2|}{n} = 0$, so the series converges for all x (by the Root Test). $R=\infty$ and $I=(-\infty,\infty)$.

26. If $a_n = \dfrac{(2x-1)^n}{5^n\sqrt{n}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}} \cdot \frac{5^n\sqrt{n}}{(2x-1)^n}\right| = \lim_{n\to\infty}\frac{|2x-1|}{5}\sqrt{\frac{n}{n+1}} = \lim_{n\to\infty}\frac{|2x-1|}{5}\sqrt{\frac{1}{1+1/n}} = \frac{|2x-1|}{5}.$$

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{(2x-1)^n}{5^n\sqrt{n}}$ converges when $\dfrac{|2x-1|}{5} < 1 \;\Leftrightarrow\; |2x-1| < 5 \;\Leftrightarrow\; \left|x-\frac{1}{2}\right| < \frac{5}{2} \;\Leftrightarrow$ $-\frac{5}{2} < x - \frac{1}{2} < \frac{5}{2} \;\Leftrightarrow\; -2 < x < 3$, so $R=\frac{5}{2}$. When $x=3$, the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$ is a divergent p-series $\left(p=\frac{1}{2}\le 1\right)$. When $x=-2$, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I=[-2,3)$.

27. If $a_n = \dfrac{\ln n}{n}x^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\ln(n+1)\,x^{n+1}}{n+1} \cdot \frac{n}{(\ln n)\,x^n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\,\frac{\ln(n+1)}{\ln n}\,x\right|$$
$$= 1 \cdot 1 \cdot |x| = |x|$$

since $\lim\limits_{n\to\infty}\dfrac{\ln(n+1)}{\ln n} \overset{\text{H}}{=} \lim\limits_{n\to\infty}\dfrac{1/(n+1)}{1/n} = \lim\limits_{n\to\infty}\dfrac{n}{n+1} = \lim\limits_{n\to\infty}\dfrac{1}{1+1/n} = 1$. By the Ratio Test, the series $\sum\limits_{n=4}^{\infty}\dfrac{\ln n}{n}x^n$

converges when $|x| < 1$, so $R = 1$. When $x = 1$, the series $\sum\limits_{n=4}^{\infty} \dfrac{\ln n}{n}$ diverges by direct comparison with the (partial) harmonic series $\sum\limits_{n=4}^{\infty} \dfrac{1}{n}$. When $x = -1$, the series $\sum\limits_{n=4}^{\infty} \dfrac{(-1)^n \ln n}{n}$ converges by the Alternating Series Test. Thus, the interval of convergence is $[-1, 1)$.

28. If $a_n = \dfrac{(-1)^n}{n \ln n} x^n$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{(-1)^n x^n} \right| = \lim_{n\to\infty} \left[\frac{n}{n+1} \frac{\ln n}{\ln(n+1)} |x| \right] = 1 \cdot 1 \cdot |x| = |x| \text{ since}$$

$\displaystyle\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \overset{\text{H}}{=} \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$. By the Ratio Test, the series $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{n \ln n} x^n$ converges when $|x| < 1$, so $R = 1$. When $x = 1$, the series $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{n \ln n}$ converges by the Alternating Series Test. When $x = -1$, the series $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n(-1)^n}{n \ln n} = \sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n}$ diverges by the Integral Test. Thus, the interval of convergence is $(-1, 1]$.

29. $a_n = \dfrac{n}{b^n}(x-a)^n$, where $b > 0$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)\,|x-a|^{n+1}}{b^{n+1}} \cdot \frac{b^n}{n\,|x-a|^n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \frac{|x-a|}{b} = \frac{|x-a|}{b}.$$

By the Ratio Test, the series converges when $\dfrac{|x-a|}{b} < 1 \quad\Leftrightarrow\quad |x-a| < b \quad [\text{so } R = b] \quad\Leftrightarrow\quad -b < x - a < b \quad\Leftrightarrow$ $a - b < x < a + b$. When $|x - a| = b$, $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} n = \infty$, so the series diverges. Thus, $I = (a - b, a + b)$.

30. $a_n = \dfrac{b^n}{\ln n}(x-a)^n$, where $b > 0$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{b^{n+1}(x-a)^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{b^n (x-a)^n} \right| = \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \cdot b\,|x-a| = b\,|x-a| \text{ since}$$

$\displaystyle\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty} \frac{\ln x}{\ln(x+1)} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{1/(x+1)} = \lim_{x\to\infty} \frac{x+1}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1}{1} = 1$. By the Ratio Test, the series $\sum\limits_{n=2}^{\infty} \dfrac{b^n}{\ln n}(x-a)^n$ converges when $b\,|x-a| < 1 \quad\Leftrightarrow\quad |x-a| < \dfrac{1}{b} \quad\Leftrightarrow\quad -\dfrac{1}{b} < x - a < \dfrac{1}{b} \quad\Leftrightarrow\quad a - \dfrac{1}{b} < x < a + \dfrac{1}{b}$, so $R = \dfrac{1}{b}$. When $x = a + \dfrac{1}{b}$, the series $\sum\limits_{n=2}^{\infty} \dfrac{1}{\ln n}$ diverges by direct comparison with the divergent p-series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$ since $\dfrac{1}{\ln n} > \dfrac{1}{n}$ for $n \geq 2$. When $x = a - \dfrac{1}{b}$, the series $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{\ln n}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I = \left[a - \dfrac{1}{b}, a + \dfrac{1}{b}\right)$.

31. If $a_n = n!\,(2x-1)^n$, then $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \left| \dfrac{(n+1)!\,(2x-1)^{n+1}}{n!(2x-1)^n} \right| = \lim\limits_{n\to\infty} (n+1)\,|2x-1| \to \infty$ as $n \to \infty$ for all $x \neq \frac{1}{2}$. Since the series diverges for all $x \neq \frac{1}{2}$, $R = 0$ and $I = \left\{\frac{1}{2}\right\}$.

32. $a_n = \dfrac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} = \dfrac{n^2 x^n}{2^n n!} = \dfrac{n x^n}{2^n (n-1)!}$, so

$\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n \to \infty} \dfrac{(n+1)\,|x|^{n+1}}{2^{n+1} n!} \cdot \dfrac{2^n (n-1)!}{n\,|x|^n} = \lim\limits_{n \to \infty} \dfrac{n+1}{n^2} \dfrac{|x|}{2} = 0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

33. If $a_n = \dfrac{(5x-4)^n}{n^3}$, then

$$\begin{aligned}\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \to \infty} \left| \frac{(5x-4)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(5x-4)^n} \right| = \lim_{n \to \infty} |5x-4| \left(\frac{n}{n+1} \right)^3 = \lim_{n \to \infty} |5x-4| \left(\frac{1}{1+1/n} \right)^3 \\ &= |5x-4| \cdot 1 = |5x-4|\end{aligned}$$

By the Ratio Test, $\sum\limits_{n=1}^{\infty} \dfrac{(5x-4)^n}{n^3}$ converges when $|5x-4| < 1 \quad \Leftrightarrow \quad \left| x - \frac{4}{5} \right| < \frac{1}{5} \quad \Leftrightarrow \quad -\frac{1}{5} < x - \frac{4}{5} < \frac{1}{5} \quad \Leftrightarrow$

$\frac{3}{5} < x < 1$, so $R = \frac{1}{5}$. When $x = 1$, the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$ is a convergent p-series ($p = 3 > 1$). When $x = \frac{3}{5}$, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^3}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I = \left[\frac{3}{5}, 1 \right]$.

34. If $a_n = \dfrac{x^{2n}}{n\,(\ln n)^2}$, then $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n \to \infty} \left| \dfrac{x^{2n+2}}{(n+1)[\ln(n+1)]^2} \cdot \dfrac{n\,(\ln n)^2}{x^{2n}} \right| = \left| x^2 \right| \lim\limits_{n \to \infty} \dfrac{n\,(\ln n)^2}{(n+1)[\ln(n+1)]^2} = x^2$.

By the Ratio Test, the series $\sum\limits_{n=2}^{\infty} \dfrac{x^{2n}}{n\,(\ln n)^2}$ converges when $x^2 < 1 \quad \Leftrightarrow \quad |x| < 1$, so $R = 1$. When $x = \pm 1$, $x^{2n} = 1$, the series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\,(\ln n)^2}$ converges by the Integral Test (see Exercise 11.3.31). Thus, the interval of convergence is $I = [-1, 1]$.

35. If $a_n = \dfrac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$, then

$\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n \to \infty} \left| \dfrac{x^{n+1}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)(2n+1)} \cdot \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{x^n} \right| = \lim\limits_{n \to \infty} \dfrac{|x|}{2n+1} = 0 < 1$. Thus, by the Ratio Test, the series $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$ converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

36. If $a_n = \dfrac{n!\,x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$, then

$\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n \to \infty} \left| \dfrac{(n+1)!\,x^{n+1}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)(2n+1)} \cdot \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!\,x^n} \right| = \lim\limits_{n \to \infty} \dfrac{(n+1)\,|x|}{2n+1} = \frac{1}{2}\,|x|$.

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} a_n$ converges when $\frac{1}{2}\,|x| < 1 \quad \Rightarrow \quad |x| < 2$, so $R = 2$. When $x = \pm 2$,

$|a_n| = \dfrac{n!\,2^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} = \dfrac{[1 \cdot 2 \cdot 3 \cdot \cdots \cdot n]\,2^n}{[1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)]} = \dfrac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} > 1$, so both endpoint series diverge by the Test for Divergence. Thus, the interval of convergence is $I = (-2, 2)$.

37. (a) We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = 4$. So by Theorem 4, it must converge for at least $-4 < x \le 4$. In particular, it converges when $x = -2$; that is, $\sum_{n=0}^{\infty} c_n(-2)^n$ is convergent.

(b) It does not follow that $\sum_{n=0}^{\infty} c_n(-4)^n$ is necessarily convergent. [See the comments after Theorem 4 about convergence at the endpoint of an interval. An example is $c_n = (-1)^n/(n4^n)$.]

38. We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = -4$ and divergent when $x = 6$. So by Theorem 4 it converges for at least $-4 \le x < 4$ and diverges for at least $x \ge 6$ and $x < -6$. Therefore:

(a) It converges when $x = 1$; that is, $\sum c_n$ is convergent.

(b) It diverges when $x = 8$; that is, $\sum c_n 8^n$ is divergent.

(c) It converges when $x = -3$; that is, $\sum c_n(-3^n)$ is convergent.

(d) It diverges when $x = -9$; that is, $\sum c_n(-9)^n = \sum(-1)^n c_n 9^n$ is divergent.

39. If $a_n = \dfrac{(n!)^k}{(kn)!} x^n$, then

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{[(n+1)!]^k\,(kn)!}{(n!)^k\,[k(n+1)]!}\,|x| = \lim_{n\to\infty}\frac{(n+1)^k}{(kn+k)(kn+k-1)\cdots(kn+2)(kn+1)}\,|x| \\
&= \lim_{n\to\infty}\left[\frac{(n+1)}{(kn+1)}\frac{(n+1)}{(kn+2)}\cdots\frac{(n+1)}{(kn+k)}\right]|x| \\
&= \lim_{n\to\infty}\left[\frac{n+1}{kn+1}\right]\lim_{n\to\infty}\left[\frac{n+1}{kn+2}\right]\cdots\lim_{n\to\infty}\left[\frac{n+1}{kn+k}\right]|x| \\
&= \left(\frac{1}{k}\right)^k|x| < 1 \quad\Leftrightarrow\quad |x| < k^k \text{ for convergence, and the radius of convergence is } R = k^k.
\end{aligned}$$

40. (a) Note that the four intervals in parts (a)–(d) have midpoint $m = \frac{1}{2}(p+q)$ and radius of convergence $r = \frac{1}{2}(q-p)$. We also know that the power series $\sum\limits_{n=0}^{\infty} x^n$ has interval of convergence $(-1, 1)$. To change the radius of convergence to r, we can change x^n to $\left(\dfrac{x}{r}\right)^n$. To shift the midpoint of the interval of convergence, we can replace x with $x - m$. Thus, a power series whose interval of convergence is (p, q) is $\sum\limits_{n=0}^{\infty}\left(\dfrac{x-m}{r}\right)^n$, where $m = \frac{1}{2}(p+q)$ and $r = \frac{1}{2}(q-p)$.

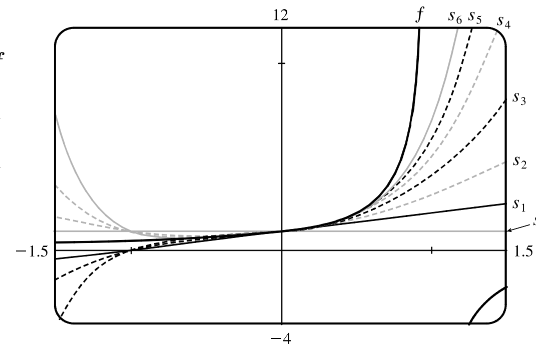
(b) Similar to Example 2, we know that $\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n}$ has interval of convergence $[-1, 1)$. By introducing the factor $(-1)^n$ in a_n, the interval of convergence changes to $(-1, 1]$. Now change the midpoint and radius as in part (a) to get $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{1}{n}\left(\dfrac{x-m}{r}\right)^n$ as a power series whose interval of convergence is $(p, q]$.

(c) As in part (b), $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}\left(\dfrac{x-m}{r}\right)^n$ is a power series whose interval of convergence is $[p, q)$.

(d) If we increase the exponent on n (to say, $n = 2$), in the power series in part (c), then when $x = q$, the power series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}\left(\dfrac{x-m}{r}\right)^n$ will converge by comparison to the p-series with $p = 2 > 1$, and the interval of convergence will be $[p, q]$.

41. No. If a power series is centered at a, its interval of convergence is symmetric about a. If a power series has an infinite radius of convergence, then its interval of convergence must be $(-\infty, \infty)$, not $[0, \infty)$.

42. The partial sums of the series $\sum_{n=0}^{\infty} x^n$ definitely do not converge to $f(x) = 1/(1-x)$ for $x \geq 1$, since f is undefined at $x = 1$ and negative on $(1, \infty)$, while all the partial sums are positive on this interval. The partial sums also fail to converge to f for $x \leq -1$, since $0 < f(x) < 1$ on this interval, while the partial sums are either larger than 1 or less than 0. The partial sums seem to converge to f on $(-1, 1)$. This graphical evidence is consistent with what we know about geometric series: convergence for $|x| < 1$, divergence for $|x| \geq 1$.



43. We use the Root Test on the series $\sum c_n x^n$. We need $\lim\limits_{n\to\infty} \sqrt[n]{|c_n x^n|} = |x| \lim\limits_{n\to\infty} \sqrt[n]{|c_n|} = c\,|x| < 1$ for convergence, or $|x| < 1/c$, so $R = 1/c$.

44. Suppose $c_n \neq 0$. Applying the Ratio Test to the series $\sum c_n (x-a)^n$, we find that

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = \lim_{n\to\infty} \frac{|x-a|}{|c_n/c_{n+1}|} (\star) = \frac{|x-a|}{\lim\limits_{n\to\infty} |c_n/c_{n+1}|}$$ (if $\lim\limits_{n\to\infty} |c_n/c_{n+1}| \neq 0$), so the series converges when $\dfrac{|x-a|}{\lim\limits_{n\to\infty} |c_n/c_{n+1}|} < 1 \iff |x-a| < \lim\limits_{n\to\infty} \left| \dfrac{c_n}{c_{n+1}} \right|$. Thus, $R = \lim\limits_{n\to\infty} \left| \dfrac{c_n}{c_{n+1}} \right|$. If $\lim\limits_{n\to\infty} \left| \dfrac{c_n}{c_{n+1}} \right| = 0$ and $|x-a| \neq 0$, then $(\star)$ shows that $L = \infty$ and so the series diverges, and hence, $R = 0$. Thus, in all cases, $R = \lim\limits_{n\to\infty} \left| \dfrac{c_n}{c_{n+1}} \right|$.

45. For $2 < x < 3$, $\sum c_n x^n$ diverges and $\sum d_n x^n$ converges. By Exercise 11.2.89, $\sum (c_n + d_n)\, x^n$ diverges. Since both series converge for $|x| < 2$, the radius of convergence of $\sum (c_n + d_n)\, x^n$ is 2.

46. Since $\sum c_n x^n$ converges whenever $|x| < R$, $\sum c_n x^{2n} = \sum c_n (x^2)^n$ converges whenever $|x^2| < R \iff |x| < \sqrt{R}$, so the second series has radius of convergence $\sqrt{R}$.

11.9 Representations of Functions as Power Series

1. If $f(x) = \sum\limits_{n=0}^{\infty} c_n x^n$ has radius of convergence 10, then $f'(x) = \sum\limits_{n=1}^{\infty} n c_n x^{n-1}$ also has radius of convergence 10 by Theorem 2.

2. If $f(x) = \sum\limits_{n=0}^{\infty} b_n x^n$ converges on $(-2, 2)$, then $\int f(x)\, dx = C + \sum\limits_{n=0}^{\infty} \dfrac{b_n}{n+1}\, x^{n+1}$ has the same radius of convergence (by Theorem 2), but may not have the same interval of convergence—it may happen that the integrated series converges at an endpoint (or both endpoints).

3. Our goal is to write the function in the form $\dfrac{1}{1-r}$, and then use Equation 1 to represent the function as a sum of a power series. $f(x) = \dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty}(-x)^n = \sum\limits_{n=0}^{\infty}(-1)^n x^n$ with $|-x| < 1 \;\Leftrightarrow\; |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

4. $f(x) = \dfrac{x}{1+x} = x\left(\dfrac{1}{1-(-x)}\right) = x\sum\limits_{n=0}^{\infty}(-x)^n$, or, equivalently, $\sum\limits_{n=0}^{\infty}(-1)^n x^{n+1}$. The series converges when $|-x| < 1 \;\Leftrightarrow\; |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

5. $f(x) = \dfrac{1}{1-x^2} = \sum\limits_{n=0}^{\infty}(x^2)^n = \sum\limits_{n=0}^{\infty}x^{2n}$. The series converges when $|x^2| < 1 \;\Leftrightarrow\; |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

6. $f(x) = \dfrac{5}{1-4x^2} = 5\left(\dfrac{1}{1-4x^2}\right) = 5\sum\limits_{n=0}^{\infty}(4x^2)^n = 5\sum\limits_{n=0}^{\infty}4^n x^{2n}$. The series converges when $|4x^2| < 1 \;\Leftrightarrow\; |x|^2 < \frac{1}{4} \;\Leftrightarrow\; |x| < \frac{1}{2}$, so $R = \frac{1}{2}$ and $I = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

7. $f(x) = \dfrac{2}{3-x} = \dfrac{2}{3}\left(\dfrac{1}{1-x/3}\right) = \dfrac{2}{3}\sum\limits_{n=0}^{\infty}\left(\dfrac{x}{3}\right)^n$ or, equivalently, $2\sum\limits_{n=0}^{\infty}\dfrac{1}{3^{n+1}}x^n$. The series converges when $\left|\dfrac{x}{3}\right| < 1$, that is, when $|x| < 3$, so $R = 3$ and $I = (-3, 3)$.

8. $f(x) = \dfrac{4}{2x+3} = \dfrac{4}{3}\left(\dfrac{1}{1+2x/3}\right) = \dfrac{4}{3}\left(\dfrac{1}{1-(-2x/3)}\right) = \dfrac{4}{3}\sum\limits_{n=0}^{\infty}\left(-\dfrac{2x}{3}\right)^n$ or, equivalently, $\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{2^{n+2}}{3^{n+1}}x^n$.

The series converges when $\left|-\dfrac{2x}{3}\right| < 1$, that is, when $|x| < \frac{3}{2}$, so $R = \frac{3}{2}$ and $I = \left(-\frac{3}{2}, \frac{3}{2}\right)$.

9. $f(x) = \dfrac{x^2}{x^4+16} = \dfrac{x^2}{16}\left(\dfrac{1}{1+x^4/16}\right) = \dfrac{x^2}{16}\left(\dfrac{1}{1-[-(x/2)]^4}\right) = \dfrac{x^2}{16}\sum\limits_{n=0}^{\infty}\left[-\left(\dfrac{x}{2}\right)^4\right]^n$ or, equivalently, $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^{4n+2}}{2^{4n+4}}$.

The series converges when $\left|-\left(\dfrac{x}{2}\right)^4\right| < 1 \;\Rightarrow\; \left|\dfrac{x}{2}\right| < 1 \;\Rightarrow\; |x| < 2$, so $R = 2$ and $I = (-2, 2)$.

10. $f(x) = \dfrac{x}{2x^2+1} = x\left(\dfrac{1}{1-(-2x^2)}\right) = x\sum\limits_{n=0}^{\infty}(-2x^2)^n$ or, equivalently, $\sum\limits_{n=0}^{\infty}(-1)^n 2^n x^{2n+1}$. The series converges when $|-2x^2| < 1 \;\Rightarrow\; |x^2| < \frac{1}{2} \;\Rightarrow\; |x| < \dfrac{1}{\sqrt{2}}$, so $R = \dfrac{1}{\sqrt{2}}$ and $I = \left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$.

11.
$$\begin{aligned} f(x) &= \frac{x-1}{x+2} = \frac{x+2-3}{x+2} = 1 - \frac{3}{x+2} = 1 - \frac{3/2}{x/2+1} = 1 - \frac{3}{2}\cdot\frac{1}{1-(-x/2)} \\ &= 1 - \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = 1 - \frac{3}{2} - \frac{3}{2}\sum_{n=1}^{\infty}\left(-\frac{x}{2}\right)^n = -\frac{1}{2} - \sum_{n=1}^{\infty}\frac{(-1)^n\,3x^n}{2^{n+1}}. \end{aligned}$$

The geometric series $\sum\limits_{n=0}^{\infty}\left(-\dfrac{x}{2}\right)^n$ converges when $\left|-\dfrac{x}{2}\right| < 1 \;\Leftrightarrow\; |x| < 2$, so $R = 2$ and $I = (-2, 2)$.

Alternatively, you could write $f(x) = 1 - 3\left(\dfrac{1}{x+2}\right)$ and use the series for $\dfrac{1}{x+2}$ found in Example 2.

12. $f(x) = \dfrac{x+a}{x^2+a^2} \quad [a>0] \quad = \dfrac{x}{a^2}\left[\dfrac{1}{1-(-x^2/a^2)}\right] + \dfrac{a}{a^2}\left[\dfrac{1}{1-(-x^2/a^2)}\right]$

$$= \frac{x}{a^2}\sum_{n=0}^{\infty}\left(-\frac{x^2}{a^2}\right)^n + \frac{1}{a}\sum_{n=0}^{\infty}\left(-\frac{x^2}{a^2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{a^{2n+1}} + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{a^{2n+1}}.$$

The geometric series $\sum\limits_{n=0}^{\infty}\left(-\dfrac{x^2}{a^2}\right)^n$ converges when $\left|-\dfrac{x^2}{a^2}\right| < 1 \quad \Leftrightarrow \quad |x| < a$, so $R = a$ and $I = (-a, a)$.

13. $f(x) = \dfrac{2x-4}{x^2-4x+3} = \dfrac{2x-4}{(x-1)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-3} \quad \Rightarrow \quad 2x-4 = A(x-3) + B(x-1)$. Let $x = 1$ to get $-2 = -2A \quad \Leftrightarrow \quad A = 1$ and $x = 3$ to get $2 = 2B \quad \Leftrightarrow \quad B = 1$. Thus,

$$\frac{2x-4}{x^2-4x+3} = \frac{1}{x-1} + \frac{1}{x-3} = \frac{-1}{1-x} + \frac{1}{-3}\left[\frac{1}{1-(x/3)}\right] = -\sum_{n=0}^{\infty}x^n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\left(-1-\frac{1}{3^{n+1}}\right)x^n.$$

We represented f as the sum of two geometric series; the first converges for $x \in (-1, 1)$ and the second converges for $x \in (-3, 3)$. Thus, the sum converges for $x \in (-1, 1) = I$.

14. $f(x) = \dfrac{2x+3}{x^2+3x+2} = \dfrac{2x+3}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} \quad \Rightarrow \quad 2x+3 = A(x+2) + B(x+1)$. Let $x = -1$ to get $1 = A$ and $x = -2$ to get $-1 = -B \quad \Leftrightarrow \quad B = 1$. Thus,

$$\frac{2x+3}{x^2+3x+2} = \frac{1}{x+1} + \frac{1}{x+2} = \frac{1}{1-(-x)} + \frac{1}{2}\left[\frac{1}{1-(-x/2)}\right]$$
$$= \sum_{n=0}^{\infty}(-x)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\left[(-1)^n\left(1+\frac{1}{2^{n+1}}\right)\right]x^n$$

We represented f as the sum of two geometric series; the first converges for $x \in (-1, 1)$ and the second converges for $x \in (-2, 2)$. Thus, the sum converges for $x \in (-1, 1) = I$.

15. (a) $f(x) = \dfrac{1}{(1+x)^2} = \dfrac{d}{dx}\left(\dfrac{-1}{1+x}\right) = -\dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty}(-1)^n x^n\right]$ [from Exercise 3]

$$= \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1} \quad \text{[from Theorem 2(i)]} \quad = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n \text{ with } R = 1.$$

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

(b) $f(x) = \dfrac{1}{(1+x)^3} = -\dfrac{1}{2}\dfrac{d}{dx}\left[\dfrac{1}{(1+x)^2}\right] = -\dfrac{1}{2}\dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ [from part (a)]

$$= -\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n \text{ with } R = 1.$$

(c) $f(x) = \dfrac{x^2}{(1+x)^3} = x^2 \cdot \dfrac{1}{(1+x)^3} = x^2 \cdot \dfrac{1}{2}\sum\limits_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ [from part (b)]

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$$

To write the power series with x^n rather than x^{n+2}, we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\dfrac{1}{2}\sum\limits_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$ with $R = 1$.

16. (a) $\displaystyle\int \frac{1}{1-x}\,dx = -\ln(1-x) + C$ and

$$\int \frac{1}{1-x}\,dx = \int (1 + x + x^2 + \cdots)\,dx = \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) + C = \sum_{n=1}^{\infty} \frac{x^n}{n} + C \text{ for } |x| < 1.$$

So $-\ln(1-x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n} + C$ and letting $x = 0$ gives $0 = C$. Thus, $f(x) = \ln(1-x) = -\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ with $R = 1$.

(b) $f(x) = x\ln(1-x) = -x \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n} = -\sum_{n=1}^{\infty} \frac{x^{n+1}}{n}$.

(c) Letting $x = \frac{1}{2}$ gives $\ln \frac{1}{2} = -\displaystyle\sum_{n=1}^{\infty} \frac{(1/2)^n}{n} \quad\Rightarrow\quad \ln 1 - \ln 2 = -\sum_{n=1}^{\infty} \frac{1^n}{n2^n} \quad\Rightarrow\quad \ln 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n}$.

17. We know that $\dfrac{1}{1+4x} = \dfrac{1}{1-(-4x)} = \displaystyle\sum_{n=0}^{\infty} (-4x)^n$. Differentiating, we get

$$\frac{-4}{(1+4x)^2} = \sum_{n=1}^{\infty} (-4)^n n x^{n-1} = \sum_{n=0}^{\infty} (-4)^{n+1}(n+1)x^n, \text{ so}$$

$$f(x) = \frac{x}{(1+4x)^2} = \frac{-x}{4} \cdot \frac{-4}{(1+4x)^2} = \frac{-x}{4} \sum_{n=0}^{\infty} (-4)^{n+1}(n+1)x^n = \sum_{n=0}^{\infty} (-1)^n 4^n (n+1) x^{n+1}$$

for $|-4x| < 1 \quad\Leftrightarrow\quad |x| < \frac{1}{4}$, so $R = \frac{1}{4}$.

18. $\dfrac{1}{2-x} = \dfrac{1}{2(1-x/2)} = \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n$. Now $\dfrac{d}{dx}\left(\dfrac{1}{2-x}\right) = \dfrac{d}{dx}\left(\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n\right) \quad\Rightarrow$

$\dfrac{1}{(2-x)^2} = \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} n x^{n-1}$ and $\dfrac{d}{dx}\left(\dfrac{1}{(2-x)^2}\right) = \dfrac{d}{dx}\left(\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} n x^{n-1}\right) \quad\Rightarrow$

$$\frac{2}{(2-x)^3} = \sum_{n=2}^{\infty} \frac{1}{2^{n+1}} n(n-1)x^{n-2} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+3}} x^n.$$

Thus, $f(x) = \left(\dfrac{x}{2-x}\right)^3 = \dfrac{x^3}{(2-x)^3} = \dfrac{x^3}{2} \cdot \dfrac{2}{(2-x)^3} = \dfrac{x^3}{2} \displaystyle\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+3}} x^n = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+4}} x^{n+3}$

for $\left|\dfrac{x}{2}\right| < 1 \quad\Leftrightarrow\quad |x| < 2$, so $R = 2$.

19. By Example 4, $\dfrac{1}{(1-x)^2} = \displaystyle\sum_{n=0}^{\infty} (n+1)x^n$. Thus,

$$\begin{aligned} f(x) = \frac{1+x}{(1-x)^2} &= \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n + \sum_{n=0}^{\infty} (n+1)x^{n+1} \\ &= \sum_{n=0}^{\infty} (n+1)x^n + \sum_{n=1}^{\infty} n x^n \qquad \text{[make the starting values equal]} \\ &= 1 + \sum_{n=1}^{\infty} [(n+1) + n]x^n = 1 + \sum_{n=1}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+1)x^n \text{ with } R = 1. \end{aligned}$$

20. By Example 4, $\dfrac{1}{(1-x)^2} = \displaystyle\sum_{n=0}^{\infty} (n+1)x^n$, so

$\dfrac{d}{dx}\left(\dfrac{1}{(1-x)^2}\right) = \dfrac{d}{dx}\left(\displaystyle\sum_{n=0}^{\infty} (n+1)x^n\right) \quad\Rightarrow\quad \dfrac{2}{(1-x)^3} = \displaystyle\sum_{n=1}^{\infty} (n+1)n x^{n-1}$. Thus,

$$f(x) = \frac{x^2+x}{(1-x)^3} = \frac{x^2}{(1-x)^3} + \frac{x}{(1-x)^3} = \frac{x^2}{2}\cdot\frac{2}{(1-x)^3} + \frac{x}{2}\cdot\frac{2}{(1-x)^3}$$

$$= \frac{x^2}{2}\sum_{n=1}^{\infty}(n+1)nx^{n-1} + \frac{x}{2}\sum_{n=1}^{\infty}(n+1)nx^{n-1} = \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^{n+1} + \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^n$$

$$= \sum_{n=2}^{\infty}\frac{n(n-1)}{2}x^n + \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^n \qquad \text{[make the exponents on } x \text{ equal by changing an index]}$$

$$= \sum_{n=2}^{\infty}\frac{n^2-n}{2}x^n + x + \sum_{n=2}^{\infty}\frac{n^2+n}{2}x^n \qquad \text{[make the starting values equal]}$$

$$= x + \sum_{n=2}^{\infty}n^2x^n = \sum_{n=1}^{\infty}n^2x^n \text{ with } R = 1.$$

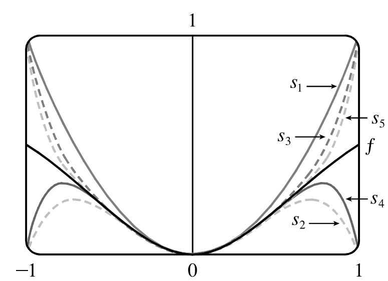
21. $f(x) = \ln(5-x) = -\int\frac{dx}{5-x} = -\frac{1}{5}\int\frac{dx}{1-x/5} = -\frac{1}{5}\int\left[\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n\right]dx$

$$= C - \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty}\frac{x^n}{n5^n}$$

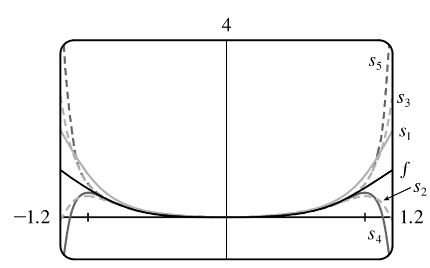
Putting $x=0$, we get $C = \ln 5$. The series converges for $|x/5| < 1 \iff |x| < 5$, so $R = 5$.

22. $f(x) = x^2\tan^{-1}(x^3) = x^2\sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n+1}}{2n+1}$ [by Example 7] $= \sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+3+2}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+5}}{2n+1}$ for $|x^3| < 1 \iff |x| < 1$, so $R = 1$.

23. $f(x) = \frac{x^2}{x^2+1} = x^2\left(\frac{1}{1-(-x^2)}\right) = x^2\sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n+2}$. This series converges when $|-x^2| < 1 \iff x^2 < 1 \iff |x| < 1$, so $R = 1$. The partial sums are $s_1 = x^2$, $s_2 = s_1 - x^4$, $s_3 = s_2 + x^6$, $s_4 = s_3 - x^8$, $s_5 = s_4 + x^{10}$, ... Note that s_1 corresponds to the first term of the infinite sum, regardless of the value of the summation variable and the value of the exponent. As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1, 1)$.



24. From Example 5, we have $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ with $|x| < 1$, so $f(x) = \ln(1+x^4) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{4n}}{n}$ with $|x^4| < 1 \iff |x| < 1$ $[R = 1]$. The partial sums are $s_1 = x^4$, $s_2 = s_1 - \frac{1}{2}x^8$, $s_3 = s_2 + \frac{1}{3}x^{12}$, $s_4 = s_3 - \frac{1}{4}x^{16}$, $s_5 = s_4 + \frac{1}{5}x^{20}$, ... Note that s_1 corresponds to the first term of the infinite sum, regardless of the value of the summation variable and the value of the exponent. As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $[-1, 1]$. (When $x = \pm 1$, the series is the convergent alternating harmonic series.)



25. $f(x) = \ln\left(\dfrac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \displaystyle\int \frac{dx}{1+x} + \int \frac{dx}{1-x} = \int \frac{dx}{1-(-x)} + \int \frac{dx}{1-x}$

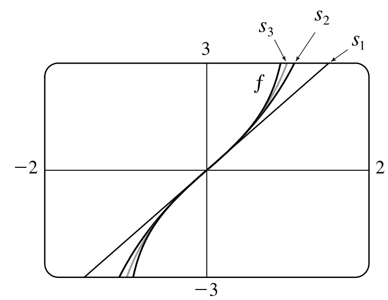
$$= \int \left[\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n\right] dx = \int [(1 - x + x^2 - x^3 + x^4 - \cdots) + (1 + x + x^2 + x^3 + x^4 + \cdots)]\, dx$$

$$= \int (2 + 2x^2 + 2x^4 + \cdots)\, dx = \int \sum_{n=0}^{\infty} 2x^{2n}\, dx = C + \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$$

But $f(0) = \ln \frac{1}{1} = 0$, so $C = 0$ and we have $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$ with $R = 1$. If $x = \pm 1$, then $f(x) = \pm 2 \displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1}$,

which both diverge by the Limit Comparison Test with $b_n = \dfrac{1}{n}$.

The partial sums are $s_1 = \dfrac{2x}{1}$, $s_2 = s_1 + \dfrac{2x^3}{3}$, $s_3 = s_2 + \dfrac{2x^5}{5}$, $\ldots$.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1, 1)$.

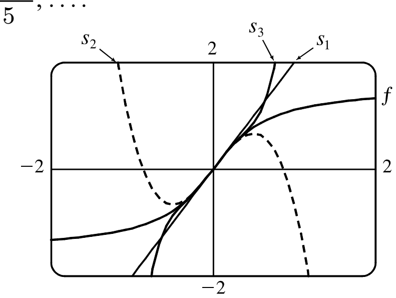
26. $f(x) = \tan^{-1}(2x) = 2\displaystyle\int \frac{dx}{1+4x^2} = 2\int \sum_{n=0}^{\infty} (-1)^n \left(4x^2\right)^n dx = 2\int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}\, dx$

$$= C + 2\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} \qquad [f(0) = \tan^{-1} 0 = 0, \text{ so } C = 0]$$

The series converges when $\left|4x^2\right| < 1 \iff |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. If $x = \pm\frac{1}{2}$, then $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ and

$f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$, respectively. Both series converge by the Alternating Series Test. The partial sums are

$s_1 = \dfrac{2x}{1}$, $s_2 = s_1 - \dfrac{2^3 x^3}{3}$, $s_3 = s_2 + \dfrac{2^5 x^5}{5}$, $\ldots$.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

27. $\dfrac{t}{1-t^8} = t \cdot \dfrac{1}{1-t^8} = t\displaystyle\sum_{n=0}^{\infty} (t^8)^n = \sum_{n=0}^{\infty} t^{8n+1} \quad \Rightarrow \quad \int \frac{t}{1-t^8}\, dt = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$. The series for $\dfrac{1}{1-t^8}$ converges when $\left|t^8\right| < 1 \iff |t| < 1$, so $R = 1$ for that series and also the series for $t/(1-t^8)$. By Theorem 2, the series for $\displaystyle\int \frac{t}{1-t^8}\, dt$ also has $R = 1$.

28. $\dfrac{t}{1+t^3} = t \cdot \dfrac{1}{1-(-t^3)} = t\sum\limits_{n=0}^{\infty}(-t^3)^n = \sum\limits_{n=0}^{\infty}(-1)^n t^{3n+1} \quad\Rightarrow\quad \displaystyle\int \frac{t}{1+t^3}\,dt = C + \sum_{n=0}^{\infty}(-1)^n \frac{t^{3n+2}}{3n+2}$. The series for $\dfrac{1}{1+t^3}$ converges when $\left|-t^3\right| < 1 \quad\Leftrightarrow\quad |t| < 1$, so $R = 1$ for that series and also for the series $\dfrac{t}{1+t^3}$. By Theorem 2, the series for $\displaystyle\int \frac{t}{1+t^3}\,dt$ also has $R = 1$.

29. From Example 5, $\ln(1+x) = \sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{x^n}{n}$ for $|x| < 1$, so $x^2\ln(1+x) = \sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{x^{n+2}}{n}$ and $\displaystyle\int x^2\ln(1+x)\,dx = C + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n+3}}{n(n+3)}$. $R = 1$ for the series for $\ln(1+x)$, so $R = 1$ for the series representing $x^2\ln(1+x)$ as well. By Theorem 2, the series for $\displaystyle\int x^2\ln(1+x)\,dx$ also has $R = 1$.

30. From Example 6, $\tan^{-1}x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$ for $|x| < 1$, so $\dfrac{\tan^{-1}x}{x} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{2n+1}$ and $\displaystyle\int \frac{\tan^{-1}x}{x}\,dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)^2}$. $R = 1$ for the series for $\tan^{-1}x$, so $R = 1$ for the series representing $\dfrac{\tan^{-1}x}{x}$ as well. By Theorem 2, the series for $\displaystyle\int\frac{\tan^{-1}x}{x}\,dx$ also has $R = 1$.

31. $\dfrac{x}{1+x^3} = x\left[\dfrac{1}{1-(-x^3)}\right] = x\sum\limits_{n=0}^{\infty}(-x^3)^n = \sum\limits_{n=0}^{\infty}(-1)^n x^{3n+1} \quad\Rightarrow$

$\displaystyle\int\frac{x}{1+x^3}\,dx = \int\sum_{n=0}^{\infty}(-1)^n x^{3n+1}\,dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+2}}{3n+2}$. Thus,

$$I = \int_0^{0.3}\frac{x}{1+x^3}\,dx = \left[\frac{x^2}{2} - \frac{x^5}{5} + \frac{x^8}{8} - \frac{x^{11}}{11} + \cdots\right]_0^{0.3} = \frac{(0.3)^2}{2} - \frac{(0.3)^5}{5} + \frac{(0.3)^8}{8} - \frac{(0.3)^{11}}{11} + \cdots.$$

The series is alternating, so if we use the first three terms, the error is at most $(0.3)^{11}/11 \approx 1.6\times10^{-7}$.

So $I \approx (0.3)^2/2 - (0.3)^5/5 + (0.3)^8/8 \approx 0.044\,522$ to six decimal places.

32. We substitute $x/2$ for x in Example 6, and find that

$$\begin{aligned}\int\arctan\frac{x}{2}\,dx &= \int\sum_{n=0}^{\infty}(-1)^n\frac{(x/2)^{2n+1}}{2n+1}\,dx = \int\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^{2n+1}(2n+1)}\,dx \\ &= C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{2^{2n+1}(2n+1)(2n+2)}\end{aligned}$$

Thus,

$$\begin{aligned}I = \int_0^{1/2}\arctan\frac{x}{2}\,dx &= \left[\frac{x^2}{2(1)(2)} - \frac{x^4}{2^3(3)(4)} + \frac{x^6}{2^5(5)(6)} - \frac{x^8}{2^7(7)(8)} + \frac{x^{10}}{2^9(9)(10)} - \cdots\right]_0^{1/2} \\ &= \frac{1}{2^3(1)(2)} - \frac{1}{2^7(3)(4)} + \frac{1}{2^{11}(5)(6)} - \frac{1}{2^{15}(7)(8)} + \frac{1}{2^{19}(9)(10)} - \cdots\end{aligned}$$

The series is alternating, so if we use four terms, the error is at most $1/(2^{19}\cdot 90) \approx 2.1\times10^{-8}$. So

$I \approx \dfrac{1}{16} - \dfrac{1}{1536} + \dfrac{1}{61{,}440} - \dfrac{1}{1{,}835{,}008} \approx 0.061\,865$ to six decimal places.

[continued]

Remark: The sum of the first three terms gives us the same answer to six decimal places, but the error is at most $1/1{,}835{,}008 \approx 5.5 \times 10^{-7}$, slightly too large to guarantee the desired accuracy.

33. We substitute x^2 for x in Example 5, and find that

$$\int x\ln(1+x^2)\,dx = \int x\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x^2)^n}{n}\,dx = \int\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+1}}{n}\,dx = C + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+2}}{n(2n+2)}$$

Thus,

$$I \approx \int_0^{0.2} x\ln(1+x^2)\,dx = \left[\frac{x^4}{1(4)} - \frac{x^6}{2(6)} + \frac{x^8}{3(8)} - \frac{x^{10}}{4(10)} + \cdots\right]_0^{0.2} = \frac{(0.2)^4}{4} - \frac{(0.2)^6}{12} + \frac{(0.2)^8}{24} - \frac{(0.2)^{10}}{40} + \cdots$$

The series is alternating, so if we use two terms, the error is at most $(0.2)^8/24 \approx 1.1 \times 10^{-7}$. So $I \approx \dfrac{(0.2)^4}{4} - \dfrac{(0.2)^6}{12} \approx 0.000\,395$ to six decimal places.

34. $$\int_0^{0.3}\frac{x^2}{1+x^4}\,dx = \int_0^{0.3} x^2\sum_{n=0}^{\infty}(-1)^n x^{4n}\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{4n+3}}{4n+3}\right]_0^{0.3} = \sum_{n=0}^{\infty}\frac{(-1)^n\,3^{4n+3}}{(4n+3)10^{4n+3}}$$
$$= \frac{3^3}{3\times 10^3} - \frac{3^7}{7\times 10^7} + \frac{3^{11}}{11\times 10^{11}} - \cdots$$

The series is alternating, so if we use only two terms, the error is at most $\dfrac{3^{11}}{11\times 10^{11}} \approx 0.000\,000\,16$. So, to six decimal places, $\displaystyle\int_0^{0.3}\frac{x^2}{1+x^4}\,dx \approx \frac{3^3}{3\times 10^3} - \frac{3^7}{7\times 10^7} \approx 0.008\,969$.

35. By Example 6, $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$, so $\arctan 0.2 = 0.2 - \dfrac{(0.2)^3}{3} + \dfrac{(0.2)^5}{5} - \dfrac{(0.2)^7}{7} + \cdots$.

The series is alternating, so if we use three terms, the error is at most $\dfrac{(0.2)^7}{7} \approx 0.000\,002$.

Thus, to five decimal places, $\arctan 0.2 \approx 0.2 - \dfrac{(0.2)^3}{3} + \dfrac{(0.2)^5}{5} \approx 0.197\,40$.

36. By Example 5, $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$, so $\ln 1.1 = \ln(1+0.1) = 0.1 - \dfrac{(0.1)^2}{2} + \dfrac{(0.1)^3}{3} - \dfrac{(0.1)^4}{4} + \cdots$.

The series is alternating, so if we use four terms, the error is at most $\dfrac{(0.1)^5}{5} = 0.000\,002$. Thus, to four decimal places, $\ln 1.1 \approx 0.1 - \dfrac{(0.1)^2}{2} + \dfrac{(0.1)^3}{3} - \dfrac{(0.1)^4}{4} \approx 0.0953$.

37. (a) $f(x) = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} \;\Rightarrow\; f'(x) = \sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{x^n}{n!} = f(x)$

(b) By Theorem 9.4.2, the only solution to the differential equation $df(x)/dx = f(x)$ is $f(x) = Ke^x$, but $f(0) = 1$, so $K = 1$ and $f(x) = e^x$.

Or: We could solve the equation $df(x)/dx = f(x)$ as a separable differential equation.

38. $f(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \;\Rightarrow\; f'(x) = \sum_{n=1}^{\infty}\frac{(-1)^n 2nx^{2n-1}}{(2n)!}$ [the first term disappears], so

$$f''(x) = \sum_{n=1}^{\infty}\frac{(-1)^n(2n)(2n-1)x^{2n-2}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2(n-1)}}{[2(n-1)]!} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n}}{(2n)!} \quad \text{[substituting } n+1 \text{ for } n\text{]}$$
$$= -\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = -f(x) \quad\Rightarrow\quad f''(x) + f(x) = 0.$$

39. (a) $J_0(x) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n\, x^{2n}}{2^{2n}(n!)^2}$, $J_0'(x) = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n\, 2nx^{2n-1}}{2^{2n}(n!)^2}$, and $J_0''(x) = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n\, 2n(2n-1)x^{2n-2}}{2^{2n}(n!)^2}$, so

$$\begin{aligned}
x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n\, 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n\, 2nx^{2n}}{2^{2n}(n!)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+2}}{2^{2n}(n!)^2} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n\, 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n\, 2nx^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, x^{2n}}{2^{2n-2}\,[(n-1)!]^2} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n\, 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n\, 2nx^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^{-1}2^2n^2x^{2n}}{2^{2n}(n!)^2} \\
&= \sum_{n=1}^{\infty} (-1)^n \left[\frac{2n(2n-1) + 2n - 2^2n^2}{2^{2n}(n!)^2}\right] x^{2n} \\
&= \sum_{n=1}^{\infty} (-1)^n \left[\frac{4n^2 - 2n + 2n - 4n^2}{2^{2n}(n!)^2}\right] x^{2n} = 0
\end{aligned}$$

(b) $$\begin{aligned}
\int_0^1 J_0(x)\,dx &= \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{2^{2n}(n!)^2}\right] dx = \int_0^1 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots\right) dx \\
&= \left[x - \frac{x^3}{3\cdot 4} + \frac{x^5}{5\cdot 64} - \frac{x^7}{7\cdot 2304} + \cdots\right]_0^1 = 1 - \frac{1}{12} + \frac{1}{320} - \frac{1}{16{,}128} + \cdots
\end{aligned}$$

Since $\frac{1}{16{,}128} \approx 0.000062$, it follows from The Alternating Series Estimation Theorem that, correct to three decimal places, $\int_0^1 J_0(x)\,dx \approx 1 - \frac{1}{12} + \frac{1}{320} \approx 0.920$.

40. (a) If $a_n = \dfrac{(-1)^n\, x^{2n+1}}{n!(n+1)!\,2^{2n+1}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(n+1)!(n+2)!\,2^{2n+3}} \cdot \frac{n!(n+1)!\,2^{2n+1}}{x^{2n+1}}\right| = \left(\frac{x}{2}\right)^2 \lim_{n\to\infty} \frac{1}{(n+1)(n+2)} = 0 \text{ for all } x.$$

So $J_1(x)$ converges for all x and its domain is $(-\infty, \infty)$.

(b) $J_1(x) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$, $J_1'(x) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n\,(2n+1)\,x^{2n}}{n!\,(n+1)!\,2^{2n+1}}$, and $J_1''(x) = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n\,(2n+1)\,(2n)\,x^{2n-1}}{n!\,(n+1)!\,2^{2n+1}}$.

$$\begin{aligned}
&x^2 J_1''(x) + xJ_1'(x) + \left(x^2 - 1\right)J_1(x) \\
&\quad= \sum_{n=1}^{\infty} \frac{(-1)^n\,(2n+1)(2n)x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n+1)x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\
&\qquad\qquad + \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+3}}{n!\,(n+1)!\,2^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\
&\quad= \sum_{n=1}^{\infty} \frac{(-1)^n\,(2n+1)(2n)x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n+1)x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\
&\qquad\qquad - \sum_{n=1}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(n-1)!\,n!\,2^{2n-1}} - \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \qquad \left[\begin{array}{c}\text{Replace } n \text{ with } n-1 \\ \text{in the third term}\end{array}\right] \\
&\quad= \frac{x}{2} - \frac{x}{2} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{(2n+1)(2n) + (2n+1) - (n)(n+1)2^2 - 1}{n!\,(n+1)!\,2^{2n+1}}\right] x^{2n+1} = 0
\end{aligned}$$

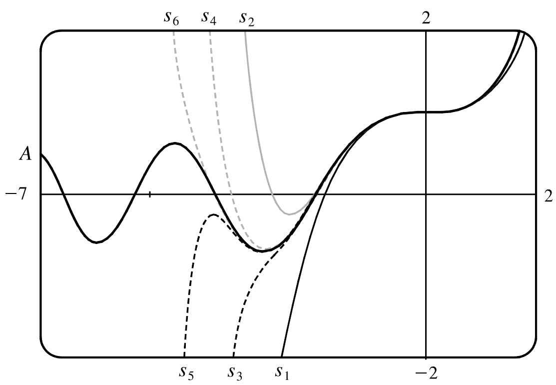
(c) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \quad \Rightarrow$

$$J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)x^{2n-1}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2(n+1)x^{2n+1}}{2^{2n+2}[(n+1)!]^2} \qquad [\text{Replace } n \text{ with } n+1]$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(n+1)!\,n!} \quad [\text{cancel 2 and } n+1\text{; take } -1 \text{ outside sum}] \quad = -J_1(x)$$

41. (a) $A(x) = 1 + \sum_{n=1}^{\infty} a_n$, where $a_n = \frac{x^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \cdots \cdot (3n-1)(3n)}$, so $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|^3 \lim_{n\to\infty} \frac{1}{(3n+2)(3n+3)} = 0$ for all x, so the domain is $\mathbb{R}$.

(b), (c)



$s_0 = 1$ has been omitted from the graph. The partial sums seem to approximate $A(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.

To plot A, we must first define $A(x)$ for the CAS. Note that for $n \geq 1$, the denominator of a_n is

$2 \cdot 3 \cdot 5 \cdot 6 \cdot \cdots \cdot (3n-1) \cdot 3n = \frac{(3n)!}{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-2)} = \frac{(3n)!}{\prod_{k=1}^{n}(3k-2)}$, so $a_n = \frac{\prod_{k=1}^{n}(3k-2)}{(3n)!}x^{3n}$ and thus

$A(x) = 1 + \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n}(3k-2)}{(3n)!}x^{3n}$. Both Maple and Mathematica are able to plot A if we define it this way.

Maple and Mathematica have two initially known Airy functions, called `AI·SERIES(z,m)` and `BI·SERIES(z,m)` from `AiryAi` and `AiryBi` in Maple and Mathematica (just `Ai` and `Bi` in older versions of Maple). However, it is very difficult to solve for A in terms of the CAS's Airy functions, although in fact $A(x) = \frac{\sqrt{3}\,\mathtt{AiryAi}(x) + \mathtt{AiryBi}(x)}{\sqrt{3}\,\mathtt{AiryAi}(0) + \mathtt{AiryBi}(0)}$.

42. $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_{n+4} = c_n$ for all $n \geq 0$. So

$$s_{4n-1} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_0 x^4 + c_1 x^5 + c_2 x^6 + c_3 x^7 + \cdots + c_3 x^{4n-1}$$

$$= \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3\right)\left(1 + x^4 + x^8 + \cdots + x^{4n-4}\right) \to \frac{c_0 + c_1 x + c_2 x^2 + c_3 x^3}{1 - x^4} \text{ as } n \to \infty$$

[by (11.2.4) with $r = x^4$] for $|x^4| < 1 \quad \Leftrightarrow \quad |x| < 1$. Also s_{4n}, s_{4n+1}, s_{4n+2} have the same limits (for example, $s_{4n} = s_{4n-1} + c_0 x^{4n}$ and $x^{4n} \to 0$ for $|x| < 1$). So if at least one of c_0, c_1, c_2, and c_3 is nonzero, then the interval of convergence is $(-1, 1)$ and $f(x) = \frac{c_0 + c_1 x + c_2 x^2 + c_3 x^3}{1 - x^4}$.

43. $f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$, where $c_{2n} = 1$ and $c_{2n+1} = 2$ for all $n \ge 0$. So

$$\begin{aligned} s_{2n-1} &= 1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \cdots + x^{2n-2} + 2x^{2n-1} \\ &= 1(1+2x) + x^2(1+2x) + x^4(1+2x) + \cdots + x^{2n-2}(1+2x) = (1+2x)(1 + x^2 + x^4 + \cdots + x^{2n-2}) \\ &= (1+2x)\frac{1-x^{2n}}{1-x^2} \quad \text{[by (11.2.3) with } r = x^2] \quad \to \frac{1+2x}{1-x^2} \text{ as } n \to \infty \text{ by (11.2.4), when } |x| < 1. \end{aligned}$$

Also $s_{2n} = s_{2n-1} + x^{2n} \to \dfrac{1+2x}{1-x^2}$ since $x^{2n} \to 0$ for $|x| < 1$. Therefore, $s_n \to \dfrac{1+2x}{1-x^2}$ since s_{2n} and s_{2n-1} both approach $\dfrac{1+2x}{1-x^2}$ as $n \to \infty$. Thus, the interval of convergence is $(-1, 1)$ and $f(x) = \dfrac{1+2x}{1-x^2}$.

44. $\dfrac{|\sin nx|}{n^2} \le \dfrac{1}{n^2}$, so $\sum\limits_{n=1}^{\infty} \dfrac{\sin nx}{n^2}$ converges by the Direct Comparison Test. $\dfrac{d}{dx}\left(\dfrac{\sin nx}{n^2}\right) = \dfrac{\cos nx}{n}$, so when $x = 2k\pi$ [k an integer], $\sum\limits_{n=1}^{\infty} f_n'(x) = \sum\limits_{n=1}^{\infty} \dfrac{\cos(2kn\pi)}{n} = \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, which diverges [harmonic series]. $f_n''(x) = -\sin nx$, so $\sum\limits_{n=1}^{\infty} f_n''(x) = -\sum\limits_{n=1}^{\infty} \sin nx$, which converges only if $\sin nx = 0$, or $x = k\pi$ [k an integer].

45. If $a_n = \dfrac{x^n}{n^2}$, then by the Ratio Test, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)^2} \cdot \dfrac{n^2}{x^n}\right| = |x| \lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^2 = |x| < 1$ for convergence, so $R = 1$. When $x = \pm 1$, $\sum\limits_{n=1}^{\infty}\left|\dfrac{x^n}{n^2}\right| = \sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ which is a convergent p-series ($p = 2 > 1$), so the interval of convergence for f is $[-1, 1]$. By Theorem 2, the radii of convergence of f' and f'' are both 1, so we need only check the endpoints. $f(x) = \sum\limits_{n=1}^{\infty}\dfrac{x^n}{n^2} \quad \Rightarrow \quad f'(x) = \sum\limits_{n=1}^{\infty}\dfrac{nx^{n-1}}{n^2} = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n+1}$, and this series diverges for $x = 1$ (harmonic series) and converges for $x = -1$ (Alternating Series Test), so the interval of convergence is $[-1, 1)$. $f''(x) = \sum\limits_{n=1}^{\infty}\dfrac{nx^{n-1}}{n+1}$ diverges at both 1 and -1 (Test for Divergence) since $\lim\limits_{n\to\infty}\dfrac{n}{n+1} = 1 \ne 0$, so its interval of convergence is $(-1, 1)$.

46. (a) $\sum\limits_{n=1}^{\infty} nx^{n-1} = \sum\limits_{n=0}^{\infty} \dfrac{d}{dx} x^n = \dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty} x^n\right] = \dfrac{d}{dx}\left[\dfrac{1}{1-x}\right] = -\dfrac{1}{(1-x)^2}(-1) = \dfrac{1}{(1-x)^2}$, $|x| < 1$.

(b) (i) $\sum\limits_{n=1}^{\infty} nx^n = x\sum\limits_{n=1}^{\infty} nx^{n-1} = x\left[\dfrac{1}{(1-x)^2}\right]$ [from part (a)] $= \dfrac{x}{(1-x)^2}$ for $|x| < 1$.

(ii) Put $x = \frac{1}{2}$ in (i): $\sum\limits_{n=1}^{\infty}\dfrac{n}{2^n} = \sum\limits_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \dfrac{1/2}{(1-1/2)^2} = 2$.

(c) (i)
$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)x^n &= x^2\sum_{n=2}^{\infty} n(n-1)x^{n-2} = x^2\frac{d}{dx}\left[\sum_{n=1}^{\infty} nx^{n-1}\right] = x^2\frac{d}{dx}\frac{1}{(1-x)^2} \\ &= x^2\frac{2}{(1-x)^3} = \frac{2x^2}{(1-x)^3} \text{ for } |x| < 1. \end{aligned}$$

(ii) Put $x = \frac{1}{2}$ in (i): $\sum\limits_{n=2}^{\infty}\dfrac{n^2-n}{2^n} = \sum\limits_{n=2}^{\infty} n(n-1)\left(\frac{1}{2}\right)^n = \dfrac{2(1/2)^2}{(1-1/2)^3} = 4$.

(iii) From (b)(ii) and (c)(ii), we have $\sum\limits_{n=1}^{\infty}\dfrac{n^2}{2^n} = \sum\limits_{n=1}^{\infty}\dfrac{n^2-n}{2^n} + \sum\limits_{n=1}^{\infty}\dfrac{n}{2^n} = 4 + 2 = 6$.

47. $f(x) = \dfrac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n \quad \Rightarrow \quad f'(x) = \dfrac{1}{(1-x)^2} = \sum\limits_{n=1}^{\infty} nx^{n-1} \quad \Rightarrow \quad f''(x) = \dfrac{2}{(1-x)^3} = \sum\limits_{n=2}^{\infty} n(n-1)x^{n-2}$

The power series representation of $h(x)$ is

$$\begin{aligned} h(x) = xf'(x) + x^2 f''(x) &= x \sum_{n=1}^{\infty} nx^{n-1} + x^2 \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \sum_{n=1}^{\infty} nx^n + \sum_{n=2}^{\infty} n(n-1)x^n \\ &= x + \sum_{n=2}^{\infty} nx^n + \sum_{n=2}^{\infty} n(n-1)x^n = x + \sum_{n=2}^{\infty} [nx^n + n(n-1)x^n] \\ &= x + \sum_{n=2}^{\infty} (1+n-1)nx^n = x + \sum_{n=2}^{\infty} n^2x^n = \sum_{n=1}^{\infty} n^2x^n \end{aligned}$$

$h(x)$ has the same radius of convergence as the power series representation of $f(x)$, that is, $R = 1$.

Now, $h\left(\dfrac{1}{2}\right) = \sum\limits_{n=1}^{\infty} \dfrac{n^2}{2^n}$ and using the function representation, we have

$h\left(\dfrac{1}{2}\right) = \dfrac{1}{2}f'\left(\dfrac{1}{2}\right) + \left(\dfrac{1}{2}\right)^2 f''\left(\dfrac{1}{2}\right) = \dfrac{1}{2} \cdot \dfrac{1}{(1-1/2)^2} + \dfrac{1}{4} \cdot \dfrac{2}{(1-1/2)^3} = 6$. Thus, $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{2^n} = 6$.

48. By Example 4, $f(x) = \dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \sum\limits_{n=1}^{\infty} nx^{n-1}$, with radius of convergence $R = 1$. We wish to substitute a value of x with $|x| < 1$ that will result in a denominator of $99^2 = 9801$. Using $x = 1/100$, we get

$\dfrac{1}{(1-1/100)^2} = 1 + 2\left(\dfrac{1}{100}\right) + 3\left(\dfrac{1}{100}\right)^2 + 4\left(\dfrac{1}{100}\right)^3 + \cdots \quad \Rightarrow$

$\dfrac{100^2}{99^2} = 1 + \dfrac{2}{100} + \dfrac{3}{100^2} + \dfrac{4}{100^3} + \cdots + \dfrac{98}{100^{97}} + \cdots \quad \Rightarrow \quad \dfrac{1}{9801} = \dfrac{1}{10^4} + \dfrac{2}{10^6} + \dfrac{3}{10^8} + \dfrac{4}{10^{10}} + \cdots + \dfrac{98}{10^{198}} + \cdots$.

Observe that the series starts with 0.0001, that is, 1 ten-thousandths, and each subsequent term adds the next integer two decimal places deeper. This gives the initial pattern 0.00 01 02 03 04 05 06 07 08 09 10 11 . . ., which will hold until the 100th integer is reached, at which point digits will "carry down" from subsequent terms. We explore this behavior below, starting from the 98th integer.

$$\begin{aligned} &\frac{98}{10^{198}} + \frac{99}{10^{200}} + \frac{100}{10^{202}} + \frac{101}{10^{204}} + \frac{102}{10^{206}} + \cdots \\ &\qquad = \frac{98}{10^{198}} + \frac{99}{10^{200}} + \frac{1}{10^{200}} + \left(\frac{100}{10^{204}} + \frac{1}{10^{204}}\right) + \left(\frac{100}{10^{206}} + \frac{2}{10^{206}}\right) + \cdots \\ &\qquad = \frac{98}{10^{198}} + \frac{100}{10^{200}} + \left(\frac{1}{10^{202}} + \frac{1}{10^{204}}\right) + \left(\frac{1}{10^{204}} + \frac{2}{10^{206}}\right) + \cdots \\ &\qquad = \frac{98}{10^{198}} + \frac{1}{10^{198}} + \frac{1}{10^{202}} + \frac{2}{10^{204}} + \cdots = \frac{99}{10^{198}} + \frac{1}{10^{202}} + \frac{2}{10^{204}} + \cdots \end{aligned}$$

Thus, we see that the 98th integer is omitted and the sequence of integers repeats after the 99th integer (due to the repeated carrying from later terms). This gives the result $\dfrac{1}{9801} = 0.\overline{00\,01\,02\,03 \ldots 96\,97\,99}$.

49. By Example 6, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ for $|x| < 1$. In particular, for $x = \dfrac{1}{\sqrt{3}}$, we have $\dfrac{\pi}{6} = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty} (-1)^n \dfrac{\left(1/\sqrt{3}\right)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{1}{3}\right)^n \dfrac{1}{\sqrt{3}} \dfrac{1}{2n+1}$, so

$$\pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

50. (a) $$\int_0^{1/2} \frac{dx}{x^2 - x + 1} = \int_0^{1/2} \frac{dx}{(x - 1/2)^2 + 3/4} \qquad \left[x - \tfrac{1}{2} = \tfrac{\sqrt{3}}{2}u,\ u = \tfrac{2}{\sqrt{3}}\left(x - \tfrac{1}{2}\right),\ dx = \tfrac{\sqrt{3}}{2}\,du\right]$$

$$= \int_{-1/\sqrt{3}}^{0} \frac{\left(\sqrt{3}/2\right) du}{(3/4)(u^2+1)} = \frac{2\sqrt{3}}{3}\Big[\tan^{-1} u\Big]_{-1/\sqrt{3}}^{0} = \frac{2}{\sqrt{3}}\left[0 - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi}{3\sqrt{3}}$$

(b) $$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} \quad \Rightarrow$$

$$\frac{1}{x^2-x+1} = (x+1)\left(\frac{1}{1+x^3}\right) = (x+1)\frac{1}{1-(-x^3)} = (x+1) \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{3n+1} + \sum_{n=0}^{\infty} (-1)^n x^{3n} \quad \text{for } |x| < 1 \quad \Rightarrow$$

$$\int \frac{dx}{x^2-x+1} = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{3n+2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} \quad \text{for } |x| < 1 \quad \Rightarrow$$

$$\int_0^{1/2} \frac{dx}{x^2-x+1} = \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{4 \cdot 8^n (3n+2)} + \frac{1}{2 \cdot 8^n (3n+1)}\right] = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \left(\frac{2}{3n+1} + \frac{1}{3n+2}\right).$$

By part (a), this equals $\dfrac{\pi}{3\sqrt{3}}$, so $\pi = \dfrac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \dfrac{(-1)^n}{8^n} \left(\dfrac{2}{3n+1} + \dfrac{1}{3n+2}\right)$.

51. Using the Ratio Test, the series $\sum_{n=0}^{\infty} c_n x^n$ converges when $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| < 1 \quad \Rightarrow$

$$\lim_{n\to\infty} \left|\frac{c_{n+1}x^{n+1}}{c_n x^n}\right| = |x| \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| < 1 \quad \Rightarrow \quad |x| < \frac{1}{\lim\limits_{n\to\infty} \left|\dfrac{c_{n+1}}{c_n}\right|} = R \quad \Rightarrow \quad \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \frac{1}{R}.$$

Now, using the Ratio Test for the series $\sum_{n=1}^{\infty} n c_n x^{n-1}$, we find

$\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{(n+1)c_{n+1}x^n}{n c_n x^{n-1}}\right| = \lim_{n\to\infty} \left|\left(1 + \dfrac{1}{n}\right) \dfrac{c_{n+1}}{c_n}\, x\right| = |x| \lim_{n\to\infty} \left|\dfrac{c_{n+1}}{c_n}\right| = \dfrac{|x|}{R}$. Hence, the series $\sum_{n=1}^{\infty} n c_n x^{n-1}$ converges when $\dfrac{|x|}{R} < 1$ or $|x| < R$. Finally, testing the series $\sum_{n=0}^{\infty} c_n \dfrac{x^{n+1}}{n+1}$ using the Ratio Test, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{c_{n+1}x^{n+2}}{n+2} \cdot \frac{n+1}{c_n x^{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(n+1)c_{n+1}x}{(n+2)c_n}\right| = |x| \lim_{n\to\infty} \left|\frac{(1+1/n)c_{n+1}}{(1+2/n)c_n}\right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \frac{|x|}{R}$$

so $\sum_{n=0}^{\infty} c_n \dfrac{x^{n+1}}{n+1}$ also converges when $|x| < R$. Thus, both $\sum_{n=1}^{\infty} n c_n x^{n-1}$ and $\sum_{n=0}^{\infty} c_n \dfrac{x^{n+1}}{n+1}$ have radius of convergence R when $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence R.

11.10 Taylor and Maclaurin Series

1. Using Theorem 5 with $\sum_{n=0}^{\infty} b_n(x-5)^n$, $b_n = \dfrac{f^{(n)}(a)}{n!}$, so $b_8 = \dfrac{f^{(8)}(5)}{8!}$.

2. (a) Using Equation 7, the Maclaurin series of f must have the form $f(0) + \dfrac{f'(0)}{1!}\,x + \dfrac{f''(0)}{2!}\,x^2 + \cdots$. Comparing to the given series, $1.1 + 0.7x^2 + 2.2x^3$, we must have $f(0) = 1.1$. But from the graph, $f(0) \approx 0.5$. Hence, the given series is *not* the Maclaurin series.

(b) Using Equation 6, a power series expansion of f at 1 must have the form $\dfrac{f'(1)}{1!}(x-1) + \dfrac{f''(1)}{2!}(x-1)^2 + \cdots$.
Comparing to the given series, $1.6 - 0.8(x-1) + 0.4(x-1)^2 - \cdots$, we must have $f'(1) = -0.8$. But from the graph, f is increasing near $x = 1$, so $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1.

(c) A power series expansion of f at 2 must have the form $f(2) + \dfrac{f'(2)}{1!}(x-2) + \dfrac{f''(2)}{2!}(x-2)^2 + \cdots$. Comparing to the given series, $2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$, we must have $\frac{1}{2}f''(2) = 1.5$; that is, $f''(2)$ is positive. But from the graph, f is concave downward near $x = 2$, so $f''(2)$ must be negative. Hence, the given series is *not* the Taylor series of f centered at 2.

3. Since $f^{(n)}(0) = (n+1)!$, Equation 7 gives the Maclaurin series
$\sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \dfrac{(n+1)!}{n!}x^n = \sum_{n=0}^{\infty}(n+1)x^n$. Applying the Ratio Test with $a_n = (n+1)x^n$ gives us
$\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{(n+2)x^{n+1}}{(n+1)x^n}\right| = |x|\lim_{n\to\infty}\dfrac{n+2}{n+1} = |x|\cdot 1 = |x|$. For convergence, we must have $|x| < 1$, so the radius of convergence $R = 1$.

4. Since $f^{(n)}(4) = \dfrac{(-1)^n\,n!}{3^n(n+1)}$, Equation 6 gives the Taylor series
$\sum_{n=0}^{\infty} \dfrac{f^{(n)}(4)}{n!}(x-4)^n = \sum_{n=0}^{\infty} \dfrac{(-1)^n\,n!}{3^n(n+1)\,n!}(x-4)^n = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{3^n(n+1)}(x-4)^n$, which is the Taylor series for f centered at 4. Apply the Ratio Test to find the radius of convergence R.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-4)^{n+1}}{3^{n+1}(n+2)}\cdot\frac{3^n(n+1)}{(-1)^n(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)(x-4)(n+1)}{3(n+2)}\right| \\ &= \frac{1}{3}\,|x-4|\lim_{n\to\infty}\frac{n+1}{n+2} = \frac{1}{3}\,|x-4|\end{aligned}$$

For convergence, $\frac{1}{3}\,|x-4| < 1 \quad\Leftrightarrow\quad |x-4| < 3$, so $R = 3$.

5.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^x	0
1	$(x+1)e^x$	1
2	$(x+2)e^x$	2
3	$(x+3)e^x$	3
4	$(x+4)e^x$	4

Using Equation 6 with $n = 0$ to 4 and $a = 0$, we get

$$\begin{aligned}\sum_{n=0}^{4}\frac{f^{(n)}(0)}{n!}(x-0)^n &= \frac{0}{0!}\,x^0 + \frac{1}{1!}\,x^1 + \frac{2}{2!}\,x^2 + \frac{3}{3!}\,x^3 + \frac{4}{4!}\,x^4 \\ &= x + x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{6}x^4\end{aligned}$$

6.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\dfrac{1}{1+x}$	$\frac{1}{3}$
1	$-\dfrac{1}{(1+x)^2}$	$-\frac{1}{9}$
2	$\dfrac{2}{(1+x)^3}$	$\frac{2}{27}$
3	$-\dfrac{6}{(1+x)^4}$	$-\frac{6}{81}$

$$\sum_{n=0}^{3} \frac{f^{(n)}(2)}{n!}(x-2)^n = \frac{\frac{1}{3}}{0!}(x-2)^0 - \frac{\frac{1}{9}}{1!}(x-2)^1 + \frac{\frac{2}{27}}{2!}(x-2)^2 - \frac{\frac{6}{81}}{3!}(x-2)^3$$
$$= \tfrac{1}{3} - \tfrac{1}{9}(x-2) + \tfrac{1}{27}(x-2)^2 - \tfrac{1}{81}(x-2)^3$$

7.

n	$f^{(n)}(x)$	$f^{(n)}(8)$
0	$\sqrt[3]{x}$	2
1	$\dfrac{1}{3x^{2/3}}$	$\frac{1}{12}$
2	$-\dfrac{2}{9x^{5/3}}$	$-\frac{2}{288}$
3	$\dfrac{10}{27x^{8/3}}$	$\frac{10}{6912}$

$$\sum_{n=0}^{3} \frac{f^{(n)}(8)}{n!}(x-8)^n = \frac{2}{0!}(x-8)^0 + \frac{\frac{1}{12}}{1!}(x-8)^1 - \frac{\frac{2}{288}}{2!}(x-8)^2 + \frac{\frac{10}{6912}}{3!}(x-8)^3$$
$$= 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2 + \tfrac{5}{20{,}736}(x-8)^3$$

8.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$1/x$	1
2	$-1/x^2$	-1
3	$2/x^3$	2
4	$-6/x^4$	-6

$$\sum_{n=0}^{4} \frac{f^{(n)}(1)}{n!}(x-1)^n = \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4$$
$$= (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4$$

9.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/6)$
0	$\sin x$	$1/2$
1	$\cos x$	$\sqrt{3}/2$
2	$-\sin x$	$-1/2$
3	$-\cos x$	$-\sqrt{3}/2$

$$\sum_{n=0}^{3} \frac{f^{(n)}(\pi/6)}{n!}\left(x-\frac{\pi}{6}\right)^n = \frac{1/2}{0!}\left(x-\frac{\pi}{6}\right)^0 + \frac{\sqrt{3}/2}{1!}\left(x-\frac{\pi}{6}\right)^1 - \frac{1/2}{2!}\left(x-\frac{\pi}{6}\right)^2 - \frac{\sqrt{3}/2}{3!}\left(x-\frac{\pi}{6}\right)^3$$
$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) - \frac{1}{4}\left(x-\frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3$$

10.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos^2 x$	1
1	$-2\cos x\,\sin x = -\sin 2x$	0
2	$-2\cos 2x$	-2
3	$4\sin 2x$	0
4	$8\cos 2x$	8
5	$-16\sin 2x$	0
6	$-32\cos 2x$	-32

$$\sum_{n=0}^{6} \frac{f^{(n)}(0)}{n!}(x-0)^n = \frac{1}{0!}x^0 - \frac{2}{2!}x^2 + \frac{8}{4!}x^4 - \frac{32}{6!}x^6$$
$$= 1 - x^2 + \tfrac{1}{3}x^4 - \tfrac{2}{45}x^6$$

11.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1-x)^{-2}$	1
1	$2(1-x)^{-3}$	2
2	$6(1-x)^{-4}$	6
3	$24(1-x)^{-5}$	24
4	$120(1-x)^{-6}$	120
$\vdots$	$\vdots$	$\vdots$

$$(1-x)^{-2} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$
$$= 1 + 2x + \tfrac{6}{2}x^2 + \tfrac{24}{6}x^3 + \tfrac{120}{24}x^4 + \cdots$$
$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = |x| \lim_{n\to\infty}\frac{n+2}{n+1} = |x|\,(1) = |x| < 1$$

for convergence, so $R = 1$.

12.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24
$\vdots$	$\vdots$	$\vdots$

$$\ln(1+x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots$$
$$= 0 + x - \tfrac{1}{2}x^2 + \tfrac{2}{6}x^3 - \tfrac{6}{24}x^4 + \tfrac{24}{120}x^5 - \cdots$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{1+1/n} = |x| < 1 \text{ for convergence,}$$

so $R = 1$.

Notice that the answer agrees with the entry for $\ln(1+x)$ in Table 1, but we obtained it by a different method. (Compare with Example 11.9.5.)

13.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
$\vdots$	$\vdots$	$\vdots$

$$\cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$
$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \qquad \text{[Agrees with (16).]}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = \lim_{n\to\infty}\frac{x^2}{(2n+2)(2n+1)} = 0 < 1$$

for all x, so $R = \infty$.

14.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{-2x}	1
1	$-2e^{-2x}$	-2
2	$4e^{-2x}$	4
3	$-8e^{-2x}$	-8
4	$16e^{-2x}$	16
$\vdots$	$\vdots$	$\vdots$

$$e^{-2x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!}\, x^n.$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-2)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{(-2)^n x^n}\right| = \lim_{n\to\infty} \frac{2\,|x|}{n+1}$$

$$= 0 < 1 \quad \text{for all } x \text{, so } R = \infty.$$

15.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$2x^4 - 3x^2 + 3$	3
1	$8x^3 - 6x$	0
2	$24x^2 - 6$	-6
3	$48x$	0
4	48	48
5	0	0
6	0	0
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(x) = 0$ for $n \geq 5$, so f has a finite Maclaurin series.

$$f(x) = 2x^4 - 3x^2 + 3$$

$$= f(0) + \frac{f'(0)}{1!}\, x + \frac{f''(0)}{2!}\, x^2 + \frac{f'''(0)}{3!}\, x^3 + \frac{f^{(4)}(0)}{4!}\, x^4$$

$$= 3 + 0x + \frac{-6}{2} x^2 + 0x^3 + \frac{48}{24} x^4 = 3 - 3x^2 + 2x^4$$

A finite series converges for all x, so $R = \infty$.

16.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin 3x$	0
1	$3\cos 3x$	3
2	$-9\sin 3x$	0
3	$-27\cos 3x$	-27
4	$81\sin 3x$	0
5	$243\cos 3x$	243
6	$729\sin 3x$	0
$\vdots$	$\vdots$	$\vdots$

$$\sin 3x = f(0) + \frac{f'(0)}{1!}\, x + \frac{f''(0)}{2!}\, x^2 + \frac{f'''(0)}{3!}\, x^3 + \frac{f^{(4)}(0)}{4!}\, x^4 + \cdots$$

$$= 3x - \frac{27}{3!}\, x^3 + \frac{243}{5!}\, x^5 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n\, 3^{2n+1}}{(2n+1)!}\, x^{2n+1}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}\, 3^{2n+3}\, x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n\, 3^{2n+1}\, x^{2n+1}}\right|$$

$$= x^2 \cdot \lim_{n\to\infty} \frac{3^2}{(2n+3)(2n+2)}$$

$$= 0 < 1 \text{ for all } x \text{, so } R = \infty.$$

17.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	2^x	1
1	$2^x(\ln 2)$	$\ln 2$
2	$2^x(\ln 2)^2$	$(\ln 2)^2$
3	$2^x(\ln 2)^3$	$(\ln 2)^3$
4	$2^x(\ln 2)^4$	$(\ln 2)^4$
$\vdots$	$\vdots$	$\vdots$

$$2^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n = \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!}\, x^n.$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(\ln 2)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{(\ln 2)^n x^n}\right|$$

$$= \lim_{n\to\infty} \frac{(\ln 2)\,|x|}{n+1} = 0 < 1 \quad \text{for all } x \text{, so } R = \infty.$$

18.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x\cos x$	0
1	$-x\sin x+\cos x$	1
2	$-x\cos x-2\sin x$	0
3	$x\sin x-3\cos x$	-3
4	$x\cos x+4\sin x$	0
5	$-x\sin x+5\cos x$	5
6	$-x\cos x-6\sin x$	0
7	$x\sin x-7\cos x$	-7
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} x\cos x &= f(0)+f'(0)x+\frac{f''(0)}{2!}\,x^2+\frac{f'''(0)}{3!}\,x^3+\frac{f^{(4)}(0)}{4!}\,x^4+\cdots \\ &= 0+1x+0-\frac{3}{3!}\,x^3+0+\frac{5}{5!}\,x^5+0-\frac{7}{7!}\,x^7+\cdots \\ &= x-\frac{1}{2!}\,x^3+\frac{1}{4!}\,x^5-\frac{1}{6!}\,x^7+\cdots \\ &= \sum_{n=0}^{\infty}(-1)^n\,\frac{1}{(2n)!}\,x^{2n+1} \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+2)!}\cdot\frac{(2n)!}{(-1)^n x^{2n+1}}\right| \\ &= \lim_{n\to\infty}\frac{x^2}{(2n+2)(2n+1)}=0<1 \quad \text{for all } x\text{, so } R=\infty. \end{aligned}$$

19.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh x$	0
1	$\cosh x$	1
2	$\sinh x$	0
3	$\cosh x$	1
4	$\sinh x$	0
$\vdots$	$\vdots$	$\vdots$

$$f^{(n)}(0)=\begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd}\end{cases} \qquad \text{so } \sinh x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}.$$

Use the Ratio Test to find R. If $a_n=\dfrac{x^{2n+1}}{(2n+1)!}$, then

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right| = x^2\cdot\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)} \\ &= 0<1 \quad \text{for all } x\text{, so } R=\infty. \end{aligned}$$

20.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cosh x$	1
1	$\sinh x$	0
2	$\cosh x$	1
3	$\sinh x$	0
$\vdots$	$\vdots$	$\vdots$

$$f^{(n)}(0)=\begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd}\end{cases} \qquad \text{so } \cosh x=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}.$$

Use the Ratio Test to find R. If $a_n=\dfrac{x^{2n}}{(2n)!}$, then

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = x^2\cdot\lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} \\ &= 0<1 \quad \text{for all } x\text{, so } R=\infty. \end{aligned}$$

21.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	x^5+2x^3+x	50
1	$5x^4+6x^2+1$	105
2	$20x^3+12x$	184
3	$60x^2+12$	252
4	$120x$	240
5	120	120
6	0	0
7	0	0
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(x)=0$ for $n\geq 6$, so f has a finite expansion about $a=2$.

$$\begin{aligned} f(x)=x^5+2x^3+x &= \sum_{n=0}^{5}\frac{f^{(n)}(2)}{n!}\,(x-2)^n \\ &= \frac{50}{0!}\,(x-2)^0+\frac{105}{1!}\,(x-2)^1+\frac{184}{2!}\,(x-2)^2+\frac{252}{3!}\,(x-2)^3 \\ &\qquad +\frac{240}{4!}\,(x-2)^4+\frac{120}{5!}\,(x-2)^5 \\ &= 50+105(x-2)+92(x-2)^2+42(x-2)^3 \\ &\qquad +10(x-2)^4+(x-2)^5 \end{aligned}$$

A finite series converges for all x, so $R=\infty$.

22.

n	$f^{(n)}(x)$	$f^{(n)}(-2)$
0	$x^6 - x^4 + 2$	50
1	$6x^5 - 4x^3$	-160
2	$30x^4 - 12x^2$	432
3	$120x^3 - 24x$	-912
4	$360x^2 - 24$	1416
5	$720x$	-1440
6	720	720
7	0	0
8	0	0
$\vdots$	$\vdots$	$\vdots$

$f(n)(x) = 0$ for $n \geq 7$, so f has a finite expansion about $a = -2$.

$$\begin{aligned} f(x) = x^6 - x^4 + 2 &= \sum_{n=0}^{6} \frac{f^{(n)}(-2)}{n!}(x+2)^n \\ &= \frac{50}{0!}(x+2)^0 - \frac{160}{1!}(x+2)^1 + \frac{432}{2!}(x+2)^2 - \frac{912}{3!}(x+2)^3 \\ &\quad + \frac{1416}{4!}(x+2)^4 - \frac{1440}{5!}(x+2)^5 + \frac{720}{6!}(x+2)^6 \\ &= 50 - 160(x+2) + 216(x+2)^2 - 152(x+2)^3 + 59(x+2)^4 \\ &\quad - 12(x+2)^5 + (x+2)^6 \end{aligned}$$

A finite series converges for all x, so $R = \infty$.

23.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	$1/x$	$1/2$
2	$-1/x^2$	$-1/2^2$
3	$2/x^3$	$2/2^3$
4	$-6/x^4$	$-6/2^4$
5	$24/x^5$	$24/2^5$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) = \ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= \frac{\ln 2}{0!}(x-2)^0 + \frac{1}{1!\,2^1}(x-2)^1 + \frac{-1}{2!\,2^2}(x-2)^2 + \frac{2}{3!\,2^3}(x-2)^3 \\ &\quad + \frac{-6}{4!\,2^4}(x-2)^4 + \frac{24}{5!\,2^5}(x-2)^5 + \cdots \\ &= \ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n-1)!}{n!\,2^n}(x-2)^n \\ &= \ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n\,2^n}(x-2)^n \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-2)^{n+1}}{(n+1)\,2^{n+1}} \cdot \frac{n\,2^n}{(-1)^{n+1}(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)(x-2)n}{(n+1)2}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\frac{|x-2|}{2} \\ &= \frac{|x-2|}{2} < 1 \quad \text{for convergence, so } |x-2| < 2 \text{ and } R = 2. \end{aligned}$$

24.

n	$f^{(n)}(x)$	$f^{(n)}(-3)$
0	$1/x$	$-1/3$
1	$-1/x^2$	$-1/3^2$
2	$2/x^3$	$-2/3^3$
3	$-6/x^4$	$-6/3^4$
4	$24/x^5$	$-24/3^5$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) = \frac{1}{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x+3)^n \\ &= \frac{-1/3}{0!}(x+3)^0 + \frac{-1/3^2}{1!}(x+3)^1 + \frac{-2/3^3}{2!}(x+3)^2 \\ &\quad + \frac{-6/3^4}{3!}(x+3)^3 + \frac{-24/3^5}{4!}(x+3)^4 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{-n!/3^{n+1}}{n!}(x+3)^n = -\sum_{n=0}^{\infty} \frac{(x+3)^n}{3^{n+1}} \end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+3)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{(x+3)^n}\right| = \lim_{n\to\infty}\frac{|x+3|}{3} = \frac{|x+3|}{3} < 1 \quad \text{for convergence,}$$

so $|x+3| < 3$ and $R = 3$.

25.

n	$f^{(n)}(x)$	$f^{(n)}(3)$
0	e^{2x}	e^6
1	$2e^{2x}$	$2e^6$
2	2^2e^{2x}	$4e^6$
3	2^3e^{2x}	$8e^6$
4	2^4e^{2x}	$16e^6$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) = e^{2x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n \\ &= \frac{e^6}{0!}(x-3)^0 + \frac{2e^6}{1!}(x-3)^1 + \frac{4e^6}{2!}(x-3)^2 \\ &\qquad + \frac{8e^6}{3!}(x-3)^3 + \frac{16e^6}{4!}(x-3)^4 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{2^n e^6}{n!}(x-3)^n \end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}e^6(x-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n e^6(x-3)^n}\right| = \lim_{n\to\infty}\frac{2\,|x-3|}{n+1} = 0 < 1 \quad \text{for all } x, \text{ so } R = \infty.$$

26.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$1/x^2$	1
1	$-2/x^3$	-2
2	$6/x^4$	6
3	$-24/x^5$	-24
4	$120/x^6$	120
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) = \frac{1}{x^2} &= f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \cdots \\ &= 1 - \frac{2}{1!}(x-1) + \frac{6}{2!}(x-1)^2 - \frac{24}{3!}(x-1)^3 + \frac{120}{4!}(x-1)^4 - \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)!}{n!}(x-1)^n = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+2)(x-1)^{n+1}}{(-1)^n(n+1)(x-1)^n}\right| = \lim_{n\to\infty}\frac{(n+2)\,|x-1|}{(n+1)} = |x-1| \lim_{n\to\infty}\frac{1+2/n}{1+1/n} \\ &= |x-1| < 1 \text{ for convergence, so } R = 1. \end{aligned}$$

27.

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\sin x$	0
1	$\cos x$	-1
2	$-\sin x$	0
3	$-\cos x$	1
4	$\sin x$	0
5	$\cos x$	-1
6	$-\sin x$	0
7	$-\cos x$	1
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) = \sin x &= \sum_{n=0}^{\infty}\frac{f^{(n)}(\pi)}{n!}(x-\pi)^n \\ &= \frac{-1}{1!}(x-\pi)^1 + \frac{1}{3!}(x-\pi)^3 + \frac{-1}{5!}(x-\pi)^5 + \frac{1}{7!}(x-\pi)^7 + \cdots \\ &= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1} \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\,(x-\pi)^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{(-1)^{n+1}\,(x-\pi)^{2n+1}}\right| \\ &= \lim_{n\to\infty}\frac{(x-\pi)^2}{(2n+3)(2n+2)} = 0 < 1 \quad \text{for all } x, \text{ so } R = \infty. \end{aligned}$$

28.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\cos x$	0
1	$-\sin x$	-1
2	$-\cos x$	0
3	$\sin x$	1
4	$\cos x$	0
5	$-\sin x$	-1
6	$-\cos x$	0
7	$\sin x$	1
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) = \cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)}{n!}\left(x-\frac{\pi}{2}\right)^n \\ &= \frac{-1}{1!}\left(x-\frac{\pi}{2}\right)^1 + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 + \frac{-1}{5!}\left(x-\frac{\pi}{2}\right)^5 + \frac{1}{7!}\left(x-\frac{\pi}{2}\right)^7 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\left(x-\frac{\pi}{2}\right)^{2n+1} \end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\left(x-\frac{\pi}{2}\right)^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{(-1)^{n+1}\left(x-\frac{\pi}{2}\right)^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\frac{\left(x-\frac{\pi}{2}\right)^2}{(2n+3)(2n+2)} = 0 < 1 \quad \text{for all } x, \text{ so } R=\infty.$$

29.

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\sin 2x$	0
1	$2\cos 2x$	2
2	$-4\sin 2x$	0
3	$-8\cos 2x$	-8
4	$16\sin 2x$	0
5	$32\cos 2x$	32
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) = \sin 2x &= f(\pi) + \frac{f'(\pi)}{1!}(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 \\ &\quad + \frac{f'''(\pi)}{3!}(x-\pi)^3 + \frac{f^{(4)}(\pi)}{4!}(x-\pi)^4 + \cdots \\ &= 0 + \frac{2}{1!}(x-\pi) + 0 - \frac{8}{3!}(x-\pi)^3 + 0 + \frac{32}{5!}(x-\pi)^5 - \cdots \\ &= \frac{2}{1!}(x-\pi) - \frac{8}{3!}(x-\pi)^3 + \frac{32}{5!}(x-\pi)^5 - \cdots \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n\,2^{2n+1}}{(2n+1)!}(x-\pi)^{2n+1} \end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\,2^{2n+3}\,(x-\pi)^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{(-1)^n\,2^{2n+1}\,(x-\pi)^{2n+1}}\right|$$

$$= (x-\pi)^2\lim_{n\to\infty}\frac{2^2}{(2n+3)(2n+2)} = 0 < 1 \text{ for all } x, \text{ so } R=\infty.$$

30.

n	$f^{(n)}(x)$	$f^{(n)}(16)$
0	$\sqrt{x}$	4
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}\cdot\frac{1}{4}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}\cdot\frac{1}{4^3}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{3}{8}\cdot\frac{1}{4^5}$
4	$-\frac{15}{16}x^{-7/2}$	$-\frac{15}{16}\cdot\frac{1}{4^7}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) = \sqrt{x} &= \sum_{n=0}^{\infty}\frac{f^{(n)}(16)}{n!}(x-16)^n \\ &= \frac{4}{0!}(x-16)^0 + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{1!}(x-16)^1 - \frac{1}{4}\cdot\frac{1}{4^3}\cdot\frac{1}{2!}(x-16)^2 \\ &\quad + \frac{3}{8}\cdot\frac{1}{4^5}\cdot\frac{1}{3!}(x-16)^3 - \frac{15}{16}\cdot\frac{1}{4^7}\cdot\frac{1}{4!}(x-16)^4 + \cdots \\ &= 4 + \frac{1}{8}(x-16) + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n 4^{2n-1}\,n!}(x-16)^n \\ &= 4 + \frac{1}{8}(x-16) + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^{5n-2}\,n!}(x-16)^n \end{aligned}$$

[continued]

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(x-16)^{n+1}}{2^{5n+3}(n+1)!}\cdot\frac{2^{5n-2}n!}{(-1)^{n-1}\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)(x-16)^n}\right|$$

$$= \lim_{n\to\infty}\frac{(2n-1)\,|x-16|}{2^5(n+1)} = \frac{|x-16|}{32}\lim_{n\to\infty}\frac{2-1/n}{1+1/n} = \frac{|x-16|}{32}\cdot 2$$

$$= \frac{|x-16|}{16} < 1 \quad \text{for convergence, so } |x-16| < 16 \text{ and } R = 16.$$

31. If $f(x) = \cos x$, then $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$. In each case, $\left|f^{(n+1)}(x)\right| \le 1$, so by Formula 9 with $a = 0$ and $M = 1$, $|R_n(x)| \le \dfrac{1}{(n+1)!}\,|x|^{n+1}$. Thus, $|R_n(x)| \to 0$ as $n \to \infty$ by Equation 10. So $\lim\limits_{n\to\infty} R_n(x) = 0$ and, by Theorem 8, the series in Exercise 13 represents $\cos x$ for all x.

32. If $f(x) = \sin x$, then $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$. In each case, $\left|f^{(n+1)}(x)\right| \le 1$, so by Formula 9 with $a = 0$ and $M = 1$, $|R_n(x)| \le \dfrac{1}{(n+1)!}\,|x-\pi|^{n+1}$. Thus, $|R_n(x)| \to 0$ as $n \to \infty$ by Equation 10. So $\lim\limits_{n\to\infty} R_n(x) \to 0$ and, by Theorem 8, the series in Exercise 27 represents $\sin x$ for all x.

33. If $f(x) = \sinh x$, then for all n, $f^{(n+1)}(x) = \cosh x$ or $\sinh x$. Since $|\sinh x| < |\cosh x| = \cosh x$ for all x, we have $\left|f^{(n+1)}(x)\right| \le \cosh x$ for all n. If d is any positive number and $|x| \le d$, then $\left|f^{(n+1)}(x)\right| \le \cosh x \le \cosh d$, so by Formula 9 with $a = 0$ and $M = \cosh d$, we have $|R_n(x)| \le \dfrac{\cosh d}{(n+1)!}\,|x|^{n+1}$. It follows that $|R_n(x)| \to 0$ as $n \to \infty$ for $|x| \le d$ (by Equation 10). But d was an arbitrary positive number. So by Theorem 8, the series represents $\sinh x$ for all x.

34. If $f(x) = \cosh x$, then for all n, $f^{(n+1)}(x) = \cosh x$ or $\sinh x$. Since $|\sinh x| < |\cosh x| = \cosh x$ for all x, we have $\left|f^{(n+1)}(x)\right| \le \cosh x$ for all n. If d is any positive number and $|x| \le d$, then $\left|f^{(n+1)}(x)\right| \le \cosh x \le \cosh d$, so by Formula 9 with $a = 0$ and $M = \cosh d$, we have $|R_n(x)| \le \dfrac{\cosh d}{(n+1)!}\,|x|^{n+1}$. It follows that $|R_n(x)| \to 0$ as $n \to \infty$ for $|x| \le d$ (by Equation 10). But d was an arbitrary positive number. So by Theorem 8, the series represents $\cosh x$ for all x.

35. $$\sqrt[4]{1-x} = [1+(-x)]^{1/4} = \sum_{n=0}^{\infty}\binom{1/4}{n}(-x)^n$$

$$= 1 + \tfrac{1}{4}(-x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}(-x)^2 + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3!}(-x)^3 + \cdots$$

$$= 1 - \frac{1}{4}x + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}(-1)^n\cdot[3\cdot 7\cdot\cdots\cdot(4n-5)]}{4^n\cdot n!}x^n$$

$$= 1 - \frac{1}{4}x - \sum_{n=2}^{\infty}\frac{3\cdot 7\cdot\cdots\cdot(4n-5)}{4^n\cdot n!}x^n$$

and $|-x| < 1 \quad\Leftrightarrow\quad |x| < 1$, so $R = 1$.

36. $\sqrt[3]{8+x} = \sqrt[3]{8\left(1+\dfrac{x}{8}\right)} = 2\left(1+\dfrac{x}{8}\right)^{1/3} = 2\sum\limits_{n=0}^{\infty}\dbinom{1/3}{n}\left(\dfrac{x}{8}\right)^n$

$$= 2\left[1+\frac{1}{3}\left(\frac{x}{8}\right)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}\left(\frac{x}{8}\right)^2+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{x}{8}\right)^3+\cdots\right]$$

$$= 2\left[1+\frac{1}{24}x+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cdot[2\cdot 5\cdot\cdots\cdot(3n-4)]}{3^n\cdot 8^n\cdot n!}x^n\right]$$

$$= 2+\frac{1}{12}x+2\sum_{n=2}^{\infty}\frac{(-1)^{n-1}[2\cdot 5\cdot\cdots\cdot(3n-4)]}{24^n\cdot n!}x^n$$

and $\left|\dfrac{x}{8}\right|<1 \quad\Leftrightarrow\quad |x|<8$, so $R=8$.

37. $\dfrac{1}{(2+x)^3} = \dfrac{1}{[2(1+x/2)]^3} = \dfrac{1}{8}\left(1+\dfrac{x}{2}\right)^{-3} = \dfrac{1}{8}\sum\limits_{n=0}^{\infty}\dbinom{-3}{n}\left(\dfrac{x}{2}\right)^n$. The binomial coefficient is

$$\binom{-3}{n} = \frac{(-3)(-4)(-5)\cdot\cdots\cdot(-3-n+1)}{n!} = \frac{(-3)(-4)(-5)\cdot\cdots\cdot[-(n+2)]}{n!}$$

$$= \frac{(-1)^n\cdot 2\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot(n+1)(n+2)}{2\cdot n!} = \frac{(-1)^n(n+1)(n+2)}{2}$$

Thus, $\dfrac{1}{(2+x)^3} = \dfrac{1}{8}\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n(n+1)(n+2)}{2}\dfrac{x^n}{2^n} = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n(n+1)(n+2)x^n}{2^{n+4}}$ for $\left|\dfrac{x}{2}\right|<1 \quad\Leftrightarrow\quad |x|<2$, so $R=2$.

38. $(1-x)^{3/4} = [1+(-x)]^{3/4} = \sum\limits_{n=0}^{\infty}\dbinom{3/4}{n}(-x)^n = 1+\dfrac{3}{4}(-x)+\dfrac{\frac{3}{4}\left(-\frac{1}{4}\right)}{2!}(-x)^2+\dfrac{\frac{3}{4}\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{3!}(-x)^3+\cdots$

$$= 1-\frac{3}{4}x+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(-1)^n\cdot 3\cdot[1\cdot 5\cdot 9\cdot\ \cdots\ \cdot(4n-7)]}{4^n\cdot n!}x^n$$

$$= 1-\frac{3}{4}x-3\sum_{n=2}^{\infty}\frac{1\cdot 5\cdot 9\cdot\ \cdots\ \cdot(4n-7)}{4^n\cdot n!}x^n$$

and $|-x|<1 \quad\Leftrightarrow\quad |x|<1$, so $R=1$.

39. $\arctan x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$, so $f(x)=\arctan(x^2) = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{\left(x^2\right)^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{1}{2n+1}x^{4n+2},\ R=1.$

40. $\sin x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{(2n+1)!}$, so $f(x)=\sin\left(\dfrac{\pi}{4}x\right) = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{\left(\frac{\pi}{4}x\right)^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}x^{2n+1}$, $R=\infty$.

41. $\cos x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{(2n)!} \quad\Rightarrow\quad \cos 2x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{(2x)^{2n}}{(2n)!} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{2^{2n}x^{2n}}{(2n)!}$, so

$f(x) = x\cos 2x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{2^{2n}}{(2n)!}x^{2n+1},\ R=\infty.$

42. $e^x = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}$, so $f(x)=e^{3x}-e^{2x} = \sum\limits_{n=0}^{\infty}\dfrac{(3x)^n}{n!}-\sum\limits_{n=0}^{\infty}\dfrac{(2x)^n}{n!} = \sum\limits_{n=0}^{\infty}\dfrac{3^nx^n}{n!}-\sum\limits_{n=0}^{\infty}\dfrac{2^nx^n}{n!} = \sum\limits_{n=0}^{\infty}\dfrac{3^n-2^n}{n!}x^n,\ R=\infty.$

43. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \Rightarrow \quad \cos\left(\frac{1}{2}x^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}x^2\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{2^{2n}\,(2n)!}$, so

$f(x) = x\cos\left(\frac{1}{2}x^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n}(2n)!}\, x^{4n+1},\ R = \infty.$

44. $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad \Rightarrow \quad \ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{3n}}{n}$, so $f(x) = x^2 \ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{3n+2}}{n}$,

$R = 1.$

45. We must write the binomial in the form (1+ expression), so we'll factor out a 4.

$$\frac{x}{\sqrt{4+x^2}} = \frac{x}{\sqrt{4(1+x^2/4)}} = \frac{x}{2\sqrt{1+x^2/4}} = \frac{x}{2}\left(1+\frac{x^2}{4}\right)^{-1/2} = \frac{x}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x^2}{4}\right)^n$$

$$= \frac{x}{2}\left[1 + \left(-\tfrac{1}{2}\right)\frac{x^2}{4} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x^2}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x^2}{4}\right)^3 + \cdots\right]$$

$$= \frac{x}{2} + \frac{x}{2}\sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2^n \cdot 4^n \cdot n!}x^{2n}$$

$$= \frac{x}{2} + \sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{n!\,2^{3n+1}}x^{2n+1} \text{ and } \frac{x^2}{4} < 1 \quad \Leftrightarrow \quad \frac{|x|}{2} < 1 \quad \Leftrightarrow \quad |x| < 2, \quad \text{so } R = 2.$$

46. $$\frac{x^2}{\sqrt{2+x}} = \frac{x^2}{\sqrt{2\,(1+x/2)}} = \frac{x^2}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-1/2} = \frac{x^2}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x}{2}\right)^n$$

$$= \frac{x^2}{\sqrt{2}}\left[1 + \left(-\tfrac{1}{2}\right)\left(\frac{x}{2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right]$$

$$= \frac{x^2}{\sqrt{2}} + \frac{x^2}{\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{n!\,2^{2n}}x^n$$

$$= \frac{x^2}{\sqrt{2}} + \sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{n!\,2^{2n+1/2}}x^{n+2} \text{ and } \left|\frac{x}{2}\right| < 1 \quad \Leftrightarrow \quad |x| < 2, \quad \text{so } R = 2.$$

47. $\sin^2 x = \frac{1}{2}(1-\cos 2x) = \frac{1}{2}\left[1 - \sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}\right] = \frac{1}{2}\left[1 - 1 - \sum_{n=1}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}\right] = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{2n-1}x^{2n}}{(2n)!}$,

$R = \infty$

48. $$\frac{x-\sin x}{x^3} = \frac{1}{x^3}\left[x - \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] = \frac{1}{x^3}\left[x - x - \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] = \frac{1}{x^3}\left[-\sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\right]$$

$$= \frac{1}{x^3}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{(2n+3)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+3)!}$$

and this series also gives the required value at $x = 0$ (namely $1/6$); $R = \infty$.

49. (a) The Maclaurin series for e^x is $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$, so

$$e^{-x} = 1 - x + \frac{1}{2!}(-x)^2 + \frac{1}{3!}(-x)^3 + \cdots = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots, \text{ and}$$

$$\begin{aligned}
\sinh x &= \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}\left[\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) - \left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots\right)\right] \\
&= \frac{1}{2}\left[(1-1) + (x+x) + \left(\frac{1}{2!}x^2 - \frac{1}{2!}x^2\right) + \left(\frac{1}{3!}x^3 + \frac{1}{3!}x^3\right) + \cdots\right] \\
&= \frac{1}{2}\left[2x + \frac{2}{3!}x^3 + \frac{2}{5!}x^5 + \cdots\right] = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

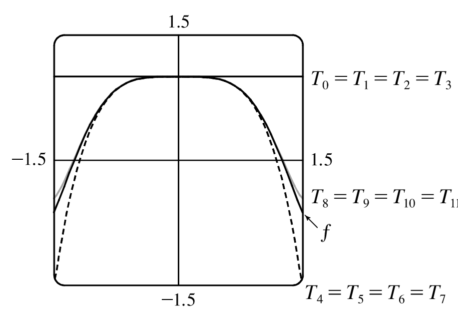
(b)
$$\begin{aligned}
\cosh x &= \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\left[\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) + \left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots\right)\right] \\
&= \frac{1}{2}\left[(1+1) + (x-x) + \left(\frac{1}{2!}x^2 + \frac{1}{2!}x^2\right) + \left(\frac{1}{3!}x^3 - \frac{1}{3!}x^3\right) + \cdots\right] \\
&= \frac{1}{2}\left[2 + \frac{2}{2!}x^2 + \frac{2}{4!}x^4 + \cdots\right] = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}
\end{aligned}$$

50. From Table (1), $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, so

$$\begin{aligned}
\tanh^{-1}x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}[\ln(1+x) - \ln(1-x)] \\
&= \frac{1}{2}\left[\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right)\right] \\
&= \frac{1}{2}\left[(x+x) + \left(\frac{x^2}{2} - \frac{x^2}{2}\right) + \left(\frac{x^3}{3} + \frac{x^3}{3}\right) + \left(\frac{x^4}{4} - \frac{x^4}{4}\right) + \cdots\right] \\
&= \frac{1}{2}\left[2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots\right] = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}
\end{aligned}$$

51. $\cos x \overset{(16)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \Rightarrow$

$$\begin{aligned}
f(x) = \cos(x^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} \\
&= 1 - \tfrac{1}{2}x^4 + \tfrac{1}{24}x^8 - \tfrac{1}{720}x^{12} + \cdots
\end{aligned}$$

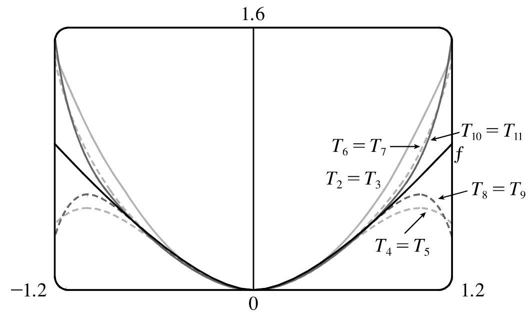


The series for $\cos x$ converges for all x, so the same is true of the series for $f(x)$, that is, $R = \infty$. Notice that, as n increases, $T_n(x)$ becomes a better approximation to $f(x)$.

52. $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} \quad \Rightarrow$

$$f(x) = \ln(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x^2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n}}{n}$$
$$= x^2 - \tfrac{1}{2}x^4 + \tfrac{1}{3}x^6 - \tfrac{1}{4}x^8 + \cdots$$

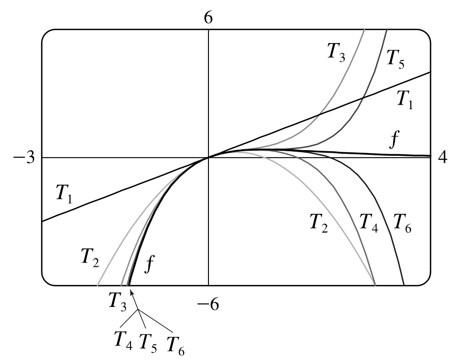
The series for $\ln(1+x)$ has $R = 1$ and $|x^2| < 1 \;\Leftrightarrow\; |x| < 1$, so the series for $f(x)$ also has $R = 1$. From the graphs of f and the first few Taylor polynomials, we see that $T_n(x)$ provides a closer fit to $f(x)$ near 0 as n increases.



53. $e^x \overset{(11)}{=} \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$, so $e^{-x} = \sum_{n=0}^{\infty} \dfrac{(-x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^n}{n!}$, so

$$f(x) = xe^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^{n+1}$$
$$= x - x^2 + \tfrac{1}{2}x^3 - \tfrac{1}{6}x^4 + \tfrac{1}{24}x^5 - \tfrac{1}{120}x^6 + \cdots$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{(n-1)!}$$

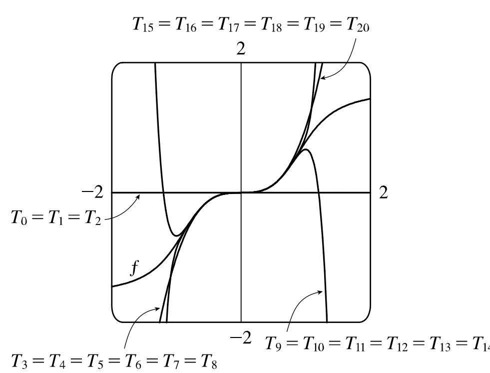
The series for e^x converges for all x, so the same is true of the series for $f(x)$; that is, $R = \infty$. From the graphs of f and the first few Taylor polynomials, we see that $T_n(x)$ provides a closer fit to $f(x)$ near 0 as n increases.



54. From Table 1, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$, so

$$f(x) = \tan^{-1}(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}$$
$$= x^3 - \tfrac{1}{3}x^9 + \tfrac{1}{5}x^{15} - \tfrac{1}{7}x^{21} + \cdots$$

The series for $\tan^{-1} x$ has $R = 1$ and $|x^3| < 1 \;\Leftrightarrow\; |x| < 1$, so the series for $f(x)$ also has $R = 1$. From the graphs of f and the first few Taylor polynomials, we see that $T_n(x)$ provides a closer fit to $f(x)$ near 0 as n increases.



55. $5^\circ = 5^\circ \left(\dfrac{\pi}{180^\circ}\right) = \dfrac{\pi}{36}$ radians and $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$, so

$\cos \dfrac{\pi}{36} = 1 - \dfrac{(\pi/36)^2}{2!} + \dfrac{(\pi/36)^4}{4!} - \dfrac{(\pi/36)^6}{6!} + \cdots$. Now $1 - \dfrac{(\pi/36)^2}{2!} \approx 0.99619$ and adding $\dfrac{(\pi/36)^4}{4!} \approx 2.4 \times 10^{-6}$ does not affect the fifth decimal place, so $\cos 5^\circ \approx 0.99619$ by the Alternating Series Estimation Theorem.

56. $1/\sqrt[10]{e} = e^{-1/10}$ and $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, so

$e^{-1/10} = 1 - \frac{1}{10} + \frac{(1/10)^2}{2!} - \frac{(1/10)^3}{3!} + \frac{(1/10)^4}{4!} - \frac{(1/10)^5}{5!} + \cdots$. Now

$1 - \frac{1}{10} + \frac{(1/10)^2}{2!} - \frac{(1/10)^3}{3!} + \frac{(1/10)^4}{4!} \approx 0.90484$ and subtracting $\frac{(1/10)^5}{5!} \approx 8.3 \times 10^{-8}$ does not affect the fifth decimal place, so $e^{-1/10} \approx 0.90484$ by the Alternating Series Estimation Theorem.

57. (a) $1/\sqrt{1-x^2} = \left[1 + (-x^2)\right]^{-1/2} = 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \cdots$

$$= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n \cdot n!} x^{2n}$$

(b) $\sin^{-1} x = \int \frac{1}{\sqrt{1-x^2}}\,dx = C + x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)2^n \cdot n!} x^{2n+1}$

$$= x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)2^n \cdot n!} x^{2n+1} \quad \text{since } 0 = \sin^{-1} 0 = C.$$

58. (a) $1/\sqrt[4]{1+x} = (1+x)^{-1/4} = \sum_{n=0}^{\infty} \binom{-\frac{1}{4}}{n} x^n = 1 - \frac{1}{4}x + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}x^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!}x^3 + \cdots$

$$= 1 - \frac{1}{4}x + \sum_{n=2}^{\infty} (-1)^n \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{4^n \cdot n!} x^n$$

(b) $1/\sqrt[4]{1+x} = 1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{15}{128}x^3 + \frac{195}{2048}x^4 - \cdots$. $1/\sqrt[4]{1.1} = 1/\sqrt[4]{1+0.1}$, so let $x = 0.1$. The sum of the first four terms is then $1 - \frac{1}{4}(0.1) + \frac{5}{32}(0.1)^2 - \frac{15}{128}(0.1)^3 \approx 0.976$. The fifth term is $\frac{195}{2048}(0.1)^4 \approx 0.000\,009\,5$, which does not affect the third decimal place of the sum, so we have $1/\sqrt[4]{1.1} \approx 0.976$. (Note that the third decimal place of the sum of the first three terms is affected by the fourth term, so we need to use more than three terms for the sum.)

59. $\sqrt{1+x^3} = (1+x^3)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (x^3)^n = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^{3n} \quad \Rightarrow \quad \int \sqrt{1+x^3}\,dx = C + \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{x^{3n+1}}{3n+1}$,

with $R = 1$.

60. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \Rightarrow \quad \sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} \quad \Rightarrow$

$x^2 \sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+4}}{(2n+1)!} \quad \Rightarrow \quad \int x^2 \sin(x^2)\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+5}}{(2n+1)!(4n+5)}$, with $R = \infty$.

61. $\cos x \overset{(16)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \Rightarrow \quad \cos x - 1 = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \Rightarrow \quad \frac{\cos x - 1}{x} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n)!} \quad \Rightarrow$

$\int \frac{\cos x - 1}{x}\,dx = C + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{2n \cdot (2n)!}$, with $R = \infty$.

62. $\arctan x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} \quad \Rightarrow \quad \arctan(x^2) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(x^2)^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{4n+2}}{2n+1} \quad \Rightarrow$

$\int \arctan(x^2)\,dx = C + \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{4n+3}}{(2n+1)(4n+3)}$, with $R = 1$.

63. $\arctan x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ for $|x| < 1$, so $x^3 \arctan x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+4}}{2n+1}$ for $|x| < 1$ and

$\int x^3 \arctan x\,dx = C + \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+5}}{(2n+1)(2n+5)}$. Since $\frac{1}{2} < 1$, we have

$\int_0^{1/2} x^3 \arctan x\,dx = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(1/2)^{2n+5}}{(2n+1)(2n+5)} = \dfrac{(1/2)^5}{1 \cdot 5} - \dfrac{(1/2)^7}{3 \cdot 7} + \dfrac{(1/2)^9}{5 \cdot 9} - \dfrac{(1/2)^{11}}{7 \cdot 11} + \cdots$. Now

$\dfrac{(1/2)^5}{1 \cdot 5} - \dfrac{(1/2)^7}{3 \cdot 7} + \dfrac{(1/2)^9}{5 \cdot 9} \approx 0.0059$ and subtracting $\dfrac{(1/2)^{11}}{7 \cdot 11} \approx 6.3 \times 10^{-6}$ does not affect the fourth decimal place,

so $\int_0^{1/2} x^3 \arctan x\,dx \approx 0.0059$ by the Alternating Series Estimation Theorem.

64. $\sin x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$ for all x, so $\sin(x^4) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{8n+4}}{(2n+1)!}$ for all x and

$\int \sin(x^4)\,dx = C + \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{8n+5}}{(2n+1)!\,(8n+5)}$. Thus,

$\int_0^1 \sin(x^4)\,dx = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{1}{(2n+1)!\,(8n+5)} = \dfrac{1}{1! \cdot 5} - \dfrac{1}{3! \cdot 13} + \dfrac{1}{5! \cdot 21} - \dfrac{1}{7! \cdot 29} + \cdots$. Now

$\dfrac{1}{1! \cdot 5} - \dfrac{1}{3! \cdot 13} + \dfrac{1}{5! \cdot 21} \approx 0.1876$ and subtracting $\dfrac{1}{7! \cdot 29} \approx 6.84 \times 10^{-6}$ does not affect the fourth decimal place, so

$\int_0^1 \sin(x^4)\,dx \approx 0.1876$ by the Alternating Series Estimation Theorem.

65. $\sqrt{1+x^4} = (1+x^4)^{1/2} = \sum\limits_{n=0}^{\infty} \dbinom{\frac{1}{2}}{n} (x^4)^n$, so $\int \sqrt{1+x^4}\,dx = C + \sum\limits_{n=0}^{\infty} \dbinom{\frac{1}{2}}{n} \dfrac{x^{4n+1}}{4n+1}$ and hence, since $0.4 < 1$,

we have

$$\begin{aligned} I &= \int_0^{0.4} \sqrt{1+x^4}\,dx = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(0.4)^{4n+1}}{4n+1} \\ &= (1)\,\frac{(0.4)^1}{0!} + \frac{\frac{1}{2}}{1!}\frac{(0.4)^5}{5} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\frac{(0.4)^9}{9} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\frac{(0.4)^{13}}{13} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}\frac{(0.4)^{17}}{17} + \cdots \\ &= 0.4 + \frac{(0.4)^5}{10} - \frac{(0.4)^9}{72} + \frac{(0.4)^{13}}{208} - \frac{5(0.4)^{17}}{2176} + \cdots \end{aligned}$$

Now $\dfrac{(0.4)^9}{72} \approx 3.6 \times 10^{-6} < 5 \times 10^{-6}$, so by the Alternating Series Estimation Theorem, $I \approx 0.4 + \dfrac{(0.4)^5}{10} \approx 0.40102$

(correct to five decimal places).

66. $\displaystyle\int_0^{0.5} x^2 e^{-x^2}\,dx = \int_0^{0.5} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+3}}{n!(2n+3)}\right]_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)2^{2n+3}}$ and since the term with $n=2$ is $\dfrac{1}{1792} < 0.001$, we use $\displaystyle\sum_{n=0}^{1} \frac{(-1)^n}{n!(2n+3)2^{2n+3}} = \frac{1}{24} - \frac{1}{160} \approx 0.0354$.

67. $$\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2} = \lim_{x\to 0} \frac{x - \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots\right)}{x^2} = \lim_{x\to 0} \frac{\frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{5}x^5 + \cdots}{x^2}$$
$$= \lim_{x\to 0}\left(\tfrac{1}{2} - \tfrac{1}{3}x + \tfrac{1}{4}x^2 - \tfrac{1}{5}x^3 + \cdots\right) = \tfrac{1}{2}$$

since power series are continuous functions.

68. $$\lim_{x\to 0} \frac{1 - \cos x}{1 + x - e^x} = \lim_{x\to 0} \frac{1 - \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)}{1 + x - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots\right)}$$
$$= \lim_{x\to 0} \frac{\frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \cdots}{-\frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 - \frac{1}{5!}x^5 - \frac{1}{6!}x^6 - \cdots}$$
$$= \lim_{x\to 0} \frac{\frac{1}{2!} - \frac{1}{4!}x^2 + \frac{1}{6!}x^4 - \cdots}{-\frac{1}{2!} - \frac{1}{3!}x - \frac{1}{4!}x^2 - \frac{1}{5!}x^3 - \frac{1}{6!}x^4 - \cdots} = \frac{\frac{1}{2} - 0}{-\frac{1}{2} - 0} = -1$$

since power series are continuous functions.

69. $$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0} \frac{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) - x + \frac{1}{6}x^3}{x^5}$$
$$= \lim_{x\to 0} \frac{\frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^5} = \lim_{x\to 0}\left(\frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} - \cdots\right) = \frac{1}{5!} = \frac{1}{120}$$

since power series are continuous functions.

70. $$\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2} = \lim_{x\to 0} \frac{\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots\right) - 1 - \frac{1}{2}x}{x^2} = \lim_{x\to 0} \frac{-\frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots}{x^2}$$
$$= \lim_{x\to 0}\left(-\tfrac{1}{8} + \tfrac{1}{16}x - \cdots\right) = -\tfrac{1}{8}$$ since power series are continuous functions.

71. $$\lim_{x\to 0} \frac{x^3 - 3x + 3\tan^{-1} x}{x^5} = \lim_{x\to 0} \frac{x^3 - 3x + 3\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right)}{x^5}$$
$$= \lim_{x\to 0} \frac{x^3 - 3x + 3x - x^3 + \frac{3}{5}x^5 - \frac{3}{7}x^7 + \cdots}{x^5} = \lim_{x\to 0} \frac{\frac{3}{5}x^5 - \frac{3}{7}x^7 + \cdots}{x^5}$$
$$= \lim_{x\to 0}\left(\tfrac{3}{5} - \tfrac{3}{7}x^2 + \cdots\right) = \tfrac{3}{5}$$ since power series are continuous functions.

72. $$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots\right) - x}{x^3} = \lim_{x\to 0} \frac{\frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots}{x^3} = \lim_{x\to 0}\left(\tfrac{1}{3} + \tfrac{2}{15}x^2 + \cdots\right) = \tfrac{1}{3}$$

since power series are continuous functions.

73. From Equation 11, we have $e^{-x^2} = 1 - \dfrac{x^2}{1!} + \dfrac{x^4}{2!} - \dfrac{x^6}{3!} + \cdots$ and we know that $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$ from Equation 16. Therefore, $e^{-x^2}\cos x = \left(1 - x^2 + \frac{1}{2}x^4 - \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots\right)$. Writing only the terms with degree ≤ 4, we get $e^{-x^2}\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - x^2 + \frac{1}{2}x^4 + \frac{1}{2}x^4 + \cdots = 1 - \frac{3}{2}x^2 + \frac{25}{24}x^4 + \cdots$.

74. $\sec x = \dfrac{1}{\cos x} \overset{(16)}{=} \dfrac{1}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots}$.

$$
\begin{array}{r|l}
 & 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots \\
\hline
1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots & 1 \\
 & 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \\
\cline{2-2}
 & \quad \frac{1}{2}x^2 - \frac{1}{24}x^4 + \cdots \\
 & \quad \frac{1}{2}x^2 - \frac{1}{4}x^4 + \cdots \\
\cline{2-2}
 & \qquad \frac{5}{24}x^4 + \cdots \\
 & \qquad \frac{5}{24}x^4 + \cdots \\
\cline{2-2}
 & \qquad\qquad \cdots
\end{array}
$$

From the long division above, $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots$.

75. $\dfrac{x}{\sin x} \overset{(15)}{=} \dfrac{x}{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots}$.

$$
\begin{array}{r|l}
 & 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \cdots \\
\hline
x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots & x \\
 & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots \\
\cline{2-2}
 & \quad \frac{1}{6}x^3 - \frac{1}{120}x^5 + \cdots \\
 & \quad \frac{1}{6}x^3 - \frac{1}{36}x^5 + \cdots \\
\cline{2-2}
 & \qquad \frac{7}{360}x^5 + \cdots \\
 & \qquad \frac{7}{360}x^5 + \cdots \\
\cline{2-2}
 & \qquad\qquad \cdots
\end{array}
$$

From the long division above, $\dfrac{x}{\sin x} = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \cdots$.

76. From Table 1, we have $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ and that $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$. Therefore,

$y = e^x \ln(1+x) = \left(1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots\right)\left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots\right)$. Writing only terms with degree ≤ 3,

we get $e^x \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + x^2 - \frac{1}{2}x^3 + \frac{1}{2}x^3 + \cdots = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$.

77. $y = (\arctan x)^2 = \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right)\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right)$. Writing only the terms with degree ≤ 6, we get $(\arctan x)^2 = x^2 - \frac{1}{3}x^4 + \frac{1}{5}x^6 - \frac{1}{3}x^4 + \frac{1}{9}x^6 + \frac{1}{5}x^6 + \cdots = x^2 - \frac{2}{3}x^4 + \frac{23}{45}x^6 + \cdots$.

78. $y = e^x \sin^2 x = (e^x \sin x)\sin x = \left(x + x^2 + \frac{1}{3}x^3 + \cdots\right)\left(x - \frac{1}{6}x^3 + \cdots\right)$ [from Example 15]. Writing only the terms with degree ≤ 4, we get $e^x \sin^2 x = x^2 - \frac{1}{6}x^4 + x^3 + \frac{1}{3}x^4 + \cdots = x^2 + x^3 + \frac{1}{6}x^4 + \cdots$.

79. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{\left(-x^4\right)^n}{n!} = e^{-x^4}$, by (11).

80. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{4n}}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x^4)^n}{n} = \ln(1+x^4)$ [from Table 1]

81. $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^{2n+1}(2n+1)}=\sum_{n=0}^{\infty}(-1)^n\frac{(x/2)^{2n+1}}{2n+1}=\tan^{-1}\left(\frac{x}{2}\right)$ [from Table 1]

82. $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^{2n+1}(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{(x/2)^{2n+1}}{(2n+1)!}=\sin\frac{x}{2}$ by (15).

83. $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$ is the Maclaurin series for e^x evaluated at $x=-1$. Thus, $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}=e^{-1}$ by (11).

84. $\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{6^{2n}(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!}=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$, by (16).

85. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{n5^n}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(3/5)^n}{n}=\ln\left(1+\frac{3}{5}\right)$ [from Table 1] $=\ln\frac{8}{5}$

86. $\sum_{n=0}^{\infty}\frac{3^n}{5^n\,n!}=\sum_{n=0}^{\infty}\frac{(3/5)^n}{n!}=e^{3/5}$, by (11).

87. $\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$, by (15).

88. $1-\ln 2+\frac{(\ln 2)^2}{2!}-\frac{(\ln 2)^3}{3!}+\cdots=\sum_{n=0}^{\infty}\frac{(-\ln 2)^n}{n!}=e^{-\ln 2}=\left(e^{\ln 2}\right)^{-1}=2^{-1}=\frac{1}{2}$, by (11).

89. $3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+\cdots=\frac{3^1}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\cdots=\sum_{n=1}^{\infty}\frac{3^n}{n!}=\sum_{n=0}^{\infty}\frac{3^n}{n!}-1=e^3-1$, by (11).

90.
$$\frac{1}{1\cdot 2}-\frac{1}{3\cdot 2^3}+\frac{1}{5\cdot 2^5}-\frac{1}{7\cdot 2^7}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)2^{2n+1}}$$
$$=\sum_{n=0}^{\infty}(-1)^n\frac{(1/2)^{2n+1}}{2n+1}=\tan^{-1}\left(\frac{1}{2}\right)\quad\text{[from Table 1]}$$

91. If p is an nth-degree polynomial, then $p^{(i)}(x)=0$ for $i>n$, so its Taylor series at a is $p(x)=\sum_{i=0}^{n}\frac{p^{(i)}(a)}{i!}(x-a)^i$.

Put $x-a=1$, so that $x=a+1$. Then $p(a+1)=\sum_{i=0}^{n}\frac{p^{(i)}(a)}{i!}$.

This is true for any a, so replace a by x: $p(x+1)=\sum_{i=0}^{n}\frac{p^{(i)}(x)}{i!}$

92. Using the geometric series from Table 1, we have

$$\frac{x}{1+x^2}=x\cdot\frac{1}{1-(-x^2)}=x\cdot\sum_{n=0}^{\infty}(-x^2)^n=x\cdot\sum_{n=0}^{\infty}(-1)^nx^{2n}=\sum_{n=0}^{\infty}(-1)^nx^{2n+1}.$$

The x^{101} term is obtained when $n=50$ and is $(-1)^{50}x^{101}=x^{101}$. So the coefficient of x^{101} is 1 which, by Equation 7, must equal $\frac{f^{(101)}(0)}{101!}$. Thus, $\frac{f^{(101)}(0)}{101!}=1\quad\Rightarrow\quad f^{(101)}(0)=101!$.

93. Using Equation 15, we have

$$x\sin(x^2) = x \cdot \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = x \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)!}$$

The x^{203} term is obtained when $n = 50$ and is $(-1)^{50}\dfrac{x^{203}}{101!} = \dfrac{1}{101!}x^{203}$. So the coefficient of x^{203} is $\dfrac{1}{101!}$, which, by Equation 7, must equal $\dfrac{f^{(203)}(0)}{203!}$. Thus, $\dfrac{f^{(203)}(0)}{203!} = \dfrac{1}{101!} \quad\Rightarrow\quad f^{(203)}(0) = \dfrac{203!}{101!}$.

94. The coefficient of x^{58} in the Maclaurin series of $f(x) = (1+x^3)^{30}$ is $\dfrac{f^{(58)}(0)}{58!}$. But the binomial series for $f(x)$ is $(1+x^3)^{30} = \sum\limits_{n=0}^{\infty} \dbinom{30}{n} x^{3n}$, so it involves only powers of x that are multiples of 3 and therefore the coefficient of x^{58} is 0.

So $f^{(58)}(0) = 0$.

95. Assume that $|f'''(x)| \le M$, so $f'''(x) \le M$ for $a \le x \le a+d$. Now $\int_a^x f'''(t)\,dt \le \int_a^x M\,dt \quad\Rightarrow$

$f''(x) - f''(a) \le M(x-a) \quad\Rightarrow\quad f''(x) \le f''(a) + M(x-a)$. Thus, $\int_a^x f''(t)\,dt \le \int_a^x [f''(a) + M(t-a)]\,dt \quad\Rightarrow$

$f'(x) - f'(a) \le f''(a)(x-a) + \frac{1}{2}M(x-a)^2 \quad\Rightarrow\quad f'(x) \le f'(a) + f''(a)(x-a) + \frac{1}{2}M(x-a)^2 \quad\Rightarrow$

$\int_a^x f'(t)\,dt \le \int_a^x \left[f'(a) + f''(a)(t-a) + \frac{1}{2}M(t-a)^2\right] dt \quad\Rightarrow$

$f(x) - f(a) \le f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}M(x-a)^3$. So

$f(x) - f(a) - f'(a)(x-a) - \frac{1}{2}f''(a)(x-a)^2 \le \frac{1}{6}M(x-a)^3$. But

$R_2(x) = f(x) - T_2(x) = f(x) - f(a) - f'(a)(x-a) - \frac{1}{2}f''(a)(x-a)^2$, so $R_2(x) \le \frac{1}{6}M(x-a)^3$.

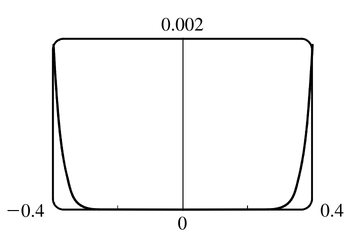
A similar argument using $f'''(x) \ge -M$ shows that $R_2(x) \ge -\frac{1}{6}M(x-a)^3$. So $|R_2(x_2)| \le \frac{1}{6}M\,|x-a|^3$.

Although we have assumed that $x > a$, a similar calculation shows that this inequality is also true if $x < a$.

96. (a) $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$ so $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \dfrac{e^{-1/x^2}}{x} = \lim\limits_{x\to 0} \dfrac{1/x}{e^{1/x^2}} = \lim\limits_{x\to 0} \dfrac{x}{2e^{1/x^2}} = 0$

(using l'Hospital's Rule and simplifying in the penultimate step). Similarly, we can use the definition of the derivative and l'Hospital's Rule to show that $f''(0) = 0$, $f^{(3)}(0) = 0$, …, $f^{(n)}(0) = 0$, so that the Maclaurin series for f consists entirely of zero terms. But since $f(x) \ne 0$ except for $x = 0$, we see that f cannot equal its Maclaurin series except at $x = 0$.

(b)



From the graph, it seems that the function is extremely flat at the origin. In fact, it could be said to be "infinitely flat" at $x = 0$, since all of its derivatives are 0 there.

97. (a) $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad \Rightarrow \quad g'(x) = \sum_{n=1}^{\infty} \binom{k}{n} nx^{n-1}$, so

$$\begin{aligned}
(1+x)g'(x) &= (1+x) \sum_{n=1}^{\infty} \binom{k}{n} nx^{n-1} = \sum_{n=1}^{\infty} \binom{k}{n} nx^{n-1} + \sum_{n=1}^{\infty} \binom{k}{n} nx^n \\
&= \sum_{n=0}^{\infty} \binom{k}{n+1} (n+1)x^n + \sum_{n=0}^{\infty} \binom{k}{n} nx^n \qquad \left[\begin{array}{c}\text{Replace } n \text{ with } n+1 \\ \text{in the first series}\end{array}\right] \\
&= \sum_{n=0}^{\infty} (n+1) \frac{k(k-1)(k-2)\cdots(k-n+1)(k-n)}{(n+1)!} x^n + \sum_{n=0}^{\infty} \left[(n) \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \right] x^n \\
&= \sum_{n=0}^{\infty} \frac{(n+1)k(k-1)(k-2)\cdots(k-n+1)}{(n+1)!} \left[(k-n) + n \right] x^n \\
&= k \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n = k \sum_{n=0}^{\infty} \binom{k}{n} x^n = kg(x)
\end{aligned}$$

Thus, $g'(x) = \dfrac{kg(x)}{1+x}$.

(b) $h(x) = (1+x)^{-k} g(x) \quad \Rightarrow$

$$\begin{aligned}
h'(x) &= -k(1+x)^{-k-1} g(x) + (1+x)^{-k} g'(x) && \text{[Product Rule]} \\
&= -k(1+x)^{-k-1} g(x) + (1+x)^{-k} \frac{kg(x)}{1+x} && \text{[from part (a)]} \\
&= -k(1+x)^{-k-1} g(x) + k(1+x)^{-k-1} g(x) = 0
\end{aligned}$$

(c) From part (b) we see that $h(x)$ must be constant for $x \in (-1, 1)$, so $h(x) = h(0) = 1$ for $x \in (-1, 1)$.

Thus, $h(x) = 1 = (1+x)^{-k} g(x) \quad \Leftrightarrow \quad g(x) = (1+x)^k$ for $x \in (-1, 1)$.

98. Using the binomial series to expand $\sqrt{1+x}$ as a power series as in Example 9, we get

$\sqrt{1+x} = (1+x)^{1/2} = 1 + \dfrac{x}{2} + \sum_{n=2}^{\infty} \dfrac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3) x^n}{2^n \cdot n!}$, so

$\left(1 - x^2\right)^{1/2} = 1 - \dfrac{1}{2} x^2 - \sum_{n=2}^{\infty} \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n \cdot n!} x^{2n}$ and

$\sqrt{1 - e^2 \sin^2 \theta} = 1 - \dfrac{1}{2} e^2 \sin^2 \theta - \sum_{n=2}^{\infty} \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n \cdot n!} e^{2n} \sin^{2n} \theta$. Thus,

$$\begin{aligned}
L &= 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta = 4a \int_0^{\pi/2} \left(1 - \frac{1}{2} e^2 \sin^2 \theta - \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n \cdot n!} e^{2n} \sin^{2n} \theta \right) d\theta \\
&= 4a \left[\frac{\pi}{2} - \frac{e^2}{2} S_1 - \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{n!} \left(\frac{e^2}{2} \right)^n S_n \right]
\end{aligned}$$

where $S_n = \displaystyle\int_0^{\pi/2} \sin^{2n} \theta \, d\theta = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n} \frac{\pi}{2}$ by Exercise 7.1.56.

[continued]

$$L = 4a\left(\frac{\pi}{2}\right)\left[1 - \frac{e^2}{2}\cdot\frac{1}{2} - \sum_{n=2}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{n!}\left(\frac{e^2}{2}\right)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\right]$$

$$= 2\pi a\left[1 - \frac{e^2}{4} - \sum_{n=2}^{\infty}\frac{e^{2n}}{2^n}\cdot\frac{1^2\cdot 3^2\cdot 5^2\cdot\cdots\cdot(2n-3)^2(2n-1)}{n!\cdot 2^n\cdot n!}\right]$$

$$= 2\pi a\left[1 - \frac{e^2}{4} - \sum_{n=2}^{\infty}\frac{e^{2n}}{4^n}\left(\frac{1\cdot 3\cdot\cdots\cdot(2n-3)}{n!}\right)^2(2n-1)\right]$$

$$= 2\pi a\left[1 - \frac{e^2}{4} - \frac{3e^4}{64} - \frac{5e^6}{256} - \cdots\right] = \frac{\pi a}{128}\left(256 - 64e^2 - 12e^4 - 5e^6 - \cdots\right)$$

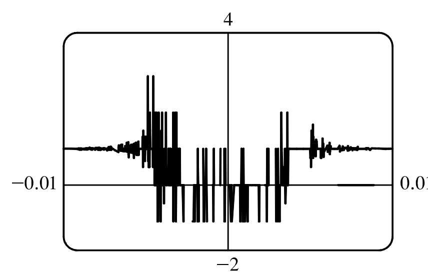
DISCOVERY PROJECT An Elusive Limit

1. $f(x) = \dfrac{n(x)}{d(x)} = \dfrac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}$

x	$f(x)$
1	1.1838
0.1	0.9821
0.01	2.0000
0.001	3.3333
0.0001	3.3333

The table of function values were obtained using Maple with 10 digits of precision. The results of this project will vary depending on the CAS and precision level. It appears that as $x \to 0^+$, $f(x) \to \frac{10}{3}$. Since f is an even function, we have $f(x) \to \frac{10}{3}$ as $x \to 0$.

2. The graph is inconclusive about the limit of f as $x \to 0$.



3. The limit has the indeterminate form $\frac{0}{0}$. Applying l'Hospital's Rule, we obtain the form $\frac{0}{0}$ six times. Finally, on the seventh application we obtain $\lim\limits_{x\to 0}\dfrac{n^{(7)}(x)}{d^{(7)}(x)} = \dfrac{-168}{-168} = 1$.

4. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\dfrac{n(x)}{d(x)} \overset{\text{CAS}}{=} \lim\limits_{x\to 0}\dfrac{-\frac{1}{30}x^7 - \frac{29}{756}x^9 + \cdots}{-\frac{1}{30}x^7 + \frac{13}{756}x^9 + \cdots}$

$$= \lim_{x\to 0}\frac{\left(-\frac{1}{30}x^7 - \frac{29}{756}x^9 + \cdots\right)/x^7}{\left(-\frac{1}{30}x^7 + \frac{13}{756}x^9 + \cdots\right)/x^7} = \lim_{x\to 0}\frac{-\frac{1}{30} - \frac{29}{756}x^2 + \cdots}{-\frac{1}{30} + \frac{13}{756}x^2 + \cdots} = \frac{-\frac{1}{30}}{-\frac{1}{30}} = 1$$

Note that $n^{(7)}(x) = d^{(7)}(x) = -\frac{7!}{30} = -\frac{5040}{30} = -168$, which agrees with the result in Problem 3.

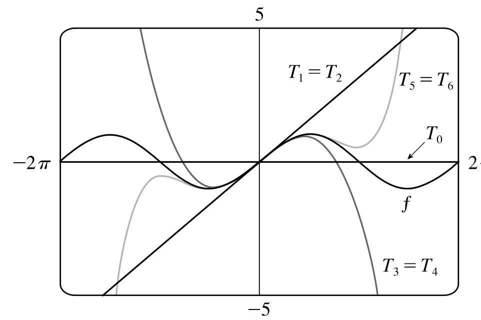
5. The limit command gives the result that $\lim\limits_{x\to 0} f(x) = 1$.

6. The strange results (with only 10 digits of precision) must be due to the fact that the terms being subtracted in the numerator and denominator are very close in value when $|x|$ is small. Thus, the differences are imprecise (have few correct digits).

11.11 Applications of Taylor Polynomials

1. (a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$
0	$\sin x$	0	0
1	$\cos x$	1	x
2	$-\sin x$	0	x
3	$-\cos x$	-1	$x - \frac{1}{6}x^3$
4	$\sin x$	0	$x - \frac{1}{6}x^3$
5	$\cos x$	1	$x - \frac{1}{6}x^3 + \frac{1}{120}x^5$



Note: $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$

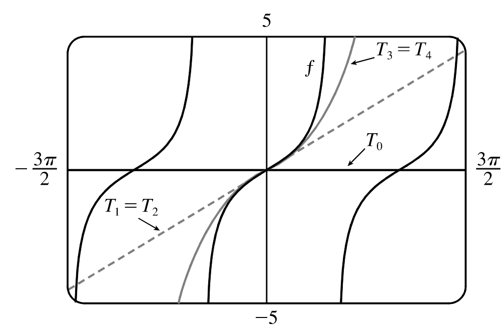
(b)

x	f	$T_0(x)$	$T_1(x) = T_2(x)$	$T_3(x) = T_4(x)$	$T_5(x)$
$\frac{\pi}{4}$	0.7071	0	0.7854	0.7047	0.7071
$\frac{\pi}{2}$	1	0	1.5708	0.9248	1.0045
π	0	0	3.1416	-2.0261	0.5240

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

2. (a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$
0	$\tan x$	0	0
1	$\sec^2 x$	1	x
2	$2\sec^2 x\,\tan x$	0	x
3	$4\sec^2 x\,\tan^2 x + 2\sec^4 x$	2	$x + \frac{1}{3}x^3$



Note: $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$

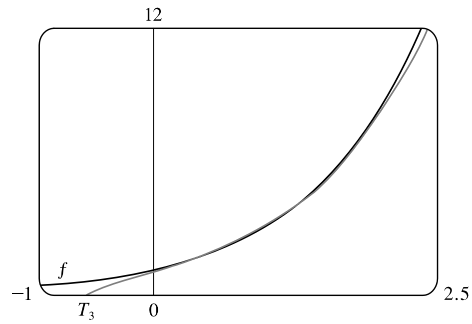
(b)

x	f	$T_0(x)$	$T_1(x) = T_2(x)$	$T_3(x)$
$\frac{\pi}{6}$	0.5774	0	0.5236	0.5714
$\frac{\pi}{4}$	1	0	0.7854	0.9469
$\frac{\pi}{3}$	1.7321	0	1.0472	1.4300

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval. Because the Taylor polynomials are continuous, they cannot approximate the infinite discontinuities at $x = \pm\pi/2$. They can only approximate $\tan x$ on $(-\pi/2, \pi/2)$.

3.

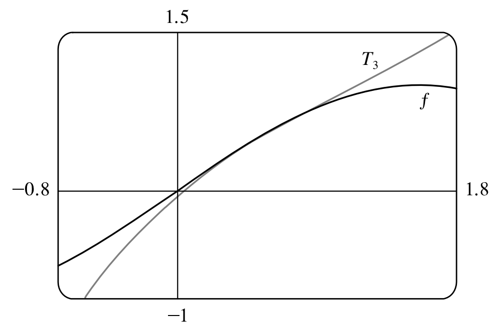
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	e^x	e
1	e^x	e
2	e^x	e
3	e^x	e



$$\begin{aligned} T_3(x) &= \sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= \frac{e}{0!}(x-1)^0 + \frac{e}{1!}(x-1)^1 + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 \\ &= e + e(x-1) + \tfrac{1}{2}e(x-1)^2 + \tfrac{1}{6}e(x-1)^3 \end{aligned}$$

4.

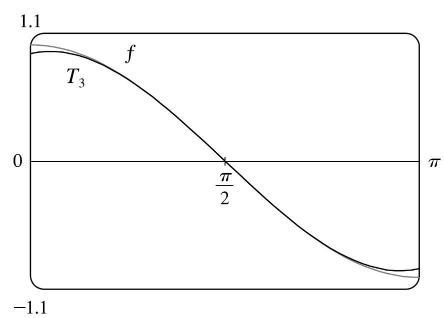
n	$f^{(n)}(x)$	$f^{(n)}(\pi/6)$
0	$\sin x$	$1/2$
1	$\cos x$	$\sqrt{3}/2$
2	$-\sin x$	$-1/2$
3	$-\cos x$	$-\sqrt{3}/2$



$$\begin{aligned} T_3(x) &= \sum_{n=0}^{3} \frac{f^{(n)}(\pi/6)}{n!}\left(x-\frac{\pi}{6}\right)^n \\ &= \frac{1/2}{0!}\left(x-\frac{\pi}{6}\right)^0 + \frac{\sqrt{3}/2}{1!}\left(x-\frac{\pi}{6}\right)^1 - \frac{1/2}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{\sqrt{3}/2}{3!}\left(x-\frac{\pi}{6}\right)^3 \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) - \frac{1}{4}\left(x-\frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3 \end{aligned}$$

5.

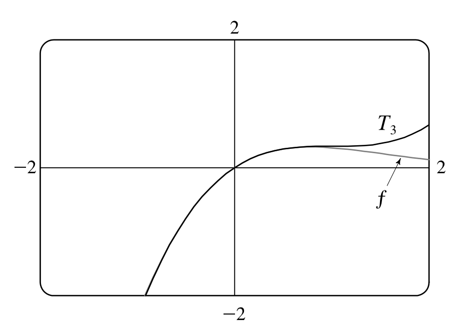
n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\cos x$	0
1	$-\sin x$	-1
2	$-\cos x$	0
3	$\sin x$	1



$$\begin{aligned} T_3(x) &= \sum_{n=0}^{3} \frac{f^{(n)}(\pi/2)}{n!}\left(x-\tfrac{\pi}{2}\right)^n \\ &= -\left(x-\tfrac{\pi}{2}\right) + \tfrac{1}{6}\left(x-\tfrac{\pi}{2}\right)^3 \end{aligned}$$

6.

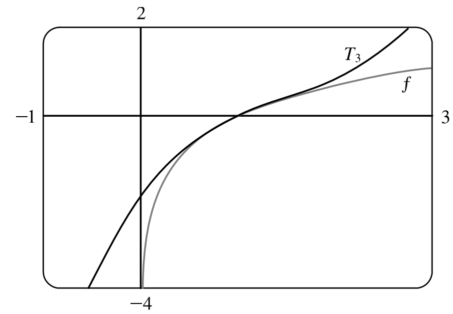
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$e^{-x}\sin x$	0
1	$e^{-x}(\cos x - \sin x)$	1
2	$-2e^{-x}\cos x$	-2
3	$2e^{-x}(\cos x + \sin x)$	2



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!}x^n = x - x^2 + \tfrac{1}{3}x^3$$

7.

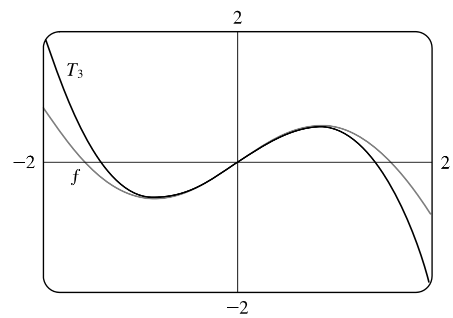
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$1/x$	1
2	$-1/x^2$	-1
3	$2/x^3$	2



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!}(x-1)^n$$
$$= 0 + \frac{1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$
$$= (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3$$

8.

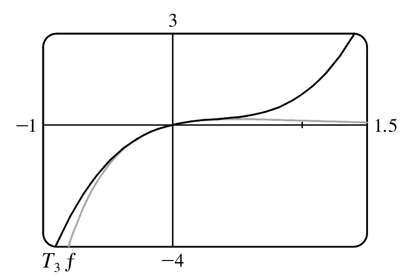
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x\cos x$	0
1	$-x\sin x + \cos x$	1
2	$-x\cos x - 2\sin x$	0
3	$x\sin x - 3\cos x$	-3



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!}x^n$$
$$= 0 + \frac{1}{1!}x + 0 + \frac{-3}{3!}x^3 = x - \frac{1}{2}x^3$$

9.

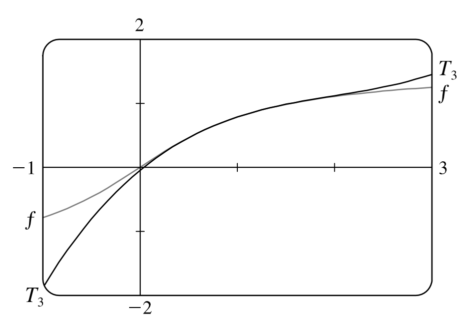
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^{-2x}	0
1	$(1-2x)e^{-2x}$	1
2	$4(x-1)e^{-2x}$	-4
3	$4(3-2x)e^{-2x}$	12



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!}x^n = \tfrac{0}{1}\cdot 1 + \tfrac{1}{1}x^1 + \tfrac{-4}{2}x^2 + \tfrac{12}{6}x^3 = x - 2x^2 + 2x^3$$

10.

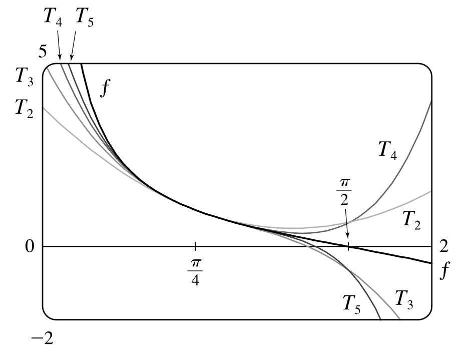
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\tan^{-1} x$	$\frac{\pi}{4}$
1	$\dfrac{1}{1+x^2}$	$\frac{1}{2}$
2	$\dfrac{-2x}{(1+x^2)^2}$	$-\frac{1}{2}$
3	$\dfrac{6x^2-2}{(1+x^2)^3}$	$\frac{1}{2}$



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!}(x-1)^n = \frac{\pi}{4} + \frac{1/2}{1}(x-1)^1 + \frac{-1/2}{2}(x-1)^2 + \frac{1/2}{6}(x-1)^3$$
$$= \tfrac{\pi}{4} + \tfrac{1}{2}(x-1) - \tfrac{1}{4}(x-1)^2 + \tfrac{1}{12}(x-1)^3$$

11. You may be able to simply find the Taylor polynomials for $f(x) = \cot x$ using your CAS. We will list the values of $f^{(n)}(\pi/4)$ for $n = 0$ to $n = 5$.

n	0	1	2	3	4	5
$f^{(n)}(\pi/4)$	1	-2	4	-16	80	-512

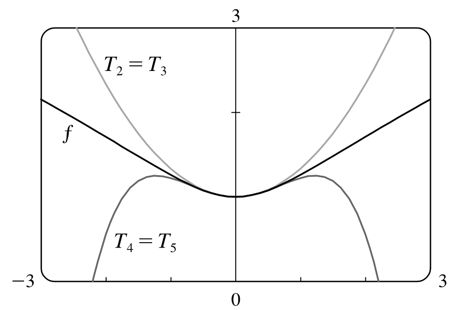


$$T_5(x) = \sum_{n=0}^{5} \frac{f^{(n)}(\pi/4)}{n!}\left(x - \tfrac{\pi}{4}\right)^n$$
$$= 1 - 2\left(x - \tfrac{\pi}{4}\right) + 2\left(x - \tfrac{\pi}{4}\right)^2 - \tfrac{8}{3}\left(x - \tfrac{\pi}{4}\right)^3 + \tfrac{10}{3}\left(x - \tfrac{\pi}{4}\right)^4 - \tfrac{64}{15}\left(x - \tfrac{\pi}{4}\right)^5$$

For $n = 2$ to $n = 5$, $T_n(x)$ is the polynomial consisting of all the terms up to and including the $\left(x - \frac{\pi}{4}\right)^n$ term.

12. You may be able to simply find the Taylor polynomials for $f(x) = \sqrt[3]{1 + x^2}$ using your CAS. We will list the values of $f^{(n)}(0)$ for $n = 0$ to $n = 5$.

n	0	1	2	3	4	5
$f^{(n)}(0)$	1	0	$\frac{2}{3}$	0	$-\frac{8}{3}$	0



$$T_5(x) = \sum_{n=0}^{5} \frac{f^{(n)}(0)}{n!}x^n = 1 + \tfrac{1}{3}x^2 - \tfrac{1}{9}x^4$$

For $n = 2$ to $n = 5$, $T_n(x)$ is the polynomial consisting of all the terms up to and including the x^n term. Note that $T_2 = T_3$ and $T_4 = T_5$.

13. (a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$1/x$	1
1	$-1/x^2$	-1
2	$2/x^3$	2
3	$-6/x^4$	

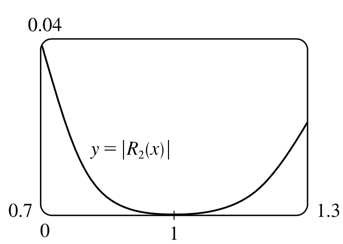
$$f(x) = 1/x \approx T_2(x)$$
$$= \frac{1}{0!}(x-1)^0 - \frac{1}{1!}(x-1)^1 + \frac{2}{2!}(x-1)^2$$
$$= 1 - (x-1) + (x-1)^2$$

(b) $|R_2(x)| \le \dfrac{M}{3!}|x-1|^3$, where $|f'''(x)| \le M$. Now $0.7 \le x \le 1.3 \;\Rightarrow\; |x-1| \le 0.3 \;\Rightarrow\; |x-1|^3 \le 0.027$.

Since $|f'''(x)|$ is decreasing on $[0.7, 1.3]$, we can take $M = |f'''(0.7)| = 6/(0.7)^4$, so

$$|R_2(x)| \le \frac{6/(0.7)^4}{6}(0.027) = 0.112\,453\,1.$$

(c)



From the graph of $|R_2(x)| = \left|\dfrac{1}{x} - T_2(x)\right|$, it seems that the error is less than $0.038\,571$ on $[0.7, 1.3]$.

14. (a)

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$x^{-1/2}$	$\frac{1}{2}$
1	$-\frac{1}{2}x^{-3/2}$	$-\frac{1}{16}$
2	$\frac{3}{4}x^{-5/2}$	$\frac{3}{128}$
3	$-\frac{15}{8}x^{-7/2}$	

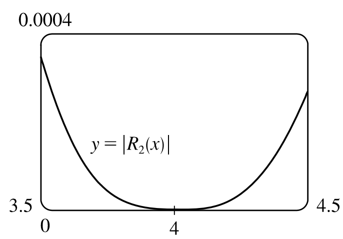
$$\begin{aligned} f(x) = x^{-1/2} &\approx T_2(x) \\ &= \frac{1/2}{0!}(x-4)^0 - \frac{1/16}{1!}(x-4)^1 + \frac{3/128}{2!}(x-4)^2 \\ &= \tfrac{1}{2} - \tfrac{1}{16}(x-4) + \tfrac{3}{256}(x-4)^2 \end{aligned}$$

(b) $|R_2(x)| \le \dfrac{M}{3!}|x-4|^3$, where $|f'''(x)| \le M$. Now $3.5 \le x \le 4.5 \Rightarrow |x-4| \le 0.5 \Rightarrow |x-4|^3 \le 0.125$.

Since $|f'''(x)|$ is decreasing on $[3.5, 4.5]$, we can take $M = |f'''(3.5)| = \dfrac{15}{8(3.5)^{7/2}}$, so

$|R_2(x)| \le \dfrac{15}{6 \cdot 8(3.5)^{7/2}}(0.125) \approx 0.000\,487$.

(c)



From the graph of $|R_2(x)| = \left|x^{-1/2} - T_2(x)\right|$, it seems that the error is less than $0.000\,343$ on $[3.5, 4.5]$.

15.

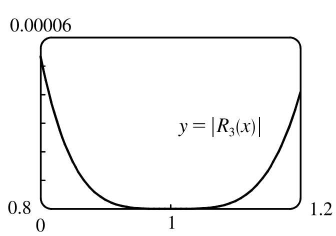
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{2/3}$	1
1	$\frac{2}{3}x^{-1/3}$	$\frac{2}{3}$
2	$-\frac{2}{9}x^{-4/3}$	$-\frac{2}{9}$
3	$\frac{8}{27}x^{-7/3}$	$\frac{8}{27}$
4	$-\frac{56}{81}x^{-10/3}$	

(a) $$\begin{aligned} f(x) = x^{2/3} \approx T_3(x) &= 1 + \tfrac{2}{3}(x-1) - \frac{2/9}{2!}(x-1)^2 + \frac{8/27}{3!}(x-1)^3 \\ &= 1 + \tfrac{2}{3}(x-1) - \tfrac{1}{9}(x-1)^2 + \tfrac{4}{81}(x-1)^3 \end{aligned}$$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0.8 \le x \le 1.2 \Rightarrow$ $|x-1| \le 0.2 \Rightarrow |x-1|^4 \le 0.0016$. Since $\left|f^{(4)}(x)\right|$ is decreasing on $[0.8, 1.2]$, we can take $M = \left|f^{(4)}(0.8)\right| = \frac{56}{81}(0.8)^{-10/3}$, so

$|R_3(x)| \le \dfrac{\frac{56}{81}(0.8)^{-10/3}}{24}(0.0016) \approx 0.000\,096\,97$.

(c)



From the graph of $|R_3(x)| = \left|x^{2/3} - T_3(x)\right|$, it seems that the error is less than $0.000\,053\,3$ on $[0.8, 1.2]$.

16.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/6)$
0	$\sin x$	$1/2$
1	$\cos x$	$\sqrt{3}/2$
2	$-\sin x$	$-1/2$
3	$-\cos x$	$-\sqrt{3}/2$
4	$\sin x$	$1/2$
5	$\cos x$	

(a) $f(x) = \sin x \approx T_4(x)$

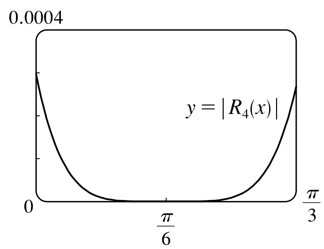
$$= \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\left(x - \tfrac{\pi}{6}\right) - \tfrac{1}{4}\left(x - \tfrac{\pi}{6}\right)^2 - \tfrac{\sqrt{3}}{12}\left(x - \tfrac{\pi}{6}\right)^3 + \tfrac{1}{48}\left(x - \tfrac{\pi}{6}\right)^4$$

(b) $|R_4(x)| \le \dfrac{M}{5!}\left|x - \frac{\pi}{6}\right|^5$, where $\left|f^{(5)}(x)\right| \le M$. Now $0 \le x \le \frac{\pi}{3} \;\Rightarrow\; -\frac{\pi}{6} \le x - \frac{\pi}{6} \le \frac{\pi}{6} \;\Rightarrow\; \left|x - \frac{\pi}{6}\right| \le \frac{\pi}{6} \;\Rightarrow$

$\left|x - \frac{\pi}{6}\right|^5 \le \left(\frac{\pi}{6}\right)^5$. Since $\left|f^{(5)}(x)\right|$ is decreasing on $\left[0, \frac{\pi}{3}\right]$, we can take $M = \left|f^{(5)}(0)\right| = \cos 0 = 1$, so

$|R_4(x)| \le \dfrac{1}{5!}\left(\dfrac{\pi}{6}\right)^5 \approx 0.000\,328.$

(c)



From the graph of $|R_4(x)| = |\sin x - T_4(x)|$, it seems that the error is less than $0.000\,297$ on $\left[0, \frac{\pi}{3}\right]$.

17.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sec x$	1
1	$\sec x\,\tan x$	0
2	$\sec x\,(2\sec^2 x - 1)$	1
3	$\sec x\,\tan x\,(6\sec^2 x - 1)$	

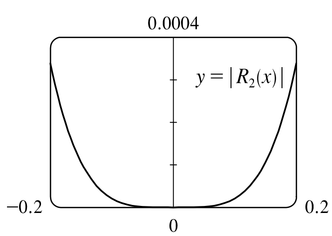
(a) $f(x) = \sec x \approx T_2(x) = 1 + \frac{1}{2}x^2$

(b) $|R_2(x)| \le \dfrac{M}{3!}|x|^3$, where $\left|f^{(3)}(x)\right| \le M$. Now $-0.2 \le x \le 0.2 \;\Rightarrow\; |x| \le 0.2 \;\Rightarrow\; |x|^3 \le (0.2)^3$.

$f^{(3)}(x)$ is an odd function and it is increasing on $[0, 0.2]$ since $\sec x$ and $\tan x$ are increasing on $[0, 0.2]$,

so $\left|f^{(3)}(x)\right| \le f^{(3)}(0.2) \approx 1.085\,158\,892$. Thus, $|R_2(x)| \le \dfrac{f^{(3)}(0.2)}{3!}(0.2)^3 \approx 0.001\,447$.

(c)



From the graph of $|R_2(x)| = |\sec x - T_2(x)|$, it seems that the error is less than $0.000\,339$ on $[-0.2, 0.2]$.

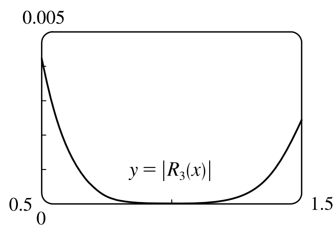
18.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(1+2x)$	$\ln 3$
1	$2/(1+2x)$	$\frac{2}{3}$
2	$-4/(1+2x)^2$	$-\frac{4}{9}$
3	$16/(1+2x)^3$	$\frac{16}{27}$
4	$-96/(1+2x)^4$	

(a) $f(x) = \ln(1+2x) \approx T_3(x)$

$$= \ln 3 + \tfrac{2}{3}(x-1) - \frac{4/9}{2!}(x-1)^2 + \frac{16/27}{3!}(x-1)^3$$

(b) $|R_3(x)| \leq \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \leq M$. Now $0.5 \leq x \leq 1.5 \Rightarrow -0.5 \leq x - 1 \leq 0.5 \Rightarrow |x-1| \leq 0.5 \Rightarrow |x-1|^4 \leq \frac{1}{16}$, and letting $x = 0.5$ gives $M = 6$, so $|R_3(x)| \leq \dfrac{6}{4!} \cdot \dfrac{1}{16} = \dfrac{1}{64} = 0.015\,625$.

(c) 

From the graph of $|R_3(x)| = |\ln(1+2x) - T_3(x)|$, it seems that the error is less than 0.005 on $[0.5, 1.5]$.

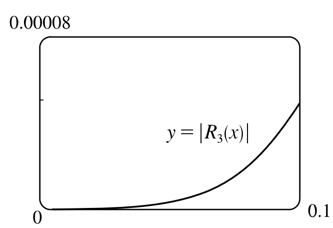
19.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{x^2}	1
1	$e^{x^2}(2x)$	0
2	$e^{x^2}(2+4x^2)$	2
3	$e^{x^2}(12x+8x^3)$	0
4	$e^{x^2}(12+48x^2+16x^4)$	

(a) $f(x) = e^{x^2} \approx T_3(x) = 1 + \dfrac{2}{2!}x^2 = 1 + x^2$

(b) $|R_3(x)| \leq \dfrac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right| \leq M$. Now $0 \leq x \leq 0.1 \Rightarrow x^4 \leq (0.1)^4$, and letting $x = 0.1$ gives

$$|R_3(x)| \leq \frac{e^{0.01}\,(12 + 0.48 + 0.0016)}{24}(0.1)^4 \approx 0.000\,053.$$

(c) 

From the graph of $|R_3(x)| = \left|e^{x^2} - T_3(x)\right|$, it appears that the error is less than $0.000\,051$ on $[0, 0.1]$.

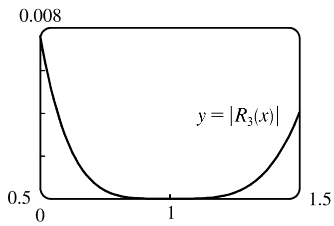
20.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x\ln x$	0
1	$\ln x + 1$	1
2	$1/x$	1
3	$-1/x^2$	-1
4	$2/x^3$	

(a) $f(x) = x\ln x \approx T_3(x) = (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3$

(b) $|R_3(x)| \leq \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \leq M$. Now $0.5 \leq x \leq 1.5 \Rightarrow |x-1| \leq \frac{1}{2} \Rightarrow |x-1|^4 \leq \frac{1}{16}$. Since $\left|f^{(4)}(x)\right|$ is decreasing on $[0.5, 1.5]$, we can take $M = \left|f^{(4)}(0.5)\right| = 2/(0.5)^3 = 16$, so $|R_3(x)| \leq \frac{16}{24}(1/16) = \frac{1}{24} = 0.041\overline{6}$.

(c)



From the graph of $|R_3(x)| = |x \ln x - T_3(x)|$, it seems that the error is less than 0.0076 on $[0.5, 1.5]$.

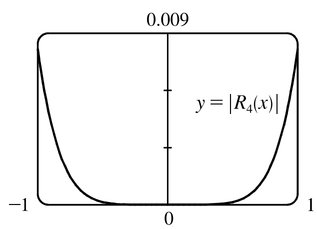
21.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x \sin x$	0
1	$\sin x + x \cos x$	0
2	$2 \cos x - x \sin x$	2
3	$-3 \sin x - x \cos x$	0
4	$-4 \cos x + x \sin x$	-4
5	$5 \sin x + x \cos x$	

(a) $f(x) = x \sin x \approx T_4(x) = \dfrac{2}{2!}(x-0)^2 + \dfrac{-4}{4!}(x-0)^4 = x^2 - \dfrac{1}{6}x^4$

(b) $|R_4(x)| \le \dfrac{M}{5!}\,|x|^5$, where $\left| f^{(5)}(x) \right| \le M$. Now $-1 \le x \le 1 \;\Rightarrow$ $|x| \le 1$, and a graph of $f^{(5)}(x)$ shows that $\left| f^{(5)}(x) \right| \le 5$ for $-1 \le x \le 1$. Thus, we can take $M = 5$ and get $|R_4(x)| \le \dfrac{5}{5!} \cdot 1^5 = \dfrac{1}{24} = 0.041\overline{6}$.

(c)



From the graph of $|R_4(x)| = |x \sin x - T_4(x)|$, it seems that the error is less than 0.0082 on $[-1, 1]$.

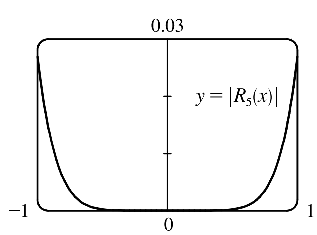
22.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh 2x$	0
1	$2 \cosh 2x$	2
2	$4 \sinh 2x$	0
3	$8 \cosh 2x$	8
4	$16 \sinh 2x$	0
5	$32 \cosh 2x$	32
6	$64 \sinh 2x$	

(a) $f(x) = \sinh 2x \approx T_5(x) = 2x + \frac{8}{3!}x^3 + \frac{32}{5!}x^5 = 2x + \frac{4}{3}x^3 + \frac{4}{15}x^5$

(b) $|R_5(x)| \le \frac{M}{6!}\,|x|^6$, where $\left| f^{(6)}(x) \right| \le M$. For x in $[-1, 1]$, we have $|x| \le 1$. Since $f^{(6)}(x)$ is an increasing odd function on $[-1, 1]$, we see that $\left| f^{(6)}(x) \right| \le f^{(6)}(1) = 64 \sinh 2 = 32(e^2 - e^{-2}) \approx 232.119$, so we can take $M = 232.12$ and get $|R_5(x)| \le \frac{232.12}{720} \cdot 1^6 \approx 0.3224$.

(c)



From the graph of $|R_5(x)| = |\sinh 2x - T_5(x)|$, it seems that the error is less than 0.027 on $[-1, 1]$.

23. From Exercise 5, $\cos x = -\left(x-\frac{\pi}{2}\right)+\frac{1}{6}\left(x-\frac{\pi}{2}\right)^3+R_3(x)$, where $|R_3(x)| \le \dfrac{M}{4!}\left|x-\frac{\pi}{2}\right|^4$ with

$\left|f^{(4)}(x)\right| = |\cos x| \le M = 1$. Now $x = 80^\circ = (90^\circ - 10^\circ) = \left(\frac{\pi}{2}-\frac{\pi}{18}\right) = \frac{4\pi}{9}$ radians, so the error is

$\left|R_3\left(\frac{4\pi}{9}\right)\right| \le \frac{1}{24}\left(\frac{\pi}{18}\right)^4 \approx 0.000\,039$, which means our estimate would *not* be accurate to five decimal places. However,

$T_3 = T_4$, so we can use $\left|R_4\left(\frac{4\pi}{9}\right)\right| \le \frac{1}{120}\left(\frac{\pi}{18}\right)^5 \approx 0.000\,001$. Therefore, to five decimal places,

$\cos 80^\circ \approx -\left(-\frac{\pi}{18}\right)+\frac{1}{6}\left(-\frac{\pi}{18}\right)^3 \approx 0.17365$.

24. From Exercise 16, $\sin x = \frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3+\frac{1}{48}\left(x-\frac{\pi}{6}\right)^4+R_4(x)$, where

$|R_4(x)| \le \dfrac{M}{5!}\left|x-\frac{\pi}{6}\right|^5$ with $\left|f^{(5)}(x)\right| = |\cos x| \le M = 1$. Now $x = 38^\circ = (30^\circ + 8^\circ) = \left(\frac{\pi}{6}+\frac{2\pi}{45}\right)$ radians,

so the error is $\left|R_4\left(\frac{38\pi}{180}\right)\right| \le \frac{1}{120}\left(\frac{2\pi}{45}\right)^5 \approx 0.000\,000\,44$, which means our estimate will be accurate to five decimal places.

Therefore, to five decimal places, $\sin 38^\circ = \frac{1}{2}+\frac{\sqrt{3}}{2}\left(\frac{2\pi}{45}\right)-\frac{1}{4}\left(\frac{2\pi}{45}\right)^2-\frac{\sqrt{3}}{12}\left(\frac{2\pi}{45}\right)^3+\frac{1}{48}\left(\frac{2\pi}{45}\right)^4 \approx 0.61566$.

25. All derivatives of e^x are e^x, so $|R_n(x)| \le \dfrac{e^x}{(n+1)!}|x|^{n+1}$, where $0 < x < 0.1$. Letting $x = 0.1$,

$R_n(0.1) \le \dfrac{e^{0.1}}{(n+1)!}(0.1)^{n+1} < 0.00001$, and by trial and error we find that $n = 3$ satisfies this inequality since

$R_3(0.1) < 0.0000046$. Thus, by adding the four terms of the Maclaurin series for e^x corresponding to $n = 0$, 1, 2, and 3,

we can estimate $e^{0.1}$ to within 0.00001. (In fact, this sum is $1.10516\overline{6}$ and $e^{0.1} \approx 1.10517$.)

26. From Table 1 in Section 11.10, $\ln(1+x) = \sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{x^n}{n}$ for $|x| < 1$. Thus, $\ln 1.4 = \ln(1+0.4) = \sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{(0.4)^n}{n}$.

Since this is an alternating series, the error is less than the first neglected term by the Alternating Series Estimation Theorem,

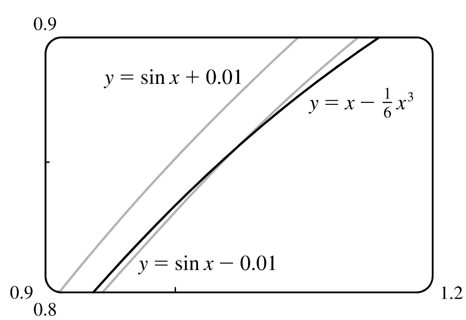
and we find that $|a_6| = (0.4)^6/6 \approx 0.0007 < 0.001$. So we need the first five (nonzero) terms of the Maclaurin series for the desired accuracy. (In fact, this sum is approximately 0.33698 and $\ln 1.4 \approx 0.33647$.)

27. $\sin x = x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 - \cdots$. By the Alternating Series Estimation Theorem, the error in the approximation

$\sin x = x - \dfrac{1}{3!}x^3$ is less than $\left|\dfrac{1}{5!}x^5\right| < 0.01 \quad\Leftrightarrow$

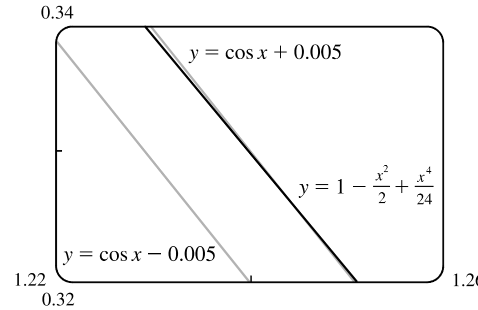
$\left|x^5\right| < 120(0.01) \quad\Leftrightarrow\quad |x| < (1.2)^{1/5} \approx 1.037$. The curves

$y = x - \frac{1}{6}x^3$ and $y = \sin x - 0.01$ intersect at $x \approx 1.043$, so the graph confirms our estimate. Since both the sine function

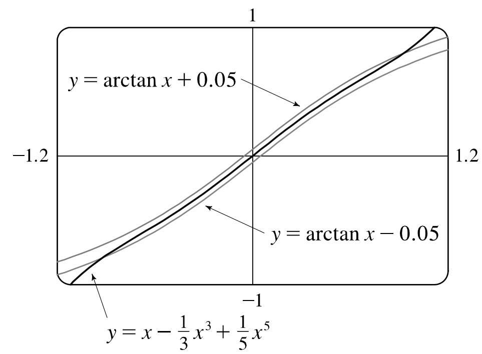


and the given approximation are odd functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.037 < x < 1.037$.

28. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$. By the Alternating Series Estimation Theorem, the error is less than $\left|-\frac{1}{6!}x^6\right| < 0.005 \quad\Leftrightarrow$ $x^6 < 720(0.005) \quad\Leftrightarrow\quad |x| < (3.6)^{1/6} \approx 1.238$. The curves $y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ and $y = \cos x + 0.005$ intersect at $x \approx 1.244$, so the graph confirms our estimate. Since both the cosine function and the given approximation are even functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.238 < x < 1.238$.



29. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. By the Alternating Series Estimation Theorem, the error is less than $\left|-\frac{1}{7}x^7\right| < 0.05 \quad\Leftrightarrow$ $\left|x^7\right| < 0.35 \quad\Leftrightarrow\quad |x| < (0.35)^{1/7} \approx 0.8607$. The curves $y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$ and $y = \arctan x + 0.05$ intersect at $x \approx 0.9245$, so the graph confirms our estimate. Since both the arctangent function and the given approximation are odd functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-0.86 < x < 0.86$.



30. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n\, n!}{3^n(n+1)\, n!}(x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)}(x-4)^n$. Now

$f(5) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)} = \sum_{n=0}^{\infty} (-1)^n b_n$ is the sum of an alternating series that satisfies (i) $b_{n+1} \le b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$, so by the Alternating Series Estimation Theorem, $|R_5(5)| = |f(5) - T_5(5)| \le b_6$, and

$b_6 = \frac{1}{3^6(7)} = \frac{1}{5103} \approx 0.000196 < 0.0002$; that is, the fifth-degree Taylor polynomial approximates $f(5)$ with error less than 0.0002.

31. Let $s(t)$ be the position function of the car, and for convenience set $s(0) = 0$. The velocity of the car is $v(t) = s'(t)$ and the acceleration is $a(t) = s''(t)$, so the second degree Taylor polynomial is $T_2(t) = s(0) + v(0)t + \frac{a(0)}{2}t^2 = 20t + t^2$. We estimate the distance traveled during the next second to be $s(1) \approx T_2(1) = 20 + 1 = 21$ m. The function $T_2(t)$ would not be accurate over a full minute, since the car could not possibly maintain an acceleration of 2 m/s^2 for that long (if it did, its final speed would be 140 m/s ≈ 313 mi/h!).

32. (a)

n	$\rho^{(n)}(t)$	$\rho^{(n)}(20)$
0	$\rho_{20}e^{\alpha(t-20)}$	ρ_{20}
1	$\alpha\rho_{20}e^{\alpha(t-20)}$	$\alpha\rho_{20}$
2	$\alpha^2\rho_{20}e^{\alpha(t-20)}$	$\alpha^2\rho_{20}$

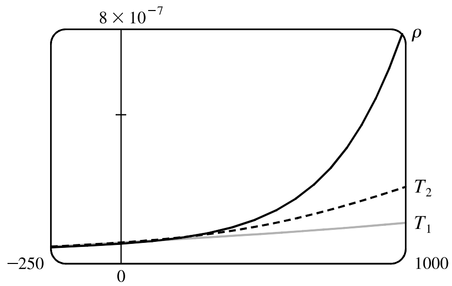
The linear approximation is

$$T_1(t) = \rho(20) + \rho'(20)(t-20) = \rho_{20}[1 + \alpha(t-20)]$$

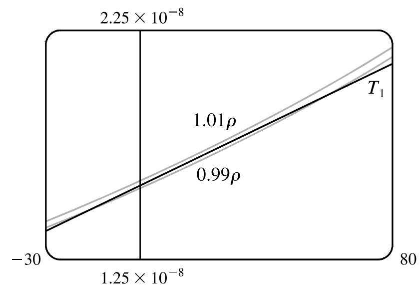
The quadratic approximation is

$$T_2(t) = \rho(20) + \rho'(20)(t-20) + \frac{\rho''(20)}{2}(t-20)^2$$
$$= \rho_{20}\left[1 + \alpha(t-20) + \tfrac{1}{2}\alpha^2(t-20)^2\right]$$

(b)



(c)



From the graph, it seems that $T_1(t)$ is within 1% of $\rho(t)$, that is, $0.99\rho(t) \leq T_1(t) \leq 1.01\rho(t)$, for $-14^\circ\text{C} \leq t \leq 58^\circ\text{C}$.

33. $E = \dfrac{q}{D^2} - \dfrac{q}{(D+d)^2} = \dfrac{q}{D^2} - \dfrac{q}{D^2(1+d/D)^2} = \dfrac{q}{D^2}\left[1 - \left(1 + \dfrac{d}{D}\right)^{-2}\right].$

We use the Binomial Series to expand $(1 + d/D)^{-2}$:

$$E = \frac{q}{D^2}\left[1 - \left(1 - 2\left(\frac{d}{D}\right) + \frac{2\cdot 3}{2!}\left(\frac{d}{D}\right)^2 - \frac{2\cdot 3\cdot 4}{3!}\left(\frac{d}{D}\right)^3 + \cdots\right)\right] = \frac{q}{D^2}\left[2\left(\frac{d}{D}\right) - 3\left(\frac{d}{D}\right)^2 + 4\left(\frac{d}{D}\right)^3 - \cdots\right]$$
$$\approx \frac{q}{D^2}\cdot 2\left(\frac{d}{D}\right) = 2qd\cdot\frac{1}{D^3}$$

when D is much larger than d; that is, when P is far away from the dipole.

34. (a) $\dfrac{n_1}{\ell_o} + \dfrac{n_2}{\ell_i} = \dfrac{1}{R}\left(\dfrac{n_2 s_i}{\ell_i} - \dfrac{n_1 s_o}{\ell_o}\right)$ [Equation 1] where

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\phi} \quad \text{and} \quad \ell_i = \sqrt{R^2 + (s_i - R)^2 + 2R(s_i - R)\cos\phi} \quad \textbf{(2)}$$

Using $\cos\phi \approx 1$ gives

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)} = \sqrt{R^2 + s_o^2 + 2Rs_o + R^2 - 2Rs_o - 2R^2} = \sqrt{s_o^2} = s_o$$

and similarly, $\ell_i = s_i$. Thus, Equation 1 becomes $\dfrac{n_1}{s_o} + \dfrac{n_2}{s_i} = \dfrac{1}{R}\left(\dfrac{n_2 s_i}{s_i} - \dfrac{n_1 s_o}{s_o}\right) \;\Rightarrow\; \dfrac{n_1}{s_o} + \dfrac{n_2}{s_i} = \dfrac{n_2 - n_1}{R}.$

(b) Using $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ in **(2)** gives us

$$\begin{aligned}\ell_o &= \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)\left(1 - \tfrac{1}{2}\phi^2\right)} \\ &= \sqrt{R^2 + s_o^2 + 2Rs_o + R^2 - 2Rs_o + Rs_o\phi^2 - 2R^2 + R^2\phi^2} = \sqrt{s_o^2 + Rs_o\phi^2 + R^2\phi^2}\end{aligned}$$

Anticipating that we will use the binomial series expansion $(1+x)^k \approx 1 + kx$, we can write the last expression for ℓ_o as

$s_o\sqrt{1 + \phi^2\left(\dfrac{R}{s_o} + \dfrac{R^2}{s_o^2}\right)}$ and similarly, $\ell_i = s_i\sqrt{1 - \phi^2\left(\dfrac{R}{s_i} - \dfrac{R^2}{s_i^2}\right)}$. Thus, from Equation 1,

$$\frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} = \frac{1}{R}\left(\frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o}\right) \quad\Leftrightarrow\quad n_1\ell_o^{-1} + n_2\ell_i^{-1} = \frac{n_2}{R}\cdot\frac{s_i}{\ell_i} - \frac{n_1}{R}\cdot\frac{s_o}{\ell_o} \quad\Leftrightarrow$$

$$\frac{n_1}{s_o}\left[1+\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]^{-1/2}+\frac{n_2}{s_i}\left[1-\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]^{-1/2}$$
$$=\frac{n_2}{R}\left[1-\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]^{-1/2}-\frac{n_1}{R}\left[1+\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]^{-1/2}$$

Approximating the expressions for ℓ_o^{-1} and ℓ_i^{-1} by the first two terms in their binomial series, we get

$$\frac{n_1}{s_o}\left[1-\tfrac{1}{2}\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]+\frac{n_2}{s_i}\left[1+\tfrac{1}{2}\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]$$
$$=\frac{n_2}{R}\left[1+\tfrac{1}{2}\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]-\frac{n_1}{R}\left[1-\tfrac{1}{2}\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]\quad\Leftrightarrow$$

$$\frac{n_1}{s_o}-\frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)+\frac{n_2}{s_i}+\frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)=\frac{n_2}{R}+\frac{n_2\phi^2}{2R}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)-\frac{n_1}{R}+\frac{n_1\phi^2}{2R}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\quad\Leftrightarrow$$

$$\frac{n_1}{s_o}+\frac{n_2}{s_i}=\frac{n_2}{R}-\frac{n_1}{R}+\frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)+\frac{n_1\phi^2}{2R}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)+\frac{n_2\phi^2}{2R}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)-\frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)$$
$$=\frac{n_2-n_1}{R}+\frac{n_1\phi^2}{2}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\left(\frac{1}{s_o}+\frac{1}{R}\right)+\frac{n_2\phi^2}{2}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\left(\frac{1}{R}-\frac{1}{s_i}\right)$$
$$=\frac{n_2-n_1}{R}+\frac{n_1\phi^2R^2}{2s_o}\left(\frac{1}{R}+\frac{1}{s_o}\right)\left(\frac{1}{R}+\frac{1}{s_o}\right)+\frac{n_2\phi^2R^2}{2s_i}\left(\frac{1}{R}-\frac{1}{s_i}\right)\left(\frac{1}{R}-\frac{1}{s_i}\right)$$
$$=\frac{n_2-n_1}{R}+\phi^2R^2\left[\frac{n_1}{2s_o}\left(\frac{1}{R}+\frac{1}{s_o}\right)^2+\frac{n_2}{2s_i}\left(\frac{1}{R}-\frac{1}{s_i}\right)^2\right]$$

From Figure 8, we see that $\sin\phi = h/R$. So if we approximate $\sin\phi$ with ϕ, we get $h = R\phi$ and $h^2 = \phi^2R^2$ and hence, Equation 4, as desired.

35. (a) If the water is deep, then $2\pi d/L$ is large, and we know that $\tanh x \to 1$ as $x \to \infty$. So we can approximate $\tanh(2\pi d/L) \approx 1$, and so $v^2 \approx gL/(2\pi) \quad\Leftrightarrow\quad v \approx \sqrt{gL/(2\pi)}$.

(b) From the table, the first term in the Maclaurin series of $\tanh x$ is x, so if the water is shallow, we can approximate $\tanh\dfrac{2\pi d}{L} \approx \dfrac{2\pi d}{L}$, and so $v^2 \approx \dfrac{gL}{2\pi}\cdot\dfrac{2\pi d}{L} \quad\Leftrightarrow\quad v \approx \sqrt{gd}$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tanh x$	0
1	$\operatorname{sech}^2 x$	1
2	$-2\operatorname{sech}^2 x\tanh x$	0
3	$2\operatorname{sech}^2 x\,(3\tanh^2 x - 1)$	-2

(c) Since $\tanh x$ is an odd function, its Maclaurin series is alternating, so the error in the approximation $\tanh\dfrac{2\pi d}{L} \approx \dfrac{2\pi d}{L}$ is less than the first neglected term, which is $\dfrac{|f'''(0)|}{3!}\left(\dfrac{2\pi d}{L}\right)^3 = \dfrac{1}{3}\left(\dfrac{2\pi d}{L}\right)^3$.

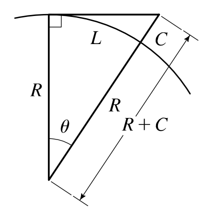
If $L > 10d$, then $\dfrac{1}{3}\left(\dfrac{2\pi d}{L}\right)^3 < \dfrac{1}{3}\left(2\pi\cdot\dfrac{1}{10}\right)^3 = \dfrac{\pi^3}{375}$, so the error in the approximation $v^2 = gd$ is less than $\dfrac{gL}{2\pi}\cdot\dfrac{\pi^3}{375} \approx 0.0132gL$.

36. First note that

$$\begin{aligned}2\left(\sqrt{d^2+R^2}-d\right) &= 2\left[\sqrt{d^2}\sqrt{1+\frac{R^2}{d^2}}-d\right]\\ &\approx 2\left[d\left(1+\frac{R^2}{d^2}\cdot\frac{1}{2}+\cdots\right)-d\right] \quad \left[\text{use the binomial series } 1+\tfrac{1}{2}x+\cdots \text{ for } \sqrt{1+x}\,\right]\\ &= 2\left[\left(d+\frac{R^2}{2d}+\cdots\right)-d\right]\approx\frac{R^2}{d}\end{aligned}$$

since for large d the other terms are comparatively small. Now $V=2\pi k_e\sigma\left(\sqrt{d^2+R^2}-d\right)\approx\dfrac{\pi k_e R^2\sigma}{d}$ by the preceding approximation.

37. (a) L is the length of the arc subtended by the angle θ, so $L=R\theta \;\Rightarrow\; \theta=L/R$. Now $\sec\theta=(R+C)/R \;\Rightarrow\; R\sec\theta=R+C \;\Rightarrow\; C=R\sec\theta-R=R\sec(L/R)-R$.



(b) First we'll find a Taylor polynomial $T_4(x)$ for $f(x)=\sec x$ at $x=0$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sec x$	1
1	$\sec x\tan x$	0
2	$\sec x(2\tan^2x+1)$	1
3	$\sec x\tan x(6\tan^2x+5)$	0
4	$\sec x(24\tan^4x+28\tan^2x+5)$	5

Thus, $f(x)=\sec x\approx T_4(x)=1+\frac{1}{2!}(x-0)^2+\frac{5}{4!}(x-0)^4=1+\frac{1}{2}x^2+\frac{5}{24}x^4$. By part (a),

$$C\approx R\left[1+\frac{1}{2}\left(\frac{L}{R}\right)^2+\frac{5}{24}\left(\frac{L}{R}\right)^4\right]-R=R+\frac{1}{2}R\cdot\frac{L^2}{R^2}+\frac{5}{24}R\cdot\frac{L^4}{R^4}-R=\frac{L^2}{2R}+\frac{5L^4}{24R^3}.$$

(c) Taking $L=100$ km and $R=6370$ km, the formula in part (a) says that

$C=R\sec(L/R)-R=6370\sec(100/6370)-6370\approx 0.785\,009\,965\,44$ km.

The formula in part (b) says that $C\approx\dfrac{L^2}{2R}+\dfrac{5L^4}{24R^3}=\dfrac{100^2}{2\cdot 6370}+\dfrac{5\cdot 100^4}{24\cdot 6370^3}\approx 0.785\,009\,957\,36$ km.

The difference between these two results is only $0.000\,000\,008\,08$ km, or $0.000\,008\,08$ m!

38. (a) $4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}} = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+\left(-k^2\sin^2 x\right)\right]^{-1/2}\,dx$

$$= 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1-\frac{1}{2}\left(-k^2\sin^2 x\right)+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(-k^2\sin^2 x\right)^2-\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\left(-k^2\sin^2 x\right)^3+\cdots\right]dx$$

$$= 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+\left(\frac{1}{2}\right)k^2\sin^2 x+\left(\frac{1\cdot 3}{2\cdot 4}\right)k^4\sin^4 x+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)k^6\sin^6 x+\cdots\right]dx$$

$$= 4\sqrt{\frac{L}{g}}\left[\frac{\pi}{2}+\left(\frac{1}{2}\right)\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)\left(\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}\right)k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2}\right)k^6+\cdots\right]$$

[split up the integral and use the result from Exercise 7.1.56]

$$= 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6+\cdots\right]$$

(b) The first of the two inequalities is true because all of the terms in the series are positive. For the second,

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6+\frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}k^8+\cdots\right]$$

$$\le 2\pi\sqrt{\frac{L}{g}}\left[1+\tfrac{1}{4}k^2+\tfrac{1}{4}k^4+\tfrac{1}{4}k^6+\tfrac{1}{4}k^8+\cdots\right]$$

The terms in brackets (after the first) form a geometric series with $a=\frac{1}{4}k^2$ and $r=k^2=\sin^2\left(\frac{1}{2}\theta_0\right)<1$.

So $T \le 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{k^2/4}{1-k^2}\right] = 2\pi\sqrt{\frac{L}{g}}\,\frac{4-3k^2}{4-4k^2}$.

(c) We substitute $L=1$, $g=9.8$, and $k=\sin(10^\circ/2)\approx 0.08716$, and the inequality from part (b) becomes

$2.01090 \le T \le 2.01093$, so $T\approx 2.0109$. The estimate $T\approx 2\pi\sqrt{L/g}\approx 2.0071$ differs by about 0.2%.

If $\theta_0=42^\circ$, then $k\approx 0.35837$ and the inequality becomes $2.07153\le T\le 2.08103$, so $T\approx 2.0763$.

The one-term estimate is the same, and the discrepancy between the two estimates increases to about 3.4%.

39. Using $f(x)=T_n(x)+R_n(x)$ with $n=1$ and $x=r$, we have $f(r)=T_1(r)+R_1(r)$, where T_1 is the first-degree Taylor polynomial of f at a. Because $a=x_n$, $f(r)=f(x_n)+f'(x_n)(r-x_n)+R_1(r)$. But r is a zero of f, so $f(r)=0$ and we have $0=f(x_n)+f'(x_n)(r-x_n)+R_1(r)$. Taking the first two terms to the left side gives us $f'(x_n)(x_n-r)-f(x_n)=R_1(r)$. Dividing by $f'(x_n)$, we get $x_n-r-\dfrac{f(x_n)}{f'(x_n)}=\dfrac{R_1(r)}{f'(x_n)}$. By the formula for Newton's method, the left side of the preceding equation is $x_{n+1}-r$, so $|x_{n+1}-r|=\left|\dfrac{R_1(r)}{f'(x_n)}\right|$. Taylor's Inequality gives us $|R_1(r)|\le\dfrac{|f''(r)|}{2!}|r-x_n|^2$. Combining this inequality with the facts $|f''(x)|\le M$ and $|f'(x)|\ge K$ gives us $|x_{n+1}-r|\le\dfrac{M}{2K}|x_n-r|^2$.

APPLIED PROJECT Radiation from the Stars

1. If we write $f(\lambda) = \dfrac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1} = \dfrac{a\lambda^{-5}}{e^{b/(\lambda T)} - 1}$, then as $\lambda \to 0^+$, it is of the form ∞/∞, and as $\lambda \to \infty$ it is of the form $0/0$, so in either case we can use l'Hospital's Rule. First of all,

$$\lim_{\lambda\to\infty} f(\lambda) \overset{\text{H}}{=} \lim_{\lambda\to\infty} \frac{a\left(-5\lambda^{-6}\right)}{-\dfrac{bT}{(\lambda T)^2} e^{b/(\lambda T)}} = 5\,\frac{aT}{b} \lim_{\lambda\to\infty} \frac{\lambda^2\lambda^{-6}}{e^{b/(\lambda T)}} = 5\,\frac{aT}{b} \lim_{\lambda\to\infty} \frac{\lambda^{-4}}{e^{b/(\lambda T)}} = 0$$

Also,
$$\lim_{\lambda\to 0^+} f(\lambda) \overset{\text{H}}{=} 5\,\frac{aT}{b} \lim_{\lambda\to 0^+} \frac{\lambda^{-4}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} 5\,\frac{aT}{b} \lim_{\lambda\to 0^+} \frac{-4\lambda^{-5}}{-\dfrac{bT}{(\lambda T)^2} e^{b/(\lambda T)}} = 20\,\frac{aT^2}{b^2} \lim_{\lambda\to 0^+} \frac{\lambda^{-3}}{e^{b/(\lambda T)}}$$

This is still indeterminate, but note that each time we use l'Hospital's Rule, we gain a factor of λ in the numerator, as well as a constant factor, and the denominator is unchanged. So if we use l'Hospital's Rule three more times, the exponent of λ in the numerator will become 0. That is, for some $\{k_i\}$, all constant,

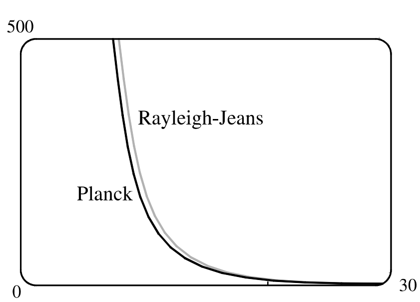
$$\lim_{\lambda\to 0^+} f(\lambda) \overset{\text{H}}{=} k_1 \lim_{\lambda\to 0^+} \frac{\lambda^{-3}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_2 \lim_{\lambda\to 0^+} \frac{\lambda^{-2}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_3 \lim_{\lambda\to 0^+} \frac{\lambda^{-1}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_4 \lim_{\lambda\to 0^+} \frac{1}{e^{b/(\lambda T)}} = 0$$

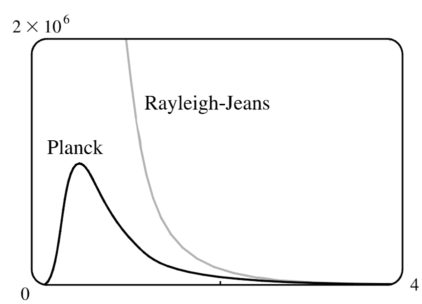
2. We expand the denominator of Planck's Law using the Taylor series $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ with $x = \dfrac{hc}{\lambda kT}$, and use the fact that if λ is large, then all subsequent terms in the Taylor expansion are very small compared to the first one, so we can approximate using the Taylor polynomial T_1:

$$f(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1} = \frac{8\pi hc\lambda^{-5}}{\left[1 + \dfrac{hc}{\lambda kT} + \dfrac{1}{2!}\left(\dfrac{hc}{\lambda kT}\right)^2 + \dfrac{1}{3!}\left(\dfrac{hc}{\lambda kT}\right)^3 + \cdots\right] - 1} \approx \frac{8\pi hc\lambda^{-5}}{\left(1 + \dfrac{hc}{\lambda kT}\right) - 1} = \frac{8\pi kT}{\lambda^4}$$

which is the Rayleigh-Jeans Law.

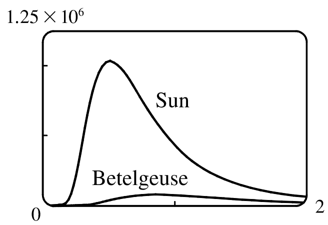
3. To convert to μm, we substitute $\lambda/10^6$ for λ in both laws. The first figure shows that the two laws are similar for large λ. The second figure shows that the two laws are very different for short wavelengths (Planck's Law gives a maximum at $\lambda \approx 0.51\ \mu$m; the Rayleigh-Jeans Law gives no minimum or maximum.).

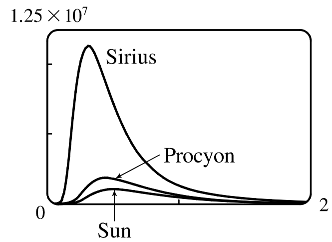




4. From the graph in Problem 3, $f(\lambda)$ has a maximum under Planck's Law at $\lambda \approx 0.51\ \mu$m.

5.





As T gets larger, the total area under the curve increases, as we would expect: the hotter the star, the more energy it emits. Also, as T increases, the λ-value of the maximum decreases, so the higher the temperature, the shorter the peak wavelength (and consequently the average wavelength) of light emitted. This is why Sirius is a blue star and Betelgeuse is a red star: most of Sirius's light is of a fairly short wavelength; that is, a higher frequency, toward the blue end of the spectrum, whereas most of Betelgeuse's light is of a lower frequency, toward the red end of the spectrum.

11 Review

TRUE-FALSE QUIZ

1. False. See the WARNING after Theorem 11.2.6.

2. False. The series $\sum\limits_{n=1}^{\infty} n^{-\sin 1} = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{\sin 1}}$ is a p-series with $p = \sin 1 \approx 0.84 \le 1$, so the series diverges.

3. True. If $\lim\limits_{n\to\infty} a_n = L$, then as $n \to \infty$, $2n+1 \to \infty$, so $a_{2n+1} \to L$.

4. True by Theorem 11.8.4.
Or: Use the Direct Comparison Test to show that $\sum c_n(-2)^n$ converges absolutely.

5. False. For example, take $c_n = (-1)^n/(n6^n)$.

6. True by Theorem 11.8.4.

7. False, since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{1}{(n+1)^3} \cdot \dfrac{n^3}{1}\right| = \lim\limits_{n\to\infty} \left|\dfrac{n^3}{(n+1)^3} \cdot \dfrac{1/n^3}{1/n^3}\right| = \lim\limits_{n\to\infty} \dfrac{1}{(1+1/n)^3} = 1.$

8. True, since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{1}{(n+1)!} \cdot \dfrac{n!}{1}\right| = \lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0 < 1.$

9. False. See the paragraph preceding "The Limit Comparison Test" in Section 11.4.

10. True, since $\dfrac{1}{e} = e^{-1}$ and $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$, so $e^{-1} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{n!}$.

11. True. See (9) in Section 11.1.

12. True, because if $\sum |a_n|$ is convergent, then so is $\sum a_n$ by the contrapositive of Theorem 11.5.3.

13. True. By Theorem 11.10.5 the coefficient of x^3 is $\dfrac{f'''(0)}{3!} = \dfrac{1}{3} \quad\Rightarrow\quad f'''(0) = 2.$
Or: Use Theorem 11.9.2 to differentiate f three times.

14. False. Let $a_n = n$ and $b_n = -n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n + b_n = 0$, so $\{a_n + b_n\}$ is convergent.

15. False. For example, let $a_n = b_n = (-1)^n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n b_n = 1$, so $\{a_n b_n\}$ is convergent.

16. True by the Monotonic Sequence Theorem, since $\{a_n\}$ is decreasing and $0 < a_n \le a_1$ for all $n \Rightarrow \{a_n\}$ is bounded.

17. True by Theorem 11.5.3. $\left[\sum (-1)^n a_n \text{ is absolutely convergent and hence convergent.}\right]$

18. True. $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1 \Rightarrow \sum a_n$ converges (Ratio Test) $\Rightarrow \lim\limits_{n\to\infty} a_n = 0$ [Theorem 11.2.6].

19. True. $0.99999\ldots = 0.9 + 0.9(0.1)^1 + 0.9(0.1)^2 + 0.9(0.1)^3 + \cdots = \sum\limits_{n=1}^{\infty} (0.9)(0.1)^{n-1} = \dfrac{0.9}{1-0.1} = 1$ by the formula for the sum of a geometric series $[S = a_1/(1-r)]$ with ratio r satisfying $|r| < 1$.

20. True. Since $\lim\limits_{n\to\infty} a_n = 2$, we know that $\lim\limits_{n\to\infty} a_{n+3} = 2$. Thus, $\lim\limits_{n\to\infty} (a_{n+3} - a_n) = \lim\limits_{n\to\infty} a_{n+3} - \lim\limits_{n\to\infty} a_n = 2 - 2 = 0$.

21. True. A finite number of terms doesn't affect convergence or divergence of a series.

22. False. Let $a_n = (0.1)^n$ and $b_n = (0.2)^n$. Then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (0.1)^n = \dfrac{0.1}{1-0.1} = \dfrac{1}{9} = A$,

$\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} (0.2)^n = \dfrac{0.2}{1-0.2} = \dfrac{1}{4} = B$, and $\sum\limits_{n=1}^{\infty} a_n b_n = \sum\limits_{n=1}^{\infty} (0.02)^n = \dfrac{0.02}{1-0.02} = \dfrac{1}{49}$, but

$AB = \frac{1}{9} \cdot \frac{1}{4} = \frac{1}{36}$.

EXERCISES

1. $\left\{\dfrac{2+n^3}{1+2n^3}\right\}$ converges since $\lim\limits_{n\to\infty} \dfrac{2+n^3}{1+2n^3} = \lim\limits_{n\to\infty} \dfrac{2/n^3+1}{1/n^3+2} = \dfrac{1}{2}$.

2. $a_n = \dfrac{9^{n+1}}{10^n} = 9 \cdot \left(\frac{9}{10}\right)^n$, so $\lim\limits_{n\to\infty} a_n = 9 \lim\limits_{n\to\infty} \left(\frac{9}{10}\right)^n = 9 \cdot 0 = 0$ by (11.1.9).

3. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n^3}{1+n^2} = \lim\limits_{n\to\infty} \dfrac{n}{1/n^2+1} = \infty$, so the sequence diverges.

4. $a_n = \cos(n\pi/2)$, so $a_n = 0$ if n is odd and $a_n = \pm 1$ if n is even. As n increases, a_n keeps cycling through the values 0, 1, 0, -1, so the sequence $\{a_n\}$ is divergent.

5. $|a_n| = \left|\dfrac{n \sin n}{n^2+1}\right| \le \dfrac{n}{n^2+1} < \dfrac{1}{n}$, so $|a_n| \to 0$ as $n \to \infty$. Thus, $\lim\limits_{n\to\infty} a_n = 0$. The sequence $\{a_n\}$ is convergent.

6. $a_n = \dfrac{\ln n}{\sqrt{n}}$. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$ for $x > 0$. Then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1/(2\sqrt{x})} = \lim\limits_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$.

Thus, by Theorem 11.1.4, $\{a_n\}$ converges and $\lim\limits_{n\to\infty} a_n = 0$.

7. $\left\{\left(1+\dfrac{3}{n}\right)^{4n}\right\}$ is convergent. Let $y=\left(1+\dfrac{3}{x}\right)^{4x}$. Then

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}4x\ln(1+3/x)=\lim_{x\to\infty}\frac{\ln(1+3/x)}{1/(4x)}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{\dfrac{1}{1+3/x}\left(-\dfrac{3}{x^2}\right)}{-1/(4x^2)}=\lim_{x\to\infty}\frac{12}{1+3/x}=12,\text{ so}$$

$$\lim_{x\to\infty}y=\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{4n}=e^{12}.$$

8. $\left\{\dfrac{(-10)^n}{n!}\right\}$ converges, since $\dfrac{10^n}{n!}=\dfrac{10\cdot10\cdot10\cdot\cdots\cdot10}{1\cdot2\cdot3\cdot\cdots\cdot10}\cdot\dfrac{10\cdot10\cdot\cdots\cdot10}{11\cdot12\cdot\cdots\cdot n}\le10^{10}\left(\dfrac{10}{11}\right)^{n-10}\to0$ as $n\to\infty$, so

$\displaystyle\lim_{n\to\infty}\frac{(-10)^n}{n!}=0$ [Squeeze Theorem]. *Or:* Use (11.10.10).

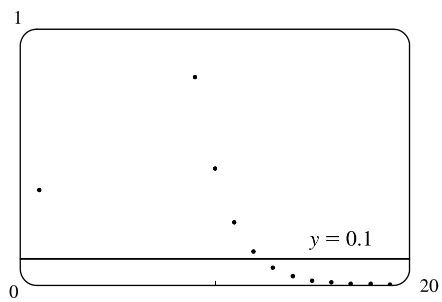
9. We use induction, hypothesizing that $a_{n-1}<a_n<2$. Note first that $1<a_2=\frac{1}{3}(1+4)=\frac{5}{3}<2$, so the hypothesis holds for $n=2$. Now assume that $a_{k-1}<a_k<2$. Then $a_k=\frac{1}{3}(a_{k-1}+4)<\frac{1}{3}(a_k+4)<\frac{1}{3}(2+4)=2$. So $a_k<a_{k+1}<2$, and the induction is complete. To find the limit of the sequence, we note that $L=\displaystyle\lim_{n\to\infty}a_n=\lim_{n\to\infty}a_{n+1}\;\Rightarrow$

$L=\frac{1}{3}(L+4)\;\Rightarrow\;L=2.$

10. $\displaystyle\lim_{x\to\infty}\frac{x^4}{e^x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{4x^3}{e^x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{12x^2}{e^x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{24x}{e^x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{24}{e^x}=0$

Then we conclude from Theorem 11.1.4 that $\displaystyle\lim_{n\to\infty}n^4e^{-n}=0$.

From the graph, it seems that $12^4e^{-12}>0.1$, but $n^4e^{-n}<0.1$ whenever $n>12$. So the smallest value of N corresponding to $\varepsilon=0.1$ in the definition of the limit is $N=12$.



11. $\dfrac{n}{n^3+1}<\dfrac{n}{n^3}=\dfrac{1}{n^2}$, so $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3+1}$ converges by the Direct Comparison Test with the convergent p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ $[p=2>1]$.

12. Let $a_n=\dfrac{n^2+1}{n^3+1}$ and $b_n=\dfrac{1}{n}$, so $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^3+n}{n^3+1}=\lim_{n\to\infty}\frac{1+1/n^2}{1+1/n^3}=1>0$.

Since $\displaystyle\sum_{n=1}^{\infty}b_n$ is the divergent harmonic series, $\displaystyle\sum_{n=1}^{\infty}a_n$ also diverges by the Limit Comparison Test.

13. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left[\frac{(n+1)^3}{5^{n+1}}\cdot\frac{5^n}{n^3}\right]=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^3\cdot\frac{1}{5}=\frac{1}{5}<1$, so $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{5^n}$ converges by the Ratio Test.

14. Let $b_n=\dfrac{1}{\sqrt{n+1}}$. Then b_n is positive for $n\ge1$, the sequence $\{b_n\}$ is decreasing, and $\displaystyle\lim_{n\to\infty}b_n=0$, so the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ converges by the Alternating Series Test.

15. Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. Then f is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies.

$$\int_2^\infty f(x)\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x\sqrt{\ln x}}\,dx \quad \left[u = \ln x,\, du = \tfrac{1}{x}\,dx\right] \quad = \lim_{t\to\infty}\int_{\ln 2}^{\ln t} u^{-1/2}\,du = \lim_{t\to\infty}\left[2\sqrt{u}\right]_{\ln 2}^{\ln t}$$

$$= \lim_{t\to\infty}\left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty,$$

so the series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\sqrt{\ln n}}$ diverges.

16. $\lim\limits_{n\to\infty} \dfrac{n}{3n+1} = \dfrac{1}{3}$, so $\lim\limits_{n\to\infty} \ln\left(\dfrac{n}{3n+1}\right) = \ln\frac{1}{3} \neq 0$. Thus, the series $\sum\limits_{n=1}^{\infty} \ln\left(\dfrac{n}{3n+1}\right)$ diverges by the Test for Divergence.

17. $|a_n| = \left|\dfrac{\cos 3n}{1+(1.2)^n}\right| \leq \dfrac{1}{1+(1.2)^n} < \dfrac{1}{(1.2)^n} = \left(\dfrac{5}{6}\right)^n$, so $\sum\limits_{n=1}^{\infty} |a_n|$ converges by direct comparison with the convergent geometric series $\sum\limits_{n=1}^{\infty} \left(\frac{5}{6}\right)^n$ $\left[r = \frac{5}{6} < 1\right]$. It follows that $\sum\limits_{n=1}^{\infty} a_n$ converges (by Theorem 11.5.3).

18. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \sqrt[n]{\left|\dfrac{n^{2n}}{(1+2n^2)^n}\right|} = \lim\limits_{n\to\infty} \dfrac{n^2}{1+2n^2} = \lim\limits_{n\to\infty} \dfrac{1}{1/n^2+2} = \dfrac{1}{2} < 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n^{2n}}{(1+2n^2)^n}$ converges by the Root Test.

19. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)}{5^{n+1}\,(n+1)!} \cdot \dfrac{5^n\,n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} = \lim\limits_{n\to\infty} \dfrac{2n+1}{5(n+1)} = \dfrac{2}{5} < 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{5^n\,n!}$ converges by the Ratio Test.

20. $\sum\limits_{n=1}^{\infty} \dfrac{(-5)^{2n}}{n^2\,9^n} = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}\left(\dfrac{25}{9}\right)^n$. Now $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{25^{n+1}}{(n+1)^2\cdot 9^{n+1}} \cdot \dfrac{n^2\cdot 9^n}{25^n} = \lim\limits_{n\to\infty} \dfrac{25n^2}{9(n+1)^2} = \dfrac{25}{9} > 1$, so the series diverges by the Ratio Test.

21. $b_n = \dfrac{\sqrt{n}}{n+1} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1}\dfrac{\sqrt{n}}{n+1}$ converges by the Alternating Series Test.

22. Use the Limit Comparison Test with $a_n = \dfrac{\sqrt{n+1}-\sqrt{n-1}}{n} = \dfrac{2}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)}$ (rationalizing the numerator) and $b_n = \dfrac{1}{n^{3/2}}$. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}} = 1$, so since $\sum\limits_{n=1}^{\infty} b_n$ converges $\left[p = \frac{3}{2} > 1\right]$, $\sum\limits_{n=1}^{\infty} a_n$ converges also.

23. Consider the series of absolute values: $\sum\limits_{n=1}^{\infty} n^{-1/3}$ is a p-series with $p = \frac{1}{3} \leq 1$ and is therefore divergent. But if we apply the Alternating Series Test, we see that $b_n = \dfrac{1}{\sqrt[3]{n}} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1}\,n^{-1/3}$ converges. Thus, $\sum\limits_{n=1}^{\infty} (-1)^{n-1}\,n^{-1/3}$ is conditionally convergent.

24. $\sum\limits_{n=1}^{\infty} \left|(-1)^{n-1}\,n^{-3}\right| = \sum\limits_{n=1}^{\infty} n^{-3}$ is a convergent p-series $[p = 3 > 1]$. Therefore, $\sum\limits_{n=1}^{\infty} (-1)^{n-1}\,n^{-3}$ is absolutely convergent.

25. $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{(-1)^{n+1}(n+2)3^{n+1}}{2^{2n+3}} \cdot \dfrac{2^{2n+1}}{(-1)^n(n+1)3^n}\right| = \dfrac{n+2}{n+1} \cdot \dfrac{3}{4} = \dfrac{1+(2/n)}{1+(1/n)} \cdot \dfrac{3}{4} \to \dfrac{3}{4} < 1$ as $n \to \infty$, so by the Ratio Test, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n(n+1)3^n}{2^{2n+1}}$ is absolutely convergent.

26. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{\ln x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/(2\sqrt{x}\,)}{1/x} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{2} = \infty$. Therefore, $\lim\limits_{n\to\infty} \dfrac{(-1)^n\sqrt{n}}{\ln n} \neq 0$, so the given series $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n\sqrt{n}}{\ln n}$ is divergent by the Test for Divergence.

27.
$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{3n}} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{(2^3)^n} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{8^n} = \frac{1}{8}\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{8^{n-1}} = \frac{1}{8}\sum_{n=1}^{\infty}\left(-\frac{3}{8}\right)^{n-1} = \frac{1}{8}\left(\frac{1}{1-(-3/8)}\right)$$
$$= \frac{1}{8} \cdot \frac{8}{11} = \frac{1}{11}$$

28. $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+3)} = \sum\limits_{n=1}^{\infty} \left[\dfrac{1}{3n} - \dfrac{1}{3(n+3)}\right]$ [partial fractions].

$s_n = \sum\limits_{i=1}^{n} \left[\dfrac{1}{3i} - \dfrac{1}{3(i+3)}\right] = \dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{9} - \dfrac{1}{3(n+1)} - \dfrac{1}{3(n+2)} - \dfrac{1}{3(n+3)}$ (telescoping sum), so

$\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+3)} = \lim\limits_{n\to\infty} s_n = \dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{9} = \dfrac{11}{18}$.

29.
$$\begin{aligned}\sum_{n=1}^{\infty}[\tan^{-1}(n+1) - \tan^{-1} n] &= \lim_{n\to\infty} s_n \\ &= \lim_{n\to\infty}[(\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + \cdots + (\tan^{-1}(n+1) - \tan^{-1}n)] \\ &= \lim_{n\to\infty}[\tan^{-1}(n+1) - \tan^{-1}1] = \tfrac{\pi}{2} - \tfrac{\pi}{4} = \tfrac{\pi}{4}\end{aligned}$$

30. $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n\pi^n}{3^{2n}\,(2n)!} = \sum\limits_{n=0}^{\infty}(-1)^n \dfrac{1}{(2n)!} \cdot \dfrac{\pi^n}{3^{2n}} = \sum\limits_{n=0}^{\infty}(-1)^n \dfrac{1}{(2n)!} \cdot \left(\dfrac{\sqrt{\pi}}{3}\right)^{2n} = \cos\left(\dfrac{\sqrt{\pi}}{3}\right)$ since $\cos x = \sum\limits_{n=0}^{\infty}(-1)^n \dfrac{x^{2n}}{(2n)!}$ for all x.

31. $1 - e + \dfrac{e^2}{2!} - \dfrac{e^3}{3!} + \dfrac{e^4}{4!} - \cdots = \sum\limits_{n=0}^{\infty}(-1)^n \dfrac{e^n}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{(-e)^n}{n!} = e^{-e}$ since $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$ for all x.

32. $4.17\overline{326} = 4.17 + \dfrac{326}{10^5} + \dfrac{326}{10^8} + \cdots = 4.17 + \dfrac{326/10^5}{1 - 1/10^3} = \dfrac{417}{100} + \dfrac{326}{99{,}900} = \dfrac{416{,}909}{99{,}900}$

33.
$$\begin{aligned}\cosh x &= \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right) \\ &= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)\right] \\ &= \frac{1}{2}\left(2 + 2\cdot\frac{x^2}{2!} + 2\cdot\frac{x^4}{4!} + \cdots\right) = 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty}\frac{x^{2n}}{(2n)!} \geq 1 + \frac{1}{2}x^2 \quad \text{for all } x\end{aligned}$$

34. $\sum\limits_{n=1}^{\infty} (\ln x)^n$ is a geometric series which converges whenever $|\ln x| < 1 \quad \Rightarrow \quad -1 < \ln x < 1 \quad \Rightarrow \quad e^{-1} < x < e$.

35. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^5} = 1 - \dfrac{1}{32} + \dfrac{1}{243} - \dfrac{1}{1024} + \dfrac{1}{3125} - \dfrac{1}{7776} + \dfrac{1}{16{,}807} - \dfrac{1}{32{,}768} + \cdots$.

Since $b_8 = \dfrac{1}{8^5} = \dfrac{1}{32{,}768} < 0.000031$, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^5} \approx \sum\limits_{n=1}^{7} \dfrac{(-1)^{n+1}}{n^5} \approx 0.9721$.

36. (a) $s_5 = \sum\limits_{n=1}^{5} \dfrac{1}{n^6} = 1 + \dfrac{1}{2^6} + \cdots + \dfrac{1}{5^6} \approx 1.017305$. The series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^6}$ converges by the Integral Test, so we estimate the remainder R_5 with (11.3.2): $R_5 \le \displaystyle\int_5^{\infty} \frac{dx}{x^6} = \left[-\frac{x^{-5}}{5}\right]_5^{\infty} = \frac{5^{-5}}{5} = 0.000064$. So the error is at most 0.000064.

(b) In general, $R_n \le \displaystyle\int_n^{\infty} \frac{dx}{x^6} = \frac{1}{5n^5}$. If we take $n = 9$, then $s_9 \approx 1.01734$ and $R_9 \le \dfrac{1}{5 \cdot 9^5} \approx 3.4 \times 10^{-6}$.

So to five decimal places, $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^5} \approx \sum\limits_{n=1}^{9} \dfrac{1}{n^5} \approx 1.01734$.

Another method: Use (11.3.3) instead of (11.3.2).

37. $\sum\limits_{n=1}^{\infty} \dfrac{1}{2+5^n} \approx \sum\limits_{n=1}^{8} \dfrac{1}{2+5^n} \approx 0.18976224$. To estimate the error, note that $\dfrac{1}{2+5^n} < \dfrac{1}{5^n}$, so the remainder term is

$R_8 = \sum\limits_{n=9}^{\infty} \dfrac{1}{2+5^n} < \sum\limits_{n=9}^{\infty} \dfrac{1}{5^n} = \dfrac{1/5^9}{1-1/5} = 6.4 \times 10^{-7}$ $\left[\text{geometric series with } a = \frac{1}{5^9} \text{ and } r = \frac{1}{5}\right]$.

38. (a) $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^{n+1}}{[2(n+1)]!} \cdot \frac{(2n)!}{n^n}\right| = \lim_{n\to\infty} \frac{(n+1)^n (n+1)^1}{(2n+2)(2n+1)n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \frac{1}{2(2n+1)}$$
$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \frac{1}{2(2n+1)} = e \cdot 0 = 0 < 1$$

so the given series $\sum\limits_{n=1}^{\infty} \dfrac{n^n}{(2n)!}$ converges by the Ratio Test.

(b) The series in part (a) is convergent, so $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n^n}{(2n)!} = 0$ by Theorem 11.2.6.

39. Use the Limit Comparison Test. $\lim\limits_{n\to\infty} \left|\dfrac{\left(\frac{n+1}{n}\right)a_n}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{n+1}{n} = \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right) = 1 > 0$.

Since $\sum |a_n|$ is convergent, so is $\sum \left|\left(\dfrac{n+1}{n}\right)a_n\right|$, by the Limit Comparison Test.

40. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{(n+1)^2\, 5^{n+1}} \cdot \dfrac{n^2 5^n}{x^n}\right| = \lim\limits_{n\to\infty} \dfrac{1}{(1+1/n)^2} \dfrac{|x|}{5} = \dfrac{|x|}{5}$, so by the Ratio Test, $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{x^n}{n^2\, 5^n}$ converges when $\dfrac{|x|}{5} < 1 \quad \Leftrightarrow \quad |x| < 5$, so $R = 5$. When $x = -5$, the series becomes the convergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ with $p = 2 > 1$. When $x = 5$, the series becomes $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. Thus, $I = [-5, 5]$.

41. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{|x+2|^{n+1}}{(n+1)\,4^{n+1}}\cdot\dfrac{n\,4^n}{|x+2|^n}\right] = \lim\limits_{n\to\infty}\left[\dfrac{n}{n+1}\,\dfrac{|x+2|}{4}\right] = \dfrac{|x+2|}{4} < 1 \quad\Leftrightarrow\quad |x+2| < 4$, so $R = 4$.

$|x+2| < 4 \quad\Leftrightarrow\quad -4 < x+2 < 4 \quad\Leftrightarrow\quad -6 < x < 2$. If $x = -6$, then the series $\sum\limits_{n=1}^{\infty}\dfrac{(x+2)^n}{n\,4^n}$ becomes

$\sum\limits_{n=1}^{\infty}\dfrac{(-4)^n}{n4^n} = \sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. When $x = 2$, the series becomes the harmonic series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$, which diverges. Thus, $I = [-6, 2)$.

42. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{2^{n+1}\,(x-2)^{n+1}}{(n+3)!}\cdot\dfrac{(n+2)!}{2^n(x-2)^n}\right| = \lim\limits_{n\to\infty}\dfrac{2}{n+3}\,|x-2| = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{2^n\,(x-2)^n}{(n+2)!}$ converges for all x. $R = \infty$ and $I = (-\infty, \infty)$.

43. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}}\cdot\dfrac{\sqrt{n+3}}{2^n(x-3)^n}\right| = 2\,|x-3|\lim\limits_{n\to\infty}\sqrt{\dfrac{n+3}{n+4}} = 2\,|x-3| < 1 \quad\Leftrightarrow\quad |x-3| < \frac{1}{2}$,

so $R = \frac{1}{2}$. $|x-3| < \frac{1}{2} \quad\Leftrightarrow\quad -\frac{1}{2} < x-3 < \frac{1}{2} \quad\Leftrightarrow\quad \frac{5}{2} < x < \frac{7}{2}$. For $x = \frac{7}{2}$, the series $\sum\limits_{n=1}^{\infty}\dfrac{2^n(x-3)^n}{\sqrt{n+3}}$ becomes

$\sum\limits_{n=0}^{\infty}\dfrac{1}{\sqrt{n+3}} = \sum\limits_{n=3}^{\infty}\dfrac{1}{n^{1/2}}$, which diverges $\left[p = \frac{1}{2} \le 1\right]$, but for $x = \frac{5}{2}$, we get $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{\sqrt{n+3}}$, which is a convergent alternating series, so $I = \left[\frac{5}{2}, \frac{7}{2}\right)$.

44. For $\sum\limits_{n=1}^{\infty}\dfrac{(2n)!}{(n!)^2}\,x^n$, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(2n+2)!\,x^{n+1}}{[(n+1)!]^2}\cdot\dfrac{(n!)^2}{(2n)!\,x^n}\right| = \lim\limits_{n\to\infty}\dfrac{(2n+2)(2n+1)}{(n+1)(n+1)}\,|x| = 4\,|x|$.

To converge, we must have $4\,|x| < 1 \quad\Leftrightarrow\quad |x| < \frac{1}{4}$, so $R = \frac{1}{4}$.

45.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{6}\right)$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$
4	$\sin x$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
\sin x &= f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{6}\right)}{3!}\left(x-\frac{\pi}{6}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{6}\right)}{4!}\left(x-\frac{\pi}{6}\right)^4 + \cdots \\
&= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{6}\right)^4 - \cdots\right] + \frac{\sqrt{3}}{2}\left[\left(x-\frac{\pi}{6}\right) - \frac{1}{3!}\left(x-\frac{\pi}{6}\right)^3 + \cdots\right] \\
&= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(x-\frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1}
\end{aligned}$$

46.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{3}\right)$
0	$\cos x$	$\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$-\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\cos x &= f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{3}\right)}{3!}\left(x-\frac{\pi}{3}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{3}\right)}{4!}\left(x-\frac{\pi}{3}\right)^4 + \cdots \\ &= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{3}\right)^4 - \cdots\right] + \frac{\sqrt{3}}{2}\left[-\left(x-\frac{\pi}{3}\right) + \frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3 - \cdots\right] \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(x-\frac{\pi}{3}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}\left(x-\frac{\pi}{3}\right)^{2n+1}\end{aligned}$$

47. $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty}(-x)^n = \sum\limits_{n=0}^{\infty}(-1)^n x^n$ for $|x|<1$ $\Rightarrow$ $\dfrac{x^2}{1+x} = \sum\limits_{n=0}^{\infty}(-1)^n x^{n+2}$ with $R=1$.

48. $\tan^{-1}x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$ with interval of convergence $[-1,1]$, so

$\tan^{-1}(x^2) = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{(x^2)^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{4n+2}}{2n+1}$, which converges when $x^2\in[-1,1]$ $\Leftrightarrow$ $x\in[-1,1]$.

Therefore, $R=1$.

49. $\displaystyle\int\frac{1}{4-x}\,dx = -\ln(4-x)+C$ and

$$\int\frac{1}{4-x}\,dx = \frac{1}{4}\int\frac{1}{1-x/4}\,dx = \frac{1}{4}\int\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n dx = \frac{1}{4}\int\sum_{n=0}^{\infty}\frac{x^n}{4^n}\,dx = \frac{1}{4}\sum_{n=0}^{\infty}\frac{x^{n+1}}{4^n(n+1)}+C.$$ So

$\ln(4-x) = -\dfrac{1}{4}\sum\limits_{n=0}^{\infty}\dfrac{x^{n+1}}{4^n(n+1)}+C = -\sum\limits_{n=0}^{\infty}\dfrac{x^{n+1}}{4^{n+1}(n+1)}+C = -\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n4^n}+C$. Putting $x=0$, we get $C=\ln 4$.

Thus, $f(x)=\ln(4-x)=\ln 4-\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n4^n}$. The series converges for $|x/4|<1$ $\Leftrightarrow$ $|x|<4$, so $R=4$.

Another solution:

$$\begin{aligned}\ln(4-x) &= \ln[4(1-x/4)] = \ln 4+\ln(1-x/4) = \ln 4+\ln[1+(-x/4)] \\ &= \ln 4+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-x/4)^n}{n} \quad \text{[from Table 1 in Section 11.10]} \\ &= \ln 4+\sum_{n=1}^{\infty}(-1)^{2n+1}\frac{x^n}{n4^n} = \ln 4-\sum_{n=1}^{\infty}\frac{x^n}{n4^n}.\end{aligned}$$

50. $e^x = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}$ $\Rightarrow$ $e^{2x} = \sum\limits_{n=0}^{\infty}\dfrac{(2x)^n}{n!}$ $\Rightarrow$ $xe^{2x} = x\sum\limits_{n=0}^{\infty}\dfrac{2^n x^n}{n!} = \sum\limits_{n=0}^{\infty}\dfrac{2^n x^{n+1}}{n!}$, $R=\infty$

51. $\sin x = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$ $\Rightarrow$ $\sin(x^4) = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n (x^4)^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^{8n+4}}{(2n+1)!}$ for all x, so the radius of convergence is ∞.

52. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow 10^x = e^{(\ln 10)x} = \sum_{n=0}^{\infty} \frac{[(\ln 10)x]^n}{n!} = \sum_{n=0}^{\infty} \frac{(\ln 10)^n x^n}{n!},\ R = \infty$

53. $f(x) = \frac{1}{\sqrt[4]{16-x}} = \frac{1}{\sqrt[4]{16(1-x/16)}} = \frac{1}{\sqrt[4]{16}\left(1-\frac{1}{16}x\right)^{1/4}} = \frac{1}{2}\left(1-\frac{1}{16}x\right)^{-1/4}$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{4}\right)\left(-\frac{x}{16}\right) + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}\left(-\frac{x}{16}\right)^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!}\left(-\frac{x}{16}\right)^3 + \cdots\right]$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{2 \cdot 4^n \cdot n! \cdot 16^n} x^n = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{2^{6n+1}\, n!} x^n$$

for $\left|-\frac{x}{16}\right| < 1 \quad \Leftrightarrow \quad |x| < 16$, so $R = 16$.

54. $(1-3x)^{-5} = \sum_{n=0}^{\infty} \binom{-5}{n} (-3x)^n = 1 + (-5)(-3x) + \frac{(-5)(-6)}{2!}(-3x)^2 + \frac{(-5)(-6)(-7)}{3!}(-3x)^3 + \cdots$

$$= 1 + \sum_{n=1}^{\infty} \frac{5 \cdot 6 \cdot 7 \cdot \cdots \cdot (n+4) \cdot 3^n x^n}{n!} \quad \text{for } |-3x| < 1 \quad \Leftrightarrow \quad |x| < \tfrac{1}{3}, \text{ so } R = \tfrac{1}{3}.$$

55. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $\frac{e^x}{x} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!} = x^{-1} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$ and

$\int \frac{e^x}{x}\,dx = C + \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$.

56. $(1+x^4)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (x^4)^n = 1 + \left(\frac{1}{2}\right)x^4 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(x^4)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(x^4)^3 + \cdots$

$$= 1 + \tfrac{1}{2}x^4 - \tfrac{1}{8}x^8 + \tfrac{1}{16}x^{12} - \cdots$$

so $\int_0^1 (1+x^4)^{1/2}\,dx = \left[x + \frac{1}{10}x^5 - \frac{1}{72}x^9 + \frac{1}{208}x^{13} - \cdots\right]_0^1 = 1 + \frac{1}{10} - \frac{1}{72} + \frac{1}{208} - \cdots$.

This is an alternating series, so by the Alternating Series Test, the error in the approximation

$\int_0^1 (1+x^4)^{1/2}\,dx \approx 1 + \frac{1}{10} - \frac{1}{72} \approx 1.086$ is less than $\frac{1}{208}$, sufficient for the desired accuracy.

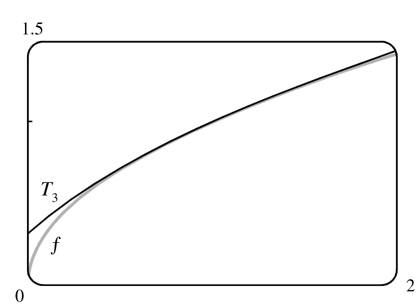
Thus, correct to two decimal places, $\int_0^1 (1+x^4)^{1/2}\,dx \approx 1.09$.

57. (a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{1/2}$	1
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}x^{-7/2}$	$-\frac{15}{16}$
$\vdots$	$\vdots$	$\vdots$

$$\sqrt{x} \approx T_3(x) = 1 + \frac{1/2}{1!}(x-1) - \frac{1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3$$
$$= 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \tfrac{1}{16}(x-1)^3$$

(b)

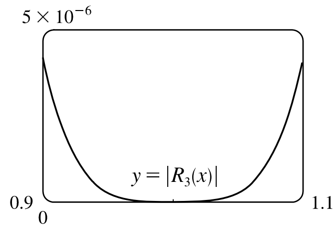


(c) $|R_3(x)| \leq \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \leq M$ with $f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$. Now $0.9 \leq x \leq 1.1 \quad\Rightarrow$

$-0.1 \leq x-1 \leq 0.1 \quad\Rightarrow\quad (x-1)^4 \leq (0.1)^4$, and letting $x = 0.9$ gives $M = \dfrac{15}{16(0.9)^{7/2}}$, so

$|R_3(x)| \leq \dfrac{15}{16(0.9)^{7/2}\,4!}(0.1)^4 \approx 0.000\,005\,648 \approx 0.000\,006 = 6 \times 10^{-6}$.

(d)



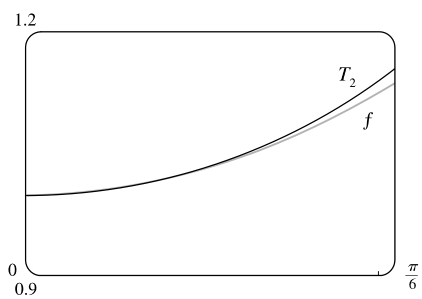
From the graph of $|R_3(x)| = |\sqrt{x} - T_3(x)|$, it appears that the error is less than 5×10^{-6} on $[0.9, 1.1]$.

58. (a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sec x$	1
1	$\sec x \tan x$	0
2	$\sec x \tan^2 x + \sec^3 x$	1
3	$\sec x \tan^3 x + 5\sec^3 x \tan x$	0
$\vdots$	$\vdots$	$\vdots$

$\sec x \approx T_2(x) = 1 + \frac{1}{2}x^2$

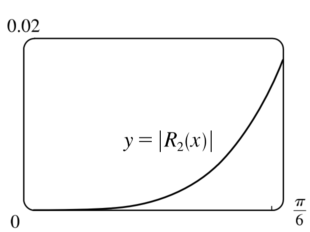
(b)



(c) $|R_2(x)| \leq \dfrac{M}{3!}|x|^3$, where $\left|f^{(3)}(x)\right| \leq M$ with $f^{(3)}(x) = \sec x\,\tan^3 x + 5\sec^3 x\,\tan x$.

Now $0 \leq x \leq \frac{\pi}{6} \quad\Rightarrow\quad x^3 \leq \left(\frac{\pi}{6}\right)^3$, and letting $x = \frac{\pi}{6}$ gives $M = \frac{14}{3}$, so $|R_2(x)| \leq \dfrac{14}{3 \cdot 6}\left(\dfrac{\pi}{6}\right)^3 \approx 0.111648$.

(d)



From the graph of $|R_2(x)| = |\sec x - T_2(x)|$, it appears that the error is less than 0.02 on $\left[0, \frac{\pi}{6}\right]$.

59. $\sin x = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, so $\sin x - x = -\dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$ and

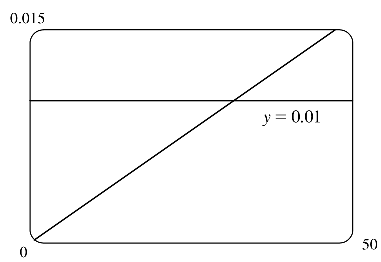
$\dfrac{\sin x - x}{x^3} = -\dfrac{1}{3!} + \dfrac{x^2}{5!} - \dfrac{x^4}{7!} + \cdots$. Thus, $\displaystyle\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\left(-\frac{1}{6} + \frac{x^2}{120} - \frac{x^4}{5040} + \cdots\right) = -\frac{1}{6}$.

60. (a) $F = \dfrac{mgR^2}{(R+h)^2} = \dfrac{mg}{(1+h/R)^2} = mg \sum\limits_{n=0}^{\infty} \dbinom{-2}{n} \left(\dfrac{h}{R}\right)^n$ [binomial series]

(b) We expand $F = mg\left[1 - 2\,(h/R) + 3\,(h/R)^2 - \cdots\right]$.

This is an alternating series, so by the Alternating Series Estimation Theorem, the error in the approximation $F = mg$ is less than $2mgh/R$, so for accuracy within 1% we want

$$\left|\frac{2mgh/R}{mgR^2/(R+h)^2}\right| < 0.01 \quad \Leftrightarrow \quad \frac{2h(R+h)^2}{R^3} < 0.01.$$



This inequality would be difficult to solve for h, so we substitute $R = 6{,}400$ km and plot both sides of the inequality. It appears that the approximation is accurate to within 1% for $h < 31$ km.

61. $f(x) = \sum\limits_{n=0}^{\infty} c_n\,x^n \quad \Rightarrow \quad f(-x) = \sum\limits_{n=0}^{\infty} c_n(-x)^n = \sum\limits_{n=0}^{\infty} (-1)^n c_n\,x^n$

(a) If f is an odd function, then $f(-x) = -f(x) \quad \Rightarrow \quad \sum\limits_{n=0}^{\infty} (-1)^n c_n x^n = \sum\limits_{n=0}^{\infty} -c_n x^n$. The coefficients of any power series are uniquely determined (by Theorem 11.10.5), so $(-1)^n\,c_n = -c_n$.

If n is even, then $(-1)^n = 1$, so $c_n = -c_n \quad \Rightarrow \quad 2c_n = 0 \quad \Rightarrow \quad c_n = 0$. Thus, all even coefficients are 0, that is, $c_0 = c_2 = c_4 = \cdots = 0$.

(b) If f is even, then $f(-x) = f(x) \quad \Rightarrow \quad \sum\limits_{n=0}^{\infty} (-1)^n\,c_n\,x^n = \sum\limits_{n=0}^{\infty} c_n\,x^n \quad \Rightarrow \quad (-1)^n\,c_n = c_n$.

If n is odd, then $(-1)^n = -1$, so $-c_n = c_n \quad \Rightarrow \quad 2c_n = 0 \quad \Rightarrow \quad c_n = 0$. Thus, all odd coefficients are 0, that is, $c_1 = c_3 = c_5 = \cdots = 0$.

62. $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} \quad \Rightarrow \quad f(x) = e^{x^2} = \sum\limits_{n=0}^{\infty} \dfrac{(x^2)^n}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{x^{2n}}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{1}{n!}\,x^{2n}$. By Theorem 11.10.6 with $a = 0$, we also have $f(x) = \sum\limits_{k=0}^{\infty} \dfrac{f^{(k)}(0)}{k!}x^k$. Comparing coefficients for $k = 2n$, we have $\dfrac{f^{(2n)}(0)}{(2n)!} = \dfrac{1}{n!} \quad \Rightarrow \quad f^{(2n)}(0) = \dfrac{(2n)!}{n!}$.

PROBLEMS PLUS

1. (a) From Formula 15a in Appendix D, with $x = y = \theta$, we get $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$, so $\cot 2\theta = \dfrac{1-\tan^2\theta}{2\tan\theta} \quad \Rightarrow$

$2\cot 2\theta = \dfrac{1-\tan^2\theta}{\tan\theta} = \cot\theta - \tan\theta$. Replacing θ by $\frac{1}{2}x$, we get $2\cot x = \cot\frac{1}{2}x - \tan\frac{1}{2}x$, or

$\tan\frac{1}{2}x = \cot\frac{1}{2}x - 2\cot x$.

(b) From part (a) with $\dfrac{x}{2^{n-1}}$ in place of x, $\tan\dfrac{x}{2^n} = \cot\dfrac{x}{2^n} - 2\cot\dfrac{x}{2^{n-1}}$, so the nth partial sum of $\sum\limits_{n=1}^{\infty}\dfrac{1}{2^n}\tan\dfrac{x}{2^n}$ is

$$\begin{aligned} s_n &= \frac{\tan(x/2)}{2} + \frac{\tan(x/4)}{4} + \frac{\tan(x/8)}{8} + \cdots + \frac{\tan(x/2^n)}{2^n} \\ &= \left[\frac{\cot(x/2)}{2} - \cot x\right] + \left[\frac{\cot(x/4)}{4} - \frac{\cot(x/2)}{2}\right] + \left[\frac{\cot(x/8)}{8} - \frac{\cot(x/4)}{4}\right] + \cdots \\ &\qquad + \left[\frac{\cot(x/2^n)}{2^n} - \frac{\cot(x/2^{n-1})}{2^{n-1}}\right] = -\cot x + \frac{\cot(x/2^n)}{2^n} \quad \text{[telescoping sum]} \end{aligned}$$

Now $\dfrac{\cot(x/2^n)}{2^n} = \dfrac{\cos(x/2^n)}{2^n\sin(x/2^n)} = \dfrac{\cos(x/2^n)}{x}\cdot\dfrac{x/2^n}{\sin(x/2^n)} \to \dfrac{1}{x}\cdot 1 = \dfrac{1}{x}$ as $n\to\infty$ since $x/2^n \to 0$

for $x \neq 0$. Therefore, if $x \neq 0$ and $x \neq k\pi$ where k is any integer, then

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\tan\frac{x}{2^n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(-\cot x + \frac{1}{2^n}\cot\frac{x}{2^n}\right) = -\cot x + \frac{1}{x}$$

If $x = 0$, then all terms in the series are 0, so the sum is 0.

2. $|AP_2|^2 = 2$, $|AP_3|^2 = 2 + 2^2$, $|AP_4|^2 = 2 + 2^2 + \left(2^2\right)^2$, $|AP_5|^2 = 2 + 2^2 + \left(2^2\right)^2 + \left(2^3\right)^2, \ldots,$

$$\begin{aligned} |AP_n|^2 &= 2 + 2^2 + \left(2^2\right)^2 + \cdots + (2^{n-2})^2 \quad \text{[for } n \geq 3\text{]} \quad = 2 + (4 + 4^2 + 4^3 + \cdots + 4^{n-2}) \\ &= 2 + \frac{4(4^{n-2}-1)}{4-1} \quad \text{[finite geometric sum with } a = 4, r = 4\text{]} \quad = \frac{6}{3} + \frac{4^{n-1}-4}{3} = \frac{2}{3} + \frac{4^{n-1}}{3} \end{aligned}$$

So $\tan\angle P_nAP_{n+1} = \dfrac{|P_nP_{n+1}|}{|AP_n|} = \dfrac{2^{n-1}}{\sqrt{\dfrac{2}{3}+\dfrac{4^{n-1}}{3}}} = \dfrac{\sqrt{4^{n-1}}}{\sqrt{\dfrac{2}{3}+\dfrac{4^{n-1}}{3}}} = \dfrac{1}{\sqrt{\dfrac{2}{3\cdot 4^{n-1}}+\dfrac{1}{3}}} \to \sqrt{3}$ as $n\to\infty$.

Thus, $\angle P_nAP_{n+1} \to \frac{\pi}{3}$ as $n \to \infty$.

3. (a) At each stage, each side is replaced by four shorter sides, each of length $\frac{1}{3}$ of the side length at the preceding stage. Writing s_0 and ℓ_0 for the number of sides and the length of the side of the initial triangle, we generate the table at right. In general, we have $s_n = 3\cdot 4^n$ and $\ell_n = \left(\frac{1}{3}\right)^n$, so the length of the perimeter at the nth stage of construction is $p_n = s_n\ell_n = 3\cdot 4^n\cdot\left(\frac{1}{3}\right)^n = 3\cdot\left(\frac{4}{3}\right)^n$.

$s_0 = 3$	$\ell_0 = 1$
$s_1 = 3\cdot 4$	$\ell_1 = 1/3$
$s_2 = 3\cdot 4^2$	$\ell_2 = 1/3^2$
$s_3 = 3\cdot 4^3$	$\ell_3 = 1/3^3$
$\vdots$	$\vdots$

(b) $p_n = \dfrac{4^n}{3^{n-1}} = 4\left(\dfrac{4}{3}\right)^{n-1}$. Since $\frac{4}{3} > 1$, $p_n \to \infty$ as $n \to \infty$.

(c) The area of each of the small triangles added at a given stage is one-ninth of the area of the triangle added at the preceding stage. Let a be the area of the original triangle. Then the area a_n of each of the small triangles added at stage n is $a_n = a \cdot \dfrac{1}{9^n} = \dfrac{a}{9^n}$. Since a small triangle is added to each side at every stage, it follows that the total area A_n added to the figure at the nth stage is $A_n = s_{n-1} \cdot a_n = 3 \cdot 4^{n-1} \cdot \dfrac{a}{9^n} = a \cdot \dfrac{4^{n-1}}{3^{2n-1}}$. Then the total area enclosed by the snowflake curve is $A = a + A_1 + A_2 + A_3 + \cdots = a + a \cdot \dfrac{1}{3} + a \cdot \dfrac{4}{3^3} + a \cdot \dfrac{4^2}{3^5} + a \cdot \dfrac{4^3}{3^7} + \cdots$. After the first term, this is a geometric series with common ratio $\dfrac{4}{9}$, so $A = a + \dfrac{a/3}{1 - \frac{4}{9}} = a + \dfrac{a}{3} \cdot \dfrac{9}{5} = \dfrac{8a}{5}$. But the area of the original equilateral triangle with side 1 is $a = \dfrac{1}{2} \cdot 1 \cdot \sin \dfrac{\pi}{3} = \dfrac{\sqrt{3}}{4}$. So the area enclosed by the snowflake curve is $\dfrac{8}{5} \cdot \dfrac{\sqrt{3}}{4} = \dfrac{2\sqrt{3}}{5}$.

4. Let the series $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$. Then every term in S is of the form $\dfrac{1}{2^m 3^n}$, $m, n \geq 0$, and furthermore each term occurs only once. So we can write

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2^m 3^n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2^m} \frac{1}{3^n} = \sum_{m=0}^{\infty} \frac{1}{2^m} \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{2}} \cdot \frac{1}{1 - \frac{1}{3}} = 2 \cdot \tfrac{3}{2} = 3$$

5. (a) Let $a = \arctan x$ and $b = \arctan y$. Then, from Formula 15b in Appendix D,

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \, \tan b} = \frac{\tan(\arctan x) - \tan(\arctan y)}{1 + \tan(\arctan x) \tan(\arctan y)} = \frac{x - y}{1 + xy}$$

Now $\arctan x - \arctan y = a - b = \arctan(\tan(a - b)) = \arctan \dfrac{x - y}{1 + xy}$ since $-\frac{\pi}{2} < a - b < \frac{\pi}{2}$.

(b) From part (a) we have

$$\arctan \tfrac{120}{119} - \arctan \tfrac{1}{239} = \arctan \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}} = \arctan \frac{\frac{28{,}561}{28{,}441}}{\frac{28{,}561}{28{,}441}} = \arctan 1 = \tfrac{\pi}{4}$$

(c) Replacing y by $-y$ in the formula of part (a), we get $\arctan x + \arctan y = \arctan \dfrac{x + y}{1 - xy}$. So

$$\begin{aligned} 4 \arctan \tfrac{1}{5} &= 2\left(\arctan \tfrac{1}{5} + \arctan \tfrac{1}{5}\right) = 2 \arctan \frac{\frac{1}{5} + \frac{1}{5}}{1 - \frac{1}{5} \cdot \frac{1}{5}} = 2 \arctan \tfrac{5}{12} = \arctan \tfrac{5}{12} + \arctan \tfrac{5}{12} \\ &= \arctan \frac{\frac{5}{12} + \frac{5}{12}}{1 - \frac{5}{12} \cdot \frac{5}{12}} = \arctan \tfrac{120}{119} \end{aligned}$$

Thus, from part (b), we have $4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4}$.

(d) From Example 11.9.6 we have $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \dfrac{x^{11}}{11} + \cdots$, so

$$\arctan \frac{1}{5} = \frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \frac{1}{9 \cdot 5^9} - \frac{1}{11 \cdot 5^{11}} + \cdots$$

This is an alternating series and the size of the terms decreases to 0, so by the Alternating Series Estimation Theorem, the sum lies between s_5 and s_6, that is, $0.197395560 < \arctan \frac{1}{5} < 0.197395562$.

(e) From the series in part (d) we get $\arctan \dfrac{1}{239} = \dfrac{1}{239} - \dfrac{1}{3 \cdot 239^3} + \dfrac{1}{5 \cdot 239^5} - \cdots$. The third term is less than 2.6×10^{-13}, so by the Alternating Series Estimation Theorem, we have, to nine decimal places, $\arctan \frac{1}{239} \approx s_2 \approx 0.004184076$. Thus, $0.004184075 < \arctan \frac{1}{239} < 0.004184077$.

(f) From part (c) we have $\pi = 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239}$, so from parts (d) and (e) we have $16(0.197395560) - 4(0.004184077) < \pi < 16(0.197395562) - 4(0.004184075) \quad \Rightarrow$ $3.141592652 < \pi < 3.141592692$. So, to 7 decimal places, $\pi \approx 3.1415927$.

6. (a) Let $a = \operatorname{arccot} x$ and $b = \operatorname{arccot} y$ where $0 < a - b < \pi$. Then

$$\cot(a-b) = \frac{1}{\tan(a-b)} = \frac{1 + \tan a \, \tan b}{\tan a - \tan b} = \frac{\dfrac{1}{\cot a} \cdot \dfrac{1}{\cot b} + 1}{\dfrac{1}{\cot a} - \dfrac{1}{\cot b}} \cdot \frac{\cot a \, \cot b}{\cot a \, \cot b}$$

$$= \frac{1 + \cot a \, \cot b}{\cot b - \cot a} = \frac{1 + \cot(\operatorname{arccot} x)\cot(\operatorname{arccot} y)}{\cot(\operatorname{arccot} y) - \cot(\operatorname{arccot} x)} = \frac{1 + xy}{y - x}$$

Now $\operatorname{arccot} x - \operatorname{arccot} y = a - b = \operatorname{arccot}(\cot(a-b)) = \operatorname{arccot} \dfrac{1+xy}{y-x}$ since $0 < a - b < \pi$.

(b) From part (a), we want $\operatorname{arccot}(n^2 + n + 1)$ to equal $\operatorname{arccot} \dfrac{1+xy}{y-x}$. Note that $1 + xy = n^2 + n + 1 \quad \Leftrightarrow$ $xy = n^2 + n = (n+1)n$, so if we let $x = n + 1$ and $y = n$, then $y - x = 1$. Therefore,

$$\operatorname{arccot}(n^2 + n + 1) = \operatorname{arccot}(1 + n(n+1)) = \operatorname{arccot} \frac{1 + n(n+1)}{(n+1) - n} = \operatorname{arccot} n - \operatorname{arccot}(n+1)$$

Thus, we have a telescoping series with nth partial sum

$s_n = [\operatorname{arccot} 0 - \operatorname{arccot} 1] + [\operatorname{arccot} 1 - \operatorname{arccot} 2] + \cdots + [\operatorname{arccot} n - \operatorname{arccot}(n+1)] = \operatorname{arccot} 0 - \operatorname{arccot}(n+1)$.

Thus, $\sum\limits_{n=0}^{\infty} \operatorname{arccot}(n^2 + n + 1) = \lim\limits_{n \to \infty} s_n = \lim\limits_{n \to \infty} [\operatorname{arccot} 0 - \operatorname{arccot}(n+1)] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$.

7. We want $\arctan\left(\dfrac{2}{n^2}\right)$ to equal $\arctan \dfrac{x-y}{1+xy}$. Note that $1 + xy = n^2 \quad \Leftrightarrow \quad xy = n^2 - 1 = (n+1)(n-1)$, so if we let $x = n + 1$ and $y = n - 1$, then $x - y = 2$ and $xy \neq -1$. Thus, from Problem 5(a),

$\arctan\left(\dfrac{2}{n^2}\right) = \arctan \dfrac{x-y}{1+xy} = \arctan x - \arctan y = \arctan(n+1) - \arctan(n-1)$. Therefore,

$$\begin{aligned}\sum_{n=1}^{k} \arctan\left(\frac{2}{n^2}\right) &= \sum_{n=1}^{k} [\arctan(n+1) - \arctan(n-1)] \\ &= \sum_{n=1}^{k} [\arctan(n+1) - \arctan n + \arctan n - \arctan(n-1)] \\ &= \sum_{n=1}^{k} [\arctan(n+1) - \arctan n] + \sum_{n=1}^{k} [\arctan n - \arctan(n-1)] \\ &= [\arctan(k+1) - \arctan 1] + [\arctan k - \arctan 0] \quad \text{[since both sums are telescoping]} \\ &= \arctan(k+1) - \tfrac{\pi}{4} + \arctan k - 0\end{aligned}$$

[continued]

Now $\sum_{n=1}^{k} \arctan\left(\frac{2}{n^2}\right) = \lim_{k\to 0} \sum_{n=1}^{k} \arctan\left(\frac{2}{n^2}\right) = \lim_{k\to\infty}\left[\arctan(k+1) - \frac{\pi}{4} + \arctan k\right] = \frac{\pi}{2} - \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$.

Note: For all $n \geq 1$, $0 \leq \arctan(n-1) < \arctan(n+1) < \frac{\pi}{2}$, so $-\frac{\pi}{2} < \arctan(n+1) - \arctan(n-1) < \frac{\pi}{2}$, and the identity in Problem 5(a) holds.

8. Let's first try the case $k = 1$: $a_0 + a_1 = 0 \quad\Rightarrow\quad a_1 = -a_0 \quad\Rightarrow$

$$\lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1}\right) = \lim_{n\to\infty}\left(a_0\sqrt{n} - a_0\sqrt{n+1}\right) = a_0 \lim_{n\to\infty}\left(\sqrt{n} - \sqrt{n+1}\right)\frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{n}+\sqrt{n+1}}$$

$$= a_0 \lim_{n\to\infty}\frac{-1}{\sqrt{n}+\sqrt{n+1}} = 0$$

In general we have $a_0 + a_1 + \cdots + a_k = 0 \quad\Rightarrow\quad a_k = -a_0 - a_1 - \cdots - a_{k-1} \quad\Rightarrow$

$$\lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}\right)$$

$$= \lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1} + \cdots + a_{k-1}\sqrt{n+k-1} - a_0\sqrt{n+k} - a_1\sqrt{n+k} - \cdots - a_{k-1}\sqrt{n+k}\right)$$

$$= a_0\lim_{n\to\infty}\left(\sqrt{n} - \sqrt{n+k}\right) + a_1\lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n+k}\right) + \cdots + a_{k-1}\lim_{n\to\infty}\left(\sqrt{n+k-1} - \sqrt{n+k}\right)$$

Each of these limits is 0 by the same type of simplification as in the case $k = 1$. So we have

$$\lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}\right) = a_0(0) + a_1(0) + \cdots + a_{k-1}(0) = 0$$

9. We start with the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, $|x| < 1$, and differentiate:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} \text{ for } |x| < 1 \quad\Rightarrow\quad \sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}$$

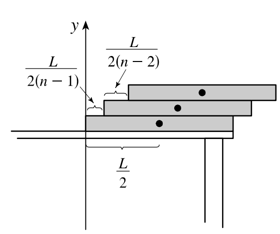
for $|x| < 1$. Differentiate again:

$$\sum_{n=1}^{\infty} n^2x^{n-1} = \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{(1-x)^2 - x\cdot 2(1-x)(-1)}{(1-x)^4} = \frac{x+1}{(1-x)^3} \quad\Rightarrow\quad \sum_{n=1}^{\infty} n^2x^n = \frac{x^2+x}{(1-x)^3} \quad\Rightarrow$$

$$\sum_{n=1}^{\infty} n^3x^{n-1} = \frac{d}{dx}\frac{x^2+x}{(1-x)^3} = \frac{(1-x)^3(2x+1) - (x^2+x)3(1-x)^2(-1)}{(1-x)^6} = \frac{x^2+4x+1}{(1-x)^4} \quad\Rightarrow$$

$\sum_{n=1}^{\infty} n^3x^n = \frac{x^3+4x^2+x}{(1-x)^4}$, $|x| < 1$. The radius of convergence is 1 because that is the radius of convergence for the geometric series we started with. If $x = \pm 1$, the series is $\sum n^3(\pm 1)^n$, which diverges by the Test For Divergence, so the interval of convergence is $(-1, 1)$.

10. Place the y-axis as shown and let the length of each book be L. We want to show that the center of mass of the system of n books lies above the table, that is, $\overline{x} < L$. The x-coordinates of the centers of mass of the books are $x_1 = \frac{L}{2}$, $x_2 = \frac{L}{2(n-1)} + \frac{L}{2}$, $x_3 = \frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \frac{L}{2}$, and so on.



[continued]

Each book has the same mass m, so if there are n books, then

$$\begin{aligned}\overline{x} &= \frac{mx_1 + mx_2 + \cdots + mx_n}{mn} = \frac{x_1 + x_2 + \cdots + x_n}{n} \\ &= \frac{1}{n}\left[\frac{L}{2} + \left(\frac{L}{2(n-1)} + \frac{L}{2}\right) + \left(\frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \frac{L}{2}\right) + \cdots \right. \\ &\qquad\qquad \left. + \left(\frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \cdots + \frac{L}{4} + \frac{L}{2} + \frac{L}{2}\right)\right] \\ &= \frac{L}{n}\left[\frac{n-1}{2(n-1)} + \frac{n-2}{2(n-2)} + \cdots + \frac{2}{4} + \frac{1}{2} + \frac{n}{2}\right] = \frac{L}{n}\left[(n-1)\frac{1}{2} + \frac{n}{2}\right] = \frac{2n-1}{2n}L < L\end{aligned}$$

This shows that, no matter how many books are added according to the given scheme, the center of mass lies above the table. It remains to observe that the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots = \frac{1}{2}\sum (1/n)$ is divergent (harmonic series), so we can make the top book extend as far as we like beyond the edge of the table if we add enough books.

11. $\ln\left(1 - \dfrac{1}{n^2}\right) = \ln\left(\dfrac{n^2-1}{n^2}\right) = \ln\dfrac{(n+1)(n-1)}{n^2} = \ln[(n+1)(n-1)] - \ln n^2$

$$\begin{aligned}&= \ln(n+1) + \ln(n-1) - 2\ln n = \ln(n-1) - \ln n - \ln n + \ln(n+1) \\ &= \ln\frac{n-1}{n} - [\ln n - \ln(n+1)] = \ln\frac{n-1}{n} - \ln\frac{n}{n+1}.\end{aligned}$$

Let $s_k = \sum\limits_{n=2}^{k} \ln\left(1 - \dfrac{1}{n^2}\right) = \sum\limits_{n=2}^{k}\left(\ln\dfrac{n-1}{n} - \ln\dfrac{n}{n+1}\right)$ for $k \geq 2$. Then

$s_k = \left(\ln\dfrac{1}{2} - \ln\dfrac{2}{3}\right) + \left(\ln\dfrac{2}{3} - \ln\dfrac{3}{4}\right) + \cdots + \left(\ln\dfrac{k-1}{k} - \ln\dfrac{k}{k+1}\right) = \ln\dfrac{1}{2} - \ln\dfrac{k}{k+1}$, so

$\sum\limits_{n=2}^{\infty} \ln\left(1 - \dfrac{1}{n^2}\right) = \lim\limits_{k\to\infty} s_k = \lim\limits_{k\to\infty}\left(\ln\dfrac{1}{2} - \ln\dfrac{k}{k+1}\right) = \ln\dfrac{1}{2} - \ln 1 = \ln 1 - \ln 2 - \ln 1 = -\ln 2$ (or $\ln\frac{1}{2}$).

12. First notice that both series are absolutely convergent (p-series with $p > 1$.) Let the given expression be called x. Then

$$\begin{aligned}x &= \frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = \frac{1 + \left(2\cdot\frac{1}{2^p} - \frac{1}{2^p}\right) + \frac{1}{3^p} + \left(2\cdot\frac{1}{4^p} - \frac{1}{4^p}\right) + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} \\ &= \frac{\left(1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots\right) + \left(2\cdot\frac{1}{2^p} + 2\cdot\frac{1}{4^p} + 2\cdot\frac{1}{6^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} \\ &= 1 + \frac{2\left(\frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{6^p} + \frac{1}{8^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = 1 + \frac{\frac{1}{2^{p-1}}\left(1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = 1 + 2^{1-p}x\end{aligned}$$

Therefore, $x = 1 + 2^{1-p}x \quad\Leftrightarrow\quad x - 2^{1-p}x = 1 \quad\Leftrightarrow\quad x(1 - 2^{1-p}) = 1 \quad\Leftrightarrow\quad x = \dfrac{1}{1 - 2^{1-p}}$.

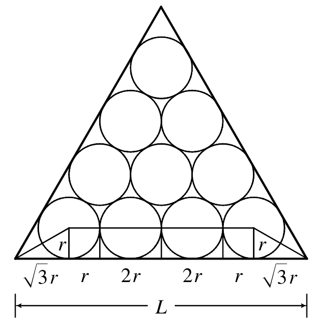
13. If L is the length of a side of the equilateral triangle, then the area is $A = \frac{1}{2}L \cdot \frac{\sqrt{3}}{2}L = \frac{\sqrt{3}}{4}L^2$ and so $L^2 = \frac{4}{\sqrt{3}}A$.

Let r be the radius of one of the circles. When there are n rows of circles, the figure shows that

$L = \sqrt{3}\,r + r + (n-2)(2r) + r + \sqrt{3}\,r = r\big(2n - 2 + 2\sqrt{3}\big)$, so $r = \dfrac{L}{2\big(n + \sqrt{3} - 1\big)}$.

The number of circles is $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$, and so the total area of the circles is

$$\begin{aligned} A_n &= \frac{n(n+1)}{2}\pi r^2 = \frac{n(n+1)}{2}\,\pi\,\frac{L^2}{4\big(n+\sqrt{3}-1\big)^2} \\ &= \frac{n(n+1)}{2}\,\pi\,\frac{4A/\sqrt{3}}{4\big(n+\sqrt{3}-1\big)^2} = \frac{n(n+1)}{\big(n+\sqrt{3}-1\big)^2}\,\frac{\pi A}{2\sqrt{3}} \quad \Rightarrow \\ \frac{A_n}{A} &= \frac{n(n+1)}{\big(n+\sqrt{3}-1\big)^2}\,\frac{\pi}{2\sqrt{3}} \\ &= \frac{1+1/n}{\big[1+\big(\sqrt{3}-1\big)/n\big]^2}\,\frac{\pi}{2\sqrt{3}} \to \frac{\pi}{2\sqrt{3}} \quad \text{as } n \to \infty \end{aligned}$$



14. Given $a_0 = a_1 = 1$ and $a_n = \dfrac{(n-1)(n-2)a_{n-1} - (n-3)a_{n-2}}{n(n-1)}$, we calculate the next few terms of the sequence:

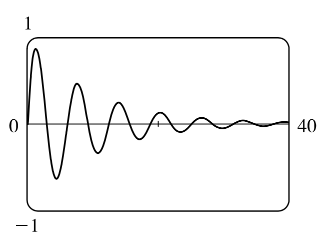
$a_2 = \dfrac{1 \cdot 0 \cdot a_1 - (-1)a_0}{2 \cdot 1} = \dfrac{1}{2}$, $a_3 = \dfrac{2 \cdot 1 \cdot a_2 - 0 \cdot a_1}{3 \cdot 2} = \dfrac{1}{6}$, $a_4 = \dfrac{3 \cdot 2 \cdot a_3 - 1 \cdot a_2}{4 \cdot 3} = \dfrac{1}{24}$. It seems that $a_n = \dfrac{1}{n!}$,

so we try to prove this by induction. The first step is done, so assume $a_k = \dfrac{1}{k!}$ and $a_{k-1} = \dfrac{1}{(k-1)!}$. Then

$$a_{k+1} = \frac{k(k-1)a_k - (k-2)a_{k-1}}{(k+1)k} = \frac{\dfrac{k(k-1)}{k!} - \dfrac{k-2}{(k-1)!}}{(k+1)k} = \frac{(k-1)-(k-2)}{[(k+1)(k)](k-1)!} = \frac{1}{(k+1)!}$$ and the induction is

complete. Therefore, $\sum\limits_{n=0}^{\infty} a_n = \sum\limits_{n=0}^{\infty} \dfrac{1}{n!} = e$.

15. (a)



The x-intercepts of the curve occur where $\sin x = 0 \iff x = n\pi$, n an integer. So using the formula for disks (and either a CAS or $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and Formula 99 to evaluate the integral), the volume of the nth bead is

$$\begin{aligned} V_n &= \pi \textstyle\int_{(n-1)\pi}^{n\pi} \big(e^{-x/10}\sin x\big)^2\,dx = \pi \int_{(n-1)\pi}^{n\pi} e^{-x/5}\sin^2 x\,dx \\ &= \tfrac{250\pi}{101}\big(e^{-(n-1)\pi/5} - e^{-n\pi/5}\big) \end{aligned}$$

(b) The total volume is

$$\pi \textstyle\int_0^\infty e^{-x/5}\sin^2 x\,dx = \sum\limits_{n=1}^{\infty} V_n = \frac{250\pi}{101}\sum\limits_{n=1}^{\infty}\big[e^{-(n-1)\pi/5} - e^{-n\pi/5}\big] = \frac{250\pi}{101} \quad \text{[telescoping sum].}$$

Another method: If the volume in part (a) has been written as $V_n = \frac{250\pi}{101}e^{-n\pi/5}\big(e^{\pi/5} - 1\big)$, then we recognize $\sum\limits_{n=1}^{\infty} V_n$ as a geometric series with $a = \frac{250\pi}{101}\big(1 - e^{-\pi/5}\big)$ and $r = e^{-\pi/5}$.

16. (a) Since P_n is defined as the midpoint of $P_{n-4}P_{n-3}$, $x_n = \frac{1}{2}(x_{n-4} + x_{n-3})$ for $n \geq 5$. So we prove by induction that $\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$. The case $n = 1$ is immediate, since $\frac{1}{2} \cdot 0 + 1 + 1 + 0 = 2$. Assume that the result holds for $n = k - 1$, that is, $\frac{1}{2}x_{k-1} + x_k + x_{k+1} + x_{k+2} = 2$. Then for $n = k$,

$$\begin{aligned}\tfrac{1}{2}x_k + x_{k+1} + x_{k+2} + x_{k+3} &= \tfrac{1}{2}x_k + x_{k+1} + x_{k+2} + \tfrac{1}{2}(x_{k+3-4} + x_{k+3-3}) \quad \text{[by above]}\\ &= \tfrac{1}{2}x_{k-1} + x_k + x_{k+1} + x_{k+2} = 2 \quad \text{[by the induction hypothesis]}\end{aligned}$$

Similarly, for $n \geq 5$, $y_n = \frac{1}{2}(y_{n-4} + y_{n-3})$, so the same argument as above holds for y, with 2 replaced by $\frac{1}{2}y_1 + y_2 + y_3 + y_4 = \frac{1}{2} \cdot 1 + 1 + 0 + 0 = \frac{3}{2}$. So $\frac{1}{2}y_n + y_{n+1} + y_{n+2} + y_{n+3} = \frac{3}{2}$ for all n.

(b) $\lim\limits_{n\to\infty} \left(\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3}\right) = \frac{1}{2}\lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} x_{n+1} + \lim\limits_{n\to\infty} x_{n+2} + \lim\limits_{n\to\infty} x_{n+3} = 2$. Since all the limits on the left hand side are the same, we get $\frac{7}{2}\lim\limits_{n\to\infty} x_n = 2 \quad \Rightarrow \quad \lim\limits_{n\to\infty} x_n = \frac{4}{7}$. In the same way, $\frac{7}{2}\lim\limits_{n\to\infty} y_n = \frac{3}{2} \quad \Rightarrow \quad \lim\limits_{n\to\infty} y_n = \frac{3}{7}$, so $P = \left(\frac{4}{7}, \frac{3}{7}\right)$.

17. By Table 1 in Section 11.10, $\tan^{-1} x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ for $|x| < 1$. In particular, for $x = \dfrac{1}{\sqrt{3}}$, we have $\dfrac{\pi}{6} = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{\left(1/\sqrt{3}\right)^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty} (-1)^n \left(\dfrac{1}{3}\right)^n \dfrac{1}{\sqrt{3}}\dfrac{1}{2n+1}$, so

$$\pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)3^n}\right) \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = \frac{\pi}{2\sqrt{3}} - 1.$$

18. (a) Using $s_n = a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$,

$$1 - x + x^2 - x^3 + \cdots + x^{2n-2} - x^{2n-1} = \frac{1\left[1 - (-x)^{2n}\right]}{1 - (-x)} = \frac{1 - x^{2n}}{1 + x}.$$

(b) $\displaystyle\int_0^1 (1 - x + x^2 - x^3 + \cdots + x^{2n-2} - x^{2n-1})\,dx = \int_0^1 \frac{1 - x^{2n}}{1 + x}\,dx \quad \Rightarrow$

$$\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{2n}\right]_0^1 = \int_0^1 \frac{dx}{1+x} - \int_0^1 \frac{x^{2n}}{1+x}\,dx \quad \Rightarrow$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \int_0^1 \frac{dx}{1+x} - \int_0^1 \frac{x^{2n}}{1+x}\,dx$$

(c) Since $1 - \dfrac{1}{2} = \dfrac{1}{1 \cdot 2}$, $\dfrac{1}{3} - \dfrac{1}{4} = \dfrac{1}{3 \cdot 4}$, $\ldots$, $\dfrac{1}{2n-1} - \dfrac{1}{2n} = \dfrac{1}{(2n-1)(2n)}$, we see from part (b) that

$\displaystyle\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(2n-1)(2n)} - \int_0^1 \frac{dx}{1+x} = -\int_0^1 \frac{x^{2n}}{1+x}\,dx$. Thus,

$$\left|\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(2n-1)(2n)} - \int_0^1 \frac{dx}{1+x}\right| = \int_0^1 \frac{x^{2n}}{1+x}\,dx < \int_0^1 x^{2n}\,dx$$

$\left[\text{since } \dfrac{x^{2n}}{1+x} < x^{2n} \text{ for } 0 < x \leq 1\right]$.

(d) Note that $\int_0^1 \frac{dx}{1+x} = \Big[\ln(1+x)\Big]_0^1 = \ln 2$ and $\int_0^1 x^{2n}\,dx = \left[\frac{x^{2n+1}}{2n+1}\right]_0^1 = \frac{1}{2n+1}$. So part (c) becomes

$\left|\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(2n-1)(2n)} - \ln 2\right| < \frac{1}{2n+1}$. In other words, the nth partial sum s_n of the given series

satisfies $|s_n - \ln 2| < \frac{1}{2n+1}$. Thus, $\lim\limits_{n\to\infty} s_n = \ln 2$, that is, $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \frac{1}{7\cdot 8} + \cdots = \ln 2$.

19. Let $f(x)$ denote the left-hand side of the equation $1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \frac{x^4}{8!} + \cdots = 0$. If $x \geq 0$, then $f(x) \geq 1$ and there are no solutions of the equation. Note that $f(-x^2) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \cos x$. The solutions of $\cos x = 0$ for $x < 0$ are given by $x = \frac{\pi}{2} - \pi k$, where k is a positive integer. Thus, the solutions of $f(x) = 0$ are $x = -\left(\frac{\pi}{2} - \pi k\right)^2$, where k is a positive integer.

20. Suppose the base of the first right triangle has length a. Then by repeated use of the Pythagorean theorem, we find that the base of the second right triangle has length $\sqrt{1+a^2}$, the base of the third right triangle has length $\sqrt{2+a^2}$, and in general, the nth right triangle has base of length $\sqrt{n-1+a^2}$ and hypotenuse of length $\sqrt{n+a^2}$. Thus, $\theta_n = \tan^{-1}\left(1/\sqrt{n-1+a^2}\right)$ and

$\sum\limits_{n=1}^{\infty} \theta_n = \sum\limits_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{\sqrt{n-1+a^2}}\right) = \sum\limits_{n=0}^{\infty} \tan^{-1}\left(\frac{1}{\sqrt{n+a^2}}\right)$. We wish to show that this series diverges.

First notice that the series $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n+a^2}}$ diverges by the Limit Comparison Test with the divergent p-series $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $\left[p = \frac{1}{2} \leq 1\right]$ since $\lim\limits_{n\to\infty} \frac{1/\sqrt{n+a^2}}{1/\sqrt{n}} = \lim\limits_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+a^2}} = \lim\limits_{n\to\infty} \sqrt{\frac{n}{n+a^2}} = \lim\limits_{n\to\infty} \sqrt{\frac{1}{1+a^2/n}} = 1 > 0$. Thus, $\sum\limits_{n=0}^{\infty} \frac{1}{\sqrt{n+a^2}}$ also diverges. Now $\sum\limits_{n=0}^{\infty} \tan^{-1}\left(\frac{1}{\sqrt{n+a^2}}\right)$ diverges by the Limit Comparison Test with $\sum\limits_{n=0}^{\infty} \frac{1}{\sqrt{n+a^2}}$ since

$$\begin{aligned}
\lim_{n\to\infty} \frac{\tan^{-1}\left(1/\sqrt{n+a^2}\right)}{1/\sqrt{n+a^2}} &= \lim_{x\to\infty} \frac{\tan^{-1}\left(1/\sqrt{x+a^2}\right)}{1/\sqrt{x+a^2}} = \lim_{y\to\infty} \frac{\tan^{-1}(1/y)}{1/y} \qquad \left[y = \sqrt{x+a^2}\,\right] \\
&= \lim_{z\to 0^+} \frac{\tan^{-1} z}{z} \quad [z = 1/y] \quad \overset{\text{H}}{=} \lim_{z\to 0^+} \frac{1/(1+z^2)}{1} = 1 > 0
\end{aligned}$$

Thus, $\sum\limits_{n=1}^{\infty} \theta_n$ is a divergent series.

21. Call the series S. We group the terms according to the number of digits in their denominators:

$$S = \underbrace{\left(\tfrac{1}{1} + \tfrac{1}{2} + \cdots + \tfrac{1}{8} + \tfrac{1}{9}\right)}_{g_1} + \underbrace{\left(\tfrac{1}{11} + \cdots + \tfrac{1}{99}\right)}_{g_2} + \underbrace{\left(\tfrac{1}{111} + \cdots + \tfrac{1}{999}\right)}_{g_3} + \cdots$$

Now in the group g_n, since we have 9 choices for each of the n digits in the denominator, there are 9^n terms.

[continued]

Furthermore, each term in g_n is less than $\frac{1}{10^{n-1}}$ [except for the first term in g_1]. So $g_n < 9^n \cdot \frac{1}{10^{n-1}} = 9\left(\frac{9}{10}\right)^{n-1}$.

Now $\sum\limits_{n=1}^{\infty} 9\left(\frac{9}{10}\right)^{n-1}$ is a geometric series with $a = 9$ and $r = \frac{9}{10} < 1$. Therefore, by the Comparison Test,

$S = \sum\limits_{n=1}^{\infty} g_n < \sum\limits_{n=1}^{\infty} 9\left(\frac{9}{10}\right)^{n-1} = \frac{9}{1 - 9/10} = 90.$

22. (a) Let $f(x) = \dfrac{x}{1 - x - x^2} = \sum\limits_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$. Then

$$\begin{aligned}
x &= (1 - x - x^2)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots) \\
x &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots \\
&\quad - c_0 x - c_1 x^2 - c_2 x^3 - c_3 x^4 - c_4 x^5 - \cdots \\
&\quad - c_0 x^2 - c_1 x^3 - c_2 x^4 - c_3 x^5 - \cdots \\
x &= c_0 + (c_1 - c_0)x + (c_2 - c_1 - c_0)x^2 + (c_3 - c_2 - c_1)x^3 + \cdots
\end{aligned}$$

Comparing coefficients of powers of x gives us $c_0 = 0$ and

$$\begin{aligned}
c_1 - c_0 &= 1 &&\Rightarrow\quad c_1 = c_0 + 1 = 1 \\
c_2 - c_1 - c_0 &= 0 &&\Rightarrow\quad c_2 = c_1 + c_0 = 1 + 0 = 1 \\
c_3 - c_2 - c_1 &= 0 &&\Rightarrow\quad c_3 = c_2 + c_1 = 1 + 1 = 2
\end{aligned}$$

In general, we have $c_n = c_{n-1} + c_{n-2}$ for $n \geq 3$. Each c_n is equal to the nth Fibonacci number, that is,

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=1}^{\infty} c_n x^n = \sum_{n=1}^{\infty} f_n x^n$$

Now $f(x) = \dfrac{x}{1 - x - x^2} = \dfrac{x}{1 - \left(x^2 + x + \frac{1}{4}\right) + \frac{1}{4}} = \dfrac{x}{\frac{5}{4} - \left(x + \frac{1}{2}\right)^2} = \dfrac{4x/5}{1 - \frac{4}{5}\left(x + \frac{1}{2}\right)^2} = \dfrac{4x}{5} \cdot \dfrac{1}{1 - \frac{4}{5}\left(x + \frac{1}{2}\right)^2}$.

The last fraction is the sum of a geometric series with $r = \frac{4}{5}\left(x + \frac{1}{2}\right)^2$, which converges when $\left|\frac{4}{5}\left(x + \frac{1}{2}\right)^2\right| < 1 \quad\Rightarrow$

$-1 < \dfrac{2}{\sqrt{5}}\left(x + \dfrac{1}{2}\right) < 1 \quad\Rightarrow\quad -\dfrac{\sqrt{5}}{2} < x + \dfrac{1}{2} < \dfrac{\sqrt{5}}{2} \quad\Rightarrow\quad \dfrac{-\sqrt{5} - 1}{2} < x < \dfrac{\sqrt{5} - 1}{2}$. By Theorem 11.8.4, the Maclaurin series of $f(x)$ converges when $-R < x < R$. (We are finding the radius of convergence of the Maclaurin series of $f(x)$ as opposed to the Taylor series centered at $x = \frac{1}{2}$.) Thus, the radius of convergence must be $R = \dfrac{\sqrt{5} - 1}{2}$, which ensures the inequality obtained for the geometric series is still satisfied $\left(\text{since } \left|\dfrac{\sqrt{5} - 1}{2}\right| < \left|\dfrac{-\sqrt{5} - 1}{2}\right|\right)$.

Another method: Applying the Ratio Test, we have $\lim\limits_{n \to \infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n \to \infty} \left|\dfrac{c_{n+1}\, x^{n+1}}{c_n x^n}\right| = |x| \lim\limits_{n \to \infty} \dfrac{c_{n+1}}{c_n}$. In Exercise 11.1.89(b), the preceding limit was shown to be $\lim\limits_{n \to \infty} \dfrac{c_{n+1}}{c_n} = \dfrac{1 + \sqrt{5}}{2} = \phi$ (the Golden Ratio). Thus, the Maclaurin series of $f(x)$ converges when $|x|\,\phi < 1$ or $|x| < \dfrac{1}{\phi}$, so the radius of convergence is $R = \dfrac{1}{\phi} = \dfrac{\sqrt{5} - 1}{2}$.

(b) Completing the square on $x^2 + x - 1$ gives us

$$\left(x^2 + x + \frac{1}{4}\right) - 1 - \frac{1}{4} = \left(x + \frac{1}{2}\right)^2 - \frac{5}{4} = \left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2$$
$$= \left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right) = \left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right)$$

So $\dfrac{x}{1 - x - x^2} = \dfrac{-x}{x^2 + x - 1} = \dfrac{-x}{\left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right)}$. The factors in the denominator are linear,

so the partial fraction decomposition is

$$\frac{-x}{\left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right)} = \frac{A}{x + \frac{1+\sqrt{5}}{2}} + \frac{B}{x + \frac{1-\sqrt{5}}{2}} \quad -x = A\left(x + \tfrac{1-\sqrt{5}}{2}\right) + B\left(x + \tfrac{1+\sqrt{5}}{2}\right)$$

If $x = \frac{-1+\sqrt{5}}{2}$, then $-\frac{-1+\sqrt{5}}{2} = B\sqrt{5} \;\Rightarrow\; B = \frac{1-\sqrt{5}}{2\sqrt{5}}$.

If $x = \frac{-1-\sqrt{5}}{2}$, then $-\frac{-1-\sqrt{5}}{2} = A\left(-\sqrt{5}\right) \;\Rightarrow\; A = \frac{1+\sqrt{5}}{-2\sqrt{5}}$. Thus,

$$\frac{x}{1-x-x^2} = \frac{\dfrac{1+\sqrt{5}}{-2\sqrt{5}}}{x + \dfrac{1+\sqrt{5}}{2}} + \frac{\dfrac{1-\sqrt{5}}{2\sqrt{5}}}{x + \dfrac{1-\sqrt{5}}{2}} = \frac{\dfrac{1+\sqrt{5}}{-2\sqrt{5}}}{x + \dfrac{1+\sqrt{5}}{2}} \cdot \frac{\dfrac{2}{1+\sqrt{5}}}{\dfrac{2}{1+\sqrt{5}}} + \frac{\dfrac{1-\sqrt{5}}{2\sqrt{5}}}{x + \dfrac{1-\sqrt{5}}{2}} \cdot \frac{\dfrac{2}{1-\sqrt{5}}}{\dfrac{2}{1-\sqrt{5}}}$$
$$= \frac{-1/\sqrt{5}}{1 + \dfrac{2}{1+\sqrt{5}}x} + \frac{1/\sqrt{5}}{1 + \dfrac{2}{1-\sqrt{5}}x} = -\frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left(-\frac{2}{1+\sqrt{5}}x\right)^n + \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left(-\frac{2}{1-\sqrt{5}}x\right)^n$$
$$= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left[\left(\frac{-2}{1-\sqrt{5}}\right)^n - \left(\frac{-2}{1+\sqrt{5}}\right)^n\right] x^n$$
$$= \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \left[\frac{(-2)^n\left(1+\sqrt{5}\right)^n - (-2)^n\left(1-\sqrt{5}\right)^n}{\left(1-\sqrt{5}\right)^n\left(1+\sqrt{5}\right)^n}\right] x^n \qquad [\text{the } n = 0 \text{ term is } 0]$$
$$= \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \left[\frac{(-2)^n\left(\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n\right)}{(1-5)^n}\right] x^n$$
$$= \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \left[\frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n}\right] x^n \qquad [(-4)^n = (-2)^n \cdot 2^n]$$

From part (a), this series must equal $\sum\limits_{n=1}^{\infty} f_n x^n$, so $f_n = \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$, which is an explicit formula for the nth Fibonacci number.

23. $u = 1 + \dfrac{x^3}{3!} + \dfrac{x^6}{6!} + \dfrac{x^9}{9!} + \cdots$, $v = x + \dfrac{x^4}{4!} + \dfrac{x^7}{7!} + \dfrac{x^{10}}{10!} + \cdots$, $w = \dfrac{x^2}{2!} + \dfrac{x^5}{5!} + \dfrac{x^8}{8!} + \cdots$.

Use the Ratio Test to show that the series for u, v, and w have positive radii of convergence (∞ in each case), so

Theorem 11.9.2 applies, and hence, we may differentiate each of these series:

$$\frac{du}{dx} = \frac{3x^2}{3!} + \frac{6x^5}{6!} + \frac{9x^8}{9!} + \cdots = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots = w$$

Similarly, $\dfrac{dv}{dx} = 1 + \dfrac{x^3}{3!} + \dfrac{x^6}{6!} + \dfrac{x^9}{9!} + \cdots = u$, and $\dfrac{dw}{dx} = x + \dfrac{x^4}{4!} + \dfrac{x^7}{7!} + \dfrac{x^{10}}{10!} + \cdots = v$.

So $u' = w$, $v' = u$, and $w' = v$. Now differentiate the left-hand side of the desired equation:

$$\begin{aligned}\frac{d}{dx}(u^3 + v^3 + w^3 - 3uvw) &= 3u^2u' + 3v^2v' + 3w^2w' - 3(u'vw + uv'w + uvw') \\ &= 3u^2w + 3v^2u + 3w^2v - 3(vw^2 + u^2w + uv^2) = 0 \quad \Rightarrow\end{aligned}$$

$u^3 + v^3 + w^3 - 3uvw = C$. To find the value of the constant C, we put $x = 0$ in the last equation and get $1^3 + 0^3 + 0^3 - 3(1 \cdot 0 \cdot 0) = C \quad \Rightarrow \quad C = 1$, so $u^3 + v^3 + w^3 - 3uvw = 1$.

24. **To prove:** If $n > 1$, then the nth partial sum $s_n = \sum\limits_{i=1}^{n} \dfrac{1}{i}$ of the harmonic series is not an integer.

Proof: Let 2^k be the largest power of 2 that is less than or equal to n and let M be the product of all the odd positive integers that are less than or equal to n. Suppose that $s_n = m$, an integer. Then $M2^k s_n = M2^k m$. Since $n \geq 2$, we have $k \geq 1$, and hence, $M2^k m$ is an even integer. We will show that $M2^k s_n$ is an odd integer, contradicting the equality $M2^k s_n = M2^k m$ and showing that the supposition that s_n is an integer must have been wrong.

$M2^k s_n = M2^k \sum\limits_{i=1}^{n} \dfrac{1}{i} = \sum\limits_{i=1}^{n} \dfrac{M2^k}{i}$. If $1 \leq i \leq n$ and i is odd, then $\dfrac{M}{i}$ is an odd integer since i is one of the odd integers that were multiplied together to form M. Thus, $\dfrac{M2^k}{i}$ is an even integer in this case. If $1 \leq i \leq n$ and i is even, then we can write $i = 2^r l$, where 2^r is the largest power of 2 dividing i and l is odd. If $r < k$, then $\dfrac{M2^k}{i} = \dfrac{2^k}{2^r} \cdot \dfrac{M}{l} = 2^{k-r}\dfrac{M}{l}$, which is an even integer, the product of the even integer 2^{k-r} and the odd integer $\dfrac{M}{l}$. If $r = k$, then $l = 1$, since $l > 1 = l \geq 2 \quad \Rightarrow$ $i = 2^k l \geq 2^k \cdot 2 = 2^{k+1}$, contrary to the choice of 2^k as the largest power of 2 that is less than or equal to n. This shows that $r = k$ only when $i = 2^k$. In that case, $\dfrac{M2^k}{i} = M$, an *odd* integer. Since $\dfrac{M2^k}{i}$ is an even integer for every i except 2^k and $\dfrac{M2^k}{i}$ is an odd integer when $i = 2^k$, we see that $M2^k s_n$ is an odd integer. This concludes the proof.