National Taiwan University - Calculus 3 Class 09 (Wed) - Quiz 2
3/27/2024 - 50 minutes

Name: ID: Department:

There are **FOUR** questions in this quiz.
Your work is graded on the quality of your writing as well as the validity of the mathematics.

1. Assume that the equation
$$x + x^7 z - yz^8 + y^3 = 0$$
defines z as a function of x, y, i.e. $z = z(x, y)$, near $(x, y, z) = (0, 1, 1)$.

   (a) (10%) Compute $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(x, y, z) = (0, 1, 1)$.

   (b) (4%) Find the maximum directional derivative of $z(x, y)$ at $(x, y) = (0, 1)$.

   (c) (4%) Estimate $z(-0.2, 0.9)$ by the linear approximation.

   (d) (6%) Let $f(t) = z(t^2 + 3t,\ e^{2t})$. Compute $f'(0)$.

Sol:

(a) Solution 1 : Let $F(x, y, z) = x + x^7 z - yz^8 + y^3$. Then by the implicit differentiation, we know that
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1 + 7x^6 z}{x^7 - 8yz^7}\ , \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-z^8 + 3y^2}{x^7 - 8yz^7}.$$
Hence, at $(x, y, z) = (0, 1, 1)$, $\frac{\partial z}{\partial x} = \frac{1}{8}$, and $\frac{\partial z}{\partial y} = \frac{1}{4}$.
(2 pts for $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$. 2 pts for $F_x = 1 + 7x^6 z$. 2 pts for $F_y = -z^8 + 3y^2$. 2 pts for $f_z = x^7 - 8yz^7$. 2 pts for evaluating at $(x, y, z) = (0, 1, 1)$.)
Solution 2: Consider the equation
$$x + x^7 z(x, y) - y(z(x, y))^8 + y^3 = 0$$
Differentiate the equation with respect to x. We obtain
$$1 + 7x^6 z + x^7 z_x - 8yz^7 z_x = 0. \qquad \text{(2 pts)}$$
Thus $z_x = -\frac{1 + 7x^6 z}{x^7 - 8yz^7}$ and at $(x, y, z) = (0, 1, 1)$, $z_x = 1/8$.
(2 pts for solving z_x and 1 pt for evaluating at $(x, y, z) = (0, 1, 1)$.) Similarly, after differentiating the equation with respect to y, we obtain
$$x^7 z_y - z^8 - 8yz^7 z_y + 3y^2 = 0. \qquad \text{(2 pts)}$$
Thus $z_= -\frac{-z^8 + 3y^2}{x^7 - 8yz^7}$ and at $(x, y, z) = (0, 1, 1)$, $z_y = 1/4$.
(2 pts for solving z_y and 1 pt for evaluating at $(x, y, z) = (0, 1, 1)$.)

(b) The maximal directional derivative of $z(x, y)$ at $(x, y) = (0, 1)$ is the length of $\nabla z(0, 1)$. (2 pts)
And $|\nabla z(0, 1)| = \sqrt{(\frac{1}{4})^2 + (\frac{1}{8})^2} = \frac{\sqrt{5}}{8}$. (2 pts)

(c) The linearization of $z(x, y)$ at $(x, y) = (0, 1)$ is $L(x, y) = z(0, 1) + z_x(0, 1)(x - 0) + z_y(0, 1)(y - 1)$. Hence,
$$z(-0.2, 0.9) \approx z(0, 1) + z_x(0, 1) \times (-0.2) + z_y(0, 1) \times (0.9 - 1) \qquad \text{(2 pts)}$$
$$= 1 - \frac{1}{8} \times 0.2 - \frac{1}{4} \times 0.1 = 0.95 \qquad \text{(2 pts)}$$

(d)

$$f'(t) = \frac{\partial z}{\partial x}(t^2+3t, e^{2t}) \times (2t+3) + \frac{\partial z}{\partial y}(t^2+3t, e^{2t}) \times 2e^{2t} \qquad \text{(4 pts)}$$

When $t=0$, $(t^2+3t, e^{2t}) = (0,1)$ and $f'(0) = \frac{1}{8} \times 3 + \frac{1}{4} \times 2 = \frac{7}{8}$. (2 pts)

2. Suppose that $\nabla f(x,y,z) = 2xy\mathbf{i} \ + \ (x^2+z)\mathbf{j} \ + \ y\mathbf{k}$ and $f(1,1,1) = 0$.

(a) (6%) Find the tangent plane to the level surface $f(x,y,z) = 0$ at $(x,y,z) = (1,1,1)$.

(b) (10%) Suppose that curve C is the intersection of the level surface $f(x,y,z) = 0$ and the plane $x+2y+z=4$. Find the tangent line of C at $(x,y,z) = (1,1,1)$.

(c) (8%) (Continued) Assume that a differentiable function $h(x,y,z)$ obtains local maximum at $(x,y,z) = (1,1,1)$ when restricted on the curve C. Circle ALL correct statement(s).

i. $\nabla h(1,1,1)$ is normal to the tangent line of C at $(x,y,z) = (1,1,1)$.

ii. $\nabla h(1,1,1)$ is parallel to the tangent line of C at $(x,y,z) = (1,1,1)$.

iii. $\nabla h(1,1,1)$ is normal to both $\nabla f(1,1,1)$ and $\mathbf{i}+2\mathbf{j}+\mathbf{k}$.

iv. $\nabla h(1,1,1)$ lies in the plane spanned by $\nabla f(1,1,1)$ and $\mathbf{i}+2\mathbf{j}+\mathbf{k}$.

Sol:

(a) $\nabla f(1,1,1) = 2\mathbf{i} \ + \ 2\mathbf{j} \ + \ \mathbf{k}$ is normal to the level surface $f(x,y,z) = 0$ at $(x,y,z) = (1,1,1)$. (3 pts) Hence the tangent plane is $2(x-1) + 2(y-1) + (z-1) = 0$ which is $2x+2y+z=5$. (3 pts)

(b) The tangent line of C lies on both the tangent plane of $f(x,y,z) = 0$ and the plane $x+2y+z=4$. Therefore the tangent line of C is both normal to $\nabla f(1,1,1)$ and $\mathbf{i} \ + \ 2\mathbf{j} \ + \ \mathbf{k}$ which means that the tangent line is parallel to $(2\mathbf{i} \ + \ 2\mathbf{j} \ + \ \mathbf{k}) \times (\mathbf{i} \ + \ 2\mathbf{j} \ + \ \mathbf{k}) = -\mathbf{j} \ + \ 2\mathbf{k}$. (6 pts)
The tangent line of C is

$$(x(t), y(t), z(t)) = (1, \ 1-t, \ 1+2t) \ , \quad t \in \mathbf{R} \qquad \text{(4 pts)}$$

(c) C is defined by two equations $f(x,y,z) = 0$ and $x+2y+z=4$. Hence by Lagrange's multiplier method, we know that at the local maximizer $(1,1,1)$, $\nabla h(1,1,1) = \lambda \nabla f(1,1,1) + \mu(\mathbf{i}+2\mathbf{j}+\mathbf{k})$ which means that $\nabla h(1,1,1)$ lies in the plane spanned by $\nabla f(1,1,1)$ and $\mathbf{i}+2\mathbf{j}+\mathbf{k}$. Moreover, if C has a parametrization $\gamma(t)$ with $\gamma(0) = (1,1,1)$, then $h(\gamma(t))$ obtains local maximum at $t=0$. Hence
$\frac{d}{dt} h(\gamma(t))|_{t=0} = \nabla h(1,1,1) \cdot \gamma'(0) = 0$ which means that $\nabla h(1,1,1)$ is normal to the tangent line of C at $(x,y,z) = (1,1,1)$.
Therefore, (1) (iv) are correct statements and (ii) (iii) are false statements.
(2 pts for choosing (i).
2 pts for not choosing (ii).
2 pts for not choosing (iii).
2 pts for choosing (iv).)

3. Find the absolute extreme value of $f(x, y, z) = ze^{-xy}$ on the region $D = \{(x, y, z) \mid x^2 + 4y^2 + z^2 \le 4\}$.

   (a) (5%) Is there any critical point of f on D?

   (b) (20%) Use Lagrange multiplier method to find the extreme value of f on the boundary of D which is $x^2 + 4y^2 + z^2 = 4$. Then find the absolute extreme values of f on D.

Sol:

(a) $f_x = -yze^{-xy}$, $f_y = -xze^{-xy}$, $f_z = e^{-xy}$. The critical points are points at which $f_x = f_y = f_z = 0$. Since $f_z = e^{-xy} > 0$ for all (x, y, z), we conclude that there are no critical points of f on D.
(3 pts for f_x, f_y, f_z. 2 pts for showing that there are no critical points)

(b) Let $g(x, y, z) = x^2 + 4y^2 + z^2$. Lagrange multiplier method suggests that we shall solve the system of equations.

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 4 \end{cases} \Longrightarrow \begin{cases} -yze^{-xy} = \lambda 2x \\ -xze^{-xy} = \lambda 8y \\ e^{-xy} = \lambda 2z \\ x^2 + 4y^2 + z^2 = 4 \end{cases} \qquad \text{(5 pts)}$$

The third equation tells that $\lambda z > 0$ i.e. λ, z are not zero and they have the same sign.

The first and second equations may have solution $x = y = 0$. Then the fourth equation derives that $z = \pm 2$. And the third equation solves $\lambda = 1/4$ when $z = 2$ and $\lambda = -1/4$ when $z = -2$. Therefore $(x, y, z, \lambda) = (0, 0, 2, 1/4)$, $(0, 0, -2, -1/4)$ are solutions and $f(0, 0, 2) = 2$, $f(0, 0, -2) = -2$. (4 pts)

If $x, y \ne 0$, we divide the first equation by the second equation and obtain that $\dfrac{y}{x} = \dfrac{x}{4y}$. Moreover, z and λ have the same sign. Thus the first equation says that x and y have different signs. Hence, $x = -2y$. Now we solve z from the first equation:

$$z = -2\lambda e^{xy}\frac{x}{y} = 4\lambda e^{xy}.$$

The third equation solves $z = \dfrac{1}{2\lambda e^{xy}}$. Therefore, $\lambda e^{xy} = \pm\dfrac{1}{2\sqrt{2}}$ and $z = \pm\sqrt{2}$. From the fourth equation, we have $x^2 = 4y^2 = 1$. Thus the solutions are
$(x, y, z, \lambda) = (1, -\frac{1}{2}, \sqrt{2}, \frac{\sqrt{2}}{4}e^{\frac{1}{2}})$, $(1, -\frac{1}{2}, -\sqrt{2}, -\frac{\sqrt{2}}{4}e^{\frac{1}{2}})$, $(-1, \frac{1}{2}, \sqrt{2}, \frac{\sqrt{2}}{4}e^{\frac{1}{2}})$, $(-1, \sqrt{2}, -\sqrt{2}, -\frac{\sqrt{2}}{4}e^{\frac{1}{2}})$
(2 pts for each solution.)
$f(1, -\frac{1}{2}, \sqrt{2}) = f(-1, \frac{1}{2}, \sqrt{2}) = \sqrt{2e}$ and $f(1, -\frac{1}{2}, -\sqrt{2}) = f(-1, \frac{1}{2}, -\sqrt{2}) = -\sqrt{2e}$. Since

$$-\sqrt{2e} < f(0, 0, -2) = -2 < f(0, 0, 2) = 2 < \sqrt{2e},$$

we know that the absolute maximum value of f on the boundary of D is $\sqrt{2e}$ and the absolute minimum value of f on the boundary of D is $-\sqrt{2e}$. (2 pts)

Because f has no critical points on D, the absolute extreme values occur on the boundary of D. Therefore the absolute maximum value of f on D is $\sqrt{2e}$ and the absolute minimum value of f on D is $-\sqrt{2e}$. (1 pt)

4. (a) (12%) Compute $\int_0^1 \int_{\arcsin y}^{\frac{\pi}{2}} \cos x \, e^{\cos^2 x} \, dxdy$.

(b) (15%) Find the volume of the tetrahedron bounded by $2x + y + z = 2$ and coordinate planes. (You must compute the volume by a double integral.)

Sol:

(a) $\int_0^1 \int_{\arcsin y}^{\frac{\pi}{2}} \cos x \, e^{\cos^2 x} \, dxdy = \iint_D \cos x e^{\cos^2 x} \, dA$ where D is bounded by $y = \sin x$, $y = 0$ and $0 \leq x \leq \frac{\pi}{2}$.
Hence

$$\int_0^1 \int_{\arcsin y}^{\frac{\pi}{2}} \cos x \, e^{\cos^2 x} \, dxdy = \int_0^{\frac{\pi}{2}} \int_0^{\sin x} \cos x \, e^{\cos^2 x} \, dydx \qquad \text{(5 pts)}$$

$$= \int_0^{\frac{\pi}{2}} \sin x \cos x \, e^{\cos^2 x} \, dx \qquad \text{(2 pts)}$$

$$\overset{u=\cos^2 x}{=} \int_1^0 e^u \left(-\frac{1}{2}\right) du = \frac{e-1}{2}. \qquad \text{(5 pts)}$$

(b) The project of the tetrahedron, E, onto the xy-plane is a triangle D bounded by the lines $x = 0$, $y = 0$ and $2x + y = 2$. Then $E = \{(x, y, z) \mid (x, y) \in D, \ 0 \leq z \leq 2 - 2x - y\}$ (3 pts)

We can write D as a type I region, $D = \{(x, y) \mid 0 \leq x \leq 1, \ 0 \leq y \leq 2 - 2x\}$. (3 pts)

Hence

$$\mathrm{V}(E) = \iiint_E 1 \, dV \qquad \text{(2 pts)}$$

$$= \iint_D \int_0^{2-2x-y} 1 \, dzdA \qquad \text{(2 pts)}$$

$$= \int_0^1 \int_0^{2-2x} 2 - 2x - y \, dydx \qquad \text{(2 pts)}$$

$$= \int_0^1 \frac{1}{2}(2-2x)^2 \, dx = \frac{2}{3} \qquad \text{(3 pts)}$$

We can also write D as a type II region, $D = \{(x, y) \mid 0 \leq y \leq 2, \ 0 \leq x \leq \frac{2-y}{2}\}$.
Then $\mathrm{V}(E) = \int_0^2 \int_0^{\frac{2-y}{2}} \int_0^{2-2x-y} 1 \, dzdxdy$.