Calculus IV Quiz 2

Department: Name: Id:

1. **Fundamental theorem of Calculus - Curl** (25 points)
   Let **F**(x,y,z) $= xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ be a vector field on $\mathbb{R}^3$.
   S be the curved surface $\{(x, y, z) | x \geq 0,\ y \geq 0,\ z \geq 0,\ z = 1 - x^2 - y^2\}$.
   C be the boundary of S, with counterclockwise direction when looking from positive z-direction to negative z-direction.
   Answer the following questions.

   (a) (5 %) Sketch S and C. Also, assuming C is the positive orientation of S, point out the orientation of S.

   (b) (5 %) Find the curl of **F**.

   (c) (15 %) Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ by using the Stoke's Theorem. (*Hint:* You may want to first parametrize the curved surface S. Simultaneously, please notice whether the orientation defined in your integral is the same as you got from problem a.)

   (a) Please refer to the Exercises 16.8 Problem 9 in the textbook.

   (b) curl $\mathbf{F} = < -y, -z, -x >$

   (c) Let $\mathbf{r}(x, y) = < x, y, 1 - x^2 - y^2 >$ be a parametrization. We have:

$$
\begin{aligned}
&0 \leq x \leq \sqrt{1 - y^2} \\
&0 \leq y \leq 1 \\
&\mathbf{r}_x = < 1, 0, -2x > . \\
&\mathbf{r}_y = < 0, 1, -2y > . \\
&\mathbf{r}_x \times \mathbf{r}_y = < 2x,\ 2y,\ 1 > .
\end{aligned}
$$

   which is the correct orientation with respect to the boundary C.
   Therefore, we can calculate the line integral by the Stoke's Theorem.

$$
\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \\
&= \int_0^1 \int_0^{\sqrt{1-y^2}} -2xy - 2y(1 - x^2 - y^2) - x \, dx \, dy \\
&= \frac{-17}{20}.
\end{aligned}
$$

2. **Fundamental theorem of Passing Calculus IV - Power Series** (20 points)

(a) (5 %) Let $f(x) = \sum_{n=1}^{\infty} c_n x^n$ be a power series, with $c_n \in \mathbb{R}$. Show that if $\lim_{n\to\infty} \left| \frac{c_n}{c_{n-1}} \right| = L$ exists and not equals to 0, then the radius of convergence for this power series is $1/L$. (*Hint:* Please don't be afraid of the problem. This is very easy if you can successfully use the right test to justify the claim.)

(b) (15 %) For what values of x is the series $\sum_{n=1}^{\infty} \frac{x^{2n+1}}{3n^{3/4}}$ convergent?

(a) Applying ratio test, to make the series converge, it must at least satisfy $\lim_{n\to\infty} \left| \frac{c_n x^n}{c_{n-1} x^{n-1}} \right| \leq 1$. Therefore $|x| \leq 1/L$, which implies the radius of convergence is $1/L$.

(b) Applying the ratio test, to make the series converge, it must at least satisfy

$$\lim_{n\to\infty} \left| \frac{x^{2n+1}}{3n^{3/4}} \right| \left| \frac{3(n-1)^{3/4}}{x^{2n-1}} \right| \leq 1$$

which gives $x^2 \leq 1$, so we can ensure for every x, satisfying $-1 < x < 1$, the series would converge.
For the endpoint,

- $x = 1$: The series would be $\sum_{n=1}^{\infty} \frac{1}{3n^{3/4}}$, which doesn't converge since $3/4 < 1$.
- $x = -1$: The series would be $\sum_{n=1}^{\infty} \frac{-1}{3n^{3/4}} = -\sum_{n=1}^{\infty} \frac{1}{3n^{3/4}}$, which doesn't converge since $\sum_{n=1}^{\infty} \frac{1}{3n^{3/4}}$ is not convergent.

Therefore, the answer is $x \in (-1, 1)$.

3. **Fundamental theorem of Calculus - Divergence** (40 points):
Let **F**(x,y,z) $= < e^x, -ye^x + xy(y^2+z^2)^{3/2}, xz(y^2+z^2)^{3/2} >$ be a vector field on $\mathbb{R}^3$.
S be part of the surface $x^2+y^2+z^2 = 16$, which lies within the cylinder $y^2+z^2 = 12$ and satisfies $x > 0$.
V be part of the ball $x^2+y^2+z^2 \leq 16$, which satisfies $x \geq 2$.
Answer the following questions.

(a) (5 %) Calculate the divergence of **F**(x,y,z).

(b) (15 %) Following the result from (a), please derive the triple integral $\iiint_V \nabla\cdot\mathbf{F}(x,y,z)\,dV$.

(c) (10%) Let S' be the disk $S' = \{(x,y,z)|\, x = 2,\, y^2+z^2 \leq 12\}$. Calculate $\iint_{S'} \mathbf{F}(x,y,z)\cdot d\mathbf{S'}$., where the direction of **S'** is pointed toward the negative x-direction.

(d) (10 %) Apply the divergence theorem, and find $\iint_S \mathbf{F}(x,y,z)\cdot d\mathbf{S}$., where the direction of **S** is always pointed outward with respect to the ball $x^2+y^2+z^2 \leq 16$.

(a)

$$\begin{aligned}
\nabla\cdot\mathbf{F}(x,y,z) &= e^x - e^x + x(y^2+z^2)^{3/2} + \frac{3}{2}xy(y^2+z^2)^{1/2}2y \\
&+ x(y^2+z^2)^{3/2} + \frac{3}{2}xz(y^2+z^2)^{1/2}2z \\
&= 5x(y^2+z^2)^{3/2}.
\end{aligned}$$

(b) Apply the cylindrical coordinate but with $x = x$, $y = r\cos\theta$, $z = r\sin\theta$.

$$\begin{aligned}
&0 \leq \theta \leq 2\pi. \\
&0 \leq r \leq \sqrt{12}. \\
&2 \leq x \leq \sqrt{16-r^2}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\iiint_V \nabla\cdot\mathbf{F}(x,y,z)\,dV &= \int_0^{2\pi}\int_0^{\sqrt{12}}\int_2^{\sqrt{16-r^2}} 5xr^3\,dx\,r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{12}} \frac{5}{2}((16-r^2)-4)r^4\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{12}} 30r^4 - \frac{5r^6}{2}\,dr\,d\theta \\
&= \int_0^{2\pi} 6\sqrt{12}\cdot 12^2 - \frac{5}{14}\sqrt{12}\cdot 12^3\,d\theta \\
&= (6\sqrt{12}\cdot 12^2 - \frac{5}{14}\sqrt{12}12^3)\cdot 2\pi \text{ (This answer can be viewed correct.)} \\
&= 12^3\sqrt{12}\cdot\pi\cdot\frac{2}{7}.
\end{aligned}$$

(c) Again apply the cylindrical parametrization ($0 \leq r \leq 2\sqrt{3}$):

$$\mathbf{r}(\theta, r) =< 2, r\cos\theta, r\sin\theta >$$

$$\mathbf{r}_\theta \times \mathbf{r}_r =< -r, 0, 0 >= r < -1, 0, 0 >$$

We have

$$\begin{aligned}
\iint_{S'} \mathbf{F}(x,y,z) \cdot \, d\mathbf{S'} &= \iint_{S'} \mathbf{F}(x,y,z) \cdot \, (\mathbf{r}_\theta \times \mathbf{r}_r) \\
&= \int_0^{2\pi} \int_0^{2\sqrt{3}} -re^2 \, dr \, d\theta \\
&= 2\pi \frac{12-0}{-2} e^2 = -12\pi e^2.
\end{aligned}$$

(d) Notice that $\mathbf{S} \bigcup \mathbf{S'}$ is the positive oriented boundary of V. Hence, we have

$$\begin{aligned}
\iint_S \mathbf{F}(x,y,z) \cdot \, d\mathbf{S} &= \iiint_V \nabla \cdot \mathbf{F}(x,y,z) \, dV - \iint_{S'} \mathbf{F}(x,y,z) \cdot \, d\mathbf{S'} \\
&= 12^3\sqrt{12} \cdot \pi \cdot \frac{2}{7} + 12\pi e^2.
\end{aligned}$$

4. **Fundamental theorem of Learning Calculus Well - Webwork** (15 points)
Let **F** be a **radial** vector field on $\mathbb{R}^3$. S_1 is a sphere of radius 5 centered at the origin, with the flux $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = 9$.
S_2 is a sphere of radius 40 centered at the origin, and consider the flux integral $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = I$
The orientation of S_1 and S_2 are always pointed away from the origin.

(a) (7 %) If the magnitude of **F** is proportional to the inverse square of the distance from the origin, find the value of I.

(b) (8 %) If the magnitude of **F** is proportional to the distance from the origin, find the value of I.

In this problem, we can let $\mathbf{F} = A(r)B(\theta, \phi)\hat{r}$.

(a) In this case, $A(r) = \dfrac{1}{r^2}$. Therefore, we can write down the flux with respect to S_1 and S_2.

$$\int_0^{2\pi} \int_0^{\pi} \frac{1}{r^2} B(\theta, \phi)\, r^2 \, \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi} B(\theta, \phi) \, \sin\phi \, d\phi \, d\theta.$$

which is independent of r (the radius of the sphere). Hence, we will get $I = 9$.

(b) In this case, $A(r) = r$. Therefore, we can write down the flux with respect to S_1 and S_2.

$$\int_0^{2\pi} \int_0^{\pi} r B(\theta, \phi)\, r^2 \, \sin\phi \, d\phi \, d\theta = r^3 \int_0^{2\pi} \int_0^{\pi} B(\theta, \phi) \, \sin\phi \, d\phi \, d\theta.$$

which is proportional to r^3. Therefore, we have

$$\frac{40^3}{5^3} = \frac{I}{9}.$$

$$I = 9 \times 8^3.$$