National Taiwan University - Calculus 4 for Class 01-09

Worksheet 1 : More on Conservative Fields

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Supplementary reference : Stewart §16.3, 16.4

First, we review the definition and the key result concerning a conservative vector field.

Review of Conservative Fields

(1) **Definition.**

A vector field $\mathbf{F}$ on $D \subseteq \mathbb{R}^2$ (or $\mathbb{R}^3$) is said to be conservative on a region D (or $\mathbb{R}^3$) if there exists a scalar function $f : D \to \mathbb{R}$ such that $\nabla f = \mathbf{F}$. In other words, a vector field is conservative on a given region if it can be recognized as the gradient of some scalar function that is well-defined on D. We call f a *scalar potential* function of $\mathbf{F}$.

(2) **Fundamental Theorem of Line Integrals.**

An important feature of conservative fields is that 'energy' is conserved across any closed curve inside the region. To be more precise,

$$\boxed{\mathbf{F} \text{ is conservative on } D.} \Leftrightarrow \boxed{\oint_C \mathbf{F}\cdot d\mathbf{r} = 0 \text{ for every closed curve } C \text{ inside } D.} \Leftrightarrow \boxed{\int_C \mathbf{F}\cdot d\mathbf{r} \text{ depends on only the starting and ending points of } C \text{ (path-independent).}}$$

(3) **Theorem.** At the outset, let $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a C^1-vector field. Recall that Clairaut's Theorem can be used to deduce that

$$\boxed{\mathbf{F} \text{ is conservative on } D.} \Rightarrow \boxed{Q_x - P_y = 0.}$$

However, the converse is incorrect. For the converse to be correct, we require the region D to be <u>simply-connected</u>. Indeed, if D is <u>simply-connected</u>, then we have

$$\boxed{\mathbf{F} \text{ is conservative on } D.} \Leftrightarrow \boxed{Q_x - P_y = 0.}$$

Reminder. A scalar function f is called C^1 if all its first order partial deriatives are continuous.

Exercise 1. Consider the vector field $\mathbf{F}(x,y) = -\dfrac{y}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j}$.

(a) Is the region $D_1 = \{(x,y) : x > 0\}$ simply-connected?

(b) Determine whether $\mathbf{F}$ is conservative on D_1. Find a scalar potential function of $\mathbf{F}$ if it is conservative.

(a) yes, the region D_1 is simply connected.

(b) $Q_x - P_y = \dfrac{2xy}{(x^2+y^2)^2} - \dfrac{2xy}{(x^2+y^2)} = 0 \Rightarrow$ conservative

$$\begin{cases} F_x = \dfrac{-y}{x^2+y^2} \\ F_y = \dfrac{x}{x^2+y^2} \end{cases} \Rightarrow F(x,y) = \int \frac{x}{x^2+y^2}\,dy = x\left(\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right) + h(x)$$

As $F_x = \dfrac{-y}{x^2+y^2} \Rightarrow \left(\tan^{-1}\left(\frac{y}{x}\right) + h(x)\right)_x = \dfrac{\frac{-y^2}{x}}{1+\left(\frac{y}{x}\right)^2} + h'(x)$

$\therefore h'(x) = 0 \Rightarrow h(x) = C \Rightarrow F = \tan^{-1}\left(\frac{y}{x}\right) + C,\ C \in \mathbb{R}$

(TBD)

Exercise 2. We continue to consider the vector field $\mathbf{F}(x,y) = -\dfrac{y}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j}$.

(a) Compute, directly, the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where C is the unit circle $x^2+y^2=1$, oriented counterclockwise.

(b) Determine whether $\mathbf{F}$ is conservative on $D_2 = \mathbb{R}^2 \setminus \{(0,0)\}$.

Moral of Exercise 1 and 2 : The conservativeness of a vector field is very sensitive to the region D. The same vector field may lose its conservativeness if we view it on a different domain.

(a) $F(x,y) = \frac{1}{x^2+y^2}\langle -y, x\rangle$

let $\vec{r}(t) = \langle r\cos t, r\sin t\rangle \Rightarrow \vec{r}\,'(t) = \langle -r\sin t, r\cos t\rangle$, $0 \le t \le 2\pi$

$\oint_C \vec{F}\cdot d\vec{r} \overset{def}{=\!=} \int_0^{2\pi} \frac{1}{r^2}\left(\langle -r\sin t, r\cos t\rangle \cdot \langle -r\sin t, r\cos t\rangle\right) dt = \int_0^{2\pi} \frac{1}{r^2} \times r^2\, dt = 2\pi$ ※

(b) if we take the curve from (a), $\oint_C \vec{F}\cdot d\vec{r} = 2\pi \neq 0$

for any conservative field, $\oint_{C_1} \vec{F}\cdot d\vec{r} = 0$ for any curve C_1

As $2\pi \neq 0 \Rightarrow F$ is not conservative on D_2

Exercise 3. Review from the lecture the following proof of the statement : if D is simply-connected, then we have

$$\boxed{Q_x - P_y = 0.} \Rightarrow \boxed{\mathbf{F} \text{ is conservative on } D.}$$

Proof. Suppose $Q_x - P_y = 0$. Let C_1 and C_2 be two paths with the same starting and ending points and for simplicity we assume that C_1 and C_2 has no other intersections. Then $C = C_1 \cup (-C_2)$ forms a simple closed curve and let R be the curve enclosed by C. By Green's Theorem, $$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R Q_x - P_y \, dA = 0 \Rightarrow \int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}.$$ This implies that line integrals of $\mathbf{F}$ are path-independent. Hence, $\mathbf{F}$ is conservative on D. **Q.E.D.**

Point out where in the proof we have used crucially the condition that D is simply-connected.

Only when D is simply-connected

we can say $Q_x - P_y = 0$ implies F is conservative

and with F is conservative $\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \iint_D Q_x - P_y \, dA = 0$

Summary. Let $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a C^1-vector field. To address the question

'*Is* $\mathbf{F}$ *conservative on* D *?*',

- if D is simply-connected, the conservativeness of $\mathbf{F}$ is completely determined by $Q_x - P_y$. In particular, in this case,

$$\boxed{\mathbf{F} \text{ is conservative on } D.} \Leftrightarrow \boxed{Q_x - P_y = 0.}$$

- if D is not simply-connected, then in this case, we only have the implication :

$$\boxed{\mathbf{F} \text{ is conservative on } D.} \Rightarrow \boxed{Q_x - P_y = 0.}$$

To determine whether $\mathbf{F}$ is conservative or not on D, sorry, we would need to stick with our old trick : to compute line integrals. In particular,

$$\boxed{\mathbf{F} \text{ is conservative on } D.} \Leftrightarrow \boxed{\begin{array}{l} \oint_C \mathbf{F}\cdot d\mathbf{r} = 0 \\ \text{for every closed curve } C \text{ inside } D. \end{array}}$$

Exercise 4. Let $\mathbf{F}(x,y) = \dfrac{y}{4x^2+9y^2}\mathbf{i} - \dfrac{x}{4x^2+9y^2}\mathbf{j}$. Let E be the ellipse $4x^2+9y^2=1$, oriented counterclockwise.

(a) Evaluate directly $\oint_E \mathbf{F}\cdot d\mathbf{r}$.

(b) Determine whether $\mathbf{F}$ is conservative on each of the following regions.

  (a) $D_1 = \mathbb{R}^2 \setminus \{(0,0)\}$

  (b) $D_2 = \{(x,y) \in \mathbb{R}^2 \,:\, x > 0\}$.

Justify your claims. In the case if $\mathbf{F}$ is conservative, find its scalar potential function.

(a) $F = \dfrac{1}{4x^2+9y^2}\langle y, -x\rangle$

1° parametrize E as $\vec{r}(t) = \langle \frac{1}{2}\cos t, \frac{1}{3}\sin t\rangle$, $0 \le t \le 2\pi$

$\vec{r}'(t) = \langle \frac{-1}{2}\sin t, \frac{1}{3}\cos t\rangle$

2° $\oint_E \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle \frac{1}{3}\sin t, \frac{-1}{2}\cos t\rangle \cdot \langle \frac{-1}{2}\sin t, \frac{1}{3}\cos t\rangle \, dt = \frac{-1}{6}\int_0^{2\pi} 1 \, dt = \frac{-\pi}{3}$

(b) $F = \left\langle \dfrac{y}{4x^2+9y^2}, \dfrac{x}{4x^2+9y^2}\right\rangle$ let $P = \dfrac{y}{4x^2+9y^2}$, $Q = \dfrac{-x}{4x^2+9y^2}$

$Q_x - P_y = \frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P = \dfrac{-(4x^2+9y^2)+x(8x)}{(4x^2+9y^2)^2} - \dfrac{(4x^2+9y^2)-y(18y)}{(4x^2+9y^2)^2} = 0$

(a) As D_1 is not simply-connected, $\oint_E \vec{F}\cdot d\vec{r} \stackrel{\text{def}}{=} \frac{-\pi}{3} \neq 0 \Rightarrow F$ is not conservative on D_1

(b) As D_2 is simply connected and $Q_x - P_y = 0 \Rightarrow F$ is conservative on D_2

$F_x = \dfrac{y}{4x^2+9y^2} \Rightarrow F(x,y) = \int \dfrac{y}{4x^2+9y^2}\,dx = \frac{1}{6}\tan^{-1}\left(\frac{2x}{3y}\right) + C$

the result is same for computing from F_y

$\therefore$ F has scalar potential function of $\frac{1}{6}\tan^{-1}\left(\frac{2x}{3y}\right) + C$, $C \in \mathbb{R}$

(TBD)

(c) Let $\mathbf{G}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a C^1-vector field on $\mathbb{R}^2 \setminus \{(0,0)\}$. It is known that $\mathbf{G}$ satisfies :

$$(1)\ Q_x = P_y, \qquad (2)\ \oint_E \mathbf{G}\cdot d\mathbf{r} = 4\pi.$$

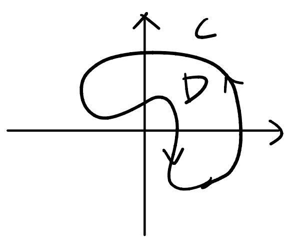
(i) Find the value of k such that $\oint_E (\mathbf{G} + k\mathbf{F})\cdot d\mathbf{r} = 0$.

$$\oint_E (\vec{G}+k\vec{F})\cdot d\vec{r} = \oint_E \vec{G}\, d\vec{r} + k\oint_E \vec{F}\cdot\vec{r}$$

$$= 4\pi + \frac{-\pi}{3}k = 0 \Rightarrow k = 12$$

(ii) For the value that we found in (c)(i), we are going to prove that $\mathbf{H} = \mathbf{G} + k\mathbf{F}$ is conservative on $\mathbb{R}^2 \setminus \{(0,0)\}$ by the following steps.

(Step I) Prove that $\oint_C \mathbf{H}\cdot \mathrm{d}\mathbf{r} = 0$ for every simple closed curve C that does not enclose the origin.



For the simply-closed curve C, the region enclosed does not enclose the origin

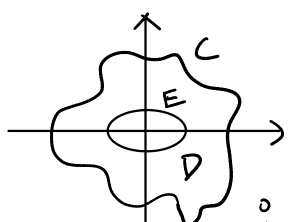
Hence, $P_1(x,y)$, $Q_1(x,y)$ are C^1 on D, wher $P_1(x,y) = \frac{y}{4x^2+9y^2}$, $Q_1(x,y) = \frac{-x}{4x^2+9y^2}$

and given that $\vec{G}$ is C^1 on $\mathbb{R}^2\setminus\{(0,0)\}$, can apply Green.

$$\oint_C \vec{H}\cdot d\vec{r} = \oint_C \vec{G}\cdot d\vec{r} + \oint_C 12\vec{F}\cdot dr \overset{\text{Green}}{=} \iint_D \underbrace{Q_x - P_y}_{0} dA + 12\iint_D \underbrace{Q_{1x} - P_{1y}}_{0} dA = 0$$

(by def 1) (by (b)) Q.E.D

(Step II) Prove that $\oint_C \mathbf{H}\cdot \mathrm{d}\mathbf{r} = 0$ for every simple closed curve C that encloses the origin.



For the ellipsoid curve E the region enclosed the origin

let C to be a simple closed curve big enough to fit outside E

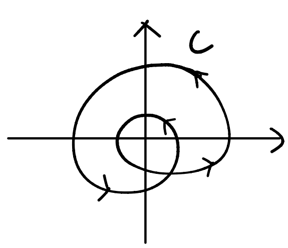
let D be the Region enclosed by C & E, which doesn't contain $(0,0)$

Hence, $P_1(x,y)$, $Q_1(x,y)$ are C^1 on D, By Generalized Green's Theorem,

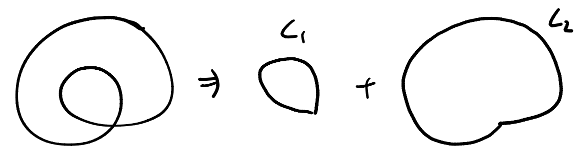
$$\iint_D \overbrace{Q_x - P_y}^{0}\, dA = \oint_C \vec{H}\cdot d\vec{r} - \oint_E \vec{H}\cdot d\vec{r} \Rightarrow \oint_C \vec{H}\cdot d\vec{r} = \oint_E \vec{H}\cdot d\vec{r} \overset{(i)}{=} 0$$

Q.E.D.

(Step III) Prove that $\oint_C \mathbf{H}\cdot \mathrm{d}\mathbf{r} = 0$ for every closed curve C (not necessarily simply) in $\mathbb{R}^2 \setminus \{(0,0)\}$.



Decompose C into two simple curves



As Both C_1 & C_2 are simple curves that encloses the origin

By (b), $\oint_C \vec{H}\cdot d\vec{r} = \oint_{C_1} \vec{H}\cdot d\vec{r} + \oint_{C_2} \vec{H}\cdot d\vec{r} = 0 + 0 = 0$

As $\oint_C \mathbf{H}\cdot \mathrm{d}\mathbf{r} = 0$ for every closed curve C in $\mathbb{R}^2 \setminus \{(0,0)\}$, we conclude that $\mathbf{H}$ is conservative on $\mathbb{R}^2 \setminus \{(0,0)\}$.