$g(x) = f^{-1}(x)$

$g'(x) = \frac{1}{f'(g(x))}$

National Taiwan University - Calculus 1 Quiz 2 (Class 09) - Quiz 2

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68

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There are **FOUR** questions in this quiz.

Your work is graded on the quality of your writing as well as the validity of the mathematics.

19 1. Let $f(x) = 2x + \tan^{-1}(x-1)$.

5 (a) (5%) Prove that $f(x)$ is one-to-one. (Hint: Use Rolle's Theorem)

14 (b) (15%) Define $g(x) = f^{-1}(x)$. Compute $g(2)$, $g'(2)$, and $g''(2)$.

0 (c) (5%) Use linear approximation to estimate $g(2.06)$ which is the root of the equation $2x + \tan^{-1}(x-1) = 2.06$.

(a) 1° $f'(x) = 2 + \frac{1}{1+(x-1)^2} = \frac{2+2(x-1)^2+1}{1+(x-1)^2} = \frac{2(x-1)^2+3}{(x-1)^2+1} > 0$

$\therefore$ f(x) is a strictly increasing function.

~~$\because$ 2x is continuous, $\tan^{-1}(x-1)$ is continuous $\Rightarrow$ f(x) is continuous~~

~~and f(x) is continuous~~ and strictly increasing $\Rightarrow$ f(x) is 1 to 1.

(b) $g(2) = f^{-1}(2) \Rightarrow 2 = 2x + \tan^{-1}(x-1) \Rightarrow x = 1 \Rightarrow g(2) = 1$ ※

$g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(1)} = \frac{1}{3} \Rightarrow$? $g'(2) = 3$ ※ -1

$g'(x) = \frac{1}{f'(g(x))} \Rightarrow g''(x) = \frac{0 - f''(g(x)) \times g'(x)}{[f'(g(x))]^2}$

$\therefore g''(2) = -\frac{f''(g(2)) \times g'(2)}{[f'(g(2))]^2} = -\frac{f''(1) \times 3}{3^2} = 0$ ※

$f''(x) = \frac{4(x-1) - [2(x-1)^2+3](2(x-1))}{((x-1)^2+1)^2} \Rightarrow f''(1) = 0$ 代入

(c) let $L(x) = f(2) + f'(2)(x-2) = 4 + \frac{\pi}{4} + \frac{5}{2}(x-2)$

$\Rightarrow L(2.06) = 4 + \frac{\pi}{4} + \frac{5}{2} \times 0.06 = 4.15 + \frac{\pi}{4}$ ※

18 2. The equation $x^2 + xy + y^3 = 1$ defines y as an implicit function of x, say $y = f(x)$, near $x = 0$.

10 (a) (10%) Compute $f(0)$, $f'(0)$, and $f''(0)$.

8 (b) (15%) Evaluate $\frac{d}{dx}\left(\sin^{-1}(\frac{1}{2}f(x)) + \tan^{-1}(f(x)) + \sec^{-1}(2 + xf(x))\right)\Big|_{x=0}$.

(a) $x^2+xy+y^3=1$, if $x=0$, $y=1 \Rightarrow f(0)=1$ ※

1° $2x+y+xy'+3y^2y'=0 \Rightarrow 1+3y'=0 \Rightarrow f'(0)=\frac{-1}{3}$ ※

2° $2+y'+y'+xy''+3(2yy'y'+y^2y'')=0$

$\Rightarrow 2-\frac{2}{3}+6\times1\times\frac{1}{9}+3\times1y''=0 \Rightarrow f''(0)=\frac{-2}{3}$ ※

(b)

$\frac{d}{dx}\left[\sin^{-1}(\frac{1}{2}f(x))+\tan^{-1}(f(x))+\sec^{-1}(2+xf(x))\right]\Big|_{x=0}$ $\frac{1}{x\sqrt{x^2-1}}$

$=\frac{1}{\sqrt{1-(\frac{1}{2}f(x))^2}}\times\frac{1}{2}f'(x)+\frac{f'(x)}{1+f^2(x)}+\frac{f(x)+xf'(x)}{(2+xf(x))\sqrt{[2+xf(x)]^2-1}}\Big|_{x=0}$

$=\frac{1}{\sqrt{1-(\frac{1}{2}f(0))^2}}\times\frac{1}{2}(f'(0))+\frac{f'(0)}{1+f^2(0)}+\frac{f(0)+0f'(0)}{2+0f(0)\sqrt{(2+0f(0))^2-1}}$

$=\frac{1}{\sqrt{1-\frac{1}{4}}}\times\frac{1}{2}\times\frac{-1}{3}+\frac{\frac{-1}{3}}{1+1}+\frac{1}{2}$

$=\frac{2}{\sqrt{3}}\times\frac{1}{2}\times\frac{-1}{3}+\frac{-1}{6}+\frac{1}{2}$

$=\frac{-1}{3\sqrt{3}}+\frac{-1}{6}+\frac{1}{2}=\frac{-2-\sqrt{3}+3\sqrt{3}}{6\sqrt{3}}=\frac{2\sqrt{3}-2}{6\sqrt{3}}=\cancel{\frac{\sqrt{3}-1}{3\sqrt{3}}}$ ※

21 ~~30~~ 3. Consider the function $f(x) = \left(\frac{x}{2e}\right)^x$ for $x > 0$.

10 (a) (10%) Compute $f'(x)$.

5 ~~4~~ (b) (5%) Find critical numbers of $f(x)$ on the interval $(1, 2e)$.

2 (c) (5%) Find the absolute maximum and minimum values of $f(x)$ on $[1, 2e]$.

4 (d) (5%) Use the linearization of $f(x)$ at $x = 2e$ to estimate $f(2e - 0.02)$.

(a) $f(x) = \left(\frac{x}{2e}\right)^x = e^{x\ln\left(\frac{x}{2e}\right)}$

$f'(x) = \left(\frac{x}{2e}\right)^x \times \left[\ln\left(\frac{x}{2e}\right) + x \times \frac{2e}{x} \times \frac{1}{2e}\right]$

$f'(x) = \left(\frac{x}{2e}\right)^x \times \left(\ln\left(\frac{x}{2e}\right) + 1\right)$ ※

(b) $\left(\frac{x}{2e}\right)^x \times \left(\ln\left(\frac{x}{2e}\right) + 1\right) = 0 \Rightarrow \ln\left(\frac{x}{2e}\right) = -1 \Rightarrow \frac{x}{2e} = \frac{1}{e} \Rightarrow x = 2$

$\because \left(\frac{x}{2e}\right)^x$ is exponetial, no x satisfly $\left(\frac{x}{2e}\right)^x = 0$

$\therefore$ only critical number is $x = 2$ +4 ※

(c) $f'(x) = \left(\frac{x}{2e}\right)^x \times \left(\ln\left(\frac{x}{2e}\right) + 1\right)$, $\left(\frac{x}{2e}\right)^x = e^{x\ln\left(\frac{x}{2e}\right)} > 0$

$f'(x) > 0 \Rightarrow \ln\left(\frac{x}{2e}\right) + 1 > 0 \Rightarrow \ln\left(\frac{x}{2e}\right) > \ln\left(\frac{1}{e}\right) \Rightarrow x > 2$

$f'(x) < 0 \Rightarrow \ln\left(\frac{x}{2e}\right) + 1 < 0 \Rightarrow \ln\left(\frac{x}{2e}\right) < \ln\left(\frac{1}{e}\right) \Rightarrow x < 2$

$\therefore$ minimum value $f(2) = e^{-2}$, maximum value $f(2e) = 1$ +2 ※

(d) let $L(x) = f(2e) + f'(2e)(x - 2e)$

$= 1 + 1(x - 2e) \Rightarrow L(2e - 0.02) = 1 + 2e - 0.02 - 2e$

+ ~~4~~ $= 0.98$ ※

10 4. Compute derivatives.

5 (a) (15%) $f(x) = (1+x^2)^{\cos x}(\ln x)^x$, for $x > 1$. Find $f'(x)$.

5 (b) (10%) Let $f(x) = \log_{2^x}(\log_{x^2}(e))$. Compute $f'(x)$.

(Hint: Use the change of base formula

$$\log_a b = \frac{\ln b}{\ln a} \quad \text{for all} \quad a, b > 0$$

to simplify $f(x)$.)

(a) $f(x) = (1+x^2)^{\cos x} \times (\ln x)^x$

$= e^{\cos x \ln(1+x^2) + x\ln(\ln x)}$

+5.

$f'(x) = (1+x^2)^{\cos x} \times (\ln x)^x \times \left[-\sin x \ln(1+x^2) + \frac{\cos x \times (2x)}{1+x^2} + \ln(\ln x) + \frac{1}{\ln x} \times \frac{1}{x}\right]$

(b) $f(x) = \log_{2^x}(\log_{x^2}(e)) = \frac{\ln(\log_{x^2} e)}{\ln 2^x} = \frac{\ln\left(\frac{\ln e}{\ln x^2}\right)}{\ln 2^x} = \frac{\ln(\ln x^2)^{-1}}{\ln 2^x}$

+3

$f'(x) = -\left(\frac{\frac{1}{\ln x^2} \times \frac{1}{x^2} \times 2x \times x\ln 2 - \ln\left(\frac{1}{\ln x^2}\right)\ln 2}{x^2 \ln^2 2}\right)$

$= -\frac{\frac{2\ln 2}{\ln x^2} + \ln 2 \ln \ln x^2}{x^2 \ln^2 2}$ +2

$= -\frac{\frac{1}{\ln x} + \ln \ln x^2}{x^2 \ln 2} = -\frac{1 + \ln x \cdot \ln \ln x^2}{x^2 \ln x \ln 2}$

27 28 1320 2640