National Taiwan University - Calculus 3 Class 09 (Thur) - Quiz 2
3/28/2024 - 50 minutes

Name: ID: Department:

There are **FOUR** questions in this quiz.
Your work is graded on the quality of your writing as well as the validity of the mathematics.

1. Assume that the equation

$$3x + x^5z^2 - yz^6 + y^3 = 0$$

defines z as a function of x, y, i.e. $z = z(x, y)$, near $(x, y, z) = (0, 1, 1)$.

(a) (10%) Compute $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(x, y, z) = (0, 1, 1)$.

(b) (4%) Find the maximum directional derivative of $z(x, y)$ at $(x, y) = (0, 1)$.

(c) (4%) Estimate $z(-0.1,\ 0.97)$ by the linear approximation.

(d) (6%) Let $f(t) = z(\sin(3t),\ 1 - t + t^2)$. Compute $f'(0)$.

Sol:

(a) Solution 1 : Let $F(x, y, z) = 3x + x^5z^2 - yz^6 + y^3$. Then by the implicit differentiation, we know that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3 + 5x^4z^2}{2x^5z - 6yz^5}\ , \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-z^6 + 3y^2}{2x^5z - 6yz^5}.$$

Hence, at $(x, y, z) = (0, 1, 1)$, $\frac{\partial z}{\partial x} = \frac{1}{2}$, and $\frac{\partial z}{\partial y} = \frac{1}{3}$.
(2 pts for $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$. 2 pts for $F_x = 3 + 5x^4z^2$. 2 pts for $F_y = -z^6 + 3y^2$. 2 pts for $f_z = 2x^5z - 6yz^5$. 2 pts for evaluating at $(x, y, z) = (0, 1, 1)$.)
Solution 2: Consider the equation

$$3x + x^5(z(x, y))^2 - y(z(x, y))^6 + y^3 = 0$$

Differentiate the equation with respect to x. We obtain

$$3 + 5x^4z^2 + 2x^5zz_x - 6yz^5z_x = 0. \qquad \text{(2 pts)}$$

Thus $z_x = -\frac{3 + 5x^4z^2}{2x^5z - 6yz^5}$ and at $(x, y, z) = (0, 1, 1)$, $z_x = 1/2$.
(2 pts for solving z_x and 1 pt for evaluating at $(x, y, z) = (0, 1, 1)$.) Similarly, after differentiating the equation with respect to y, we obtain

$$2x^5zz_y - z^6 - 6yz^5z_y + 3y^2 = 0. \qquad \text{(2 pts)}$$

Thus $z_= -\frac{-z^6 + 3y^2}{2x^5z - 6yz^5}$ and at $(x, y, z) = (0, 1, 1)$, $z_y = 1/3$.
(2 pts for solving z_y and 1 pt for evaluating at $(x, y, z) = (0, 1, 1)$.)

(b) The maximal directional derivative of $z(x, y)$ at $(x, y) = (0, 1)$ is the length of $\nabla z(0, 1)$. (2 pts)
And $|\nabla z(0, 1)| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{3})^2} = \frac{\sqrt{13}}{6}$. (2 pts)

(c) The linearization of $z(x, y)$ at $(x, y) = (0, 1)$ is $L(x, y) = z(0, 1) + z_x(0, 1)(x - 0) + z_y(0, 1)(y - 1)$. Hence,

$$\begin{aligned} z(-0.1, 0.97) &\approx z(0, 1) + z_x(0, 1) \times (-0.1) + z_y(0, 1) \times (0.97 - 1) \qquad \text{(2 pts)} \\ &= 1 - \frac{1}{2} \times 0.1 - \frac{1}{3} \times 0.03 = 0.94 \qquad \text{(2 pts)} \end{aligned}$$

(d)

$$f'(t) = \frac{\partial z}{\partial x}(\sin(3t),\ 1-t+t^2) \times 3\cos(3t) + \frac{\partial z}{\partial y}(\sin(3t),\ 1-t+t^2) \times (-1+2t) \qquad \text{(4 pts)}$$

When $t = 0$, $(\sin(3t),\ 1-t+t^2) = (0,1)$ and $f'(0) = \frac{1}{2} \times 3 + \frac{1}{3} \times (-1) = \frac{7}{6}$. (2 pts)

2. Suppose that $\nabla f(x,y,z) = 2xy^2\mathbf{i} + (2x^2y - z^2)\mathbf{j} - 2yz\mathbf{k}$ and $f(1,1,1) = 0$.

   (a) (6%) Find the tangent plane to the level surface $f(x,y,z) = 0$ at $(x,y,z) = (1,1,1)$.

   (b) (10%) Suppose that curve C is the intersection of the level surface $f(x,y,z) = 0$ and the plane $x+2y+z = 4$. Find the tangent line of C at $(x,y,z) = (1,1,1)$.

   (c) (8%) (Continued) Assume that a differentiable function $h(x,y,z)$ obtains local minimum at $(x,y,z) = (1,1,1)$ when restricted on the curve C. Circle ALL correct statement(s).

      i. $\nabla h(1,1,1)$ is normal to the tangent line of C at $(x,y,z) = (1,1,1)$.

      ii. $\nabla h(1,1,1)$ is parallel to the tangent line of C at $(x,y,z) = (1,1,1)$.

      iii. $\nabla h(1,1,1)$ is normal to both $\nabla f(1,1,1)$ and $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

      iv. $\nabla h(1,1,1)$ lies in the plane spanned by $\nabla f(1,1,1)$ and $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

Sol:

(a) $\nabla f(1,1,1) = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is normal to the level surface $f(x,y,z) = 0$ at $(x,y,z) = (1,1,1)$. (3 pts) Hence the tangent plane is $2(x-1) + (y-1) - 2(z-1) = 0$ which is $2x + y - 2z = 1$. (3 pts)

(b) The tangent line of C lies on both the tangent plane of $f(x,y,z) = 0$ and the plane $x + 2y + z = 4$. Therefore the tangent line of C is both normal to $\nabla f(1,1,1)$ and $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ which means that the tangent line is parallel to $(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$. (6 pts)
The tangent line of C is

$$(x(t), y(t), z(t)) = (1+5t,\ 1-4t,\ 1+3t)\ , \quad t \in \mathbf{R} \qquad \text{(4 pts)}$$

(c) C is defined by two equations $f(x,y,z) = 0$ and $x + 2y + z = 4$. Hence by Lagrange's multiplier method, we know that at the local minimizer $(1,1,1)$, $\nabla h(1,1,1) = \lambda \nabla f(1,1,1) + \mu(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ which means that $\nabla h(1,1,1)$ lies in the plane spanned by $\nabla f(1,1,1)$ and $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Moreover, if C has a parametrization $\gamma(t)$ with $\gamma(0) = (1,1,1)$, then $h(\gamma(t))$ obtains local minimum at $t = 0$. Hence
$\frac{d}{dt}\, h(\gamma(t))|_{t=0} = \nabla h(1,1,1) \cdot \gamma'(0) = 0$ which means that $\nabla h(1,1,1)$ is normal to the tangent line of C at $(x,y,z) = (1,1,1)$.
Therefore, (1) (iv) are correct statements and (ii) (iii) are false statements.
(2 pts for choosing (i).
2 pts for not choosing (ii).
2 pts for not choosing (iii).
2 pts for choosing (iv).)

3. Find the absolute extreme value of $f(x,y,z) = xe^{-yz}$ on the region $D = \{(x,y,z) \mid x^2+4y^2+z^2 \le 10\}$.

(a) (5%) Is there any critical point of f on D?

(b) (20%) Use Lagrange multiplier method to find the extreme value of f on the boundary of D which is $x^2+4y^2+z^2 = 10$. Then find the absolute extreme values of f on D.

Sol:

(a) $f_x = e^{-yz}$, $f_y = -xze^{-yz}$, $f_z = -xye^{-yz}$. The critical points are points at which $f_x = f_y = f_z = 0$. Since $f_x = e^{-yz} > 0$ for all (x,y,z), we conclude that there are no critical points of f on D.
(3 pts for f_x, f_y, f_z. 2 pts for showing that there are no critical points)

(b) Let $g(x,y,z) = x^2+4y^2+z^2$. Lagrange multiplier method suggests that we shall solve the system of equations.

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 10 \end{cases} \Longrightarrow \begin{cases} e^{-yz} = \lambda 2x \\ -xze^{-yz} = \lambda 8y \\ -xye^{-yz} = \lambda 2z \\ x^2+4y^2+z^2 = 10 \end{cases} \qquad \text{(5 pts)}$$

The first equation tells that $\lambda x > 0$ i.e. λ, x are not zero and they have the same sign.

The second and the third equations may have solution $y = z = 0$. Then the fourth equation derives that $x = \pm\sqrt{10}$. And the first equation solves $\lambda = \dfrac{1}{2\sqrt{10}}$ when $x = \sqrt{10}$ and $\lambda = -\dfrac{1}{2\sqrt{10}}$ when $x = -\sqrt{10}$. Therefore $(x,y,z,\lambda) = (\sqrt{10}, 0, 0, 1/2\sqrt{10})$, $(-\sqrt{10}, 0, 0, -1/2\sqrt{10})$ are solutions and $f(\sqrt{10},0,0) = \sqrt{10}$, $f(-\sqrt{10},0,0) = -\sqrt{10}$. (4 pts)

If $y, z \ne 0$, we divide the second equation by the third equation and obtain that $\dfrac{z}{y} = \dfrac{4y}{z}$. Moreover, x and λ have the same sign. Thus the first equation says that y and z have different signs. Hence, $z = -2y$. Now we solve x from the second equation:

$$x = -8\lambda e^{yz}\frac{y}{z} = 4\lambda e^{yz}.$$

The first equation solves $x = \dfrac{1}{2\lambda e^{yz}}$. Therefore, $\lambda e^{yz} = \pm\dfrac{1}{2\sqrt{2}}$ and $x = \pm\sqrt{2}$. From the fourth equation, we have $z^2 = 4y^2 = 4$. Thus the solutions are
$(x,y,z,\lambda) = (\sqrt{2}, 1, -2, \dfrac{e^2}{2\sqrt{2}})$, $(\sqrt{2}, -1, 2, \dfrac{e^2}{2\sqrt{2}})$, $(-\sqrt{2}, 1, -2, -\dfrac{e^2}{2\sqrt{2}})$, $(-\sqrt{2}, -1, 2, -\dfrac{e^2}{2\sqrt{2}})$
(2 pts for each solution.)
$f(\sqrt{2},1,-2) = f(\sqrt{2},-1,2) = \sqrt{2e^2}$ and $f(-\sqrt{2},1,-2) = f(-\sqrt{2},-1,2) = -\sqrt{2e^2}$. Since

$$-\sqrt{2e^2} < f(-\sqrt{10},0,0) = -\sqrt{10} < f(\sqrt{10},0,0) = \sqrt{10} < \sqrt{2e^2},$$

we know that the absolute maximum value of f on the boundary of D is $\sqrt{2e^2}$ and the absolute minimum value of f on the boundary of D is $-\sqrt{2e^2}$. (2 pts)

Because f has no critical points on D, the absolute extreme values occur on the boundary of D. Therefore the absolute maximum value of f on D is $\sqrt{2e^2}$ and the absolute minimum value of f on D is $-\sqrt{2e^2}$. (1 pt)

4. (a) (12%) Compute $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1+2x^3}\, dxdy$.

(b) (15%) Find the volume of the tetrahedron bounded by $x + 2y + z = 4$ and coordinate planes. (You must compute the volume by a double integral.)

Sol:

(a) $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1+2x^3}\, dxdy = \iint_D \sqrt{1+2x^3}\, dA$ where D is bounded by $y = x^2$, $y = 0$ and $x = 1$.
Hence

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1+2x^3}\, dxdy == \int_0^1 \int_0^{x^2} \sqrt{1+2x^3}\, dydx \qquad \text{(5 pts)}$$

$$= \int_0^1 x^2\sqrt{1+2x^3}\, dx \qquad \text{(2 pts)}$$

$$\overset{u=1+2x^3}{=} \int_1^3 \sqrt{u}\frac{1}{6}\, du = \frac{1}{9}(3\sqrt{3}-1). \qquad \text{(5 pts)}$$

(b) The project of the tetrahedron, E, onto the xy-plane is a triangle D bounded by the lines $x = 0$, $y = 0$ and $x + 2y = 4$. Then $E = \{(x, y, z) \mid (x, y) \in D,\ 0 \le z \le 4 - x - 2y\}$ (3 pts)

We can write D as a type II region, $D = \{(x, y) \mid 0 \le y \le 2,\ 0 \le x \le 4 - 2y\}$. (3 pts)

Hence

$$\mathrm{V}(E) = \iiint_E 1\, dV \qquad \text{(2 pts)}$$

$$= \iint_D \int_0^{4-x-2y} 1\, dzdA \qquad \text{(2 pts)}$$

$$= \int_0^2 \int_0^{4-2y} 4 - x - 2y\, dxdy \qquad \text{(2 pts)}$$

$$= \int_0^2 \frac{1}{2}(4-2y)^2\, dy = \frac{16}{3} \qquad \text{(3 pts)}$$

We can also write D as a type I region, $D = \{(x, y) \mid 0 \le x \le 4,\ 0 \le y \le \frac{4-x}{2}\}$.

Then $\mathrm{V}(E) = \int_0^4 \int_0^{\frac{4-x}{2}} \int_0^{4-x-2y} 1\, dzdydx$.