National Taiwan University - Calculus 1 for Class 01-09
Worksheet 2 : The Precise Definition of a Limit

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Deriving the Precise Definition of a Limit.
The intuitive definition of a limit, $\lim_{x\to a} f(x) = L$, says that
we can make the values of $f(x)$ **arbitrarily** *close to L by restricting x to be sufficiently close to a but not equal to a.*

Verifying the above statement is a back and forth, offensive and defensive process. When one claims $\lim_{x\to a} f(x) = L$, he/she is responsible to answer the following challenges:
How close to a does x have to be so that $|f(x) - L| < 0.1$?
How close to a does x have to be so that $|f(x) - L| < 0.01$? And so on...Thus he/she must once and for all answer:

How close to a does x have to be so that $|f(x) - L| < \epsilon$, where ϵ is an arbitrarily small positive number?

In general, given smaller ϵ, one may need to restrict x closer to a. Hence, the above answer (the distance between x and a) depends on ϵ. Now we conclude that

$\lim_{x\to a} f(x) = L$ **means that**
for any number $\epsilon > 0$ there is a $\delta > 0$ (depending on ϵ) such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

The last statement is the **precise definition** of $\lim_{x\to a} f(x) = L$. In this worksheet we will have hands-on experiences of using this definition.

Exercise 1.

$$\text{Consider } f(x) = \begin{cases} \dfrac{2x^2 - x - 1}{x - 1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$$

- Simplify $f(x)$, $f(x) =$ $2x+1$ for $x \neq 1$. Guess the limit, $\lim_{x\to 1} f(x) =$ 3.
- When you wrote down $\lim_{x\to 1} f(x) = L$, you have made a very *strong* statement! You claimed that
  We can make $f(x)$ **arbitrarily** *close to L by restricting x to be sufficiently close to 1 but not equal to 1.*
  One may ask you to show him :
  How close to 1 does x have to be so that $|f(x) - L| < 0.1, 0.01, 0.001...$?
  Starting from the goal inequality $|f(x) - L|$, derive an equivalent inequality regarding $|x - 1|$.

  Fill in the blank.

  If $0 < |x - 1| <$ 0.05, $|f(x) - L| < 0.1$.

  If $0 < |x - 1| <$ 0.005, $|f(x) - L| < 0.01$.

  If $0 < |x - 1| <$ $\frac{\epsilon}{2}$, $|f(x) - L| < \epsilon$, where ϵ is any positive number.

- What happens at $x = 1$? Obviously, $|f(1) - L| = |0 - L| > 0.1$. Does this violate the statement $\lim_{x \to 1} f(x) = L$?

1° $f(1) = 0$

2° $\lim_{x \to 1} f(x) = 3$

Although $f(1) = 0$

However, this doesn't violate the fact that $\lim_{x \to 1} f(x) = L = 3$

- For any $\epsilon > 0$, we find $\delta =$ $\frac{\epsilon}{2}$ such that if $0 < |x - 1| < \delta$, then $|f(x) - L| < \epsilon$. This proves that indeed $\lim_{x \to 1} f(x) = L$.

Exercise 2.

(a) Show that $\lim_{x \to 0} \sqrt[3]{x} = 0$.

To prove the limit is 0, for any $\epsilon > 0$, we need to find a $\delta > 0$ such that if $0 < |x - 0| < \delta$ then $|\sqrt[3]{x} - 0| < \epsilon$.

- Starting from the desired inequality $|\sqrt[3]{x} - 0| < \epsilon$, derive an inequality for $|x - 0|$.

$|\sqrt[3]{x} - 0| < \epsilon$

$(|\sqrt[3]{x} - 0|)^3 = x < \epsilon^3$

- For a given ϵ, find such δ and show that if $0 < |x - 0| < \delta$ then $|\sqrt[3]{x} - 0| < \epsilon$.

$0 < |x - 0| < \delta$

$|\sqrt[3]{x} - 0| < \sqrt[3]{\delta} = \epsilon$

$\therefore \delta = \epsilon^3$

(b) Imitating the precise definition of a limit, write down precise definitions of one-sided limits.

- $\lim_{x \to a^+} f(x) = L \Leftrightarrow$ $\forall \epsilon > 0$, $\exists \delta > 0$ (ϵ and δ is related) that $a < x < a + \delta$ then $|f(x) - L| < \epsilon$

- $\lim_{x \to a^-} f(x) = L \Leftrightarrow$ $\forall \epsilon > 0$, $\exists \delta > 0$ (ϵ and δ is related) that $a - \delta < x < a$ then $|f(x) - L| < \epsilon$

Precise Definition of an Infinite Limit.

The intuitive definition of an infinite limit, $\lim_{x\to a} f(x) = \infty$, says that

we can make the values of $f(x)$ **arbitrarily** *large by restricting x to be sufficiently close to a but not equal to a.*

Again, when one claims $\lim_{x\to a} f(x) = \infty$, he/she is responsible to answer the following challenges:

How close to a does x have to be so that $f(x) > 100$?

How close to a does x have to be so that $f(x) > 1000$? And so on...Thus he/she must once and for all answer:

How close to a does x have to be so that $f(x) > N$, where N is an arbitrarily large positive number?

Hence we conclude that

$$\lim_{x\to a} f(x) = \infty \text{ means that}$$

for any number $N > 0$ there is a $\delta > 0$ (depending on N) such that if $0 < |x - a| < \delta$ then $f(x) > N$.

The last statement is the **precise definition** of $\lim_{x\to a} f(x) = \infty$.

Exercise 3.

(a) Imitating the above definition, write down precise definitions of other limits regarding infinity.

- $\lim_{x\to a^+} f(x) = -\infty \Leftrightarrow$ $\forall N<0$, $\exists\ \delta>0$ (ε and N is related) that $0<|x-a^+|<\delta$ then $f(x)<N$
- $\lim_{x\to\infty} f(x) = L \Leftrightarrow$ $\forall \varepsilon>0$, $\exists N>0$ (ε and N is related) that $x>N$ then $|f(x)-L|<\varepsilon$
- $\lim_{x\to-\infty} f(x) = \infty \Leftrightarrow$ $\forall N>0$, $\exists M<0$ (N and M is related) that $x<M$ then $f(x)>N$

(b) Show that $\lim_{x\to 0^+} \ln x = -\infty$.

To prove that the limit is negative infinity, for any negative $N < 0$, we need to find a $\delta > 0$ such that if $0 < x - 0 < \delta$ then $\ln x < N$.

- Starting from the desired inequality $\ln x < N$, derive an inequality for $x - 0$.

$\ln x < N = \ln e^N$

$\Rightarrow 0 < x < e^N$

$\Rightarrow 0 < x - 0 < e^N$

- For a given $N < 0$, find such δ and show that if $0 < x - 0 < \delta$ then $\ln x < N$.

$\ln x < N = \ln e^N$

$\Rightarrow 0 < x < e^N$

$\Rightarrow 0 < x - 0 < e^N = \delta$

$\therefore \delta = e^N$

Exercise 4 (Optional).

We can use the precise definition of a limit to prove limit laws and corollaries about limits. Try to prove the following statements.

(a) Prove that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then $\lim_{x\to a}(f(x)+g(x)) = L+M$.
(Hint: For any $\epsilon > 0$ we need to find a $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|(f(x)+g(x)) - (L+M)| < \epsilon$. However, we have $|(f(x)+g(x)) - (L+M)| \le |f(x)-L| + |g(x)-M|$. Moreover, since $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, we can let $|f(x)-L| < \epsilon/2$ and $|g(x)-M| < \epsilon/2$ if x is sufficiently close to a.)

1° $\lim_{x\to a} f(x) = L \Rightarrow \forall \varepsilon > 0,\ \exists\, \delta_1 > 0$, that $0 < |x-a| < \delta_1$ then $|f(x)-L| < \frac{\varepsilon}{2}$

2° $\lim_{x\to a} g(x) = M \Rightarrow \forall \varepsilon > 0,\ \exists\, \delta_2 > 0$, that $0 < |x-a| < \delta_2$ then $|g(x)-M| < \frac{\varepsilon}{2}$

3° $\forall \varepsilon > 0,\ \exists\, \delta > 0$ that $0 < |x-a| < \delta$ then $|f(x)+g(x)-(L+M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

(b) Prove that if $\lim_{x\to a} f(x) = 0$ and $|g(x)| < M$ for all x where $M > 0$ is a constant then $\lim_{x\to a} f(x)g(x) = 0$.

1° $0 \le |f(x)g(x)| = |f(x)| \cdot |g(x)| < |f(x)| \cdot M$

$\lim_{x\to a} |f(x)| \cdot M = M \lim_{x\to a} |f(x)| = 0$

2° by squeeze, $\lim_{x\to a} f(x)g(x) = 0$

(c) Prove that if $\lim_{x\to a} f(x) = L > 0$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a} f(x)g(x) = \infty$.

1° $\lim_{x\to a} f(x) = L \Rightarrow \forall \varepsilon > 0,\ \exists\, \delta_1 > 0$ that $0 < |x-a| < \delta_1$ then $|f(x)-L| < \varepsilon$

2° $\lim_{x\to a} g(x) = \infty \Rightarrow \forall M > 0, \exists\, \delta_2 > 0$ that $0 < |x-a| < \delta_2$ then $g(x) > M$

3° let $\delta = \min\{\delta_1, \delta_2\}$

$\Rightarrow \begin{cases} \forall M > 0,\ \exists\, \delta > 0 \text{ that } 0 < |x-a| < \delta \text{ then } g(x) > M \\ \forall \varepsilon > 0,\ \exists\, \delta > 0 \text{ that } 0 < |x-a| < \delta \text{ then } |f(x)-L| < \varepsilon \end{cases} \Rightarrow \begin{cases} g(x) > M \\ L-\varepsilon < f(x) < L+\varepsilon \end{cases} \Rightarrow f(x)g(x) > M(L-\varepsilon) = M$

$\therefore \lim_{x\to a} [g(x) \times f(x)] = \infty$

(d) Prove that if $f(x) < g(x)$ for all $x \neq a$ and the limits $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.
(Hint: If $\lim_{x\to a} f(x) > \lim_{x\to a} g(x)$ can you derive a contradiction?)

prove $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$ is equivalent to prove $\lim_{x\to a} [g(x) - f(x)] \ge 0$

1° consider $h(x) = g(x) - f(x)$, given that ① $f(x) < g(x) \Rightarrow h(x) > 0\ \ \forall x \in \mathbb{R}\ \&\ x \neq a$

② $\lim_{x\to a} f(x)$ exists, $\lim_{x\to a} g(x)$ exists $\Rightarrow \lim_{x\to a} h(x)$ exists

2° assume $\lim_{x\to a} h(x) = L < 0 \Rightarrow \forall \varepsilon > 0, \exists\, \delta > 0$ that $0 < |x-a| < \delta$ then $|h(x)-L| < \varepsilon$

take $\varepsilon = \frac{|L|}{2} \Rightarrow |h(x) - L| < \varepsilon = \frac{|L|}{2} \Rightarrow L - \frac{|L|}{2} < h(x) < L + \frac{|L|}{2}$

$\because L + \frac{|L|}{2} < 0 \Rightarrow$ contradicts condition ①

$\therefore \lim_{x\to a} h(x) \ge 0 \Rightarrow \lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$

(d) Prove that if $f(x) < g(x)$ for all $x \neq a$ and the limits $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.
(Hint: If $\lim_{x\to a} f(x) > \lim_{x\to a} g(x)$ can you derive a contradiction?)

What we want to prove is equivalent to $\lim_{x\to a} g(x) - f(x) \geq 0$

Hence, $\boxed{\text{consider } h(x) = g(x) - f(x)}$

Given that, ① $f(x) < g(x) \Rightarrow h(x) > 0 \quad \forall x \neq a$

② $\begin{cases} \lim_{x\to a} f(x) \text{ exists} \\ \lim_{x\to a} g(x) \text{ exists} \end{cases} \Rightarrow \lim_{x\to a} h(x) \text{ exists}$

Assume that $\lim_{x\to a} h(x) = L < 0$

↳ ∵ limit exists

Which means $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t if $|x-a| < \delta$ $|h(x) - L| < \varepsilon$

Then we take $\varepsilon = \frac{|L|}{2} > 0 \Rightarrow |h(x) - L| < \varepsilon = \frac{|L|}{2}$

$\Rightarrow L - \frac{|L|}{2} < h(x) < L + \frac{|L|}{2}$

However, $L + \frac{|L|}{2} < 0$ (since $L < 0$), which contradicts the given condition ①

Hence $\lim_{x\to a} h(x) \geq 0 \Rightarrow \lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$ ※ proved by contradiction.