National Taiwan University - Calculus 1 for Class 01-09

Worksheet 1 : Inverse Trigonometric Functions

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Introduction.

Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. This simple function sends a number to the cube of it. For example,

$$f(1) = 1, \quad f(2) = 8, \quad f(3) = 27, \cdots$$

What we want to do now is to find another function $g(x)$ such that $g(x)$ will *undo the effect of* $f(x)$. In this case, we can simply take $g(x) = \sqrt[3]{x}$: the cubic root function. Then one can observe that $g(x)$ 'cancels' the effect of $f(x)$ and lead us back to the original input : $g(1) = 1, \quad g(8) = 2, \quad g(27) = 3, \cdots$ Indeed, more generally we have :

Definition. Let $f : D \to \mathbb{R}$. Suppose $g(x)$ is a function such that

$$g(f(x)) = x \quad \text{for all } x \in D.$$

In this case, we say that $g(x)$ is the **inverse function of** $f(x)$ and denote it by $f^{-1}(x)$.

Warning. One should not confuse the inverse function with the reciprocal function :

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

They are **DIFFERENT**!

Exercise 1. For each of the following function $f(x)$,

(a) write down its inverse function,

(b) write down the domain and range of the inverse function,

(c) sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same diagram.

What do you observe from (b) and (c) ?

$y = x^2$ $\sqrt{y}$

(i) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x$ (ii) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 3x - 4$ (iii) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad f(x) = x^2$

(i)

(a) g: inverse of f, $f(x) = 2x$

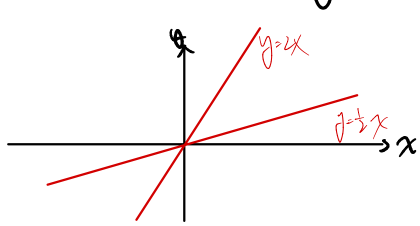
$g(f(x)) = x, \ \forall x \in \mathbb{R}$

$\Downarrow$

$g(2x) = x, \ g(x) = \frac{1}{2}x$

(b) $\text{domain}(f^{-1}) = \mathbb{R} = \text{range}(f^{-1})$

(c)



(ii) g: inverse of f, $f(x) = 3x-4$

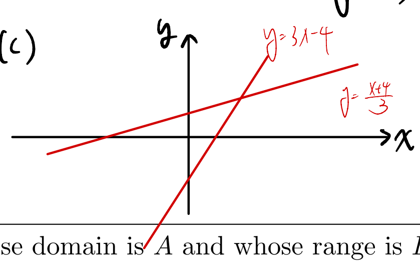
$g(f(x)) = x, \ \forall x \in \mathbb{R}$

$\Downarrow$

$g(3x-4) = x, \ g(x) = \frac{x+4}{3}$

(b) $\text{domain}(f^{-1}) = \mathbb{R} = \text{range}(f^{-1})$

(c)



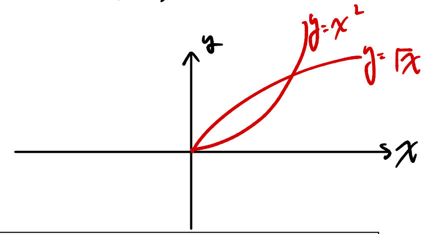
(iii) g: inverse of f, $f(x) = x^2$, $\{x \in \mathbb{R} \mid x \geq 0\}$

$g(f(x)) = x, \ \forall x \in \mathbb{R}$ 且 $x \geq 0$

$g(x^2) = x, \ g(x) = \sqrt{x}$

(b) $\text{domain}(f^{-1}) = \mathbb{R}^+ = \text{range}(f^{-1})$

(c)



Theorem. Let $f(x)$ be a function whose domain is A and whose range is B and let $f^{-1}(x)$ be its inverse function.

(a) The domain of $f^{-1}(x)$ is B and its range is A.

(b) The graphs $y = f(x)$ and $y = f^{-1}(x)$ are symmetric along the line $y = x$.

Bad news : Inverse may not exist

Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. What is its inverse function ? One may naively guess that $g(x) = \sqrt{x}$ is its inverse function. However, this is not true because one can check that

$$g(f(-3)) = g(9) = 3 \neq -3.$$

Indeed, in this case, the function $f(x)$ does not have an inverse at all. The underlying reason is that $f(x)$ does not satisfy the so-called **one-to-one** property in the sense that two different inputs may lead to the same output : for example, $f(-3) = f(3) = 9$.

Consider a function $f : D \to \mathbb{R}$.
Definition. f is called one-to-one if for any distinct inputs $x_1, x_2 \in D$, we have $f(x_1) \neq f(x_2)$.
Theorem. The inverse function $f^{-1}(x)$ exists if and only if f is one-to-one.

Exercise 2.

(a) Explain briefly why the above discussion does not contradict with your findings in Exercise 1 (iii).

(b) Explain briefly why if a function is strictly increasing or strictly decreasing (but not both), then it is a one-to-one function (and hence its inverse exists).

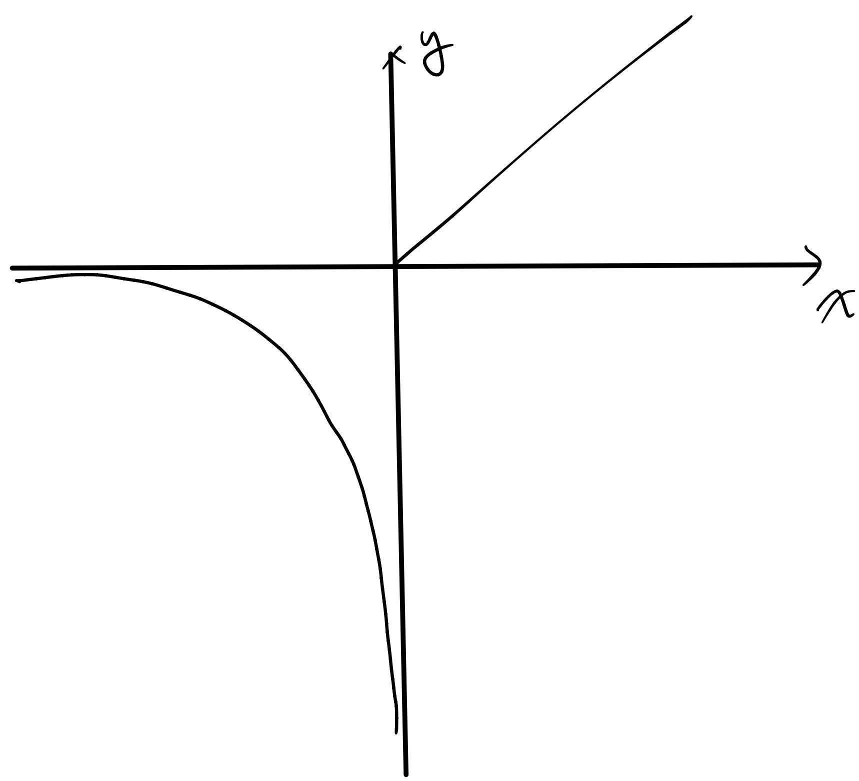
(c) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is neither strictly increasing or strictly decreasing but it is one-to-one. Graph your example.

(a) because in exercise 1 all of the functions are one-to-one

(b) If a function is strictly increasing or decreasing there won't be a chance for the function to possess to two different x occupied at the same y level.

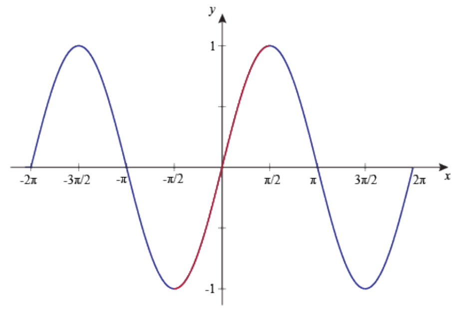
(c)

$$f(x) = \begin{cases} x & (x > 0) \\ \frac{1}{x} & (x < 0) \end{cases}$$



Inverse trigonometric functions.

Our goal is to define the inverse function of trigonometric functions. Let us take $f(x) = \sin x$. If we graph it,



we immediately encounter an issue : the sine function is not one-to-one so its inverse does not exist. Therefore, in order to define its inverse function, we will have to *sacrifice parts of its domain.* Indeed, if we, for example, consider the sine function with restricted domain

$$f : \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \to [-1, 1], \quad f(x) = \sin x \text{ (indicated in red above).}$$

Then this part of the sine curve is strictly increasing so it is one-to-one, and hence its inverse function exists !
We are going to denote its inverse function by $f^{-1}(x) = \sin^{-1}(x)$ and read it as the 'arcsine function'. Note that in some textbook this function may also be denoted by $f^{-1}(x) = \arcsin(x)$.

Warning (again!). One should not confuse the inverse function with the reciprocal function :

$$\sin^{-1}(x) \neq \frac{1}{\sin(x)}.$$

Exercise 3.

(a) What are the domain and the range of $f^{-1}(x) = \sin^{-1}(x)$?

(b) Write down the following values : $\sin^{-1}(1)$, $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$, $\sin^{-1}\left(-\frac{1}{2}\right)$, $\sin^{-1}\left(\sin\left(-\frac{\pi}{12}\right)\right)$, $\sin^{-1}(\sin \pi)$

(c) Sketch the graph $y = \sin^{-1}(x)$.

(a) $\text{domain}(f^{-1}(x)) = \{-1 \le x \le 1\}$

$\text{range}(f^{-1}(x)) = \{\frac{-\pi}{2} \le f^{-1}(x) \le \frac{\pi}{2}\}$

(b)

$\sin^{-1}(1) = \frac{\pi}{2}$

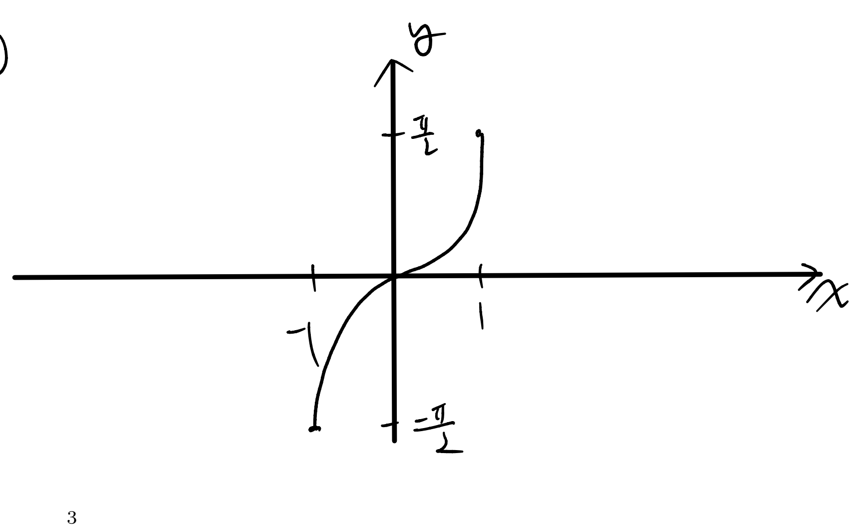
$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$\sin^{-1}\left(-\frac{1}{2}\right) = \frac{-\pi}{6}$

$\sin^{-1}\left(\sin\left(\frac{-\pi}{12}\right)\right) = \frac{-\pi}{12}$

$\sin^{-1}(\sin\pi) = 0$

(c)



In a similar manner, one can define the 'arccosine' and 'arctangent' functions by 'restricting the domains'. The following table summarizes the domains and ranges of these inverse functions.

Function	Domain	Range	Function	Domain	Range
$f(x) = \sin^{-1}(x)$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$f(x) = \sec^{-1}(x)$	$[-\infty,-1) \cup (1,\infty]$	$[0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi]$
$f(x) = \cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$	$f(x) = \csc^{-1}(x)$	$[-\infty,-1) \cup (1,\infty]$	$[\frac{-\pi}{2},0) \cup (0,\frac{\pi}{2}]$
$f(x) = \tan^{-1}(x)$	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$f(x) = \cot^{-1}(x)$	$(-\infty,\infty)$	$(0,\pi)$

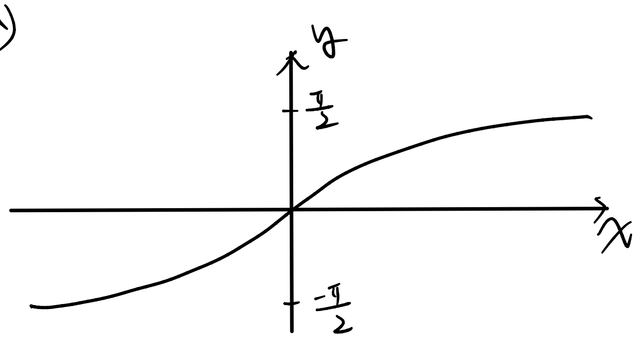
Exercise 4.

(a) Sketch the graph $y = \tan^{-1}(x)$

(b) From (a), write down $\lim_{x\to\infty} \tan^{-1}(x)$ and $\lim_{x\to-\infty} \tan^{-1}(x)$.

(c) Look up the relevant section in the textbook (Stewart), write down the domains and ranges of the 'inverse reciprocal trigonometric functions' on the right hand side of the above table.

(a)



(b) $\lim_{x\to\infty} \tan^{-1}x = \frac{\pi}{2}$, $\lim_{x\to-\infty} \tan^{-1}x = \frac{-\pi}{2}$

(c) 如上

Exercise 5. Consider the function $f(x) = \tan(\sin^{-1}(x))$

(a) Let $\theta = \sin^{-1}(x)$. For $x > 0$, draw a right angle triangle with θ being one of the angle such that $\sin(\theta) = x$.

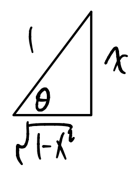
(b) Hence, guess a simplified expression of $f(x)$.

(c) Using a similar method, simplify the expression $\cos\left(\tan^{-1}\frac{1}{\sqrt{x^2-1}}\right)$. Hence, evaluate

$$\lim_{x\to1^+} \frac{1}{\sqrt{x^2-1}} \cos\left(\tan^{-1}\frac{1}{\sqrt{x^2-1}}\right).$$

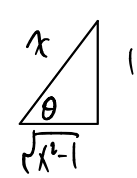
$f(x) = \tan(\sin^{-1}(x))$

(a) $\theta = \sin^{-1}(x)$



(b) $f(x) = \frac{x}{\sqrt{1-x^2}}$

(c) 1° $\theta = \tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$



$\cos\theta = \frac{\sqrt{x^2-1}}{x}$

2° $\lim_{x\to1^+} \frac{1}{\sqrt{x^2-1}} \times \cos\left(\tan^{-1}\frac{1}{\sqrt{x^2-1}}\right)$

$= \lim_{x\to1^+}\left(\frac{1}{\sqrt{x^2-1}} \times \frac{\sqrt{x^2-1}}{x}\right) = 1$