Calculus II Quiz I

Department: Name: Id:

*Minor error (-1)

1. **A taste to integration...** (30 points):
   Evaluate the following indefinite integral:

   (a) (5 %) $\displaystyle\int \frac{x^3 - 2x^{1/3} + 9x^{2/3}}{x^{4/3}}\,dx.$ (*Hint: This is very easy.*)

   (b) (10 %) $\displaystyle\int (\ln x)^2\,dx.$

   (c) (15 %) $\displaystyle\int \frac{12 + 6x}{3x^2 + 3}\,dx.$

   Sol:

   (a)

$$\begin{aligned}
\int \frac{x^3 - 2x^{1/3} + 9x^{2/3}}{x^{4/3}}\,dx &= \int \frac{x^3}{x^{4/3}}\,dx + \int \frac{-2x^{1/3}}{x^{4/3}}\,dx + \int \frac{9x^{2/3}}{x^{4/3}}\,dx \\
&= \int x^{5/3}\,dx + \int \frac{-2}{x}\,dx + \int 9x^{-2/3}\,dx \\
&= \frac{3}{8}x^{8/3} - 2\ln|x| + 27x^{1/3} + C
\end{aligned}$$

+2 (一個錯誤扣1)

+2 +1

   (b) Let $u = (\ln x)^2$, $dv = dx$. We have

+2

$$\begin{aligned}
\int (\ln x)^2\,dx &= (\ln x)^2 x - \int x 2\ln x \frac{1}{x}\,dx \\
&= (\ln x)^2 x - 2\int \ln x\,dx.
\end{aligned}$$

+2

Do the integration by part again by letting $u = \ln x$ and $dv = dx$. We have:

+2 +2

$$\begin{aligned}
(\ln x)^2 x - 2\int \ln x\,dx &= (\ln x)^2 x - 2x\ln x + 2\int x\frac{1}{x}\,dx \\
&= (\ln x)^2 x - 2x\ln x + 2x + C
\end{aligned}$$

+1 +1

   (c)

$$\begin{aligned}
\int \frac{4 + 2x}{x^2 + 1}\,dx &= \int \frac{4}{x^2 + 1}\,dx + \int \frac{2x}{x^2 + 1}\,dx \\
&= 4\tan^{-1} x + \int \frac{2x}{x^2 + 1}\,dx
\end{aligned}$$

+5

+2

For the second integral, make a subtitution by letting $u = x^2 + 1$, $du = 2x\,dx$.

$$\begin{aligned}
4\tan^{-1} x + \int \frac{2x}{x^2 + 1}\,dx &= 4\tan^{-1} x + \int \frac{1}{u}\,du \\
&= 4\tan^{-1} x + \ln|u| + C \\
&= 4\tan^{-1} x + \ln\left|x^2 + 1\right| + C
\end{aligned}$$

+2

+5 +1

2. **A more difficult integration...** (25 points)

(a) (15 %) Find a function f and a number a such that $-7 + \int_a^{x^3} \frac{f(t)}{t^4}\, dt = 2x^{-1}$, for all $x > 0$.

(b) (10 %) Simplify the integral $\int x f''(x)\, dx$ in terms of powers of x (or x^2, x^3, ...), $f(x)$ and $f'(x)$. (Your answer cannot include either integrals nor $f''(x)$.)

For example:

i. "$f'(x) + xf(x) + 3x^2 + C$ ", is a valid answer.

ii. "$f'(x) + \int xf(x)\, dx + 3x^2 + C$ ", is **not** a valid answer.

Sol:

+3

i. Let $x = a^{1/3}$. We have $7 + \int_a^a \frac{f(t)}{t^4}\, dt = 2a^{-1/3} = 7 + 0 = \frac{2}{a^{1/3}}$. Therefore, $a = (\frac{2}{7})^3$.

For f, we can differentiate the whole equation:

+2 +5

$$\frac{d}{dx}\left(-7 + \int_a^{x^3} \frac{f(t)}{t^4}\, dt\right) = \frac{d}{dx} 2x^{-1} \Longrightarrow 3x^2 \frac{f(x^3)}{x^{12}} = -2x^{-2}$$

+2

Therefore, $f(x^3) = -\frac{2}{3}x^8$. And $f(x) = -\frac{2}{3}x^{\frac{8}{3}}$ +3

ii. Let $u = x$, $dv = f''(x)\, dx$. We have:

$$\int x f''(x)\, dx = xf'(x) - \int f'(x)\, dx = xf'(x) - f(x) + C$$

+3 +3 +3 +1

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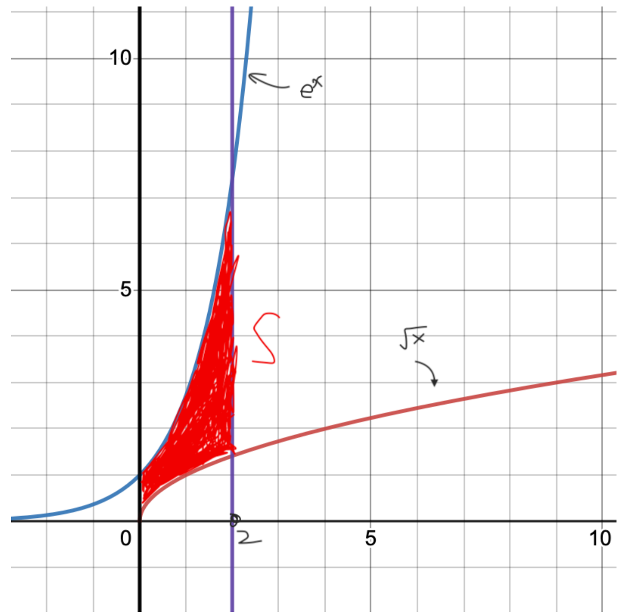
3. **Applications of integration...** (30 points)
Let "S" be the region which is above $g(x) = \sqrt{x}$ and below $f(x) = e^x$, with $x \in [0, 2]$. (See the figure below.) Calculate the volume of the solid obtained by:

(a) (15 %) rotating "S" about "x-axis".

(b) (15 %) rotating "S" about "$x = -1$".



Sol:

(a)

$$\begin{aligned} Volume &= \int_0^2 \pi(f(x)^2 - g(x)^2)\,dx \\ &= \int_0^2 \pi(e^{2x} - x)\,dx \\ &= \pi(\frac{1}{2}e^{2x} - \frac{1}{2}x^2)\Big|_0^2 \\ &= \pi(\frac{e^4 - 5}{2}). \end{aligned}$$

+1 +3 +1 +1 +1 +3 +2 +3

$\int xe^x dx = xe^x - \int e^x dx$
$= (x-1)e^x$

$2\pi\left(xe^x + e^x - x^{\frac{3}{2}} - x^{\frac{1}{2}}\right)$

(b)

$$\begin{aligned} Volume &= \int_0^2 2\pi(x+1)(f(x) - g(x))\,dx \\ &= \int_0^2 2\pi(x+1)(e^x - \sqrt{x})\,dx \\ &= 2\pi(2e^2 - \frac{44\sqrt{2}}{15}). \end{aligned}$$

+1 +2 +1 +1 +1 +1 +3 +2 +3

$= 2\pi\left(xe^x - \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right)\Big|_0^2$

4. **Finally! Your best friend L'H and integration...** (15 points)
Try to find the following limit. (*Hint: Remember to use your best friend L'Hospital's Rule to conquer it.*

$$\lim_{x\to 0} \frac{x^2}{\int_{x^2}^{x^3} (216 - t^3)^{1/3}\, dt}$$

+2

$$\begin{aligned}
(L'H \longrightarrow) \quad & \lim_{x\to 0} \frac{2x}{3x^2(216-(x^3)^3)^{1/3} - 2x(216-(x^2)^3)^{1/3}} \\
= \quad & \lim_{x\to 0} \frac{2}{3x(216-x^9)^{1/3} - 2(216-x^6)^{1/3}} \\
= \quad & -\frac{1}{6}.
\end{aligned}$$

+2 +2 +2 +2

+5