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M09 Calculus 4 Quiz 1

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1. For each of the following vector fields, find its curl and determine whether it is a gradient field.

(a) $\vec{F} = (4xy + x^3)\vec{i} + (2x^2 + z^2)\vec{j} + (2yz - z)\vec{k}$ [10%]

(b) $\vec{G} = (2yz)\vec{i} + (z^2 - 2xz)\vec{j} + (2xy + 2yz)\vec{k}$ [10%]

(a) $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy+x^3 & 2x^2+z^2 & 2yz-z \end{vmatrix} = (2z-2z)\vec{i} + (0-0)\vec{j} + (4x-4x)\vec{k} = 0$

+10

and $\vec{F}$ is simply connected in any region in $\mathbb{R}^3$,

$\therefore$ $\vec{F}$ is a gradient vector field ※

(b) $\operatorname{curl}(\vec{G}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz & z^2-2xz & 2xy+2yz \end{vmatrix}$

+10

$= (2x+2z-(2z-2x))\vec{i} + (2y-2y)\vec{j} + (-2z-2z)\vec{k} \neq 0$

$\therefore$ $\vec{G}$ is not a gradient vector field ※

$x^2 + \frac{y^2}{4} = 1$

$x^2 + (\frac{y}{2})^2 = 1$

$\int_C$ $\quad \int_C P\,dx + Q\,dy$

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2. Evaluate the line integral $\int_C \left(-x - y + \frac{y^2}{2}\right) dx + (x + 2xy + 3)\, dy$ along the given curve C.

P $\qquad Q$

(a) C is the line segment from $(0, -1)$ to $(0,1)$. [10%]
(b) C is the half ellipse $4x^2 + y^2 = 4$, $x \geq 0$, from $(0,1)$ to $(0, -1)$.[15%]

(a) parametric $r(t) = (0, t)$; $-1 \le t \le 1$

+10 $\qquad r'(t) = (0,1)$

$\vec{F} = (-x-y+\frac{y^2}{2})\,\vec{i} + (x+2xy+3)\,\vec{j}$

$\int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1} \langle -t+\frac{t^2}{2}, 3\rangle \cdot (0,1)\, dt$

$= \int_{-1}^{1} 3\, dt = 6$ #

(b) $x^2+4y^2=4 \Rightarrow (\frac{x}{2})^2 + y^2 = 1 \Rightarrow \frac{x}{2} = \cos\theta \Rightarrow x = 2\cos\theta$

+15 $\qquad y = \sin\theta \Rightarrow y = \sin\theta$

$(2\cos\theta, \sin\theta)$

let E be opposite orientation of C.

$\int_E \vec{F}\cdot d\vec{r} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \langle -2\cos\theta - \sin\theta + \frac{\sin^2\theta}{2}, 2\cos\theta + 2\sin\theta\cos\theta + 3\rangle \cdot \langle -2\sin\theta, \cos\theta\rangle\, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cancel{4\sin\theta\cos\theta} + 2\sin^2\theta - \cancel{\sin^3\theta} + 2\cos^2\theta + \cancel{2\sin\theta\cos^2\theta} + 3\cos\theta\right) d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2+3\cos\theta)\, d\theta = (2\theta + 3\sin\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\pi + 6$

$\int_C \vec{F}\cdot d\vec{r} = \int_{-E} \vec{F}\cdot d\vec{r} = -\int_E \vec{F}\cdot d\vec{r} = -2\pi - 6$ #

$\frac{1}{\cos\theta}$ $\frac{1}{\cos\theta}$

$\sqrt{2}$ $\sqrt{2}$

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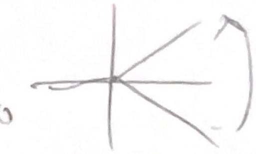


3. Consider the vector field defined by $\vec{G}(x,y) = (3x^2+y)\vec{i} + (2x^2y - x)\vec{j}$, $(x,y) \in \mathbb{R}^2$

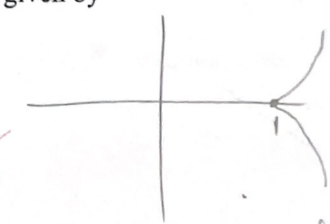
(a) Find a function $\mu(x)$ with $\mu(1) = 1$ such that $\mu(x)G(x,y)$ is conservative on its domain. [10%]

(b) Set $\vec{F}(x,y)$ to be the conservative vector field in (a). Find the potential function $f(x,y)$ of $\vec{F}$ with $f(1,0) = 3$. [10%]

(c) Let C be the curve with defining equation in polar coordinate given by

$$r = \sec\theta,\ \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right].$$

Evaluate the integral $\int_C \vec{F} \cdot d\vec{r}$. [5%]



(a) identify $P(x,y) = \mu(x)(3x^2+y)$, $Q(x,y) = \mu(x)(2x^2y-x)$

+6

$Q_x = \mu'(x)(2x^2y-x) + \mu(x)(4xy-1)$, $P_y = \mu(x)$

for $Q_x - P_y = 0 \Rightarrow \mu'(x)(2x^2y-x) + \mu(x)(4xy-2) = 0$

$\mu(1) = 1 \Rightarrow \mu'(1)(2y-1) + 2(2y-1) = 0 \Rightarrow (\mu'(1)+2)(2y-1) = 0$

$\mu'(1) = -2$

$\mu(1) = 1$

$\mu(x) = \cancel{ax+b}$

(b) $F = (-2x+3)(3x^2+y)\vec{i} + (-2x+3)(2x^2y-x)\vec{j}$

+0

$= (-6x^3 - 2xy + 9x^2 + 3y)\vec{i} + (-4x^3y + 2x^2 + 6x^2y - 3x)\vec{j}$

$f_x = -6x^3 - 2xy + 9x^2 + 3y \Rightarrow f = -\frac{6}{4}x^4 - yx^2 + 3x^3 + 3xy + A(y)$

$f_y = -4x^3y + 2x^2 + 6x^2y - 3x \Rightarrow f = -2x^3y^2 + 2x^2y + 3x^2y^2 - 3xy + B(x)$

(c) +0

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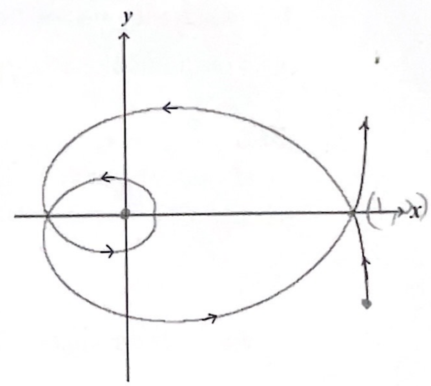
4. Let $\vec{F}(x,y) = \dfrac{-4y}{x^2+y^2}\vec{i} + \dfrac{4x}{x^2+y^2}\vec{j}$.

(a) Verify that $\vec{F}$ is conservative on the right half plane $x > 0$. Find the potential function of $\vec{F}$ on the right half plane. [15%]

(b) Evaluate $\oint_{C_1} \vec{F}\cdot d\vec{r}$ where C_1 is the curve with polar equation $r = e^{|\theta|}$, $-\frac{9}{4}\pi \le \theta \le \frac{9}{4}\pi$. [15%]



(a) identify $P(x,y) = \dfrac{-4y}{x^2+y^2}$, $Q(x,y) = \dfrac{4x}{x^2+y^2}$

+14 $Q_x = \dfrac{4(x^2+y^2) - 4x(2x)}{(x^2+y^2)^2} = \dfrac{-4x^2+4y^2}{x^2+y^2}$, $P_y = \dfrac{-4(x^2+y^2)+4y(2y)}{(x^2+y^2)^2} = \dfrac{-4x^2+4y^2}{(x^2+y^2)^2}$

for $Q_x - P_y = 0$ and $\{(x,y)\in\mathbb{R}^2, x>0\}$ is simply connected

this implies that F is conservative on the right half plane $x>0$ #

take $f = 4\tan^{-1}(\frac{y}{x}) \Rightarrow \nabla f = \left(\dfrac{\frac{-4y}{x^2}}{1+(\frac{y}{x})^2}, \dfrac{\frac{4}{x}}{1+(\frac{y}{x})^2}\right) = \vec{F}$

Hence, potential function $f = 4\tan^{-1}(\frac{y}{x}) + C$ #

(b) decompose curve to :

+12 positive-oriented

for any closed curve enclosed the origin, let curve C_ε be a circle small enough to fit in

then F is conservative in the region enclosed ⇒ can apply Green

$\iint_D Q_x - P_y \, dA = 0 \overset{\text{Green}}{=\!=} \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_\varepsilon} \vec{F}\, dr \Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_\varepsilon} \vec{F}\, dr$

$\vec{r}(\theta) = (r\cos\theta, r\sin\theta)$, $\vec{r}'(\theta) = (-r\sin\theta, r\cos\theta) \Rightarrow \oint_{C_\varepsilon} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \frac{4}{r^2}(-r\sin\theta, r\cos\theta)\cdot(-r\sin\theta, r\cos\theta)\, d\theta$
$= 8\pi$

$\therefore \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_3} \vec{F}\cdot dr = 8\pi$

又 因 for $x>0 \Rightarrow F$ is conservative $\Rightarrow \int_{C_4} \vec{F}\cdot d\vec{r} = \underbrace{f(0) - f(-\tfrac{9}{4}\pi)}_{9\pi}$, $\int_{C_5} \vec{F}\cdot d\vec{r} = \underbrace{f(\tfrac{9}{4}\pi) - f(0)}_{9\pi}$

$\therefore \oint_{C_1} \vec{F}\cdot d\vec{r} = 8\pi + 8\pi + 9\pi + 9\pi = 34\pi$ #