National Taiwan University - Calculus 1 for Class 01-09

Worksheet 3 : Related Rates

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Introduction.

Everything changes with time. The most natural intuition for derivatives is thinking of how variables change over time, in other words, think of every variable as a function of time.

With Chain Rule we can understand how y changes with time if $y = f(x)$ and the rate of change for x is given.

$$y = f(x) \;\Rightarrow\; y(t) = f(x(t)) \;\Rightarrow\; \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Problems of Related Rates ask students to first identify all geometric or algebraic relations between variables. They are presented as equations. The next step is similar to Implicit Differentiation but we differentiate with respect to t and apply Chain Rule for every variable. The resulting equation showcases relations between derivatives, hence the name Related Rates.

Example. Consider a square with side length x. Let y be the area of the square. Then

$$y = x^2 \;\Rightarrow \frac{dy}{dt} = 2x \cdot \frac{dx}{dt}.$$

If the side length is shrinking at a rate of 1 cm/s when the side length is equal to 10 cm, then the area must be shrinking as well. The rate of change of the area is -20 cm^2/s, meaning that the area is shrinking at the rate of 20 cm^2/s at that exact moment.

Remark. The above discussion also works for functions that are implicitly defined by an equation.

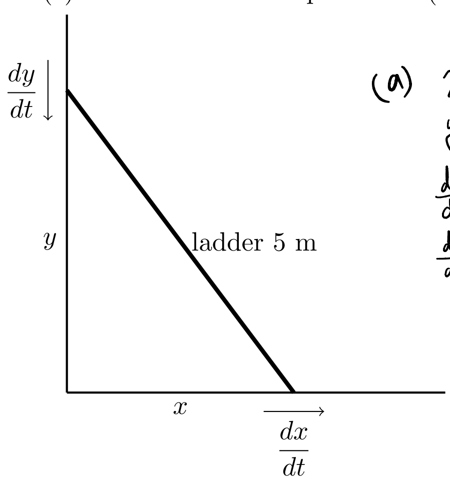
Exercise 1. A ladder 5 m long rests against a vertical wall. The bottom of the ladder slides away from the wall at a rate of 1 m/s. We want to find out how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 m from the wall.

$x=3$

(a) According to the diagram, describe in words the meanings of the variables x, y and the derivatives $\frac{dx}{dt}$, $\frac{dy}{dt}$.

(b) Find an equation that relates x and y.

(c) Differentiate the equation in (b) with respect to t and hence solve the problem.



(a) x: the length between the wall and the bottom of the ladder

y: the length between the floor and the top of the ladder

$\frac{dx}{dt}$: the variation of x with respect to t $\Rightarrow$ the velocity of increasing x

$\frac{dy}{dt}$: the variation of y with respect to t $\Rightarrow$ the velocity of decreasing y

(b) $x^2+y^2=5^2$

(c) 1° $x^2+y^2=25$

2° $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$

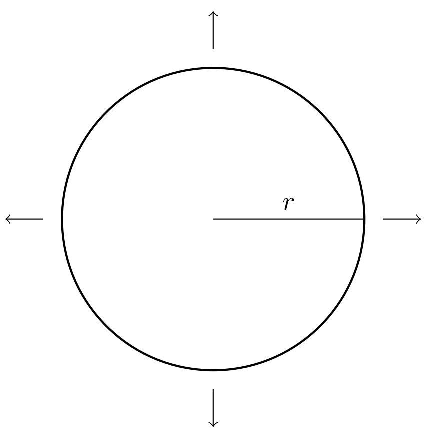
3° given that $\frac{dx}{dt}=1$, $x=3$ $\Rightarrow$ $y=4$

$\Rightarrow 3\cdot 1+4\cdot\frac{dy}{dt}=0 \Rightarrow \frac{dy}{dt}=\frac{-3}{4}$ ※

Ans: $\frac{-3}{4}$ m/s

Why Related Rates?
In the real world, some quantities are easier to measure than others. By studying their related rates, we can measure the rate of change of a certain quantity in terms of that of another quantity which might be easier to measure. The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to the time variable.

Exercise 2. Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast must the area of the spill increasing when the radius is 20 m?



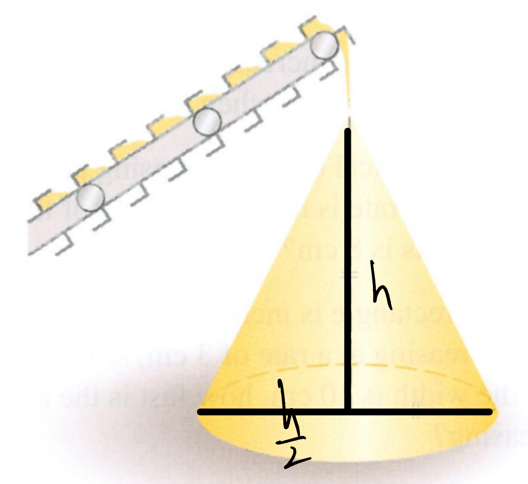
1° let Area = A $\Rightarrow$ $A = \pi r^2$

2° $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$, and given that $\frac{dr}{dt} = 1$

$= 2 \cdot \pi \cdot 20 \cdot 1 = 40\pi$

Ans: 40π m^2/s

Exercise 3. Gravel is being dumped from a conveyor belt at a rate of 3 m^3/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 3 m high?



let h be the height of the pile

1° $V = \frac{1}{3} \times \pi \left(\frac{h}{2}\right)^2 \times h = \frac{\pi}{12} h^3$

2° $\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$, given that $h = 3$

$\Rightarrow 3 = \frac{\pi}{4} \times 9 \times \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{4}{3\pi}$ (m/min)

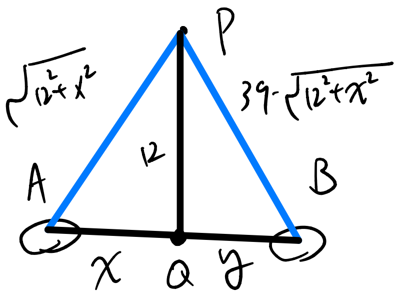
Ans: $\frac{4}{3\pi}$ m/min

Summary.

Let us tackle more complicated problems with the following problem solving strategy.

1. Read the problem carefully and draw a diagram.
2. Introduce notation and write equation(s) that relate the variables.
3. Use the Chain Rule to differentiate both sides and plug in the given information.

Exercise 4. Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P. The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q?



1° $y^2+12^2 = 39^2+12^2+x^2-2\cdot 39\cdot\sqrt{12^2+x^2}$

2° $2y\frac{dy}{dt} = 2x\frac{dx}{dt} - 39\left(12^2+x^2\right)^{-\frac{1}{2}}\cdot 2x\cdot\frac{dx}{dt}$

$$= \left(2x-\frac{78x}{\sqrt{12^2+x^2}}\right)\frac{dx}{dt}$$

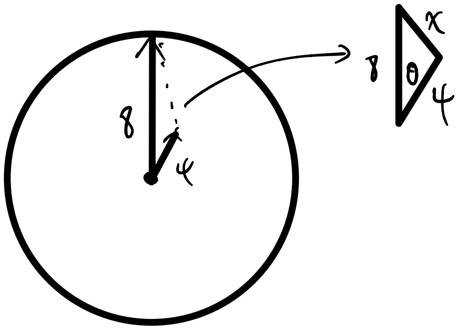
3° given that $x=5$, $\frac{dx}{dt}=2 \Rightarrow y=\sqrt{26^2-12^2}=2\sqrt{133}$

$$\Rightarrow 2\cdot 2\sqrt{133}\cdot\frac{dy}{dt} = \left(2\cdot 5-\frac{78\cdot 5}{13}\right)\cdot 2$$

$$\Rightarrow \frac{dy}{dt} = \frac{-10\sqrt{133}}{133}$$

Ans: $\frac{10\sqrt{133}}{133}$ ft/min

Exercise 5. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?



1° $\dfrac{8^2+4^2-x^2}{2\cdot 8\cdot 4} = \cos\theta \Rightarrow 80-x^2 = 64\cos\theta$

2° $-2x\dfrac{dx}{dt} = -64\sin\theta\dfrac{d\theta}{dt} \Rightarrow \dfrac{dx}{dt} = 32\sin\theta\dfrac{d\theta}{dt}\times\dfrac{1}{x}$

3° given that

① current tim is 1:00

$\therefore\ \theta = \frac{\pi}{6} \Rightarrow x = 4\sqrt{5-2\sqrt{3}}$

$\Downarrow$

$\sin\theta = \frac{1}{2}$

② the angular velocity of hour hand is $\frac{2\pi}{12}$ rad/hr

the angular velocity of minute hand is $\frac{2\pi}{1}$ rad/hr

$\therefore$ the variation of the angular velocity of both hand

is $\frac{2\pi}{1} - \frac{2\pi}{12} = \frac{22\pi}{12}$ ~~for~~ $\frac{3}{60}$

$\Rightarrow \dfrac{d\theta}{dt} = \dfrac{22\pi}{12}$ rad/hr

from ① and ②, $\dfrac{dx}{dt} = 32\times\dfrac{1}{2}\times\dfrac{22\pi}{12}\times\dfrac{1}{4\sqrt{5-2\sqrt{3}}}$

≈ 18.6

Ans: -18.6 mm/hr