Calculus II Quiz I

Department: Name: Id:

* Minor error (-1)

1. **A taste to integration...** (30 points):
   Evaluate the following indefinite integral:

   (a) (5 %) $\displaystyle\int \frac{x^2 + 3x^{1/3} + 8x^{2/3}}{x^{4/3}}\, dx.$ (*Hint: This is very easy.*)

   (b) (10 %) $\displaystyle\int x(\ln x)^2\, dx.$

   (c) (15 %) $\displaystyle\int \frac{12 + 2x}{4x^2 + 1}\, dx.$

   Sol:

   (a)

$$
\begin{aligned}
\int \frac{x^2 + 3x^{1/3} + 8x^{2/3}}{x^{4/3}}\, dx &= \int \frac{x^2}{x^{4/3}}\, dx + \int \frac{3x^{1/3}}{x^{4/3}}\, dx + \int \frac{8x^{2/3}}{x^{4/3}}\, dx \\
&= \underline{\int x^{2/3}\, dx + \int \frac{3}{x}\, dx + \int 8x^{-2/3}\, dx} \quad {+2} \\
&= \underbrace{\frac{3}{5}x^{5/3} + 3\ln|x| + 24x^{1/3}}_{+2} + \underbrace{C}_{+1}.
\end{aligned}
$$

+2 (-一個錯誤扣1)

   (b) Let $u = (\ln x)^2$, $dv = x dx$. We have

$$
\begin{aligned}
\int x(\ln x)^2\, dx &= \overset{+2}{\underline{\frac{1}{2}(\ln x)^2 x^2}} - \underline{\int x^2 \ln x \frac{1}{x}\, dx} \quad {+2} \\
&= \frac{1}{2}(\ln x)^2 x^2 - \int x \ln x\, dx \\
&= \frac{1}{2}(\ln x)^2 x^2 - \overset{+2}{\underline{\frac{1}{2}x^2 \ln x}} + \overset{+2}{\underline{\frac{1}{2}\int x^2 \frac{1}{x}\, dx}} \\
&= \frac{1}{2}(\ln x)^2 x^2 - \frac{1}{2}x^2 \ln x + \underbrace{\frac{1}{4}x^2}_{+1} + \underbrace{C}_{+1}.
\end{aligned}
$$

   (c)

$$
\begin{aligned}
\int \frac{12 + 2x}{4x^2 + 1}\, dx &= \int \frac{12}{4x^2 + 1}\, dx + \int \frac{2x}{4x^2 + 1}\, dx \\
&= \overset{+5}{\underline{6\tan^{-1} 2x}} + \int \frac{2x}{4x^2 + 1}\, dx
\end{aligned}
$$

   For the second integral, make a subtitution by letting $\overset{+2}{\underline{u = 4x^2 + 1,\ du = 8x\, dx}}$.

$$
\begin{aligned}
6\tan^{-1} 2x + \int \frac{2x}{4x^2 + 1}\, dx &= 6\tan^{-1} 2x + \underline{\frac{1}{4}\int \frac{1}{u}\, du} \quad {+2} \\
&= 6\tan^{-1} 2x + \frac{1}{4}\ln|u| + C \\
&= 6\tan^{-1} 2x + \underbrace{\frac{1}{4}\ln\left|4x^2 + 1\right|}_{+5} + \underbrace{C}_{+1}
\end{aligned}
$$

2. **A more difficult integration...** (25 points)

(a) (15 %) Find a function f and a number a such that $3+\int_{x^{\frac{1}{3}}}^{a} \frac{f(t)}{t^3}\,dt = 4x^{-1}$, for all $x>0$.

(b) (10 %) Simplify the integral $\int x f''(x)\,dx$ in terms of powers of x (or x^2, x^3, ...), $f(x)$ and $f'(x)$. (Your answer cannot include either integral nor $f''(x)$.)

For example:

i. "$f'(x)+xf(x)+3x^2+C$ ", is a valid answer.

ii. "$f'(x)+\int xf(x)\,dx+3x^2+C$ ", is **not** a valid answer.

Sol:

+3

(a) Let $x=a^3$. We have $3+\int_a^a \frac{f(t)}{t^3}\,dt = 4a^{-3} = 3+0 = \frac{4}{a^3}$. Therefore, $a=(\frac{4}{3})^{\frac{1}{3}}$. For f, we can differentiate the whole equation:

+2 +5

$$\frac{d}{dx}\left(3+\int_{x^{\frac{1}{3}}}^{a} \frac{f(t)}{t^3}\,dt\right) = \frac{d}{dx}4x^{-1} \Longrightarrow -\frac{1}{3}x^{-\frac{2}{3}}\frac{f(x^{\frac{1}{3}})}{x} = -4x^{-2}$$

+2

Therefore, $f(x^{\frac{1}{3}}) = -12x^{-\frac{1}{3}}$. And $f(x) = -\frac{12}{x}$ +3

(b) Let $u=x$, $dv=f''(x)\,dx$. We have:

$$\int xf''(x)\,dx = xf'(x) - \int f'(x)\,dx = xf'(x) - f(x) + C$$

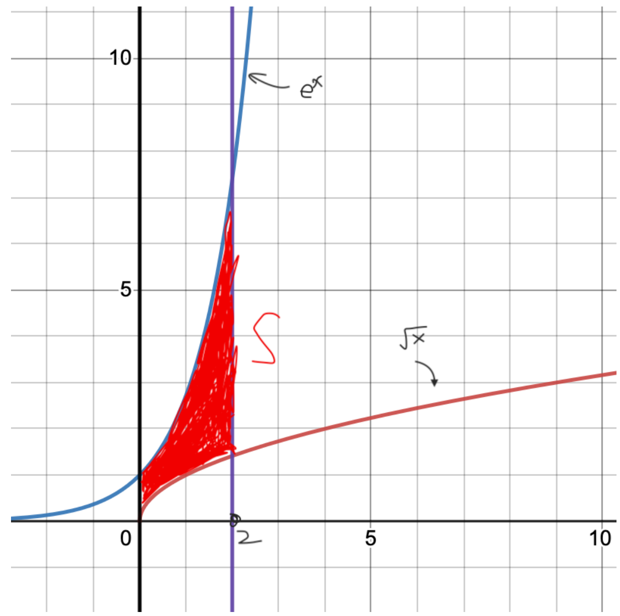
+3 +3 +3 +1

.

3. **Applications of integration...** (30 points)
Let "S" be the region which is above $g(x) = \sqrt{x}$ and below $f(x) = e^x$, with $x \in [0, 2]$. (See the figure below.) Calculate the volume of the solid obtained by:

(a) (15 %) rotating "S" about "y-axis".

(b) (15 %) rotating "S" about "$y = -2$".



Sol:

(a)

$$\begin{aligned} Volume &= 2\pi \int_0^2 x(f(x) - g(x))\, dx \\ &= 2\pi \int_0^2 x(e^x - \sqrt{x})\, dx \\ &= 2\pi(xe^x - e^x - \frac{2}{5}x^{\frac{5}{2}})\Big|_0^2 \\ &= 2\pi(1 - \frac{8\sqrt{2}}{5} + e^2). \end{aligned}$$

+3 +1 +1 +1 +1 +3 +2 +3

(b)

$$\begin{aligned} Volume &= \int_0^2 \pi((f(x) + 2)^2 - (g(x) + 2)^2)\, dx \\ &= \int_0^2 \pi(e^{2x} + 4e^x - 4\sqrt{x} - x)\, dx \\ &= \pi(\frac{1}{2}e^{2x} + 4e^x - \frac{8}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2)\Big|_0^2 \\ &= \pi(\frac{e^4}{2} + 4e^2 - \frac{16}{3}\sqrt{2} - \frac{13}{2}). \end{aligned}$$

+1 +1 +3 +1 +1 +3 +2 +3

4. **Finally! Your best friend L'H and integration...** (15 points)
Try to find the following limit. (*Hint: Remember to use your best friend L'Hopital's Rule to conquer it.*

$$\lim_{x\to 0} \frac{x^2}{\int_{\cos x - 1}^{x^5} (125 - t^3)^{1/3}\, dt}$$

Sol:
(We cannot use L'H to solve the original problem. Therefore, we discussed not to count this problem as a score. Your overall score will be Score in Problem1 to Problem3 $\times\dfrac{100}{85}$. The problem here is the modified one.)

$$\begin{aligned}
(L'H \longrightarrow) \quad & \lim_{x\to 0} \frac{2x}{5x^4(125-(x^5)^3)^{1/3} + \sin x(125-(\cos x - 1)^3)^{1/3}} \\
= \quad & \lim_{x\to 0} \frac{2}{5x^3(125-(x^5)^3)^{1/3} + \frac{\sin x}{x}(125-(\cos x - 1)^3)^{1/3}} \\
= \quad & \frac{2}{5}.
\end{aligned}$$