National Taiwan University - Calculus 4 for Class 01-09

Worksheet : Parametric Surfaces

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Reference : Stewart §16.6

Parametrization of surfaces.

In Calculus 3, we learned double integrals for which integrations were defined over a planar region. We now aim to define integrals over an arbitrary surface (which is not necessarily flat). Similar to defining line integrals, we should begin with parametrizing the surfaces.

Example 1. (Graph of a function)

A parametrization of part of a graph $S = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\}$ whose projection on xy-plane is D is given by

$$\mathbf{r}(x, y) = \langle x,\, y,\, f(x, y)\rangle,\ x, y \in D.$$

Example 2. (Sphere of radius R)

Fix $R > 0$. A parametrization of the sphere $x^2 + y^2 + z^2 = R^2$ is given in *spherical coordinates* by

$$\mathbf{r}(\theta, \phi) = \langle R\sin\phi\cos\theta,\, R\sin\phi\sin\theta,\, R\cos\phi\rangle,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

Example 3. (Surface of revolution)

Consider the surface obtained by rotating the curve $y = g(z) \ge 0$, $z_1 \le z \le z_2$ about the z-axis. A parametrization of this surface is given in *cylindrical coordinates* by

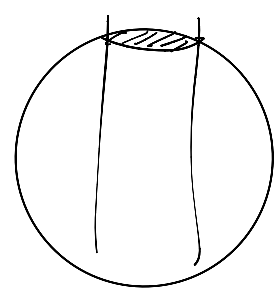
$$\mathbf{r}(z, \theta) = \langle g(z)\cos\theta,\, g(z)\sin\theta,\, z\rangle,\ 0 \le \theta \le 2\pi,\ z_1 \le z \le z_2.$$

Exercise 1. Consider the surface S that is part of the sphere $x^2+y^2+z^2 = 4$ that lies within the cylinder $x^2+y^2 = 1$ and above the xy-plane.

(a) Parametrize S as a graph.

(b) Parametrize S in cylindrical coordinates.

(c) Parametrize S in spherical coordinates.



(a)

$r_a = \langle x, y, \sqrt{4-x^2-y^2} \rangle$, $x^2+y^2=1$

(b)

$r_b(z,\theta) = \langle \cos\theta, \sin\theta, z \rangle$, $0 \le \theta \le 2\pi$, $\sqrt{3} \le z \le 2$

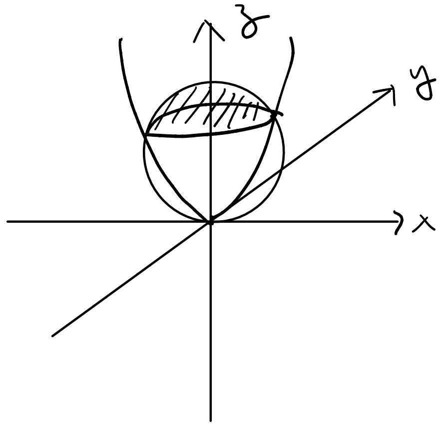
(c)

$r_c(\phi,\theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi \rangle$ $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{6}$

Exercise 2. Parametrize the following surfaces.

(a) The part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.

1°



$$x^2+y^2+z^2=4z \Rightarrow x^2+y^2+(z-2)^2 = 4$$
$$\Rightarrow (z-2)^2 = 4-x^2-y^2$$
$$\Rightarrow z = 2+\sqrt{4-x^2-y^2}$$

2° let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

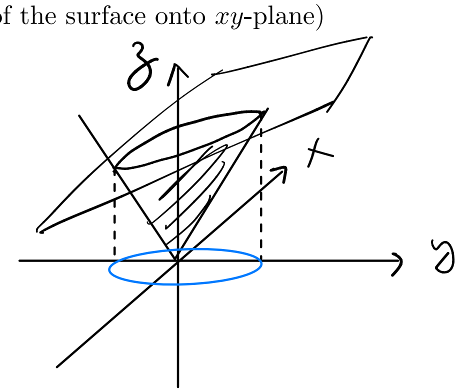
$$\begin{cases} x^2+y^2 = z \Rightarrow z = r^2 \\ x^2+y^2+z^2=4z \Rightarrow r^2+z^2=4z \Rightarrow z^2-3z=0 \Rightarrow z=0 \text{ or } z=3 \end{cases}$$

3° $r_A(r,\theta) = \langle r\cos\theta, r\sin\theta, 2+\sqrt{4-r^2} \rangle$

$0 \le r \le \sqrt{3}, \quad 0 \le \theta \le 2\pi$

(b) The part of the cone $z = \sqrt{2(x^2 + y^2)}$ that lies below the plane $z = 1 + y$. (Hint. First find the projection of the surface onto xy-plane)

1°



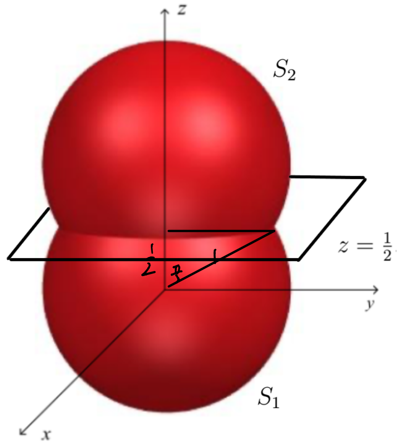
let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\vec{r}_A(r,\theta) = \langle r\cos\theta, r\sin\theta, \sqrt{2}r \rangle$$

$$z = 1+r = 1+r\sin\theta = \sqrt{2}r \Rightarrow r = \frac{1}{\sqrt{2}-\sin\theta}$$

$$\therefore \vec{r}_A(r,\theta) = \langle r\cos\theta, r\sin\theta, \sqrt{2}r \rangle, \quad 0 \le r \le \frac{1}{\sqrt{2}-\sin\theta}, \quad 0 \le \theta \le 2\pi$$

Exercise 3. In this question, a surface is formed by a union of two spherical surfaces.



(a) The lower part S_1 of the surface is part of the sphere $x^2 + y^2 + z^2 = 1$ that lies below the plane $z = \frac{1}{2}$. Find a parametrization of S_1.

(b) The upper part S_2 of the surface is part of the sphere $x^2 + y^2 + (z-1)^2 = 1$ that lies above the plane $z = \frac{1}{2}$. Find a parametrization of S_2.

(a) $\vec{r}_a(\phi, \theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$, $\frac{\pi}{3} \le \phi \le \pi$, $0 \le \theta \le 2\pi$

(b) $\vec{r}_b(r, \theta) = \langle r\cos\theta, r\sin\theta, 1 + \sqrt{1-r^2} \rangle$, $0 \le r \le \frac{\sqrt{3}}{2}$, $0 \le \theta \le 2\pi$

$1 + \sqrt{1-r^2} = \frac{1}{2} \Rightarrow \sqrt{1-r^2} = \frac{-1}{2} \Rightarrow 1 - r^2 = \frac{1}{4}$

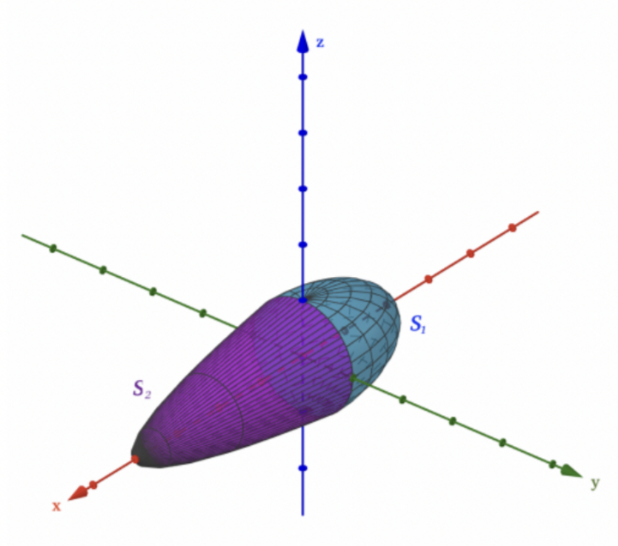
$\frac{3}{4} = r^2 \Rightarrow r = \frac{\sqrt{3}}{2}$

or

$\vec{r}_b(\phi, \theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, 1 + \cos\phi \rangle$, $0 \le \phi \le \frac{2\pi}{3}$, $0 \le \theta \le 2\pi$

Exercise 4. The shell of a spacecraft is a union of two surfaces :

- S_1 is part of $x^2 + 4y^2 + 4z^2 = 4$ where $x \leq 0$.
- S_2 is part of $x = 4 - 4y^2 - 4z^2$ where $x \geq 0$.



(a) Find a parametrization of S_1.

(b) Find a parametrization of S_2.

(a) $x^2+4y^2+4z^2=4$, $x \leq 0$

let $\begin{cases} x = 2\sin\phi\cos\theta \\ 2y = 2\sin\phi\sin\theta \\ 2z = 2\cos\phi \end{cases} \Rightarrow \vec{r}_a(\phi,\theta) = \langle 2\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$

where $0 \leq \phi \leq \pi$, $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

(b) $x = 4-4y^2-4z^2$, $x \geq 0$

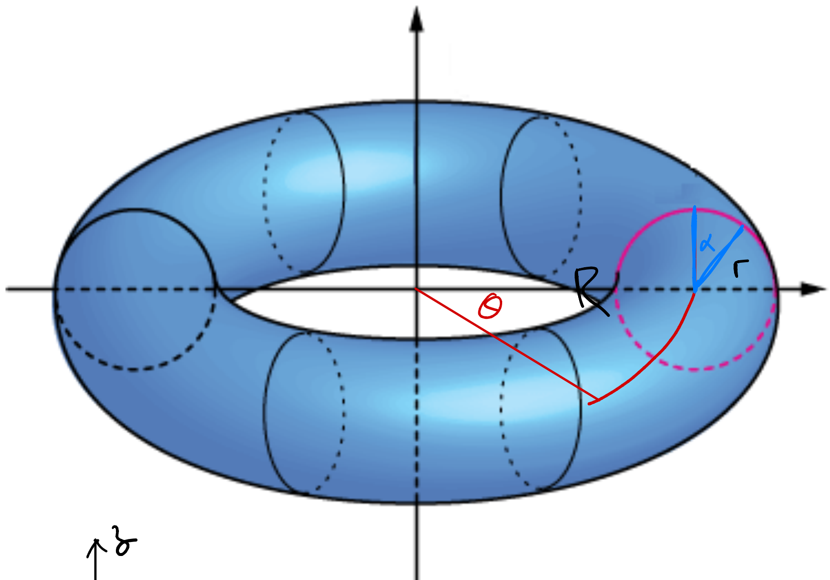
$4(y^2+z^2) = 4-x \Rightarrow y^2+z^2 = 1-\frac{x}{4}$

let $\begin{cases} x = t \\ y = \sqrt{1-\frac{t}{4}}\cos\theta \\ z = \sqrt{1-\frac{t}{4}}\sin\theta \end{cases}$

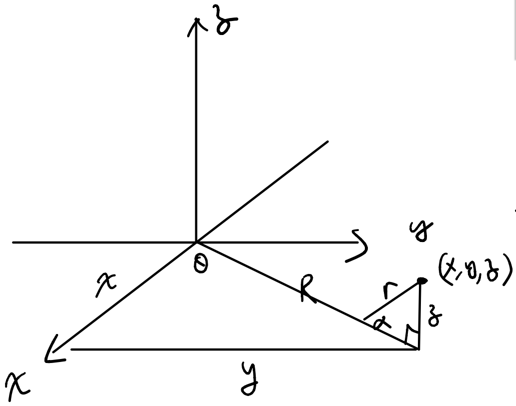
$\vec{r}_b(t,\theta) = \langle t, \sqrt{1-\frac{t}{4}}\cos\theta, \sqrt{1-\frac{t}{4}}\sin\theta \rangle$

where $0 \leq t \leq 4$, $0 \leq \theta \leq 2\pi$

Exercise 5. **(Optional)** A torus is formed by revolving a small circle of radius r along the circumference of a bigger circle of radius R. Find a parametrization of this torus.



1°



$$\Rightarrow \begin{cases} x = (R + r\cos\alpha)\cos\theta \\ y = (R + r\cos\alpha)\sin\theta \\ z = r\sin\alpha \end{cases}$$

$$\vec{r}(\theta, \alpha) = \langle (R + r\cos\alpha)\cos\theta,\ (R + r\cos\alpha)\sin\theta,\ r\sin\alpha \rangle$$

where $\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \alpha \le 2\pi \end{cases}$