Calculus 3 Quiz 1 Solution

Examination range: WS1, 12.6, 14.1-14.4

2024/03/13 Wed.

Department:____________ Name:____________ Id:____________

Total score: 100

Please write your argument or computation as detailed as possible.

1. (15%) Consider the curve with parametrization $r(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle$.
   (a) (5%) Determine the unit tangent vector at $t = 0$.
   (b) (10%) Compute the arc length of $r(t)$ with $0 \leq t \leq 1$.

Solution.

(a) Recall that the tangent vector of a curve $r(t) = \langle x(t), y(t), z(t)\rangle$ at t_0 is given by $\langle x'(t_0), y'(t_0), z'(t_0)\rangle$. Therefore, by direct computation, the tangent vector of $r(t)$ at $t = 0$ is $\langle \sqrt{2}, e^0, -e^0\rangle = \langle \sqrt{2}, 1, -1\rangle$. To obtain the unit tangent vector, we need to divide $r'(0)$ by its length, which is $\langle \frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2}\rangle$.

(b) Recall that the arc length of $r(t) = \langle x(t), y(t), z(t)\rangle$ for $a \leq t \leq b$ is defined by

$$s = \int_a^b \|r'(t)\|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt.$$

Therefore, by direct computation, we get

$$\begin{aligned} s &= \int_0^1 \|r'(t)\|\, dt = \int_0^1 \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2}\, dt \\ &= \int_0^1 \sqrt{(e^t + e^{-t})^2}\, dt = \int_0^1 e^t + e^{-t}\, dt = e^t - e^{-t}\Big|_0^1 \\ &= (e - e^{-1}). \end{aligned}$$

2. (20%) Determine whether the following limits exist. If the limit exists, compute it. If the limit doesn't exist, prove it.

   (a) (10%) $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$.

   (b) (10%) $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2 + y^2}}$.

Solution.

(a) Notice that on the x-axis, we get 0, so the limit along x-axis is 0. However, on the direction $r(t) = \langle t^2, t^2, t\rangle$, we have

$$\frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4} = \frac{t^4 + t^4 + t^4}{t^4 + t^4 + t^4} = \frac{3t^4}{3t^4} = 1,$$

so the limit along $r(t)$ is 1. Since $0 \neq 1$, we know that the limit DNE.

(b) Consider the polar coordinate substitution, we have

$$\frac{xy}{\sqrt{x^2 + y^2}} = \frac{r^2 \cos(\theta)\sin(\theta)}{r} = r\cos(\theta)\sin(\theta).$$

Now, notice that

$$0 \leq |r\cos(\theta)\sin(\theta)| \leq |r|$$

and it is clear that $|r| \to 0$ as $r \to 0$. Therefore, by squeeze theorem, the limit is 0.

3. (40%) Consider the function

$$f(x,y) = \begin{cases} \frac{x^2+y^2}{4}\ln(x^2+y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) (15%) Compute $f_x(x,y)$ for both $(x,y) \neq (0,0)$ and $(x,y) = (0,0)$.

(b) (10%) Is $f(x,y)$ continuous at $(0,0)$?

(c) (15%) Compute the linearization of $f(x,y)$ near $(0,0)$ and deduce that $f(x,y)$ is differentiable at $(0,0)$.

Solution.

(a) For $(x,y) \neq (0,0)$, by direct computation, we have

$$f_x(x,y) = \frac{x}{2}\ln(x^2+y^2) + \frac{x^2+y^2}{4}\cdot\frac{2x}{x^2+y^2} = \frac{x}{2}(\ln(x^2+y^2)+1).$$

For $(x,y) = (0,0)$, by the definition of partial derivative, we have

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{h^2}{4}\ln(h^2)-0}{h} = \frac{1}{2}\lim_{h\to 0} h\ln(|h|) \\ &= \frac{1}{2}\lim_{h\to 0}\frac{\ln(|h|)}{h^{-1}} = \frac{1}{2}\lim_{h\to 0}\frac{h^{-1}}{-h^{-2}} = \frac{1}{2}\lim_{h\to 0} -h = 0. \end{aligned}$$

(b) Take the polar coordinate substitution, we have $f(r,\theta) = \frac{r^2}{4}\ln(r^2)$. Therefore,

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{r\to 0^+} f(r,\theta) = \lim_{r\to 0^+}\frac{r^2}{4}\ln(r^2) = \frac{1}{2}\lim_{r\to 0^+}\frac{\ln(r)}{r^{-2}} \\ &= \frac{1}{2}\lim_{r\to 0^+}\frac{r^{-1}}{(-2)r^{-3}} = \frac{1}{2}\lim_{r\to 0^+}\frac{r^2}{-2} = 0 \end{aligned}$$

(c) We first compute the partial derivatives at $(0,0)$. By (a), we have $f_x(0,0) = 0$. By symmetricity, we also have $f_y(0,0) = 0$. So, the linearization of $f(x,y)$ at $(0,0)$ is

$$L(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y = 0.$$

We have already computed the polar coordinate of f, which is $f(r,\theta) = \frac{r^2}{4}\ln(r^2)$. Then limit yields

$$\lim_{r\to 0}\frac{|f(r,\theta)-L(r,\theta)|}{r} = \lim_{r\to 0}\frac{\left|\frac{r^2}{4}\ln(r^2)\right|}{r} = \lim_{r\to 0}\left|\frac{r}{4}\ln(r^2)\right| = 0.$$

Since the limit is 0, by definition of differentiability, $f(x,y)$ is differentiable at $(0,0)$.

4. (25%) Consider the function $z = f(x, y) = \sqrt{x^2 + 9y^2}$

   (a) (5%) Sketch the graph of the function (you should label the axis).

   (b) (10%) Compute the equation for tangent plane of the surface at $(4, 1, 5)$.

   (c) (10%) Use the linear approximation to estimate $\sqrt{(4.01)^2 + 9(1.01)^2}$

Solution.

(a) One should label the ratio of the width of the surface along x and y-axis.

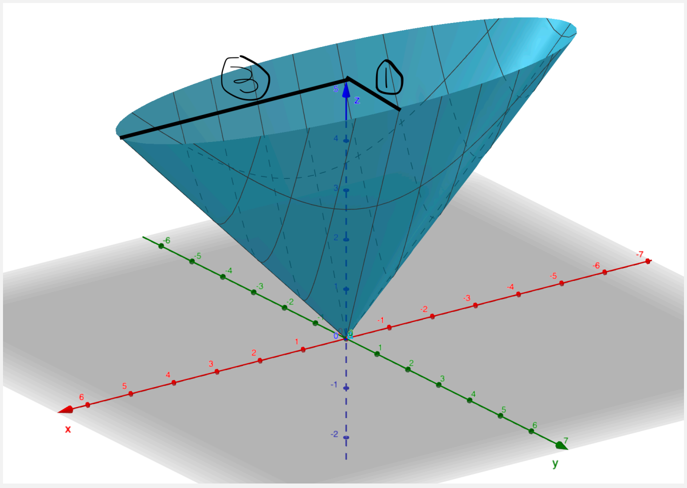

Figure 1: $z = \sqrt{x^2 + 9y^2}$

(b) Recall that the tangent plane of $z = f(x, y)$ at (x_0, y_0, z_0) is given by $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. We first compute the partial derivatives

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + 9y^2}}, \quad f_y(x, y) = \frac{9y}{\sqrt{x^2 + 9y^2}},$$

It follows that $f_x(4, 1) = \frac{4}{5}$ and $f_y(4, 1) = \frac{9}{5}$. Therefore, the equation for tangent plane at $(4, 1, 5)$ is

$$z = 5 + \frac{4}{5}(x - 4) + \frac{9}{5}(y - 1).$$

(c) By (b), the linear approximation $L(x, y)$ of $f(x, y)$ is

$$L(x, y) = 5 + \frac{4}{5}(x - 4) + \frac{9}{5}(y - 1).$$

Therefore, $\sqrt{(4.01)^2 + 9(1.01)^2} = f(4.01, 1.01) \approx L(4.01, 1.01) = 5 + \frac{4}{5}(4.01 - 4) + \frac{9}{5}(1.01 - 1) = 5.026$.