CSM
Spring 2019

Electrical Engineering 16A
Worksheet 3

1. Subspace Drills

Determine if the following describe subspaces.

(a) $\{\vec{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T : x_i \geq 0 \; \forall i = 1, \cdots, n\}$

(b) $\{\vec{0}\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

(d) $\text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

(e) **(PRACTICE)**
$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1 \right\}$

(f) **(PRACTICE)**
$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \right\}$

(g) **(CHALLENGE PRACTICE)**
Let V be the vector space of $n \times n$ matrices, and $M \in V$ a fixed matrix. We define the set

$$U = \{N \in V \,|\, NM = MN\}$$

as the *centralizer* of M in V. Show that U is a subspace of V.

(h) **(CHALLENGE PRACTICE)**
Suppose U and V are both subspaces of a vector space S, show that $U \cap V$ is also a subspace of S.

2. **Null Space Drill**

In this question, we explore intuition about null spaces and a recipe to compute them. Recall that the nullspace of a matrix $\mathbf{M}$ is the set of all vectors, $\vec{x}$ such that $\mathbf{M}\vec{x} = \vec{0}$.

(a) First, we begin by proving that a null space is indeed a subspace. Show that any nullspace of a matrix $\mathbf{M}$ with n rows and n columns is a subspace.

(b) Now we will explore a recipe to compute null spaces. Let's start with some 3×3 matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix}$$

$\mathbf{A}'$ is the row reduced matrix $\mathbf{A}$.

$$\mathbf{A}' = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -18 \end{bmatrix}$$

Compute the nullspace of $\mathbf{A}$.

(c) Consider another matrix

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 4 & -2 \\ -2 & 2 & -4 \end{bmatrix}$$

$\mathbf{B}'$ is row reduced $\mathbf{B}$.

$$\mathbf{B}' = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 8 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

What is the null space of $\mathbf{B}$? What is the dimension of the row space of $\mathbf{B}$?

(d) In the previous part, we chose one of the variables and set it to be a free variable. Can we choose any variable as our free variable?

(e) How can we know which variables can be used as free variables?

(f) Now consider another matrix, $\mathbf{C} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 2 & 4 & 12 & -17 \\ 1 & -4 & -12 & 22 \end{bmatrix}$ Without doing any math, will this matrix have a trivial nullspace, i.e. consisting of only $\vec{0}$?

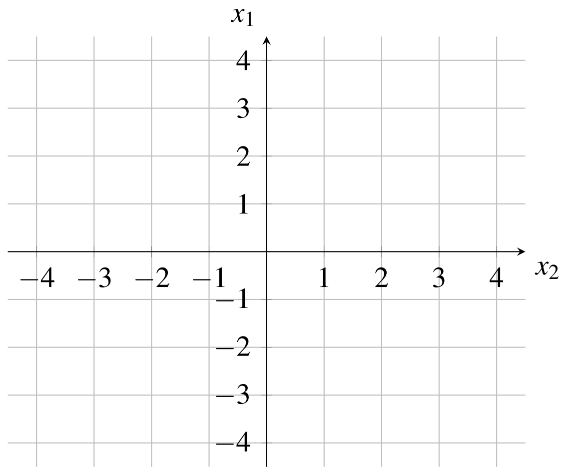
(g) **(PRACTICE)**

Consider another matrix, $\mathbf{D} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Find vector(s) that span the nullspace.

(h) **(PRACTICE)**

Consider one final matrix, $\mathbf{E} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. What are the vector(s) that span this nullspace?

3. Range Intuition



$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(a) Draw the space on the figure above that is represented by Col($\mathbf{A}$). Also draw the space for the Row($\mathbf{A}$) (which is the same as Col($\mathbf{A}^\top$). What dimension are these spaces?

(b) Plot the point $\vec{x}$, then plot $\mathbf{A}\vec{x}$

(c) Consider some arbitrary vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ Write out the product $\mathbf{A}\vec{v}$ in terms of v_1, v_2, and the columns of $\mathbf{A}$.

(d) We have talked about how matrices like $\mathbf{A}$ have no inverse. Give a geometric explanation for why this is the case.

(e) Consider all points $\vec{y}$ such that $\mathbf{A}\vec{y} = 0$ Draw the space that the $\vec{y}$'s will make up. What do you notice geometrically? What is the dimension of this space?

4. Mechanical Eigenvectors and Eigenvalues

(a) Solve for the eigenvalue-eigenvector pairs for the following 2 by 2 matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

(b) Find the eigenvectors for matrix A given that we know that $\lambda_1 = 4, \lambda_2 = \lambda_3 = -2$ and that

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

(c) Find the eigenvalues for matrix **A** given that we know that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ are the eigenvectors of **A**, and that

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

5. (PRACTICE) Nullspaces and Projections

Assume that the vector $\vec{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$. For each of the following matrices $\mathbf{A} \in \mathbb{R}^{n \times m}$, answer the following:

- Compute the matrix product $\mathbf{A}\vec{x}$. Explain in words how the matrix transforms the vector.
- Suppose you know that A transforms $\vec{x}$ to give $\vec{y}$. Given $\vec{y}$, can you find what the original vector $\vec{x}$ was?
- Is the matrix **A** invertible? How do you know? If it is invertible, find the inverse.
- Verify that (dimension of nullspace) + (dimension of column space) = min(n, m)

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \vec{y} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$

(d) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$