1. Eigenvalues Everywhere

(a) What is the one eigenvalue and eigenvector of the matrix that you can see without solving any equations?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

(b) What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{B} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(c) What are the eigenvalues of

$$\mathbf{C} = \begin{bmatrix} 2 & 0 \\ 3 & 4 \\ 1 & 3 \end{bmatrix}?$$

(d) Consider a matrix that rotates a vector in $\mathbb{R}^2$ by 45° counterclockwise. For instance, it rotates any vector along the x-axis to orient towards the $y = x$ line. Find its eigenvalues and corresponding eigenvectors. This matrix is given as

$$\mathbf{D} = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(e) What are the eigenvalues of the following matrix?

$$\mathbf{E} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

(f) Can you find an eigenvalue of the following matrix without solving any equations?

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

(g) Show that a matrix and its transpose have the same eigenvalues
Hint: The determinant of a matrix is the same as the determinant of its transpose

(h) Consider a matrix whose columns sum to one. What is one possible eigenvalue of this matrix?

2. Eigenvalues and Flow

In this question, we will examine how the eigenvalues of a matrix relate to how it actually acts on vectors. We will also try to interpret this in terms of flow of water between reservoirs.

For each of the following parts, assume that you have a matrix $\mathbf{A}$ with the listed eigenvalues and eigenvectors. Give the output of $\lim_{n\to\infty} \mathbf{A}^n\vec{x}$ for the provided vectors $\vec{x}$ which have all nonnegative entries (i.e. they represents some distribution of water in reservoirs). How can you interpret each of these matrices in terms of reservoirs and pumps? What must be true of the sums of elements in the columns?

(a)

$$\lambda_1 = \frac{1}{2}, \vec{v}_1 = \begin{bmatrix}-2 & 1 & 0\end{bmatrix}^T$$
$$\lambda_2 = \frac{1}{3}, \vec{v}_2 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$$
$$\lambda_3 = \frac{1}{4}, \vec{v}_3 = \begin{bmatrix}-1 & 1 & -2\end{bmatrix}^T$$
$$\vec{x} = \begin{bmatrix}0 & 5 & 1\end{bmatrix}^T$$

(b)

$$\lambda_1 = 2, \vec{v}_1 = \begin{bmatrix}-1 & -1 & 0\end{bmatrix}^T$$
$$\lambda_2 = 1, \vec{v}_2 = \begin{bmatrix}2 & 3 & 2\end{bmatrix}^T$$
$$\lambda_3 = \frac{1}{2}, \vec{v}_3 = \begin{bmatrix}0 & -1 & -1\end{bmatrix}^T$$
$$\vec{x}_1 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T \text{ and } \vec{x}_2 = \begin{bmatrix}2 & 2 & 1\end{bmatrix}^T$$

(c) **(PRACTICE)**

$$\lambda_1 = -1, \vec{v}_1 = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^T$$
$$\lambda_2 = \frac{1}{2}, \vec{v}_2 = \begin{bmatrix}3 & 1 & 1\end{bmatrix}^T$$
$$\lambda_3 = \frac{1}{4}, \vec{v}_3 = \begin{bmatrix}2 & 1 & 4\end{bmatrix}^T$$
$$\vec{x} = \begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$$

3. StateRank Car Rentals (21 points)

You are an analyst at StateRank Car Rentals, which operates in California, Oregon, and Nevada. You are hired to analyze the number of rental cars going into and out of each of the three states (CA, OR, and NV). The number of cars in each state on day $n \in \{0, 1, \ldots\}$ can be represented by the state vector $\vec{s}[n] = \begin{bmatrix} s_{\text{CA}}[n] \\ s_{\text{OR}}[n] \\ s_{\text{NV}}[n] \end{bmatrix}$.

The state vector follows the state evolution equation $\vec{s}[n+1] = \mathbf{A}\vec{s}[n], \forall n \in \{0,1,\dots\}$, where the transition matrix, $\mathbf{A}$, of this linear dynamic system is

$$\mathbf{A} = \begin{bmatrix} \frac{7}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{6}{10} & \frac{1}{10} \\ \frac{2}{10} & \frac{3}{10} & \frac{8}{10} \end{bmatrix}.$$

(a) We denote the eigenvalue/eigenvector pairs of the matrix $\mathbf{A}$ by

$$\left(\lambda_1 = 1, \vec{u}_1 = \begin{bmatrix} 50 \\ 40 \\ 110 \end{bmatrix}\right), \left(\lambda_2, \vec{u}_2 = \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}\right), \text{ and } \left(\lambda_3, \vec{u}_3 = \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix}\right).$$

Find the eigenvalues λ_2 and λ_3 corresponding to the eigenvectors $\vec{u}_2$ and $\vec{u}_3$, respectively. Note that since $\lambda_1 = 1$ is given, you don't have to calculate it.

(b) Suppose that the initial number of rental cars in each state on day 0 is

$$\vec{s}[0] = \begin{bmatrix} 7000 \\ 5000 \\ 8000 \end{bmatrix} = 100\vec{u}_1 - 100\vec{u}_2 - 200\vec{u}_3,$$

where $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$ are the eigenvectors from part (a).
After a very large number of days n, how many rental cars will there be in each state?
That is, i) calculate

$$\vec{s}^* = \lim_{n\to\infty} \vec{s}[n]$$

<u>and</u> ii) show that the system will indeed converge to $\vec{s}^*$ as $n \to \infty$ if it starts from $\vec{s}[0]$.
Hint: If you didn't solve part (a), the eigenvalues satisfy $\lambda_1 = 1, |\lambda_2| < 1$ and $|\lambda_3| < 1$.

4. Diagonalization and Other Things Related (ish)

(a) When is an $n \times n$ matrix diagonalizable, or able to be represented in the form $\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{D}$ is a diagonal matrix?

(b) Given eigenvalues $\lambda = 1, 2$, diagonalize this matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

(c) Consider a pump system with transition matrix $\mathbf{A}$, diagonalized as $\mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. Find the system state $\vec{s}[n]$ given state $\vec{s}[0]$

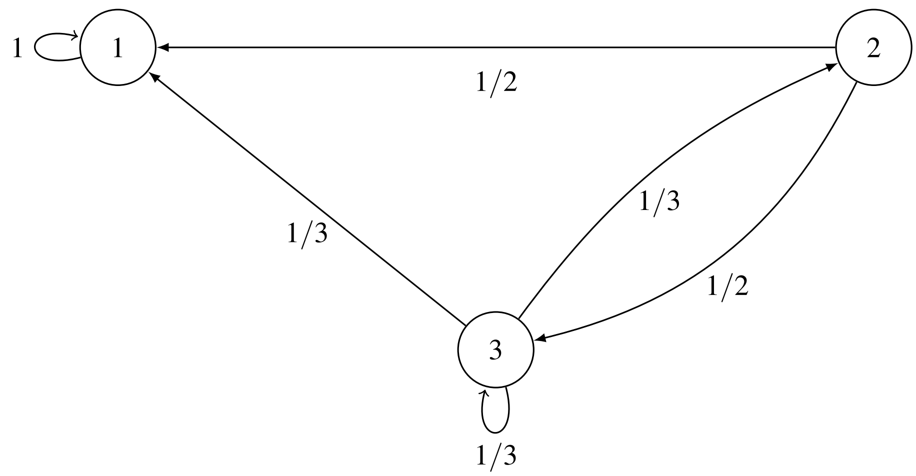
(d) Is there a relationship between invertibility and diagonizability?

i. First, let us consider: does invertibility imply diagonizability? Give a brief explanation or counterexample. (Hint: think about how linear independence plays a role in whether or not a matrix is invertible or diagonizable).

ii. Does diagonizability imply invertibility? (Hint: think about the invertibility of each individual matrix that constitutes the diagonalized matrix.)

5. (PRACTICE) Page Rank

Now suppose we have a network consisting of 3 websites connected as shown below. Each of the weights on the edges represent the probability of a user taking that edge.



(a) Write down the probability transition matrix for this graph, and call it $\mathbf{P}$. Can you say something about the eigenalues/eigenvectors of $\mathbf{P}^T$? (*Hint: Try to recall the properties of transition matrices*).

(b) We want to rank these webpages in order of importance. But first, find the eigenvector of $\mathbf{P}$ corresponding to eigenvalue 1.

(c) Now looking at the matrix $\mathbf{P}$, can you identify what its other eigenvalues are?

(d) Suppose that we start with 90 users evenly distributed among the websites. What is the steady-state number of people who will end up at each website?