CSM
Spring 2019

Electrical Engineering 16A
Worksheet 3

Meta:

Definition: Vector Space/Subspace

A *vector space* V is a set of elements that is

(a) closed under vector addition.
$\forall \vec{v}_1, \vec{v}_2 \in V$
$\vec{v}_1 + \vec{v}_2 \in V$

(b) closed under scalar multiplication.
$\forall \vec{v} \in V, \alpha \in \mathbb{R}$
$\alpha \vec{v} \in V$

An important implication of these two properties is that a vector space (or subspace) contains a zero vector.

For the sake of clarity and brevity, we won't formally describe the properties of vector addition (associative, commutative, additive identity, and additive inverse) and scalar multiplication (associative, multiplicative identity, distributive in vector addition, and distributive in scalar addition) upon which the above definition is based.

Meta:

Review: Four Fundamental Subspaces

Any $m \times n$ matrix $\mathbf{A}$ has four subspaces:

- **Column Space:** $C(\mathbf{A})$ consists of all combinations of the columns of $\mathbf{A}$ and is a vector space in $\mathbb{R}^m$
- **Nullspace:** $N(\mathbf{A})$ consists of all solutions $\vec{x}$ to the equation $\mathbf{A}\vec{x} = 0$, and lies in $\mathbb{R}^n$.
- **Row Space:** $C(\mathbf{A^T})$ The combinations of the row vectors of $\mathbf{A}$, lies in $\mathbb{R}^n$. The rowspace is equal to, and represented as, the column space of the transpose of $\mathbf{A}$.
- **Left Nullspace:** $N(\mathbf{A^T})$ consists of all solutions $\vec{x}$ such that $\mathbf{A^T}\vec{x} = 0$, and lies in $\mathbb{R}^m$.

Meta:

Review: Eigenvalues, Steady State, and Convergence

It helps to think of eigenspaces before thinking of eigenvalues. Given some matrix $\mathbf{A}$, there are vectors which, when multiplied by $\mathbf{A}$, just return scaled versions of those vectors.

$$\mathbf{A}\vec{v} = \lambda \vec{v}$$

The eigenvalue λ just tells you how much it scales by. The physical significance of λ is only really apparent when talking about steady state analysis. Say you're working with some system (not necessarily pumps, but the idea of having a vector describe the state is the same) which undergoes a transformation $\mathbf{A}$ every timestep. We'll take a look at a few cases:

(a) $\lambda = 1$: Any vector in the eigenspace associated with eigenvalue $\lambda = 1$ is a "steady state". In other words,

$$\mathbf{A}\vec{v}_{\lambda=1} = \vec{v}_{\lambda=1}$$

So if we keep applying $\mathbf{A}$ to $\vec{v}_{\lambda=1}$, the state doesn't change.

$$\lim_{n\to\infty} (\mathbf{A}^n \vec{v}_{\lambda=1}) = \vec{v}_{\lambda=1}$$

(b) $\lambda = -1$: Say it has an eigenspace containing $\vec{v}_{\lambda=-1}$. If we keep applying $\mathbf{A}$ to this vector, it just bounces back and forth between the positive and negative versions of the vector:

$$\lim_{n\to\infty} (\mathbf{A}^n \vec{v}_{\lambda=-1}) = \lim_{n\to\infty} (-1)^n \vec{v}_{\lambda=-1}$$

The limit doesn't exist! It just goes back and forth forever and ever, meaning if you start out the system in state $\vec{v}_{\lambda=-1}$, it will never converge. This shows up in controls when people talk about instability.

(c) $|\lambda| < 1$: We'll say this eigenvalue has an eigenspace containing $\vec{v}_{<1}$. Repeating the process from above:

$$\begin{aligned}\lim_{n\to\infty} (\mathbf{A}^n \vec{v}_{<1}) &= \lim_{n\to\infty} \lambda^n \vec{v}_{<1} \\ &= \vec{0}\end{aligned}$$

Because $|\lambda| < 1$, multiplying it by itself repeatedly just makes the magnitude smaller and smaller. This holds whether λ is positive or negative, and what this means is that if the system starts out in $\vec{v}_{<1}$, after an infinite amount of time the state will approach $\vec{0}$. You can think of it as asymptotically approaching zero. For physical systems, (not in scope of the class) this is related to damping (overdamped being $\lambda > 0$ and underdamped being $\lambda < 0$).

(d) $|\lambda| > 1$: Now we'll say this eigenvalue has an eigenspace containing $\vec{v}_{>1}$.

$$\lim_{n\to\infty} (\mathbf{A}^n \vec{v}_{>1}) = \lim_{n\to\infty} \lambda^n \vec{v}_{>1}$$

In this case, the state vector just keeps increasing in size to infinity, meaning if you start out the system in state $\vec{v}_{>1}$, your system will never converge to a steady state.

1. Subspace Drills

Determine if the following describe subspaces.

Meta:

- Emphasize the importance of *generalization* when determining whether a given set is a subspace (and for proofs in general). It is crucial to show that the two properties above hold for *all* vectors within the set, not just one or two numerical examples.
- Clarify the mathematical notation, e.g. $\forall$ means "for all" and : means "such that". Vocalizing what you're writing down as you write it will help keep you from moving too quickly while clarifying any confusing notation.
- Graphical intuition tends to help students who don't have a strong linear algebra background.

(a) $\{\vec{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T : x_i \geq 0 \; \forall i = 1, \cdots, n\}$

Meta: Tell the students what this means graphically!

Solution:

We proceed to test if the set is closed under vector addition and scalar multiplication.
Let's check if this set is closed under vector addition and scalar multiplication.

- *Vector addition*: Let's consider two vectors $\vec{v}$ and $\vec{u}$, both in the set.

$$\vec{v} \doteq \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^T$$
$$\vec{u} \doteq \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}^T$$

where $v_i, u_i \geq 0 \; \forall i = 1, \cdots, n$, i.e. all the elements of $\vec{v}, \vec{u}$ are nonnegative. To see that V is closed under vector addition, we need $\vec{v} + \vec{u}$ to be contained within the set as well.

$$\vec{v} + \vec{u} = \begin{bmatrix} v_1 + u_1 & \cdots & v_n + u_n \end{bmatrix}^T$$

Because $v_i, u_i \geq 0$, then $v_i + u_i \geq 0$, and so $\vec{v} + \vec{u}$ is closed under vector addition.

- *Scalar multiplication*: Suppose we have a vector $\vec{v} \in V$ and a number $\alpha \in \mathbb{R}$. Notice that α can be negative!
To prove something false, we need only come up with a counterexample; so without loss of generality, let's say $\alpha = -1$. This means $\alpha\vec{v} = -\vec{v} = \begin{bmatrix} -v_1 & \cdots & -v_n \end{bmatrix}^T$, which does *not* necessarily have nonnegative entries! Thus, the set is not closed under scalar multiplication.

The set does not describe a vector subspace because it is not closed under scalar multiplication □

(b) $\{\vec{0}\}$

Meta: When teaching this question to your students, make sure you mention the significance of a single-element set (the fact that $\vec{0}$ is the **only** vector in the set), which greatly simplifies the subspace test process.

Solution:

- *Vector addition*: Since $\vec{0}$ is the only vector in this set, $\vec{0} + \vec{0} = \vec{0} \in V$.
- *Scalar multiplication*: Since $a \cdot \vec{0} = \vec{0} \; \forall a \in \mathbb{R}$, we have that $a \cdot \vec{0} \in V$.

The set is closed under vector addition and scalar multiplication, and so it is a vector space! □

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Meta: Emphasize the difference between a set containing 2 vectors and a set that is the **span** of 2 vectors. The former **only contains 2 vectors**, while the latter **contains all linear combinations of the 2 vectors**.

Solution:

- *Vector addition*: We can see that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which is not in the set of vectors.

- *Scalar multiplication*: Taking either of the vectors and scaling them by $\alpha \neq 1$ returns a vector not in the set, and so it is not closed under scalar multiplication.

This is *not* a vector subspace. □

(d) span$\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)$

Meta: Reiterate the distinction between a single vector (like what's seen in part (c)) and its span.

Solution: By definition, the span of N vectors is the set of the linear combinations of those N vectors. For this problem, we can rewrite the set as:

$$\left\{a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}:a,b\in\mathbb{R}\right\}$$

With the new definition, we see that the span of this set is $\mathbb{R}^2$, which is a vector subspace. For the sake of completeness, however, we'll go through the standard tests:

- *Vector addition*: Let's define the following $\vec{v}_1$ and $\vec{v}_2$, both in the set.

$$\begin{aligned}\vec{v}_i &= a_i\begin{bmatrix}1\\0\end{bmatrix}+b_i\begin{bmatrix}0\\1\end{bmatrix}, i=1,2\\ \vec{v}_1+\vec{v}_2 &= \left(a_1\begin{bmatrix}1\\0\end{bmatrix}+b_1\begin{bmatrix}0\\1\end{bmatrix}\right)+\left(a_2\begin{bmatrix}1\\0\end{bmatrix}+b_2\begin{bmatrix}0\\1\end{bmatrix}\right)\\ &=(a_1+a_2)\begin{bmatrix}1\\0\end{bmatrix}+(b_1+b_2)\begin{bmatrix}0\\1\end{bmatrix}\\ &=a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\end{aligned}$$

- *Scalar multiplication*:

$$\begin{aligned}\vec{v} &= a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\\ \alpha\vec{v} &= \alpha a\begin{bmatrix}1\\0\end{bmatrix}+\alpha b\begin{bmatrix}0\\1\end{bmatrix}\\ &=\hat{a}\begin{bmatrix}1\\0\end{bmatrix}+\hat{b}\begin{bmatrix}0\\1\end{bmatrix}\end{aligned}$$

Hence, the set is a vector subspace $\square$

(e) **(PRACTICE)**

$$\left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\in\mathbb{R}^3 : x_1+x_2+x_3=1\right\}$$

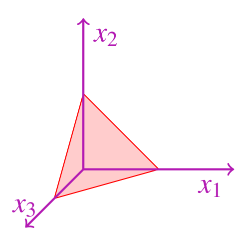
Meta: Some semesters they've been introduced to the notion of *affine* spaces. If they've seen it, feel free to mention the term.

Ask your students about the graphical interpretation of this set, then draw it out.

Solution:

Graphical Understanding

This set describes all points on the plane that passes through $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.



(If it's not immediately clear why this is true, try thinking of the line $x_1 + x_2 = 1$.)

We know from the properties of vector spaces that one implication of closure under addition and scalar multiplication is that the vector space must contain the zero vector. Furthermore, using any two vectors in the space and adding them together disproves closure under vector addition; scaling a point in the plane by $\alpha \neq 1$ disproves closure under scalar multiplication.

Quick Counterexample Using the endpoints

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and testing closure under scalar multiplication and addition is sufficient.

(f) **(PRACTICE)**

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \right\}$$

Meta:

The process for finding the spanning vectors is the same as what they'll have seen in discussion for finding the null space.

Solution: Similar to part (e), we can gain some understanding of what this is by considering the fact that it describes a plane which passes through the origin. However, this is not sufficient to prove that this (spoiler alert) is a vector space. We can do this by seeing if we can find a set of vectors whose span is the set.

The process is identical to finding a null space given a set of linear equations.

Looking at the equation we're given

$$x_1 + x_2 + x_3 = 0$$

we have one equation but three unknowns, meaning we can choose two free variables. Any two will work, but we'll say that x_1 and x_2 are free. That means we want to write x_3 in terms of the other two variables:

$$x_3 = -x_1 - x_2$$

So rewriting the original vector in terms of our free variables x_1 and x_2:

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} x_1 \\ x_2 \\ -x_1 - x_2 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

We can now rewrite the set as $\text{span}\left(\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T\right)$. We know from part (d) that this set is indeed a vector space! $\square$

(g) **(CHALLENGE PRACTICE)**

Let V be the vector space of $n \times n$ matrices, and $M \in V$ a fixed matrix. We define the set

$$U = \{N \in V \mid NM = MN\}$$

as the *centralizer* of M in V. Show that U is a subspace of V.

Meta: This is a good example to show students that vector space/subspace doesn't have to strictly contain only vectors and that the subspace test process still remains the same!

Solution:

- *Vector Addition:*
  Note that the "vectors" in this case are actually matrices! Suppose we have 2 $n \times n$ matrices $A, B \in U$, we want to show that $A + B \in U$, meaning that it must satisfy $(A+B)M = M(A+B)$.
  Starting with $(A+B)M$, we expand it out into $AM + BM$. Now, since both A and B are in U, this means that $AM = MA, BM = MB$. Hence, $(A+B)M = AM + BM = MA + MB = M(A+B)$.
- *Scalar Multiplication:*
  Suppose we have $N \in U$, and $c \in \mathbb{R}$. Similar to the process above, we want to in the end show that $(cN)M = M(cN)$.
  Since $N \in U$, we have that $NM = MN$. Since c is a real scalar, by the rule of matrix multiplication with scalars, we can move the scalar around within the expression. Hence, $(cN)M = cNM = c(NM) = c(MN) = cMN = McN = M(cN)$.

Hence, we can now conclude that U is indeed a subspace of V! $\square$

(h) **(CHALLENGE PRACTICE)**

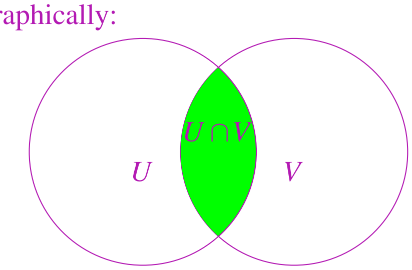
Suppose U and V are both subspaces of a vector space S, show that $U \cap V$ is also a subspace of S.

Meta: NB: $A \cap B$ means the intersection between sets A and B.

This is one of the more abstract examples on vector subspaces that also involve some understanding on simple set relations. Make sure to show the students that the subspace test process still remains mostly unchanged.

Solution:

- *Vector Addition*: Consider 2 vectors $\vec{x}, \vec{y} \in U \cap V$. We want to show that $\vec{x} + \vec{y} \in U \cap V$.
  To show that this is true, it seems like a direct approach might be a bit hard since it seems unclear what exactly $U \cap V$ contains in terms of their properties. However, making use of the fact that $U \cap V \subset U, V$ (**The intersection of U and V is a subset of U and V**) will be crucial to the proof.
  It may help to consider this graphically:



  We've marked in green the intersection between sets U and V.
  If $\vec{x}, \vec{y} \in U \cap V$, both fall in the center region of the venn diagram. This means $\vec{x}, \vec{y} \in U$ as well as V! The implication goes both ways; if some vector $\vec{z}$ falls in both U and V (the left and right circles), then it necessarily falls into their intersection, $U \cap V$.
  We're already told that V and U are vector spaces, meaning $\vec{x} + \vec{y} \in U$ and V separately, so $\vec{x} + \vec{y} \in U \cap V$.
- *Scalar Multiplication:*
  Consider a vector $\vec{x} \in U \cap V$, and a real-number scalar $c \in \mathbb{R}$.
  Again, since $\vec{x} \in U \cap V \implies \vec{x} \in V$, and V is already a vector subspace, so we know that by the property of scalar multiplication for a subspace, it must be true that $c\vec{x} \in V$.
  Applying the same logic again for U, we can see $c\vec{x} \in U$.

Since we've shown that the set is closed under vector addition and scalar multiplication, $U \cap V$ is also a vector subspace of S! $\square$

2. Null Space Drill

Meta: Prereq: Introduction to nullspaces. A mini-lecture gets you ready.
Description: First a proof about nullspaces, and then lots of mechanical practice on nullspaces.

In this question, we explore intuition about null spaces and a recipe to compute them. Recall that the nullspace of a matrix $\mathbf{M}$ is the set of all vectors, $\vec{x}$ such that $\mathbf{M}\vec{x} = \vec{0}$.

(a) First, we begin by proving that a null space is indeed a subspace. Show that any nullspace of a matrix $\mathbf{M}$ with n rows and n columns is a subspace.

Solution:

i. A nullspace of a matrix with n rows and n columns must contain vectors of n elements. These vectors clearly form a subset of $\mathbb{R}^n$.

ii. $\mathbb{R}^n$ is a known vector space.

iii. Closures and 0

  i. Consider two elements, $\vec{x_1}$ and $\vec{x_2}$ in the nullspace of $\mathbf{M}$. By definition, we know that $\mathbf{M}\vec{x_1} = \vec{0}$ and $\mathbf{M}\vec{x_2} = \vec{0}$. Now consider the vector $\vec{x_1} + \vec{x_2}$. Is $\mathbf{M}(\vec{x_1} + \vec{x_2}) \stackrel{?}{=} \vec{0}$. $\mathbf{M}\vec{x_1} + \mathbf{M}\vec{x_2} = \vec{0} + \vec{0} = \vec{0}$. Done.

  ii. Consider another element, $\vec{x_3}$ in the nullspace of $\mathbf{M}$. Consider a scalar $a \in \mathbb{R}$. Is $a\vec{x_2}$ in the nullspace of $\mathbf{M}$, i.e., is $\mathbf{M}(a\vec{x_3}) \stackrel{?}{=} \vec{0}$. Yes, because $a\mathbf{M}\vec{x_3} = 0$ since $\vec{x_3}$ is in the nullspace.

  iii. Is $\vec{0}$ in the nullspace of $\mathbf{M}$? Yes, because $\mathbf{M}\vec{0} = \vec{0}$.

Therefore, a nullspace is indeed a subspace.

(b) Now we will explore a recipe to compute null spaces. Let's start with some 3×3 matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix}$$

$\mathbf{A}'$ is the row reduced matrix $\mathbf{A}$.

$$\mathbf{A}' = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -18 \end{bmatrix}$$

Compute the nullspace of $\mathbf{A}$.

Solution: Since the row reduced matrix $\mathbf{A}'$ has a pivot in every column, the matrix has a trivial nullspace. The nullspace is the vector $\vec{0}$.

Let's look at this in more depth, however. Remember that a nullspace is the set of vectors such that $\mathbf{A}\vec{x} = \vec{0}$. Let's solve this as linear equations.

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve this, we row reduce. This results in

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let's convert this back to linear equations:

$$x_1 - 3x_2 + x_3 = 0$$
$$-x_2 + x_3 = 0$$
$$-18x_3 = 0$$

The third equation is only satisfied by $x_3 = 0$. The second equation implies that $x_2 = x_3 = 0$. And finally, the first equation is also only satisfied by $x_1 = 0$. Therefore, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is the only vector which satisfies these equations

(c) Consider another matrix

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 4 & -2 \\ -2 & 2 & -4 \end{bmatrix}$$

$\mathbf{B}'$ is row reduced $\mathbf{B}$.

$$\mathbf{B}' = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 8 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

What is the null space of $\mathbf{B}$? What is the dimension of the row space of $\mathbf{B}$?

Solution: Think of this as linear equations once again. Let the first column correspond to x, the second to y and the third to z. In equation form, the row reduced matrix becomes

$$x - y + 2z = 0 \tag{1}$$
$$8y - 10z = 0 \tag{2}$$
$$0x + 0y + 0z = 0 \tag{3}$$

Equation 3 gives us no information – it is always true. So we ignore it.

Equation 2 says that $4y = 5z$. Let's set $z = t$ (let z be a free variable that can take on any value). Then $y = \frac{5}{4}t$.

Equation 1 is then $x - \frac{5}{4}t + 2t = 0 \implies x = \frac{-3}{4}t$. The nullspace is then all vectors of the form $t\begin{bmatrix} \frac{-3}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix}$, where t is any real number. Another way to say this is that the nullspace is spanned by the vector

$$\begin{bmatrix} \frac{-3}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix} \tag{4}$$

The dimension of the nullspace, i.e., the minimum number of vectors required to span it is 1.
From the rank-nullity theorem, we know that Dim(Rowspace($\mathbf{B}$)) + Dim(Nullspace($\mathbf{B}$)) = Number of columns in $\mathbf{B}$. Therefore, the dimension of the rowspace of $\mathbf{B}$ is 2.

Meta: Mentors: State the rank nullity theorem without proof. For any matrix, $\mathbf{A}$, Rank($\mathbf{A}$) + Nullity($\mathbf{A}$) = number of columns in $\mathbf{A}$. Rank($\mathbf{A}$) = dim(colspace($\mathbf{A}$)) = dim(rowspace($\mathbf{A}$)). Nullity($\mathbf{A}$) = dim(nullspace($\mathbf{A}$))

(d) In the previous part, we chose one of the variables and set it to be a free variable. Can we choose any variable as our free variable?

Solution: Let's investigate this question by choosing each variable as a free variable. We know z works from the solution to the previous part.

Let's consider y. If we set $y = t$ instead, then we get, from Equation (2) $5z = 4t \implies z = \frac{4}{5}t$.

Equation (1) then gives us $x - t + 2\frac{4}{5}t = 0 \implies x - \frac{5t}{5} + \frac{8t}{5} = 0 \implies x = \frac{-3t}{5}$.

The nullspace is then spanned by the vector $\begin{bmatrix} \frac{-3}{5} \\ 1 \\ \frac{4}{5} \end{bmatrix}$

Note that this vector is $\frac{4}{5}$ times the vector we found in (4).

Meta: At this point, stress that the choice of free variable doesn't change the null space. Since subspaces are closed under scalar multiplication, the fact that this vector is a multiple of the previous shouldn't be a surprise to students. If it is, explain why this is the case.

Solution:

Now, let's see what happens if we set $x = t$ instead. We can't use Equation (2) yet, so let's try using Equation (1). $t - y + 2z = 0$. Now what? How do we find the value for y or z in terms of t? ...We can't. So x does not work...

(e) How can we know which variables can be used as free variables?

Solution: Pick your free variables are by looking at columns with no pivots. Although, sometimes, other variables might work (like y above), the variables with no pivots will always work!

(f) Now consider another matrix, $\mathbf{C} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 2 & 4 & 12 & -17 \\ 1 & -4 & -12 & 22 \end{bmatrix}$ Without doing any math, will this matrix have a trivial nullspace, i.e. consisting of only $\vec{0}$?

Solution: No! A 3x4 matrix can simply not have 4 pivots. So at least one of the variables will need to be free!

(g) **(PRACTICE)**

Consider another matrix, $\mathbf{D} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Find vector(s) that span the nullspace.

Solution: In terms of equations, let the variables for cols 1-4 be a to d respectively. Column 3 does not have a pivot. So c is free. Let $c = t$. At this point, we should feel comfortable reading the matrix as its equations without explicitly writing the equations!

Row 3 of the matrix says that $-d = 0$, or that $d = 0$.

Row 2 says that $-2b - 6c + 10d = 0 \implies -2b - 6t = 0 \implies b = -3t$.

Row 1 says that $a - 2b - 6c + 12d = 0 \implies a + 6t - 6t = 0 \implies a = 0$.

The vector that spans this nullspace is $\begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$

Meta: Students could be confused about 'pivots'. Column 3 doesn't have a pivot because it has a 0 in the place of the 'diagonal' instead. Also, ask them which column doesn't have a pivot in this case: $\begin{bmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & -1 & 9 \end{bmatrix}$ (note that the last row is different.) Basically, column 4 doesn't a pivot, because that pivot would have been in the 4th row which doesn't exist. *Make sure these concepts about pivots settle in.*

(h) **(PRACTICE)**

Consider one final matrix, $\mathbf{E} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. What are the vector(s) that span this nullspace?

Solution: Again, let the variables for the columns be a to d respectively. Columns 3 and 4 don't have pivots. So let's set both of them to be free!

Let $c = t, d = s$.

Row 2 says $-2b - 6c + 10d = 0 \implies -2b = 6t - 10s \implies b = -3t + 5s$.

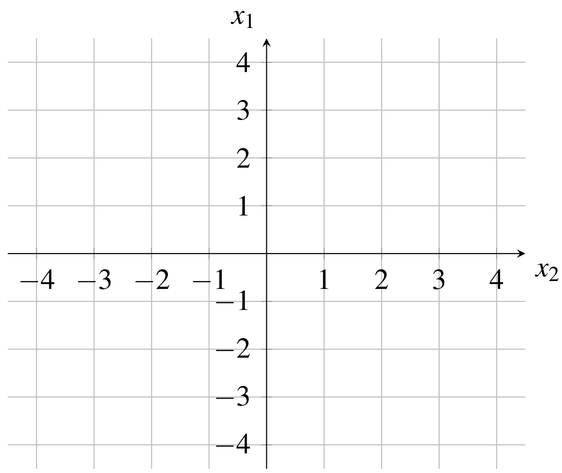
Row 1 says $a - 2b - 6c + 10d = 0 \implies a = 0$.

The general form of vectors in the nullspace is then $\begin{bmatrix} 0 \\ -3t + 5s \\ t \\ s \end{bmatrix}$. This needs to be rewritten by splitting the free variables $s \begin{bmatrix} 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$.

Finally we conclude that the vectors that span the nullspace are $\begin{bmatrix} 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$.

Observation: notice that the number of free variables = number of columns without pivot = number of vectors required to span the nullspace = dimension of the nullspace!

3. Range Intuition



$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(a) Draw the space on the figure above that is represented by Col($\mathbf{A}$). Also draw the space for the Row($\mathbf{A}$) (which is the same as Col($\mathbf{A}^\top$). What dimension are these spaces?

Solution: The 1 dimensional space for the column space is a line on the $x_2 = x_1$ axis (so a diagonal line that goes perfectly northeast, intersecting the origin along the way). The 1 dimensional space for the row space is the line $x_2 = 2x_1$

Meta: Remember that the lines that are drawn are infinite, make sure to make a point of that.

(b) Plot the point $\vec{x}$, then plot $\mathbf{A}\vec{x}$

Solution: $\mathbf{A}\vec{x}$ should be on the line that represents $\mathrm{Col}(A)$, specifically it should be $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$

Meta: Make a point of mentioning or having students mention that the result is on the line represented by A, and that this will be true for ALL possible vectors x, to be proved in the next part.

(c) Consider some arbitrary vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ Write out the product $\mathbf{A}\vec{v}$ in terms of v_1, v_2, and the columns of $\mathbf{A}$.

Solution:

$$\mathbf{A}\vec{v} = v_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + v_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Meta: Make sure to make sure students notice the implications of the above answer, that this assures that the result of Av is ALWAYS in the $\mathrm{Col}(A)$. Specifically, make sure they see it written as a sum of scalar-vector products so it's clear that we're only using components in the column space.

(d) We have talked about how matrices like $\mathbf{A}$ have no inverse. Give a geometric explanation for why this is the case.

Solution: If we are given some point on the line for the colspace from part (a), we do not know where it came from. For example, if you gave the point given by $\mathbf{A}\vec{x}$, you have no way of knowing that it came from $\vec{x}$

Meta: Make sure students see the ties between independence/invertibility and the geometry.

(e) Consider all points $\vec{y}$ such that $\mathbf{A}\vec{y} = 0$ Draw the space that the $\vec{y}$'s will make up. What do you notice geometrically? What is the dimension of this space?

Solution: This line should be a straight line that is perpendicular to the line for the row space from part (a). It is a space of dimension 1

Meta: Discuss relationship between colspace and dimension, like how they always add up to the total size of the space.

4. Mechanical Eigenvectors and Eigenvalues

Meta: This problem is supposed to be straightforward, so make sure to stress the technique used in part (a), and let the students work amongst themselves for the rest of the question. [Notice]: Mentors, please go through this question quickly as there are a lot of other questions you will need to cover.

(a) Solve for the eigenvalue-eigenvector pairs for the following 2 by 2 matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

Solution: To solve for eigenvalues and eigenvectors, let's go back and review the definition of eigenvectors and eigenvalues:
If $\vec{x}$ and λ are the eigenvector and eigenvalue of $\mathbf{A}$, respectively, then the following equation holds:

$$\mathbf{A}\vec{x} = \lambda\vec{x}$$

Since the (appropriately sized) identity matrix is analogous to multiplying by 1 in arithmetic, we can say:

$$\mathbf{A}\vec{x} = \lambda\mathbf{I}\vec{x}$$

Rearranging, we get:

$$\mathbf{A}\vec{x} - (\lambda\mathbf{I})\vec{x} = \vec{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})\vec{x} = \vec{0}$$

What does this look like? It looks similar to solving for the nullspace of $(\mathbf{A} - \lambda\mathbf{I})$!
Assuming that there is a nontrivial nullspace, that also means that $\mathbf{det}(\mathbf{A} - \lambda\mathbf{I}) = 0$!
Let's solve for λ first:

$$\begin{aligned}(\mathbf{A} - \lambda\mathbf{I}) &= \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix} \\ \mathbf{det}(\mathbf{A} - \lambda\mathbf{I}) &= (3-\lambda)(4-\lambda) - 2 \\ &= 10 - 7\lambda + \lambda^2 \\ &= (\lambda - 5)(\lambda - 2)\end{aligned}$$

By factoring:

$$\lambda = 5, 2$$

Let's check: We've just solved for the eigenvalues. But what about the eigenvectors?
To do that, we plug in λ into $(\mathbf{A} - \lambda\mathbf{I})$ and solve for the nullspace!
For $\lambda = 5$:

$$(\mathbf{A} - \lambda\mathbf{I})\vec{x} = \vec{0}$$

$$\begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

By row reduction:

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$x_1 = x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$$

So the first pair is

$$\lambda = 5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Repeating for $\lambda = 2$,

$$\begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$x_1 = -2x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2$$

So, the second pair is

$$\lambda = 2, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

(b) Find the eigenvectors for matrix A given that we know that $\lambda_1 = 4, \lambda_2 = \lambda_3 = -2$ and that

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

Solution: Once the eigenvalues of a matrix have been found, we can find the eigenvectors by Gaussian Elimination.

Step 1: For each eigenvalue λ, we have

$$(\mathbf{A} - \lambda \mathbf{I})\vec{x} = 0$$

where $\vec{x}$ is the eigenvector associated with eigenvalue λ.

Step 2: Find in the nullspace of $(\mathbf{A} - \lambda \mathbf{I})$ by plugging in a value of λ and using Gaussian elimination to solve.

Case 1: $\lambda = 4$. First, form the matrix $\mathbf{A} - 4\mathbf{I}$:

$$\mathbf{A} - 4\mathbf{I} = \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}$$

To make our numbers nicer, first let's divide our first row by -3

$$R_1 = R_1 \cdot \frac{-1}{3}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}$$

$$R_2 = R_2 - 3 \cdot R_1$$

$$R_3 = R_3 - 6 \cdot R_1$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{bmatrix}$$

$$R_3 = R_3 - R_2$$

$$R_2 = R_2 \cdot \frac{1}{6}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we see that we have reached a row of 0s, which means that our last variable x_3 is the free variable in our system. Now, we can expand this matrix by putting it into a system of linear equations and solving for all the variables in terms of our free variable x_3

$$x_1 + x_2 - x_3 = 0$$

$$-2x_2 + x_3 = 0$$

$$x_2 = \frac{x_3}{2}$$

$$x_1 + \frac{x_3}{2} - x_3 = 0$$

$$x_1 = \frac{x_3}{2}$$

$$\vec{x} = \begin{bmatrix} \frac{x_3}{2} \\ \frac{x_3}{2} \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}, \forall x_3 \in \mathbb{R}$$

So the eigenvector for when $\lambda = 4$ is $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Now, let's use this same technique to find the eigenvector for $\lambda = -2$

Meta: Here might be a good time to ask your students how many eigenvectors the next value of lambda yields considering that there are two lambda values that are equal to it

Solution: Case 2: Now let's plug in $\lambda = -2$ into $\mathbf{A} - \lambda\mathbf{I}$ to get

$$\mathbf{A} + \mathbf{2I} = \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}$$

And, just like before, let's use Gaussian elimination to reduce the matrix. We can see that this will only take a few steps.

$$R_2 = R_2 - R_1$$

$$R_3 = R_3 - 2 \cdot R_1$$

$$R_1 = R_1 \cdot \frac{1}{3}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As we can see here, we have two rows of 0s, which means that we have two free variables (x_2 and x_3). Now we can take this matrix and write it as a linear system to get

$$x_1 + x_2 - x_3 = 0 \Rightarrow x_1 = x_3 - x_2$$

Thus,

$$\vec{x} = \begin{bmatrix} x_3 - x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Which are the two eigenvectors associated with $\lambda = -2$

(c) Find the eigenvalues for matrix $\mathbf{A}$ given that we know that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ are the eigenvectors of $\mathbf{A}$, and that

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution: There are 2 ways to go about solving this problem. Either you can plug each eigenvector $\vec{v}_i$ into $\mathbf{A}v = \lambda v$ or the nullspace equation to come up with 3 equations and solve. As you have had a lot of practice with the latter, we will use the former to try to answer this question.

Let's plug in the first eigenvector and solve for the first eigenvalue.

$$\mathbf{A}\vec{v}_1 = \lambda_1 \vec{v}_1$$

$$\begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So, we can see that $\lambda_1 = 1$. Similarly, we can do this for the other two eigenvectors.

$$\begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

So, we can see that $\lambda_2 = 2$.

$$\begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}$$

So, we can see that $\lambda_3 = 3$.

5. (PRACTICE) Nullspaces and Projections

Assume that the vector $\vec{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$. For each of the following matrices $\mathbf{A} \in \mathbb{R}^{n \times m}$, answer the following:

- Compute the matrix product $\mathbf{A}\vec{x}$. Explain in words how the matrix transforms the vector.
- Suppose you know that A transforms $\vec{x}$ to give $\vec{y}$. Given $\vec{y}$, can you find what the original vector $\vec{x}$ was?
- Is the matrix $\mathbf{A}$ invertible? How do you know? If it is invertible, find the inverse.
- Verify that (dimension of nullspace) + (dimension of column space) = min(n, m)

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Solution:

- $\mathbf{A}\vec{x} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$. We can see that this matrix keeps x_0 and turns x_1 to 0. In other words, the matrix **projects** the vector $\vec{x}$ to the x-axis.
- No. We can figure out x_0, since the value does not change, but the transformation converts all the x_1 values to 0. Given 0 as an output, x_1 could have been any value, so we cannot determine the original vector $\vec{x}$. As a rule of thumb, if there isn't a one-to-one mapping of inputs to outputs for the elements of $\vec{x}$, we cannot find the original vector after the transformation.
- No, $\mathbf{A}$ is not invertible, since the columns are linearly dependent, or there is no way to turn it into an upper triangular matrix using Gaussian elimination. Intuitively, we cannot retrieve a vector to its original state after applying the matrix, and therefore we cannot "invert" the operation.
- The dimension of the nullspace is 1, or the number of elements that cannot be retrieved after the matrix transformation. The column space is dimension 1, since there is only one pivot in the matrix. Adding both together, we get 1 + 1 = 2, which is equal to min(2, 2) = 2.

(b) $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solution:

- $\mathbf{A}\vec{x} = \frac{1}{2}\begin{bmatrix} x_0 + x_1 \\ x_0 + x_1 \end{bmatrix}$ The first and second entries of the resulting vector have the same value; $x_0 = x_1$. We can consider the first entry to be equivalent to "x" and the second to "y", Hence, it is a projection onto y = x. Note that the constants are $\frac{1}{2}$ so that the x vector is not scaled.
- No, we cannot retrieve the original values because when we write out the equations represented by the transformation, the two equations that are the same (linearly dependent), so we only have one equation to work with. This is not enough information to retrieve the original information; we are trying to solve for two variables with one equation.
- No; this goes hand-in-hand with the whether or not we can retrieve the original values; the matrix has linearly dependent columns so we cannot invert it. We also know that we cannot invert matrices with determinant equal to 0, and it is clear that this transformation matrix has determinant 0.
- The column space of the matrix is 1 since we retrieved one linearly independent equation from earlier. To calculate the nullspace, we first row reduce the matrix, giving us $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then we solve for $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
This gives us the equation $x_0 + x_1 = 0$. Solving this equation, we get $x_0 = -x_1$, so the nullspace is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence, we have one vector in the column space and one vector in the nullspace; 1 + 1 = 2, the rank of the original transformation, so everything is accounted for.

(c) $\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \vec{y} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$

Solution:

- $\mathbf{A}\vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} x_0 - x_1 \\ x_0 + x_1 \end{bmatrix}$. However, this does not offer us much insight into what the matrix actually **does**. For that, we notice that this is actually the rotation matrix corresponding to an angle of $\theta = \frac{\pi}{4}$. $\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$ So, we can interpret this matrix as being a transformation that takes a vector and shifts it counterclockwise by 45°.
- Yes. We know that $x_0 - x_1 = 2$ and $x_0 + x_1 = 0$. Solving these, we get $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This makes sense, since we can think of the vector $\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$ as being a 45-degree-rotated version of the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
- Yes, the matrix is invertible. We know this because the columns are linearly independent (replace row 2 with row 2 - row 1 to get an upper triangular matrix). This also makes sense intuitively, because we know that we can reverse a rotation by applying its inverse rotation (a clockwise rotation by 45 degrees). So, in this case, the inverse matrix will be $\mathbf{A}^{-1} = \begin{bmatrix} \cos -\frac{\pi}{4} & -\sin -\frac{\pi}{4} \\ \sin -\frac{\pi}{4} & \cos -\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. You can verify that $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Because this matrix is invertible, it has a nullspace of dimension 0. And since it is full rank (rows are lin. ind.), its column space has dimension 2. Their sum is equal to min(2, 2) = 2. Thus the equation is verified.

(d) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Solution:

- $\mathbf{A}\vec{x} = \begin{bmatrix} x_0 \\ 2x_1 \end{bmatrix}$. This is a diagonal matrix, and performs component-wise scaling. We can interpret this as a transformation that scales the first component by 1, and the second component by 2.
- Yes. We know that $x_0 = 2$ and $2x_1 = 4$. Solving these, we get $\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. This makes sense, because the components are getting scaled by 1 and 2 respectively.
- Yes, the matrix is invertible. We know this because the columns are linearly independent (the matrix is already in upper triangular form). We also know that all diagonal matrices with nonzero entries are invertible. This also makes sense intuitively, because we know that we can reverse each component's scaling by applying its inverse scaling. So, in this case, the inverse matrix will be $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. You can verify that $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Because this matrix is invertible, it has a nullspace of dimension 0. And since it is full rank (rows are lin. ind.), its column space has dimension 2. Their sum is equal to min(2, 2) = 2. Thus the equation is verified.

Meta:

- Get the students to intuitively understand what each of these transformations mean. These are all fundamental transformations, so it is important that students understand what they mean physically
- Another main point is that matrices are invertible when they represent 'reversible' operations. Try to get students to understand why the examples in (a) and (b) are not invertible, while those in (c) and (d) are.
- For part (b), DO NOT mention the determinant, because students will have not covered that yet. Instead, just talk about how the rows are linearly dependent.