1. Eigenvalues Everywhere

Meta: Prereq: All of linear algebra basically, including page rank etc.
Description: Meant to be an intuition problem on eigenvalues and eigenvectors.

(a) What is the one eigenvalue and eigenvector of the matrix that you can see without solving any equations?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Solution: Since this matrix is clearly not-invertible, it must have an eigenvalue 0.

$$\mathbf{A}\vec{x} = \lambda\vec{x}$$
$$\mathbf{A}\vec{x} = 0\vec{x}$$
$$\mathbf{A}\vec{x} = \vec{0}$$

This equation is precisely the equation for computing the nullspace of $\mathbf{A}$. Therefore, any $\vec{x} \in \text{Nullspace}(\mathbf{A})$ works.

Meta: This problem is relatively fast to do so please try to go through it. The point of this problem is not to find the eigenvalues mechanically, but instead use properties of the matrix that you can eyeball to figure out some eigenvalues and eigenvectors. Don't spend time mechancially computing the eigenvalues.

(b) What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{B} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution: This is a scaling matrix. It scales any vector by a factor of 3. What this means is that any vector $\vec{x} \in \mathbb{R}^3$ when post-multiplied by $\mathbf{A}$ will output $3\vec{x}$. This matrix has only one eigenvalue, $\lambda = 3$ and any $\vec{x} \in \mathbb{R}^3$ is an eigenvector.

(c) What are the eigenvalues of

$$\mathbf{C} = \begin{bmatrix} 2 & 0 \\ 3 & 4 \\ 1 & 3 \end{bmatrix}?$$

Solution: A non-square matrix (say $m \times n$) maps a vector of dimension n to a vector of dimension m. So, it is impossible for a non-square matrix to have eigenvalues, because the output cannot be a scaled version of the input. In fact, eigenvalues are defined only for square matrices. For similar reasons, the determinant of a matrix is only well-defined if the matrix is square.

(d) Consider a matrix that rotates a vector in $\mathbb{R}^2$ by $45°$ counterclockwise. For instance, it rotates any vector along the x-axis to orient towards the $y = x$ line. Find its eigenvalues and corresponding eigenvectors. This matrix is given as

$$\mathbf{D} = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Meta: Please draw a picture to show what the matrix does to a vector. Also remember we are only considering real eigenvalues, as written in the prompt of the problem.

Solution: Remember that the equation $\mathbf{A}\vec{x} = \lambda\vec{x}$ geometrically means that for the matrix $\mathbf{A}$, there exist some special vectors $\vec{x}$ that are merely scaled by λ when post-multiplied by $\mathbf{A}$. For a matrix that takes a vector and rotates it by $45°$, there are no real-valued vectors that it can simply scale. This means that there are no real eigenvalues for this matrix either.

(e) What are the eigenvalues of the following matrix?

$$\mathbf{E} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Solution: Remember that for upper triangular matrices, the eigenvalues can be read from the diagonal. $1, \frac{1}{2}, \frac{1}{3}$ are the three eigenvalues.

(f) Can you find an eigenvalue of the following matrix without solving any equations?

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Solution: This is a matrix whose rows sum to 1, therefore, it has an eigenvalue 1.

This is proven by letting $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be a potential eigenvector of the matrix $\mathbf{F}$. Looking at the column view of matrix-vector multiplication –

$$\mathbf{F} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{2} \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$

$$\mathbf{F}\vec{x} = 1 \cdot \vec{x}$$

since the rows sum to one.

Therefore, 1 is an eigenvalue with corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Meta: Make sure students see why this works generally. Essentially $\mathbf{A} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 + 1 \cdot \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, where $\vec{v}_i$ are the columns of $\mathbf{A}$, and the sum equals $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ because each row sums to one.

(g) Show that a matrix and its transpose have the same eigenvalues

Hint: The determinant of a matrix is the same as the determinant of its transpose

Solution: For any matrix $\mathbf{M}$,

$$det(\mathbf{M}) = det(\mathbf{M}^T)$$

Eigenvalues are found by solving the equation $det(\mathbf{M} - \lambda \mathbf{I}) = 0$.

Note that $(\mathbf{M} - \lambda \mathbf{I})^T = \mathbf{M}^T - \lambda \mathbf{I}^T = \mathbf{M}^T - \lambda \mathbf{I}$.

Let $\mathbf{M} - \lambda \mathbf{I} = \mathbf{G}$.

$$det(\mathbf{G}) = det(\mathbf{G}^T)$$

$$det(\mathbf{M} - \lambda \mathbf{I}) = det(\mathbf{M}^T - \lambda \mathbf{I})$$

If we set the left hand side to 0 to solve for the lambdas, we also extract the lambdas corresponding to the right hand side. Therefore, $\mathbf{M}$ and its transpose have the same eigenvalues.

(h) Consider a matrix whose columns sum to one. What is one possible eigenvalue of this matrix?

Solution: We showed that for any matrix like $\mathbf{F}$ whose rows sum to 1, one eigenvalue is 1. We also showed that a matrix and its transpose have the same eigenvalues. Consider $\mathbf{F}^T$. It has columns summing to 1. Therefore, 1 is an eigenvalue of $\mathbf{F}^T$ too, and by extension of all matrices whose columns sum to one.

2. Eigenvalues and Flow

In this question, we will examine how the eigenvalues of a matrix relate to how it actually acts on vectors. We will also try to interpret this in terms of flow of water between reservoirs.

For each of the following parts, assume that you have a matrix $\mathbf{A}$ with the listed eigenvalues and eigenvectors. Give the output of $\lim_{n\to\infty} \mathbf{A}^n \vec{x}$ for the provided vectors $\vec{x}$ which have all nonnegative entries (i.e. they represents some distribution of water in reservoirs). How can you interpret each of these matrices in terms of reservoirs and pumps? What must be true of the sums of elements in the columns?

(a)

$$\lambda_1 = \frac{1}{2}, \vec{v}_1 = \begin{bmatrix} -2 & 1 & 0 \end{bmatrix}^T$$
$$\lambda_2 = \frac{1}{3}, \vec{v}_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$$
$$\lambda_3 = \frac{1}{4}, \vec{v}_3 = \begin{bmatrix} -1 & 1 & -2 \end{bmatrix}^T$$
$$\vec{x} = \begin{bmatrix} 0 & 5 & 1 \end{bmatrix}^T$$

Solution: We can write the vector $\vec{x}$ as a linear combination of the eigenvectors of $\mathbf{A}$. Since we know how $\mathbf{A}$ acts on each of the eigenvectors, we can use this to get an expression for $\mathbf{A}\vec{x}$.

Let $\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} = a \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$

We have the following equations:

$$-2a + b - c = 0$$

$$a+b+c=5$$

$$b-2c=1$$

Solving them, we get that $a=1, b=3$, and $c=1$. So, $\begin{bmatrix}0\\5\\1\end{bmatrix} = \begin{bmatrix}-2\\1\\0\end{bmatrix} + \begin{bmatrix}3\\3\\3\end{bmatrix} + \begin{bmatrix}-1\\1\\-2\end{bmatrix}$

And,

$$\begin{aligned}
\mathbf{A}^n\vec{x} &= \mathbf{A}^n \begin{bmatrix}-2\\1\\0\end{bmatrix} + \mathbf{A}^n \begin{bmatrix}3\\3\\3\end{bmatrix} + \mathbf{A}^n \begin{bmatrix}-1\\1\\-2\end{bmatrix} \\
&= \lambda_1^n \begin{bmatrix}-2\\1\\0\end{bmatrix} + \lambda_2^n \begin{bmatrix}3\\3\\3\end{bmatrix} + \lambda_3^n \begin{bmatrix}-1\\1\\-2\end{bmatrix} \\
&= (\frac{1}{2})^n \begin{bmatrix}-2\\1\\0\end{bmatrix} + (\frac{1}{3})^n \begin{bmatrix}3\\3\\3\end{bmatrix} + (\frac{1}{4})^n \begin{bmatrix}-1\\1\\-2\end{bmatrix}
\end{aligned}$$

All three components will decay to 0 as $\to \infty$, because $(\frac{1}{2})^n$, $(\frac{1}{3})^n$, and $(\frac{1}{4})^n$ are (exponentially) decreasing functions.

How can we interpret this in terms of pipes and reservoirs? In this case, we have leaky pipes, since at each timestep the amount of water (and the entries of the vector) decreases, and given long enough, all the water will drain out of the system.

Meta: Mentors: See if you want out the fact that no matter what, if $\vec{x} \in \mathbb{R}^3$, it will always go to $\vec{0}$. This is because any $\vec{x} \in \mathbb{R}^3$ can be written as a linear combination of the three linearly independent eigenvectors.

(b)

$$\lambda_1 = 2, \vec{v}_1 = \begin{bmatrix}-1 & -1 & 0\end{bmatrix}^T$$
$$\lambda_2 = 1, \vec{v}_2 = \begin{bmatrix}2 & 3 & 2\end{bmatrix}^T$$
$$\lambda_3 = \frac{1}{2}, \vec{v}_3 = \begin{bmatrix}0 & -1 & -1\end{bmatrix}^T$$
$$\vec{x}_1 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T \text{ and } \vec{x}_2 = \begin{bmatrix}2 & 2 & 1\end{bmatrix}^T$$

Solution: We can write the vector $\vec{x}_1$ as a linear combination of the eigenvectors of $\mathbf{A}$. Since we know how $\mathbf{A}$ acts on each of the eigenvectors, we can use this to get an expression for $\mathbf{A}\vec{x}_1$.

Let $\begin{bmatrix}1\\1\\1\end{bmatrix} = a\begin{bmatrix}-1\\-1\\0\end{bmatrix} + b\begin{bmatrix}2\\3\\2\end{bmatrix} + c\begin{bmatrix}0\\-1\\-1\end{bmatrix}$

We have the following equations:

$$-a+2b=1$$

$$-a+3b-c=1$$

$$2b-c=1$$

Solving them, we get that $a = b = c = 1$. So, $\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\\0\end{bmatrix} + \begin{bmatrix}2\\3\\2\end{bmatrix} + \begin{bmatrix}0\\-1\\-1\end{bmatrix}$

And,

$$\begin{aligned}
\mathbf{A}^n\vec{x}_1 &= \mathbf{A}^n\begin{bmatrix}-1\\-1\\0\end{bmatrix} + \mathbf{A}^n\begin{bmatrix}2\\3\\2\end{bmatrix} + \mathbf{A}^n\begin{bmatrix}0\\-1\\-1\end{bmatrix}\\
&= \lambda_1^n\begin{bmatrix}-1\\-1\\0\end{bmatrix} + \lambda_2^n\begin{bmatrix}2\\3\\2\end{bmatrix} + \lambda_3^n\begin{bmatrix}0\\-1\\-1\end{bmatrix}\\
&= (2)^n\begin{bmatrix}-1\\-1\\0\end{bmatrix} + (1)^n\begin{bmatrix}2\\3\\2\end{bmatrix} + (\frac{1}{2})^n\begin{bmatrix}0\\-1\\-1\end{bmatrix}
\end{aligned}$$

The reason why we are finding $\lim_{n\to\infty}\mathbf{A}^n x$ is because each multiplication by A represents one step forward in time, and taking the limit as $n \to \infty$ gives us the 'steady-state' distribution of water in the reservoirs.

The first component will become larger as $\to \infty$, because 2^n is an (exponentially!) increasing function. The second component will stay the same as $\to \infty$, because $1^n = 1$. The third component will decay to 0 as $\to \infty$, because $\frac{1}{2}^n$ is an (exponentially) decreasing function.

Therefore, in this case, the first term will dominate, and all components in the resulting sum will go to infinity. However, what if the input vector did not have a component in the direction of $\vec{v}_1$? Then, the vector would not blow up. Instead, one component would decay to 0, while the other would stay the same.

How can we interpret this in terms of pipes and reservoirs? In the case of a 'stochastic matrix', where all the columns sum to 1, we saw that none of the eigenvalues can have absolute value greater than 1. In this case, we know that there must be at least one column that sums to more than 1. This can be interpreted as pipes that add some extra water to the system at each timestep (the opposite of leaky pipes). This is the only way that the total amount of water in the system (and therefore the sum of the entries of the vector) can increase over time.

Here is an example of a matrix which has the provided eigenvalues (but not the same eigenvectors!): $\begin{bmatrix}2 & 1 & 1\\0 & 1 & 1\\0 & 0 & \frac{1}{2}\end{bmatrix}$ In this case, in fact, all the columns sum to a number greater than 1!

For $\vec{x}_2 = \begin{bmatrix}2 & 2 & 1\end{bmatrix}^T$

We can follow the same procedure as for $\vec{x}_1$

Let $\begin{bmatrix}2\\2\\1\end{bmatrix} = a\begin{bmatrix}-1\\-1\\0\end{bmatrix} + b\begin{bmatrix}2\\3\\2\end{bmatrix} + c\begin{bmatrix}0\\-1\\-1\end{bmatrix}$

We have the following equations:

$$-a + 2b = 2$$

$$-a + 3b - c = 2$$

$$2b - c = 1$$

Solving them, we get that $a = 0$, $b = 1$, and $c = 1$. So, $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$

And,

$$\mathbf{A}^n \vec{x}_2 = \mathbf{A}^n \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \mathbf{A}^n \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} + \mathbf{A}^n \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

However, recall that the coefficient of the first vector is 0. Hence, we actually have

$$\begin{aligned} \mathbf{A}^n \vec{x}_2 &= \lambda_2^n \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} + \lambda_3^n \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \\ &= (1)^n \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} + (\frac{1}{2})^n \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \end{aligned}$$

We no longer have a 2^n term that will approach infinity as n approaches infinity. $\frac{1}{2}^n$ still approaches 0, and $1^n = 1$ as n approaches infinity, so plugging this in, we get that

$$\mathbf{A}^n \vec{x}_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

as n approaches infinity; the system converges.

(c) **(PRACTICE)**

$\lambda_1 = -1, \vec{v}_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$
$\lambda_2 = \frac{1}{2}, \vec{v}_2 = \begin{bmatrix} 3 & 1 & 1 \end{bmatrix}^T$
$\lambda_3 = \frac{1}{4}, \vec{v}_3 = \begin{bmatrix} 2 & 1 & 4 \end{bmatrix}^T$
$\vec{x} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$

Solution: We can write the vector x as a linear combination of the eigenvectors of $\mathbf{A}$. Since we know how $\mathbf{A}$ acts on each of the eigenvectors, we can use this to get an expression for $\mathbf{A}\vec{x}$.

Let $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

We have the following equations:

$$a + 3b + 2c = 1$$

$$a + b + c = 2$$

$$b + 4c = 3$$

Solving them, we get that $a = 2, b = -1$, and $c = 1$. So, $\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}2\\2\\0\end{bmatrix} + \begin{bmatrix}-3\\-1\\-1\end{bmatrix} + \begin{bmatrix}2\\1\\4\end{bmatrix}$

And,

$$\begin{aligned}
\mathbf{A}^n\vec{x}_1 &= \mathbf{A}^n \begin{bmatrix}2\\2\\0\end{bmatrix} + \mathbf{A}^n \begin{bmatrix}-3\\-1\\-1\end{bmatrix} + \mathbf{A}^n \begin{bmatrix}2\\1\\4\end{bmatrix} \\
&= \lambda_1^n \begin{bmatrix}2\\2\\0\end{bmatrix} + \lambda_2^n \begin{bmatrix}-3\\-1\\-1\end{bmatrix} + \lambda_3^n \begin{bmatrix}2\\1\\4\end{bmatrix} \\
&= (-1)^n \begin{bmatrix}2\\2\\0\end{bmatrix} + (\frac{1}{2})^n \begin{bmatrix}-3\\-1\\-1\end{bmatrix} + (\frac{1}{4})^n \begin{bmatrix}2\\1\\4\end{bmatrix}
\end{aligned}$$

The first component will oscillate between a negative and positive value because of the $(-1)^n$. The next two terms will decay to 0 as $\to \infty$, because $(\frac{1}{2})^n$ and $(\frac{1}{4})^n$ are (exponentially) decreasing functions. The first term will therefore dominate, resulting in the entire system oscillating as n approaches infinity.

How can we interpret this in terms of pipes and reservoirs? Consider a pump system of two pumps, **A** and **B**, where at each cycle, all the water from **A** goes to **B** and all the water from **B** goes to **A**. The system never converges; as time approaches infinity, the system will oscillate between all the water being stored in tank **A** and all the water being stored in tank **B**.

Here is an example of the matrix that would represent such a system (eigenvalues and eigenvectors are not associated with the provided examples) $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

3. StateRank Car Rentals (21 points)

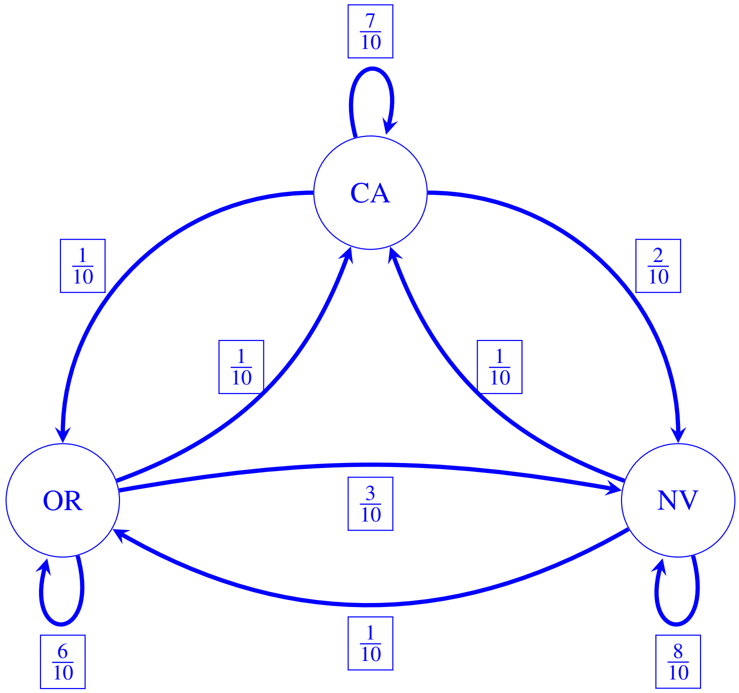
You are an analyst at StateRank Car Rentals, which operates in California, Oregon, and Nevada. You are hired to analyze the number of rental cars going into and out of each of the three states (CA, OR, and NV).

The number of cars in each state on day $n \in \{0, 1, \ldots\}$ can be represented by the state vector $\vec{s}[n] = \begin{bmatrix}s_{\text{CA}}[n]\\s_{\text{OR}}[n]\\s_{\text{NV}}[n]\end{bmatrix}$.

The state vector follows the state evolution equation $\vec{s}[n+1] = \mathbf{A}\vec{s}[n], \forall n \in \{0, 1, \ldots\}$, where the transition matrix, **A**, of this linear dynamic system is

$$\mathbf{A} = \begin{bmatrix}\frac{7}{10} & \frac{1}{10} & \frac{1}{10}\\ \frac{1}{10} & \frac{6}{10} & \frac{1}{10}\\ \frac{2}{10} & \frac{3}{10} & \frac{8}{10}\end{bmatrix}.$$

Meta: Depending on the familiarity level of the students with flow matrices, the instructor may wish to first draw out the flow diagram. This is also a good time to go over definitions like column-stochastic matrix; but most importantly, there should be an overview of how eigenvalues and eigenvectors control the stability, or the end result, of the system.



(a) We denote the eigenvalue/eigenvector pairs of the matrix $\mathbf{A}$ by

$$\left(\lambda_1 = 1, \vec{u}_1 = \begin{bmatrix} 50 \\ 40 \\ 110 \end{bmatrix}\right), \left(\lambda_2, \vec{u}_2 = \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}\right), \text{ and } \left(\lambda_3, \vec{u}_3 = \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix}\right).$$

Find the eigenvalues λ_2 and λ_3 corresponding to the eigenvectors $\vec{u}_2$ and $\vec{u}_3$, respectively. Note that since $\lambda_1 = 1$ is given, you don't have to calculate it.

Solution:

Recall that if $\vec{u}, \lambda$ are an eigenpair of a matrix $\mathbf{A}$, then $\mathbf{A}\vec{u} = \lambda\vec{u}$. By left-multiplying the eigenvectors by $\mathbf{A}$, we get:

i. $\begin{bmatrix} \frac{7}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{6}{10} & \frac{1}{10} \\ \frac{2}{10} & \frac{3}{10} & \frac{8}{10} \end{bmatrix} \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix} = 0.5 \cdot \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}$, which means that $\lambda_2 = 0.5 = \frac{1}{2}$.

ii. $\begin{bmatrix} \frac{7}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{6}{10} & \frac{1}{10} \\ \frac{2}{10} & \frac{3}{10} & \frac{8}{10} \end{bmatrix} \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 6 \end{bmatrix} = 0.6 \cdot \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix}$, which means that $\lambda_3 = 0.6 = \frac{3}{5}$.

(b) Suppose that the initial number of rental cars in each state on day 0 is

$$\vec{s}[0] = \begin{bmatrix} 7000 \\ 5000 \\ 8000 \end{bmatrix} = 100\vec{u}_1 - 100\vec{u}_2 - 200\vec{u}_3,$$

where $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$ are the eigenvectors from part (a).

After a very large number of days n, how many rental cars will there be in each state?

That is, i) calculate

$$\vec{s}^* = \lim_{n\to\infty} \vec{s}[n]$$

and ii) show that the system will indeed converge to $\vec{s}^*$ as $n \to \infty$ if it starts from $\vec{s}[0]$.
Hint: If you didn't solve part (a), the eigenvalues satisfy $\lambda_1 = 1, |\lambda_2| < 1$ and $|\lambda_3| < 1$.

Solution:
We know that $\vec{s}[n] = \mathbf{A}^n \vec{s}[0]$. We also know that $\vec{s}[0] = 100\vec{u}_1 - 100\vec{u}_2 - 200\vec{u}_3$.
Therefore, we can write:

$$\begin{aligned}\vec{s}[n] &= \mathbf{A}^n \vec{s}[0] \\ &= \mathbf{A}^n \left(100\vec{u}_1 - 100\vec{u}_2 - 200\vec{u}_3\right) \\ &= 100\mathbf{A}^n\vec{u}_1 - 100\mathbf{A}^n\vec{u}_2 - 200\mathbf{A}^n\vec{u}_3 \\ &= 100\lambda_1^n\vec{u}_1 - 100\lambda_2^n\vec{u}_2 - 200\lambda_3^n\vec{u}_3\end{aligned}$$

From that, we can write:

$$\lim_{n\to\infty} \vec{s}[n] = \lim_{n\to\infty} 100\lambda_1^n\vec{u}_1 - 100\lambda_2^n\vec{u}_2 - 200\lambda_3^n\vec{u}_3$$

Since $|\lambda_2| < 1$ and $|\lambda_3| < 1$, we know that:

$$\lim_{n\to\infty} \left(-100\lambda_2^n\vec{u}_2 - 200\lambda_3^n\vec{u}_3\right) = \vec{0}$$

From that and the fact that $\lambda_1 = 1$, we are left with:

$$\begin{aligned}\lim_{n\to\infty} \vec{s}[n] &= \lim_{n\to\infty} 100\lambda_1^n\vec{u}_1 \\ &= \lim_{n\to\infty} 100 \cdot 1^n\vec{u}_1 \\ &= 100\vec{u}_1 \\ &= \begin{bmatrix} 5000 \\ 4000 \\ 11000 \end{bmatrix}\end{aligned}$$

4. Diagonalization and Other Things Related (ish)

(a) When is an $n \times n$ matrix diagonalizable, or able to be represented in the form $\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{D}$ is a diagonal matrix?

Solution: An $n \times n$ matrix is diagonalizable when it has n linearly independent eigenvectors, or when the matrix formed by the eigenvectors is full rank. (Note that the eigenvalues do not need to be unique)

(b) Given eigenvalues $\lambda = 1, 2$, diagonalize this matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Meta: [Notice]: A mini lecture may be required before going over the diagonalization problem because students may not have seen this in lecture yet. For students interested, eigenvalues can be calculated by solving for when $det(A - \lambda I) = 0$ through cofactor expansion (which may not yet taught)

$$det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix} = (2-\lambda)^2(1-\lambda) = 0$$

$$\lambda = 1, \lambda = 2 \text{ (2 values)}$$

Solution:

Step 1: Find linearly independent eigenvectors of A by solving for the nullspace of $(A - \lambda I)$ for each value of λ

$$\lambda = 1: \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\hat{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda = 2: \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$\hat{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Step 2: Arrange the eigenvectors and eigenvalues into the **P** and **D** matrices. Note: Make sure to match the row and column of the eigenvalues in the **D** matrix with the column of the eigenvectors in the **P** matrix.

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(c) Consider a pump system with transition matrix **A**, diagonalized as $\mathbf{PDP}^{-1}$. Find the system state $\vec{s}[n]$ given state $\vec{s}[0]$

Solution:

Use the formula $\vec{s}[n] = \mathbf{A}^n \vec{s}[0]$. We want to calculate $(\mathbf{PDP}^{-1})^n \vec{s}[0]$, or $\mathbf{PDP}^{-1}\mathbf{PDP}^{-1} \ldots \mathbf{PDP}^{-1} \vec{s}[0]$. We see that the inner $\mathbf{P}^{-1}$ and **P**'s cancel out to the identity matrix, leaving us with $\mathbf{PD}^n\mathbf{P}^{-1}\vec{s}[0]$.

Consider a sizable matrix **A**, eg 10×10, and a large exponent n, eg 7. It would generally be computationally simpler to diagonalize the matrix and compute $\mathbf{PD}^n\mathbf{P}^{-1}\vec{s}[0]$ than to compute $\mathbf{A}^n\vec{s}[0]$ because $\mathbf{D}^n$ involves just raising each number in the diagonal to the nth power.

(d) Is there a relationship between invertibility and diagonizability?

i. First, let us consider: does invertibility imply diagonizability? Give a brief explanation or counterexample. (Hint: think about how linear independence plays a role in whether or not a matrix is invertible or diagonizable).

ii. Does diagonizability imply invertibility? (Hint: think about the invertibility of each individual matrix that constitutes the diagonalized matrix.)

Solution:

i. No, invertibility does not imply diagonalizability. A square matrix is invertible if and only if it has linearly independent columns. For example, take the matrix A and its inverse:

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}, \ \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ 0 & \frac{1}{2} \end{bmatrix}$$

However, when we solve for the eigenvectors, we only get one: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

We need two linearly independent vectors to form a diagonalizable matrix; hence, we have found a matrix that is invertible but not diagonalizable.

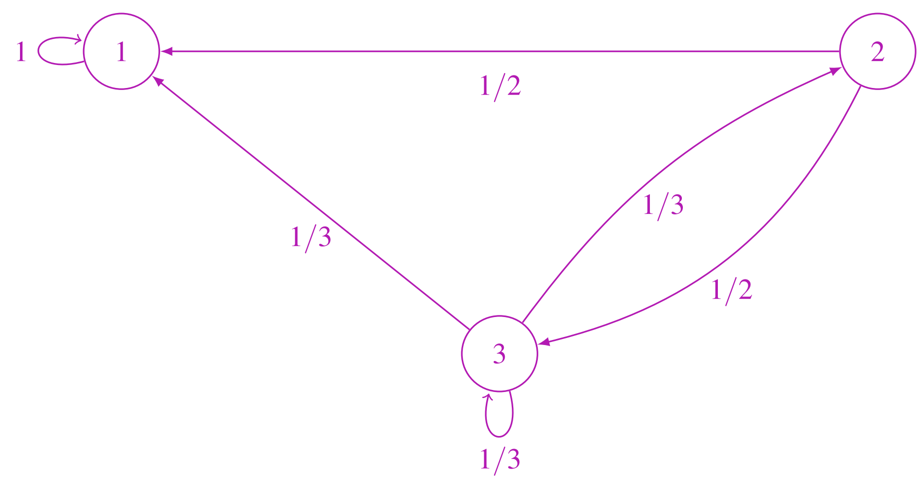
Note: remember that one eigenvalue can map to multiple eigenvectors that are linearly independent, so do not confuse the number of eigenvalues with the number of eigenvectors.

ii. False again! Consider the following matrix M:

However, we also see that the columns (and rows, for that matter) of M are linearly dependent, and so M is not invertible!

(e) (PRACTICE) Page Rank

Now suppose we have a network consisting of 3 websites connected as shown below. Each of the weights on the edges represent the probability of a user taking that edge.



(a) Write down the probability transition matrix for this graph, and call it $\mathbf{P}$. Can you say something about the eigenalues/eigenvectors of $\mathbf{P}^T$? (*Hint: Try to recall the properties of transition matrices*).

Meta: Mentors: Explain how $\mathbf{P}$ being a transition matrix relates to $\mathbf{P}^T$ as a transition matrix, and depending on how comfortable students are with eigenvalues, mention that the eigenvalues of transposed matrices are always the same as original matrices.

Solution: The transition matrix is:

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

We know that the columns of a probability transition matrix must sum to 1. This means that the rows of $\mathbf{P}^T$ must sum to 1. So, we have that the matrix-vector product $\mathbf{P}^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. This means that 1 must be an eigenvalue of the matrix $\mathbf{P}^T$, and therefore from part (a), it must also be an eigenvalue of $\mathbf{P}$. This is true for any probability transition matrix.

(b) We want to rank these webpages in order of importance. But first, find the eigenvector of $\mathbf{P}$ corresponding to eigenvalue 1.

Meta: This is largely a mechanical question, so ensure that students understand (i) the purpose of this calculation, and (ii) the techniques involved in it.

Solution:

$$P - I = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & -\frac{1}{2} & \frac{1}{3} \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}$$
$$\xrightarrow[R1 \leftrightarrow R2, R2 \leftrightarrow R3]{R1 \to R1+R2+R3} \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{3} \\ 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

We can see that the pivots lie in the second and third columns. So, we want to solve the equation

$$\begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{3} \\ 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$
$$-\frac{1}{2}x_2 + \frac{1}{3}x_3 = 0 \text{ and } -\frac{2}{3}x_3 = 0$$
$$\implies x_3 = 0 \text{ and } x_2 = 0$$

This means that the eigenvector is of the form $\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1$. And since x_1 is a free variable, the eigenvectors corresponding to eigenvalue 1 must belong in span$\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$

(c) Now looking at the matrix $\mathbf{P}$, can you identify what its other eigenvalues are?

Meta: Ask students about why the eigenvalues of an upper-triangular matrix are at its diagonals, and optionally walk though a sketch proof.

Solution: $\mathbf{P}$ is an upper-triangular matrix, which means that the diagonal elements are the eigenvalues. So, the eigenvalues are $1, \frac{1}{2}, and \frac{1}{3}$ (we already found the eigenvalue 1 in part (b) through a different method).

(d) Suppose that we start with 90 users evenly distributed among the websites. What is the steady-state number of people who will end up at each website?

Meta: Ensure that the students understand why components with smaller eigenvalues will die out. Optionally, also explain why in any transition matrix no eigenvalue can be greater than 1, and so every component without eigenvalue one will die out.

Solution: The initial vector of people is $\vec{x} = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix}$. We know that since the other eigenvalues are less than 1, those components will die out as we keep applying $\mathbf{P}$ to $\vec{x}$. So we only care about the component of $\vec{x}$ that is in the direction of $\begin{bmatrix}1\\0\\0\end{bmatrix}$. This is just the first component of the vector, which is $\begin{bmatrix}30\\0\\0\end{bmatrix}$. However, the total number of people must be conserved, so we multiply by 3 so that the total is 90, the same as before. So, the steady-state distribution is $\begin{bmatrix}90\\0\\0\end{bmatrix}$