CSM
Spring 2019

Electrical Engineering 16A
Worksheet 5

1. Passive (Aggressive) Sign Convention

For the following components, label the V_{element} or I_{element} given the I_{element} or V_{element}, respectively. *Hint: The value of the voltage and current sources shouldn't affect passive sign convention—remember that voltage and current can be negative!*

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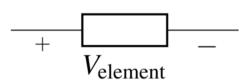

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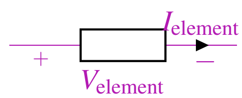

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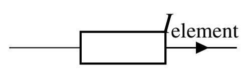

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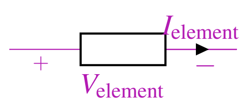

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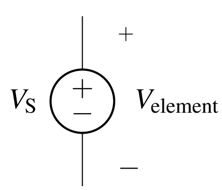

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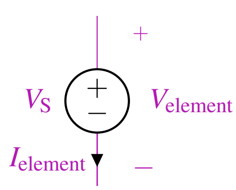

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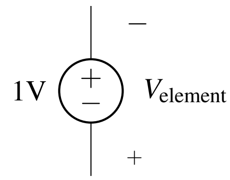

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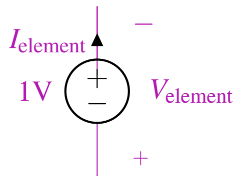

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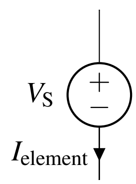

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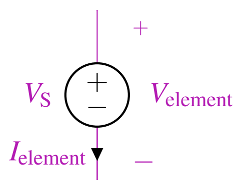

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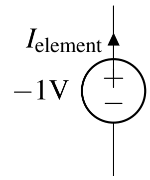

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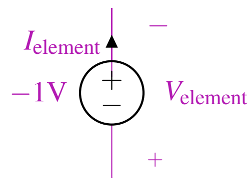

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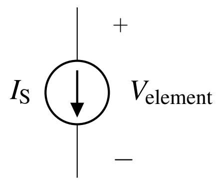

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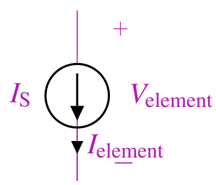

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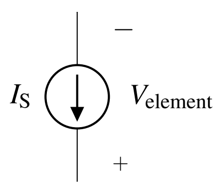

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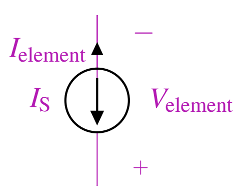
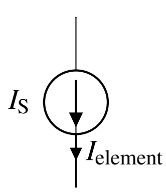


(i) **(PRACTICE)**



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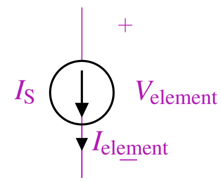

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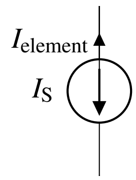

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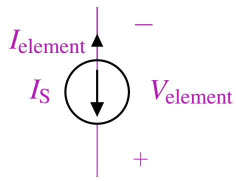

(k) **(PRACTICE)**

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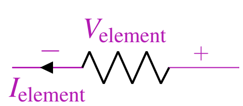

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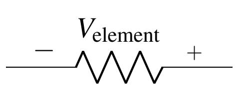

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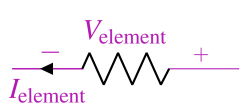


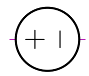
2. Basic Circuit Components

In this problem, we will introduce the fundamental circuit components.

(a) What is a voltage source?

Solution: Firstly, a voltage source is represented in this manner:

V

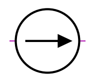


Essentially, a voltage source **guarantees** that the potential at its positive end will be V more than the potential at its negative end, no matter what.

(b) What is a current source?

Solution: A current source is represented in this manner:

I



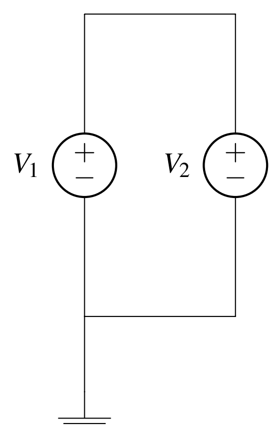
A current source **guarantees** that the current passing through the unit in the direction of the arrow will be its designated value.

(c) What is voltage? What is a voltage drop?

Solution: For our discussion, it suffices to think of voltage as a kind of driver for current. Current is the movement of charges. A voltage difference forces current to move from the point (node) that has higher voltage, to the point that has lower voltage.

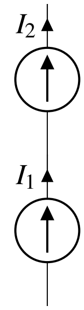
Voltage drop is the voltage lost (decline of nodal voltage) across a circuit component.

(d) What happens in this case if $V_1 \neq V_2$?



Solution: Let us designate the potential at the positive end of V_1 to be V_1^+, the potential at the negative end of V_1 to be V_1^-, the potential at the positive end of V_2 to be V_2^+, and the potential at the negative end of V_2 to be V_2^-. V_1^- and V_2^- are equal to 0 because of the ground. Then, the potential across V_1 is V_1^+, and the potential across V_2 is V_2^+. Since V_1^+ and V_2^+ are connected by a wire, they must be the same voltage; we know that a wire does not affect a circuit's behavior, so the voltage must stay constant across it. This means that $V_1^+ = V_2^+$. However, we know that the voltage potential $V_1^+ - V_1^-$ is not equal to $V_2^+ - V_1^-$ as given in the question. Hence, we see that we cannot have two voltage sources connected in this configuration.

(e) What happens in this case if $I_1 \neq I_2$?



Solution: The current source at the bottom guarantees that through that wire there will be I_1 current going through, and the current source at the top guaranteed that I_2 current goes through that wire. This is a contradiction, and is not theoretically possible in a circuit.

Also, look at the point in between the two current sources. I_1 enters on one end, and I_2 leaves on the other end. This is impossible.

(f) What is a resistor?

Solution: A resistor is represented in this manner:

A resistor is a circuit unit designed to 'resist' the flow of current. Following convention, there is a "voltage drop" across a resistor from the positive end to the negative end. The voltage drop across a resistor is $V_R = I_R R$, where V_R is the voltage drop, I_R is the current through the resistor and R is the resistance of the resistor.

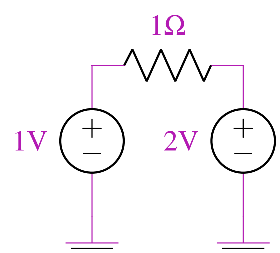
(g) What is power?

Solution: Power is the rate at which work is done, where work is in terms of electrical energy.

For circuits, the power *consumed* or *dissipated* by a device is $P = IV$, where the current and voltage abide by passive sign convention.

Common Misconceptions:

- Active components do not necessarily dissipate negative power! Consider the following circuit:



When calculating the power dissipated by the LHS voltage source, we see the current flows counterclockwise about the circuit. With passive sign convention, we calculate the power $P_{V_1} = 1\text{V} \times 1\text{A}$, which is positive! The left-side voltage source is dissipating power.

3. **Warning! High Resistivity Zone Ahead!**

Resistivity is a physical property that quantifies how much the material opposes the flow of electric current.

Assume that in an ideal case, the cross-section and physical composition of the wire are uniform, We can find its resistivity with the equation below:

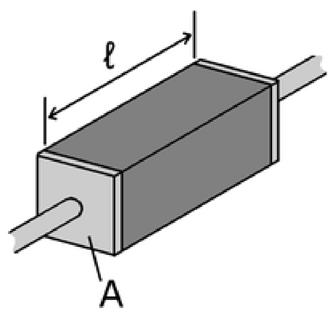
$$\rho = R\frac{A}{l}$$

Here, ρ stands for the resistivity of the wire, R stands for its resistance, A stands for the area of the cross section of the wire, and l stands for the length of the wire. Using this equation, we can also solve for the resistance of a wire:

$$R = \rho\frac{l}{A}$$

(a) Now, consider the rectangular copper wire below. Given that the cross-section of the wire is a square and has a cross section area of A, determine the overall resistance of the wire in terms of ρ_{cu}, L, and A.
Solution: By the resistivity equation, the resistance of the wire is equal to:

$$R = \rho_{cu}\frac{L}{A}$$



(b) Suppose we have N such wires and align them side by side to form a mega-wire in the following fashion. Find the overall resistance of this mega-wire. What is this configuration similar to?

Solution: Since we have all N wires aligned side by side, we are essentially expanding the cross-section area while merging all the wires together by their lengths. This means that the new mega-wire will have a new cross-section area of NA (since we are merging N such wires), while its length remains the same (L). Hence, the overall resistance of the mega-wire will be:

$$R_{mega} = \rho_{cu}\frac{L}{NA}$$

(c) Again, with N identical wires, what's a configuration that can achieve the highest resistance possible? What is this configuration similar to?
Solution: The key of this question is to start from the resistance equation:

$$R = \rho\frac{l}{A}$$

Algebraically, we want to maximize the value of R for this question. Since ρ is just a physical constant, we can't really manipulate its value. Observing the fraction $\frac{l}{A}$, we can see that the overall length of the mega-wire should be as great as possible, while its cross section area should be kept as small as possible. How can we arrange N wires in a way so that the overall new wire is as long as possible?
We arrange in a single long line! This also helps minimize the cross section area to A, and we can't really go lower than that since we can't split up a wire into two (thereby splitting the cross section area).
Hence, for this new mega-wire, it has a length of NL and a cross section area of A. Applying the resistance equation, we have:

$$R_{mega} = \rho_{cu} \frac{NL}{A}$$

This configuration is exactly the same as a series circuit where resistors are connected in one long line. If you think in terms of the equivalent resistance for a series circuit, it also makes sense since we are summing up all the resistances.

(d) Consider part (b) again, but this time, instead of N copper wires, we split the number evenly between aluminum wires and copper wires, and we again align them side by side to form a mega-wire (with the left half all aluminum , and right half all copper). What's the overall resistance of this wire? (In terms of ρ_{cu}, ρ_{Al}, L, and a)

Solution: As we can see from part (b), when we are aligning the wires side by side, we are essentially arranging the wires to be parallel to each other. Instead of thinking in terms of individual wires, we can consider the overall mega-wire to be a mega copper wire and a mega aluminum wire in parallel! For both wires, they will have a length of L and an overall cross section area of $(N/2)A$ (since we have $N/2$ wires for each category). Hence, applying the resistance equation again, we can find that:

$$R_{cu-mega} = \rho_{cu} \frac{L}{\frac{N}{2}A} = \rho_{cu} \frac{2L}{NA}$$

$$R_{Al-mega} = \rho_{Al} \frac{L}{\frac{N}{2}A} = \rho_{Al} \frac{2L}{NA}$$

Now, since these 2 mega wires are parallel to each other, by equivalent resistance, we can find the overall resistance of the wire to be:

$$R_{overall} = \frac{1}{\frac{1}{R_{cu-mega}} + \frac{1}{R_{Al-mega}}} = \left(\frac{\rho_{cu}\rho_{Al}}{\rho_{cu} + \rho_{Al}} \right) \frac{2L}{NA}$$

If you look closely at the equation, you can actually see how the resistivities of both materials are arranged algebraically as if they are in parallel!

(e) Instead of having all the $N/2$ copper wires on the right side, now all the wires are mixed together (a copper wire can be aligned right next to an aluminum wire), does the overall resistance of this new mega-wire change?

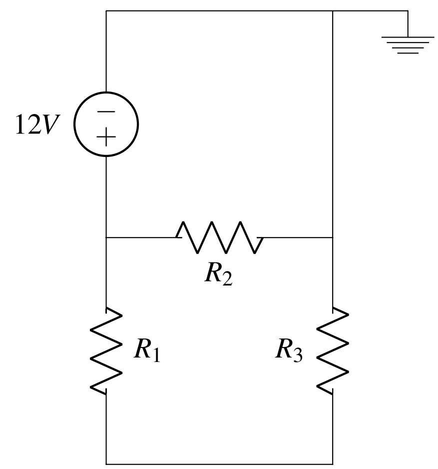
Solution: The key to this question is to realize that, we are essentially aligning all wires in parallel, so we can rearrange all the copper wires to be together on one side and all the aluminum wires to be together on another side, and now we are back to the previous question! Hence, the overall resistance of this new mega-wire doesn't change.

4. Introduction to KVL, KCL

KVL: Kirchhoff's Voltage Law says that the sum of voltage drops going around a closed network (starting and ending at the same point) is 0

KCL: Kirchhoff's Current Law says that the sum of all currents flowing into a node are equal to the sum of all currents flowing out of a node. This is equivalent to saying that the sum of all currents flowing out of a node (representing flows in opposite direction as negative) is equal to the sum of all currents flowing into a node (again, representing flows in opposite direction as negative) which is equal to zero

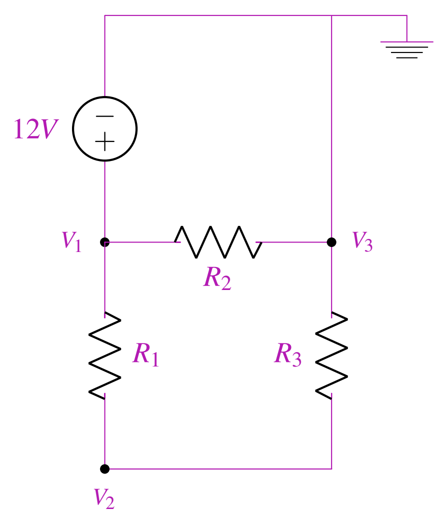
Consider the following circuit, with $R_1 = 2\Omega$, $R_2 = 3\Omega$, and $R_3 = 4\Omega$:



We will solve this picture using nothing but KCL and KVL in the following steps.

(a) First, draw a reference point (ground) on the circuit, and label all the nodes V_1, V_2, etc.

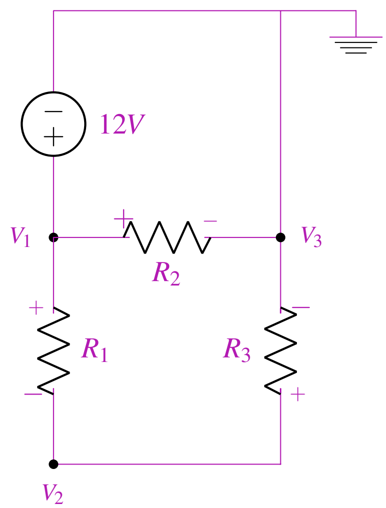
Solution: You should draw a ground on either side of the voltage source, and the 3 nodes (labelled any way) as shown below. We'll continue using this labeling convention for the rest of the problem.



(b) Next, draw the +s and -s on the resistors.

Solution: While there is no incorrect way to do this (as long as you are consistent with the equations in future parts), the way of doing so that is consistent with the method used in this problem is:

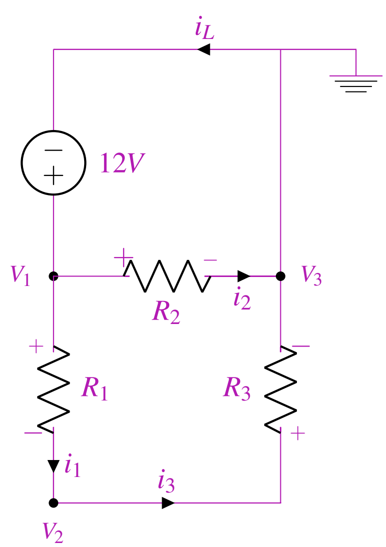
For R_1, have the + be on the northside and the - on the southside.
For R_2, have the + be on the leftside and the - on the rightside.
For R_3, have the + be on the southside and the - on the northside.



(c) Finally, draw arrows indicating the direction of current, labeling them as I_1, I_2, etc.

Solution: Once again, as long as you are consistent for the next part, there isn't a wrong answer, but our equations use:
I_1 is an arrow going south between the source and V_1.
I_2 is an arrow going east between V_1 and V_3 around R_2.
I_3 is an arrow going east between V_2 and V_3 around R_3.
I_L is an arrow going west between ground and the negative terminal of the voltage source.

Our drawn on circuit should now look something like:



(d) Now write all the equations given by Ohm's law, KCL and any voltages you can find relative to ground

Solution: Ohm's Law ($\Delta V = RI$):

$$V_1 - V_2 = 2I_1$$
$$V_2 - V_3 = 4I_3$$
$$V_1 - V_3 = 3I_2$$

We also see that V_1 is $12V$ above ground and V_3 is connected to the same node as the ground, giving us:

$$V_1 = 12V$$
$$V_3 = 0V$$

And here are our KCL equations:

$$I_1 = I_3$$
$$I_L = I_1 + I_2$$
$$I_L = I_2 + I_3$$

(e) Now that you have all equations written out, solve them to find all unknown voltages and currents.
Solution: We can start off by finding equivalent values. For simplicity, let's keep everything in terms of I_1. If we substitute all of these into the Ohm's Law formula, we have the current equations:
Now, we have the following values:

$$V_1 = 12V$$
$$V_2 = 8V$$
$$V_3 = 0V$$
$$I_1 = 2A$$
$$I_2 = 4A$$
$$I_3 = 2A$$

(f) Now, we're going to solve the same circuit with Nodal Analysis. Notice that in the previous part, you had many equations are variables (corresponding to both voltage and current). This may make the system more tedious to solve. Instead, Nodal Analysis allows us to write a system of equations that only involves the node voltages.
Start by using the ground and voltage source to find any node voltages which are immediately known.
Solution: Because V_3 belongs to the same node as ground, we know $V_3 = 0V$. Also, the voltage drop across the source is $12V$, which equals $V_1 - 0$. Therefore, $V_1 - 12 = 0 \implies V_1 = 12$. So, we know that:

$$V_1 = 12V$$

$$V_3 = 0V$$

(g) Then, Ohm's law to write a KCL equation for each node purely in terms of voltages and resistances.

Solution: Applying KCL at the node V_2, we have

$$\frac{V_1 - V_2}{R_1} + \frac{V_3 - V_2}{R_3} = 0$$

Note that we have just written each of the incoming currents in terms of the voltage drop across the corresponding resistors. At this point, we actually have enough information to solve the equation, because V_1 and V_3 are known quantities. Plugging them in, we get

$$\frac{12 - V_2}{2} + \frac{0 - V_2}{4} = 0 \implies 6 - \frac{V_2}{2} - \frac{V_2}{4} = 0 \implies \frac{3}{4}V_2 = 6 \implies V_2 = 8V$$

Now that we've solved for all the node voltages, we have implicitly solved the entire circuit! Because we can always use the node voltages to find any currents we want. In particular,

$$I_1 = \frac{V_1 - V_2}{R_1} = \frac{12 - 8}{2} = 2A$$

$$I_2 = \frac{V_1 - V_3}{R_2} = \frac{12 - 0}{3} = 4A$$

$$I_3 = \frac{V_2 - V_3}{R_3} = \frac{8 - 0}{4} = 2A$$

And the current through the top loop is $I_2 + I_3 = 6A$ anticlockwise (applying KCL at the node V_3).

(h) When doing nodal analysis, would it have helped to write a KCL equation at nodes V_1 and V_3? Why or why not?

Solution: No, it would not have helped. We would have had to introduce an extra variable for the current through the top loop, and use this in each of our KCL equations. So it's ultimately the same problem, just with more equations and variables.

There is also the issue that we now have a variable for current which we cannot turn into a voltage (becuase there is no resistor in the top loop. This is not desirable, since in nodal analysis we want all our equations to be in terms of voltages. No currents anywhere!

It turns out that when doing in nodal analysis you can skip KCL for nodes which are directly adjacent to components like voltage sources. However, we don't lose any information, because we still incorporated the source voltage into the process of finding V_1 and V_3 in the beginning!

5. **Never Fail at Resistor Equivalence**

(a) Using what you learned about resistance above ($R = \frac{\rho \cdot l}{A}$), explain what would happen if you connect two resistors in series.

Solution: Intuitively, the amount of impedance the current faces by going through the two resistors in series would just be the sum of the impedance faced by the current going through each individual of the resistor. Thus, for n resistors in series,

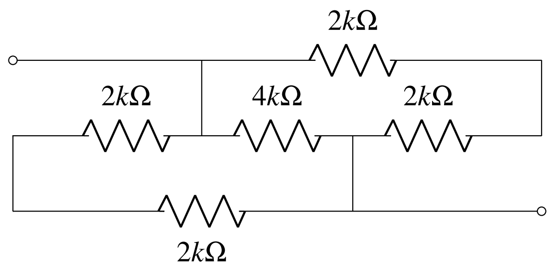
$$R_{eq} = R_1 + R_2 + \cdots + R_n$$

(b) Using the same resistance equation from part (a), explain what would happen if you connect two resistors in parallel.

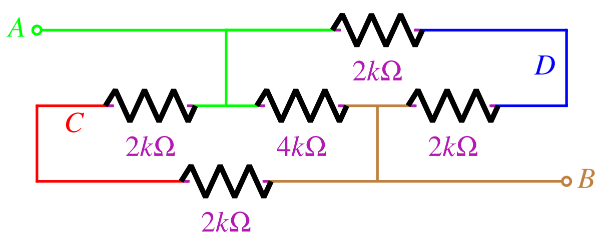
Solution: Having resistors in parallel is like taking taking two boxes and combining them via their sides. This increases the cross-sectional area in the front. So intuitively, the equivalent resistance of resistors in parallel would be smaller than the resistance of the original resistors. In addition, intuitively, more current would choose to go to the side with less resistance. Thus, for n resistors in parallel,

$$R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}}$$

(c) Now let's apply some of these ideas to a circuit. For the circuit below, mark all the nodes.



Solution:



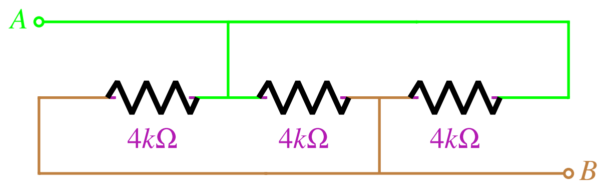
(d) Mark which nodes are 2-nodes and multi-nodes. 2 nodes are connected to only 2 components, and multi-nodes are connected to 3 or more components.
Note: *Whenever nodes are marked across which equivalent resistance must be found, those are considered 'components' because something could be connected there.*

Solution: Nodes A and B are multi-nodes. Nodes C and D are 2-nodes.

(e) Resistors that are connected to 2-nodes are considered to be in series. Redraw the circuit, and find the 2 and multi-nodes again after combining the resistors connected to 2 nodes.

Solution:

Nodes C and D disappear and become nodes A and B.



(f) Now we should be left with only multi-nodes. So far, we have seen that any resistors connected to 2-nodes are in series. We will see what happens to resistors connected to multi-nodes. Begin by writing out the 2 nodes that each resistor is connected to.

Solution: The resistor on the left is connected to nodes A and B. The resistor on the middle is connected to nodes A and B. The resistor on the right is connected to nodes A and B.

(g) If you have 2 or more resistors that are connected to the same 2 nodes, then they are in parallel. What does this mean for the 3 remaining resistors?

Solution: Since all 3 resistors are connected to nodes A and B, they are all in parallel. The equivalent resistance is $\frac{4}{3}k\Omega$. Quick tip: if you have 2 resistors in parallel, both of whose resistance is R, then the equivalent resistance is $\frac{R}{2}$. Similarly, if you have 3 resistors in parallel, all three of whose resistance is R, then the equivalent resistance is $\frac{R}{3}$, and so on with 4 or more resistors of the same value.