RECURSION, TREE RECURSION Meta

COMPUTER SCIENCE MENTORS 61A

September 23 – September 27, 2024

Example Timeline

- Tree Recursion Mini Lecture [8 Mins]

  1. Consider creating a simple example to demonstrate recursion. Example: A recursive accumulation function (ie: next # = $\sum$ previous #s)

- Problems: Recursion [20 Mins]
  - Q1 - FizzBuzz
  - Q2 - Sum Prime digits
  - Q3 - Near
- Problems: Tree Recursion [20 Mins]
  - Q4 - Copy Machine
  - Q5 - Mario Number
  - Q6 - Fast Modular Exponentiation

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1 Recursion

1. Implement a recursive version of `fizzbuzz`.

```
def fizzbuzz(n):
    """Prints the numbers from 1 to n. If the number is divisible by 3, it
    instead prints 'fizz'. If the number is divisible by 5, it instead
       prints
    'buzz'. If the number is divisible by both, it prints 'fizzbuzz'. You
       must do this recursively!

    >>> fizzbuzz(15)
    1
    2
    fizz
    4
    buzz
    fizz
    7
    8
    fizz
    buzz
    11
    fizz
    13
    14
    fizzbuzz
    """

    if n == 1:
        print(n)
    else:
        fizzbuzz(n - 1)
        if n % 3 == 0 and n % 5 == 0:
            print('fizzbuzz')
        elif n % 3 == 0:
            print('fizz')
        elif n % 5 == 0:
            print('buzz')
        else:
            print(n)
```

It may be beneficial to reiterate the recursive leap of faith! Hypothetically, if fizzbuzz works for n, what would fizzbuzz(n - 1) output?

This is an intro example of recursion - Feel free to skip if your students are clear on the concept of recursion.
⇒ Suggested Time: 3 min; Difficulty (Official): Medium; Difficulty (Adjusted): Easy/Intro

2. Complete the definition for `sum_prime_digits`, which returns the sum of all the prime digits of `n`. Recall that 1 is not prime. Assume you have access to a function `is_prime`; `is_prime(n)` returns `True` if `n` is prime, and `False` otherwise.

```
def sum_prime_digits(n):
    """
    >>> sum_prime_digits(12345)
    10 # 2 + 3 + 5
    >>> sum_prime_digits(4681029)
    2 # 2 is the only prime number
    """
    if ____________________________________________:

        return ____________________________________

    if ________________________________________:

        return ______________________________

    return ____________________________________
```

```
    if n == 0:
        return 0
    if is_prime(n % 10):
        return n % 10 + sum_prime_digits(n // 10)
    return sum_prime_digits(n // 10)
```

- Again, this problem is likely skippable if you have advanced students.
- Make sure your students understand why you are calling `sum_prime_digits` twice.

Suggested Time: 5 min; Difficulty (Official): Medium; Difficulty (Adjusted): Easy → Medium Mezzanine

3. Fill in `near`, which takes in a non-negative integer `n` and returns the largest, non-consecutively repeating, near increasing sequence of digits within n as an integer. The arguments `smallest` and `d` are part of the implementation; you must determine their purpose. You may **not** use any values except integers and booleans (`True` and `False`) in your solution (no lists, strings, etc.).

   A sequence is *near increasing* if each element but the last two is smaller than all elements following its subsequent element. That is, element i must be smaller than elements $i+2$, $i+3$, $i+4$, etc. A *non-consecutively repeating* number is one that do not have two of the same digits next to each other. [Adapted from CS61A Fa18 Final Q3(c)]

```
def near(n, smallest=10, d=10):
    """
    >>> near(123)
    123
    >>> near(153)
    153
    >>> near(1523)
    153
    >>> near(15123)
    153
    >>> near(985357)
    537
    >>> near(11111111)
    1
    >>> near(14735476)
    143576
    >>> near(14735476)
    1234567
    """
    if n == 0:

        return ______________________________

    no = near(n//10, smallest, d)

    if (smallest > ______________) and (________________):

        yes = ______________________________

        return ______________________________(yes, no)

    return ______________________________
```

```python
def near(n, smallest=10, d=10):
    if n == 0:
        return 0

    no = near(n//10, smallest, d)

    if (smallest > n % 10) and (n % 10 != d):
        yes = 10 * near(n//10, min(smallest, d), n%10) + n%10
        # OR yes = 10 * near(n//10, d, min(d, n%10)) + n%10
        return max(yes, no)

    return no
```

`smallest` = smallest digit

`d` = previous digit

For a video walkthrough of the unadapted exam problem, see https://youtu.be/NnE6qFZsoGo. This should give you a good idea how to approach the adapted version.

- Demonstrating with the doc tests is very important - the problem description can be confusing.
- This is probably a decently tricky problem – even for an exam level problem. I think even mentors should watch the walk through video as that gives you some good ideas on how to teach this problem.
- Essentially it boils down to figuring out what `smallest` & `d` is and how to decide whether to use the last digit or not in the computation (when you apply recursion to this, you can construct the entire number of the solution).

⇒ Suggested Time: 20 min; Difficulty (Official): Hard – Exam level; Difficulty (Adjusted): Hard – Exam level

2 Tree Recursion

1. James wants to print this week's discussion handouts for all the students in CS 61A. However, both printers are broken! The first printer only prints multiples of `n` pages, and the second printer only prints multiples of `m` pages. Help James figure out whether or not it's possible to print exactly `total` number of handouts!

```
def has_sum(total, n, m):
    """
    >>> has_sum(1, 3, 5)
    False
    >>> has_sum(5, 3, 5) # 0 * 3 + 1 * 5 = 5
    True
    >>> has_sum(11, 3, 5) # 2 * 3 + 1 * 5 = 11
    True
    """
    if ________________________________________________:

        return ________________________________

    elif ______________________________________________:

        return ________________________________

    return ____________________________________________
```

```
def has_sum(total, n, m):
    if total == 0:
        return True
    elif total < 0: # you could also put total < min(m, n)
        return False
    return has_sum(total - n, n, m) or has_sum(total - m, n, m)
```

An alternate solution you could write that may be slightly faster in certain cases:

```
def has_sum(total, n, m):
    if total == 0 or total % n == 0 or total % m == 0:
        return True
    elif total < 0: # you could also put total < min(m, n)
        return False
    return has_sum(total - n, n, m) or has_sum(total - m, n, m)
```

(Solution continues on the next page)

When thinking about the recursive calls, we need to think about how each step of the problem works. Tree recursion allows us to explore the two options we have while printing: either print `m` papers at this step or print `n` papers at this step and can combine the results after exploring both options. Inside the recursive call for `has_sum(total - n, n, m)`, which represents printing `n` papers, we again consider printing either `n` or `m` papers.

Once we have these recursive calls we need to think about how to put them together. We know the return should be a boolean so we want to use either **`and`** or **`or`** to combine the values for a final result. Given that we only need one of the calls to work, we can use **`or`** to reach our final answer.

In the base cases we also need to make sure we return the correct data type. Given that the final return should be a boolean we want to return booleans in the base cases.

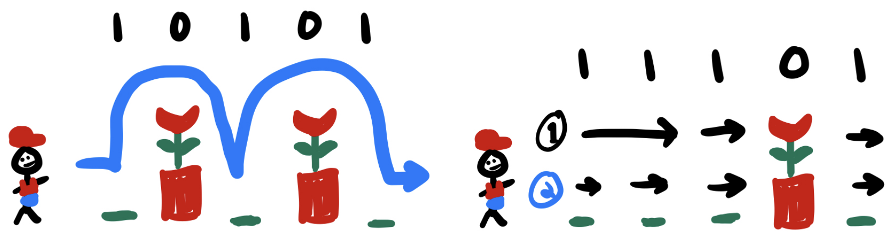
Another alternate base case would be: `total == 0 or total % n == 0 or total % m == 0`. This solution would also work! You would just be stopping the recursion early, since the total can be a multiple of `n` or `m` in order to trigger the base case - it doesn't have to be 0 anymore. Just be sure to still include the `total == 0` check, just in case someone inputs 0 as the total into the function.

Teaching Tips

- Some leading questions:
  - What are the base cases (when to return True/False)?
  - How can we reduce this problem into smaller subproblems (recursive step)?
  - What does the value of a recrusive call tell us?
  - How can we put recursive calls together to get a final answer?
- Ask students about the simplest possible cases to identify base cases; make sure they realize `(total == n)` or `(total == m)` is incorrect because `(total == 0)` is a simpler True case.
- Point out the fact that tree recursion problems usually have you consider multiple "options" or "possibilities," and they should all be explored when you are writing your recursive cases.
- Out of all possible combinations of `n` and `m`, we only need 1 way for `n` and `m` to sum to the total for the function to return True, which implies **`or`** is an appropriate way to aggregate our recursive calls.
- This is a good mini-lecture problem to use as a demo since this is probably the closest thing the students have seen to what they demo-ed in lecture.

⇒ Suggested Time: 6 min; Difficulty (Official): Medium; Difficulty (Adjusted): Easy

2. Mario needs to get from one end of a level to the other, but there are deadly Piranha plants in his way! Mario only moves forward and can either *step* (move forward one space) or *jump* (move forward two spaces) from each position. A level is represented as a series of ones and zeros, with zeros denoting the location of Piranha plants. Mario can step on ones but not on zeros. How many different ways can Mario traverse a level without stepping or jumping into a Piranha plant? Assume that every level begins with a 1 (where Mario starts) and ends with a 1 (where Mario must end up).



Hint: Does it matter whether Mario goes from left to right or right to left? Which one is easier to check?

```
def mario_number(level):
    """
    >>> mario_number(10101)
    1
    >>> mario_number(11101)
    2
    >>> mario_number(100101)
    0
    """
    if _______________________:

        _______________________

    elif _____________________:

        _______________________

    else:

        ____________________________________________________
```

```python
def mario_number(level):
    if level == 1:
        return 1
    elif level % 10 == 0:
        return 0
    else:
        return mario_number(level // 10) + mario_number((level // 10) //
            10)
```

You can think about this tree recursion problem as testing out all of the possible ways Mario can traverse the level, and adding 1 every time you find a possible traversal.

Here it doesn't matter whether Mario goes left to right or right to left; either way we'll end up with the same number of ways to traverse the `level`. In that case, we can simply choose for Mario to start from the right, and then we can process the level like we process other numbers in digit-parsing related questions by using floor division (`//`) and modulo (`%`)

At each point in time, Mario can either step or jump. We use a single floor division (`//`) of `level` by 10 to represent taking one step (if we took a step, then the entire `level` would be left except for the last number), while two floor divisions by 10 (or equivalently one floor division by 100) corresponds to a jump at this point in the `level` (if we took a jump, then the entire `level` would be left except for the last two numbers).

To think of the base cases, you can consider the two ways that Mario ends his journey. The first, corresponding to `level == 1`, means that Mario has successfully reached the end of the level. You can **`return`** `1` here, because this means you've found one additional path to the end. The second, corresponding to `level % 10 == 0`, means that Mario has landed on a Piranha plant. This returns 0 because it's a failed traversal of the `level`, so you don't want to add anything to your result.

In tree recursion, you need to find a way to combine separate recursive calls. In this case, because `mario_number` returns an integer and the base cases are integers and you're trying to count the total number of ways of traversal, it makes sense to add your recursive calls.

Teaching Tips

- Some leading questions:
  - What are our base cases? (When do we know we've reached the end of the level? When do we know that we've failed?)
  - Is there any difference between going left to right or right to left in terms of the number of ways to traverse the level?
  - What are our two options at each step?
  - What do those look like in a recursive call?
  - How should we combine our recursive calls? (and, or, addition, etc.)
- Try leaning into the narrative of the question! It's fun and can help rephrase recursive calls "in plain english" I also like drawing the problem out along with the doctests to visualize the different steps Mario can take :D !
- It's very useful to draw a tree diagram! Each function call has one branch for stepping once and another branch for stepping twice. Each branch then has 2 branches of their own (until a base case is reached).

- Teach students that recursive calls can be treated as numbers using the recursive leap of faith, so combining the two recursive call branches with addition is really just adding two numbers.

⇒ Suggested Time: 10 min; Difficulty (Official): Medium; Difficulty (Adjusted): Easy → Medium (Mezzanine)

The classic mario-number problem from CSM. A worksheet wouldn't be complete without it!

3. **Fast Modular Exponentiation:** In many computing applications, we need to quickly compute $n^x \mod z$ where $n > 0$, and x and z are arbitary whole numbers. Computing $n^x \mod z$ for large numbers can get extremely slow if we repeatedly multiply n for x times. We can implement the following recursive algorithm to help us speed up the exponentiation operation.

$$x^n \mod z = \begin{cases} x * (x^2)^{(n-1)/2} \,\%\, z & \text{if } n \text{ is odd} \\ (x^2)^(n/2) \,\%\, z & \text{if } n \text{ is even} \end{cases}$$

This is an example of a "divide & conquer" algorithm and follows the same train of thought as tree-recursion problems (you are dividing some complex problem into smaller parts and performing both options).

```
def modular_exponentiation(base, exponent, modulus):
    """
    >>> modular_exponentiation(2, 2, 2)
    0
    >>> modular_exponentiation(4, 2, 3)
    1
    """
    if ____________________:

        return __________________________________

    if ____________________:

        half_power = __________________________________

        return __________________________________ % modulus

    else:

        return ______________________________________ % modulus
```

Note: The algorithm you just implemented is a key part of modern day cryptography techniques such as RSA and Diffie-Hellman key exchange. In some cases, the exact operations you just implemented is used in modern day, state of the art, programs (if you are curious, Google "Right-to-left binary method"). You will learn more about RSA in CS70. If you want to learn more about computer security, consider taking CS161 after CS61C.

```python
def modular_exponentiation(base, exponent, modulus):
# Base case: exponent is 0
if exponent == 0:
    return 1

# Recursive case
if exponent % 2 == 0:
    half_power = modular_exponentiation(base, exponent // 2,modulus)
    return (half_power * half_power) % modulus
else:
    return (base * modular_exponentiation(base, exponent - 1,
        modulus)) % modulus
```

- This problem doesn't exactly follow the scheme of tree recursion discussed in lecture, but it does follow the general philosophy behind tree recursion. You probably want to reiterate this.

  I think it is important for students to relate "tree recursion" which is a SICP concept to something in the real world (whether that'd be a type of ~~leetcode~~ algorithms problem or a real world use). Here real world use made more sense as a lot of algorithms problems with tree recursion involved binary trees (and trees in general), which hasn't been introduced in lecture yet.
- This problem is not meant to be tricky or as much of a "drill" problem as the other ones. This is more of a "here is what tree recursion can do" type – so feel free to walk through this problem if you want.

Suggested Time: 10 min; Difficulty (Official): Medium; Difficulty (Adjusted): Medium