Recursion, Tree Recursion Meta

Computer Science Mentors 61A

February 13–February 17, 2022

Recommended Timeline

- Tree recursion mini lecture: 8 minutes
- Gibonacci: 15 minutes – skippable if your students feel especially comfortable with tree recursion; otherwise please do this.
- FizzBuzz: 8 minutes – could be used as a step between Gibonacci and Selective Sum.
- Selective Sum: 10 minutes
- Has Sum: 10 minutes
- Mario Number: 10 minutes
- Collapse: 10–20 minutes

As a reminder, there is no expectation that you get through all problems in a section. Choose questions that will best benefit your students.

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Tree Recursion vs Recursion

In most recursive problems we've seen so far, the solution function contains only one call to itself. However, some problems will require multiple recursive calls – we colloquially call this type of recursion "tree recursion," since the propagation of function frames reminds us of the branches of a tree. "Tree recursive" or not, these problems are still solved the same way as those requiring a single function call: a base case, the recursive leap of faith on a subproblem, and solving the original problem with the solution to our subproblems. The difference? We simply may need to use multiple subproblems to solve our original problem.

Tree recursion will often be needed when solving counting problems (how many ways are there of doing something?) and optimization problems (what is the maximum or minimum number of ways of doing something?), but remember there are all sorts of problems that may need multiple recursive calls! Always come back to the recursive leap of faith.

Two rules that are often useful in solving counting problems:

1. If there are x *ways* of doing something and y *ways* of doing another thing, there are xy *ways* of doing **both** at the same time.
2. If there are x *ways* of doing one thing and y *ways* of doing another, but we can't do both things at the same time, there are $x + y$ ways of doing either the first thing **or** the second thing.

Teaching Tips

- Stress the power of tree recursion: it lets us find a single solution among 14,000,605 futures.
- Try dividing tree recursion questions into three parts: base cases, recursive calls, and combining recursive calls.
    1. What are the simplest possible arguments for the function?
        - There may be hints for base cases in doc tests. Run through simple examples!
    2. What options should be recursively explored?
        - Drawing tree diagrams can help a lot for this section.
    3. How should the answers of subproblems be combined?
        - Trust recursive calls to return the correct values (recursive leap of faith!) and combine them with mathematical or logical operators.

1. The *Gibonacci sequence* is a recursively defined sequence of integers; we denote the nth Gibonacci number g_n. The first three terms of the sequence are $g_0 = 0, g_1 = 1, g_2 = 2$. For $n \geq 3$, g_n is defined as the sum of the previous three terms in the sequence.

   Complete the function `gib`, which takes in an integer `n` and returns the nth Gibonacci number, g_n. Also, identify the three parts of recursive function design as they are used in your solution.

```
def gib(n):
    """
    >>> gib(0)
    0
    >>> gib(1)
    1
    >>> gib(2)
    2
    >>> gib(3) # gib(2) + gib(1) + gib(0) = 3
    3
    >>> gib(4) # gib(3) + gib(2) + gib(1) = 6
    6
    """
    if _____________________________________:

        return _____________________________________

    return _____________________________________
```

```
def gib(n):
    if n <= 2:
        return n
    return gib(n - 1) + gib(n - 2) + gib(n - 3)
```

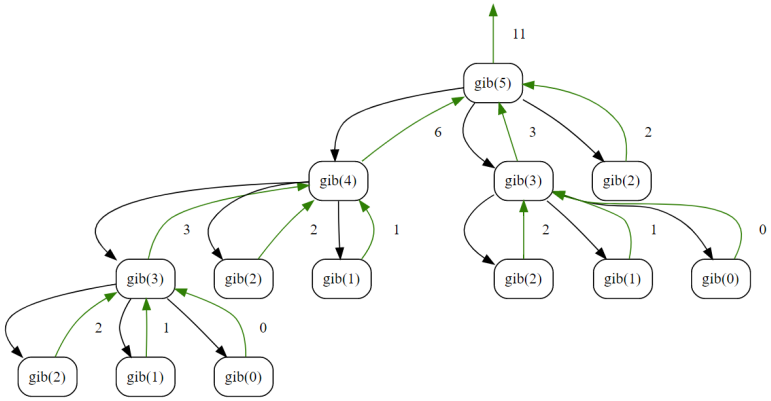
- Base case: `if n <= 2: return n`
- Recursive calls: `gib(n - 1),gib(n - 2),gib(n - 3)`
- Solving the larger problem: adding the results of the three recursive calls.

- This problem is very similar to Fibonacci, which students should be familiar with from lecture.
- Ensure that students understand how the minor alteration to the sequence slightly modifies the recursive calls and the base case

2. Including the original call, how many calls are made to `gib` when you evaluate `gib(5)`?

   13.

   There are a few ways you could determine this. First, you could draw out a call graph that shows all of the calls that are made. `https://tinyurl.com/gibsol`



   You could also build things from the ground up using recursive intuition.

   - `gib(0)` takes one call because it's a base case.
   - `gib(1)` takes one call because it's a base case.
   - `gib(2)` takes one call because it's a base case.
   - `gib(3)` takes four calls: one for the original call, one for `gib(0)`, one for `gib(1)`, and one for `gib(2)`.
   - `gib(4)` takes seven calls: one for the original call, four for `gib(3)`, one for `gib(2)`, and one for `gib(1)`.
   - `gib(5)` takes 13 calls: one for the original call, seven for `gib(4)`, four for `gib(3)`, and one for `gib(2)`.

   This second method is much faster, especially for higher values of `n`.

   In general, I believe that drawing out call graphs can be confusing for students, especially for more difficult recursive problems. The whole idea behind recursion, after all, is that you don't have to worry about the dense tree of calls that result from a single call to a recursive function—you can just trust that the recursive calls you do make return the correct result and don't have to worry about how they specifically got that result. However, I believe that it is useful to do this sort of thing once because it demonstrates how recursion actually works in practice and why the recursion is valid and useful.

   When teaching this problem, I recommend first going over the call tree. I would

emphasize a few ideas. First, our base cases are at the bottom of the call graph, and every call eventually terminates at one of the base cases. They literally serve as the foundation of the tree. They are the "truth" upon which the function's validity for higher inputs rests. I would also point out that every call that's not a base case has three calls underneath it because each of these calls makes three recursive calls. Hopefully this makes the idea of recursion more concrete for the students and qualms perennial concerns that the logic behind recursion is circular.

I would then going over the "smarter" way of doing the problem. I would point out how doing the problem in this way allows us to avoid the repetitive, confusing, and error prone work of considering every call in the function. I would note that if we tried to do the same thing with `gib(1000)`, the graph would be untenable. The lesson I want them to walk away with is: abstracting away the huge number of calls is a valuable tool when working with recursion, and that they should try to avoid thinking about these calls if at all possible, as it only gets more confusing for more complicated functions.

3. Implement a recursive `fizzbuzz`.

```
def fizzbuzz(n):
    """Prints the numbers from 1 to n. If the number is
       divisible by 3, it
    instead prints 'fizz'. If the number is divisible by 5,
       it instead prints
    'buzz'. If the number is divisible by both, it prints
       'fizzbuzz'.

    >>> fizzbuzz(15)
    1
    2
    fizz
    4
    buzz
    fizz
    7
    8
    fizz
    buzz
    11
    fizz
    13
    14
    fizzbuzz
    """


    if n == 1:
        print(n)
    else:
        fizzbuzz(n - 1)
        if n % 3 == 0 and n % 5 == 0:
            print('fizzbuzz')
        elif n % 3 == 0:
            print('fizz')
        elif n % 5 == 0:
            print('buzz')
        else:
            print(n)
```

4. Write a function `selective_sum`, which takes in an integer `n` and a predicate function `cond`. `selective_sum` returns the sum of all positive integers up to `n` for which `cond(n)` is true.

```
def selective_sum(n, cond):
    """
    >>> is_odd = lambda x: x % 2 == 1
    >>> selective_sum(5, is_odd) # 5 + 3 + 1 = 9
    9
    >>> bigger_than_10 = lambda x: x > 10
    >>> selective_sum(13, bigger_than_10) # 13 + 12 + 11 = 36
    36
    >>> selective_sum(-1, is_odd) # no positive integers <= 1
    0
    """
    if _____:

        return _____

    if _____:

        return _____

    return _____
```

```
def selective_sum(n, cond):
    if n <= 0:
        return 0
    if cond(n):
        return n + selective_sum(n - 1, cond)
    return selective_sum(n - 1, cond)
```

Teaching Tips

- Make sure to explain what predicate functions are (functions that return either True or False).
- Start with the base case! If there are no numbers that satisfy `cond` what do we return?
- Then move to the recursive step. How do we keep only the numbers that satisfy `cond`?

5. In an alternate universe, 61A software is not that good (inconceivable!). Tyler is in charge of assigning students to discussion sections, but sections.cs61a.org only knows how to list sections with either `m` or `n` number of students (the two most popular sizes). Given a `total` number of students, can Tyler create sections with only sizes of either `m` or `n` and not have any leftover students? Return `True` if he can and `False` otherwise.

```
def fit_sections(total, n, m):
    """
    >>> fit_sections(1, 3, 5)
    False
    >>> fit_sections(5, 3, 5) # 0 * 3 + 1 * 5 = 5
    True
    >>> fit_sections(11, 3, 5) # 2 * 3 + 1 * 5 = 11
    True
    >>> fit_sections(61, 11, 15) # can't express 61 as a
       * 11 + b * 15
    False
    """
    if ____________________________________________________:

        return True

    elif
       ____________________________________________________:

        return False

    return ____________________________________________
```

```
def fit_sections(total, n, m):
    if total == 0:
        return True
    elif total < 0: # you could also put total < min(m, n)
        return False
    return fit_sections(total - n, n, m) or
       fit_sections(total - m, n, m)
```

An alternate solution you could write that may be slightly faster in certain cases:

```
def fit_sections(total, n, m):
    if total == 0 or total % n == 0 or total % m == 0:
        return True
    elif total < 0: # you could also put total < min(m, n)
        return False
    return fit_sections(total - n, n, m) or
       fit_sections(total - m, n, m)
```

(Solution continues on the next page)

When thinking about the recursive calls, we need to think about how each step of the problem works. Tree recursion allows us to explore the two options we have: either create a new `m`-person discussion at this step or create a new `n`-person discussion at this step and can combine the results after exploring both options. Inside the recursive call for `fit_sections(total - n, n, m)`, which represents accommodating `n` students, we again consider adding either `n` or `m` students to the next section.

Once we have these recursive calls we need to think about how to put them together. We know the return should be a boolean so we want to use either **`and`** or **`or`** to combine the values for a final result. Given that we only need one of the calls to work, we can use **`or`** to reach our final answer.

In the base cases we also need to make sure we return the correct data type. Given that the final return should be a boolean we want to return booleans in the base cases.

Another alternate base case would be: `total == 0 or total % n == 0 or total % m == 0`. This solution would also work! You would just be stopping the recursion early, since the total can be a multiple of `n` or `m` in order to trigger the base case - it doesn't have to be 0 anymore. Just be sure to still include the `total == 0` check, just in case someone inputs 0 as the total into the function.

Teaching Tips

- Some leading questions:
  - What are the base cases (when to return True/False)?
  - How can we reduce this problem into smaller subproblems (recursive step)?
  - What does the value of a recursive call tell us?
  - How can we put recursive calls together to get a final answer?
- Ask students about the simplest possible cases to identify base cases; make sure they realize `(total == n)` or `(total == m)` is incorrect because `(total == 0)` is a simpler True case.
- Point out the fact that tree recursion problems usually have you consider multiple "options" or "possibilities", and they should all be explored when you are writing your recursive cases.
- Out of all possible combinations of `n` and `m`, we only need 1 way for `n` and `m` to sum to the total for the function to return True, which implies **`or`** is an appropriate way to aggregate our recursive calls.

6. Mario needs to get from one end of a level to the other, but there are deadly Piranha plants in his way! Mario only moves forward and can either *step* (move forward one space) or *jump* (move forward two spaces) from each position. A level is represented as a series of ones and zeros, with zeros denoting the location of Piranha plants. Mario can step on ones but not on zeros. How many different ways can Mario traverse a level without stepping or jumping into a Piranha plant? Assume that every level begins with a 1 (where Mario starts) and ends with a 1 (where Mario must end up).

   Hint: Does it matter whether Mario goes from left to right or right to left? Which one is easier to check?

```
def mario_number(level):
    """
    >>> mario_number(10101)
    1
    >>> mario_number(11101)
    2
    >>> mario_number(100101)
    0
    """
    if ____________________________:

        ____________________________

    elif ____________________________:

        ____________________________

    else:

        ____________________________________________________________
```

```
def mario_number(level):
    if level == 1:
        return 1
    elif level % 10 == 0:
        return 0
    else:
        return mario_number(level // 10) +
           mario_number((level // 10) // 10)
```

You can think about this tree recursion problem as testing out all of the possible ways Mario can traverse the level, and adding 1 every time you find a possible traversal.

Here it doesn't matter whether Mario goes left to right or right to left; either way we'll end up with the same number of ways to traverse the `level`. In that case, we can simply choose for Mario to start from the right, and then we can process the level like we process other numbers in digit-parsing related questions by using floor division (`//`) and modulo (`%`)

At each point in time, Mario can either step or jump. We use a single floor division (`//`) of `level` by 10 to represent taking one step (if we took a step, then the entire `level` would be left except for the last number), while two floor divisions by 10 (or equivalently one floor division by 100) corresponds to a jump at this point in the `level` (if we took a jump, then the entire `level` would be left except for the last two numbers).

To think of the base cases, you can consider the two ways that Mario ends his journey. The first, corresponding to `level == 1`, means that Mario has successfully reached the end of the level. You can **`return`** `1` here, because this means you've found one additional path to the end. The second, corresponding to `level % 10 == 0`, means that Mario has landed on a Piranha plant. This returns 0 because it's a failed traversal of the `level`, so you don't want to add anything to your result.

In tree recursion, you need to find a way to combine separate recursive calls. In this case, because `mario_number` returns an integer and the base cases are integers and you're trying to count the total number of ways of traversal, it makes sense to add your recursive calls.

Teaching Tips

- Some leading questions:
  - What are our base cases? (When do we know we've reached the end of the level? When do we know that we've failed?)
  - Is there any difference between going left to right or right to left in terms of

the number of ways to traverse the level?

- What are our two options at each step?
- What do those look like in a recursive call?
- How should we combine our recursive calls? (and, or, addition, etc.)

- Try leaning into the narrative of the question! It's fun and can help rephrase recursive calls "in plain english" I also like drawing the problem out along with the doctests to visualize the different steps Mario can take :D !
- It's very useful to draw a tree diagram! Each function call has one branch for stepping once and another branch for stepping twice. Each branch then has 2 branches of their own (until a base case is reached).
- Teach students that recursive calls can be treated as numbers using the recursive leap of faith, so combining the two recursive call branches with addition is really just adding two numbers.

7. Fill in `collapse`, which takes in a non-negative integer `n` and returns the number resulting from removing all digits that are equal to an adjacent digit, i.e. the number has no adjacent digits that are the same.

```
def collapse(n):
    """
    >>> collapse(12234441)
    12341
    >>> collapse(11200000013333)
    12013
    """
    rest, last = n // 10, n % 10

    if _____________________________________:

        _____________________________________

    elif _____________________________________:

        _____________________________________
    else:

        _____________________________________
```

```
def collapse(n):
    rest, last = n // 10, n % 10
    if rest == 0:
        return last
    elif last == rest % 10:
        return collapse(rest)
    else:
        return collapse(rest) * 10 + last
```

Teaching Tips

- Demonstrating with the doc tests is very important- the problem description can be confusing.
- How are we going to traverse the number?
  - As always, we traverse right to left, since traversing left to right will only produce the same results for more work.
  - Knowing this, we can figure `rest` stands for the remaining number and `last` stands for the last digit
- What are the base cases?
  - The simplest possible case is if `n` is one digit, at which there is nothing to compare it to.
- What are the recursive cases?
  - Either we remove a digit or we do not.
  - How do we structure the recursive call when we **don't** want to get rid of `last`? We need to add it back on **after** the recursive call.
- Why does this work?
  - Remind students that part of the recursive leap of faith is to trust that calling `collapse(left)` will remove identical left adjacent numbers.
  - All they need to care about in the moment is whether or not `last` should be removed or not.