RECURSION, TREE RECURSION Solutions

COMPUTER SCIENCE MENTORS 61A

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Tree Recursion vs Recursion

In most recursive problems we've seen so far, the solution function contains only one call to itself. However, some problems will require multiple recursive calls – we colloquially call this type of recursion "tree recursion," since the propagation of function frames reminds us of the branches of a tree. "Tree recursive" or not, these problems are still solved the same way as those requiring a single function call: a base case, the recursive leap of faith on a subproblem, and solving the original problem with the solution to our subproblems. The difference? We simply may need to use multiple subproblems to solve our original problem.

Tree recursion will often be needed when solving counting problems (how many ways are there of doing something?) and optimization problems (what is the maximum or minimum number of ways of doing something?), but remember there are all sorts of problems that may need multiple recursive calls! Always come back to the recursive leap of faith.

Two rules that are often useful in solving counting problems:

1. If there are x *ways* of doing something and y *ways* of doing another thing, there are xy *ways* of doing **both** at the same time.
2. If there are x *ways* of doing one thing and y *ways* of doing another, but we can't do both things at the same time, there are $x + y$ ways of doing either the first thing **or** the second thing.

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1. The *Gibonacci sequence* is a recursively defined sequence of integers; we denote the nth Gibonacci number g_n. The first three terms of the sequence are $g_0 = 0, g_1 = 1, g_2 = 2$. For $n \geq 3$, g_n is defined as the sum of the previous three terms in the sequence.

   Complete the function `gib`, which takes in an integer `n` and returns the nth Gibonacci number, g_n. Also, identify the three parts of recursive function design as they are used in your solution.

```
def gib(n):
    """
    >>> gib(0)
    0
    >>> gib(1)
    1
    >>> gib(2)
    2
    >>> gib(3) # gib(2) + gib(1) + gib(0) = 3
    3
    >>> gib(4) # gib(3) + gib(2) + gib(1) = 6
    6
    """
    if ______________________________:

        return ______________________________

    return ______________________________
```

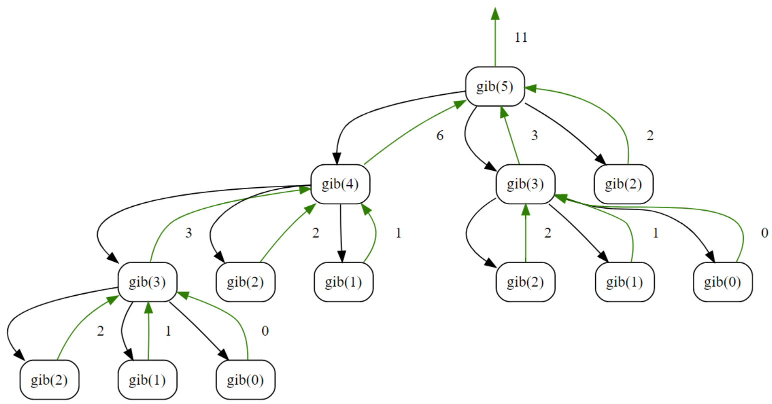
```
def gib(n):
    if n <= 2:
        return n
    return gib(n - 1) + gib(n - 2) + gib(n - 3)
```

- Base case: `if n <= 2: return n`
- Recursive calls: `gib(n - 1),gib(n - 2),gib(n - 3)`
- Solving the larger problem: adding the results of the three recursive calls.

2. Including the original call, how many calls are made to `gib` when you evaluate `gib(5)`?

   13.

   There are a few ways you could determine this. First, you could draw out a call graph that shows all of the calls that are made. `https://tinyurl.com/gibsol`



   You could also build things from the ground up using recursive intuition.

   - `gib(0)` takes one call because it's a base case.
   - `gib(1)` takes one call because it's a base case.
   - `gib(2)` takes one call because it's a base case.
   - `gib(3)` takes four calls: one for the original call, one for `gib(0)`, one for `gib(1)`, and one for `gib(2)`.
   - `gib(4)` takes seven calls: one for the original call, four for `gib(3)`, one for `gib(2)`, and one for `gib(1)`.
   - `gib(5)` takes 13 calls: one for the original call, seven for `gib(4)`, four for `gib(3)`, and one for `gib(2)`.

   This second method is much faster, especially for higher values of `n`.

3. Implement a recursive `fizzbuzz`.

```
def fizzbuzz(n):
    """Prints the numbers from 1 to n. If the number is
       divisible by 3, it
    instead prints 'fizz'. If the number is divisible by 5,
       it instead prints
    'buzz'. If the number is divisible by both, it prints
       'fizzbuzz'.

    >>> fizzbuzz(15)
    1
    2
    fizz
    4
    buzz
    fizz
    7
    8
    fizz
    buzz
    11
    fizz
    13
    14
    fizzbuzz
    """

    if n == 1:
        print(n)
    else:
        fizzbuzz(n - 1)
        if n % 3 == 0 and n % 5 == 0:
            print('fizzbuzz')
        elif n % 3 == 0:
            print('fizz')
        elif n % 5 == 0:
            print('buzz')
        else:
            print(n)
```

4. Write a function `selective_sum`, which takes in an integer `n` and a predicate function `cond`. `selective_sum` returns the sum of all positive integers up to `n` for which `cond(n)` is true.

```
def selective_sum(n, cond):
    """
    >>> is_odd = lambda x: x % 2 == 1
    >>> selective_sum(5, is_odd) # 5 + 3 + 1 = 9
    9
    >>> bigger_than_10 = lambda x: x > 10
    >>> selective_sum(13, bigger_than_10) # 13 + 12 + 11 = 36
    36
    >>> selective_sum(-1, is_odd) # no positive integers <= 1
    0
    """
    if ______________________________________________:

        return ______________________________________________

    if ______________________________________________:

        return ______________________________________________

    return ______________________________________________
```

```
def selective_sum(n, cond):
    if n <= 0:
        return 0
    if cond(n):
        return n + selective_sum(n - 1, cond)
    return selective_sum(n - 1, cond)
```

5. In an alternate universe, 61A software is not that good (inconceivable!). Tyler is in charge of assigning students to discussion sections, but sections.cs61a.org only knows how to list sections with either `m` or `n` number of students (the two most popular sizes). Given a `total` number of students, can Tyler create sections with only sizes of either `m` or `n` and not have any leftover students? Return `True` if he can and `False` otherwise.

```
def fit_sections(total, n, m):
    """
    >>> fit_sections(1, 3, 5)
    False
    >>> fit_sections(5, 3, 5) # 0 * 3 + 1 * 5 = 5
    True
    >>> fit_sections(11, 3, 5) # 2 * 3 + 1 * 5 = 11
    True
    >>> fit_sections(61, 11, 15) # can't express 61 as a
       * 11 + b * 15
    False
    """
    if ________________________________________:

        return True

    elif
       ________________________________________:

        return False

    return ____________________________________
```

```python
def fit_sections(total, n, m):
    if total == 0:
        return True
    elif total < 0: # you could also put total < min(m, n)
        return False
    return fit_sections(total - n, n, m) or
       fit_sections(total - m, n, m)
```

An alternate solution you could write that may be slightly faster in certain cases:

```python
def fit_sections(total, n, m):
    if total == 0 or total % n == 0 or total % m == 0:
        return True
    elif total < 0: # you could also put total < min(m, n)
        return False
    return fit_sections(total - n, n, m) or
       fit_sections(total - m, n, m)
```

(Solution continues on the next page)

When thinking about the recursive calls, we need to think about how each step of the problem works. Tree recursion allows us to explore the two options we have: either create a new `m`-person discussion at this step or create a new `n`-person discussion at this step and can combine the results after exploring both options. Inside the recursive call for `fit_sections(total - n, n, m)`, which represents accommodating `n` students, we again consider adding either `n` or `m` students to the next section.

Once we have these recursive calls we need to think about how to put them together. We know the return should be a boolean so we want to use either **`and`** or **`or`** to combine the values for a final result. Given that we only need one of the calls to work, we can use **`or`** to reach our final answer.

In the base cases we also need to make sure we return the correct data type. Given that the final return should be a boolean we want to return booleans in the base cases.

Another alternate base case would be: `total == 0 or total % n == 0 or total % m == 0`. This solution would also work! You would just be stopping the recursion early, since the total can be a multiple of `n` or `m` in order to trigger the base case - it doesn't have to be 0 anymore. Just be sure to still include the `total == 0` check, just in case someone inputs 0 as the total into the function.

6. Mario needs to get from one end of a level to the other, but there are deadly Piranha plants in his way! Mario only moves forward and can either *step* (move forward one space) or *jump* (move forward two spaces) from each position. A level is represented as a series of ones and zeros, with zeros denoting the location of Piranha plants. Mario can step on ones but not on zeros. How many different ways can Mario traverse a level without stepping or jumping into a Piranha plant? Assume that every level begins with a 1 (where Mario starts) and ends with a 1 (where Mario must end up).

   Hint: Does it matter whether Mario goes from left to right or right to left? Which one is easier to check?

```
def mario_number(level):
    """
    >>> mario_number(10101)
    1
    >>> mario_number(11101)
    2
    >>> mario_number(100101)
    0
    """
    if ________________________:

        ________________________

    elif ______________________:

        ________________________

    else:

        ________________________________________________________
```

```
def mario_number(level):
    if level == 1:
        return 1
    elif level % 10 == 0:
        return 0
    else:
        return mario_number(level // 10) +
           mario_number((level // 10) // 10)
```

You can think about this tree recursion problem as testing out all of the possible ways Mario can traverse the level, and adding 1 every time you find a possible traversal.

Here it doesn't matter whether Mario goes left to right or right to left; either way we'll end up with the same number of ways to traverse the `level`. In that case, we can simply choose for Mario to start from the right, and then we can process the level like we process other numbers in digit-parsing related questions by using floor division (`//`) and modulo (`%`)

At each point in time, Mario can either step or jump. We use a single floor division (`//`) of `level` by 10 to represent taking one step (if we took a step, then the entire `level` would be left except for the last number), while two floor divisions by 10 (or equivalently one floor division by 100) corresponds to a jump at this point in the `level` (if we took a jump, then the entire `level` would be left except for the last two numbers).

To think of the base cases, you can consider the two ways that Mario ends his journey. The first, corresponding to `level == 1`, means that Mario has successfully reached the end of the level. You can **`return`** `1` here, because this means you've found one additional path to the end. The second, corresponding to `level % 10 == 0`, means that Mario has landed on a Piranha plant. This returns 0 because it's a failed traversal of the `level`, so you don't want to add anything to your result.

In tree recursion, you need to find a way to combine separate recursive calls. In this case, because `mario_number` returns an integer and the base cases are integers and you're trying to count the total number of ways of traversal, it makes sense to add your recursive calls.

7. Fill in `collapse`, which takes in a non-negative integer `n` and returns the number resulting from removing all digits that are equal to an adjacent digit, i.e. the number has no adjacent digits that are the same.

```
def collapse(n):
    """
    >>> collapse(12234441)
    12341
    >>> collapse(11200000013333)
    12013
    """
    rest, last = n // 10, n % 10

    if ______________________________:

        ______________________________

    elif ____________________________:

        ______________________________

    else:

        ______________________________
```

```
def collapse(n):
    rest, last = n // 10, n % 10
    if rest == 0:
        return last
    elif last == rest % 10:
        return collapse(rest)
    else:
        return collapse(rest) * 10 + last
```