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CSM 70 Discrete Mathematics and Probability Theory

Spring 2016 More Counting, Discrete Probability, Monty **Solutions 7**

Key Terms: Combinatorial Proofs, Monty Hall, Probability space, Halting/Uncomputability, Rules of Counting, Sample space

I. More Counting

Exercises

Poker

We grab a deck of cards and it's poker time. Remember, in poker, order doesn't matter.

a. How many ways can we have a hand with exactly one pair? This means a hand with ranks (a, a, b, c, d)
   i. = 13 * 4C2 * 12C3 * 4^3
   ii. There are 13 value options for a (2, 3, 4, …, K, A). We then need to choose 2 out of the 4 possible suits. Now we need to choose b, c, and d. There are 12 values left (must be different from a). Finally, there are 4 suit options for each of the values chosen for b, c, d.

b. How many ways can we have a hand with four of a kind? This means a hand with ranks (a, a, a, a, b)
   i. = 13 * 12 * 4

c. How many ways can we have a straight ? A straight is 5 consecutive cards, that don't all necessarily have the same suit.
   i. straight can be (2, 3, 4, 5, 6); (3, 4, 5, 6, 7); … ; (10, J, Q, K, A)
   ii. can start from 2 - 10, which is 9 possibilities
   iii. each number in hand has 4 possibilities (suits)
   iv. so = 9 * (4^5)

d. How many ways can we have a hand of all of the same suit?
   i. 4 * (13 C 5)
   ii. for each of the 4 suits, there are 13 C 5 different combinations of 5 cards among 13 to choose from

e. How many ways can we have a straight flush? This means we have a consecutive-rank hand of the same suit. For examples, (2, 3, 4, 5, 6), all of spades is a straight flush, while (2, 3, 5, 7, 8) of all spades is NOT, as the ranks are not consecutive.
   i. For each of 4 suits, there are 9 number combinations (as shown in c, starting from 2 to starting from 10). Each number combination is unique, because there is only one number per suit

ii. = 4 * 9 = 36

Solving Equations

[Discussion] How many solutions does x + y + z = 10 have, if all variables must be positive integers?

We know no number can be greater than 8, because all are positive.

So fostion: y can take on any value from 1→ 8, and z will just be whatever is left (y can onlr each of 8 values of x (1→ 8), we solve y + z = 10 - x

Take example x = 1, then y + z = 9. Because each value >= 1, there are 8 solutions for this quey take on 8 values because z >= 1).

So we can see that in general, there are 10-x-1 solutions for each value of x.

so when x = 1, 8 solutions; when x = 2, 7 solutions, etc. for a total of 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 (1 happens when x=8 and y and z both must = 1)

total = 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36 solutions

Easier to think in terms of stars and bars. Bars can't be next to each other since variables are all positive integers. So n = 10 stars, k = 3 bars. Answer = (n-1 choose k-1) = (9 choose 2) = 36

Starbucks.

At Starbucks, you can choose either a Tall, a Grande, or a Venti drink. Further, you can choose whether you want an extra shot of espresso or not. Furthermore, you can choose whether you want a Latte, a Cappuccino, an Americano, or a Frappuccino. a) How many different drink combinations can you order?

3 * 2 * 4 (# sizes * espresso or not * # types of coffee)

Arranging letters.

How many ways are there to arrange the letters of the word "SUPERMAN"

a) On a straight line?

8!

b) On a straight line, such that "SUPER" occurs as a substring?

4! (treat "SUPER" as one character)

c) On a straight line, such that "SUPER" occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

3! * (8 choose 3)

This reduces to a stars and bars problem--the S U P E R are bars, and we want to put M A N somewhere in the sequence. Once we do so, there can be any permutation of M A N within the bars.

Equivalently, we can arrange the letters of SUPERMAN (8! ways), but divide by 5! because we have arranged SUPER in any of 5! ways, when we only want one way. This gives us 8! / 5!, which is equal to 3! * 8! / (5! 3!) = 3! * (8 choose 3)

d) On a circle?

7!

Anchor one element, arrange the other 7 in a line around it

e) On a circle, such that “SUPER” occurs as a substring?

3!

Anchor one element, arrange the other 7 around in a line, but treat SUPER as a single character, so it’s arranging the other 3 around in a line

f) On a circle, such that “SUPER” occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

2! * (7 choose 2)

Anchor one element (for simplicity, choose M, A, or N). Then following the same procedure as part c), we have 5 bars and 2 stars, where the two stars can be ordered any way.

Challenge Question

A 2×2 square grid is constructed with four 1×1 squares. The square on the upper left is labeled *A*, the square on the upper right is labeled *B*, the square in the lower left is labeled *C*, and the square on the lower right is labeled *D*. The four squares are to be painted such that 2 are blue, 1 is red, and 1 is green. In how many ways can this be done?

We choose two squares to be blue and one to be red; then the green's position is forced. There are $\binom{4}{2}\binom{2}{1} = 12$ ways to do this.

Equivalently, we could choose one square to be red and one square to be green, then blue is forced: $\binom{4}{1}\binom{3}{1} = 12$.

Combinatorial Proofs

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Choose a team of k players where one of the players is the captain.

LHS: Pick a team with k players. This is n choose k. Then make one of the players the captain. There are k options for the captain so we get k * n choose k.

RHS: Pick the captain. There are n choices for the captain. Now pick the last k - 1 players on the team. There are now n - 1 people to choose from. So we get n * n-1 choose k-1.

$$n! = \binom{n}{k}k!(n-k)!$$

Arrange n items

LHS: # ways to order n items

RHS: Choose k items without ordering. Order these k items. Order the remaining n - k items.

$$\sum_{k=1}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$$

Number of ordered triplets of the form (i, j, k) where i and j are less than or equal to k for every k from 0 to n

LHS: For each k there are k options for i and k options for j so k^2 options for all.

RHS: Consider the case where i = j. Then we must choose two numbers from {0, .., n} which amounts to n+1 CHOOSE 2.

If i != j then we choose 3 numbers from n+1. But i can be less than j or greater than j so we must multiply by 2.

Prove $a(n-a)\binom{n}{a} = n(n-1)\binom{n-2}{a-1}$ by a combinatorial proof.

Suppose that you have a group of n players. The lefthand side is the number of ways to pick a team of a of these players, designate one member of the team as captain, and then pick one reserve player from the remaining n−a people. The righthand side is the number of ways to pick the captain, then the reserve player, and then the other a−1 members of the team.

Challenge Question

a) Prove the Hockey Stick Theorem:

$$\sum_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1}$$ where n, t are natural numbers and n > t

In the homework you were asked to show the following:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Plug this in for:

$$\binom{t+1}{k+1}$$

$$\binom{t+1}{k+1} = \binom{t}{k} + \binom{t}{k+1} \Longrightarrow \binom{t}{k} = \binom{t+1}{k+1} - \binom{t}{k+1}$$

Now do the summation:

$$\sum_{t=k}^{n} \binom{t}{k} = \sum_{t=k}^{n} \binom{t+1}{k+1} - \sum_{t=k}^{n} \binom{t}{k+1}$$

Split apart the two summations as follows: take out the last term in the first summation and the last term in the second summation

$$\sum_{t=k}^{n} \binom{t}{k} = \left(\sum_{t=k}^{n-1} \binom{t+1}{k+1} + \binom{n+1}{k+1} \right) - \left(\sum_{t=k+1}^{n} \binom{t}{k+1} + \binom{k}{k+1} \right)$$

Look at all subsets of {1, 2, 3, …, 2015} that have 1000 elements. Choose the least element from each subset. Find the average of all least elements.

Since k CHOOSE k+1 is 0 we can just remove this term:

$$\sum_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1} + \sum_{t=k}^{n-1} \binom{t+1}{k+1} - \sum_{t=k+1}^{n} \binom{t}{k+1}$$

We want the summations to "match". So change t such that the new t, t', goes from k+1 to n-1. So let t' = t - 1. Then t = k+1 → t' = k and t=n → t'=n-1.

$$\sum_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1} + \sum_{t=k}^{n-1} \binom{t+1}{k+1} - \sum_{t'=k}^{n-1} \binom{t'+1}{k+1}$$

Notice that the summations cancel out. We are left with the statement we were trying to prove.

b) Let S be the set $\{1, 2, 3, \ldots 2015\}$. Look at all 1000 element subsets of S. Find the average of all smallest elements the subsets.

Let us denote the mean by M. There are 2015 choose 1000 subsets with 1000 elements.

So $\binom{2015}{1000}M$ must be the sum of all smallest elements. Let us try to write an explicit formula for the sum of all smallest elements.

What can the smallest elements be?

1, 2, 3, 4, …, 1016

Now we need to find out for how many subsets is each of the smallest elements the smallest element?

If 1 is the smallest element, there are 2014 other numbers that can fill in the other 999 spots.

If 2 is the smallest element, there are 2013 other numbers that can fill in the other 999 spots. And so on until:

If 1016 is the smallest element, there are 999 other numbers that can fill in the other 999 spots.

So we get the following formula:

$$\binom{2015}{1000}M = 1\cdot\binom{2014}{999} + 2\cdot\binom{2013}{999} + \cdots + 1016\cdot\binom{999}{999}$$

Now rearrange the terms as follows:

$$\begin{aligned}
&= \binom{2014}{999} + \binom{2013}{999} + \cdots + \binom{999}{999} \\
&+ \binom{2013}{999} + \binom{2012}{999} + \cdots + \binom{999}{999} \\
&\cdots \\
&+ \binom{999}{999}
\end{aligned}$$

We can apply the Hockey Stick Theorem!

$$= \binom{2015}{1000} + \binom{2014}{1000} + \cdots + \binom{1000}{1000}$$

Apply it again!

$$= \binom{2016}{1001}$$

Use the definition of n CHOOSE k to get:

$$M = \frac{2016}{1001} = \frac{288}{143}.$$

II. <u>Discrete Probability</u>

What is a sample space?

set of all possible outcomes

What is a probability space?

sample space with a probability assigned for each point

Probability: The probability of any event A is defined as follows

$$\Pr[A] = \frac{\#\text{ of sample points in } A}{\#\text{ of sample points in } \Omega} = \frac{|A|}{|\Omega|}.$$

Analogy: Imagine that you are throwing a dart randomly at a dart board. Each point in A is a spot on the board that you want to hit, the points in omega are all the points on the board. The more points there are in A, the better your odds are.

(This is why we learned counting before we learned discrete probability!)

Example: Probability of drawing a flush in poker is:

(#Of hands that are a flush)/(Total # of hands I can draw from the deck)

Balls and Bins: Suppose we have k balls and n bins and throw the balls into the bins.
What is the sample space?

$$\{(b_1, \ldots, b_k) \text{ such that } 1 \le b_i \le n\}$$

Hint: How many ways can the balls go into the bins? Are the balls distinguishable (does it matter what balls go into which bin)?

Multiplying: Suppose that you want the probability that a bunch of things happen together. For example:

P[I eat Pizza and Take CS70] = P[I eat Pizza]*P[I take CS70]

In **some** cases, you can actually multiply the probabilities like above. **It isn't always OK though!** We'll get more into this when we do independent and dependent events, so start thinking about why those would matter here.

TLDR: Sometimes you can multiply, but not always. We haven't learned why and when.

Exercises:

Probably Poker (reference Counting section above):

a) What is the probability of drawing a hand with a pair?
   ia /(52 C 5) = ((52 * 3 * 48 * 44 * 40) / (5!))/(52 C 5)
b) What is the probability of drawing a hand with four of a kind?

ib/(52 C 5)

c) What is the probability of drawing a straight ?

ic/(52 C 5)

d) What is the probability of drawing a hand of all of the same suit?

id/(52 C 5)

e) What is the probability of drawing a straight house?

ie/(52 C 5)

Probably Piazza

toppings = [pepperoni, sausage, pineapple, bacon, mushrooms]

We have 5 pizzas each with a topping from above. Each pizza has 8 slices. If you are given one slice at random, what are the following probabilities?

Pr(S = pepperoni)	8*1/8*5
Pr(S = sausage)	8*1/8*5
Pr(S = pineapple or bacon)	8*2/8*5

On the sausage and pineapple pizzas, someone adds bell peppers to 4 slices each. What are the following probabilities?

Pr(S has bell peppers)	8/40
Pr(S has bell peppers and sausage)	4/40
Pr(S does not have bell peppers and is not sausage)	28/40

Now suppose Alex's roommate eats 7 slices of pepperoni, 6 slices of bacon, and 5 slices of mushrooms. What are the following probabilities?

Pr(S = pepperoni)	1/22
Pr(S = bacon)	2/22
Pr(S = mushrooms or sausage)	(3+8)/22

Probably Parked Priuses

Suppose you arrange 12 different cars in a parking lot, uniformly at random. Three of the cars are Priuses, four of the cars are Teslas, and the other five are Nissan Leaves. What is the probability that the three Prius's are all together?

Answer: There are 12! possible ways to arrange the 12 cars. Now, there are $12-3+1 = 10$ different places the Prius's could go (positions 1,2,3, positions 2,3,4, all the way until positions 10,11,12). For each of those 10 places, there are 3! ways to arrange the Prius's and 9! ways to arrange the other 9 cars in the 9 remaining spots. So, in total, there are $10 \cdot 3! \cdot 9!$ ways of arranging the cars so that the 3 Prius's are together. So the probability we get what we want is $10 \cdot 3! \cdot 9! / 12! \approx .0455$

III. <u>Monty Hall</u>

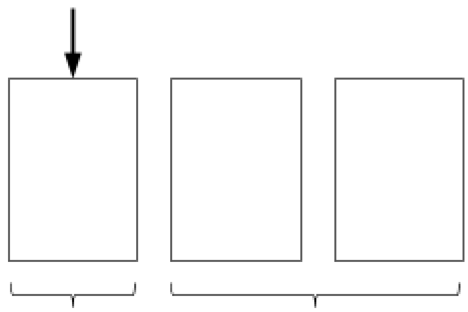
The Problem:

Suppose a contestant is shown 3 doors. There is a car behind one of them and goats behind the rest. Then they do the following:

1) Contestant chooses a door.
2) Host opens a door with a goat behind it.
3) Contestant can choose to switch or stick to original choice

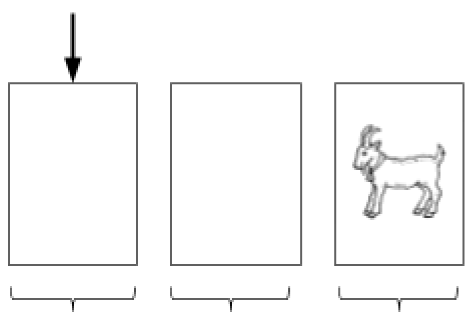
Is the contestant more likely to win if they switch?

At step 1, what is the probability that the car is behind the door the contestant chose? What is the probability that the car is behind the other two doors?



$\frac{1}{3}$ ______ $\frac{2}{3}$ ______

After the host opens a door with a goat, what are the probabilities of the car being behind each door?



$\frac{1}{3}$ ______ $\frac{2}{3}$ ______ 0 ______

Exercises:

Grouping Doors

Now we have 6 doors. You pick 1 and the other 5 doors are divided into two groups: one with 2 doors and the other with 3 doors. He removes doors until each group has 1 door left. Do you switch? What do you switch to?

Choose the door that was part of the group of three doors. When the doors were split up into three groups, those groups each had a probability of 1/6, 2/6, and 3/6. These probabilities do not change when doors are removed, so three remaining doors each have probability 1/6, 2/6, and 3/6.

Macs and Monty

Suppose instead of the normal Monty Hall scenario in which we have two empty doors and a car residing behind the third we have a car behind one door, a Mac behind another, and nothing behind the third.

Let us assume that the contestant makes an initial pick at his/her discretion (random) and the host proceeds to ALWAYS open the empty door. When the contestant's initial choice corresponds to the empty door, the host will say so and the contestant must switch.

Does the typical Monty Hall paradox of 2/3 chance of obtaining the car by switching versus a 1/3 chance of obtaining the car by staying apply in this particular case?

No. The normal Monty Hall paradox holds because when another door is opened, you learn nothing about your door, so the chance that you picked the right door from the beginning remains 1/n (which means the chance of the other door must be n-1/n). However, in this case, you may learn something about your door--you may be told that it must be wrong for example--thus shattering the paradox.

These are all of the possible scenarios:

Initial Pick	Switch?	Win?
Car	Yes	Lose
Car	No	Win
Mac	Yes	Win
Mac	No	Lose
Empty	Yes (to car)	Win
Empty	Yes (to Mac)	Lose

Switching results in a win 50% of the time, so there is a 50% chance of winning regardless of strategy.

Generalizing Monty

Now say we have n doors and there is a car behind one of them. Monty opens k doors, where $0 \leq k \leq n - 2$. Should you switch?

Yes.

$$\frac{n-1}{n} \cdot \frac{1}{n-k-1} = \frac{1}{n} \cdot \frac{n-1}{n-k-1} \geq \frac{1}{n}$$