MARKOV CHAINS 12

COMPUTER SCIENCE MENTORS 70

November 26 - 30, 2018

1 Markov Chains

P is a **transition probability matrix** if:

1. All of the entries are non-negative.
2. The sum of entries in each row is 1.

A **Markov chain** is defined by four things: ($\mathscr{X}$, π_0, P, $\{X_n\}_{n=0}^{\infty}$)

$\mathscr{X}$	Set of states
π_0	Initial probability distribution (the chance you are in any of the states)
P	Transition probability matrix
$\{X_n\}_{n=0}^{\infty}$	Sequence of random variables where:

$$\mathrm{P}[X_0 = i] = \pi_0(i), i \in \mathscr{X}$$
$$\mathrm{P}[X_{n+1} = j | X_n = i, X_{n-1}, \ldots, X_0] = \mathrm{P}(i,j), \forall n \geq 0, \forall i, j \in \mathscr{X}$$

A Markov chain is **irreducible** if we can go from any state to any other state, possibly in multiple steps.

Periodicity has to do with the period of occurrence of a state. If a state s has period 2, the Markov chain can be in s at every other time point. If a state has period 1, it's aperiodic; otherwise, it's periodic. More quantitatively, define value $d(i)$ for each state i as:

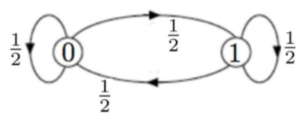
$$d(i) := g.c.d\{n > 0 | P^n(i,i) = \mathrm{P}[X_n = i | X_0 = i] > 0\}, i \in \mathscr{X}$$

If $d(i) = 1$, then the Markov chain is **aperiodic**. If $d(i) \neq 1$, then the Markov chain is periodic and its **period** is $d(i)$.

A distribution π is **invariant** for the transition probability P if it satisfies the following **balance equations**

$$\pi \cdot P = \pi.$$

Theorem 1: A finite irreducible Markov chain has a unique invariant distribution.
Theorem 2: All irreducible and aperiodic Markov chains converge to the unique invariant distribution - i.e, after a large amount of steps, the chance you are in any given state is given by the invariant distribution.
Theorem 3: If a Markov chain is finite and reducible, the amount of time spent in each state approaches the invariant distribution as n grows large
Equations that model what will happen at the next step are called **first step equations**

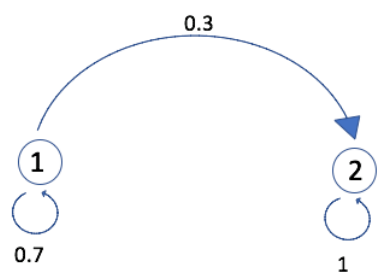


Denote $\beta(i,j)$ as the expected amount of time it would take to move from i to j. $\beta(0,1) = 1 + \frac{1}{2} \cdot \beta(0,1)$ $\beta(1,1) = 0$

1.1 Questions

1. **Types of Markov Chains**

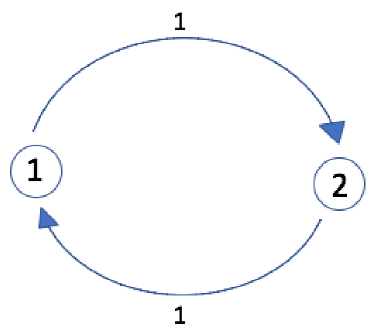
   (a) Draw a Markov Chain that is reducible.



Solution:

   (b) Draw a periodic Markov Chain (note that when we talk about periodicity, we imply that the Markov chain is irreducible). Does it have an invariant distribution? Does it always converge to this invariant distribution? Does the invariant distribution represent the average amount of time spent in each state in this chain?

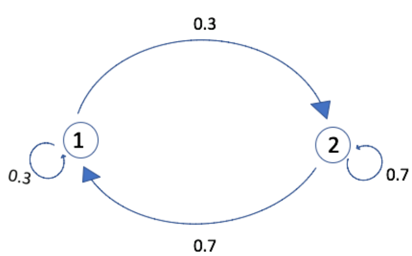
Solution: For the following example, we see that this chain is periodic because in order to get back to a state, you must travel some multiple of 2 steps. For these types of Markov chains, it **does not always converge** to the invariant distribution. Clearly, in this case the invariant distribution is $\pi = [0.50.5]$. However, if we start with the distribuion $\pi_0 = [10]$, this distribution will never converge to π. The invariant distribution **does represent the average amount of time spent in each state**, because this chain is finite and



irreducible.

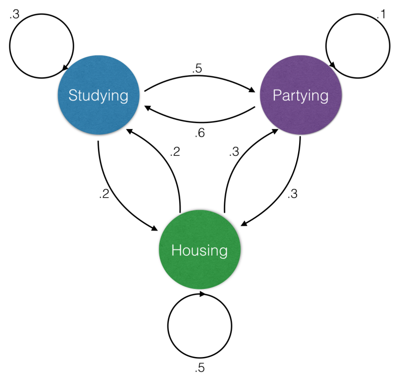
(c) Draw an aperiodic Markov Chain. Does it have an invariant distribution? Does it converge to this invariant distribution? Does the invariant distribution represent the average amount of time spent in each state in this chain?

Solution: For these types of Markov chains, it **always** converges to the invariant distribution. The invariant distribution **does represent the average amount of time spent in each state**, because this chain is finite and irreducible.



2. **Life of Alex**
   Alex is enjoying college life. She spends a day either studying, partying, or looking for housing for the next year. If she is studying, the chances of her studying the next day are 30%, the chances of her partying the next day are 50%, and the chances of her looking for housing the next day are 20%. If she is partying, the chances of her partying the next day are 10%, the chances of her studying the next day are 60%, and the chances of her looking for housing the next day are 30%. If she is looking for housing, the chances of her looking for housing the next day are 50%, the chances of her partying the next day are 30% and the chances of her studying the next day are 20%.

   (a) Draw a Markov chain to visualize Alex's life.

   Solution:

   (b) Write out a matrix to represent this Markov chain.

   Solution: In this solution (and in CS 70), the rows represent the source and the columns represent the destination.

$$\begin{bmatrix} .3 & .5 & .2 \\ .6 & .1 & .3 \\ .2 & .3 & .5 \end{bmatrix}$$

   (c) If Alex studies on Monday, what is the chance that she is partying on Friday? (Don't do the math, just write out the expression that you would use to find it.)

   Solution: If P is the matrix above, then it is $[1, 0, 0] \cdot P^4$

   (d) What percentage of her time should Alex expect to use looking for housing?

Solution: Solve the following system of equations: (first step equations)

$$S = .3S + .6P + .2H$$
$$P = .5S + .1P + .3H$$
$$H = .2S + .3P + .5H$$
$$S + P + H = 1$$

(e) If Alex parties on Monday, what is the chance of Alex partying again before studying?

Solution: Set up the following equations:

$$H_1 = 0$$
$$H_2 = .6(H_1) + .1(1) + .3(H_3)$$
$$H_3 = .2(H_1) + .3(1) + .5(H_3)$$

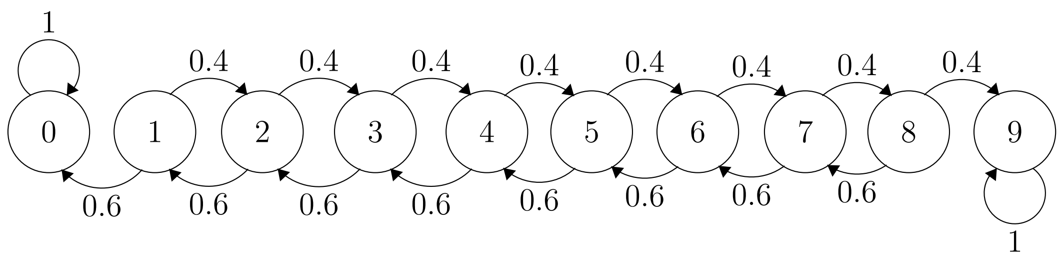
Solving for H_2, we get 0.28.

3. **Bet On It**

Smith is in jail and has 3 dollars; he can get out on bail if he has 8 dollars. A guard agrees to make a series of bets with him. If Smith bets A dollars, he wins A dollars with probability 0.4 and loses A dollars with probability 0.6.

a) Find the probability that he wins 8 dollars before losing all of his money if he bets 1 dollar each time. (Don't solve the system of equations; your mentor will give you the final probabilities.)

Solution: The Markov chain $(X_n, n = 0, 1, \dots)$ representing the evolution of Smith's money has diagram:



Let $\phi(i)$ be the probability that the chain reaches state 8 before reaching state 0, starting from state i. In other words, if S_j is the first $n \leq 0$ such that $X_n = j$,

$$P_i(S_8 < S_0) = P(S_8 < S_0 | X_0 = i) \tag{1}$$

Using first-step analysis (viz. the Markov property at time $n = 1$), we have

$$\begin{aligned} &\phi(i) = 0.4\phi(i+1) + 0.6\phi(i-1),\ i = 1, 2, 3, 4, 5, 6, 7 \\ &\phi(0) = 0 \\ &\phi(8) = 1. \end{aligned}$$

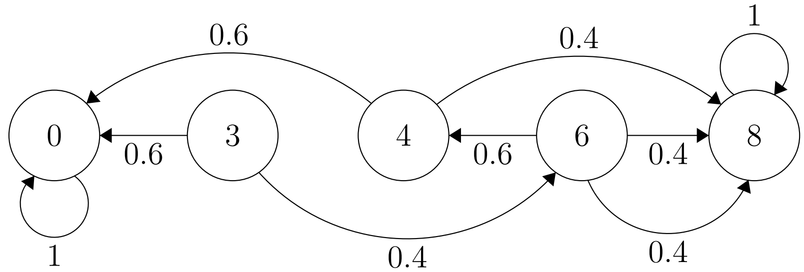
We solve this system of linear equations and find

$$\begin{aligned} \phi &= (\phi(1), \phi(2), \phi(3), \phi(4), \phi(5), \phi(6), \phi(7)) \\ &= (0.0203, 0.0508, 0.0964, 0.1649, 0.2677, 0.4219, 0.6531, 1). \end{aligned}$$

E.g. the probability that the chain reaches state 8 before reaching state 0, starting from state 3, is the third component of the vector and is equal to 0.0964. Note that $\phi(i)$ is increasing in i, which was expected.

b) Find the probability that he wins 8 dollars before losing all of his money if he bets, each time, as much as possible but not more than necessary to bring his fortune up to 8 dollars.

Solution: Now the chain is:



and the equations are:

$$\phi(3) = 0.4\phi(6)$$
$$\phi(6) = 0.4\phi(8) + 0.6\phi(4)$$
$$\phi(4) = 0.4\phi(8)$$
$$\phi(0) = 0$$
$$\phi(8) = 1$$

We solve and find

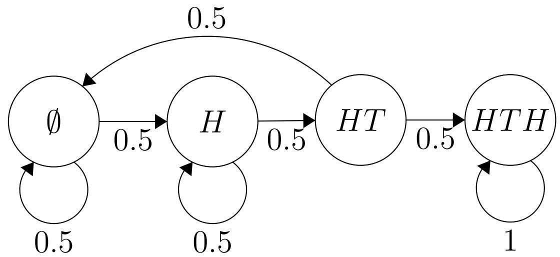
$$\phi(3) = 0.256, \phi(4) = 0.4, \phi(6) = 0.64$$

c) Which strategy gives Smith the better chance of getting out of jail?

Solution: By comparing the third components of the vector ϕ we find that the strategy in b) gives Smith a better chance to get our of jail.

4. **Tossing Coins**
A fair coin is tossed repeatedly and independently. Find the expected number of tosses till the pattern HTH appears.

Solution: Call HTH our target. Consider a chain that starts from a state called nothing ($\emptyset$) and is eventually absorbed at HTH. If we first toss H then we move to state H because this is the first letter of our target. If we toss a T then we move back to $\emptyset$ having expended 1 unit of time. Being in state H we either move to a new state HT if we bring T and we are 1 step closer to the target or, if we bring H, we move back to H: we have expended 1 unit of time, but the new H can be the beginning of a target. When in state HT we either move to HTH and we are done or, if T occurs then we move to $\emptyset$. The transition diagram is



Let $\phi(i)$ be the expected number of steps to reach HTH starting from i. We have

$$\begin{aligned}
\phi(HT) &= 1 + \frac{1}{2}\phi(\emptyset) \\
\phi(H) &= 1 + \frac{1}{2}\phi(H) + \frac{1}{2}\phi(HT) \\
\phi(\emptyset) &= 1 + \frac{1}{2}\phi(\emptyset) + \frac{1}{2}\phi(H)
\end{aligned}$$

We solve and find $\phi(\emptyset) = 10$.