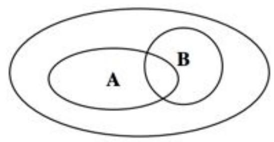
Worksheet 8

I. Conditional Probability

P[A|B]: What is the probability that the event A happens *given* that event B happened?

Let the following image denote the entire sample space **Ω**. What are we trying to find? What is our new sample space?



Let A and B be events in the same probability space. Then we can calculate the conditional probability of A given B:

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

II. Product Rule

Pr[A and B] = Pr[A|B] * Pr[B]

III. Total Probability Rule

$$\Pr[B] = \Pr[A \cap B] + \Pr[\overline{A} \cap B] = \Pr[B|A]\Pr[A] + \Pr[B|\overline{A}](1 - \Pr[A])$$

IV. Bayes' Rule

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{\Pr[B|A]\Pr[A]}{\Pr[B]}$$

Exercises

Basic Probability

You want to put 5 fruits into a salad and pick them randomly from a group of 10 strawberries and 6 bananas. Find the probability that the salad has

a) 3 strawberries and 2 bananas

b) 4 strawberries and 1 bananas

c) 5 strawberries

There are ${}_{16}C_5$ ways to select 5 fruits out of a total of 16 fruits (strawberries and bananas)

ways to select 3 strawberries out of 10: ${}_{10}C_3$

ways to select 2 bananas out of 6: ${}_{6}C_2$

ways to select 3 strawberries out of 10 AND 2 bananas out of 6: ${}_{10}C_3 * {}_{6}C_2$

a) P(3S and 2B) = ${}_{10}C_3 * {}_{6}C_2 \,/\, {}_{16}C_5 = 0.412087$
b) P(4S and 1B) = ${}_{10}C_4 * {}_{6}C_1 \,/\, {}_{16}C_5 = 0.288461$
c) P(5S) = ${}_{10}C_5 * {}_{6}C_0 \,/\, {}_{16}C_5 = 0.0576923$ (in ${}_{6}C_0$ the 0 is for no bananas)

Conditional Probability and Bayes' Rule

Donuts

You are feeling hungry, so you decide to visit Bayes's Donuts. There are three kinds of donuts: old-fashioned donuts — half of which are chocolate, and half of which are glazed — regular donuts — which have an equal probability of being chocolate, glazed, powdered, and cinnamon — and filled chocolate donuts. 40% of the total are old-fashioned, 40% regular, and 20% filled.

a. If you ask for a random donut, what is the probability it is a glazed regular donut?

   0.4 * 0.25 = 0.1

b. You decide you want a chocolate donut, so you ask the cashier for a random chocolate donut. What is the probability it is old-fashioned?

   Pr(old-fashioned and chocolate) = 0.4 * 0.5 = 0.2

   Pr(chocolate) = 0.4 * 0.5 + 0.4 * 0.25 + 0.2 = 0.2 + 0.1 + 0.2 = 0.5

   Pr(old-fashioned | chocolate) = Pr(old-fashioned and chocolate) / Pr(chocolate) = 0.2 / 0.5 = 0.4

Mentor Meeting

The CSM 70 mentors are in a meeting and trying to choose where to hold meetings. The probability that a mentor goes to the meeting depends on which side of campus they live. The probability that a mentor living on south side goes to a meeting in Soda is 9/10. The probability that a mentor goes to a meeting in Soda given that they live on North side is 1. The probability that a mentor lives on South side is ½ (and for some reason the only other place they can live is North side).

What is the probability that any one mentor attends the meeting?

Pr(attend) = Pr(south side)*Pr(attend|south) + Pr(north)*Pr(attend|north) =
(½*9/10)+(½*1)

What is the probability that all 10 mentors attend?

Pr(attend)^10 = [Pr(attend|south)Pr(south) + Pr(attend|north)Pr(north)]^10
((½*9/10)+(½*1))^10

All of a sudden, everyone moves to South side! Now what is the probability that any one mentor attends the meeting?

Pr(attend|south) = 9/10

Now what is the probability that everyone attends the meeting?

(9/10)^10

Making Midterms Might Make You Mad

Fahad is making questions for a CS70 midterm, and Fahad makes a question that is hard with probability 0.7. The head GSI, Katya, of the class looks at all of Fahad's questions and only gives it to students if she believes it to be an easy question. Katya believes a hard question is easy with probability 0.5 and an easy question is hard with probability 0.3. What's the probability that a hard question actually appears to the students?

We are trying to find the probability that a hard question appears to the students,that is, a hard question appears given that the GSI thinks it is easy

P(It's hard | GSI Says it's easy) = P(GSI says it's easy | It's hard)P(It'shard) / P(GSI says it's easy) by Bayes Rule.

P(GSI says it's easy) = P(GSI says it's easy | It's hard)P(It's hard) + P(GSI says it's easy | It's easy)P(It's easy) by the law of Total Probability.

Hence, overall, when we plug in numbers, we get

(.5)(.7) / ((.5)(.7) + (.3)(.7))

= (.35) / (.56)

= .625

707 Not Found

70% of the 707 airplanes that disappear while in flight are subsequently discovered. Of the aircraft that are discovered, 60% have an emergency locator, whereas 90% of the aircraft not discovered do not have a locator. Suppose that a 707 has disappeared. If it has an emergency locator, what is the probability that it will be discovered?

A: Let D = event light aircraft is discovered if and when it disappears when in flight and E = event aircraft has an emergency locator Given: P(D) = 0.70 < P(D1) = 1- P(D) = 1 - 0.70 = 0.30 P(E|D) = 0.60 P(E1|D1) = 0.90 < P(E|D1) = 1 - P(E1|D1) = 1 - 0.90 = 0.10 Then, P(D|E) = P(D and E)/P(E) = P(D and E)/P((E and D) OR (E and D1)) = P(D and E)/{P(E and D) + P(E and D1)} = {P(D) × P(E|D)}/{(P(D) × P(E|D)) + (P(D1) × P(E|D1))} = (0.70 × 0.60)/((0.70×0.60) + (0.30 × 0.10)) = 0.42/(0.42+0.03) = 0.42/0.45 = 0.93 http://sites.stat.psu.edu/~lsimon/stat250/homework/chapter3/bayes.pdf

Challenge Problem

Peaceful rooks

A friend of yours, Eithen Quinn, is fascinated by the following problem: placing m rooks on an $n \times n$ chessboard, so that they are in peaceful harmony (i.e. no two threaten each other). Each rook is a chess piece, and two rooks threaten each other if and only if they are in the same row or column. You remind your friend that this is so simple that a baby can accomplish the task. You forget however that babies cannot understand instructions, so when you give the m rooks to your baby niece, she simply puts them on random places on the chessboard. She however, never puts two rooks at the same place on the board.

1. Assuming your niece picks the places uniformly at random, what is the chance that she actually accomplishes the task and does not prove you wrong?

   Solution: After having placed i rooks in a peaceful position, i of the rows and i of the columns are taken. So for the next rook we have $n-i$ choices for the row and $n-i$ choices for the column in order to remain in a peaceful position. The total number of board cells left is n^2-i. So the chance that the next rook keeps the peace is $\frac{(n-i)^2}{n^2-i}$.
   The product over $i = 0, \ldots, m-1$ gives us the final answer. So the answer is

$$\prod_{i=0}^{m-1} \frac{(n-i)^2}{n^2-i} = \frac{(n!)^2(n^2-m)!}{(n^2)!((n-m)!)^2}$$

2. If you were using checker pieces as a replacement for rooks (so that they can be stacked on top of each other), then what would be the probability that your niece's placements result in peace? Assume that two pieces stacked on top of each other threaten each other.

Solution: The only thing that changes from the previous part is that when placing the i-th piece, we no longer have $n^2 - i$ possibilities, but n^2 possibilities. So the answer changes to

$$\prod_{i=0}^{m-1} \frac{(n-i)^2}{n^2} = \frac{(n!)^2}{((n-m)!)^2 n^{2m}}$$

Independence

Two events A,B in the same probability space are *independent* if $\Pr[A \cap B] = \Pr[A] \times \Pr[B]$.

Three people independently roll a die. Let $R_{n,m}$ be the event that person m and person n roll the same number. $R_{1,2,3}$ is the event that all three rolled the same number.

a. Are $R_{1,3}$ and $R_{1,2}$ independent?

   Yes.

b. Are $R_{1,2}$ and $R_{1,2,3}$ independent?

   No. Given that 1, 2, 3 rolled the same number we are 100% sure that 1,2 rolled the same number.

Union of Events

$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \text{ and } B]$

There are three groups working on the CS70 homework. There is a 50% chance of Group A finishing the homework, a 40% chance of Group B finishing, and a 30% chance of Group C finishing.

a. What is the probability that at least one group finishes?

   1.20 - .20 - .12 - .15 + .06 = 79%

   May be easier to think as 1 - Pr(none finish) = 1 - (0.5 * 0.6 * 0.7) = 0.79

b. What is the probability that exactly one finishes?

   1.20 - .40 - .24 - .30 + .18 = 44%

c. What is the probability that exactly two finish?

   Same logic as above. 29%

Supplementary Review

Combinatorial Proofs

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Choose a team of k players where one of the players is the captain.

LHS: Pick a team with k players. This is n choose k. Then make one of the players the captain. There are k options for the captain so we get k * n choose k.

RHS: Pick the captain. There are n choices for the captain. Now pick the last k - 1 players on the team. There are now n - 1 people to choose from. So we get n * n-1 choose k-1.

$$n! = \binom{n}{k} k!(n-k)!$$

Arrange n items

LHS: # ways to order n items

RHS: Choose k items without ordering. Order these k items. Order the remaining n - k items.

$$\sum_{k=1}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$$

Number of ordered triplets of the form (i, j, k) where i and j are less than or equal to k for every k from 0 to n

LHS: For each k there are k options for i and k options for j so k^2 options for all.

RHS: Consider the case where i = j. Then we must choose two numbers from {0, .., n} which amounts to n+1 CHOOSE 2.

If i != j then we choose 3 numbers from n+1. But i can be less than j or greater than j so we must multiply by 2.

Prove $a(n-a)\binom{n}{a} = n(n-1)\binom{n-2}{a-1}$ by a combinatorial proof.

Suppose that you have a group of n players. The lefthand side is the number of ways to pick a team of a of these players, designate one member of the team as captain, and then pick one reserve player from the remaining n−a people. The righthand side is the number of ways to pick the captain, then the reserve player, and then the other a−1 members of the team.

Monty Hall

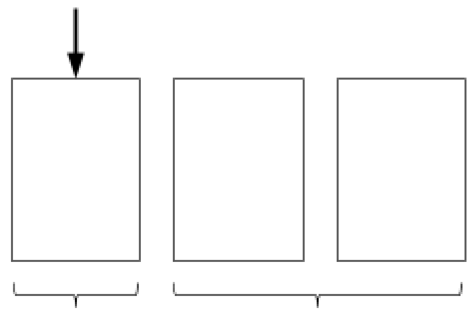
The Problem:

Suppose a contestant is shown 3 doors. There is a car behind one of them and goats behind the rest. Then they do the following:

1) Contestant chooses a door.
2) Host opens a door with a goat behind it.
3) Contestant can choose to switch or stick to original choice

Is the contestant more likely to win if they switch?

At step 1, what is the probability that the car is behind the door the contestant chose? What is the probability that the car is behind the other two doors?



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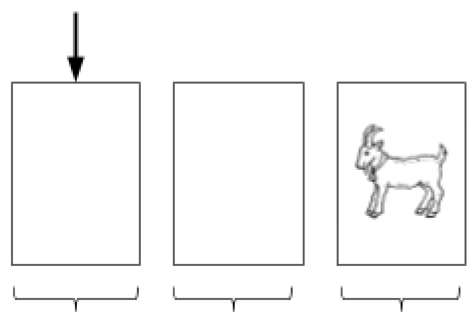
$\frac{1}{3}$ $\frac{2}{3}$

After the host opens a door with a goat, what are the probabilities of the car being behind each door?



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$\frac{1}{3}$ $\frac{2}{3}$ 0

Exercises:

Grouping Doors

Now we have 6 doors. You pick 1 and the other 5 doors are divided into two groups: one with 2 doors and the other with 3 doors. He removes doors until each group has 1 door left. Do you switch? What do you switch to?

Initially, the probability that the car is behind my door is ⅙. The probability the car is in the 2-group is ⅓. The probability the car is in the 3-group is ½. After eliminating the described doors, the probabilities have coalesced. There is still ⅙ chance my door has the car. There is still a ⅓ chance the car is in the 2-group. There is still a ½ chance the car is in the 3-group. since each group now only has 1 door, all the probabilities are for that one door. Thus, it is best to switch to the only remaining door in the 3-group.

Macs and Monty

Suppose instead of the normal Monty Hall scenario in which we have two empty doors and a car residing behind the third we have a car behind one door, a Mac behind another, and nothing behind the third.

Let us assume that the contestant makes an initial pick at his/her discretion (random) and the host proceeds to ALWAYS open the empty door. When the contestant's initial choice corresponds to the empty door, the host will say so and the contestant must switch.

Does the typical Monty Hall paradox of 2/3 chance of obtaining the car by switching versus a 1/3 chance of obtaining the car by staying apply in this particular case?

No. Essentially, we are making the probability that we select an empty door impossible. Thus, the only events in play are the event of getting a Mac or the event of getting a car. The probability is the same whether we switch or not--50-50.

Generalizing Monty

Now say we have n doors and there is a car behind one of them. Monty opens k doors, where $0 \leq k \leq n - 2$. Should you switch? Write an explicit formula for the probability of winning if you switch.

Initially, my door has 1/n chance of winning. The probability that the car is behind one of the other doors is (n-1)/n.

After revealing k other doors, that (n-1)/n has coalesced into (n-1)-k doors. Thus, the probability that the car is in behind each one of those doors is (n-1)/(n(n-k-1)). This is the probability we win if we switch.

Let us compare 1/n and (n-1)/(n(n-k-1)) by finding the condition under which it is best to switch:

$1/n < (n-1)/(n(n-k-1))$

$n-k-1 < n-1$

$k + 1 > 1$

$k > 0$

So we should switch if Monty opens a door ($k > 0$). Otherwise, we should stay.