GRAPHS 2

COMPUTER SCIENCE MENTORS 70

September 10 to September 14, 2018

1 Graph Theory

1.1 Introduction

1. Let $G = (V, E)$ be an undirected graph. Match the term with the definition.

Path/Simple Path	Tour	Walk	Tournament	Cycle	Eulerian Tour

______________Sequence of edges.
______________Sequence of edges that does not repeat vertices.
______________Sequence of edges that starts and ends at the same vertex.
______________Sequence of edges that starts and ends on the same vertex and does not repeat any other vertices.
______________Sequence that uses each edge exactly once and starts and ends at the same vertex.
______________Directed graph in which every pair of distinct vertices is connected by a single directed edge.

Note: In CS 70, we typically assume paths are simple paths.

Additional Note: The questions below do not cover Eulerian tours, but they are an important topic included in the optional practice that you should review on your own.

1.2 Build-up Error

In this section we will work through an example of buildup error.
Faulty Claim: If a graph has average degree k, more than half the vertices must have degree at most k.
Proof: We use induction on the number of vertices n.
Base Case: A graph with just 1 vertex has average degree 0. 1 out of 1 vertices, or more than half of the vertices have degree 0.
Inductive Hypothesis: For a graph with n vertices that has average degree k, more than half of the vertices have degree at most k.
Inductive Step: Consider a graph of n vertices that has average degree k. By our inductive hypothesis, we claim that at least $\frac{n}{2}$ vertices have degree at most k. Add another vertex to this graph. In order for the graph to still have average degree k, we need to connect the new vertex to exactly $\frac{k}{2}$ vertices. Now we have an $n+1$ vertex graph with at least $\frac{n}{2}+1$ vertices with at most degree k. $\frac{n}{2}+1 \geq \frac{n+1}{2}$ as desired.

1. Give a counter-example to show the claim is false.

2. Since the claim is false, there must be an error in the proof. Explain the error.

1.3 Questions

1. Given a graph G with n vertices, where n is even, prove that if every vertex has degree $\frac{n}{2}+1$, then G must contain a 3-cycle.

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2. Every tournament has a Hamiltonian path. (Recall that a Hamiltonian path is a path that visits each vertex exactly once.)

2 Trees

2.1 Introduction

If complete graphs are maximally connected, then trees are the opposite: Removing just a single edge disconnects the graph! Formally, there are a number of equivalent definitions for identifying a graph $G = (V, E)$ as a tree.

Assume G is connected. There are 3 other properties we can use to define it as a tree.

1. G contains __________ cycles.

2. G has __________ edges.

3. Removing any additional edge will ________________

One additional definition:

4. G is a tree if it has no cycles and ____________________________

Theorem: G is connected and contains no cycles if and only if G is connected and has $n - 1$ edges.

2.2 Questions

1. Now show that if a graph satisfies either of these two properties then it must be a tree:

   a If for every pair of vertices in a graph they are connected by exactly one simple path, then the graph must be a tree.

   b If the graph has no simple cycles but has the property that the addition of any single edge (not already in the graph) will create a simple cycle, then the graph is a tree.

2. A **spanning tree** of a graph G is a subgraph of G that contains all the vertices of G and is a tree.
   Prove that a graph $G = (V, E)$ if connected if and only if it contains a spanning tree.

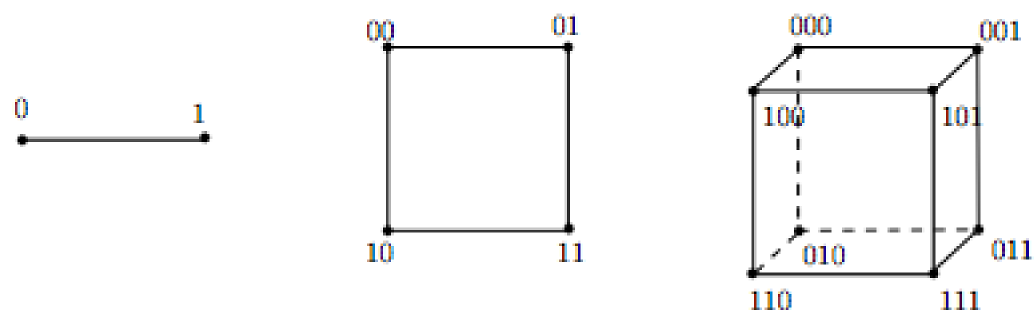
3 Hypercubes

3.1 Introduction

What is an n dimensional hypercube?

Bit definition: Two vertices x and y are adjacent and only if x and y differ in exactly one bit position.

Recursive definition: Define the 0-subcube as the $(n-1)$ dimensional hypercube with vertices labeled 0x (x is an element of $(0,1)^{n-1}$). Do the same for the 1-subcube with vertices labeled 1x. Then an n dimensional hypercube is created by placing an edge between 0x and 1x in the 0-subcube and 1-subcube respectively.



3.2 Questions

1. How many vertices and edges does an n dimensional hypercube have?

2. How many edges do you need to cut from a hypercube to isolate one vertex in an n-dimensional hypercube?

3. Prove that any cycle in an n-dimensional hypercube must have even length.