RANDOM VARIABLES, EXPECTATION, VARIANCE

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1 Expectation of Random Variables

1.1 Introduction

Random variable: a function $X : \omega \to R$ that assigns a real number to every outcome ω in the probability space.

Expectation: The expectation of a random variable X is defined as

$$\mathrm{E}(X) = \sum_{\alpha \in A} a * \mathrm{P}[X = a]$$

where the sum is over all possible values taken by the random variable. Expectation is usually denoted with the symbol μ.

Linearity of Expectation: For any random variables $X_1, X_2, ...X_n$, expectation is linear, i.e.:

$$E(X_1 + X_2 + ... + X_n) = E(X_1) + E(X_2) + ... + E(X_n)$$

This is true even when these random variables aren't independent.

1.2 Questions

1. You win the lottery with probability .01. If you win, you get \$10000. You lose with probability .99 and get no money. Define a random variable to represent this.

> **Solution:** We define a random variable X that has value 10000 with probability .01 (the event that you win the lottery) and value 0 with probability .99 (the event that you do not).

2. Given the random variable X defined as taking on the value 1 with probability 0.25, 2 with probability 0.5, and 20 with probability 0.25, what is the expectation of X?

> **Solution:** $E(X) = 0.25 * 1 + 0.5 * 2 + 0.25 * 20 = 6.25$

3. Show that $E(aX + b) = aE[X] + b$ where X is any random variable.

> **Solution:** First, for the expectation:
>
> $$E[aX + b] = \sum_{x \in X}(aX + b)P(X = x) = a\sum_{x \in X} X \cdot P(X = x) + \sum_{x \in X} bP(X = x)$$
>
> $$= aE[X] + b\sum_{x \in X} P(X = x) = aE[X] + b$$
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> .

4. Suppose X is a random variable. Does X always have to take on the value $E(X)$ at some point?

> **Solution:** No. Consider X with uniform probability space $\{0, 1\}$. The expectation of X is $\frac{1}{2}$, but X never takes on the value $\frac{1}{2}$.

5. An urn contains n balls numbered $1, 2, \ldots, n$. We remove k balls at random (without replacement) and add up their numbers. Find the mean of the total.

> **Solution:** The required total is $T = \sum_{i=1}^{k} X_i$, where X_i is the number shown on the ith ball. Hence $E(T) = k * E(X_1) = \frac{1}{2} * k * (n + 1)$.

6. A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence "book" appears?

Solution: There are $1000000 - 4 + 1 = 999997$ places where "book" can appear, each with a (nonindependent) probability of $\frac{1}{26}^4$ of happening. If A is the random variable that tells how many times "book" appears, and A_i is the indicator variable that is 1 if "book" appears starting at the ith letter, then

$$E[A] = E[A_1 + ... + A_9 99997] = E[A_1] + ... + E[A_9 99997]$$

$$= \frac{999997}{26^4}$$

7. In an arcade, you play game A 10 times and game B 20 times. Each time you play game A, you win with probability $\frac{1}{3}$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability $\frac{1}{5}$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

Solution: Let A_i be the indicator you win the ith time you play game A and B_i be the same for game B. The expected value of A_i and B_i are:

$$E[A_i] = 1 * \frac{1}{3} + 0 * \frac{2}{3} = \frac{1}{3}$$

$$E[B_i] = 1 * \frac{1}{5} + 0 * \frac{4}{5} = \frac{1}{5}$$

Let T_A be the random variable for the number of tickets you win in game A, and T_B be the number of tickets you win in game B.

$$E[T_A + T_B] = 3E[A_1] + ... + 3E[A_{10}] + 4E[B_1] + ... + 4E[B_{20}]$$

$$= 10 * (3 * \frac{1}{3}) + 20 * (4 * \frac{1}{5}) = 26$$

2   Variance

2.1 Introduction

Variance: The variance of a random variable X is defined as

$$\mathrm{Var}(X) = \mathrm{E}((X - \mathrm{E}(X))^2) \ = \mathrm{E}(X^2) - \mathrm{E}(X)^2$$

The latter version of variance is the one we usually use in computations.

The square root of $\mathrm{Var}(X)$ is called the standard deviation of X. It is usually denoted with the variable σ.

Important property of variance: for some constant c,

$$\mathrm{Var}(cX) = c^2 * \mathrm{Var}(X)$$

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2.2 Questions

1. **True or False?**
   Assume that X is a discrete random variable. If $Var(X) = 0$, then X is a constant.

   Solution: TRUE. Let $\mu = \mathrm{E}[X]$. By definition,

   $$0 = \mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2] = \sum \mathrm{P}[\omega](X(\omega) - \mu)^2$$

   The RHS is the sum of non-negative numbers, so if the sum is 0, each term must be 0. So

   $$\mathrm{P}[\omega] > 0 \rightarrow (X(\omega) - \mu)^2 = 0 = X(\omega) = \mu.$$

   Therefore X is constant (equal to $\mu = \mathrm{E}[X]$).

2. Show that $\mathrm{Var}(X) = \mathrm{E}((X - \mu)^2) = \mathrm{E}(X^2) - \mu^2$ where X is any random variable and $\mu = \mathrm{E}[X]$.

   Solution:

   $$\mathrm{Var}(X) = \mathrm{E}((X - \mu^2) = \mathrm{E}[X^2 - 2\mu X + \mu^2]$$
   $$= \mathrm{E}[X^2] - 2\mu\mathrm{E}[X] + \mu^2 \text{ (due to linearity of expectation)}$$
   $$= \mathrm{E}[X^2] - \mu^2$$

3. Show that $Var(aX + b) = a^2 Var(X)$ where X is any random variable.

Solution:

$$Var(aX + b) = \mathrm{E}(aX + b - \mathrm{E}(aX + b))^2$$
$$= \mathrm{E}(aX + b - a\mathrm{E}[X] - b)^2$$
$$= \mathrm{E}(aX - a\mathrm{E}[X])^2$$
$$= a^2\mathrm{E}(X - \mathrm{E}[X])^2 = a^2 Var(X)$$

4. Let's consider the classic problems of flipping coins and rolling dice. Let X be a random variable for the number of coins that land on heads and Y be the value of the die roll.

   (a) What is the expected value of X after flipping 3 coins? What is the variance of X?

   Solution:

   $$\mathrm{E}(X) = 0 * \frac{1}{8} + 1 * \frac{3}{8} + 2 * \frac{3}{8} + 3 * \frac{1}{8} = \frac{3}{2}$$
   $$\mathrm{E}(X^2) = 0^2 * \frac{1}{8} + 1^2 * \frac{3}{8} + 2^2 * \frac{3}{8} + 3^2 * \frac{1}{8} = \frac{24}{8} = 3$$
   $$\mathrm{E}(X)^2 = \frac{9}{4}$$
   $$\mathrm{Var}(X) = 3 - \frac{9}{4} = \frac{3}{4}$$

   (b) Let Y be the sum of rolling a dice 1 time. What is the expected value of Y?

   Solution: $\mathrm{E}(Y) = \frac{1}{6} * (1 + 2 + 3 + 4 + 5 + 6) = \frac{7}{2}$

   (c) What is the variance of Y?

   Solution: $\mathrm{E}(Y^2) = [\frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2)] = \frac{91}{6}$ $\mathrm{Var}(Y) = \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 = \frac{91}{6}\frac{7}{2}^2 = \frac{35}{12}$

5. You are at a party with n people where you have prepared a red solo cup labeled with their name. Before handing red cups to your friends, you pick up each cup and put a sticker on it with probability $\frac{1}{2}$ (independently of the other cups). Then you hand back the cups according to a uniformly random permutation. Let X be the number of people who get their own cup back AND it has a sticker on it.

(a) Compute the expectation $E(X)$.

Solution: Define $X_i = 1$ if the i-th person gets their own cup back and it has a sticker on it and $X_i = 0$ otherwise. Hence $E(X) = E(\sum_{i=1}^{n}(X_i)) = \sum_{i=1}^{n} E(X_i)$ $E(X_i) = P[X_i = 1] = \frac{1}{2n}$ since the i-th student will get his/her cup with probability $\frac{1}{n}$ and has a sticker on it with probability $\frac{1}{2}$ and stickers are put independently. Hence $E(X) = n \cdot \frac{1}{2n} = \frac{1}{2}$.

(b) Compute the variance $\text{Var}(X)$

Solution: To calculate $\text{Var}(X)$, we need to know $E(X^2)$

$$E(X^2) = E(X_1 + X_2 + \ldots + X_n)^2 = E(\sum_{i,j}(X_i * X_j)) = \sum_{i,j}(E(X_i * X_j)$$

(by linearity of expectation)

Then we consider two cases, either $i = j$ or $i \neq j$. Hence

$$\sum_{i,j} E(X_i * X_j) = \sum_{i} E(X_i^2) + \sum_{i \neq j} E(X_i * X_j)$$

$E(X_i^2) = \frac{1}{2n}$ for all i. To find $E(X_i * X_j)$, we need to calculate $P[X_i X_j = 1]$. $P[X_i * X_j = 1] = P[X_i = 1]P[X_j = 1 | X_i = 1] = \frac{1}{2n} * \frac{1}{2*(n-1)}$ since if student i has received his/her own cup, student j has $n - 1$ choices left. Hence

$$E(X^2) = n * \frac{1}{2n} + n * (n-1) * \frac{1}{2n} * \frac{1}{2*(n-1)} = \frac{3}{4}$$

$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$.