| | |
|---|---|
| **Key Terms**<br>Incident<br>Adjacent/Neighbors<br>Degree of a vertex | Path, walk, cycle, tour<br>Eulerian tour<br>Tree<br>Hypercube |

## I. Graph Theory

Let G = (V, E) be an undirected graph. Match the term with the definition.

Word Bank.

Walk Cycle Tour Path

_________________ Walk that starts and ends at the same node

_________________ Sequence of edges.

_________________ Sequences of edges with possibly repeated vertex or edge.

_________________ Sequence of edges that starts and ends on the same vertex and does not repeat vertices (except the first and last)

What is a simple path?

____________________________________________________________

Exercises

1. Given a graph G with n vertices, where n is even, prove by induction that if every vertex has degree n/2+1, then G must contain a 3-cycle.

## II. Trees

If complete graphs are "maximally connected," then trees are the opposite: Removing just a single edge disconnects the graph! Formally, there are a number of equivalent definitions of when a graph G = (V,E) is a tree, including:

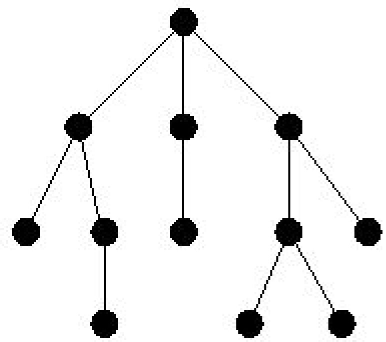
What are four ways to describe trees?

1) G is ________________ and contains ___________________
2) G is ________________ and has ________________________
3) G is ________________ and _____________________________________
4) G has no ________________ and __________________________ creates ________________

Theorem: G is connected and contains no cycles if and only if G is connected and has n - 1 edges.

We saw in the notes on page 8 that 1 and 2 above were saying the same thing- that is, stated rigorously, 1 ⇔ 2. We will now prove that 1 ⇔ 3:

Recall from the notes that a **rooted tree** is a tree with a particular node designated as the root, and the other nodes arranged in levels, “growing down” from the root. An alternative, recursive, definition of rooted tree is the following:
A rooted tree consists of a single node, the root, together with zero or more “branches,” each of which is itself a rooted tree. The root of the larger tree is connected to the root of each branch



*A rooted tree.*

2. Prove that given any tree, selecting any node to be the root produces a rooted tree according to the definition above.

A **spanning tree** of a graph G is a subgraph of G that contains all the vertices of G and is a tree.

3. Prove that a graph G = (V, E) is connected if and only if it contains a spanning tree.

4. How many distinct spanning trees does $K_3$ have? How many does $K_4$ have?

## III. Hypercubes

What is an n dimensional hypercube?

The bit definition: Two _________ x and y are ___________________if and only if ___ and ____ differ in __________________ bit position.

Recursive definition: Define the 0-___________ as the (n-1) dimensional _________________ with vertices labeled 0x (x is ___________________(hint: how many remaining bits are there?). Do the same for the 1-____________ with vertices labeled ____. Then an n dimensional ___________________ is created by placing an edge between ____ and ____ in the __________ and __________ respectively.

Exercises

1. How many vertices does an n dimensional hypercube have?

2. How many edges does an n dimensional hypercube have?

3. How many edges do you need to cut from a hypercube to isolate one vertex in an n-dimensional hypercube?

4. The hypercube is a popular architecture for parallel computation. Let each vertex of the hypercube represent a processor and each edge represent a communication link. Suppose we want to send a packet for vertex x to vertex y. Consider the following "bit-fixing" algorithm:
   In each step, the current processor compares its address to the destination address of the packet. Let's say that the two addresses match up to the first k positions. The processor then forwards the packet and the destination address on to its neighboring processor whose address matches the destination address in at least the first k+1 positions. This process continues until the packet arrives at its destination.

   Consider the following example where n = 4: Suppose that the source vertex is (1001) and the destination vertex is (0100). Give the sequence of processors that the packet is forwarded to using the bit-fixing algorithm.

Bonus Exercises:

Let $v$ be an odd degree node. Consider the longest walk starting at $v$ that does not repeat any edges (though it may omit some). Let $w$ be the final node of the walk . Show that $v \neq w$.

Prove that undirected connected graph with $|V| >= 2$, 2 nodes have same degree

Prove that every undirected finite graph where every vertex has degree of at least 2 has a cycle.

Prove that every undirected finite graph where every vertex has degree of at least 3 has a cycle of even length.