

# STABLE MARRIAGE, WELL ORDERING PRINCIPLE, OPTIMALITY, GRAPHS 1

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COMPUTER SCIENCE MENTORS 70

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## 1 Stable Marriage

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### 1.1 Introduction

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Given a set of  $n$  men and  $n$  women, and a list of each person's preferences, how can we create pairs of one man and one woman, such that everyone is content with their pairing?

**The Algorithm:**

1. **Every Morning:** Each man proposes to the most preferred woman on his list who has not yet rejected him.
2. **Every Afternoon:** Each woman collects all the proposals she received in the morning; to the man she likes best, she responds maybe, come back tomorrow (she now has him on a string), and to the others, she says never.
3. **Every Evening:** Each rejected man crosses off the woman who rejected him from his list. The above loop is repeated each successive day until each woman has a man on a string; on this day, each woman marries the man she has on a string.

**Definitions:**

- M and W are a rogue couple if they prefer to be with each other as opposed to the people they are paired with
- A pairing is stable if there are no rogue couples

**Lemmas:**

- The algorithm halts.
- "The Improvement Lemma" — If man M proposes to woman W on the kth day, then on every subsequent day W has someone on a string whom she likes at least as much as M.

## 1.2 Questions

1. Lemma: Algorithm terminates with a pairing.

**Solution:**

*Proof.* Contradiction

1. Negate the lemma: Suppose, there is a man who is unpaired when the algorithm ends.
2. How many women has he proposed to?  $\underline{n}$
3. By the Improvement Lemma, each woman has someone on a string since M proposed to each of them.
4. How many women have at least one man on a string?  $\underline{n}$
5. Man  $M$  is not on anyone's string.
6. At least how many men are there?  $\underline{n+1}$  (Contradiction!)

□

2. Lemma: The pairing is stable.

**Solution:**

*Proof.* Direct proof!

Let  $(M, W)$  be a couple in the pairing produced by SMA. Suppose  $M$  prefers  $W^*$  to  $W$ .

$W^*$  is higher in  $M$ 's preference list than  $W$ .

$M$  must have proposed to  $W^*$  before he proposed to  $W$ .

$W^*$  must have rejected  $M$  to someone she prefers more, call him  $M^*$ .

By the Improvement Lemma  $W^*$  likes her final partner at least as much as  $M^*$ .

$W^*$  prefers her final partner to  $M$ .

Thus, the pairing is stable.

□

## 2 Well Ordering Principle

### 2.1 Introduction

The well-ordering principle states that for every non-empty subset  $S$  of the set of natural numbers  $N$ , there is a smallest element  $x \in S$ ; i.e.  $\exists x : \forall y \in S : x \leq y$

### 2.2 Questions

1. In this question, we will go over how the well-ordering principle can be derived from (strong) induction.
  1. What is the significance of  $S$  being non-empty? Does WOP hold without it? Assuming that  $S$  is not empty is equivalent to saying that there exists some number  $z$  in it.

**Solution:** If there are no elements in  $S$ , clearly there can be no minimal element. The main point of the rest of the proof is to show that having a minimal element does not only require that  $S$  be non-empty, it is equivalent. We will show that if there is no minimal element, then  $S$  must be empty.

2. Induction is always stated in terms of a property that can only be a natural number. What should the induction be based on?

**Solution:** The number  $x$ , i.e. the least element in the set  $S$  as defined in the problem. If induction using this variable seems a bit circular, note that we are not assuming  $x$  exists (that *would* be circular logic). We're only saying that, if a least element did exist, we could call it  $x$ .

3. Now that the induction variable is clear, formally state the induction hypothesis.

**Solution:** Let  $A$  be a non-empty subset of  $\mathbb{N}$ . Our inductive hypothesis is the following predicate:

**P(n):** If  $n \in A$ , then  $A$  has a least element.

Note that the hypothesis  $P$  represents an implication, so it doesn't actually say anything about whether  $n$  is in  $A$ ! It only states that, *if*  $n$  happens to be in  $A$ ,  $A$  must have a least element. This distinction is the key to why the proof is general for any set, since we don't actually make any assumptions about which elements are in  $A$ .  $P(n)$  : If  $n \in A$ , then  $A$  has a least element.

4. Verify the base case.

**Solution:** We can immediately state  $P(0)$  is true. If 0 is in an arbitrary set of natural numbers  $A$ ,  $A$  must contain a least element, since  $0 \leq n \ \forall n \in \mathbb{N}$ , and will always be that least element.

5. Now prove that the induction works, by writing the inductive step.

**Solution:** We will use strong induction, so we want to show that  $[P(0) \wedge P(1) \wedge \dots \wedge P(n)] \implies P(n+1)$ . Intuitively what does this implication mean? The left side means that  $\forall k \leq n$ , if any one of them is in  $A$ , then  $A$  has a least element. The right side means that if  $n+1$  is in  $A$ , then  $A$  has a least element. The implication should hold, because intuitively there should be no special  $n+1$  that suddenly breaks our hypothesis. In fact if any one of those elements is in  $A$ , then we can default to the inductive hypothesis (left side) without considering  $n+1$  at all. However, if elements  $0 \dots n$  are not in  $A$ , then we have a separate case where we need to explicitly show  $P(n+1)$ . Let's formalize this intuition:

**Case 1:**  $A$  contains  $n+1$  and at least one element less than  $n+1$ .

$$\exists k(k \in A \wedge k < n+1)$$

By the inductive hypothesis, if  $k \in A$  and  $0 < k \leq n$ ,  $A$  contains a least element. Therefore  $P(n+1)$  is true for this case as well.

**Case 2:**  $A$  contains  $n+1$  and possibly larger elements.

$$\neg \exists k(k \in A \wedge k < n+1)$$

An immediate consequence of the above is that  $\forall x \in A \quad n+1 \leq x$  (this can be shown using proof by contradiction). By definition of least element,  $n+1$  is the least element in  $A$ , so  $P(n+1)$  is true for this case as well.

We have covered every case in which  $A$  contains  $n+1$ , and can now state  $P(n+1)$  is true in general.

6. What should you change so that the proof works by simple induction (as opposed to strong induction)?

**Solution:** We would use a contradiction to start off the proof: Suppose  $S$  has no minimal element. Then  $n = 1 \notin S$ , because otherwise  $n$  would be minimal. Similarly  $n = 2 \notin S$ , because then 2 would be minimal, since  $n = 1$  is not in  $S$ . Suppose none of  $1, 2, \dots, n$  is in  $S$ . Then  $n+1 \notin S$ , because otherwise it would be minimal. Then by induction  $S$  is empty, a contradiction.

### 3 Optimal, Pessimal

#### 3.1 Introduction

For any given person, their optimal partner is their most preferred partner among possible partners in stable pairings. A male optimal pairing is a pairing in which all males are paired with their optimal women.

Lemma: If a pairing is male optimal, then it is also female pessimal.

#### 3.2 Questions

1. Theorem: The pairing produced by the stable marriage algorithm is male optimal.

**Solution:** Contradiction. (hint: Well Ordering Principle)

- Proof.*
1. Negate the theorem: The pairing produced by the stable marriage algorithm is not male optimal.
  2. Therefore there must exist a man who was not paired with his optimal partner.
  3. Then there exists a day on which that man was rejected by his optimal partner.
  4. By the Well Ordering Principle, there is a first such day.
  5. Choose your notation: On day  $k$  man  $M_1$  was rejected by his optimal, woman  $W_1$  in favor of another man  $M_2$ . (WOP)
  6. By definition of optimal partner, there exists a stable pairing  $T$  where  $M_1$  and  $W_1$  are paired.
  7. Write down the pairing  $T$  using the notation chosen in 3:  

$$T := (M_1, W_1), (M_2, W_2), \dots$$
  8. Fact:  $W_1$  prefers  $M_2$  to  $M_1$ .  
 Why? By part 5, she rejected  $M_1$  in favor of  $M_2$ .
  9. Fact:  $M_2$  has not been rejected by his optimal woman on day  $k - 1$ .  
 Why? By assumption the first day that a man was rejected by his optimal woman was day  $k$  (part 5).
  10.  $M_2$  likes  $W_1$  at least as much as his optimal woman.
  11. Therefore  $M_2$  likes  $W_1$  at least as much as  $W_2$ .
  12.  $(M_2, W_1)$  is a rogue couple in stable pairing  $T$ .

□

It should be noted that even though the TMA creates pairings that are male optimal, these pairings can also be male pessimal. Consider the following preferences:

- $M_1: W_1 > W_2$
- $M_2: W_2 > W_1$
- $W_1: M_1 > M_2$
- $W_2: M_2 > M_1$

In this case, there is only one possible stable pairing  $\{(M_1, W_1), (M_2, W_2)\}$ . That means this pairing is both the best possible and worst possible, for both the men and women.

## 4 More Practice

### 4.1 Questions

1. Imagine that in the context of stable marriage all men have the same preference list. That is to say there is a global ranking of women, and men's preferences are directly determined by that ranking. Use any method of proof to answer the following questions.

1. Prove that the first woman in the ranking has to be paired with her first choice in any stable pairing.

**Solution:** If the first woman is not paired with her first choice, then she and her first choice would form a rogue couple, because her first choice prefers her over any other woman, and vice versa.

2. Prove that the second woman has to be paired with her first choice if that choice is not the same as the first woman's first choice. Otherwise she has to be paired with her second choice.

**Solution:** If the first and second women have different first choices, then the second woman must be matched to her first choice. Otherwise she and her first choice would form a rogue couple (since her first choice is not matched to the first woman, he would prefer the second woman over his current match). If the first choices are the same, then the second woman must be paired with her second choice, otherwise she and her second choice would form a rogue couple (neither of them are matched to their first choices, and they are each other's second choice).

3. Continuing this way, assume that we have determined the pairs for the first  $k - 1$  women in the ranking. Who should the  $k$ -th woman be paired with?

**Solution:** The  $k$ -th woman should be paired with the first man on her list who has not been matched yet (with the first  $k - 1$  women). If she's not matched to him, they would form a rogue couple. This is because the man would have to be matched to a woman ranked worse than  $k$ , so he would prefer the  $k$ -th woman over his current partner, and the  $k$ -th woman obviously prefers him to whoever she's matched with.

4. Prove that there is a unique stable pairing.

**Solution:** In the previous parts, we saw that for each woman, given the pairs for the lower-ranked women, her pair would be determined uniquely. So there is only one stable pairing. This can be stated and proved more rigorously using induction. Namely that there is a unique pairing for the first  $k$  women, assuming stability. An induction on  $k$  would prove this.

## 5 Graph Theory

### 5.1 Introduction

1. Let  $G = (V, E)$  be an undirected graph. Match the term with the definition.

Walk	Cycle	Tour	Path
_____	_____	_____	_____
_____	_____	_____	_____
_____	_____	_____	_____
_____	_____	_____	_____
_____	_____	_____	_____

**Solution:**

tour Walk that starts and ends at the same node  
path Sequence of edges.  
walk Sequences of edges with possibly repeated vertex or edge.  
cycle Sequence of edges that starts and ends on the same vertex and does not repeat vertices (except the first and last)

2. How can you represent a tournament using a graph?



**Solution:** A directed graph for which any pair of vertices,  $u, v$  either have an edge  $u \rightarrow v$  or  $v \rightarrow u$ . It can be used to represent a tournament in which every pair of player play at least once.

3. What is a simple path?

**Solution:** A sequence of edges where the vertices are necessarily distinct.