QUANTIFIERS, METHODS OF PROOF 0

COMPUTER SCIENCE MENTORS 70

Independent review

1 Quantifiers

1.1 Questions

1. Let $P(x, y)$ denote some proposition involving x and y. For each statement below, either prove that the statement is correct or provide a counterexample if it is false.

   a. $\forall x \forall y P(x, y) \implies \forall y \forall x P(x, y)$.

   Solution: True. The first statement $\forall x \forall y\ P(x, y)$ means for all x and y in our universe, the proposition $P(x, y)$ holds. The second statement $\forall y \forall x\ P(x, y)$ has the same meaning, so they are in fact equivalent (the implication goes both ways). In general, you can interchange the order of any consecutive sequence of $\forall$.

   b. $\exists x \exists y P(x, y) \implies \exists y \exists x P(x, y)$.

   Solution: True. Both statements mean there exist x and y in our universe that make $P(x, y)$ true, so both statements are equivalent. In general, you can interchange the order of any consecutive sequence of $\exists$.

   c. $\forall x \exists y P(x, y) \implies \exists y \forall x P(x, y)$.

Solution: False. Take the universe to be R (or any set with at least 2 elements), and take $P(x, y)$ to be the statement $x = y$. Then the first statement $\forall x \exists y$ $P(x, y)$ claims for all $x \in R$ we can find $y \in R$ such that $x = y$, which is true because we can take y to be x. However, the second statement $\exists y \forall x\ P(x, y)$ claims there exists $y \in R$ such that $x = y$ for all $x \in R$, which is false because a real number y cannot simultaneously be equal to all other real numbers x. Thus, the implication is false.

d. $\exists x \forall y P(x, y) \rightarrow \forall y \exists x P(x, y)$.

Solution: True. Suppose the first statement $\exists x \forall y\ P(x, y)$ is true, which means there is a special element $x^* \in R$ such that $P(x^*, y)$ is true for all $y \in R$. The second statement claims that for all $y \in R$ we can find an element $x \in R$ (which may depend on y) such that $P(x, y)$ is true. But from our first statement we know that we can choose the same value $x = x^*$ for all y. We conclude that the implication holds. However, the implication is only one way. In particular, note that part 4 is the converse to part 3, which we have seen is false.

2 Contrapositive and Contradiction

2.1 Questions

1. Write the contrapositive of the following statements and, if applicable, the statement in mathematical notation. (Using quantifiers, etc.)

   a If a quadrilateral is not a rectangle, then it does not have two pairs of parallel sides. (Skip mathematical notation for this problem, just write the contrapositive)

   Solution: If a quadrilateral has two pairs of parallel sides, then it is a rectangle.

   b For all natural numbers a where a^2 is even, a is even.

   Solution: $a \in N$ and a odd $\implies$ a^2 odd original: $a \in N$, a^2 even $\implies$ a even

   c Negate this statement: For all integers x, there exists an integer y such that $x^2+y = 16$.

Solution: $\exists x : \forall y, x^2 + y \neq 16$

2. Prove or disprove: If $P \implies Q$ and $R \implies \neg Q$, then $P \implies \neg R$.

Solution: Use the contrapositive of the second statement $P \implies Q \implies \neg R$

Alternatively, we can use proof by contradiction. Assume P does not imply $\neg R$. This means both P and R can be true at the same time. But, we know that if P is true, Q is true and if R is true, $\neg Q$ is true. This is a contradiction, as Q and $\neg Q$ cannot both be true. Hence, if P is true, not R must be true. Hence, $P \implies \neg R$.

3 Proof by Cases

3.1 Questions

1. For any integer x, x^2 has remainder 1 or 0 when divided by 3.

Solution: We approach the solution by cases.
Case 1: $x = 3k$ for some integer k.
$x^2 = 9k^2 = 3(3k^2)$. Thus x^2 has remainder 0 when divided by 3.

Case 2: $x = 3k + 1$ for some integer k
$x^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. Thus, x^2 has remainder 1 when divided by 3.

Case 3: $x = 3k + 2$ for some integer k
$x^2 = 9k^2 + 12k + 1 = 3(3k^2 + 4k) + 1$. Thus, x^2 has remainder 1 when divided by 3.
Since every integer falls into one of the three cases, it must be the case that the square of any integer has remainder 1 or 0 when divided by 3.

4 Induction

4.1 Questions

1. What are the three steps of induction?

Solution:

1. Base case
2. Inductive hypothesis

3. Inductive step

2. Prove that $\Sigma_{i=0}^{n} i * i! = (n+1)! - 1$ for $n \geq 1$ where $n \in N$.

Solution: Base case: For $n = 1$, we get $0*0!+1*1! = 0+1 = 1 = (1+1)!-1 = 2-1$, which is true as $1 = 1$

Induction Hypothesis: Assume the following is true for some k.

$$\sum_{i=0}^{k} i * i! = (k+1)! - 1$$

Inductive Step: Let's prove this for $k+1$.

$$\sum_{i=0}^{k+1} i * i! = \sum_{i=1}^{k} i * i! + (k+1) * (k+1)!$$

$= (k+1)! - 1 + (k+1)(k+1)!$ by the inductive hypothesis,
$= (k+1)!(1+k+1) - 1$
$= (k+1)!(k+2) - 1$
$= (k+2)! - 1$
Hence, we have proven it for $k+1$, and it is true for all n.

5 More Practice

Use any method of proof to answer the following questions.

1. Let x be a positive real number. Prove that if x is irrational (i.e., not a rational number), then $\sqrt{x}$ is also irrational.

   Solution: Proof by Contradiction: Suppose an irrational number x such that $\sqrt{x}$ is rational. By definition of rational, $\sqrt{x} = \frac{a}{b}$ for some integers a and b with $b \neq 0$. Then $x = (\sqrt{x})^2 = (\frac{a}{b})^2 = \frac{a^2}{b^2}$. But a^2 and b^2 are both integers (being products of integers), and $b \neq 0$ by the zero product property. Hence, x is rational (by definition of rational). This contradicts the supposition that x is irrational, and so the supposition is false. Therefore, the square root of an irrational number is irrational.

2. McDonalds sells chicken McNuggets only in 6, 9, and 20 piece packages. This means that you cannot purchase exactly 8 pieces, but can purchase 15. The Chicken McNugget Theorem states that the largest number of pieces you cannot purchase is 43. Formally state the Chicken McNugget Theorem using quantifiers.

   Solution: The theorem has two components: (i) you cannot purchase 43 pieces, and (ii) you can purchase any number larger than 43 pieces. The correct translation of the theorem is:

   $$\nexists a, b, c \in N, 6a + 9b + 20c = 43 \forall n \geq 44, \exists a, b, c \in N, 6a + 9b + 20c = n$$

3. Prove or disprove the following statement: If n is a positive integer such that $\frac{n}{3}$ leaves a remainder of 2, then n is not a perfect square.

> **Solution:** This is easiest done by looking at the contrapositive of the statement. We shall prove that if n is a perfect square, then $\frac{n}{3}$ does not leave a remainder of 2.
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> Any number divided by 3 can only have a remainder of 0, 1, or 2. Let's examine every case.
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> Case 1: Let's say $n = (3m)^2$, where m is some integer, then $n = 9m^2$, which is divisible by 3 and leaves no remainder.
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> Case 2: Let's say $n = (3m + 1)^2$, then $n = 9m^2 + 6m + 1$, which is equal to $3(3m^2 + 2m) + 1$, which is a number divisible by 3 plus one. Hence, when we take n to be this number and divide it by 3, we will have a remainder of 1.
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> Case 3: Let's say $n = (3m + 2)^2$, then $n = 9m^2 + 12m + 4$, which is equal to $3(3m^2 + 4m + 1) + 1$, which is a number divisible by 3 plus 1. When we set n to be this number and divide it by 3, we obtain a remainder of 1.
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> Hence, in every case, we either obtain a remainder of 0 or a remainder of 1, but never a remainder of 2. We have proved the contrapositive to be true and hence, the original statement is True.

4. Suppose that there are $2n + 1$ airports where n is a positive integer. The distances between any two airports are all different. For each airport, there is exactly one airplane departing from it, and heading towards the closest airport. Prove by induction that there is an airport which none of the airplanes are heading towards.

> **Solution:** We can do this with induction.
> Base case ($n = 1$): If there are 3 airports, the closest pair of airports will exchange planes. The third airport will then send a plane to one of the first two, meaning this third airport will have no incoming planes.
> Induction Hypothesis: Assume for some k, in $2k + 1$ airports one airport has no incoming planes.
> Induction Step: For $2(k + 1) + 1 = 2k + 3$ airports, consider the two airports that are closest to one another. This is necessarily unique, given the problem statement.

These airports will exchange planes, thus reducing the problem to $2k+1$ airports. We know the $2k+1$ case holds from the induction hypothesis. QED.