CSM 70 Discrete Mathematics and Probability Theory

Spring 2016 More Induction, Stable Marriage

Worksheet 2 Solutions

I. Induction Again.

The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, … is defined by:

$F_1 = 1, F_2 = 1, F_n = F_{n-2} + F_{n-1}$

Using induction, prove that $F_1 + \ldots + F_n = F_{n+2} - 1$

Induction on n.

Base case: n = 1

$F_1 = 1$

$F_3 - 1 = 2 - 1 = 1$

Inductive Hypothesis: For $k \geq 1$, $F_1 + \ldots + F_k = F_{k+2} - 1$

Inductive Step: We want $F_1 + \ldots + F_{k+1} = F_{k+3} - 1$. Start with the sum on the LHS to avoid build up error.

$$\begin{aligned} F_1 + \ldots + F_{k+1} &= F_1 + \ldots + F_k + F_{k+1} \\ &= F_{k+2} - 1 + F_{k+1} \\ &= F_{k+3} - 1 \quad \blacksquare \end{aligned}$$

II. Stable Marriage: The Facts

What is a rogue couple?

M and W are a rogue couple if they prefer to be with each other as opposed to the people they are paired with

What is a stable pairing?

A pairing is stable if there are no rogue couples

III. Stable Marriage Lemmas

Lemma: The algorithm halts.

Proof: 1. Each day the algorithm does not halt, what must at least one man do?

At least one man must eliminate some woman from his list

2. How many women are on each list? n

3. How many lists are there? n

4. What is an upper bound for the number of days? n^2 ■

Improvement Lemma: If man M proposes to woman W on the kth day, then on every subsequent day W has someone on a string whom she likes at least as much as M.

Proof: Induction on the day j.

1. Base case: j=k. Suppose man M proposes to W on the kth day.

   Proof by Cases: (hint: who could be on W's string on the kth day?)

   (i) W has no one on a string, then she says maybe to M

   (ii) W has M' on a string so she will say maybe to M only if she prefers M to M'

   Therefore on day k W has someone on a string she likes at least as much as M.

2. Inductive hypothesis: Suppose that for $j \geq k$ W has someone on a string she likes just as much as M'

3. Inductive step: Prove for j + 1

   Proof by Cases (hint: what could happen to W on day j + 1?)

   (i) No one proposes to W (she keeps M)

   (ii) Man M'' proposes to W. She says maybe to M'' only if she prefers him to M'

   In either case W has someone she likes as much as M'. ■

Extra proof (optional)

Lemma: Algorithm terminates with a pairing.

Proof: Contradiction

1. Negate the lemma: Suppose, there is a man who is unpaired when the algorithm ends.
2. How many women has he proposed to? n
3. By the Improvement Lemma, each woman has someone on a string since M proposed to each of them.
4. How many women have at least one man on a string? n
5. Man M is not on anyone's string.
6. At least how many men are there? n+1 ⇒⇐ (Contradiction!) ■

Extra proof (optional)

Lemma: The pairing is stable.
Proof: Direct proof!

1. Let (M, W) be a couple in the pairing produced by SMA. Suppose M prefers W* to W.
2. W* is higher/lower in M's preference list than W. (circle one)
3. M must have proposed to W* before he proposed to W.
4. W* must have rejected M to someone she prefers more, call him M*.
5. By the Improvement Lemma W* likes her final partner at least as much as M*
6. W* prefers her final partner to M.
7. The pairing is stable. ■

Stable Marriage Exercise: You can't please everyone

A person x is said to prefer a matching A to a matching A' if x strictly prefers her/his partner in A to her/his partner in A' . Given two stable matchings A and A' , a person may prefer one to the other or be indifferent if she/he is matched with the same person in both. Suppose now that A and A' are stable matchings, and suppose that m and w are partners in A but not in A' . Prove that one of m and w prefers A to A' , and the other prefers A' to A.

(Solution written by Jerome: this problem is challenging, and I decided to be as lucid as possible at the expense of conciseness.)

We suppose that A and A' are stable pairings. We also suppose that m and w are partners in A but m and w are not partners in A'. We wish to prove that *one of m and w prefers A to A' and that the other one prefers A' to A.*

How many possible cases do we have such that m and w prefer A or A'? I'm talking about the total possibilities, not just the ones that fit the criteria in the italicized sentence above. Well, m has two choices: prefer A to A', or prefer A' to A. Ditto with w: w can prefer A to A', or prefer A' to A. Combining these, we have four total possible cases we want to consider:

1. m prefers A to A' and w prefers A to A'
2. m prefers A' to A and w prefers A' to A
3. m prefers A to A' and w prefers A' to A
4. m prefers A' to A and w prefers A to A'

Of these four cases, (1) and (2) do not satisfy the criteria that *one of m and w prefers A to A' and that the other one prefers A' to A*. (3) and (4) satisfy the criteria.

So the proof boils down to: prove that (1) and (2) are false. We have already established that one of (1), (2), (3), or (4) must be true. If (1) and (2) are false, then (3) or (4) must be true, which is exactly what we want: *one of m and w prefers A to A' and that the other one prefers A' to A.*

So let's get down to it. We start by proving that (1) is false, then prove that (2) is also false. The second one is trickier.

Proof that (1) is false. We give a direct proof.
Assume that m prefers A to A' and w prefers A to A'.

A: m-w A': m-w' w-m'

In A, m-w is a pair. In A', m-w is not a pair, but m and w must have some other pairs, which we denote as w' and m'. But both m and w prefer A to A', which means, by the definition of preferring another stable matching (defined in the question), that m prefers his partner in A rather than his partner in A', and that w prefers her partner in A rather than her partner in A'. This directly translates to m prefers w' to w and w prefers m to m'. So the pairs m-w' and w-m' in A' results in a rogue pair, m-w. This contradicts the given that A' is a stable pairing, and therefore (1) is false.

Proof that (2) is false. We give an inductive proof after some opening remarks.
Assume that m prefers A' to A and w prefers A' to A.

A: m-w   A': m-w'   w-m'

Let's start off the proof with this same diagram as the first proof. Now we assume that m prefers A' to A and w prefers A' to A. Since m prefers A' to A, he must prefer his partner in A' over his partner in A. So he must prefer w' to w. Ditto for w: since w prefers A' to A, she must prefer m' to m. So we have:

m prefers w' to w
w prefers m' to m

With this new knowledge, we now know that the pairing m-w in A happened not because m and w love each other with a burning passion such that each were each other's top choices, but that it was really a lukewarm compromise. In other words, m prefers w' to w but had to settle for w because w' got with someone else, call him m''. Same with w: w prefers m' to m but had to settle for m because m' got with someone else, call her w''. We can sketch out this additional insight:

A: m-w   w'-m''   m'-w''

m' prefers w'' to w
w' prefers m'' to m

What are the implications of these new findings? We have a recursive implication: the pairing m-w' in A', despite the fact that w' prefers m'' to m, must mean that m'' is with someone better, call her w'''. The pairing w-m' in A', despite the fact that m' prefers w'' to w, must mean that w'' is with someone better, call him m'''. So we know that

m'' prefers w''' to w'
w'' prefers m''' to m'

This goes on and on! We can summarize all of the implications as follows:

m^k prefers w^{k+1} to w^{k-1}
w^k prefers m^{k+1} to m^{k-1}
for all *k*

We are almost done. The key insight here is that there are only a finite number of people in any given pairing. Let's say there are *n* men and *n* women in the pairing. Then m^n must be paired with someone, but whom? Not w^{n+1}, because w^{n+1} doesn't exist. So m^n is paired with some w^r such that $r < n$, and w^n is paired with some m^t such that $t < n$. But m^n must prefer w^n to w^r, and vice versa, so m^n and w^n is a rogue couple. m^n must prefer w^n to w^r because $r < n$; whatever r is, it is less than n.

IV. Well Ordering Principle

A. In this question, we will go over how the well-ordering principle can be derived from (strong) induction. Remember the well-ordering principle states the following: For every non-empty subset S of the set of natural numbers N, there is a smallest element $x \in S$; i.e. $\exists x : \forall y \in S : x \leq y$

(a) What is the significance of S being non-empty? Does WOP hold without it?

Assuming that S is not empty is equivalent to saying that there exists some number z in it. WOP does not hold with S being an empty set. Think of it as an implication: If S is a nonempty set of natural numbers then there is a smallest element. If the predicate is false then the entire implication is true regardless of whether the conclusion is true or not.

(b) Induction is always stated in terms of a property that can only be a natural number. What should the induction be based on?

Induct on the number x.

[Hints: The size of the set S? The number x? The number y? The number z?]

(c) Now that the induction variable is clear, state the induction hypothesis. Be very precise. Do not leave out dangling symbols other than the induction variable. Ideally you should be able to write this in mathematical notation.

Let A be a non-empty subset of N. We wish to show that A has a least element, that is, that there is an element $a \in A$ such that $a \leq n$ for all $n \in A$. We will do this by strong induction on the following predicate:

$P(n)$: "If $n \in A$, then A has a least element."

(d) Verify the base case.

[Hint: Note that your base case does not just consist of a single set S.]

$P(0)$ is clearly true, since $0 \leq n$ for all $n \in N$.

(e) Now prove that the induction works, by writing the inductive step.

We want to show that $[P(0) \wedge P(1) \wedge \cdots \wedge P(n)] \rightarrow P(n + 1)$.

To this end, suppose that $P(0), P(1), \cdots, P(n)$ are all true and that $n + 1 \in A$.

We consider two cases.

CASE 1: $\neg\exists m(m \in A \wedge m < n + 1)$.

In this case, $n + 1$ is the least element of A.

CASE 2: $\exists m(m \in A \wedge m < n + 1)$.

In this case, since $P(m)$ is true, A has a least element.

Either way, we conclude that $P(n + 1)$ is true.

(f) What should you change so that the proof works by simple induction (as opposed to strong induction)?

We would use a contradiction to start off the proof: Suppose S has no minimal element. Then $n=1 \notin S$, because otherwise n would be minimal. Similarly $n=2 \notin S$, because then 2 would be minimal, since $n=1$ is not in S. Suppose none of

$1,2,\cdots,n$ is in S. Then $n+1 \notin S$, because otherwise it would be minimal. Then by induction S is empty, a contradiction.

B. Solution: We prove this by the well-ordering principle. We call a graph colorable with k colors if it can be colored with at most k colors such that no two adjacent vertices end up having the same color. Suppose the statement is not true. Let G be the graph with the smallest number of vertices (all of degree at most d) that is not colorable with d+1 colors. We know that G exists by the well-ordering principle. Note that the number of vertices in G is greater than 1 since a graph of 1 vertex is always colorable with 1 color. Now take any vertex v in G and remove it. Then we know that the resulting graph is colorable with d+1 colors because G is the smallest graph that is not colorable. However, we can then add v back to the graph and assign it a color. Since v is connected to at most d edges, we can choose a color different from the colors of the vertices it is connected to. But this means that G is colorable with d+1 colors. We have arrived at a contradiction, and therefore proved that all graphs with degrees at most d are colorable with d+1 colors

C. [Back up] Suppose that I have k envelopes, numbered $0,1,\ldots,k-1$ such that envelope i contains 2^i dollars. Using the well-ordering principle, prove the following claim. Claim: for any integer $0 \le n < 2^k$, there is a set of envelopes that contain exactly n dollars between them.

V. <u>Optimal/Pessimal</u>

Define optimal partner:

A person's optimal partner is their most preferred partner among possible partners in stable pairings.

Define male optimal:

A male optimal pairing is a pairing in which all males are paired with his optimal woman.

Theorem: The pairing produced by the stable marriage algorithm is male optimal.

Proof: Contradiction. (hint: Well Ordering Principle)

1. Negate the theorem: The pairing produced by the stable marriage algorithm is not male optimal.
2. Therefore there must exist a man who was not paired with his optimal partner.
3. Then there exists a first day on which that man was rejected by his optimal partner.
4. By the Well Ordering Principle, there is a first such day

5. Choose your notation: On day k man M1 was rejected by his optimal, woman W1 in favor of another man M2. (WOP)

6. By definition of optimal partner, there exists a stable pairing T where M1 and W1 are paired.

7. Write down the pairing T using the notation chosen in 3:

$$T := \{ (M1, W1) , (M2, W2) , \ . \ . \ . \ \}$$

8. Fact: W1 prefers M2 to M1.

Why? By part 5, she rejected M1 in favor of M2.

9. Fact: M2 has not been rejected by his optimal woman on day k-1.

Why? By assumption the first day that a man was rejected by his optimal woman was day k (part 5).

10. M2 likes W1 at least as much as his optimal woman.

11. Therefore M2 likes W1 at least as much as W2.

12. (M2, W1) is a rogue couple in stable pairing T. $\Rightarrow\Leftarrow$ (Contradiction!) ■

Theorem: If a pairing is male optimal, then it is also female pessimal

Proof: Contradiction

1. Negate the theorem: There exists a male optimal pairing that is not also female pessimal
2. Let T = { … (M, W), … } be the pairing produced by the Stable Marriage Algorithm.

   T is male optimal.
3. Suppose there exists another stable pairing: S = { … , (M*, W), …, (M, W'), … } such that M* is lower/~~higher~~ on W's list than M. (hint: use assumption from 1)
4. Fact: W prefers M to M*.

   Why? M* is lower on her list than M (part 3).
5. Fact: M prefers W to W'.

   Why? M and W are paired in the male-optimal pairing T.
6. (M, W) is a rogue couple. ⇒⇐ (Contradiction!) ■