INEQUALITIES, LLSE 11

COMPUTER SCIENCE MENTORS 70

November 13-17, 2017

1 Inequalities

1.1 Introduction

Markov's Inequality
For a non-negative random variable X with expectation $\mathrm{E}(X) = \mu$, and any $\alpha > 0$:

$$\mathrm{P}[X \geq \alpha] \leq \frac{\mathrm{E}(X)}{\alpha}$$

Chernoff's Inequality

$$\mathrm{P}[|X \geq a] \leq \min_{\theta > 0} \frac{\mathrm{E}(e^{\theta X})}{e^{\theta a}}$$

Chebyshev's Inequality
For a random variable X with expectation $\mathrm{E}(X) = \mu$, and any $\alpha > 0$:

$$\mathrm{P}[|X - \mu| \geq \alpha] \leq \frac{\mathrm{Var}(X)}{\alpha^2}$$

1.2 Questions

1. Use Markov's to prove Chebyshev's Inequality:

2. Let X be the sum of 20 i.i.d. Poisson random variables $X_1, \ldots, X_{20}$ with $\mathrm{E}(X_i) = 1$. Find an upper bound of $\mathrm{P}[X \geq 26]$ using,
   (a) Markov's inequality:

   (b) Chebyshev's inequality:

3. **Bound It**
   A random variable X is always strictly larger than -100. You know that $\mathrm{E}(X) = -60$. Give the best upper bound you can on $\mathrm{P}[X \geq -20]$.

4. The citizens of the country USD (the United States of Drumpf) vote in the following manner for their presidential election: if the country is liberal, then each citizen votes for a liberal candidate with probability p and a conservative candidate with probabil-

ity $1p$, while if the country is conservative, then each citizen votes for a conservative candidate with probability p and a liberal candidate with probability $1p$. After the election, the country is declared to be of the party with the majority of the votes.

(a) Assume that $p = \frac{3}{4}$ and suppose that 100 citizens of USD vote in the election and that USD is known to be conservative. Provide a tight bound on the probability that it is declared to be a Liberal country.

(b) Now let p be unknown; we wish to estimate it. Using the CLT, determine the number of voters necessary to determine p within an error of 0.01, with probability at least 0.95.

5. **Squirrel Standard Deviation**
As we all know, Berkeley squirrels are extremely fat and cute. The average squirrel is 40% body fat. The standard deviation of body fat is 5%. Provide an upper bound on the probability that a randomly trapped squirrel is either too skinny or too fat? A skinny squirrel has less than 27.5% body fat, and a fat squirrel has more than 52.5% body fat?

6. Consider a random variable Y with expectation μ whose maximum value is $\frac{3\mu}{2}$, prove that the probability that Y is 0 is at most $\frac{1}{3}$.

2 Linear Least Squares Estimator

Theorem: Consider two random variables, X, Y with a given distribution $\mathrm{P}[X = x, Y = y]$. Then

$$\mathrm{L}[Y|X] = \mathrm{E}(Y) + \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}(X - \mathrm{E}(X))$$

2.1 Questions

1. Assume that
$$Y = \alpha X + Z$$
where X and Z are independent and $\mathrm{E}(X) = \mathrm{E}(Z) = 0$. Find $\mathrm{L}[X|Y]$.

2. The figure below shows the six equally likely values of the random pair (X, Y). Specify the functions of:
   - $L[Y \mid X]$
   - $E(X \mid Y)$
   - $L[X \mid Y]$
   - $E(Y \mid X)$