CS 70 Discrete Mathematics and Probability Theory

Spring 2016 More Induction, Stable Marriage

Worksheet 2

Key Terms

- Steps of Induction
- Rogue/stable/proposal
- Halting Lemma
- Improvement Lemma
- Stability Lemma
- Pessimal/optimal
- Well Ordering Principle

I. Induction Again.

The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, … is defined by:

$$F_1 = 1, F_2 = 1, F_n = F_{n-2} + F_{n-1}$$

Using induction, prove that $F_1 + \ldots + F_n = F_{n+2} - 1$

II. Stable Marriage: The Facts

The Algorithm:

1. **Every Morning:** Each man proposes to the most preferred woman on his list who has not yet rejected him.
2. **Every Afternoon**: Each woman collects all the proposals she received in the morning; to the man she likes best, she responds "maybe, come back tomorrow" (she now has him on a string), and to the others, she says "never".
3. **Every Evening**: Each rejected man crosses off the woman who rejected him from his list.

   The above loop is repeated each successive day until **each** woman has a man on a string; on this day, each woman marries the man she has on a string.

What is a rogue couple?

____

What is a stable pairing?

____

III. Stable Marriage Lemmas

Lemma: The algorithm halts.

Proof: 1. Each day the algorithm does not halt, what must at least one man do?

____

2. How many women are on each list? _________
3. How many lists are there? ________
4. What is an upper bound for the number of days? ________ ■

Improvement Lemma: If man M proposes to woman W on the kth day, then on every subsequent day W has someone on a string whom she likes at least as much as M.

Proof: Induction on the day j.

1. Base case: j=k. Suppose man M proposes to W on the kth day.

   Proof by Cases: (hint: who could be on W's string on the kth day?)

   (i) ____

   (ii) ____

   Therefore on day k W has someone on a string she likes at least as much as M.
2. Inductive hypothesis: ____
3. Inductive step: Prove for j + 1

   Proof by Cases (hint: what could happen to W on day j + 1?)

   (i) ____

   (ii) ____

   In either case W has someone she likes as much as M. ■

Extra proof (optional)

Lemma: Algorithm terminates with a pairing.

Proof: Contradiction

1. Negate the lemma: Suppose, ____
2. How many women has he proposed to? ___
3. By the __________________________, each woman _______________________

   ____
4. How many women have at least one man on a string? ___
5. Man M is not on anyone's string.
6. At least how many men are there? ____ ⇒⇐ (Contradiction!) ■

Extra proof (optional)

Lemma: The pairing is stable.
Proof: Direct proof!

1. Let (M, W) be a couple in the pairing produced by SMA. Suppose M prefers W* to W.
2. W* is higher/lower in M's preference list than W. (circle one)

__ . The pairing is stable. ■

Stable Marriage Exercise: You can't please everyone

A person x is said to prefer a matching A to a matching A' if x strictly prefers her/his partner in A to her/his partner in A' . Given two stable matchings A and A' , a person may prefer one to the other or be indifferent if she/he is matched with the same person in both. Suppose now that A and A' are stable matchings, and suppose that m and w are partners in A but not in A' . Prove that one of m and w prefers A to A' , and the other prefers A' to A.

IV. **<u>Well Ordering Principle</u>**

Every non empty set of natural numbers contains a smallest element.

A. In this question, we will go over how the well-ordering principle can be derived from (strong) induction. Remember the well-ordering principle states the following: For every non-empty subset S of the set of natural numbers N, there is a smallest element $x \in S$; i.e. $\exists x : \forall y \in S : x \leq y$

(a) What is the significance of S being non-empty? Does WOP hold without it? Assuming that S is not empty is equivalent to saying that there exists some number z in it.

(b) Induction is always stated in terms of a property that can only be a natural number. What should the induction be based on?

(c) Now that the induction variable is clear, formally state the induction hypothesis.

(d) Verify the base case.

(e) Now prove that the induction works, by writing the inductive step.

(f) What should you change so that the proof works by simple induction (as opposed to strong induction)?

B. Suppose that the degrees of the vertices in a graph are all at most d. Prove, using the well-ordering principle, that one can color the vertices of the graph using at most d +1 colors so that no two adjacent vertices end up having the same color.

VI. Optimal/Pessimal

Define optimal partner:

Define male optimal pairing:

Theorem: The pairing produced by the stable marriage algorithm is male optimal
Proof: Contradiction. (hint: Well Ordering Principle)

1. Negate the theorem:____

2. Therefore there must exist a man who ________________________________

3. Then there exists a day on which that man was ____________ by his _______________

4. By the Well Ordering Principle, there is a ______ such day

5. Choose your notation: On day __ man __ was ____________ by his _______________, woman ___ in favor of ___. (WOP)

6. By _________________, there exists a stable pairing T where ___ and ___ are paired.

7. Write down the pairing T using the notation chosen in 3:

   T := ____

8. Fact: ___ prefers ___ to ___.

   Why?____

   ____

9. Fact: ___ has not been rejected by his optimal woman on day ___.

   Why? ____

   ____

10. ___ likes ___ at least as much as his optimal woman.

11. Therefore ___ likes ___ at least as much as ___.

12. ______ is a _________ couple in stable pairing T. $\Rightarrow\Leftarrow$ (Contradiction!) ■

Theorem: If a pairing is male optimal, then it is also female pessimal
Proof: Contradiction

1. Negate the theorem: ____

2. Let T = { … (M, W), … } be the pairing produced by the Stable Marriage Algorithm.

   T is __________________.

3. Suppose there exists another stable pairing: S = { … , (M*, W), …, (M, W'), … } such that M* is lower/higher on W's list than M. (hint: use assumption from 1)

4. Fact: W prefers ___ to ___.

   Why? ____

   ____

5. Fact: M prefers ___ to ___.

   Why? ____

   ____

6. ______ is a _________ couple. $\Rightarrow\Leftarrow$ (Contradiction!) ■