General Errors, Uncountability, Self Reference, Counting 5

Computer Science Mentors 70

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1 General Errors (Berlekamp and Welch)

1.1 Introduction

Now instead of losing packets, we know that k packets are corrupted. Furthermore, we do not know which k packets are changed. Instead of sending k additional packets, we will send an additional $2k$.

Solomon-Reed Codes

1. Identical to erasure errors: Alice creates $n-1$ degree polynomial $P(x)$.

$$P(x) = p_0 + p_1 x + \ldots + p_{n-1}x^n$$

2. Alice sends $P(1), \ldots, P(n+2k)$

3. Bob receives $R(1), \ldots, R(n+2k)$

For how many points does $R(x) = P(x)$?

Solution: $n + k$

True or false: $P(x)$ is the unique degree $n-1$ polynomial that goes through at least $n+k$ of the received points.

Solution: True

Write the matrix view of encoding the points $P(1), \ldots, P(n+2k)$

Solution:

$$\begin{bmatrix} P(1) \\ P(2) \\ P(3) \\ \vdots \\ P(n+2k) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1^2 & \cdot & 1 \\ 1 & 2 & 2^2 & \cdot & 2^{n-1} \\ 1 & 3 & 3^2 & \cdot & 3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & (n+2k) & (n+2k)^2 & \cdot & (n+2k)^{n-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \end{bmatrix}$$

Berlekamp Welch

How do we find the original polynomial $P(x)$?

Suppose that $m_1, \ldots, m_k$ are the corrupted packets. Let $E(x) = (x - m_1) \ldots (x - m_k)$

Then $P(i) * E(i) = r_i * E(i)$ for any i greater than 1 and less than $n + 2k$. Why?

Solution: Case 1: i is corrupted

$$E(i) = 0 \rightarrow 0 = 0$$

Case 2: i is not corrupted

$$P(i) = ri \rightarrow P(i) * E(i) = r_i E(i)$$

Let $Q(i) = P(i)E(i)$ So we have $Q(i) = P(i)E(i) = r_i * E(i)$ where $1 \leq i \leq 2k + n$ What degree is $Q(i)$?

Solution: $n + k - 1$

How many coefficients do we need to describe $Q(i)$?

Solution: $n + k$

What degree is $P(i)$?

Solution: k

How many unknown coefficients do we need to describe $E(i)$?

Solution: k (we know that the first coefficient has to be 1)

We can write $Q(i) = r_i E(i)$ for every$i$ that is $1 \leq i \leq 2k + n$.

How many equations do we have? How many unknowns?

Solution: $n + 2k$

Once we have the above described equations, how do we determine what $P(i)$ is?

Solution: Solve the equations to get the coefficients for $E(i)$ and $Q(i)$. Then divide $\frac{E(i)}{Q(i)}$ to get $P(i)$.

1.2 Questions

1. (a) Alice sends Bob a message of length 3 on the Galois Field of 5 (modular space of mod 5). Bob receives the following message: (3, 2, 1, 1, 1). Assuming that Alice is sending messages using the proper general error message sending scheme, set up the linear equations that, when solved, give you the $Q(x)$ and $E(x)$ needed to find the original $P(x)$.

Solution: Set up 5 equations for the five values of x that we have, such that

$$Q(x) \mod 5 = r_x * (x - b) \mod 5$$

where

$$Q(x) = a_3 * x_i^3 + a2 * x_i^2 + a_1 * x_i + a_0$$

(r_i = ith received number)
In the form, for

$$x = x_i, a3 * x_i^3 + a_2 * x_i^2 + a_1 * x_i + a_0 = r_i * (x - b)$$

$$x = 1, (a_3 + a_2 + a_1 + a_0) \mod 5 = 3(1 - b) \mod 5$$
$$x = 2, (3a_3 + 4a_2 + 2a_1 + a_0) \mod 5 = 2(2 - b) \mod 5$$
$$x = 3, 2a_3 + 4a_2 + 3a_1 + a_0 \mod 5 = 1(3 - b) \mod 5$$
$$x = 4, 4a_3 + 1a_2 + 4a_1 + a_0 \mod 5 = 1(4 - b) \mod 5$$
$$x = 5, 0 + 0 + 0 + a0 \mod 5 = 1(0 - b) \mod 5$$

Solving for these equations, which you should give your students after setting up the above, you get the following:

$$b = 3$$

$$a_3 = 1$$
$$a_2 = 3$$
$$a_1 = 3$$
$$a_0 = 2$$

(b) What is the encoded message that Alice actually sent? What was the original message? Which packet(s) were corrupted?

Solution: Now we have $P(x) = \frac{Q(x)}{E(x)}$. Plug in the coefficients for Q and E and divide (using polynomial long division) and you get $P(x) = (x^2)+x+1$. Using the $P(x)$ that we found, we plug in the values 1, 2, 3, 4, 5 to find the encoded 5 packet message. $P(1) = 3, P(2) = 2, P(3) = 3, P(4) = 1, P(5) = 1$. The original message is the first 3 $P(1), P(2), P(3)$. Using the value of b, we know that the 3rd packet was corrupted, which we can confirm in the message that was received.

2 Secret Sharing

2.1 Questions

1. You want to send a super secret message consisting of 10 packets to the space station through some astronauts. You are afraid that some malicious spy people are going to tell the wrong message and make the space station go spiraling out of orbit. Assuming that up to 5 of the astronauts are malicious, design a scheme so that the group of astronauts (including the malicious ones) still find the correct message that you want to send. You can send any number of astronauts, but try to make the number that you have to send as small as possible. Astronauts can only carry one packet with them.

   Solution: We can use the scheme provided in the notes for general errors. If there are up to 5 malicious astronauts, then we have up to 5 general errors in our transmission. To account for those errors and to send the original message we have to send at least the length of the original message plus the twice the possible number of general errors. Thus, we have to send 20 astronauts or more. Create a polynomial of degree 9 (which is the length of the message minus 1), such that $P(0), P(1), \ldots, P(10)$ is the original message, and give each of the 20 astronauts unique points on that polynomial. Then use the methods we used in the previous question to find the message.

2. An officer stored an important letter in her safe. In case she is killed in battle, she decides to share the password with her troops. Everyone knows there are 3 spies among the troops, but no one knows who they are except for the three spies themselves.The 3 spies can coordinate with each other and they will either lie and make people not able to open the safe, or will open the safe themselves if they can. Therefore, the officer would like a scheme to share the password that satisfies the following conditions:

   1. When M of them get together, they are guaranteed to be able to open the safe even if they have spies among them.
   2. The 3 spies must not be able to open the safe all by themselves.

   Please help the officer to design a scheme to share her password. What is the scheme? What is the smallest M? Show your work and argue why your scheme works and any smaller M couldnt work.

   Solution: The key insight is to realize that both polynomial-based secret-sharing and polynomial-based error correction work on the basis of evaluating an underlying polynomial at many points and then trying to recover that polynomial.

Hence they can be easily combined. Suppose the password is s. The officer can construct a polynomial $P(x)$ such that $s = P(0)$ and share $(i, P(i))$ to the i-th person in her troops. Then the problem is: what should the degree of $P(x)$ be and what is the smallest M? First, the degree of polynomial d should not be less than 3. It is because when $d < 3$, the 3 spies can decide the polynomial $P(x)$ uniquely. Thus, n will be at least 4 symbols. Lets choose a polynomial $P(x)$ of degree 3 such that $s = P(0)$. We now view the 3 spies as 3 general errors. Then the smallest $M = 10$ since n is at least 4 symbols and we have $k = 3$ general errors, leading us to a codeword of $4 + 2 \cdot 3 = 10$ symbols (or people in our case). Even though the 3 spies are among the 10 people and try to lie on their numbers, the 10 people can still be able to correct the $k = 3$ general errors by the Berlekamp-Welch algorithm and find the correct $P(x)$.

Alternative solution: Another valid approach is making $P(x)$ of degree $M1$ and adding 6 public points to deal with 3 general errors from the spies. In other words, in addition to their own point $(i, P(i))$, everyone also knows the values of 6 more points, $(t+1, P(t+1)), (t+2, P(t+2)), \ldots, (t+6, P(t+6))$, where t is the number of the troops. The spies have access to total of $3 + 6 = 9$ points so the degree $M - 1$ must be at least 9 to prevent the spies from opening the safe by themselves. Therefore, the minimum M is 10.

3 Uncountability

3.1 Introduction

1. (a.) What does it mean for a set to be countably infinite?

> **Solution:** There is a bijection between the set and the natural numbers (or any other countable set)

(b.) Does $\mathbb{N}$ and $\mathbb{Z}^+$ have the same cardinality? Does adding one element change cardinality?

> **Solution:** Do the hotel argument. Just take each person starting at some arbitrary point k, and slide them over by one. In other words, map everybody at some point $p \geq k$ to $p + 1$, and then slide the newcomer into spot k.
>
> No, adding in one more point did not change the cardinality of the set.

(c.) Cantor-Bernstein Theorem: Suppose there is an injective function from set A to set B and there is an injective function from set B to set A. Then there is a bijection between A and B. Use this theorem to prove that Q is countable

> **Solution:** This is used when proving that rational numbers are countable. We know that $|N| \leq |Q|$ because every natural number is a rational number. Just need to show that $|Q| \leq |N|$. Do the spiral proof.

3.2 Questions

1. Are these sets countably infinite/ uncountable infinite/ finite? If finite, what is the order of the set?

(a) Finite bit strings of length n.

> **Solution:** Finite. There are 2 choices (0 or 1) for each bit, and n bits, so there are $2 \times 2 \times \ldots \times 2 = 2^n$ such bit strings.

(b) All finite bit strings of length 1 to n.

Solution: Finite. By part (1), there are 2^1 bit strings of length 1, 2^2 of length 2, etc. Thus, there are $2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 2$.

(c) All finite bit strings

Solution: Countably infinite. We can list these strings as follows: $\{0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, \ldots\}$. This gives us a bijection between the (countable) natural numbers, so these are countably infinite.

(d) All infinite bit strings

Solution: Uncountably infinite. We can construct a bijection between this set and the set of real numbers between 0 and 1. We can represent these real numbers using binary e.g. they are of the form $0.0110001010110\ldots$. The bijection between By diagonalization, the set of real numbers between 0 and 1 is uncountably infinite; therefore, so is this set.

(e) All finite or infinite bit strings.

Solution: Uncountably infinite. This is the union of a countably infinite set (part 3) and an uncountably infinite set (part 4), so it is uncountably infinite.

2. Find a bijection between N and the set of all integers congruent to $1 \mod n$, for a fixed n.

Solution: The set of integers congruent to 1 mod n is $A = \{1 + kn \mid k \in Z\}$. Define $g : Z \to A$ by $g(x) = 1 + x \times n$; this is a bijection because it is clearly one-to-one, and is onto by the definition of A. We can combine this with the bijective mapping $f : N \to Z$ from the notes, defined by $f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{-(x+1)}{2} & \text{if } x \text{ is odd} \end{cases}$ Then $f \circ g$ is a function from N to A, which is a bijection.

3. **True/False**

(a) Every infinite subset of a countable set is countable

Solution: True. Define E as the subset. Define function f where $f(1) = min(E)$, $f(k) = k$th smallest element of E. We see a bijective mapping clearly exists between f and E. and since the x-values in f are just the natural numbers, there is a mapping between E and N.

(b) If A and B are both countable, then $A \times B$ is countable

Solution: True. Can draw a bijection where first elem of A maps to 1, first elem of B maps to 2, second elem of A maps to 3, etc. This will include $A \times B$ at the end, and because there is a bijection from A to N and B to N, there is a bijection here from $A \times B$ to N. There is clearly mapping from $A \times B$ to $N \times N$, and $N \times N$ to N, so there is a bijective mapping from $A \times B$ to N

(c) Every infinite set that contains an uncountable set is uncountable.

Solution: True Let A be an uncountable subset of B. Assume that B is countable. $f : N \to B$ is a bijection. There must be a subset of N such that $f : M \to A$ is a bijection Then A is countable. This is a contradiction. So B must be uncountable.

4 Self Reference

4.1 Introduction

The Halting Problem: Does a given program ever halt when executed on a given input?

$$\texttt{TestHalt(P,x)} = \begin{cases} \text{“yes”}, & \text{if program } P \text{ halts on input } x \\ \text{“no”}, & \text{if program } P \text{ loops on input } x \end{cases}$$

How do we prove that `TestHalt` doesnt exist? Lets assume that it does, and hope we reach a contradiction.
Define another program:

```
Turing(P)
  if TestHalt(P,P) = "yes" then loop forever
  else halt
```

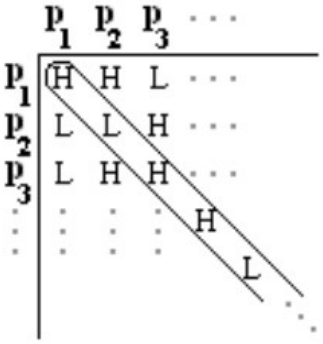
What happens when we call `Turing(Turing)`?

Solution:

Case 1 : It halts If `Turing(Turing)` halts then `TestHalt(Turing, Turing)` must have returned no. But `TestHalt(Turing Turing)` calls `Turing(Turing)` and calling `Turing(Turing)` must loop. But we assumed that `Turing(Turing)` halted. Contradiction.

Case 2 : It loops This implies that `TestHalt(Turing, Turing)` returned yes, which by the way that `TestHalt` is defined implies that `Turing` halted. But we assumed that `Turing(Turing)` looped. Contradiction.

How is this just a reformulation of proof by diagonalization?



Solution: List all possible programs as rows and columns. The rows are the programs and the columns are the inputs. Turing must be one of the rows, say row n. If entry (n, n) is H then Turing will loop by definition. If entry (n, n) is L then Turing will halt by definition. Therefore Turing cannot be on the list of all programs and therefore it does not exist.

Therefore the Halting Problem is unsolvable. We can use this to prove that other problems are also unsolvable. Say we are asked if program M is solvable. To prove it is not, we just need to prove the following claim: If we can compute program M, then we could also compute the halting problem.
This would then prove that M can not exist, since the halting problem is not computable. This amounts to proof by contradiction.

4.2 Questions

1. Say that we have a program M that decides whether any input program halts as long as it prints out the string ABC as the first operation that it carries out. Can such a program exist?

   Solution: No. Such a program M can not exist. We proceed as follows: we show that if such a program existed, the halting problem would be computable.

   Consider any program P. If we wanted to decide if P halted, we could simply create a new program P' where P' first prints out ABC, then proceed to do exactly what P would. However, if M existed, we could determine whether any program P halted by converting it to a P' and feeding it into M. This would solve the halting problem- but this is a contradiction, since by diagonalization we can prove that the halting problem is not solvable.

   Therefore, M can not exist!

5 Intro to Counting

5.1 Introduction

Rules of counting:
1. If an event is composed of different independent events, then we can multiply together the probabilities of the independent events.
2. If the order does not matter, then count the number of ways to arrange the situation with order and then divide by the number of orderings/sorted objects.

5.2 When Order Matters

1. (a) You have 15 chairs in a room and there are 9 people. How many different ways can everyone sit down?

Solution: $\frac{15!}{6!}$ There are 15 places to put the first person, then 14 places to put the second person, 13 places to put the third person, etc all the way to the last person who has 7 places to sit Another way to think about this is like the anagram example above. We have 9 unique letters and 6 repeats (our empty spaces). We divide by the repeats giving us: $15*14*13*12*11*10*9*8*7 = \frac{15!}{6!}$

(b) How many ways are there to fill 9 of the 15 chairs? (We dont care who sits in them)

Solution: $\binom{15}{9} = \frac{15!}{9!(15-9)!}$ In this example, we dont care about the uniqueness of each person, so we can just count each person as a repeat. So like the anagram example well divide for every repeat. We have 9 human repeats, and 6 empty space repeats. Hence $\frac{15!}{9!6!}$

2. **Identical Digits** The numbers 1447, 1005, and 1231 have something in common. Each of them is a four digit number that begins with 1 and has two identical digits. How many numbers like this are there?

Solution: Case 1: the identical digits are 1 (e.g. $11xy, 1x1y, 1xy1$)
Since there can only be two numbers that are identical, x and y cannot be 1 and $x \neq y$.
So [Possible formats] * [Possible x values] * [Possible y values] $= 3 * 9 * 8 = 216$
Case 2: identical digits are not 1 (e.g. $1xxy, 1xyx, 1yxx$).
So [Possible formats] * [Possible x values] * [Possible y values] $= 3 * 9 * 8 = 216$
Add both cases to arrive at the final result. $216 + 216 = 432$

5.3 More Practice

1. At Starbucks, you can choose either a Tall, a Grande, or a Venti drink. Further, you can choose whether you want an extra shot of espresso or not. Furthermore, you can choose whether you want a Latte, a Cappuccino, an Americano, or a Frappuccino.

   How many different drink combinations can you order?

Solution: $3 \cdot 2 \cdot 4$ (# sizes * espresso or not * # types of coffee)

2. Let's grab a deck of cards it's poker time! Remember, in poker, order doesnt matter. By ranks, we refer to the face value of cards (i.e. the number or K/Q/J/A), not the suit.

(a) How many ways can we have a hand with exactly one pair? This means a hand with ranks (a, a, b, c, d).

> **Solution:** $= 13 * \binom{4}{2} * \binom{12}{3} * 4^3$ There are 13 value options for a (2, 3, 4, ..., K, A). We then need to choose 2 out of the 4 possible suits. Now we need to choose b, c, and d. There are 12 values left (must be different from a). Finally, there are 4 suit options for each of the values chosen for b, c, d.

(b) How many ways can we have a hand with four of a kind? This means a hand with ranks (a, a, a, a, b)

> **Solution:** $= 13 * 12 * 4$

(c) How many ways can we have a straight? A straight is 5 consecutive cards,

> **Solution:** A straight can begin from any number from 2-10: (2, 3, 4, 5, 6); (3, 4, 5, 6, 7); ; (10, J, Q, K, A). That gives us 9 possibilities. Each number in hand has 4 possibilities (suits), so we have $9 * (4^5)$

(d) How many ways can we have a hand of all of the same suit?

> **Solution:** $4 * \binom{13}{5}$ For each of the 4 suits, there are $\binom{13}{5}$ different combinations of 5 cards among 13 to choose from.

(e) How many ways can we have a straight flush? This means we have a consecutive-rank hand of the same suit. For examples, (2, 3, 4, 5, 6), all of spades is a straight flush, while (2, 3, 5, 7, 8) of all spades is NOT, as the ranks are not consecutive.

> **Solution:** For each of 4 suits, there are 9 number combinations (as shown in c, starting from 2 to starting from 10). Each number combination is unique, because there is only one number per suit $= 4 * 9 = 36$

3. How many solutions does $x+y+z = 10$ have, if all variables must be positive integers?

> **Solution:** We know no number can be greater than 8, because all are positive. So position: y can take on any value from $1 \rightarrow 8$, and z will just be whatever is left (y can only each of 8 values of x ($1 \rightarrow 8$), we solve $y + z = 10 - x$ Take example $x = 1$, then $y + z = 9$. Because each value ≥ 1, there are 8 solutions for this equality take

on 8 values because $z \geq 1$). So we can see that in general, there are $10 - x - 1$ solutions for each value of x. so when $x = 1$, 8 solutions; when $x = 2$, 7 solutions, etc. for a total of $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$ (1 happens when $x = 8$ and y and z both must $= 1$) total $= 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36$ solutions

It's easier to think in terms of stars and bars. Bars can't be next to each other since variables are all positive integers, and this would imply that one of the values is 0. So $n = 10$ stars, $k = 3$ bars. Answer = $\binom{n-1}{k-1} = \binom{9}{2} = 36$

4. How many ways are there to arrange the letters of the word SUPERMAN

   (a) On a straight line?

   Solution: 8!

   (b) On a straight line, such that SUPER occurs as a substring?

   Solution: 4! (treat SUPER as one character)

   (c) On a straight line, such that SUPER occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

   Solution: 3! * (8 choose 3) This reduces to a stars and bars problem–the S U P E R are bars, and we want to put M A N somewhere in the sequence. Once we do so, there can be any permutation of M A N within the bars. Equivalently, we can arrange the letters of SUPERMAN (8! ways), but divide by 5! because we have arranged SUPER in any of 5! ways, when we only want one way. This gives us 8! / 5!, which is equal to $3! * 8!/(5!3!) = 3! * \binom{8}{3}$

   (d) On a circle?

   Solution: 7! Anchor one element, arrange the other 7 in a line around it

   (e) On a circle, such that SUPER occurs as a substring?

   Solution: 3! Anchor one element, arrange the other 7 around in a line, but treat SUPER as a single character, so its arranging the other 3 around in a line

(f) On a circle, such that SUPER occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

Solution: $3! * \binom{7}{3}$ Anchor one element (for simplicity, choose M, A, or N). Then follow the same procedures as in (c)