DISCRETE AND CONDITIONAL PROBABILITY 7

COMPUTER SCIENCE MENTORS 70

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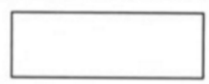
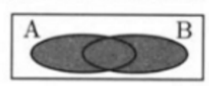
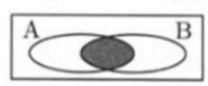
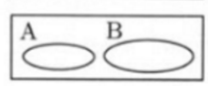
1 Discrete Probability

1.1 Introduction

Figure 1: From Pitman

Event language	Set language	Set notation	Venn diagram
outcome space	universal set	Ω	
event	subset of Ω	A, B, C, etc.	
impossible event	empty set	$\emptyset$	
not A, opposite of A	complement of A	A^c	
either A or B or both	union of A and B	$A \cup B$	
both A and B	intersection of A and B	AB, $A \cap B$	
A and B are mutually exclusive	A and B are disjoint	$AB = \emptyset$	
if A then B	A is a subset of B	$A \subseteq B$	

1.2 Questions

1. **Probably Poker**

   (a) What is the probability of drawing a hand with a pair (two cards of matching rank)?

   Solution: $\frac{13*\binom{4}{2}*\binom{12}{3}*4^3}{\binom{52}{5}}$

   (b) What is the probability of drawing a hand with four of a kind (four cards of the same rank)?

   Solution: $\frac{13*12*4}{\binom{52}{5}}$

   (c) What is the probability of drawing a straight (five cards in numerical order)?

   Solution: $\frac{9*4^5}{\binom{52}{5}}$

   (d) What is the probability of drawing a hand of all of the same suit?

   Solution: $\frac{4*\binom{13}{5}}{\binom{52}{5}}$

   (e) What is the probability of drawing a straight flush (five cards in numerical order, all of the same suit)?

   Solution: $\frac{4*9}{\binom{52}{5}}$

2. Suppose you arrange 12 different cars in a parking lot, uniformly at random. Three of the cars are Priuses, four of the cars are Teslas, and the other five are Nissan Leaves. What is the probability that the three Priuses are all together?

   Solution: There are $12!$ possible ways to arrange the 12 cars. Now, there are $12 - 3 + 1 = 10$ different places the Priuses could go (positions 1,2,3, positions 2,3,4, all the way until positions 10,11,12). For each of those 10 places, there are $3!$ ways to arrange the Priuses and 9! ways to arrange the other 9 cars in the 9 remaining spots. So, in total, there are $10 * 3! * 9!$ ways of arranging the cars so that the 3 Priuses are together. So the probability we get what we want is $\frac{10*3!*9!}{12!} = .0455$.

2 Conditional Probability

2.1 Introduction

Bayes' Rule

$$\mathrm{P}[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{\mathrm{P}[B|A]P[A]}{\mathrm{P}[B]}$$

Total Probability Rule

$$\mathrm{P}[B] = \mathrm{P}[A \cap B] + \mathrm{P}[\bar{A} \cap B] = \mathrm{P}[B|A] * \mathrm{P}[A] + \mathrm{P}[B|\bar{A}] * (1 - \mathrm{P}[A])$$

2.2 Questions

1. Say that we have a bag with 2 coins. One of the coins is fake: it has heads on both faces. The other is a normal coin. You pull a coin out of the bag, and flip it. It comes up heads. What is the probability that this coin is the fake one?

Solution: Define the event A as drawing the fake coin. Define the event B as the coin that we flipped coming up heads. We are looking for the probability of $\mathrm{P}[A|B]$ - the probability that we drew the fake coin, given that the flipped coin came up heads. We can directly plug this into Bayes' rule: $\mathrm{P}[A|B] = \frac{\mathrm{P}[A \cap B]}{\mathrm{P}[B]}$. $\mathrm{P}[A \cap B] = \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{2}$. $\mathrm{P}[B] = \frac{1}{2} \cdot \frac{1}{1} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$. So, we get $\frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$.

2. A lie detector is known to be 80% reliable when the person is guilty and 95% reliable when the person is innocent. If a suspect is chosen from a group of suspects where only 1% have ever committed a crime, and the test indicates that the person is guilty, what is the probability they are innocent?

Solution: Let I and G be the events that the person is innocent and guilty respectively, and let L_I and L_G be the events that the test says innocent or guilty.

$$P(I|L_G) = \frac{P(L_G|I) * P(I)}{P(L_G|I) * P(I) + P(L_G|G) * P(G)} = \frac{0.05 * 0.99}{0.05 * 0.99 + 0.8 * 0.01} = 0.86$$

3 Combinations of Events

3.1 Introduction

Often we are interested in the probabilities of combinations of events, whether it be their union or intersection.

Independence

Two events A, B in the same probability space are independent if

$$P[A \cap B] = P[A] * P[B]$$

Note that things get more complicated when we have more than two events. The independence of a pair of events (pairwise independence) doesn't necessarily imply the events are all mutually independent.

Intersections

When events are independent, we compute the probability of their intersection by simply multiplying the probabilities of each event. When events aren't necessarily independent, we must use the product rule.

$$P[A \cap B] = P[A] * P[B|A]$$

This rule can be extended to more than two events.

$$P[\cap_{i=1}^{n} A_i] = P[A_1] * P[A_2|A_1] * ... * P[A_{n-1}| \cap_{i=1}^{n-2} A_i] * P[A_n| \cap_{i=1}^{n-1} A_i]$$

Unions

If events are disjoint, computing the probability of their union is simple; just add up the probabilities of each. Unfortunately, many events are not disjoint (independent events in particular cannot be: why?). So, we must use the Principle of Inclusion/Exclusion.

$$P[\cup_{i=1}^{n}] = \sum_{n=1}^{n} P[A_i] - \sum_{\{i,j\}} P[A_i \cap A_j] + \sum_{\{i,j,k\}} P[A_i \cap A_j \cap A_k] - \pm P[\cap_{i=1}^{n}]$$

This sum is always less than adding up the individual probabilities of the events; this concept is known as the union bound.

$$P[\cup_{i=1}^{n}] \leq \sum_{i=1}^{n} P[A_i]$$

3.2 Questions

1. Find an example of 3 events A, B, and C such that each pair of them are independent, but they are not mutually independent.

> **Solution:** Consider a fair 4-sided die. Let A be the event that 1 or 2 appears in a die roll, B be the event that 1 or 3 appears, and C be the event that 1 or 4 appears. Then,
> $$Pr(A) = Pr(B) = Pr(C) = \frac{1}{2}$$
> Furthermore,
> $$Pr(A \cap B) = Pr(1 \text{ appears}) = \frac{1}{4} = Pr(A)Pr(B).$$
> So A and B are pairwise independent. Similarly (A, C) and (B, C) are pairwise independent. However,
> $$Pr(A \cap B \cap C) = Pr(1 \text{ appears}) = \frac{1}{4} \neq Pr(A)Pr(B)Pr(C) = \frac{1}{8}$$
> So these 3 events are not mutually independent.The answer is not unique; any other valid answer is acceptable.

2. You have a deck of 52 cards. What is the probability of:

   (a) Drawing 2 Kings with replacement?

   > **Solution:** There are $52 * 52$ combinations of two cards. There are $4 * 4$ that are 2 Kings, so the probability is $\frac{1}{13^2}$.

   (b) Drawing 2 Kings without replacement?

   > **Solution:** There are $\binom{52}{2} = 52*51$ pairs of cards possible without replacement. Of these $4 * 3$ represent pairs of kings. Therefore the probability is $\frac{4*3}{52*51}$. We can also use conditional probability (which we will have to use in the next part):
   > $$P(K \text{ on 2nd and } K \text{ on 1st}) = P(\text{K on 2nd}|\text{K on 1st})P(\text{K on 1st}) = \frac{3}{51} * \frac{4}{52}$$
   > .

   (c) The second card is a King without replacement?

Solution:

$$\begin{aligned} P(\text{K on 2nd}) &= P(K \text{ on 2nd } | K \text{ on 1st}) P(K \text{ on 1st}) \\ &+ P(K \text{ on 2nd } | \text{ no } K \text{ on 1st}) P(\text{no } K \text{ on 1st}) \\ &= \frac{3}{51} * \frac{4}{52} + \frac{4}{51} * \frac{48}{52} \\ &= \frac{4}{52} \end{aligned} \tag{1}$$

Note that this is the same as the $P(K \text{ on 1st})$, because a K is equally likely to be anywhere in the deck.

(d) The nth card is a King without replacement ($n < 52$)?

Solution: The last problem is a hint that we can argue this by symmetry. Since we have no information about what any of the preceding cards were before the nth card, it is equally likely that the nth card is any of the 52 possible cards, so the probability that it is a King is $\frac{4}{52}$.

3. You own a pizzeria. You observe that one of your customers, Andy, buys a cheese pizza on Saturday with probability 0.3 and on Sunday with probability 0.6.

(a) If Andy's pizza purchasing on Sunday is independent from his pizza purchasing on Saturday, what is the probability that he buys pizza on a given weekend?

Solution: We use the Inclusion-Exclusion principle. If event A is the situation Andy buys a pizza on Saturday, and event B is the situation in which Andy buys a pizza on Sunday, then the probability that Andy buys pizza a given weekend is $P(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B) = 0.3 + 0.6 - 0.3 * 0.6 = 0.72$.

(b) If Andy buying a pizza on Saturday means that he will not buy pizza on Sunday, what is the probability that he buys pizza on a given weekend (i.e if he buys pizza on one day, he is guaranteed to not buy a pizza the next day)?

Solution: Now we know that $P(A \cap B) = 0$. Use the Inclusion-Exclusion principle: $P(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B) = 0.3 + 0.6 - 0 = 0.9$.

(c) Given that Andy buys pizza on a given weekend with probability 0.65, what is the probability that he buys pizza both days?

> **Solution:** By the Inclusion-Exclusion principle, we know that $\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$. Solving, we find that $\mathrm{P}(A \cap B) = 0.25$.

4. Say that we pick a number at random from 1 to 90, inclusive. What is the probability that is divisible by $2, 3$, or 5?

> **Solution:** We can just apply the Inclusion-Exclusion principle here. If events A, B, and C represent the events of being divisible by $2, 3$, and 5 respectfully, then $\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C) - \mathrm{P}(B \cap C) + \mathrm{P}(A \cap B \cap C) = \frac{45}{90} + \frac{30}{90} + \frac{18}{90} - \frac{15}{90} - \frac{9}{90} - \frac{6}{90} + \frac{3}{90} = \frac{66}{90} = \frac{11}{15}$.

5. Given n bins and m balls find the largest value of m such that the probability that there is no collision is above $\frac{1}{2}$? *Hint: Use the union bound to approximate. Recall, the union bound states that* $P(A_1 \cup A_2 \cup ... \cup A_n) \leq P(A_1) + P(A_2) + ... + P(A_n)$.

> **Solution:** Let A be the event that there are no collisions. We know we have $\binom{m}{2} = k$ pairs of balls. Let A_i be the probability that pair i collides. $\Pr[A_i]$ is simply $\frac{1}{n}$, so by the union bound, we have
>
> $$\Pr[\overline{A}] \leq \sum_{i=1}^{k} \Pr[A_i] = k\left(\frac{1}{n}\right) = \frac{m(m-1)}{2n} \approx \frac{m^2}{2n}$$
>
> So we have that $\frac{m^2}{2n} = \frac{1}{2}$ and so $m \leq \sqrt{n}$.