CSM Fall 2018 Interview Materials

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Format

The format of the interview will be as follows:

- **Part 1 (∼4 min):** You will be provided with 6 relatively difficult questions beforehand. You will also have access to bare bones solutions to these questions. Pick one question and present your solution to it (does not have to follow the same solution that we provide). You do not have to take up the full time allotted, but make your answer is as thorough as you can make it (pretend that you are teaching it to your section).
- **Part 2 (∼4 min):** We will then present a problem and a made-up student's attempt at solving the problem. The made-up student will have made a common mistake or misconception while solving the problem, and you will be expected to find the mistake and explain the correct solution to the question. These questions will not be difficult and will be randomly selected. You will not be aware of these questions beforehand.
- **Part 3: (Remaining time):** With whatever remaining time there is, we will ask you some soft questions to get to you you better!

Prepared Question/Solutions:

Question 1:

Use a combinatorial proof to argue

$$\sum_{i=0}^{n} i^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$$

Solution 1:

This is the number of ways to choose triplets (a, b, c) such that $c \geq a, b$, for every c from 0 to n.$\binom{n+1}{2}$ covers the case when $a = b$, and $2\binom{n+1}{3}$ covers the case when $a \neq b$.

Question 2:

Prove that if p is prime, then there are no solutions to $x^2 \equiv p \bmod p^2$.

Solution 2:

We can rewrite the equation as

$$x^2 - p \equiv 0 \bmod p^2$$

This implies that there exists some integer k s.t

$$x^2 - p = kp^2$$

Divide both sides by p to see that

$$\frac{x^2}{p} - 1 = kp$$

This means that p divides x^2. However, p cannot divide x^2

Question 3:

You draw randomly from a hat that has 100 slips of paper in it, labeled 1-100. You can either choose to keep the dollar amount equal to the number you drew, or you can pay 1 dollar, place the number back in the hat, and draw again. Determine the optimal strategy to maximize your expected value from playing the game. Hint: Think about how we calculated expected value for a geometric series.

Solution 3:

Note: In your presented solutions, you have to show how you would go about maximizing something, but you don't have to actually calculate derivatives/solve for roots.

Let P be your profit from the game. The optimal strategy is to have some threshold X, such that when the spin is greater than or equal to X you take the money and when it is less X chooses to re-draw. We see that:

$$\mathbf{E}[P] = \frac{X-1}{100}(\mathbf{E}[P]-1) + (1 - \frac{X-1}{100}) \cdot \frac{X+100}{2}$$

Maximizing the above with respect to X, we find the optimal value for $X = 86.85$. The game only uses discrete values, so using $X = 87$ gives us an expected value of 86.37. Alternatively,

$$\mathbf{E}[P] = \frac{100+X}{2} - \frac{100}{101-X}.$$

Maximizing this with respect to X gives us the same results as the above.

Question 4:

You are practicing your max range shot put. Each day for 30 consecutive days, you throw the shot as far as you can. The shotput lands distance D away from you, where $D \sim Exp(\lambda)$ yards. Define a "Personal Record" as a throw that is farther than all the throws on the days before it. Let P be the number of days where you achieve a personal record. Calculate $\mathbf{E}[P]$.

Solution 4:

Let $P_i = 1$ if you set a PR on day i, and 0 otherwise.

$$\begin{aligned}\mathbf{E}[P] &= \sum_{i=1}^{n} \mathbf{E}[P_i] \\ &= \sum_{i=1}^{n} \frac{1}{i} \\ &\approx \ln 30 + \gamma\end{aligned}$$

Question 5:

You and your friend each spin a spinner which returns a uniform, continuous random value between 0 and 1. Your team score S is calculated by taking the max of you and your friend's results and squaring it. For example, if you spun 0.5 and your friend spun 0.1, then your score S would equal $0.5^2 = 0.25$. Calculate the pdf of S.

Solution 5: Let your score be denoted by $X \sim U[0,1]$ and your friend's denoted by $Y \sim U[0,1]$. Let $T = \max(X, Y)$. $S = T^2$

$$P(T < t) = P(X < x \cap Y < y) = t^2 \text{ so } f_t(t) = 2t$$
$$\text{Then } P(S < s) = P(T < \sqrt{s}) = s, \text{ so}$$
$$f_s(s) = 1.$$