CSM 70 — Discrete Mathematics and Probability Theory

Spring 2016 — Methods of Proof

Worksheet 1 Solutions

Key Terms

Universal and existential quantifiers

Contrapositive/Converse

Methods of Proof: Direct, Contradiction, Contrapositive, Cases, Induction

Quantifiers (15 min)

Let P(x, y) denote some proposition involving x and y. For each statement below, either prove that the statement is correct or provide a counterexample if it is false.

1. $\forall x \forall y\, P(x, y)$ implies $\forall y \forall x\, P(x, y)$.

   True. The first statement "$\forall x \forall y\, P(x, y)$" means for all x and y in our universe, the proposition P(x, y) holds. The second statement "$\forall y \forall x\, P(x, y)$" has the same meaning, so they are in fact equivalent (the implication goes both ways). In general, you can interchange the order of any consecutive sequence of $\forall$.

2. $\exists x \exists y\, P(x, y)$ implies $\exists y \exists x\, P(x, y)$.

   True. Both statements mean there exist x and y in our universe that make P(x, y)true, so both statements are equivalent. In general, you can interchange the order of any consecutive sequence of $\exists$.

3. $\forall x \exists y\, P(x, y)$ implies $\exists y \forall x\, P(x, y)$.

   False. Take the universe to be R (or any set with at least 2 elements), and take P(x, y) to be the statement "$x = y$." Then the first statement "$\forall x \exists y\, P(x, y)$" claims for all $x \in R$ we can find $y \in R$ such that $x = y$, which is true because we can take y to be x. However, the second statement "$\exists y \forall x\, P(x, y)$" claims there exists $y \in R$ such that $x = y$ for all $x \in R$, which is false because a real number y cannot simultaneously be equal to all other real numbers x. Thus, the implication is false.

4. $\exists x \forall y\, P(x, y)$ implies $\forall y \exists x\, P(x, y)$.

   True. Suppose the first statement "$\exists x \forall y\, P(x, y)$" is true, which means there is a special element $x^* \in R$ such that $P(x^*, y)$ is true for all $y \in R$. The second statement claims that for all $y \in R$ we can find an element $x \in R$ (which may depend on y) such that P(x, y) is true. But from our first statement we know that we can choose the same value $x = x^*$ for all y. We conclude that the implication holds. However, the implication is only one way. In particular, note that part 4 is the converse to part 3, which we have seen is false.

Contradiction and Contraposition (20 min)

Write the contrapositive of the following statements and, if applicable, the statement in mathematical notation.

If a quadrilateral is not a rectangle, then it does not have two pairs of parallel sides.

If a quadrilateral has two pairs of parallel sides, then it is a rectangle.

For all natural numbers a where a^2 is even, a is even.

$\forall a \in N,\ a\ odd \rightarrow a^2\ odd$

original: $\forall a \in N,\ a^2\ even \rightarrow a\ even$

There are no integer solutions to the equation $x^2 - y^2 = 10$.

$\exists\ x, y \in Z, x^2 - y^2 = 10$

original: $\forall\ x, y \in Z, x^2 - y^2 \neq 10$

Prove that an irrational number to the power of an irrational number may be rational.

We proceed by cases. Note that the statement of the theorem is quantified by an existential quantifier: Thus, to prove our claim, it suffices to demonstrate a single x and y such that x y is rational. To do so, let x = √ 2 and y = √ 2. Let us divide our proof into two cases, exactly one of which must be true: (a) √ 2^√ 2 is rational, or (b) √ 2^√ 2 is irrational.

(Case (a))

Assume first that √ 2^√ 2 is rational. But this immediately yields our claim, since x and y are irrational numbers such that x y is rational.

(Case (b))

Assume now that √ 2 √ 2 is irrational. Our first guess for x and y was not quite right, but now we have a new irrational number to play with, √ 2^√ 2 . So, let's try setting x = √ 2 √ 2 and y = √ 2. Then, x y = √ 2^√ 2^ √ 2 = √ 2^(√ 2* √ 2) = √ 2^2 = 2, where the second equality follows from the axiom (x^y)^z = x^yz. But now we again started with two irrational numbers x and y and obtained rational x y . Since one of case (a) or case (b) must hold, we thus conclude that the statement is true.

Prove or disprove: If P→Q and R→¬Q, then P→¬R.

Use the contrapositive of the second statement P → Q → not R

Alternatively, we can use proof by contradiction. Assume P does not imply -R. This means both P and R can be true at the same time. But, we know that if P is true, Q is true and if R is true, -Q is true. This is a contradiction, as Q and -Q cannot both be true. Hence, if P is true, not R must be true. Hence, P → -R.

Induction (10 min)

What are the three simple steps of induction?

Base case
Inductive hypothesis
Inductive step

Prove that Σ from i=0 to n of i!i is equal to (n+1)! - 1 for n >= 1 and n in the natural numbers.

Base case: For n = 1, we get 0 + 1*1! = (1+1)! - 2, which is true as 1 = 1
Induction Hypothesis: Assume that this is true for n = k.
Inductive Step: Let's prove this for k+1.

$$\Sigma_{\text{from } i=0 \text{ to } k+1} (i)!*i = (\Sigma_{\text{from } i=1 \text{ to } k} i!*i) + (k+1)(k+1)!$$

= (k+1)! - 1 + (k+1)(k+1)! by the inductive hypothesis,
= (k+1)!(k+2) - 1
= (k+2)! - 1
= (k+1+1)! - 1

Hence, we have proven it for k+1, and it is true for all n.

More Practice (remaining time ~35min)

Let x be a positive real number. Prove that if x is irrational (i.e., not a rational number), then √ x is also irrational.

Proof by Contradiction:
Suppose ∃ an irrational number x such that √x is rational. By definition of rational, √x = a/b for some integers a and b with b != 0. Then x = (√x)2 = $(a/b)^2 = a^2/b^2$. But a^2 and b^2 are both integers (being products of integers), and b != 0 by the zero product property. Hence, x is rational (by definition of rational). This contradicts the supposition that x is irrational, and so the supposition is false. Therefore, the square root of an irrational number is irrational.

McDonald's sells chicken McNuggets only in 6, 9, and 20 piece packages. This means that you cannot purchase exactly 8 pieces, but can purchase 15. The Chicken McNugget Theorem states that the largest number of pieces you cannot purchase is 43. Formally state the Chicken McNugget Theorem using quantifiers.

Answer: The theorem has two components: (i) you cannot purchase 43 pieces, and (ii) you can purchase any number larger than 43 pieces. The correct translation of the theorem is: $\nexists$ a,b, c ∈ N, 6a+9b+20c = 43 ⋀ ∀ n ≥ 44, ∃ a,b, c ∈ N, 6a+9b+20c = n

Prove of disprove the following statement: If n is a positive integer such that n / 3 leaves a remainder of 2, then n is not a perfect square.

This is easiest done by looking at the contrapositive of the statement. We shall prove that if n is a perfect square, then n / 3 does not leave a remainder of 2.

Any number divided by 3 can only have a remainder of 0, 1, or 2. Let's examine every case.

Case 1: Let's say n = (3m)^2, where m is some integer, then n = 9m^2, which is divisible by 3 and leaves no remainder.

Case 2: Let's say n = (3m + 1)^2, then n = 9m^2 + 6m + 1, which is equal to 3(3m^2 + 2m)+1, which is a number divisible by 3 plus one. Hence, when we take n to be this number and divide it by 3, we will have a remainder of 1.

Case 3: Let's say n = (3m + 2)^2, then n = 9m^2 + 12m + 4, which is equal to 3(3m^2 + 4m + 1) + 1, which is a number divisible by 3 plus 1. When we set n to be this number and divide it by 3, we obtain a remainder of 1.

Hence, in every case, we either obtain a remainder of 0 or a remainder of 1, but never a remainder of 2. We have proved the contrapositive to be true and hence, the original statement is True.

In a large field, n people are standing so that for each person, the distances to every other person is different. At a given signal, each person fires a water pistol and hits the person who is closest to them. When n is odd, prove that there is at least one person who is left dry.

Do this by induction:
Base case : 1 person is trivial
Inductive Hypothesis: Assume that for k odd, there is one person left dry.
Inductive Step: For (k+2) people, consider the two people closest to each other. This is unique (given by the problem), and these two people necessarily shoot each other. Therefore, the problem reduces to k people, which we know must have one person left dry by hypothesis.