Stable Marriage, Well Ordering Principle, Optimality, Graphs 1

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1 Stable Marriage

1.1 Introduction

The Algorithm:

1. **Every Morning**: Each man proposes to the most preferred woman on his list who has not yet rejected him.
2. **Every Afternoon**: Each woman collects all the proposals she received in the morning; to the man she likes best, she responds maybe, come back tomorrow (she now has him on a string), and to the others, she says never.
3. **Every Evening**: Each rejected man crosses off the woman who rejected him from his list. The above loop is repeated each successive day until each woman has a man on a string; on this day, each woman marries the man she has on a string.

Definitions:

M and W are a rogue couple if they prefer to be with each other as opposed to the people they are paired with

A pairing is stable if there are no rogue couples

Lemmas:

The algorithm halts.

If man M proposes to woman W on the kth day, then on every subsequent day W has someone on a string whom she likes at least as much as M.

1.2 Questions

1. Lemma: Algorithm terminates with a pairing.

Solution:

Proof. Contradiction

1. Negate the lemma: Suppose, there is a man who is unpaired when the algorithm ends.
2. How many women has he proposed to? n
3. By the Improvement Lemma, each woman has someone on a string since M proposed to each of them.
4. How many women have at least one man on a string? n
5. Man M is not on anyone's string.
6. At least how many men are there? $n+1$ (Contradiction!)

□

2. Lemma: The pairing is stable.

Solution:

Proof. Direct proof!

Let (M, W) be a couple in the pairing produced by SMA. Suppose M prefers $W*$ to W.

$W*$ is higher/lower in Ms preference list than W. (circle one)

M must have proposed to $W*$ before he proposed to W.

$W*$ must have rejected M to someone she prefers more, call him $M*$.

By the Improvement Lemma $W*$ likes her final partner at least as much as $M*$

$W*$ prefers her final partner to M.

The pairing is stable.

□

2 Well Ordering Principle

2.1 Questions

1. Every non empty set of natural numbers contains a smallest element.
   In this question, we will go over how the well-ordering principle can be derived from (strong) induction. Remember the well-ordering principle states the following: For every non-empty subset S of the set of natural numbers N, there is a smallest element $x \in S$; i.e. $\exists x : \forall y \in S : x \leq y$

   1. What is the significance of S being non-empty? Does WOP hold without it? Assuming that S is not empty is equivalent to saying that there exists some number z in it.

      Solution: If there are no elements in S, clearly there can be no minimal element. The main point of the rest of the proof is to show that having a minimal element does not only require that S be non-empty, it is equivalent. We will show that if there is no minimal element, then S must be empty.

   2. Induction is always stated in terms of a property that can only be a natural number. What should the induction be based on?

      Solution: The number x, i.e. the least element in the set S as defined in the problem. If induction using this variable seems a bit circular, note that we are not assuming x exists (that *would* be circular logic). We're only saying that, if a least element did exist, we could call it x.

   3. Now that the induction variable is clear, formally state the induction hypothesis.

      Solution: Let A be a non-empty subset of N. Our inductive hypothesis is the following predicate:

      P(n): If $n \in A$, then A has a least element.

      Note that the hypothesis P represents an implication, so it doesn't actually say anything about whether n is in A! It only states that, *if* n happens to be in

A, A must have a least element. This distinction is the key to why the proof is general for any set, since we don't actually make any assumptions about which elements are in A. $P(n)$: If $n \in A$, then A has a least element.

4. Verify the base case.

Solution: We can immediately state $P(0)$ is true. If 0 is in an arbitrary set of natural numbers A, A must contain a least element, since $0 \leq n \forall n \in \mathbb{N}$, and will always be that least element.

5. Now prove that the induction works, by writing the inductive step.

Solution: We will use strong induction, so we want to show that $[P(0) \wedge P(1) \wedge \ldots \wedge P(n)] \implies P(n+1)$. Intuitively what does this implication mean? The left side means that $\forall k \leq n$, if any one of them is in A, then A has a least element. The right side means that if $n+1$ is in A, then A has a least element. The implication should hold, because intuitively there should be no special $n+1$ that suddenly breaks our hypothesis. In fact if any one of those elements is in A, then we can default to the inductive hypothesis (left side) without considering $n+1$ at all. However, if elements $0 \ldots n$ are not in A, then we have a separate case where we need to explicitly show $P(n+1)$. Let's formalize this intuition:

Case 1: A contains $n+1$ and at least one element less than $n+1$.

$$\exists k(k \in A \wedge k < n+1)$$

By the inductive hypothesis, if $k \in A$ and $0 < k \leq n$, A contains a least element. Therefore $P(n+1)$ is true for this case as well.

Case 2: A contains $n+1$ and possibly larger elements.

$$\neg \exists k(k \in A \wedge k < n+1)$$

An immediate consequence of the above is that $\forall x \in A \quad n+1 \leq x$ (this can be shown using proof by contradiction). By definition of least element, $n+1$ is the least element in A, so $P(n+1)$ is true for this case as well.

We have covered every case in which A contains $n+1$, and can now state $P(n+1)$ is true in general.

6. What should you change so that the proof works by simple induction (as opposed to strong induction)?

> **Solution:** We would use a contradiction to start off the proof: Suppose S has no minimal element. Then $n = 1 \notin S$, because otherwise n would be minimal. Similarly $n = 2 \notin S$, because then 2 would be minimal, since $n = 1$ is not in S. Suppose none of $1, 2, ..., n$ is in S. Then $n + 1 \notin S$, because otherwise it would be minimal. Then by induction S is empty, a contradiction.

3 Optimal, Pessimal

3.1 Introduction

A persons optimal partner is their most preferred partner among possible partners in stable pairings.
A male optimal pairing is a pairing in which all males are paired with his optimal woman.
Lemma: If a pairing is male optimal, then it is also female pessimal

3.2 Questions

1. Theorem: The pairing produced by the stable marriage algorithm is male optimal

> **Solution:** Contradiction. (hint: Well Ordering Principle)
>
> *Proof.*
> 1. Negate the theorem: The pairing produced by the stable marriage algorithm is not male optimal.
> 2. Therefore there must exist a man who was not paired with his optimal partner.
> 3. Then there exists a day on which that man was rejected by his optimal partner
> 4. By the Well Ordering Principle, there is a first such day
> 5. Choose your notation: On day k man M_1 was rejected by his optimal, woman W_1 in favor of another man M_2. (WOP)
> 6. By definition of optimal partner, there exists a stable pairing T where M_1 and W_1 are paired.

7. Write down the pairing T using the notation chosen in 3:
   T := $(M_1, W_1), (M_2, W_2), \ldots$
8. Fact: W_1 prefers M_2 to M_1.
   Why? By part 5, she rejected M_1 in favor of M_2.
9. Fact: M_2 has not been rejected by his optimal woman on day $k-1$.
   Why? By assumption the first day that a man was rejected by his optimal woman was day k (part 5).
10. M_2 likes W_1 at least as much as his optimal woman.
11. Therefore M_2 likes W_1 at least as much as W_2.
12. (M_2, W_1) is a rogue couple in stable pairing T.

□

4 More Practice

4.1 Questions

1. Imagine that in the context of stable marriage all men have the same preference list. That is to say there is a global ranking of women, and men's preferences are directly determined by that ranking. Use any method of proof to answer the following questions.

    1. Prove that the first woman in the ranking has to be paired with her first choice in any stable pairing.

        Solution: If the first woman is not paired with her first choice, then she and her first choice would form a rogue couple, because her first choice prefers her over any other woman, and vice versa.

    2. Prove that the second woman has to be paired with her first choice if that choice is not the same as the first woman's first choice. Otherwise she has to be paired with her second choice.

        Solution: If the first and second women have different first choices, then the second woman must be matched to her first choice. Otherwise she and her first choice would form a rogue couple (since her first choice is not matched to the first woman, he would prefer the second woman over his current match). If the first choices are the same, then the second woman must be paired with her second choice, otherwise she and her second choice would form a rogue couple (neither of them are matched to their first choices, and they are each other's second choice).

    3. Continuing this way, assume that we have determined the pairs for the first $k-1$ women in the ranking. Who should the k-th woman be paired with?

        Solution: The k-th woman should be paired with the first man on her list who has not been matched yet (with the first $k-1$ women). If she's not matched to him, they would form a rogue couple. This is because the man would have to be matched to a woman ranked worse than k, so she would prefer the k-th woman over his current partner, and the k-th woman obviously prefers him to whoever she's matched with.

    4. Prove that there is a unique stable pairing.

Solution: In the previous parts, we saw that for each woman, given the pairs for the lower-ranked women, her pair would be determined uniquely. So there is only one stable pairing. This can be stated and proved more rigorously using induction. Namely that there is a unique pairing for the first k women, assuming stability. An induction on k would prove this.

5 Graph Theory

5.1 Introduction

1. Let $G = (V, E)$ be an undirected graph. Match the term with the definition.

   Walk Cycle Tour Path

   ________________ Walk that starts and ends at the same node
   ________________ Sequence of edges.
   ________________ Sequences of edges with possibly repeated vertex or edge.
   ________________ Sequence of edges that starts and ends on the same vertex and does not repeat vertices (except the first and last)

   Solution:
   tour Walk that starts and ends at the same node
   path Sequence of edges.
   walk Sequences of edges with possibly repeated vertex or edge.
   cycle Sequence of edges that starts and ends on the same vertex and does not repeat vertices (except the first and last)

2. What is a tournament?

   Solution: A directed graph for which any pair of vertices, u, v either have an edge $u \to v$ or $v \to u$. It can be used to represent a tournament in which every pair of player play at least once.

3. What is a simple path?

   Solution: Sequence of edges where the vertices are distinct