CSM 70 — Discrete Mathematics and Probability Theory

Spring 2016 — More Counting, Discrete Probability, Monty — Worksheet 7

Key Terms: Combinatorial Proofs, Monty Hall, Probability space, Halting/Uncomputability, Rules of Counting, Sample space

I. More Counting

Poker

We grab a deck of cards and it's poker time. Remember, in poker, order doesn't matter.

a. How many ways can we have a hand with exactly one pair? This means a hand with ranks (a, a, b, c, d)

b. How many ways can we have a hand with four of a kind? This means a hand with ranks (a, a, a, a, b)

c. How many ways can we have a straight ? A straight is 5 consecutive cards, that don't all necessarily have the same suit.

d. How many ways can we have a hand of all of the same suit?

e. How many ways can we have a straight flush? This means we have a consecutive hand of the same rank? For examples, (2, 3, 4, 5, 6), all of spades is a straight flush, while (2, 3, 5, 7, 8) of all spades is NOT, as the ranks are not consecutive.

Solving Equations

How many solutions does $x + y + z = 10$ have, if all variables must be positive integers?

Starbucks.

At Starbucks, you can choose either a Tall, a Grande, or a Venti drink. Further, you can choose whether you want an extra shot of espresso or not. Furthermore, you can choose whether you want a Latte, a Cappuccino, an Americano, or a Frappuccino. How many different drink combinations can you order?

Arranging letters.

How many ways are there to arrange the letters of the word "SUPERMAN"

a) On a straight line?

b) On a straight line, such that "SUPER" occurs as a substring?

c) On a straight line, such that "SUPER" occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

d) On a circle?

e) On a circle, such that "SUPER" occurs as a substring?

f) On a circle, such that "SUPER" occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

Challenge Question

A 2×2 square grid is constructed with four 1×1 squares. The square on the upper left is labeled *A*, the square on the upper right is labeled *B*, the square in the lower left is labeled *C*, and the square on the lower right is labeled *D*. The four squares are to be painted such that 2 are blue, 1 is red, and 1 is green. In how many ways can this be done?

Combinatorial Proofs

1. $$k\binom{n}{k} = n\binom{n-1}{k-1}$$

2. $n! = \binom{n}{k} k!(n-k)!$

3. $\sum_{k=1}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$

4. Prove $a(n-a)\binom{n}{a} = n(n-1)\binom{n-2}{a-1}$ by a combinatorial proof.

Challenge Question (*SKIP*)

a) Prove the Hockey Stick Theorem:

$$\sum_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1}$$ where n, t are natural numbers and n > t

Hint: (You can use this fact from homework) $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

b) Let S be the set {1, 2, 3, … 2015}. Look at all 1000 element subsets of S. Find the average of all smallest elements the subsets.

II. <u>Discrete Probability</u>

What is a sample space?

What is a probability space?

Probability: The probability of any event A is defined as follows $$\Pr[A] = \frac{\text{\# of sample points in } A}{\text{\# of sample points in } \Omega} = \frac{\lvert A \rvert}{\lvert \Omega \rvert}.$$

Example: What is the probability of drawing a flush in poker?

(#Of hands that are a flush)/(Total # of hands I can draw from the deck)

Balls and Bins: Suppose we have k balls and n bins and throw the balls into the bins.
What is the sample space?
Hint: How many ways can the balls go into the bins? Are the balls distinguishable (does it matter what balls go into which bin)?

Multiplying: Suppose that you want the probability that a bunch of things happen together. For example:

P[I eat Pizza and Take CS70] = P[I eat Pizza]*P[I take CS70]

In **some** cases, you can actually multiply the probabilities like above. **It isn't always OK though!**

Exercises:

Probably Poker (reference Counting section):

a) What is the probability of drawing a hand with a pair?

b) What is the probability of drawing a hand with four of a kind?

c) What is the probability of drawing a straight ?

d) What is the probability of drawing a hand of all of the same suit?

e) What is the probability of drawing a straight house?

Probably Piazza

toppings = [pepperoni, sausage, pineapple, bacon, mushrooms]

We have 5 pizzas each with a topping from above. Each pizza has 8 slices. If you are given one slice at random, what are the following probabilities?

Pr(S = pepperoni)

Pr(S = sausage)

Pr(S = pineapple or bacon)

On the sausage and pineapple pizzas, someone adds bell peppers to 4 slices each. What are the following probabilities?

Pr(S has bell peppers)

Pr(S has bell peppers and sausage)

Pr(S does not have bell peppers and is not sausage)

Now suppose Alex's roommate eats 7 slices of pepperoni, 6 slices of bacon, and 5 slices of mushrooms. What are the following probabilities?

Pr(S = pepperoni)

Pr(S = bacon)

Pr(S = mushrooms or sausage)

Probably Parked Priuses
Suppose you arrange 12 different cars in a parking lot, uniformly at random. Three of the cars are Priuses, four of the cars are Teslas, and the other five are Nissan Leaves. What is the probability that the three Prius's are all together?

III. <u>Monty Hall</u>

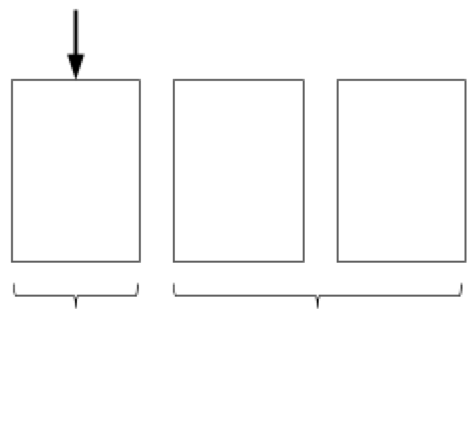
The Problem:

Suppose a contestant is shown 3 doors. There is a car behind one of them and goats behind the rest. Then they do the following:

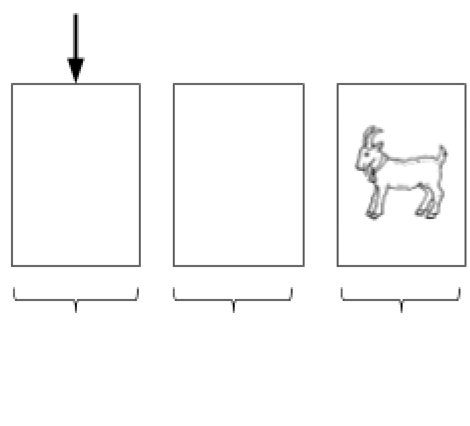
1) Contestant chooses a door.
2) Host opens a door with a goat behind it.
3) Contestant can choose to switch or stick to original choice

Is the contestant more likely to win if they switch?

At step 1, what is the probability that the car is behind the door the contestant chose? What is the probability that the car is behind the other two doors?



After the host opens a door with a goat, what are the probabilities of the car being behind each door?



Exercises:

Grouping Doors

Now we have 6 doors. You pick 1 and the other 5 doors are divided into two groups: one with 2 doors and the other with 3 doors. He removes doors until each group has 1 door left. Do you switch? What do you switch to?

Macs and Monty

Suppose instead of the normal Monty Hall scenario in which we have two empty doors and a car residing behind the third we have a car behind one door, a Mac behind another, and nothing behind the third.

Let us assume that the contestant makes an initial pick at his/her discretion (random) and the host proceeds to ALWAYS open the empty door. When the contestant's initial choice corresponds to the empty door, the host will say so and the contestant must switch.

Does the typical Monty Hall paradox of 2/3 chance of obtaining the car by switching versus a 1/3 chance of obtaining the car by staying apply in this particular case?

Generalizing Monty

Now say we have n doors and there is a car behind one of them. Monty opens k doors, where $0 \leq k \leq n - 2$. Should you switch? Write an explicit formula for the probability of winning if you switch.