CS 70 Discrete Mathematics and Probability Theory
Spring 2017

FINAL

INSTRUCTIONS

- You have 3 hours to complete the exam.
- Mark your answers **on the exam itself**. We will *not* grade answers written on scratch paper.

Last name	
First name	
Student ID number	
BearFacts email (`_@berkeley.edu`)	
TA	
Room	
Seat	
Name of the person to your left	
Name of the person to your right	
All the work on this exam is my own. **(please sign)**	

1. (5 points) Minions Using Error Correction Codes

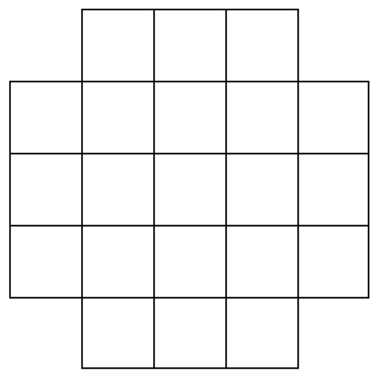
(a) Kevin wants to wants to send a message of 4 packets to Stuart and guard against 1 lost packet. Working over GF(7), he finds the unique polynomial P(x) that passes through the points he wants to send, and sends Stuart 5 packets: (0,P(0)), (1,P(1)), (2,P(2)), (3,P(3)), (4,P(4)). Stuart receives the following packets: (0,3), (1,0), (2,0), (4,0). What is the value of the missing packet? Calculate the exact value.

(b) Kevin wants to send a message of 60 ordered packets to Stuart. Those packets will first go through a hungry dragon who will eat at most one fifth of all packets (do not ask why the dragon loves eating packets). The remaining packets will then go through a powerful but malicious witch who will change the data of at most one eighth of all packets. Given this scenario, how many packets should Kevin send so that Stuart can recover the message?

2. (5 points) One, two, three

(a) How many ways can we roll 3 dice, such that they are in a strictly decreasing sequence?

(b) How many rectangles are in the following grid?



3. (5 points) High Expectations

Prove that $E[E[X|Y;Z]|Y] = E[X|Y]$.

4. (5 points) Parting Ways

Consider a finite undirected graph $G = (V, E)$, and a particle traversing this graph. At each time step, the particle on some node will transition to one of the node's neighbors with uniform probability. Notice that this is a Markov Chain. Consider a distribution π, with a probability for each node v, where $\pi[v] = \frac{d(v)}{2|E|}$ ($d(v)$ is the degree of v). Prove that π is a stationary distribution of this Markov Chain

5. (5 points) Bomb Bounds

(a) Let X be the sum of 20 i.i.d. Poisson Random Variables $X_1 \dots X_{20}$ with $\mathrm{E}[X_i] = 1$. Use Markov's inequality and Chebyshev's inequality to find two upper bounds on $Pr(X \geq 26)$

(b) Let Y be a Binomial Random Variable with n trials and an unknown probability of success p. Given that we dont know p, give the tightest bound possible using Chebyshev's on the probability that $Y \geq 5$

(c) Let Z be a normally distributed Random Variable (with mean μ and SD σ). Use Chebyshev's inequality to bound as tightly as possible the probability of falling more than $k \in R$ standard deviations above the mean.

6. (5 points) Continuously Raising Expectations

(a) A target is made of 3 concentric circles of radii $\frac{1}{\sqrt{3}}$, 1 and $\sqrt{3}$ feet. Shots within the inner circle are given 4 points, shots within the next ring are given 3 points, and shots within the third ring are given 2 points. Shots outside the target are given 0 points.Let X be the distance of the hit from the center (in feet), and let the probability density function of X be $f(x) = \frac{2}{\pi(1+x^2)}, x > 0, 0$ otherwise. What is the expected value of the score of a single shot? Dont solve; just set it up.

(b) Consider the following game. Spin a wheel and wait until it comes to rest at some x between 0 and 359. The amount of money won is $\frac{x}{36} - 6$ dollars. Let Y be a random variable for your winnings. First, define a probability density function. From there, calculate the expectation and variance.

7. (5 points) Linearly Estimate Me

The random variables X, Y, Z are i.i.d. $N(0, 1)$ (Recall the normal distribution has mean 0 and variance 1).

(a) Find $L[X^2 + Y^2 | X + Y]$

(b) Find $L[X + 2Y | X + 3Y + 4Z]$

8. (5 points) Diving Into Distributions

(a) Find the distribution of:

i. $Min(U_1, U_2)$ where $U_1, U_2 \sim Uniform[0, 1]$

ii. The sum of N i.i.d Geometric random variables with parameter p

(b) Expectations of distributions

i. $E[U_1|U_1 < U_2]$ where $U_1, U_2 \sim Uniform[0, 1]$

ii. $E[U_1|U_1 > U_2]$ where $U_1, U_2 \sim Uniform[0, 1]$

iii. Show that $\mathrm{E}[(X - t)^2] = \mathrm{E}[(X - \mu)^2] + (t - \mu)^2 = Var(X) + (t - \mu)^2$

iv. Find t such that the quantity $g(t) = \mathrm{E}[(X - t)^2]$ is minimized.