COUNTING, DISCRETE AND CONDITIONAL PROBABILITY 6

COMPUTER SCIENCE MENTORS 70

March 5 to March 9, 2018

1 Intro to Counting

1.1 Introduction

Counting:
In this class, the basic premise of counting is determining the total number of possible ways something can be done. Reaching a particular outcome requires a number of specific choices to be made. In counting, we determine the number of ways we can make these choices and reach different outcomes. To figure out the total number of outcomes, we multiply together the number of potential choices at each step.

1. You're getting ready in the morning, and you have to choose your outfit for the day.

   (a) You need to wear a necklace, a vest, and a sweater. Depending on the day, you decide whether it is worth wearing your watch. If you have 3 necklaces, 2 vests, and 4 sweaters, how many different combinations do you choose from each morning?

   Solution: $3 \cdot 2 \cdot 4 \cdot 2$ (# necklaces · # vests · # sweaters · wearing watch or not), These are all independent choices, so we can simply multiply the number of ways to make each choice together.

   (b) Now say that the order in which you put on your necklace, vest, and sweater matters. Specifically, your look after putting on necklace n, vest v, and then sweater s is different than if you put on vest v first, then necklace n, and then sweater s. Assume that when you put on your watch is irrelevant. Now how many options do you have?

Solution: $3 \cdot 2 \cdot 4 \cdot 2 \cdot 3! = 144$. We cannot simply multiply the outcomes like we did in the previous problem, because then we don't differentiate between (vest v, necklace n, sweater s) and (necklace n, vest v, sweater s) - we count them as the same case. We must count the number of ways to select a combination of 3 items, and then the number of ways to order each selection - 3!.

Ordering and Combinations:
An important idea of counting is dealing with situations in which all of our choices must be drawn from the same set. Here is a chart which walks you through how to solve problems relating to this idea:

Order matters, with replacement Example: How many 3 letter "words" can we make with the letters a, b, c, and d assuming we can repeat letters? Answer: $4^3 = 64$ General problem: From a set of n items, how many ways can we choose k of them, assuming that we can choose the same item multiple times and the order in which we choose the items matters? General Form: n^k	**Order matters, without replacement** Example: How many 3 letter "words" can we make with the letters a, b, c, d, e, and f using each letter exactly once? Answer: $\frac{6!}{(6-3)!} = 120$ General problem: From a set of n items, how many ways can we choose k of them, assuming that we can choose a given item exactly once and the order in which we choose the items matters? Answer: $P(n,k) = \frac{n!}{(n-k)!}$
Order doesn't matter, without replacement Example: How many ways can I pick a team of 3 from 7 possible people? Answer: $\frac{7!}{(7-3)!(7-4)!} = 35$ General problem: From a set of n items, how many ways can we choose k of them, assuming that we can choose a given item exactly once and the order that we choose the items doesn't matter? General Form: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$	**Special note: Sequencing** Example: How many different orderings are there of the letters in "CAT"? Answer: 3! How many different orderings are there of the letters in "BOOKKEEPER?"? Answer: $\frac{10!}{2!2!3!}$

1. How many ways are there to arrange the letters of the word SUPERMAN
   (a) On a straight line?

   Solution: 8!

   (b) On a straight line, such that SUPER occurs as a substring?

   Solution: 4! (treat SUPER as one character)

   (c) On a circle?

   Solution: 7! Anchor one element, arrange the other 7 around in a line

   (d) On a circle, such that SUPER occurs as a substring?

Solution: 3! Treat SUPER as a single character, anchor one element, arrange the other 3 around in a line

2. (a) You have 15 chairs in a room and there are 9 people. How many different ways can everyone sit down?

Solution: $\frac{15!}{6!}$ There are 15 places to put the first person, then 14 places to put the second person, 13 places to put the third person, etc. all the way to the last person who has 7 places to sit. Another way to think about this is like the anagram example above. We have 9 unique letters and 6 repeats (our empty spaces). We divide by the number of repeats giving us: $15 * 14 * 13 * 12 * 11 * 10 * 9 * 8 * 7 = \frac{15!}{6!}$

(b) How many ways are there to fill 9 of the 15 chairs? (We do not care who sits in them.)

Solution: $\binom{15}{9} = \frac{15!}{9!(15-9)!}$ In this example, we do not care about the uniqueness of each person, so we can just count each person as a repeat. So like the anagram example we will divide for every repeat. We have 9 human repeats, and 6 empty space repeats. Hence $\frac{15!}{9!6!}$.

3. The numbers 1447, 1005, and 1231 have something in common. Each of them is a four digit number that begins with 1 and has two identical digits. How many numbers like this are there?

Solution: Case 1: the identical digits are 1 (e.g. $11xy, 1x1y, 1xy1$)
Since there can only be two numbers that are identical, x and y cannot be 1 and $x \neq y$.
So [Possible formats] * [Possible x values] * [Possible y values] $= 3 * 9 * 8 = 216$
Case 2: identical digits are not 1 (e.g. $1xxy, 1xyx, 1yxx$)
So [Possible formats] * [Possible x values] * [Possible y values] $= 3 * 9 * 8 = 216$
Add both cases to arrive at the final result: $216 + 216 = 432$

4. We grab a deck of cards and it is poker time. Remember, in poker, order does not matter.

   (a) How many ways can we have a hand with exactly one pair? This means a hand with ranks (a, a, b, c, d).

   Solution: $= 13 * \binom{4}{2} * \binom{12}{3} * 4^3$. There are 13 value options for a (2, 3, 4, . . . , K, A). We then need to choose 2 out of the 4 possible suits. Now we need to choose b, c, and d. There are 12 values left (must be different from a). Finally, there are 4 suit options for each of the values chosen for b, c, and d.

   (b) How many ways can we have a hand with four of a kind? This means a hand with ranks (a, a, a, a, b).

   Solution: $= 13 * 12 * 4$

   (c) How many ways can we have a straight? A straight is 5 consecutive cards, that do not all necessarily have the same suit.

   Solution: A straight can begin at any number from 2-10: (2, 3, 4, 5, 6); (3, 4, 5, 6, 7)...(10, J, Q, K, A). That gives us 9 possibilities. Each number in hand has 4 possibilities (suits) $= 9 * (4^5)$ total possibilities.

   (d) How many ways can we have a hand of all of the same suit?

   Solution: $4 * \binom{13}{5}$. For each of the 4 suits, there are $\binom{13}{5}$ different combinations of 5 cards among 13 to choose from.

   (e) How many ways can we have a straight flush? This means we have a consecutive-rank hand of the same suit. For example, (2, 3, 4, 5, 6), all of spades, is a straight flush, while (2, 3, 5, 7, 8), all of spades, is NOT, as the ranks are not consecutive.

   Solution: For each of 4 suits, there are 9 number combinations (as shown in c, starting from 2 to starting from 10). Each number combination is unique, because there is only one number per suit. $= 4 * 9 = 36$.

5. The local library got itself a Twitter account! It appears to have been active for only a short time, but while it was running it diligently tweeted one 140-character message per day. Assume that it uses only the 26 letters of the English alphabet (plus the

period, comma, and space) and that any 140-character combination is possible (e.g. asdfasdf. . .as and ,,,,,,. . . ,,,, are both perfectly valid).

(a) How many possible tweets are there?

Solution: 29^{140}

(b) How many tweets use no spaces?

Solution: 28^{140}

(c) How many tweets consist entirely of whitespace?

Solution: 1

(d) Let T be some particular tweet. How many tweets differ from T by exactly one character?

Solution: $140 * 28 = 3920$

(e) How many have exactly six spaces and five commas?

Solution: $\binom{140}{6} * \binom{134}{5} * 27^{129} = \frac{140!}{6!5!128!} * 27^{129}$

2 Counting

2.1 Introduction

Balls and Bins:

Example Question: How many ways can we distribute 7 dollar bills amongst 3 students?

Approaching this with the approaches we currently know fails: There are 7 possible options for the number of bills you give to the first student, but the number of bills you choose to give the first student has a *direct* effect on the numbers of bills you can give to the second student - previously, if I had 7 options for the first student and choose one of the options, the second student always had 6 options to choose from. However, this is not the case in our example: if I choose to give the first student 5 dollars, for example, the second student can only get 1 or 2 dollar bills.

How do we solve this problem? We need to format it slightly differently: put the dollar bills on a line, and then try to insert 2 dividers. Everything to the left of the first divider is given to the first student. Everything in the middle of the 2 dividers is given to the second student. And everything to the right of the second divider is given to the 3rd student:
$| $$$$| $$
In the above example, the first student gets 1 dollar, the second 4, and the 3rd 2 dollars.

So we can see that the idea is to count how many ways we can arrange the 7 identical dollar bills and 2 identical dividers. Every permutation leads to some valid, distinct distribution of the money! From the previous sections we can see that we will have $\frac{9!}{7!2!} = \binom{9}{2} = 36$ ways to arrange the bills and dividers, and therefore 36 ways to distribute the money.
This tactic of using dividers is commonly referred to *stars and bars* or *sticks and stones*.

General problem: We want to distribute n indistinguishable items amongst k people.
General solution: $\binom{n+k-1}{k-1}$

Balls and Bins:
Distributing n distinguishable balls amongst k distinguishable bins: Each ball has k possible bins to go into, and there are n balls. Solution: k^n
Distributing n indistinguishable balls amongst k distinguishable bins: Classic stars-and-bars. Solution: $\binom{n+k-1}{k-1}$
Note: Distributing balls among indistinguishable bins is not covered in CS 70!

2.2 Questions

1. How many ways are there to arrange the letters of the word SUPERMAN
   (a) On a straight line, such that SUPER occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

   Solution: 3! * $\binom{8}{3}$ This reduces to a stars and bars problem–the S U P E R are bars, and we want to put M A N somewhere in the sequence. Once we do so, there can be any permutation of M A N within the bars. Equivalently, we can arrange the letters of SUPERMAN 8! ways, but divide by 5! because we have arranged SUPER in any of 5! ways, when we only want one way. This gives us 8! / 5!, which is equal to 3! * 8! / (5! 3!) = 3! * $\binom{8}{3}$.

   (b) On a circle, such that SUPER occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

   Solution: 3! * $\binom{7}{3}$. Anchor one element (for simplicity, choose M, A, or N). Then follow the same procedures as earlier.

2. How many ways can you give 10 cookies to 4 friends?

   Solution: Count the number of ways to give 10 cookies to 4 friends if some can get no cookies. The number of ways is $\binom{13}{3} = 286$.

3. How many solutions does $x+y+z = 10$ have, if all variables must be positive integers?

   Solution: We can think of this in terms of stars and bars. We have two bars between the variables x, y, and z, and our stars are the 10 1s we have to distribute among them. Since all variables must be positive integers, x, y, and z will each be at least 1. So, we have 7 1s left to distribute. So $n = 7$ stars, $k = 2$ bars. Answer = $\binom{n+k}{k} = \binom{9}{2} = 36$.

4. How many 5-digit sequences have the digits in non-decreasing order?

   Solution: This can be framed as a stars and bars problem. We have 9 bars between the numbers 0 through 9 and must place 5 stars in these slots. The location of a star represents the value of its associated digit. This ensures the 5 numbers in our sequence are either repeated or increasing. So, the answer is $\binom{14}{9} = 2002$. Note that if the question asked about increasing order, we would not use the stars and bars approach.

5. How many ways can you deal 13 cards to each of 4 players so that each player gets one card of each of the 13 values (A, 2, 3. . .K)?

Solution: There are $4!$ ways to distribute the aces to the 4 players, $4!$ ways to distribute the twos, and so on, so the number of ways to deal the cards in this manner is $4!^{13} = 24^{13}$.

3 Combinatorial Proofs

3.1 Questions

1. $n! = \binom{n}{k}k!(n-k)!$

Solution: Arrange n items.
LHS: Number of ways to order n items.
RHS: Choose k items without ordering. Order these k items. Order the remaining $n-k$ items.

2. $\sum_{k=0}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$

Solution: Number of ordered triplets of the form (i, j, k) where i and j are less than or equal to k for every k from 0 to n.
LHS: For each k there are k options for i and k options for j so k^2 options for all.
RHS: Consider the case where $i = j$. Then we must choose two numbers from $\{0, \ldots, n\}$ which amounts to $\binom{n+1}{2}$. If $i \neq j$ then we choose 3 numbers from $n+1$. But i can be less than j or greater than j so we must multiply by 2.

3. Prove $a(n-a)\binom{n}{a} = n(n-1)\binom{n-2}{a-1}$ by a combinatorial proof.

Solution: Suppose that you have a group of n players.
LHS: Number of ways to pick a team of a of these players, designate one member of the team as captain, and then pick one reserve player from the remaining $n-a$ people.
RHS: The right-hand side is the number of ways to pick the captain, then the reserve player, and then the other $a-1$ members of the team.

3.2 Challenge

1. Prove the Hockey Stick Theorem:

$$\sum_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1}$$

where n, t are natural numbers and $n > t$.

Solution: You have previously shown the following:
$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
Plug this in:
$\binom{t+1}{k+1} = \binom{t}{k} + \binom{t}{k+1} \implies \binom{t}{k} = \binom{t+1}{k+1} - \binom{t}{k+1}$
Now do the summation:
$\sum\limits_{t=k}^{n} \binom{t}{k} = \sum\limits_{t=k}^{n} \binom{t+1}{k+1} - \sum\limits_{t=k}^{n} \binom{t}{k+1}$
Split apart the two summations as follows: take out the last term in the first summation and the last term in the second summation.
$\sum\limits_{t=k}^{n} \binom{t}{k} = \left(\sum\limits_{t=k}^{n-1} \binom{t+1}{k+1} + \binom{n+1}{k+1}\right) - \left(\sum\limits_{t=k+1}^{n} \binom{t}{k+1} + \binom{k}{k+1}\right)$
Look at all subsets of $\{1, 2, 3, \ldots, 2015\}$ that have 1000 elements. Choose the smallest element from each subset. Find the average of all least elements. Since $\binom{k}{k+1}$ is 0 we can just remove this term:
$\sum\limits_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1} + \sum\limits_{t=k}^{n-1} \binom{t+1}{k+1} - \sum\limits_{t=k+1}^{n} \binom{t}{k+1}$
We want the summations to match. So change t such that the new t, t' goes from $k+1$ to $n-1$. So let t' = t - 1. Then $t = k+1 \rightarrow t' = k$ and $t = n \rightarrow t' = n-1$.
$\sum\limits_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1} + \sum\limits_{t=k}^{n-1} \binom{t+1}{k+1} - \sum\limits_{t'=k}^{n-1} \binom{t'+1}{k+1}$
Notice that the summations cancel out. We are left with the statement we were trying to prove.

4 Discrete Probability

4.1 Introduction

1. What is a sample (event, outcome) space?

   Solution: The set of all possible outcomes.

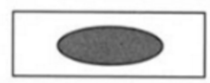
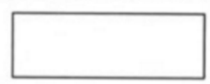
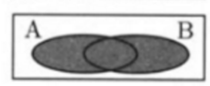
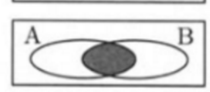
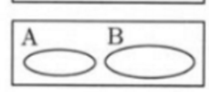
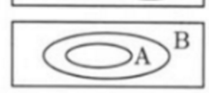
2. What is an event?

   Solution: A partition of the sample space.

3. Given a uniform probability space Ω such that $|\Omega| = N$, how many events are possible?

   Solution: 2^N. Each point is either included or excluded from any particular subset, and the number of subsets is the number of events.

Figure 1: From Pitman

Event language	Set language	Set notation	Venn diagram
outcome space	universal set	Ω	
event	subset of Ω	A, B, C, etc.	
impossible event	empty set	$\emptyset$	
not A, opposite of A	complement of A	A^c	
either A or B or both	union of A and B	$A \cup B$	
both A and B	intersection of A and B	AB, $A \cap B$	
A and B are mutually exclusive	A and B are disjoint	$AB = \emptyset$	
if A then B	A is a subset of B	$A \subseteq B$	

4.2 Questions

1. **Probably Poker**

   (a) What is the probability of drawing a hand with a pair?

   Solution: $\frac{13*\binom{4}{2}*\binom{12}{3}*4^3}{\binom{52}{5}}$

   (b) What is the probability of drawing a hand with four of a kind?

   Solution: $\frac{13*12*4}{\binom{52}{5}}$

   (c) What is the probability of drawing a straight?

   Solution: $\frac{9*4^5}{\binom{52}{5}}$

   (d) What is the probability of drawing a hand of all of the same suit?

   Solution: $\frac{4*\binom{13}{5}}{\binom{52}{5}}$

   (e) What is the probability of drawing a straight flush?

   Solution: $\frac{4*9}{\binom{52}{5}}$

2. Suppose you arrange 12 different cars in a parking lot, uniformly at random. Three of the cars are Priuses, four of the cars are Teslas, and the other five are Nissan Leaves. What is the probability that the three Priuses are all together?

   Solution: There are $12!$ possible ways to arrange the 12 cars. Now, there are $12 - 3 + 1 = 10$ different places the Priuses could go (positions 1,2,3, positions 2,3,4, all the way until positions 10,11,12). For each of those 10 places, there are 3! ways to arrange the Priuses and 9! ways to arrange the other 9 cars in the 9 remaining spots. So, in total, there are $10 * 3! * 9!$ ways of arranging the cars so that the 3 Priuses are together. So the probability we get what we want is $\frac{10*3!*9!}{12!} = .0455$.

5 Conditional Probability

5.1 Introduction

Bayes' Rule

$$\mathrm{P}[A|B] = \frac{\mathrm{P}[B|A]P[A]}{\mathrm{P}[B]}$$

Total Probability Rule

$$\mathrm{P}[B] = \mathrm{P}[A \cap B] + \mathrm{P}[\bar{A} \cap B] = \mathrm{P}[B|A] * \mathrm{P}[A] + \mathrm{P}[B|\bar{A}] * (1 - \mathrm{P}[A])$$

Independence
Two events A, B in the same probability space are independent if

$$\mathrm{P}[A \cap B] = \mathrm{P}[A] * \mathrm{P}[B]$$

5.2 Questions

1. Say that we have a bag with 2 coins. One of the coins is fake: it has heads on both faces. The other is a normal coin. You pull a coin out of the bag, and flip it. It comes up heads. What is the probability that this coin is the fake one?

   Solution: Define the event A as drawing the fake coin. Define the event B as the coin that we flipped coming up heads. We are looking for the probability of $\mathrm{P}[A|B]$ - the probability that we drew the fake coin, given that the flipped coin came up heads. We can directly plug this into Bayes' rule: $\mathrm{P}[A|B] = \frac{\mathrm{P}[A \cap B]}{\mathrm{P}[B]}$. $\mathrm{P}[A \cap B] = \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{2}$. $\mathrm{P}[B] = \frac{1}{2} \cdot \frac{1}{1} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$. So, we get $\frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$.

2. You have a deck of 52 cards. What is the probability of:

   (a) Drawing 2 Kings with replacement?

   Solution: There are $52 * 52$ combinations of two cards. There are $4 * 4$ that are 2 Kings, so the probability is $\frac{1}{13^2}$.

   (b) Drawing 2 Kings without replacement?

Solution: There are $\binom{52}{2} = 52*51$ pairs of cards possible without replacement. Of these $4*3$ represent pairs of kings. Therefore the probability is $\frac{4*3}{52*51}$. We can also use conditional probability (which we will have to use in the next part):

$$P(K \text{ on 2nd and } K \text{ on 1st}) = P(\text{K on 2nd}|\text{K on 1st})P(\text{K on 1st}) = \frac{3}{51} * \frac{4}{52}$$

.

(c) The second card is a King without replacement?

Solution:

$$\begin{aligned} P(\text{K on 2nd}) &= P(K \text{ on 2nd } | K \text{ on 1st})P(K \text{ on 1st}) \\ &+ P(K \text{ on 2nd } | \text{ no } K \text{ on 1st})P(\text{no } K \text{ on 1st}) \\ &= \frac{3}{51} * \frac{4}{52} + \frac{4}{51} * \frac{48}{52} \\ &= \frac{4}{52} \end{aligned} \tag{1}$$

Note that this is the same as the $P(K \text{ on 1st})$, because a K is equally likely to be anywhere in the deck.

(d) **[EXTRA]** The nth card is a King without replacement ($n < 52$)?

Solution: The last problem is a hint that we can argue this by symmetry. Since we have no information about what any of the preceding cards were before the nth card, it is equally likely that the nth card is any of the 52 possible cards, so the probability that it is a King is $\frac{4}{52}$.

3. Find an example of 3 events A, B, and C such that each pair of them are independent, but they are not mutually independent.

Solution: Consider a fair 4-sided die. Let A be the event that 1 or 2 appears in a die roll, B be the event that 1 or 3 appears, and C be the event that 1 or 4 appears. Then,

$$Pr(A) = Pr(B) = Pr(C) = \frac{1}{2}$$

Furthermore,

$$Pr(A \cap B) = Pr(1 \text{ appears}) = \frac{1}{4} = Pr(A)Pr(B).$$

So A and B are pairwise independent. Similarly (A, C) and (B, C) are pairwise independent. However,

$$Pr(A \cap B \cap C) = Pr(1 \text{ appears}) = \frac{1}{4} = Pr(A)Pr(B)Pr(C) = \frac{1}{8}$$

So these 3 events are not mutually independent.The answer is not unique; any other valid answer is acceptable.