Counting, Discrete and Conditional Probability 6

Computer Science Mentors 70

March 5 to March 9, 2018

1 Intro to Counting

1.1 Introduction

Counting:
In this class, the basic premise of counting is determining the total number of possible ways something can be done. Reaching a particular outcome requires a number of specific choices to be made. In counting, we determine the number of ways we can make these choices and reach different outcomes. To figure out the total number of outcomes, we multiply together the number of potential choices at each step.

1. You're getting ready in the morning, and you have to choose your outfit for the day.

   (a) You need to wear a necklace, a vest, and a sweater. Depending on the day, you decide whether it is worth wearing your watch. If you have 3 necklaces, 2 vests, and 4 sweaters, how many different combinations do you choose from each morning?

   (b) Now say that the order in which you put on your necklace, vest, and sweater matters. Specifically, your look after putting on necklace n, vest v, and then sweater s is different than if you put on vest v first, then necklace n, and then sweater s. Assume that when you put on your watch is irrelevant. Now how many options do you have?

Ordering and Combinations:
An important idea of counting is dealing with situations in which all of our choices must be drawn from the same set. Here is a chart which walks you through how to solve problems relating to this idea:

Order matters, with replacement Example: How many 3 letter "words" can we make with the letters a, b, c, and d assuming we can repeat letters? Answer: $4^3 = 64$ General problem: From a set of n items, how many ways can we choose k of them, assuming that we can choose the same item multiple times and the order in which we choose the items matters? General Form: n^k	**Order matters, without replacement** Example: How many 3 letter "words" can we make with the letters a, b, c, and d using each letter exactly once? Answer: 4!/(4-3)! = 24 General problem: From a set of n items, how many ways can we choose k of them, assuming that we can choose a given item exactly once and the order in which we choose the items matters? Answer: $P(n,k) = n!/(n-k)!$
Order doesn't matter, without replacement Example: How many ways can I pick a team of 3 from 7 possible people? Answer: 7!/((7-3)!(7-4)!) = 35 General problem: From a set of n items, how many ways can we choose k of them, assuming that we can choose a given item exactly once and the order that we choose the items doesn't matter? General Form: $C(n,k) = n!/((n-k)!k!)$	**Special note: Sequencing** Example: How many different orderings are there of the letters in "CAT"? Answer: 3! How many different orderings are there of the letters in "BOOKKEEPER?"? Answer: 10!/(2!2!3!)

1. How many ways are there to arrange the letters of the word SUPERMAN
   (a) On a straight line?
   (b) On a straight line, such that SUPER occurs as a substring?
   (c) On a circle?
   (d) On a circle, such that SUPER occurs as a substring?

2. (a) You have 15 chairs in a room and there are 9 people. How many different ways can everyone sit down?
   (b) How many ways are there to fill 9 of the 15 chairs? (We do not care who sits in them.)

3. The numbers 1447, 1005, and 1231 have something in common. Each of them is a four digit number that begins with 1 and has two identical digits. How many numbers like this are there?

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4. We grab a deck of cards and it is poker time. Remember, in poker, order does not matter.
   (a) How many ways can we have a hand with exactly one pair? This means a hand with ranks (a, a, b, c, d).
   (b) How many ways can we have a hand with four of a kind? This means a hand with ranks (a, a, a, a, b).
   (c) How many ways can we have a straight? A straight is 5 consecutive cards, that do not all necessarily have the same suit.
   (d) How many ways can we have a hand of all of the same suit?
   (e) How many ways can we have a straight flush? This means we have a consecutive-rank hand of the same suit. For example, (2, 3, 4, 5, 6), all of spades, is a straight flush, while (2, 3, 5, 7, 8), all of spades, is NOT, as the ranks are not consecutive.

5. The local library got itself a Twitter account! It appears to have been active for only a short time, but while it was running it diligently tweeted one 140-character message per day. Assume that it uses only the 26 letters of the English alphabet (plus the period, comma, and space) and that any 140-character combination is possible (e.g. asdfasdf. . .as and ,,,,,,. . . ,,,, are both perfectly valid).
   (a) How many possible tweets are there?
   (b) How many tweets use no spaces?
   (c) How many tweets consist entirely of whitespace?
   (d) Let T be some particular tweet. How many tweets differ from T by exactly one character?
   (e) How many have exactly six spaces and five commas?

2 Counting

2.1 Introduction

Balls and Bins:

Example Question: How many ways can we distribute 7 dollar bills amongst 3 students?

Approaching this with the approaches we currently know fails: There are 7 possible options for the number of bills you give to the first student, but the number of bills you choose to give the first student has a *direct* effect on the numbers of bills you can give to the second student - previously, if I had 7 options for the first student and choose one of the options, the second student always had 6 options to choose from. However, this is not the case in our example: if I choose to give the first student 5 dollars, for example, the second student can only get 1 or 2 dollar bills.

How do we solve this problem? We need to format it slightly differently: put the dollar bills on a line, and then try to insert 2 dividers. Everything to the left of the first divider is given to the first student. Everything in the middle of the 2 dividers is given to the second student. And everything to the right of the second divider is given to the 3rd student:
$| $$$$| $$
In the above example, the first student gets 1 dollar, the second 4, and the 3rd 2 dollars.

So we can see that the idea is to count how many ways we can arrange the 7 identical dollar bills and 2 identical dividers. Every permutation leads to some valid, distinct distribution of the money! From the previous sections we can see that we will have $\frac{9!}{7!2!} = \binom{9}{2} = 36$ ways to arrange the bills and dividers, and therefore 36 ways to distribute the money.
This tactic of using dividers is commonly referred to *stars and bars* or *sticks and stones*.

General problem: We want to distribute n indistinguishable items amongst k people.
General solution: $\binom{n+k-1}{k}$

Balls and Bins:
Distributing n distinguishable balls amongst k distinguishable bins: Each ball has k possible bins to go into, and there are n balls. Solution: k^n
Distributing n indistinguishable balls amongst k distinguishable bins: Classic stars-and-bars. Solution: $\binom{n+k-1}{k}$
Note: Distributing balls among indistinguishable bins is not covered in CS 70!

2.2 Questions

1. How many ways are there to arrange the letters of the word SUPERMAN
   (a) On a straight line, such that SUPER occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?
   (b) On a circle, such that SUPER occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

2. How many ways can you give 10 cookies to 4 friends?

3. How many solutions does $x+y+z = 10$ have, if all variables must be positive integers?

4. How many 5-digit sequences have the digits in non-decreasing order?

5. How many ways can you deal 13 cards to each of 4 players so that each player gets one card of each of the 13 values (A, 2, 3. . .K)?

3 Combinatorial Proofs

3.1 Questions

1. $n! = \binom{n}{k}k!(n-k)!$

2. $\sum\limits_{k=0}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$

3. Prove $a(n-a)\binom{n}{a} = n(n-1)\binom{n-2}{a-1}$ by a combinatorial proof.

3.2 Challenge

1. Prove the Hockey Stick Theorem:

$$\sum_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1}$$

where n, t are natural numbers and $n > t$.

4 Discrete Probability

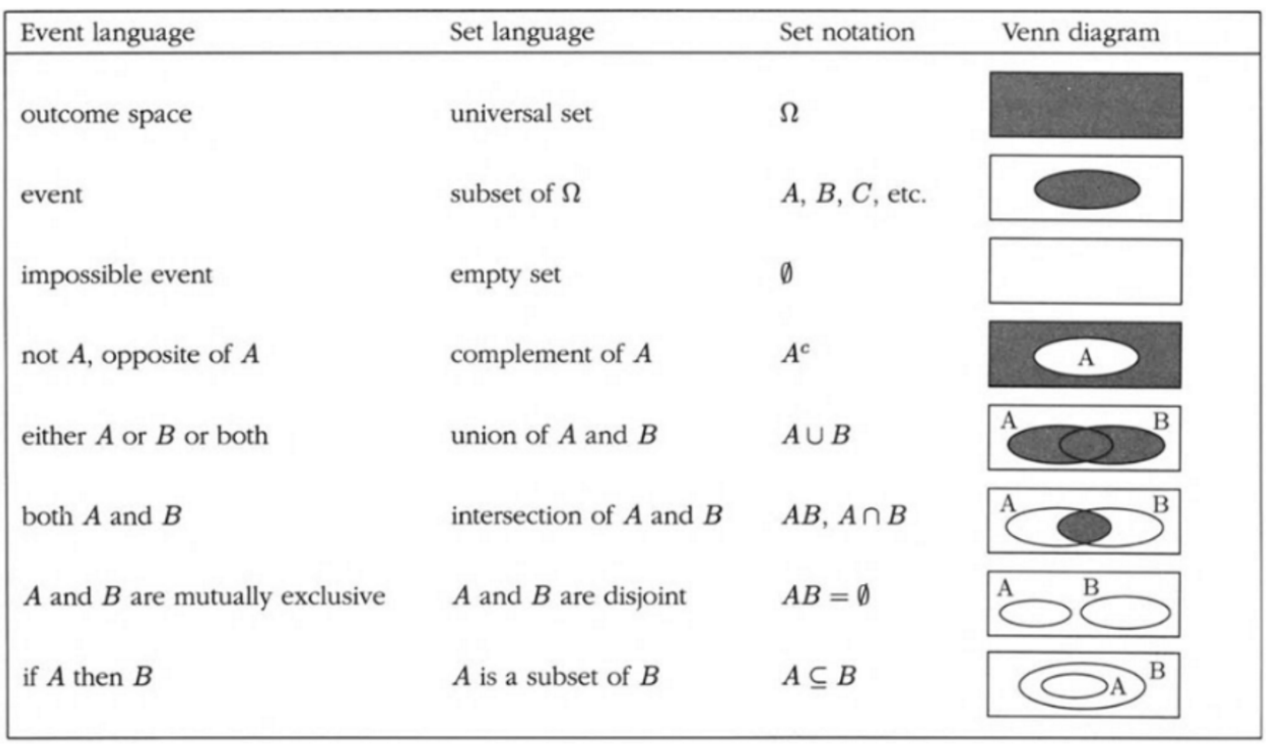
4.1 Introduction

1. What is a sample (event, outcome) space?

2. What is an event?

3. Given a uniform probability space Ω such that $|\Omega| = N$, how many events are possible?

Figure 1: From Pitman



Event language	Set language	Set notation	Venn diagram
outcome space	universal set	Ω	
event	subset of Ω	A, B, C, etc.	
impossible event	empty set	$\emptyset$	
not A, opposite of A	complement of A	A^c	A
either A or B or both	union of A and B	$A \cup B$	A B
both A and B	intersection of A and B	AB, $A \cap B$	A B
A and B are mutually exclusive	A and B are disjoint	$AB = \emptyset$	A B
if A then B	A is a subset of B	$A \subseteq B$	A B

4.2 Questions

1. **Probably Poker**

   (a) What is the probability of drawing a hand with a pair?

   (b) What is the probability of drawing a hand with four of a kind?

   (c) What is the probability of drawing a straight?

   (d) What is the probability of drawing a hand of all of the same suit?

   (e) What is the probability of drawing a straight house?

2. Suppose you arrange 12 different cars in a parking lot, uniformly at random. Three of the cars are Priuses, four of the cars are Teslas, and the other five are Nissan Leaves. What is the probability that the three Priuses are all together?

5 Conditional Probability

5.1 Introduction

Bayes' Rule

$$\mathrm{P}[A|B] = \frac{\mathrm{P}[B|A]P[A]}{\mathrm{P}[B]}$$

Total Probability Rule

$$\mathrm{P}[B] = \mathrm{P}[A \cap B] + \mathrm{P}[\bar{A} \cap B] = \mathrm{P}[B|A] * \mathrm{P}[A] + \mathrm{P}[B|\bar{A}] * (1 - \mathrm{P}[A])$$

Independence
Two events A, B in the same probability space are independent if

$$\mathrm{P}[A \cap B] = \mathrm{P}[A] * \mathrm{P}[B]$$

5.2 Questions

1. Say that we have a bag with 2 coins. One of the coins is fake: it has heads on both faces. The other is a normal coin. You pull a coin out of the bag, and flip it. It comes up heads. What is the probability that this coin is the fake one?

2. You have a deck of 52 cards. What is the probability of:
   (a) Drawing 2 Kings with replacement?
   (b) Drawing 2 Kings without replacement?
   (c) The second card is a King without replacement?
   (d) **[EXTRA]** The nth card is a King without replacement ($n < 52$)?

3. Find an example of 3 events A, B, and C such that each pair of them are independent, but they are not mutually independent.