STABLE MARRIAGE, WELL ORDERING PRINCIPLE, OPTIMALITY, GRAPHS 1

COMPUTER SCIENCE MENTORS 70

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1 Stable Marriage

1.1 Introduction

Given a set of n men and n women, and a list of each person's preferences, how can we create pairs of one man and one woman, such that everyone is content with their pairing?

The Algorithm:

1. **Every Morning**: Each man proposes to the <u>most preferred</u> woman on his list who has not yet rejected him.
2. **Every Afternoon**: Each woman collects all the proposals she received in the morning; to the man she likes best, she responds maybe, come back tomorrow (she now has him <u>on a string</u>), and to the others, she says never.
3. **Every Evening**: Each rejected man crosses off the woman who rejected him from his list. The above loop is repeated each successive day until <u>each</u> woman has a man on a string; on this day, each woman marries the man she has on a string.

Definitions:

- M and W are a rogue couple if they prefer to be with each other as opposed to the people they are paired with
- A pairing is stable if there are no rogue couples

Lemmas:

- The algorithm halts.
- "The Improvement Lemma" — If man M proposes to woman W on the kth day, then on every subsequent day W has someone on a string whom she likes at least as much as M.

1.2 Questions

1. Lemma: Algorithm terminates with a pairing.

2. Lemma: The pairing is stable.

2 Well Ordering Principle

2.1 Introduction

The well-ordering principle states that for every non-empty subset S of the set of natural numbers N, there is a smallest element $x \in S$; i.e. $\exists x : \forall y \in S : x \leq y$

2.2 Questions

1. In this question, we will go over how the well-ordering principle can be derived from (strong) induction.
   1. What is the significance of S being non-empty? Does WOP hold without it? Assuming that S is not empty is equivalent to saying that there exists some number z in it.
   2. Induction is always stated in terms of a property that can only be a natural number. What should the induction be based on?
   3. Now that the induction variable is clear, formally state the induction hypothesis.

4. Verify the base case.

5. Now prove that the induction works, by writing the inductive step.

6. What should you change so that the proof works by simple induction (as opposed to strong induction)?

3 Optimal, Pessimal

3.1 Introduction

For any given person, their optimal partner is their most preferred partner among possible partners in stable pairings. A male optimal pairing is a pairing in which all males are paired with their optimal women.
Lemma: If a pairing is male optimal, then it is also female pessimal.

3.2 Questions

1. Theorem: The pairing produced by the stable marriage algorithm is male optimal.

It should be noted that even though the TMA creates pairings that are male optimal, these pairings can also be male pessimal. Consider the following preferences:

- M_1: $W_1 > W_2$
- M_2: $W_2 > W_1$
- W_1: $M_1 > M_2$
- W_2: $M_2 > M_1$

In this case, there is only one possible stable pairing $\{(M_1, W_1), (M_2, W_2)\}$. That means this pairing is both the best possible and worst possible, for both the men and women.

4 More Practice

4.1 Questions

1. Imagine that in the context of stable marriage all men have the same preference list. That is to say there is a global ranking of women, and men's preferences are directly determined by that ranking. Use any method of proof to answer the following questions.

    1. Prove that the first woman in the ranking has to be paired with her first choice in any stable pairing.

    2. Prove that the second woman has to be paired with her first choice if that choice is not the same as the first woman's first choice. Otherwise she has to be paired with her second choice.

    3. Continuing this way, assume that we have determined the pairs for the first $k-1$ women in the ranking. Who should the k-th woman be paired with?

4. Prove that there is a unique stable pairing.

5 Graph Theory

5.1 Introduction

1. Let $G = (V, E)$ be an undirected graph. Match the term with the definition.

Walk	Cycle	Tour	Path

________________Walk that starts and ends at the same node
________________Sequence of edges.
________________Sequences of edges with possibly repeated vertex or edge.
________________Sequence of edges that starts and ends on the same vertex and does not repeat vertices (except the first and last)

2. Suppose we want to represent a round-robin tennis tournament in which every player plays one match against every other player. How might we represent this using a graph?

3. What is a simple path?