CONTINUOUS PROBABILITY, MARKOV CHAINS, CONDITIONAL EXPECTATION

COMPUTER SCIENCE MENTORS 70

November 28 to December 2, 2016

1 Continuous Probability

1.1 Questions

1. Given the following density functions, identify if they are valid random variables. If yes, find the expectation and variance. If not, what rules does the variable violate?

   (a) $f(x) = \begin{cases} \frac{1}{4} & \text{if } x \in \{\frac{1}{2}, \frac{9}{2}\} \\ 0 & \text{otherwise} \end{cases}$

   Solution: Yes. Is non-negative and area sums to 1. $\text{E}[X] = \frac{5}{2}$ $\text{Var}[X] = \frac{4}{3}$

   (b) $f(x) = \begin{cases} x - \frac{1}{2} & x \in \{0, \infty\} \end{cases}$

   Solution: No. Has negative values on $(0, \frac{1}{2})$

2. For a discrete random variable X we have $\Pr[X \in [a, b]]$ that we can calculate directly by finding how many points in the probability space fall in the interval and how many total points are in the probability space. How do we find $\Pr[X \in [a, b]]$ for a continuous random variable?

   Solution: For a continuous RV with probability density function $f(x)$, the probability that X takes on a value between a and b is the area under the pdf from a to b, which is the integral from a to b of $f(x)$.

3. Are there any values of a, b for the following functions which gives a valid pdf? If not, why? If yes, what values?

   (a) $f(x) = -1,\ a < x < b$

   Solution: No. $f(x) \geq 0$ must be true.

   (b) $f(x) = 0,\ a < x < b$

   Solution: No. $\forall a, b.\ \int_a^b 0 = 0$.

   (c) $f(x) = 10000,\ a < x < b$

   Solution: Yes, $\int_0^a 10000 = 1 = 10000a - 0 = 1 \implies a = \frac{1}{10000}$

4. For what values of the parameters are the following functions probability density functions? What is the expectation and variance of the random variable that the function represents?

   (a) $f(x) = \begin{cases} ax & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

   Solution: For a function to represent a probability density function, we need to have that the integral of the function from negative infinity to positive infinity to equal 1 and for f(x) to be greater than or equal to 0. So we need integral over $(-\infty, \infty)$ of $f(x) = 1 = \int_0^1 ax = \frac{ax^2}{2} \mid_0^1 = 1 \iff \frac{a}{2} - 0 = 1 \iff a = 2$
   For RV Y with pdf = $f(x)$, E$(Y) = \int_{-\infty}^{\infty} x \times f(x) = \int_0^1 x \times 2x = \frac{2x^3}{3} \mid_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$
   Var$(Y) = \int_{-\infty}^{\infty} x^2 \times f(x) - \text{E}[Y]^2 = \int_0^1 x^2 \times 2x - \frac{4}{9} = \int_0^1 2x^3 - \frac{4}{9} = \frac{x^4}{2} \mid_0^1 = \frac{1}{2} - 0 - \frac{4}{9} = \frac{1}{18}$

   (b) $f(x) = \begin{cases} -2x & \text{if } a < x < b\ (a = 0 \vee b = 0) \\ 0 & \text{otherwise} \end{cases}$

   Solution: Again we need $f(x) \geq 0$, so here $a, b \leq 0$, so $b = 0$. Then $\int_a 0 f(x) = 1 = \int_a^0 -2x = \frac{-2x^2}{2} \mid_a^0$
   $= 0 - (\frac{-2a^2}{2}) = \frac{2a^2}{2} = 1 \iff a^2 = 1 \iff a$
   $= \pm 1 \implies a = -1$.

For RV Y with pdf = $f(x)$,
E$(Y) = \int_{-\infty}^{\infty} x \times f(x) = \int_{-1}^{0} x \times (-2x) = \frac{-2x^3}{3} \mid_{-1} 0 = 0 - (\frac{(-2)(-1)^3}{3}) = -\frac{2}{3}$.
$Var(Y) = \int_{\infty}^{-\infty} x^2 * f(x) = \int_{0}^{-1} x^2 * (-2x) = -x^4/2|_0^{-1} = 0 - (-(-1)4)/2 = \frac{1}{2}$
$f(x) = \begin{cases} c & -30 < x < -20 \vee -5 < x < 5 \vee 60 < x < 70 \\ 0 & \text{otherwise} \end{cases}$
We need $\int_{-\infty}^{\infty} f(x) = 1$ and $f(x) \geq 0$. So $c \geq 0$
$\int_{\infty}^{\infty} f(x) = 1 = \int_{-30}^{-20} c + \int_{-5}^{5} c + \int_{60}^{70} c = cx \mid_{-30}^{-20} + cx \mid_{-5}^{5} + cx \mid_{60}^{70}$
$= 10c + 10c + 10c = 30c = 1 \implies \frac{1}{30}$ For RV Y with pdf = $f(x)$,
Dont worry too much about calculations, but you should be able to set up the equations

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} x * f(x) \\ &= \int_{-30}^{-20} xc + \int_{-5}^{5} xc + \int_{60}^{70} xc \\ &= \frac{x^2c}{2} \mid_{-30}^{-20} + \frac{x^2c}{2} \mid_{-5}^{5} + \frac{x^2c}{2} \mid_{60}^{70} \\ &= \frac{(-30)^2c}{2} - \frac{(-20)^2c}{2} + \frac{5^2c}{2} - \frac{(-5)^2c}{2} + \frac{70^2c}{2} - \frac{60^2c}{2} \\ &= 900c = \frac{900}{30} = 30 \end{aligned}$$

$$\text{Var}(Y) = \int_{-\infty}^{\infty} x^2 f(x)$$

$$\begin{aligned} &= \int_{-20} {}^{3}0x^2c + \int_{-5}^{5} x^2c + \infty_{60}^{70} x^2c \\ &= \frac{x^3c}{3} \mid_{-30}^{20} + \frac{x^3c}{3} \mid_{-5}^{5} + \frac{x^3c}{3} \mid 60^{70} \\ &= \frac{(-30)^3c}{3} - \frac{(-20)^3c}{3} + \frac{5^3c}{3} - \frac{(-5)^3c}{3} + \frac{70^3c}{3} - \frac{60^3c}{3} \\ &= \frac{108250c}{3} = 1202.77\ldots \end{aligned}$$

5. Define a continuous random variable R as follows: we pick a random point on a disk of radius 1; the value of R is distance of this point from the center of the disk. We will find the probability density function of this random variable.
   (a) What is (should be) the probability that R is between 0 and $\frac{1}{2}$? Why?

Solution: $\frac{1}{4}$, because the area of the circle with distance between 0 and $\frac{1}{2}$ is $(\pi(\frac{1}{2})^2 = \frac{\pi}{4})$, and the area of the entire circle is π.

(b) What is (should be) the probability that R is between a and b, for any $0 \leq a \leq b \leq 1$?

Solution: The area of the region containing these points is the area of the outer circle minus the area of the inner circle, or $\pi b^2 - \pi a^2 = \pi(b^2 - a^2)$. The probability that a point is within this region, rather than the entire circle, is $\frac{\pi(b^2-a^2)}{\pi} = b^2 - a^2$.

(c) What is a function $f(x)$, for which $\int_a^b f(x)dx$ satisfies these same probabilities?

Solution: $f(x) = 2x$ because $\int_a^b f(x)dx = [x^2]_a^b = b^2 - a^2$.

2 Continuous Distributions

2.1 Introduction

Uniform Distribution: $U(a, b)$ This is the distribution that represents an event that randomly happens at any time during an interval of time.

- $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$
- F(x) = 0 for $x < a$, $\frac{x-a}{b-a}$ for $a < x < b$, 1 for $x > b$
- $\mathrm{E}(x) = \frac{a+b}{2}$
- $\mathrm{Var}(x) = \frac{1}{12}(b-a)^2$

Exponential Distribution: $Expo(\lambda)$ This is the continuous analogue of the geometric distribution, meaning that this is the distribution of how long it takes for something to happen if it has a rate of occurrence of λ.

- memoryless
- $f(x) = \lambda * e^{-\lambda * x}$
- $F(x) = 1 - e^{\lambda x}$
- $\mathrm{E}(x) = \frac{1}{\lambda}$

Gaussian (Normal) Distribution: $N(\mu, \sigma^2)$

- The CLT states that any unspecified distribution of events will converge to the Gaussian as n increases
- Mean: μ
- Variance: σ^2
- $f(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{(2\pi)}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

2.2 Questions

1. There are on average 8 office hours in a day. The scores of an exam followed a normal distribution with an average of 50 and standard deviation of 6. If a student waits until an office hour starts, what is the expected value of the sum of the time they wait in hours and their score on the exam?

   Solution: E(waiting time) = E(exp(lambda=8/24)) = 3
   E(score) = E(normal(50, 36)) = 50
   By linearity of expectation, the sum is 53.

2. Every day, 100,000,000,000 cars cross the Bay Bridge, following an exponential distribution.

   (a.) What is the expected amount of time between any two cars crossing the bridge?

Solution: $\frac{1}{100{,}000{,}000{,}000}$ days

(b.) Given that you havent seen a car cross the bridge for 5 minutes, how long should you expect to wait before the next car crosses?

Solution: $\frac{1}{100{,}000{,}000{,}000}$ days

3. There are certain jellyfish that dont age called hydra. The chances of them dying is purely due to environmental factors, which well call λ. On average, 2 hydras die within 1 day.

(a) What is the probability you have to wait at least 5 days for a hydra dies?

Solution: $\lambda = 2, X \sim Exp(2)$
$P(X \leq 5) = \int_5^\infty \lambda e^{-\lambda x} dx = \int_5^\infty 2e^{-2x} dx = -e^{-2x}|_5^\infty = e^{-10} = \frac{1}{e^1 0}$

(b) Let X and Y be two independent discrete random variables. Derive a formula for expressing the distribution of the sum S = X + Y in terms of the distributions of X and of Y.

Solution: $P(S = m) = \sum_{i=-\infty}^{\infty} P(X = i)P(Y = m - i)$

(c) Use your formula in part (a) to compute the distribution of S = X +Y if X and Y are both discrete and uniformly distributed on 1,...,K.

Solution: $P(S = m) = \sum_{i=0}^{m} (1/K)(1/K) = m/K^2$

(d) Suppose now X and Y are continuous random variables with densities f and g respectively (X,Y still independent). Based on part (a) and your understanding of continuous random variables, give an educated guess for the formula of the density of S = X +Y in terms of f and g.

Solution: $h(t) = \int_{-\infty}^{\infty} f(s)g(t - s)ds$

(e) Use your formula in part (c) to compute the density of S if X and Y have both uniform densities on [0, a].

Solution: Since $f(s)$ is $\frac{1}{a}$ only when $s \in [0, a]$, and 0 everywhere else, we can simplify it to $h(t) = \int_0^a \frac{1}{a} g(t-s)ds$. Consider the case where $t \in [0, a]$. Then $g(t-s)$ will be nonzero (and equal to $\frac{1}{a}$ only when $s \leq t$), so we can further simplify $h(t) = \int_0^t \frac{1}{a}\frac{1}{a} ds = \frac{t}{a^2}$.
Now consider the case where $t \in (a, 2a]$. If so, then $g(t-s)$ is always $\frac{1}{a}$ if $t - s \geq 0$ and $t - s \leq a$ and 0 otherwise. Equivalently, we make sure that $s \leq t$ and $s \geq t - a$. However, recall that we already assumed that $s \leq a$ (or else $f(s) = 0$), so we must restrict ourselves further. Thus, we get $h(t) = \in_{t-a}^{a}$ $\frac{1}{a^2} ds = \frac{1}{a^2}(2a-t)$. So overall, $h(t) = \frac{t}{a^2}$ if $t \in [0, a]$, and $h(t) = 2a-t$ if $t \in (a, 2a]$, and $h(t) = 0$ everywhere else

3 Markov Chains

P is a **transition probability matrix** if:

1. All of the entries are non-negative.
2. The sum of entries in each row is 1.

A **Markov chain** is defined by four things: ($\mathscr{X}$, π_0, P, $\{X_n\}_{n=0}^{\infty}$)

$\mathscr{X}$	Set of states
π_0	Initial probability distribution
P	Transition probability matrix
$\{X_n\}_{n=0}^{\infty}$	Sequence of random variables where:

$\mathrm{P}[X_0 = i] = \pi_0(i), i \in \mathscr{X}$
$\mathrm{P}[X_{n+1} = j | X_n = i, X_{n-1}, \ldots, X_0] = \mathrm{P}(i,j), \forall n \geq 0, \forall i, j \in \mathscr{X}$

A Markov chain is **irreducible** if we can go from any state to any other state, possibly in multiple steps.

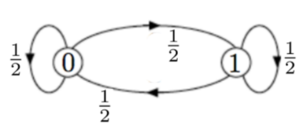
Define value $d(i)$ for each state i as:

$$d(i) := g.c.d\{n > 0 | P^n(i,i) = \mathrm{P}[X_n = i | X_0 = i] > 0\}, i \in \mathscr{X}$$

If $d(i) = 1$, then the Markov chain is **aperiodic**. If $d(i) \neq 1$, then the Markov chain is periodic and its **period** is $d(i)$.

A distribution π is **invariant** if $\pi \cdot P = \pi$.

Theorem 24.3: A finite irreducible Markov chain has a unique invariant distribution.
Theorem 24.4: All irreducible and aperiodic Markov chains converge to the unique invariant distribution. If a Markov chain is finite and reducible, the amount of time spent in each state approaches the invariant distribution as n grows large
Equations that model what will happen at the next step are called **first step equations**

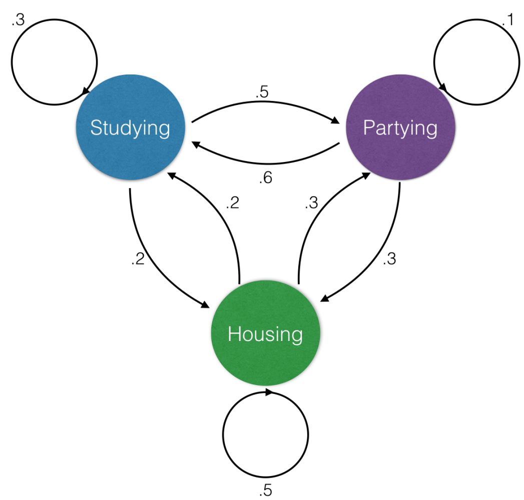


Denote $\beta(i,j)$ as the expected amount of time it would take to move from i to j. $\beta(0,1) = 1 + \frac{1}{2} \cdot \beta(0,1)$ $\beta(1,1) = 0$

3.1 Questions

1. **Life of Alex**
Alex is enjoying college life. She spends a day either studying, partying, or looking for housing for the next year. If she is studying, the chances of her studying the next day are 30%, the chances of her partying the next day are 50%, and the chances of her looking for housing the next day are 20%. If she is partying, the chances of her partying the next day are 10%, the chances of her studying the next day are 60%, and the chances of her looking for housing the next day are 30%. If she is looking for housing, the chances of her looking for housing the next day are 50%, the chances of her partying the next day are 30% and the chances of her studying the next day are 20%.

   (a) Draw a Markov chain to visualize Alexs life.

   Solution:

   (b) Write out a matrix to represent this Markov chain.

   Solution:

$$\begin{bmatrix} .3 & .5 & .2 \\ .6 & .1 & .3 \\ .2 & .3 & .5 \end{bmatrix}$$

   (c) If Alex studies on Monday, what is the chance that she is partying on Friday? (Don't do the math, just write out the expression that you would use to find it.)

Solution: If P is the matrix above, then it is $[1, 0, 0] \cdot P^4$

(d) What percentage of her time should Alex expect to use looking for housing?

Solution: Solve the following system of equations: (first step equations)

$$S = .3S + .6P + .2H$$
$$P = .5S + .1P + .3H$$
$$H = .2S + .3P + .5H$$
$$S + P + H = 1$$

(e) If Alex parties on Monday, what is the chance of Alex partying again before studying?

Solution: Set up the following equations:

$$H1 = 0$$
$$H2 = .6(H1) + .1(1) + .3(H3)$$
$$H3 = .2(H1) + .3(1) + .5(H3)$$

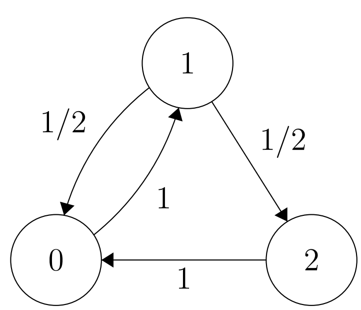
Solving for $H2$, we get 0.28

2. **Prehistoric States**

A prehistoric civilization survives by hunting game in the forests near their home. At the beginning of the hunting season, all the young men go out to the forest. After the first day, those who have a kill, which happens with probability 1/2, return home. Everyone who has been out for two days, even if without a kill, returns home for rest. And everyone who goes home goes back out the next day.

1. What are the states in this scenario? Draw a Markov chain.

Solution: We need distinct states for at home, day 1, and day 2, because the probabilities of changing depend on these, and only these. Letting 0, 1, and 2 represent these states, we have



2. What is the transition matrix? The initial vector?

Solution: $P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$

$\pi = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ because everyone starts at home.

3. Is this Markov chain reducible? Is it periodic?

Solution: This chain is irreducible, because there is a path from any node to any other. It is aperiodic (I THINK), because, for instance, weight that starts at node 0 can return in either 2 or 3 iterations.

4. What is the invariant vector?

Solution: $\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} = \pi = \pi P = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\pi_2}{2} + \pi_3 & p_1 & \frac{\pi_2}{2} \end{bmatrix}$

Thus $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$

5. What are the distributions after one week?

Solution: $P^2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $p^4 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ $P^6 = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$ $P^7 = \begin{bmatrix} \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \\ \frac{7}{16} & \frac{3}{8} & \frac{3}{16} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \end{bmatrix}$

Thus we have $\pi p^7 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \\ \frac{7}{16} & \frac{3}{8} & \frac{3}{16} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{bmatrix}$

6. What is the expected length of hunting trip?

Solution: Pr[takes 2 days] = $\frac{1}{2}$. Pr[takes 3 days] = $\frac{1}{2}$. Expected Value = $2(\frac{1}{2}) + 3(\frac{1}{2}) = 2.5$

3. **Stanford Cinema**

You have a database of an infinite number of movies. Each movie has a rating that is uniformly distributed in 0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5 independent of all other movies. You want to find two movies such that the sum of their ratings is greater than

7.5 (7.5 is not included).

a) A Stanford student chooses two movies each time and calculates the sum of their ratings. If is less than or equal to 7.5, the student throws away these two movies and chooses two other movies. The student stops when he/she finds two movies such that the sum of their ratings is greater than 7.5. What is the expected number of movies that this student needs to choose from the database?

Solution: Each time when the Stanford student chooses two movies, there are $11^2 = 121$ different possible pairs of ratings. By simple counting, we know that there are 15 pairs whose sum is greater than 7.5. Therefore, the probability that in a single trial, the Stanford student gets two movies such that the sum of their ratings is greater than 7.5 is $\frac{15}{121}$. Then the number of times that the student needs to pick movies is geometrically distributed with mean $\frac{121}{15}$. Then the expected number of movies that the student needs to choose is $\frac{242}{15} \approx 16.13$.

b) A Berkeley student chooses movies from the database one by one and keeps the movie with the highest rating. The student stops when he/she finds the sum of the ratings of the last movie that he/she has chosen and the movie with the highest rating among all the previous movies is greater than 7.5. What is the expected number of movies that the student will have to choose?

Solution: We use a Markov chain to represent to process that the Berkeley student gets two desired movies. There are 11 possible ratings:

$$S = \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5\}$$

We divide the set of ratings into 2 subsets, $L = \{0, 0.5, 1, 1.5, 2, 2.5\}$ and $H = \{3, 3.5, 4, 4.5, 5\}$. Since the goal of the student is to get two movies that the sum of their rating is greater than 7.5, the movies whose ratings are in the set L have no contribution to this goal. Then we can use 7 states $\{s_L, s_3, s_{3.5}, s_4, s_{4.5}, s_5, s_E\}$ to represent the progress to get two movies such that the sum of their ratings is greater than 7.5. The state s_L denotes the cases when highest movie rating that the Berkeley student has got is in L. The states $s_i, 3 \leq i \leq 5$, denote the cases when the highest movie rating that the student has got is i. The state s_E denotes the case when the student has got two movies such the sum of their ratings is greater than 7.5 and the choosing process ends. We can see that the process is a Markov chain with probability transition matrix P as follows:

$$P = \frac{1}{11}\left[\begin{array}{ccccccc|c} 6 & 1 & 1 & 1 & 1 & 1 & 0 & s_L \\ 0 & 7 & 1 & 1 & 1 & 0 & 1 & s_3 \\ 0 & 0 & 8 & 1 & 0 & 0 & 2 & s_{3.5} \\ 0 & 0 & 0 & 8 & 0 & 0 & 3 & s_4 \\ 0 & 0 & 0 & 0 & 7 & 0 & 4 & s_{4.5} \\ 0 & 0 & 0 & 0 & 0 & 6 & 5 & s_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 11 & s_E \\ \hline s_L & s_3 & s_4 & s_{4.5} & s_5 & s_E & & \end{array}\right]$$

Let F_s be the expected time to get to state s_E, starting from state s, $F = \begin{bmatrix} F_{s_L} & F_{s_3} & F_{s_{3.5}} & F_{s_4} & F_{s_{4.5}} & F_{s_5} \end{bmatrix}^T$ P' be the sub-matrix of P consisting of the first 6 columns and rows of P (which has 7), and U be an all-one column vector with length 6. We have the first step equations

$$F = P'F + U$$

Solving these linear equations, we get
$F = \begin{bmatrix} 6.0164 & 5.5764 & 4.8889 & 3.666667 & 2.75 & 2.2 \end{bmatrix}^T$
The initial state is s_L with probability $\frac{6}{11}$ and is $s_i, 3 \leq i \leq 5$ with probability $\frac{1}{11}$ respectively. Then we know that the expected number of movies that the student needs to choose is

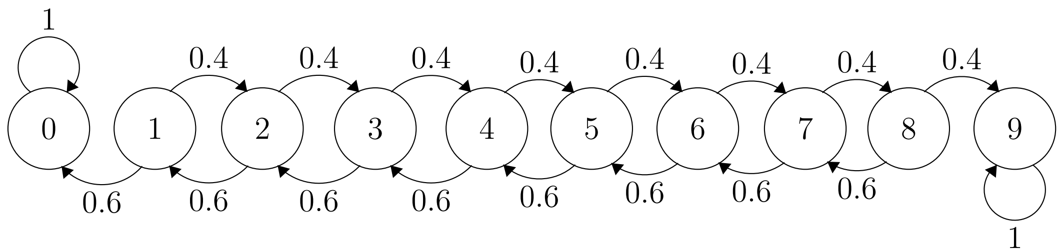
$$1 + \frac{6}{11}F_{s_L} + \frac{1}{11}(F_{s_3} + F_{s_{3.5}} + F_{s_4} + F_{s_{4.5}} + F_{s_5}) = 6.02$$

This shows that the Berkeley student is smarter than the Stanford student.

4. **Bet On It**
Smith is in jail and has 3 dollars; he can get out on bail if he has 8 dollars. A guard agrees to make a series of bets with him. If Smith bets A dollars, he wins A dollars with probability 0.4 and loses A dollars with probability 0.6.

   a) Find the probability that he wins 8 dollars before losing all of his money if he bets 1 dollar each time.

Solution: The Markov chain $(X_n, n = 0, 1, \dots)$ representing the evolution of Smith's money has diagram



Let $\phi(i)$ be the probability that the chain reaches state 8 before reaching state 0, starting from state i. In other words, if S_j is the first $n \leq 0$ such that $X_n = j$,

$$P_i(S_8 < S_0) = P(S_8 < S_0 | X_0 = i) \tag{1}$$

Using first-step analysis (viz. the Markov property at time $n = 1$), we have

$$\begin{aligned} \phi(i) &= 0.4\phi(i+1) + 0.6\phi(i-1),\ i = 1, 2, 3, 4, 5, 6, 7 \\ \phi(0) &= 0 \\ \phi(8) &= 1. \end{aligned}$$

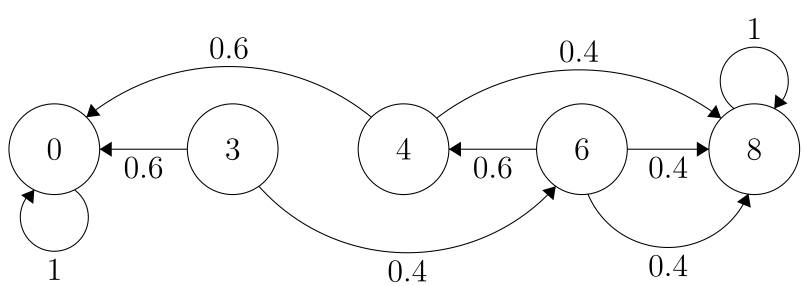
We solve this system of linear equations and find

$$\begin{aligned} \phi &= (\phi(1), \phi(2), \phi(3), \phi(4), \phi(5), \phi(6), \phi(7)) \\ &= (0.0203, 0.0508, 0.0964, 0.1649, 0.2677, 0.4219, 0.6531, 1). \end{aligned}$$

E.g. the probability that the chain reaches state 8 before reaching state 0, starting from state 3 is the third component of the vector and is equal to 0.0964. Note that $\phi(i)$ is increasing in i, which was expected.

b) Find the probability that he wins 8 dollars before losing all of his money if he bets, each time, as much as possible but not more than necessary to bring his fortune up to 8 dollars

Solution: Now the chain is



and the equations are:

$$\begin{aligned}
\phi(3) &= 0.4\phi(6) \\
\phi(6) &= 0.4\phi(8) + 0.6\phi(4) \\
\phi(4) &= 0.4\phi(8) \\
\phi(0) &= 0 \\
\phi(8) &= 1
\end{aligned}$$

We solve and find

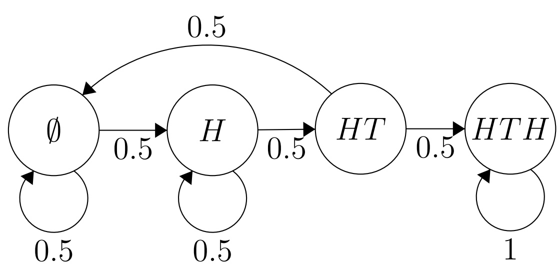
$$\phi(3) = 0.256, \phi(4) = 0.4, \phi(6) = 0.64$$

c) Which strategy gives Smith the better chance of getting out of jail?

Solution: By comparing the third components of the vector ϕ we find that the bold strategy gives Smith a better chance to get our of jail.

5. **Tossing Coins**
A fair coin is tossed repeatedly and independently. Find the expected number of tosses till the pattern HTH appears.

Solution: Call HTH our target. Consider a chain that starts from a state called nothing ($\emptyset$) and is eventually absorbed at HTH. If we first toss H then we move to state H because this is the first letter of our target. If we toss a T then we move back to $\emptyset$ having expended 1 unit of time. Being in state H we either move to a new state HT if we bring T and we are 1 step closer to the target or, if we bring H, we move back to H: we have expended 1 unit of time, but the new H can be the beginning of a target. When in state HT we either move to HTH and we are done or, if T occurs then we move to $\emptyset$. The transition diagram is



Let $\phi(i)$ be the expected number of steps to reach HTH starting from i. We have

$$\phi(HT) = 1 + \frac{1}{2}\phi(\emptyset)$$
$$\phi(H) = 1 + \frac{1}{2}\phi(H) + \frac{1}{2}\phi(HT)$$
$$\phi(\emptyset) = 1 + \frac{1}{2}\phi(\emptyset) + \frac{1}{2}\phi(H)$$

We solve and find $\phi(\emptyset) = 10$.

4 Conditional Expectation

4.1 Introduction

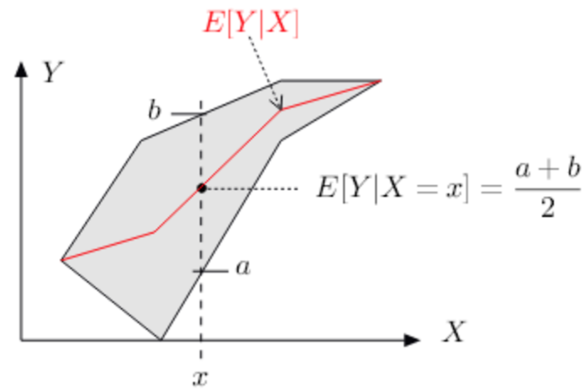
The **conditional expectation** of Y given X is defined by

$$\mathrm{E}[Y|X=x] = \sum_y y \cdot \mathrm{P}[Y=y|X=x] = \sum_y y \cdot \frac{\mathrm{P}[X=x, Y=y]}{\mathrm{P}[X=x]}$$

Properties of Conditional Expectation

$$\mathrm{E}(a|Y)) = a$$
$$\mathrm{E}(aX + bZ|Y) = a \cdot \mathrm{E}(X|Y) + b \cdot \mathrm{E}(Z|Y)$$
$$\mathrm{E}(X|Y) \geq 0 \text{ if } X \geq 0$$
$$\mathrm{E}(X|Y) = \mathrm{E}(X) \text{ if } X, Y \text{ independent}$$
$$\mathrm{E}(\mathrm{E}(X|Y)) = \mathrm{E}(X)$$

Solution: Here is a picture that shows that conditioning creates a new random variable with a new distribution. Figure 9 of note 26 does so.



4.2 Questions

1. Prove $\mathrm{E}(h(X) \cdot Y|X) = h(X) \cdot \mathrm{E}(Y|X)$

Solution:

$$\begin{aligned}
\mathrm{E}(h(X) \cdot Y|X) &= \sum_y h(X) \cdot y \cdot \mathrm{P}[Y = y|X] \\
&= h(X) \sum_y y \cdot \mathrm{P}[Y = y|X] \\
&= h(X) \cdot \mathrm{E}[Y|X]
\end{aligned}$$

2. Prove $\mathrm{E}(\mathrm{E}(Y|X)) = \mathrm{E}(Y)$

Solution:

$$\begin{aligned}
\mathrm{E}(\mathrm{E}(Y|X)) &= \sum_x \mathrm{E}(Y|X = x) \cdot \mathrm{P}[X = x] \\
&= \sum_x (\sum_y y \cdot \mathrm{P}[Y = y|X = x]) \cdot \mathrm{P}[X = x] \\
&= \sum_y y \cdot \sum_x y \cdot \mathrm{P}[X = x|Y = y]) \cdot \mathrm{P}[Y = y] \\
&= \sum_y y \cdot \mathrm{P}[Y = y] \cdot \sum_x \mathrm{P}[X = x|Y = y]) \\
&= \sum_y y \cdot \mathrm{P}[Y = y] = E[Y]
\end{aligned}$$

3. Consider the random variables Y and X with the following probabilities

This table gives the probability distribution for $P[X \cap Y]$

		X		
		0	1	2
Y	0	0	.1	.2
	1	.1	.2	.1
	2	.2	.1	0

Find:

(a) $\mathrm{E}(Y|X=0)$

Solution:

$$\begin{aligned}\mathrm{E}(Y|X=0) &= \mathrm{P}[Y=0|X=0]\cdot 0 + \mathrm{P}[Y=1|X=0]\cdot 1 + \mathrm{P}[Y=2|X=0]\cdot 2\\ &= \frac{0}{0+.1+.2}\cdot 0 + \frac{.1}{0+.1+.2}\cdot 1 + \frac{.2}{0+.1+.2}\cdot 2\\ &= \frac{20}{12} = \frac{5}{3}\end{aligned}$$

(b) $\mathrm{E}(Y|X=1)$

Solution:

$$\begin{aligned}\mathrm{E}(Y|X=1) &= \mathrm{P}[Y=0|X=1]\cdot 0 + \mathrm{P}[Y=1|X=1]\cdot 1 + \mathrm{P}[Y=2|X=1]\cdot 2\\ &= \frac{0.1}{0.1+0.2+0.1}\cdot 0 + \frac{0.2}{0.1+0.2+0.1}\cdot 1 + \frac{0.1}{0.1+0.2+0.1}\cdot 2\\ &= 0 + \frac{0.2}{0.4} + \frac{0.1\cdot 2}{0.4}\\ &= 0.5 + 0.5 = 1\end{aligned}$$

(c) $\mathrm{E}(Y|X=2)$

Solution:

$$\begin{aligned}\mathrm{E}(Y|X=2) &= \mathrm{P}[Y=0|X=2]\cdot 0 + \mathrm{P}[Y=1|X=2]\cdot 1 + \mathrm{P}[Y=2|X=2]\cdot 2\\ &= \frac{0.2}{0.2+0.1+0}\cdot 0 + \frac{0.1}{0.2+0.1+0}\cdot 1 + \frac{0}{0.2+0.1+0}\cdot 2\\ &= 0 + \frac{0.1}{0.3} + 0 = \frac{1}{3}\end{aligned}$$

(d) $\mathrm{E}(Y)$

Solution: These events are disjoint, so to find $\mathrm{E}[Y]$, we can just sum up individual probabilities (note that sum of all probabilities is the sum of 1)

$$
\begin{aligned}
\mathrm{E}(Y) &= \mathrm{E}(Y|X=0)\cdot \mathrm{P}[X=0] + \mathrm{E}(Y|X=1)\cdot \mathrm{P}[X=1] + \mathrm{E}(Y|X=2)\cdot \mathrm{P}[X=2] \\
&= \frac{20}{12}\cdot(0+0.1+0.2) + 1\cdot(0.1+0.2+0.1) + \frac{1}{3}\cdot(0.2+0.1+0) \\
&= \frac{20}{12}\cdot\frac{3}{10} + 0.4 + \frac{0.3}{3} \\
&= \frac{60}{120} + \frac{2}{5} + \frac{1}{10} \\
&= \frac{60}{120} + \frac{48}{120} + \frac{12}{120} = \frac{120}{120} = 1
\end{aligned}
$$