

# 1 Markov Chains

$P$  is a **transition probability matrix** if:

1. All of the entries are non-negative.
2. The sum of entries in each row is 1.

A **Markov chain** is defined by four things:  $(\mathcal{X}, \pi_0, P, \{X_n\}_{n=0}^\infty)$

$\mathcal{X}$	Set of states
$\pi_0$	Initial probability distribution (the chance you are in any of the states)
$P$	Transition probability matrix
$\{X_n\}_{n=0}^\infty$	Sequence of random variables where:
	$P[X_0 = i] = \pi_0(i), i \in \mathcal{X}$
	$P[X_{n+1} = j   X_n = i, X_{n-1}, \dots, X_0] = P(i, j), \forall n \geq 0, \forall i, j \in \mathcal{X}$

A Markov chain is **irreducible** if we can go from any state to any other state, possibly in multiple steps.

Periodicity has to do with the period of occurrence of a state. If a state  $s$  has period 2, the Markov chain can be in  $s$  at every other time point. If a state has period 1, it's aperiodic; otherwise, it's periodic. More quantitatively, define value  $d(i)$  for each state  $i$  as:

$$d(i) := \text{g.c.d}\{n > 0 | P^n(i, i) = P[X_n = i | X_0 = i] > 0\}, i \in \mathcal{X}$$

If  $d(i) = 1$ , then the Markov chain is **aperiodic**. If  $d(i) \neq 1$ , then the Markov chain is periodic and its **period** is  $d(i)$ .

A distribution  $\pi$  is **invariant** for the transition probability  $P$  if it satisfies the following **balance equations**

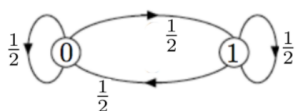
$$\pi \cdot P = \pi.$$

**Theorem 1:** A finite irreducible Markov chain has a unique invariant distribution.

**Theorem 2:** All irreducible and aperiodic Markov chains converge to the unique invariant distribution - i.e, after a large amount of steps, the chance you are in any given state is given by the invariant distribution.

**Theorem 3:** If a Markov chain is finite and reducible, the amount of time spent in each state approaches the invariant distribution as  $n$  grows large

Equations that model what will happen at the next step are called **first step equations**



Denote  $\beta(i, j)$  as the expected amount of time it would take to move from  $i$  to  $j$ .  $\beta(0, 1) = 1 + \frac{1}{2} \cdot \beta(0, 1)$   $\beta(1, 1) = 0$

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## 1.1 Questions

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### 1. Types of Markov Chains

- (a) Draw a Markov Chain that is reducible.
  
- (b) Draw a periodic Markov Chain (note that when we talk about periodicity, we imply that the Markov chain is reducible). Does it have an invariant distribution? Does it always converge to this invariant distribution? Does the invariant distribution represent the average amount of time spent in each state in this chain?
  
- (c) Draw an aperiodic Markov Chain. Does it have an invariant distribution? Does it converge to this invariant distribution? Does the invariant distribution represent the average amount of time spent in each state in this chain?

### 2. Life of Alex

Alex is enjoying college life. She spends a day either studying, partying, or looking for housing for the next year. If she is studying, the chances of her studying the next day are 30%, the chances of her partying the next day are 50%, and the chances of her looking for housing the next day are 20%. If she is partying, the chances of her partying the next day are 10%, the chances of her studying the next day are 60%, and the chances of her looking for housing the next day are 30%. If she is looking for housing, the chances of her looking for housing the next day are 50%, the chances of her partying the next day are 30% and the chances of her studying the next day are 20%.

- (a) Draw a Markov chain to visualize Alex's life.

- (b) Write out a matrix to represent this Markov chain.
- (c) If Alex studies on Monday, what is the chance that she is partying on Friday?  
(Don't do the math, just write out the expression that you would use to find it.)
- (d) What percentage of her time should Alex expect to use looking for housing?
- (e) If Alex parties on Monday, what is the chance of Alex partying again before studying?

### 3. Stanford Cinema

You have a database of an infinite number of movies. Each movie has a rating that is uniformly distributed in 0, 1, 2, 3, 4, 5 independent of all other movies. You want to find two movies such that the sum of their ratings is greater than 7.5 (7.5 is not included).

- a) A Stanford student chooses two movies each time and calculates the sum of their ratings. If it is less than or equal to 7.5, the student throws away these two movies and chooses two other movies. The student stops when he/she finds two movies such that the sum of their ratings is greater than 7.5. What is the expected number of movies that this student needs to choose from the database?

- b) A Berkeley student chooses movies from the database one by one and keeps the movie with the highest rating. The student stops when he/she finds the sum of the ratings of the last movie that he/she has chosen and the movie with the highest rating among all the previous movies is greater than 7.5. What is the expected number of movies that the student will have to choose?

#### 4. Bet On It

Smith is in jail and has 3 dollars; he can get out on bail if he has 8 dollars. A guard agrees to make a series of bets with him. If Smith bets  $A$  dollars, he wins  $A$  dollars with probability 0.4 and loses  $A$  dollars with probability 0.6.

- a) Find the probability that he wins 8 dollars before losing all of his money if he bets 1 dollar each time.

b) Find the probability that he wins 8 dollars before losing all of his money if he bets, each time, as much as possible but not more than necessary to bring his fortune up to 8 dollars.

c) Which strategy gives Smith the better chance of getting out of jail?

### 5. Tossing Coins

A fair coin is tossed repeatedly and independently. Find the expected number of tosses till the pattern HTH appears.