

INEQUALITIES, LLSE 11

COMPUTER SCIENCE MENTORS 70

April 16-18, 2018

1 Inequalities

1.1 Introduction

Markov's Inequality

For a non-negative random variable X with expectation $E(X) = \mu$, and any $\alpha > 0$:

$$P[X \geq \alpha] \leq \frac{E(X)}{\alpha}$$

Proof of Markov's Inequality

$$\begin{aligned} E(X) &= \sum_a a * Pr[X = a] \\ &\geq \sum_{a \geq \alpha} a * Pr[X = a] \\ &\geq \alpha \sum_{a \geq \alpha} Pr[X = a] \\ &= \alpha Pr[X \geq \alpha] \end{aligned}$$

Alternate Proof of Markov's Inequality

Consider the indicator random variable Y which equals 1 if $X \geq a$ and 0 otherwise. Now consider the relationship between X and aY

- If $X < a$, then $Y = 0$, which means $aY = a * 0 = 0$
Because X is a non-negative random variable, $X \geq 0$, so $aY \leq X$ in this case.
- If $X \geq a$, then $Y = 1$, which means $aY = a * 1 = a \leq X$

Thus, we have $aY \leq X$.

We take expectation on both sides to get:

$$\begin{aligned} E[aY] &\leq E[X] \\ aE[Y] &\leq E[X] \\ E[Y] &\leq \frac{E[X]}{a} \end{aligned}$$

Now we note that the expectation of an indicator random variable is the probability that it is equal to 1 and we have the proof:

$$P(X \geq a) \leq \frac{E[X]}{a}$$

For a random variable X with expectation $E(X) = \mu$, and any $\alpha > 0$:

$$\mathbb{P}[|X - \mu| \geq \alpha] \leq \frac{\text{Var}(X)}{\alpha^2}$$

1. Use Markov's to prove Chebyshev's Inequality:

- (a) Markov's inequality:

(b) Chebyshev's inequality:

A random variable X is always strictly larger than -100 . You know that $E(X) = -60$. Give the best upper bound you can on $P[X \geq -20]$.

4. The citizens of the country USD (the United States of Drumpf) vote in the following manner for their presidential election: if the country is liberal, then each citizen votes for a liberal candidate with probability p and a conservative candidate with probability $1 - p$, while if the country is conservative, then each citizen votes for a conservative candidate with probability p and a liberal candidate with probability $1 - p$. After the election, the country is declared to be of the party with the majority of the votes.

(a) Assume that $p = \frac{3}{4}$ and suppose that 100 citizens of USD vote in the election and that USD is known to be conservative. Provide a tight bound on the probability that it is declared to be a Liberal country.

(b) Now let p be unknown; we wish to estimate it. Using the CLT, determine the number of voters necessary to determine p within an error of 0.01, with probability at least 0.95.

5. Squirrel Standard Deviation

As we all know, Berkeley squirrels are extremely fat and cute. The average squirrel is 40% body fat. The standard deviation of body fat is 5%. Provide an upper bound on the probability that a randomly trapped squirrel is either too skinny or too fat? A skinny squirrel has less than 27.5% body fat, and a fat squirrel has more than 52.5% body fat?

6. Consider a random variable Y with expectation μ whose maximum value is $\frac{3\mu}{2}$, prove that the probability that Y is 0 is at most $\frac{1}{3}$.

7. Let X_1, X_2, \dots, X_n be n iid Geometric random variables with parameter p . Using Chebyshev's inequality, provide an upper-bound on: $P(|\frac{X_1+X_2+\dots+X_n}{n} - \frac{1}{p}| \geq a)$
Recall the variance for a Geometric Distribution with parameter p is $\frac{1-p}{p^2}$

1.3 Covariance

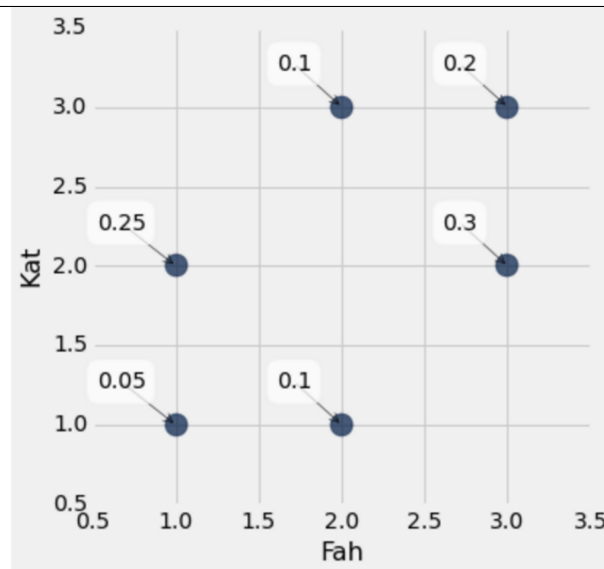
1.4 Introduction

The **covariance** of two random variables X and Y is defined as:

$$\text{Cov}(X, Y) := E((X - E(X)) \cdot (Y - E(Y)))$$

1.5 Questions

1. Prove that $\text{Cov}(X, X) = \text{Var}(X)$:
2. Consider the following distribution with random variables Fah and Kat:



Find the covariance of Fah and Kat.

3. Prove that if X and Y are independent, then $\text{Cov}(X, Y) = 0$:

4. Prove that the converse is not necessarily true. In other words, give an example of 2 random variables whose covariance is 0 but are not independent.

5. Roll 2 dice. Let A be the number of 6's you get, and B be the number of 5's, find $\text{Cov}(A, B)$

6. Prove that $\text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$:

2 Linear Least Squares Estimator

Theorem: Consider two random variables, X, Y with a given distribution $P[X = x, Y = y]$. Then

$$L[Y|X] = E(Y) + \frac{\text{Cov}(X, Y)}{\text{Var}(X)}(X - E(X))$$

2.1 Questions

1. Assume that

$$Y = \alpha X + Z$$

where X and Z are independent and $E(X) = E(Z) = 0$. Find $L[X|Y]$.

2. The figure below shows the six equally likely values of the random pair (X, Y) . Specify the functions of:
- $L[Y | X]$
 - $E(X | Y)$
 - $L[X | Y]$
 - $E(Y | X)$

