# COVARIANCE, LLSE, CONDITIONAL EXPECTATION, MARKOV CHAINS

COMPUTER SCIENCE MENTORS 70

November 14 to November 18, 2016

# 1 Covariance

## 1.1 Introduction

The **covariance** of two random variables $X$ and $Y$ is defined as:

$$\text{Cov}(X, Y) := \text{E}((X - \text{E}(X)) \cdot (Y - \text{E}(Y)))$$

## 1.2 Warm Up

1. Prove that $\text{Cov}(X, X) = \text{Var}(X)$:

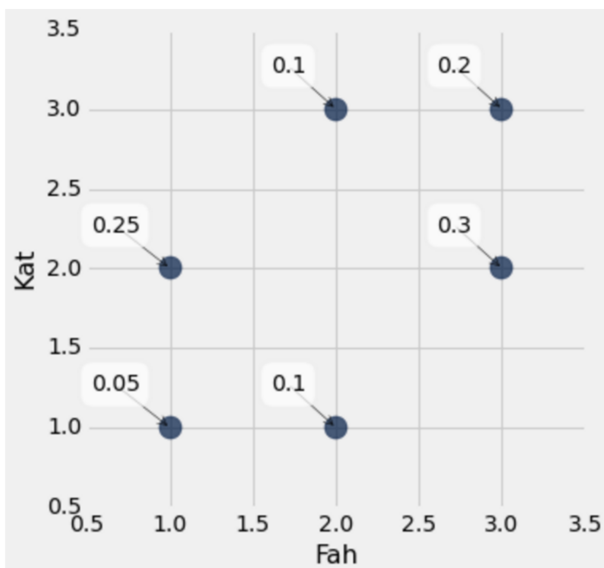
2. Prove that if $X$ and $Y$ are independent, then $\text{Cov}(X, Y) = 0$:

3. Prove that $\text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$:

### 1.3 Questions

1. Roll 2 dice. Let $A$ be the number of 6's you get, and $B$ be the number of 5's, find $\text{Cov}(A, B)$

2. Consider the following distribution with random variables Fah and Kat:




Find the covariance of Fah and Kat.

# 2 LLSE

## 2.1 Introduction

**Theorem**: Consider two random variables, $X, Y$ with a given distribution $\text{P}[X = x, Y = y]$. Then

$$\text{L}[Y|X] = \text{E}(Y) + \frac{\text{Cov}(X,Y)}{\text{Var}(X)}(X - \text{E}(X))$$

## 2.2 Questions

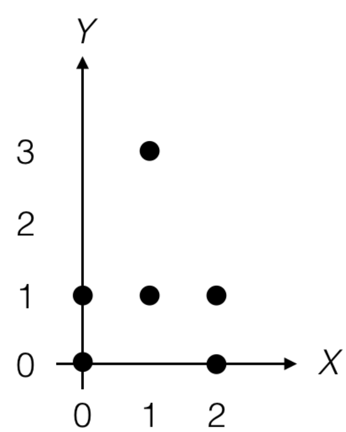
1. Assume that

$$Y = \alpha X + Z$$

where $X$ and $Z$ are independent and $\text{E}(X) = \text{E}(Z) = 0$. Find $\text{L}[X|Y]$.

2. The figure below shows the six equally likely values of the random pair $(X, Y)$. Specify the functions of:

- $L[Y \mid X]$
- $E(X \mid Y)$
- $L[X \mid Y]$
- $E(Y \mid X)$



# 3 Conditional Expectation

## 3.1 Introduction

The **conditional expectation** of $Y$ given $X$ is defined by

$$\mathrm{E}[Y|X=x] = \sum_y y \cdot \mathrm{P}[Y=y|X=x] = \sum_y y \cdot \frac{\mathrm{P}[X=x, Y=y]}{\mathrm{P}[X=x]}$$

**Properties of Conditional Expectation**

$$\begin{aligned}
\mathrm{E}(a|Y)) &= a \\
\mathrm{E}(aX+bZ|Y) &= a \cdot \mathrm{E}(X|Y) + b \cdot \mathrm{E}(Z|Y) \\
\mathrm{E}(X|Y) &\geq 0 \text{ if } X \geq 0 \\
\mathrm{E}(X|Y) &= \mathrm{E}(X) \text{ if } X, Y \text{ independent} \\
\mathrm{E}(\mathrm{E}(X|Y)) &= \mathrm{E}(X)
\end{aligned}$$

## 3.2 Questions

1. Prove $\mathrm{E}(\mathrm{E}(Y|X)) = \mathrm{E}(Y)$

2. Prove $\mathrm{E}(h(X) \cdot Y|X) = h(X) \cdot \mathrm{E}(Y|X)$

3. Consider the random variables $Y$ and $X$ with the following probabilities

   This table gives the probability distribution for $P[X \cap Y]$

| | | X | | |
|---|---|---|---|---|
| | | 0 | 1 | 2 |
| Y | 0 | 0 | .1 | .2 |
| | 1 | .1 | .2 | .1 |
| | 2 | .2 | .1 | 0 |

   Find:

   (a) $\mathrm{E}(Y|X=0)$

(b) $\mathrm{E}(Y|X=1)$

(c) $\mathrm{E}(Y|X=2)$

(d) $\mathrm{E}(Y)$

# 4 Markov Chains

P is a **transition probability matrix** if:

1. All of the entries are non-negative.
2. The sum of entries in each row is 1.

A **Markov chain** is defined by four things: ($\mathscr{X}$, $\pi_0$, $P$, $\{X_n\}_{n=0}^{\infty}$)

| | |
|---|---|
| $\mathscr{X}$ | Set of states |
| $\pi_0$ | Initial probability distribution |
| $P$ | Transition probability matrix |
| $\{X_n\}_{n=0}^{\infty}$ | Sequence of random variables where: $\text{P}[X_0 = i] = \pi_0(i), i \in \mathscr{X}$ <br> $\text{P}[X_{n+1} = j \mid X_n = i, X_{n-1}, \ldots, X_0] = \text{P}(i,j), \forall n \geq 0, \forall i,j \in \mathscr{X}$ |

A Markov chain is **irreducible** if we can go from any state to any other state, possibly in multiple steps.

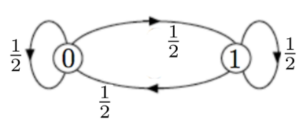
Define value $d(i)$ for each state $i$ as:

$$d(i) := g.c.d\{n > 0 \mid P^n(i,i) = \text{P}[X_n = i \mid X_0 = i] > 0\}, i \in \mathscr{X}$$

If $d(i) = 1$, then the Markov chain is **aperiodic**. If $d(i) \neq 1$, then the Markov chain is periodic and its **period** is $d(i)$.

A distribution $\pi$ is **invariant** if $\pi \cdot P = \pi$.

**Theorem 24.3:** A finite irreducible Markov chain has a unique invariant distribution.
**Theorem 24.4:** All irreducible and aperiodic Markov chains converge to the unique invariant distribution. If a Markov chain is finite and reducible, the amount of time spent in each state approaches the invariant distribution as n grows large
Equations that model what will happen at the next step are called **first step equations**




Denote $\beta(i,j)$ as the expected amount of time it would take to move from $i$ to $j$. $\beta(0,1) = 1 + \frac{1}{2} \cdot \beta(0,1)$ $\beta(1,1) = 0$

### 4.1 Questions

1. **Life of Alex**

   Alex is enjoying college life. She spends a day either studying, partying, or looking for housing for the next year. If she is studying, the chances of her studying the next day are 30%, the chances of her partying the next day are 50%, and the chances of her looking for housing the next day are 20%. If she is partying, the chances of her partying the next day are 10%, the chances of her studying the next day are 60%, and the chances of her looking for housing the next day are 30%. If she is looking for housing, the chances of her looking for housing the next day are 50%, the chances of her partying the next day are 30% and the chances of her studying the next day are 20%.

   (a) Draw a Markov chain to visualize Alexs life.

   (b) Write out a matrix to represent this Markov chain.

   (c) If Alex studies on Monday, what is the chance that she is partying on Friday? (Don't do the math, just write out the expression that you would use to find it.)

(d) What percentage of her time should Alex expect to use looking for housing?

(e) If Alex parties on Monday, what is the chance of Alex partying again before studying?