You Cant Please Everyone

Walkthrough

Problem Statement:
A person x is said to prefer a matching A to a matching A` if x strictly prefers her/his partner in A to her/his partner in A`. Given two stable matchings A and A`, a person may prefer one to the other or be indifferent if she/he is matched with the same person in both. Suppose now that A and A` are stable matchings, and suppose that m and w are partners in A but not in A`. Prove that one of m and w prefers A to A`, and the other prefers A` to A.

Before thinking about the solution, lets parse this information!
Read the problem statement and answer the following questions:

1. What is different about the matchings in A and A`?

w and m are matched in A
w and m are NOT matched in A`.
A and A` are both stable (by assumption)

2. What does it mean for w to prefer matching A to A`?

w will prefer A to A` if they are matched with someone higher on their preference list in A as opposed to A`.

3. What are we trying to prove?

Show that w and m must prefer different matchings.

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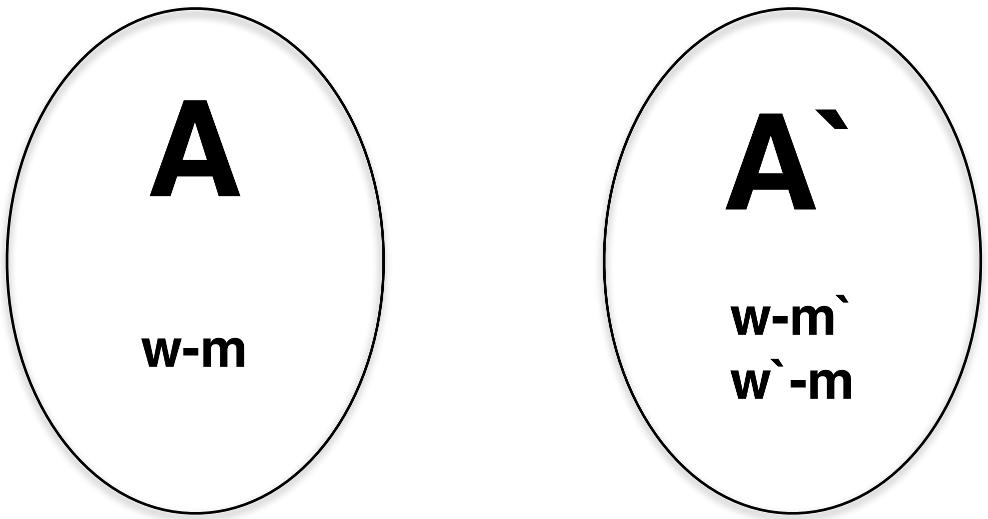
Summary of previous slide



We want to show that w and m must prefer different matchings. Take a step back. How many total possible ways can w and m prefer the matchings? For example, one way would be for w to prefer A and for m to prefer A\`.

w likes A and m likes A\`
w likes A and m likes A
w likes A\` and m likes A\`
w likes A\` and m likes A\`

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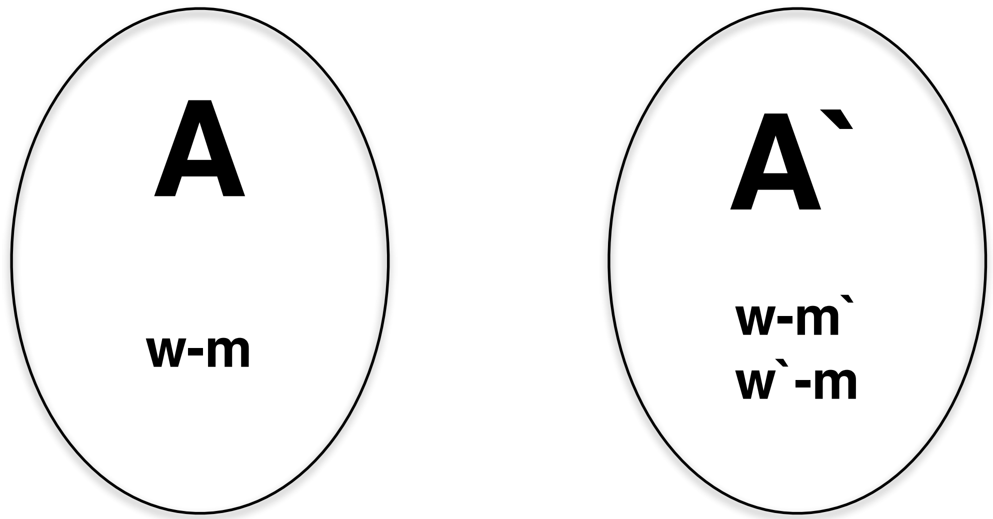

We want to show that w and m must prefer different matchings. Take a step back. How many total possible ways can w and m prefer the matchings? For example, one way would be for w to prefer A and for m to prefer A`.

w likes A and m likes A`
w likes A and m likes A
w likes A` and m likes A`
w likes A` and m likes A`

Note that these are ALL of the possible ways w and m can prefer A to A` and vice versa. So at least **one** of these must be true.

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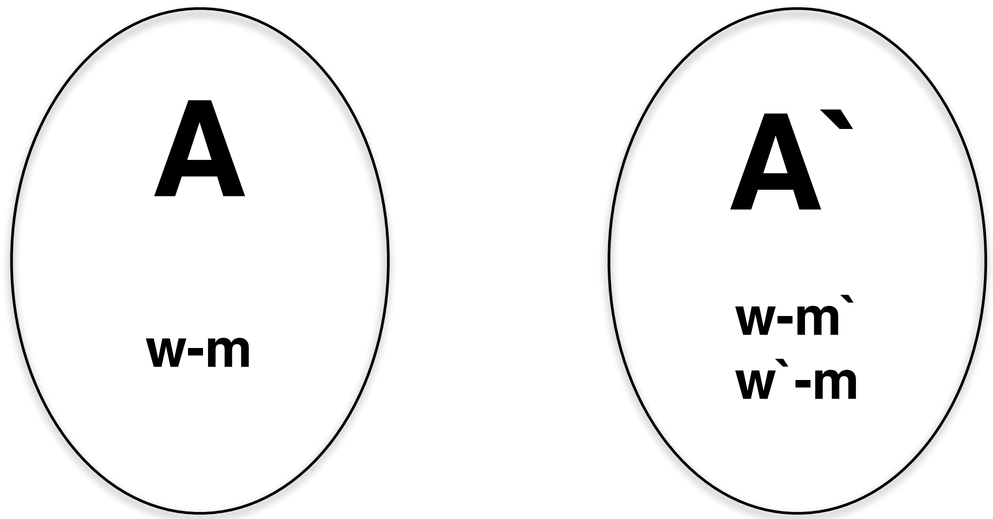

Which of the cases do we want to be true?

w likes A and m likes A`
w likes A and m likes A
w likes A` and m likes A`
w likes A` and m likes A

It's kind of hard to prove that these two cases are true **directly**. What other proof technique can we use to demonstrate that those two cases are true? Recall that the above cases cover ALL possible situations. At least one of them MUST be true. Hint: What if we can eliminate some of the cases?

Show that the middle cases are False!! Then the outer cases must be True. (or at least one of them must be true). We don't care which since either of them satisfies the bolded statement in the problem definition

Continue only if you understand *why* we are proving that the middle cases are False.

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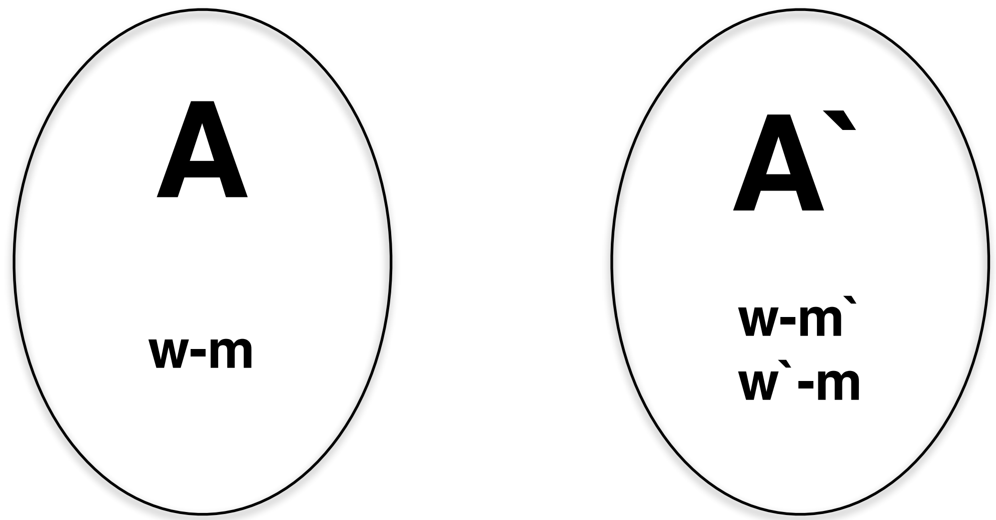

So we must prove that the following two cases are **False**:
w prefers A and m prefers A
w prefers A\` and m prefers A\`

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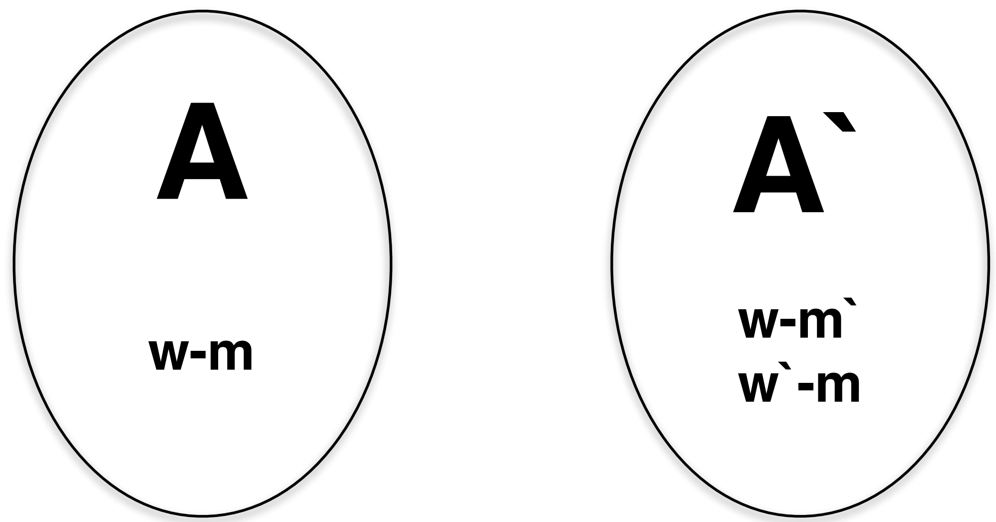

Lets examine the first case: **w prefers A and m prefers A**

What can we say about w and m's preference list? On w's preference list, is m higher or is m` higher? On m's preference list, is w higher or is w` higher? Why?

Preference lists:

w: …, m, …, m`, …
m: …, w, …, w`, …

We know that w prefers A to A`. This means that whoever she is paired with in A, she must like more than whoever she is paired with in A`. The same argument works for m.

Can you spot the contradiction? Hint: Look at the preference list and the pairings in box A`.

w and m are a rogue couple in A`. They both prefer each other over who they ended up with. But A` is a stable matching by assumption. We have reached a contradiction. So w and m cannot both prefer A.

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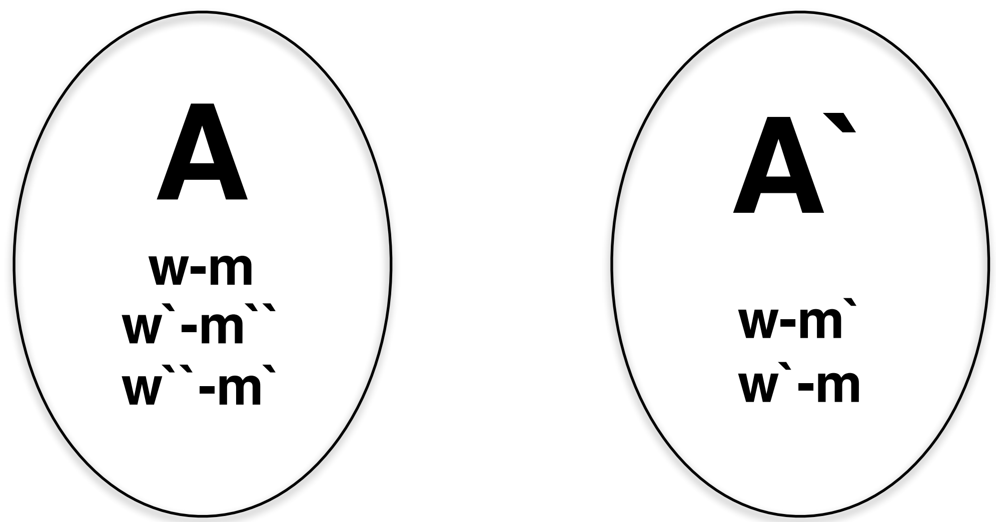

Lets examine the second case: **w prefers A` and m prefers A`**

Again, examine the preference lists of w and m. Since they both prefer A`, who do they rank higher?

w: …, m`, …, m, …
m: …, w`, …, w, …

So we know that w prefers m` to m. But then m` and w` are paired with someone different in A. Assume that m` is paired with w``, and w` is paired with m``. What can we say about the preference list of m`? Does he like w or w`` more? Hint: Assume one of the options and reach a contradiction.

m`: …, w``, …, w, …

If m` prefers w to w`` and w prefers m` to m, then we have a rogue couple in A (w-m`)

We can use the same reasoning to construct the preference list of w`.

w`: …, m``, …, m, …

If w` prefers m to m`` and m prefers w` to w, then we have a rogue couple in A (w`-m)

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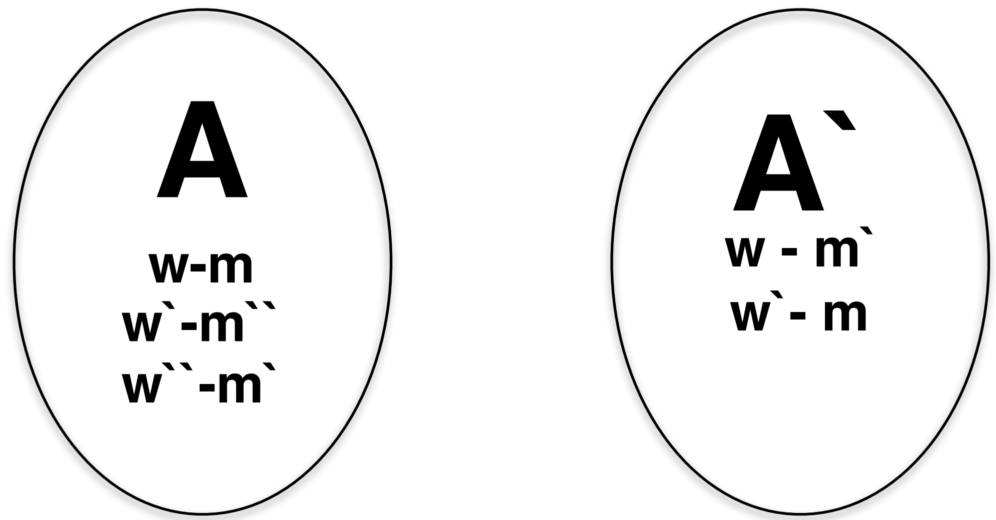

Lets examine the second case: **w prefers A` and m prefers A`**

So now we know the following:

w: …, m`, …, m, …
m: …, w`, …, w, …

w`: …, m``, …, m, …
m`: …, w``, …, w, …

Lets continue examining this trend.
m`` must be paired with someone different from w` in the A` matching. Call her w```

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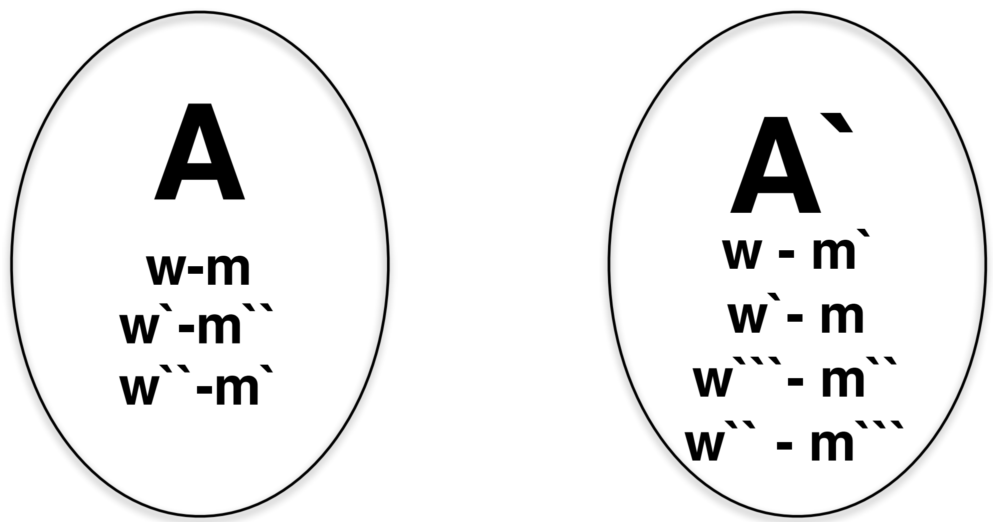

Lets examine the second case: **w prefers A` and m prefers A`**

So now we know the following:

w: …, m`, …, m, …
m: …, w`, …, w, …

w`: …, m``, …, m, …
m`: …, w``, …, w, …

Lets continue examining this trend.
m`` must be paired with someone different from w` in the A` matching. Call her w```
What can we say about m`` preferences? Does he prefer w``` or w`? What about w``? Does she prefer m``` or m`? Hint: similar argument as previous slide.

Again assume that m`` prefers w`. We know that w` prefers m`` (from previous slide). This would make m``- w` a rogue couple in A` which is a contradiction. A similar argument can be made for w``.

So we get the following preference lists for m`` and w``.

w``: …, m```, …, m`, …
m``: …, w```, …, w`, …

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Lets examine the second case: **w prefers A\` and m prefers A\`**

Take a step back and look at the preference lists we have constructed so far. Do you notice a pattern? Write a general formula for the preference list of w^k and m^k

w: …, m\`, …, m, …	m\`: …, w\`\`, …, w, …	w\`\`: …, m\`\`\`, …, m\`, …
m: …, w\`, …, w, …	w\`: …, m\`\`, …, m, …	m\`\`: …, w\`\`\`, …, w\`, …

w^k: …, m^{k+1}, …, m^{k-1}
m^k: …, w^{k+1}, …, w^{k-1}

Cool. We have a relationship between the preferences. But remember, there is a finite number of people. Say we have n people. According to this formula, what would happen to the last people, call them m^n and w^n? They obviously cannot be matched with m^{n+1} and w^{n+1} since m^{n+1} and w^{n+1} don't exist (there are n people).

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Lets examine the second case: **w prefers A` and m prefers A`**

w^k: …, m^{k+1}, …, m^{k-1}
m^k: …, w^{k+1}, …, w^{k-1}

A

w-m
w``- m`
w`- m``

A`

w-m`
w`-m
w```- m``
w``- m```

m^n needs to be matched with someone. Recall the matchings we did so far:

We are interested in what will happen to m^n and w^n.
Try continuing the pattern. Note that, w^k is matched with m^{k+1} in one matching and with m^{k-1} in the other (ex: w`-m`` in A and w` - m in A`). We must examine **both** cases. It doesn't matter where w is matched with m^{k+1} and where w is matched with m^{k-1}.
Who could w^{n-1} be matched with?

Without loss of generality, assume that w^{n-1} is matched with m^{n-2} in A, and with m^n in A`. We can repeat the same argument by switching A and A`. Draw out the known pairings given the above information.

A

w^{n-1} — m^{n-2}
w^{n-2} — m^{n-1}
w^{n} — m^{n}

A`

w^{n-1} — m^{n}
w^{n} — m^{n-1}

What do we know about the preference lists of w^{n-1} and m^{n-1}?
Hint: use the above relationship.

w^{n-1}: …, m^n, …, m^{n-2}
m^{n-1}: …, w^n, …, w^{n-2}

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Lets examine the second case: **w prefers A\` and m prefers A\`**

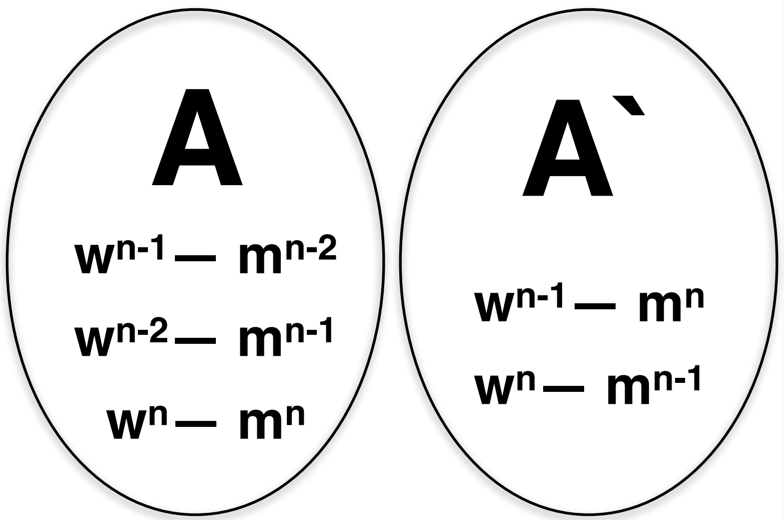
Now we need to speculate about the possible preference lists of w^n and m^n. We don't know them, but there is a limited number of possible choices!

Why can't we use our usual formula to write the preference list of w^n and m^n?

This would require the existence of w^{n+1} and m^{n+2} which don't exist.

What are the possible preference lists of w^n and m^n regarding the ordering of w^{n-1}, w^n, m^{n-1} and m^n?

w^n: …, m^{n-1}, …, m^n or …, m^n, …, m^{n-1}
m^n: …, w^{n-1}, …, w^n or …, w^n, …, w^{n-1}



w^{n-1}: …, m^n, …, m^{n-2}
m^{n-1}: …, w^n, …, w^{n-2}

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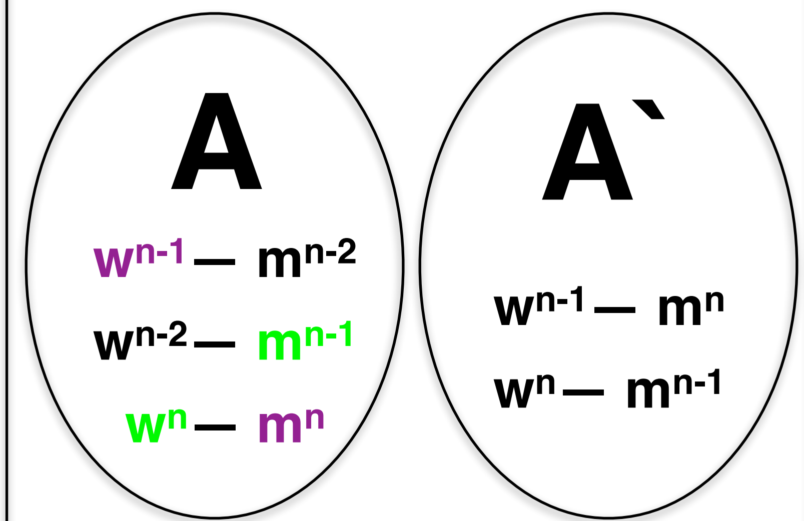

w^{n-1}: …, m^n, …, m^{n-2}
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w^n: …, ~~m^{n-1}, …, m^n~~ or …, m^n, …, m^{n-1}
m^n: …, ~~w^{n-1}, …, w^n~~ or …, w^n, …, w^{n-1}

Look at the matchings in A and the preference lists of w^{n-1} and m^{n-1}. Can you eliminate some of the above options? Hint: Some of them cause a rogue couple in A. The colors correspond to the preferences.

if w^n preferred m^{n-1}, then w^n and m^{n-1} would be a rogue couple.
if m^n preferred w^{n-1}, then w^{n-1} and m^n would be a rogue couple.

So we know that if A must be stable, then we can cross off the first option for a preference list.
But what do the remaining preference lists imply about A`? See if you can spot the contradiction!

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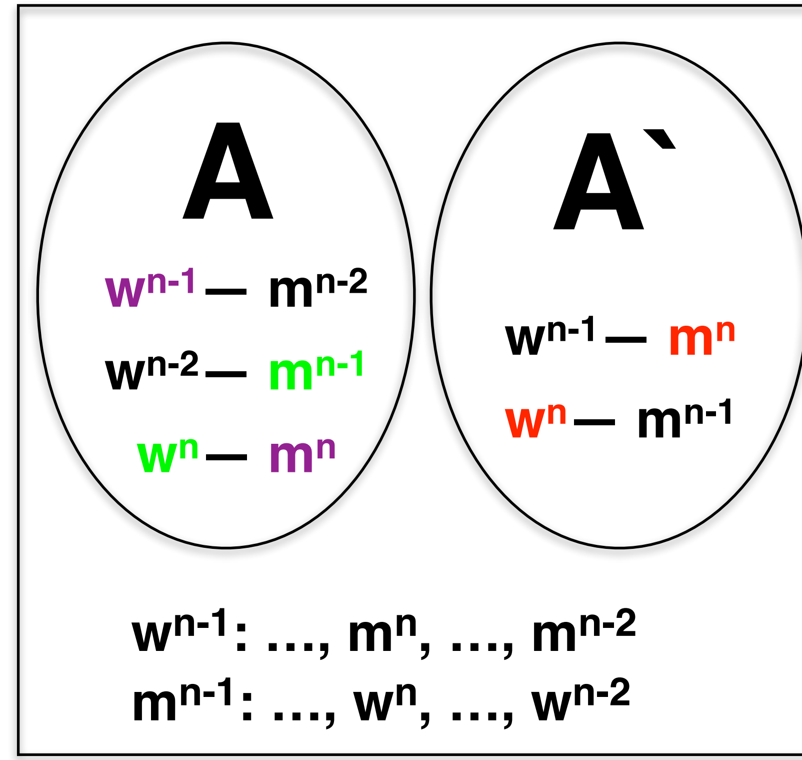

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What are the possible preference lists of w^n and m^n regarding the ordering of w^{n-1}, w^n, m^{n-1} and m^n?

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m^n: …, ~~w^{n-1}, …, w^n~~ or …, w^n, …, w^{n-1}

Look at the matchings in A and the preference lists of w^{n-1} and m^{n-1}. Can you eliminate some of the above options? Hint: Some of them cause a rogue couple in A. The colors correspond to the preferences.

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The red couple is rogue!!!

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Summary of case 2:

1. Established a relationship for the preference list: **w^k: …, m^{k+1}, …, m^{k-1}**
   m^k: …, w^{k+1}, …, w^{k-1}
2. Notice that in either world woman w^k is matched with m^{k+1} or m^{k-1}.
3. Examine the last woman and last man to be matched. (using point 2 assign them to A/A\`)
4. Write the preference list for the next to last man and woman (using point 1)
5. Examine the possible preference lists of the last man and woman.
6. All cases lead to a contradiction that A and A\` are both stable pairings.
7. Therefore it is impossible for both m and w to prefer A\` to A.

This is a very tricky proof, don't feel bad if you don't get it initially! Keep at it and ask your mentor if you have any questions!