Graphs, Bijections, FLT, Modular Arithmetic 2

Computer Science Mentors 70

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1 Graph Theory

1.1 Introduction

1. Let $G = (V, E)$ be an undirected graph. Match the term with the definition.

Walk	Cycle	Tour	Path

____________________Walk that starts and ends at the same node
____________________Sequence of edges.
____________________Sequences of edges with possibly repeated vertex or edge.
____________________Sequence of edges that starts and ends on the same vertex and does not repeat vertices (except the first and last)

2. Suppose we want to represent a round-robin tennis tournament in which every player plays one match against every other player. How might we represent this using a graph?

3. What is a simple path?

1.2 Buildup Error

In this question we will work through the canonical example of buildup error. Recall that a graph is **connected** iff there is a path between every pair of its vertices.

False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof. We use induction on the number of vertices $n \geq 1$. let $P(n)$ be the proposition that if every vertex in an n-vertex graph has positive degree, then the graph is connected.

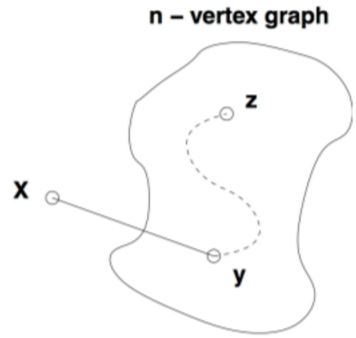
Base case: A graph with 1 vertex doesn't have any positive-degree vertices so $P(1)$ is true vacuously.

Inductive Hypothesis: Assume $P(n)$ holds. We want to show this implies $P(n+1)$.

Inductive Step: Consider an n vertex graph that has positive degree. By the asumption $P(n)$, this graph is connected and there is a path from every vertex to every other vertex. Now add a new vertex to create an $n+1$ vertex graph. All that remains is to check that there is a path from v to every other vertex. Suppose we add this vertex v to an existing vertex u. Since the graph was previously connected, we already know there is a path from u to every other vertex in the graph. Therefore, when we connect v to u, we know there will be a path from v to every other vertex in the graph. This proves the claim for $P(n+1)$. □

1. Give a counter-example to show the claim is false.

2. Since the claim is false, there must be an error in the proof. Explain the error.



3. How can we avoid this mistake?

4. What happens in the inductive step when you apply the fix?

1.3 Questions

1. Given a graph G with n vertices, where n is even, prove that if every vertex has degree $\frac{n}{2} + 1$, then G must contain a 3-cycle.

2. Every tournament has a Hamiltonian path. (Recall that a Hamiltonian path is a path that visits each vertex exactly once)

2 Eulerian Tour

2.1 Introduction

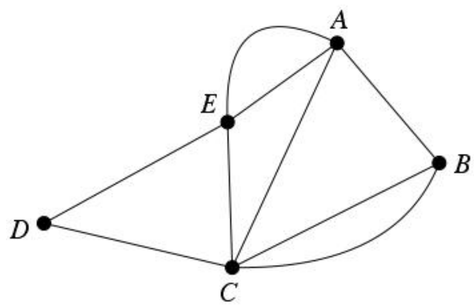
An **Eulerian path** is a path that uses every edge exactly once.

An **Eulerian tour** is a path that uses each edge exactly once and starts and ends at the same vertex.

Euler's Theorem: An undirected graph $G = (V, E)$ has an Eulerian tour if and only if G is even degree and connected (except possibly for isolated vertices).

2.2 Questions

1. Is there an Eulerian Tour? If so, find one. Repeat for an Eulerian Path.



2. If every node has even degree except two nodes that have odd degree, prove that the graph has a Eulerian path.

3 Trees

3.1 Introduction

If complete graphs are maximally connected, then trees are the opposite: Removing just a single edge disconnects the graph! Formally, there are a number of equivalent definitions for identifying a graph $G = (V, E)$ as a tree.

Assume G is connected. There are 3 other properties we can use to define it as a tree.

1. G contains ________ cycles.

2. G has ________ edges.

3. Removing any additional edge will ______________

One additional definition:

4. G is a tree if it has no cycles and ________________________

Theorem: G is connected and contains no cycles if and only if G is connected and has $n - 1$ edges.

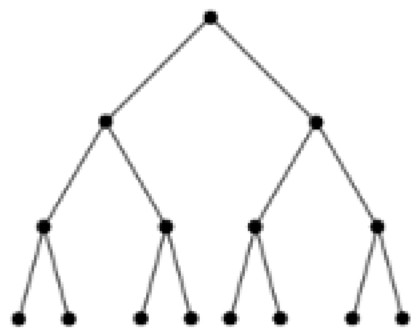
3.2 Questions

1. Now show that if a graph satisfies either of these two properties then it must be a tree:

   a If for every pair of vertices in a graph they are connected by exactly one simple path, then the graph must be a tree.

b If the graph has no simple cycles but has the property that the addition of any single edge (not already in the graph) will create a simple cycle, then the graph is a tree.

2. Recall from the notes that a **rooted tree** is a tree with a particular node designated as the root, and the other nodes arranged in levels, growing down from the root. An alternative, recursive, definition of rooted tree is the following: A rooted tree consists of a single node, the root, together with zero or more branches, each of which is itself a rooted tree. The root of the larger tree is connected to the root of each branch.



Prove that given any tree, selecting any node to be the root produces a rooted tree according to the definition above.

3. A **spanning tree** of a graph G is a subgraph of G that contains all the vertices of G and is a tree.
Prove that a graph $G = (V, E)$ if connected if and only if it contains a spanning tree.

4. Show that the edges of a complete graph on n vertices for even n can be partitioned into $\frac{n}{2}$ edge disjoint spanning trees.
   Hint: Recall that a complete graph is an undirected graph with an edge between every pair of vertices. The complete graph has $\frac{n*(n-1)}{2}$ edges. A spanning tree is a tree on all n vertices – so it has $n-1$ edges. So the complete graph has enough edges (for even n) to create exactly $\frac{n}{2}$ edge disjoint spanning trees (i.e. each edge participates in exactly one spanning tree). You have to show that this is always possible.

4 Hypercubes

4.1 Introduction

What is an n dimensional hypercube?

Bit definition: Two ____________ x and y are ____________ and only if _____ and _____ differ in __________________________ bit position.

Recursive definition: Define the 0-____________ as the $(n-1)$ dimensional ____________

with vertices labeled 0x (x is an element of _________________ (hint: how many remaining bits are there?). Do the same for the 1-____________ with vertices labeled _________. Then an n dimensional ____________ is created by placing an edge between _________ and _________ in the _______________ and _______________ respectively.

4.2 Questions

1. How many vertices does an n dimensional hypercube have?
2. How many edges does an n dimensional hypercube have?
3. How many edges do you need to cut from a hypercube to isolate one vertex in an n-dimensional hypercube?

4. Prove that any cycle in an n-dimensional hypercube must have even length.

5. **Coloring Hypercubes**
   Let $G = (V, E)$ be an undirected graph. G is said to be k-vertex-colorable if it is possible to assign one of k colors to each vertex of G so that no two adjacent vertices receive the same color. G is k-edge-colorable if it is possible to assign one of k colors to each edge of G so that no two edges incident on the same vertex receive the same color.
   Show that the n-dimensional hypercube is 2-vertex-colorable for every n.

5 Extra Practice

5.1 Questions

1. Prove that every undirected finite graph where every vertex has degree of at least 2 has a cycle.

2. Prove that every undirected finite graph where every vertex has degree of at least 3 has a cycle of even length.

6 Bijections

1. Draw an example of each of the following situations

One to one AND NOT onto (injective but not surjective)	Onto AND NOT one to one (surjective but not injective)	One to one AND onto (bijection, i.e. injective AND surjective)

2. Are the following functions **injections** from Z_{12} to Z_{24}?

   a. $f(x) = 2x$

b. $f(x) = 6x$

c. $f(x) = 2x + 4$

3. Are the following functions **surjections** from Z_{12} to Z_6? (Note: that $\lfloor x \rfloor$ is the floor operation on x)

   a. $f(x) = \lfloor \frac{x}{2} \rfloor$

   b. $f(x) = x$

   c. $f(x) = \lfloor \frac{x}{4} \rfloor$

4. Are the following functions **bijections** from Z_{12} to Z_{12}?

   a. $f(x) = 7x$

   b. $f(x) = 3x$

   c. $f(x) = x - 6$

7 Fermat's Little Theorem

7.1 Introduction

Fermat's Little Theorem: For any prime p and any $a \in \{1, 2, \ldots, p-1\}$, we have $a^{p-1} \equiv 1 \mod p$

1. Prove Fermat's Little Theorem.

7.2 Questions

1. Find $3^{5000} \mod 11$

2. Show that $n^7 - n$ is divisible by 42 for any integer n