

LLSE, MARKOV CHAINS 12

COMPUTER SCIENCE MENTORS 70

April 24-28, 2017

1 Linear Least Squares Estimator

Theorem: Consider two random variables, X, Y with a given distribution $P[X = x, Y = y]$. Then

$$L[Y|X] = E(Y) + \frac{\text{Cov}(X, Y)}{\text{Var}(X)}(X - E(X))$$

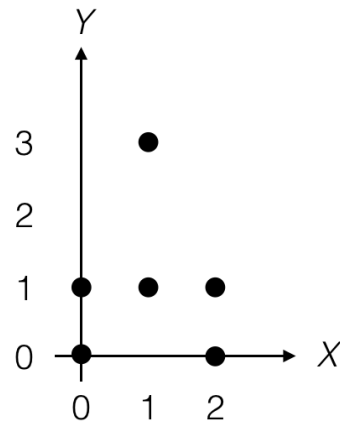
1.1 Questions

1. Assume that

$$Y = \alpha X + Z$$

where X and Z are independent and $E(X) = E(Z) = 0$. Find $L[X|Y]$.

2. The figure below shows the six equally likely values of the random pair (X, Y) . Specify the functions of:
- $L[Y | X]$
 - $E(X | Y)$
 - $L[X | Y]$
 - $E(Y | X)$



2 Markov Chains

P is a **transition probability matrix** if:

1. All of the entries are non-negative.
2. The sum of entries in each row is 1.

A **Markov chain** is defined by four things: $(\mathcal{X}, \pi_0, P, \{X_n\}_{n=0}^\infty)$

\mathcal{X} Set of states

π_0 Initial probability distribution

P Transition probability matrix

$\{X_n\}_{n=0}^\infty$ Sequence of random variables where:

$$P[X_0 = i] = \pi_0(i), i \in \mathcal{X}$$

$$P[X_{n+1} = j | X_n = i, X_{n-1}, \dots, X_0] = P(i, j), \forall n \geq 0, \forall i, j \in \mathcal{X}$$

A Markov chain is **irreducible** if we can go from any state to any other state, possibly in multiple steps.

Define value $d(i)$ for each state i as:

$$d(i) := \text{g.c.d}\{n > 0 | P^n(i, i) = P[X_n = i | X_0 = i] > 0\}, i \in \mathcal{X}$$

If $d(i) = 1$, then the Markov chain is **aperiodic**. If $d(i) \neq 1$, then the Markov chain is periodic and its **period** is $d(i)$.

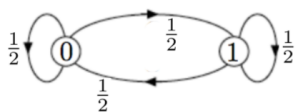
A distribution π is **invariant** for the transition probability P if it satisfies the following **balance equations**

$$\pi \cdot P = \pi.$$

Theorem 24.3: A finite irreducible Markov chain has a unique invariant distribution.

Theorem 24.4: All irreducible and aperiodic Markov chains converge to the unique invariant distribution. If a Markov chain is finite and reducible, the amount of time spent in each state approaches the invariant distribution as n grows large

Equations that model what will happen at the next step are called **first step equations**



Denote $\beta(i, j)$ as the expected amount of time it would take to move from i to j . $\beta(0, 1) = 1 + \frac{1}{2} \cdot \beta(0, 1)$
 $\beta(1, 1) = 0$

2.1 Questions

1. Life of Alex

Alex is enjoying college life. She spends a day either studying, partying, or looking for housing for the next year. If she is studying, the chances of her studying the next day are 30%, the chances of her partying the next day are 50%, and the chances of her looking for housing the next day are 20%. If she is partying, the chances of her partying the next day are 10%, the chances of her studying the next day are 60%, and the chances of her looking for housing the next day are 30%. If she is looking for housing, the chances of her looking for housing the next day are 50%, the chances of her partying the next day are 30% and the chances of her studying the next day are 20%.

(a) Draw a Markov chain to visualize Alex's life.

(b) Write out a matrix to represent this Markov chain.

(c) If Alex studies on Monday, what is the chance that she is partying on Friday?
(Don't do the math, just write out the expression that you would use to find it.)

(d) What percentage of her time should Alex expect to use looking for housing?

(e) If Alex parties on Monday, what is the chance of Alex partying again before studying?

2. Stanford Cinema

You have a database of an infinite number of movies. Each movie has a rating that is uniformly distributed in $0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5$ independent of all other movies. You want to find two movies such that the sum of their ratings is greater than 7.5 (7.5 is not included).

a) A Stanford student chooses two movies each time and calculates the sum of their ratings. If it is less than or equal to 7.5, the student throws away these two movies and chooses two other movies. The student stops when he/she finds two movies such that the sum of their ratings is greater than 7.5. What is the expected number of movies that this student needs to choose from the database?

b) A Berkeley student chooses movies from the database one by one and keeps the movie with the highest rating. The student stops when he/she finds the sum of the ratings of the last movie that he/she has chosen and the movie with the highest rating among all the previous movies is greater than 7.5. What is the