

CONFIDENCE INTERVALS, CONDITIONAL EXPECTATION, LLSE, MARKOV CHAINS

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COMPUTER SCIENCE MENTORS 70

April 23 - 25, 2017

1 Confidence Intervals

1.1 Questions

1. Define i.i.d. variables $A_k \sim \text{Bern}(p)$ where $k \in \{1, \dots, n\}$. Assume we can declare that $P[|\frac{1}{n} \sum_{k=1}^n A_k - p| > 0.25] = 0.01$.

(a) Please give a 99% confidence interval for p if given A_k .

- (b) We know that the variables X_i , for i from 1 to n , are i.i.d. random variables and have variance σ^2 . We also have a value (an observation) of $A_n = \frac{X_1 + \dots + X_n}{n}$. We want to guess the mean, μ , of each X_i .

Prove that we have 95% confidence that μ lies in the interval $[A_n - 4.5 \frac{\sigma}{\sqrt{n}}, A_n + 4.5 \frac{\sigma}{\sqrt{n}}]$

That is, $P[\mu \in [A_n - 4.5 \frac{\sigma}{\sqrt{n}}, A_n + 4.5 \frac{\sigma}{\sqrt{n}}]] \geq 95\%$

(c) Give the 99% confidence interval for μ .

2. We have a die with 6 faces of values 1, 2, 3, 4, 5, 6.

(a) Develop a 99% confidence interval for the value of n samples.

(b) Now, we say the die's face values are consecutive integers, but we do not know the starting number. The values are shifted over by some k ; for example, if $k = 6$, the die faces would take on the values 7, 8, 9, 10, 11, 12. If we observe that the average of the n samples is 15.5, develop a 99% confidence interval for the value of k .

2 Conditional Expectation

2.1 Introduction

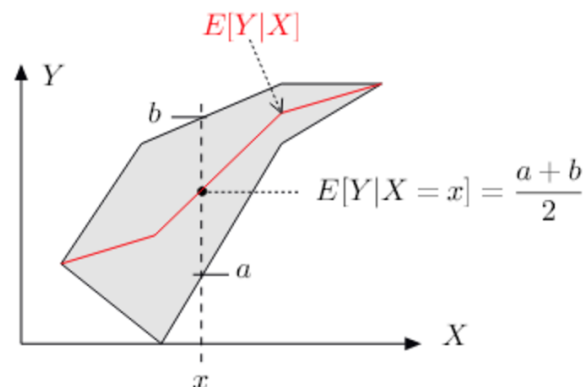
The **conditional expectation** of Y given X is defined by

$$E[Y|X = x] = \sum_y y \cdot P[Y = y|X = x] = \sum_y y \cdot \frac{P[X = x, Y = y]}{P[X = x]}$$

Properties of Conditional Expectation

$$\begin{aligned} E(a|Y) &= a \\ E(aX + bZ|Y) &= a \cdot E(X|Y) + b \cdot E(Z|Y) \\ E(X|Y) &\geq 0 \text{ if } X \geq 0 \\ E(X|Y) &= E(X) \text{ if } X, Y \text{ independent} \\ E(E(X|Y)) &= E(X) \end{aligned}$$

Here is a picture that shows that conditioning creates a new random variable with a new distribution, taken from Figure 9 of note 26.



2.2 Questions

- Consider the random variables Y and X with the following probabilities

This table gives the probability distribution for $P[X \cap Y]$

		X		
		0	1	2
Y	0	0	.1	.2
	1	.1	.2	.1
	2	.2	.1	0

Find:

(a) $E(Y|X = 0)$

(b) $E(Y|X = 1)$

(c) $E(Y|X = 2)$

(d) $E(Y)$

3 Linear Least Squares Estimator

Theorem: Consider two random variables, X, Y with a given distribution $P[X = x, Y = y]$. Then

$$L[Y|X] = E(Y) + \frac{\text{Cov}(X, Y)}{\text{Var}(X)}(X - E(X))$$

3.1 Questions

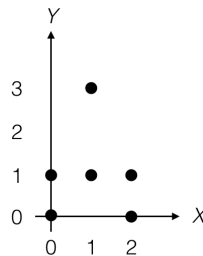
1. Assume that

$$Y = \alpha X + Z$$

where X and Z are independent and $E(X) = E(Z) = 0$. Find $L[X|Y]$.

2. The figure below shows the six equally likely values of the random pair (X, Y) . Specify the functions of:

- $L[Y | X]$
- $E(X | Y)$
- $L[X | Y]$
- $E(Y | X)$



4 Markov Chains

P is a **transition probability matrix** if:

1. All of the entries are non-negative.
2. The sum of entries in each row is 1.

A **Markov chain** is defined by four things: $(\mathcal{X}, \pi_0, P, \{X_n\}_{n=0}^\infty)$

\mathcal{X} Set of states

π_0 Initial probability distribution

P Transition probability matrix

$\{X_n\}_{n=0}^\infty$ Sequence of random variables where:

$$P[X_0 = i] = \pi_0(i), i \in \mathcal{X}$$

$$P[X_{n+1} = j | X_n = i, X_{n-1}, \dots, X_0] = P(i, j), \forall n \geq 0, \forall i, j \in \mathcal{X}$$

A Markov chain is **irreducible** if we can go from any state to any other state, possibly in multiple steps.

Periodicity has to do with the period of occurrence of a state. If a state s has period 2, the Markov chain can be in s at every other time point. If a state has period 1, it's aperiodic; otherwise, it's periodic. More quantitatively, define value $d(i)$ for each state i as:

$$d(i) := \text{g.c.d}\{n > 0 | P^n(i, i) = P[X_n = i | X_0 = i] > 0\}, i \in \mathcal{X}$$

If $d(i) = 1$, then the Markov chain is **aperiodic**. If $d(i) \neq 1$, then the Markov chain is periodic and its **period** is $d(i)$.

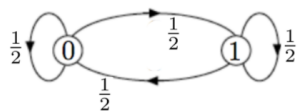
A distribution π is **invariant** for the transition probability P if it satisfies the following **balance equations**

$$\pi \cdot P = \pi.$$

Theorem 24.3: A finite irreducible Markov chain has a unique invariant distribution.

Theorem 24.4: All irreducible and aperiodic Markov chains converge to the unique invariant distribution. If a Markov chain is finite and reducible, the amount of time spent in each state approaches the invariant distribution as n grows large

Equations that model what will happen at the next step are called **first step equations**



Denote $\beta(i, j)$ as the expected amount of time it would take to move from i to j . $\beta(0, 1) = 1 + \frac{1}{2} \cdot \beta(0, 1)$
 $\beta(1, 1) = 0$

4.1 Questions

1. Life of Alex

Alex is enjoying college life. She spends a day either studying, partying, or looking for housing for the next year. If she is studying, the chances of her studying the next day are 30%, the chances of her partying the next day are 50%, and the chances of her looking for housing the next day are 20%. If she is partying, the chances of her partying the next day are 10%, the chances of her studying the next day are 60%, and the chances of her looking for housing the next day are 30%. If she is looking for housing, the chances of her looking for housing the next day are 50%, the chances of her partying the next day are 30% and the chances of her studying the next day are 20%.

- (a) Draw a Markov chain to visualize Alex's life.

- (b) Write out a matrix to represent this Markov chain.

- (c) If Alex studies on Monday, what is the chance that she is partying on Friday?
(Don't do the math, just write out the expression that you would use to find it.)

- (d) What percentage of her time should Alex expect to use looking for housing?

- (e) If Alex parties on Monday, what is the chance of Alex partying again before studying?

You have a database of an infinite number of movies. Each movie has a rating that is uniformly distributed in 0, 1, 2, 3, 4, 5 independent of all other movies. You want to find two movies such that the sum of their ratings is greater than 7.5 (7.5 is not included).

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3. **Bet On It**

Smith is in jail and has 3 dollars; he can get out on bail if he has 8 dollars. A guard agrees to make a series of bets with him. If Smith bets A dollars, he wins A dollars with probability 0.4 and loses A dollars with probability 0.6.

- a) Find the probability that he wins 8 dollars before losing all of his money if he bets 1 dollar each time.

- b) Find the probability that he wins 8 dollars before losing all of his money if he bets, each time, as much as possible but not more than necessary to bring his fortune up to 8 dollars.

- c) Which strategy gives Smith the better chance of getting out of jail?

4. **Tossing Coins**

A fair coin is tossed repeatedly and independently. Find the expected number of tosses till the pattern HTH appears.