VARIANCE, JOINT DISTRIBUTIONS, COVARIANCE 9

COMPUTER SCIENCE MENTORS 70

Oct 29 - Nov 2, 2018

1 Variance

1.1 Introduction

Variance: The variance of a random variable X is defined as

$$\text{Var}(X) = \text{E}((X - \text{E}(X))^2) \ = \text{E}(X^2) - \text{E}(X)^2$$

The latter version of variance is the one we usually use in computations.

The square root of Var(X) is called the standard deviation of X. It is usually denoted with the variable σ.
Important property of variance: for some constant c,

$$\text{Var}(cX) = c^2 * \text{Var}(X)$$

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1.2 Questions

1. **True or False?**
   Assume that X is a discrete random variable. If $Var(X) = 0$, then X is a constant.

   Solution: TRUE. Let $\mu = \text{E}[X]$. By definition,
   $$0 = \text{Var}(X) = \text{E}[(X - \mu)^2] = \sum \text{P}[\omega](X(\omega) - \mu)^2$$

The RHS is the sum of non-negative numbers, so if the sum is 0, each term must be 0. So

$$\mathrm{P}[\omega] > 0 \to (X(\omega) - \mu)^2 = 0 = X(\omega) = \mu.$$

Therefore X is constant (equal to $\mu = \mathrm{E}[X]$).

2. Show that $\mathrm{Var}(X) = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$ where X is any random variable and $\mu = \mathrm{E}[X]$.

Solution:

$$\mathrm{Var}(X) = \mathrm{E}((X - \mu^2) = \mathrm{E}[X^2 - 2\mu X + \mu^2]$$
$$= \mathrm{E}[X^2] - 2\mu\mathrm{E}[X] + \mu^2 \text{ (due to linearity of expectation)}$$
$$= \mathrm{E}[X^2] - \mu^2$$

3. Show that $Var(aX+b) = a^2 Var(X)$ where X is any random variable.

Solution:

$$Var(aX+b) = \mathrm{E}(aX + b - \mathrm{E}(aX+b))^2$$
$$= \mathrm{E}(aX + b - a\mathrm{E}[X] - b)^2$$
$$= \mathrm{E}(aX - a\mathrm{E}[X])^2$$
$$= a^2\mathrm{E}(X - \mathrm{E}[X])^2 = a^2 Var(X)$$

4. Let's consider the classic problems of flipping coins and rolling dice. Let X be a random variable for the number of coins that land on heads and Y be the value of the die roll.

(a) What is the expected value of X after flipping 3 coins? What is the variance of X?

Solution:

$$\mathrm{E}(X) = 0 * \frac{1}{8} + 1 * \frac{3}{8} + 2 * \frac{3}{8} + 3 * \frac{1}{8} = \frac{3}{2}$$
$$\mathrm{E}(X^2) = 0^2 * \frac{1}{8} + 1^2 * \frac{3}{8} + 2^2 * \frac{3}{8} + 3^2 * \frac{1}{8} = \frac{24}{8} = 3$$
$$\mathrm{E}(X)^2 = \frac{9}{4}$$
$$\mathrm{Var}(X) = 3 - \frac{9}{4} = \frac{3}{4}$$

(b) Let Y be the sum of rolling a dice 1 time. What is the expected value of Y?

Solution: $\mathrm{E}(Y) = \frac{1}{6} * (1+2+3+4+5+6) = \frac{7}{2}$

(c) What is the variance of Y?

Solution: $\mathrm{E}(Y^2) = [\frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2)] = \frac{91}{6}$ $\mathrm{Var}(Y) = \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 = \frac{91}{6}\frac{7}{2}^2 = \frac{35}{12}$

5. You are at a party with n people where you have prepared a red solo cup labeled with their name. Before handing red cups to your friends, you pick up each cup and put a sticker on it with probability $\frac{1}{2}$ (independently of the other cups). Then you hand back the cups according to a uniformly random permutation. Let X be the number of people who get their own cup back AND it has a sticker on it.

(a) Compute the expectation $\mathrm{E}(X)$.

Solution: Define $X_i = 1$ if the i-th person gets their own cup back and it has a sticker on it and $X_i = 0$ otherwise. Hence $\mathrm{E}(X) = \mathrm{E}(\sum_{i=1}^{n}(X_i)) = \sum_{i=1}^{n}\mathrm{E}(X_i)$ $\mathrm{E}(X_i) = \mathrm{P}[X_i = 1] = \frac{1}{2n}$ since the i-th student will get his/her cup with probability $\frac{1}{n}$ and has a sticker on it with probability $\frac{1}{2}$ and stickers are put independently. Hence $\mathrm{E}(X) = n \cdot \frac{1}{2n} = \frac{1}{2}$.

(b) Compute the variance $\mathrm{Var}(X)$

Solution: To calculate $\mathrm{Var}(X)$, we need to know $\mathrm{E}(X^2)$

$$\mathrm{E}(X^2) = \mathrm{E}(X_1 + X_2 + \ldots + X_n)^2 = \mathrm{E}(\sum_{i,j}(X_i * X_j)) = \sum_{i,j}(\mathrm{E}(X_i * X_j)$$

(by linearity of expectation)

Then we consider two cases, either $i = j$ or $i \neq j$. Hence

$$\sum_{i,j}\mathrm{E}(X_i * X_j) = \sum_{i}\mathrm{E}(X_i^2) + \sum_{i \neq j}\mathrm{E}(X_i * X_j)$$

$E(X_i^2) = \frac{1}{2n}$ for all i. To find $\mathrm{E}(X_i * X_j)$, we need to calculate $\mathrm{P}[X_i X_j = 1]$. $\mathrm{P}[X_i * X_j = 1] = \mathrm{P}[X_i = 1]\mathrm{P}[X_j = 1|X_i = 1] = \frac{1}{2n} * \frac{1}{2*(n-1)}$ since if student i has received his/her own cup, student j has $n-1$ choices left. Hence

$$\mathrm{E}(X^2) = n * \frac{1}{2n} + n * (n-1) * \frac{1}{2n} * \frac{1}{2*(n-1)} = \frac{3}{4}$$

$\text{Var}(X) = \text{E}(X^2) - \text{E}(X)^2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$.

2 Independence, Joint Distributions

2.1 Introduction

Joint Distribution: We use a joint distribution to describe a probability distribution over multiple random variables. A joint distribution is characterized by its joint probability mass function:

$$p_{xy}(x, y) = \text{P}(X = x \cap Y = y).$$

We can recover the *marginal* distributions:

$$p_x(x) = p(X = x) = \sum_y \text{P}(X = x, Y = y) = \sum_y p_{xy}(x, y)$$

$$p_y(y) = \text{P}(Y = y) = \sum_x \text{P}(X = x, y) = \sum_x p_{xy}(x, y).$$

Note that in general, we can't get the joint distribution just by knowing the marginal distributions: in order to get the joint distribution, we also have to know how the variables relate to each other. However, if two random variables are **independent**, then we know that

$$p_{xy}(x, y) = \text{P}(X = x \cap Y = y) = \text{P}(X = x)\text{P}(Y = y) = p_x(x)p_y(y).$$

We can say the following about the expectations and variances of independent variables:

$$\text{E}[XY] = \text{E}[X]\text{E}[Y]$$

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

These facts are proved in the notes.

2.2 Questions

1. Mr. and Mrs. Brown decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let G denote the numbers of girls that the Browns have. Let C be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

Solution: The sample space is the set of all possible sequences of children that the Browns can have: $\omega = \{g, bg, bbg, bbbg\}$. The probabilities of these sample points are:

$$P(g) = \frac{1}{2}$$
$$P(bg) = \frac{1}{2} + \frac{1}{2} = \frac{1}{4}$$
$$P(bbg) = \frac{1}{2}^3 = \frac{1}{8}$$
$$P(bbb) = \frac{1}{2}^3 = \frac{1}{8}$$

(b) Compute the joint distribution of G and C. Fill in the table below.

	C = 1	**C = 2**	**C = 3**
G = 0			
G = 1			

Solution:

	C = 1	**C = 2**	**C = 3**
G = 0	0	0	P(bbb) = 1/8
G = 1	P(g) = 1/2	P(bg) = 1/4	P(g) = 1/8

(c) Use the joint distribution to compute the marginal distributions of G and C and confirm that the values are as you'd expect. Fill in the tables below.

P(G = 0)	
P(G = 1)	

P(C = 1)	**P(C = 2)**	**P(C = 3)**

Solution: Marginal distribution for G:

$$P(G = 0) = 0 + 0 + \frac{1}{8} = \frac{1}{8}$$
$$P(G = 1) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

Marginal distribution for C:

$$P(C = 1) = 0 + \frac{1}{2} = \frac{1}{2}$$

$$P(C = 2) = 0 + \frac{1}{4} = \frac{1}{4}$$
$$P(C = 3) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

(d) Are G and C independent?

Solution: No, G and C are not independent. If two random variables are independent, then

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

To show this dependence, consider an entry in the joint distribution table, such as $P(G = 0, C = 3) = \frac{1}{8}$. This is not equal to $P(G = 0)P(C = 3) = \frac{1}{8} * \frac{1}{4} = \frac{1}{32}$, so the random variables are not independent.

(e) What is the expected number of girls the Browns will have? What is the expected number of children that the Browns will have?

Solution: We can apply the definition of expectation directly for this problem, since we?ve computed the marginal distribution for both random variables.

$$E(G) = 0 * P(G = 0) + 1 * P(G = 1) = \frac{7}{8}$$

$$E(C) = 1 * P(C = 1) + 2 * P(C = 2) + 3 * P(C = 3) = 1 * \frac{1}{2} + 2 * \frac{1}{4} + 3 * \frac{1}{4} = \frac{7}{4}$$

2. Define a probability distribution over a set of events S to be *uniform* if the probability of each event in the set occurring is equal. Find a joint distribution on X and Y such that X and Y are each uniform on the set $\{1, 2, 3, 4\}$, but (X, Y) is not uniform on the set $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$.

Solution: $P(X = x, Y = y) = \frac{1}{4}$, for $x = y$.
$P(X = x, Y = y) = 0$, for $x \neq y$

3. Say you're playing a game with a coin and die, where you flip the coin 3 times and roll the die once. In this game, your score is given by the number of heads that show multiplied with the die result. What is the expected value of your score? What's the

variance?

Solution: $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y) = \frac{21}{4}$ since X and Y are independent. $\mathrm{Var}(XY) = \mathrm{E}(X^2Y^2) - \mathrm{E}(XY)^2 = \mathrm{E}(X^2)\mathrm{E}(Y^2) - \mathrm{E}(X)^2\mathrm{E}(Y)^2 = 3 * \frac{91}{6} - \frac{3}{2}^2 * \frac{7}{2}^2 = \frac{91}{2} - \frac{9}{4} * \frac{49}{4} = 17.9375 = \frac{287}{16}$

3 Covariance, Correlation

3.1 Introduction

The **covariance** of two random variables X and Y is defined as:

$$\mathrm{Cov}(X,Y) := \mathrm{E}((X - \mathrm{E}(X)) \cdot (Y - \mathrm{E}(Y)))$$

If $\mathrm{Cov}(X,Y) > 0$, X and Y are *positively* linearly correlated.
If $\mathrm{Cov}(X,Y) < 0$, X and Y are *negatively* linearly correlated.

3.2 Questions

1. Prove that $\mathrm{Cov}(X,Y) = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]$.

Solution:

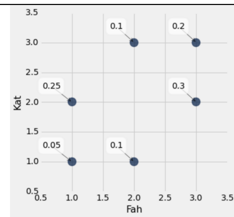
$$\begin{aligned} \mathrm{Cov}(X,Y) &:= \mathrm{E}((X - \mathrm{E}(X)) \cdot (Y - \mathrm{E}(Y))) \\ &= \mathrm{E}(X \cdot Y - X \cdot \mathrm{E}(Y) - \mathrm{E}(X) \cdot Y + \mathrm{E}(X) \cdot \mathrm{E}(Y)) \\ &= \mathrm{E}(X \cdot Y) - \mathrm{E}(X) \cdot \mathrm{E}(Y) - \mathrm{E}(X) \cdot \mathrm{E}(Y) - \mathrm{E}(X) \cdot \mathrm{E}(Y) + \mathrm{E}(X) \cdot \mathrm{E}(Y) \\ &= \mathrm{E}(X \cdot Y) - \mathrm{E}(X) \cdot \mathrm{E}(Y) \end{aligned}$$

2. Prove that $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$.

Solution:

$$\mathrm{Cov}(X,X) = \mathrm{E}(X \cdot X) - \mathrm{E}(X) \cdot \mathrm{E}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$$

3. Consider the following distribution with random variables Fah and Kat:



Find the covariance of Fah and Kat.

Solution: E(Fah) $= 1 \cdot .3 + 2 \cdot .2 + 3 \cdot .5 = \mathbf{2.2}$
E(Kat) = $1 \cdot .15 + 2 \cdot .55 + 3 \cdot .3 = \mathbf{2.15}$
E(KatFah) = $1 \cdot 1 \cdot .05 + 1 \cdot 2 \cdot .25 + 2 \cdot 1 \cdot .1 + 2 \cdot 3 \cdot .1 + 3 \cdot 2 \cdot .3 + 3 \cdot 3 \cdot .2 = \mathbf{4.95}$
cov(Kat, Fah) = $4.95 - 2.2 \cdot 2.15 = \mathbf{0.22}$

4. (a) Prove that if X and Y are independent, then $\text{Cov}(X, Y) = 0$.

Solution:
$$\text{Cov}(X, Y) = \text{E}(X \cdot Y) - \text{E}(X) \cdot \text{E}(Y)$$
Remember that a property of expectation is that if X and Y are independent, then $\text{E}(XY) = \text{E}(X) \cdot \text{E}(Y)$, so we get 0 when we subtract.

(b) Prove that the converse is not necessarily true. In other words, give an example of 2 random variables whose covariance is 0 but are not independent.

Solution: There are a number of examples that can be used.
One possible solution: Let X be a random variable that with uniform probability space {-1, 1}. Let Y be a random variable such that $Y = 0$ if $X = -1$ and Y is -1 or 1 with equal probability if $X = 1$.

$$\text{Cov}(X, Y) = \text{E}(XY) - \text{E}(X) \cdot \text{E}(Y)$$
$\text{E}(X)$ and $\text{E}(Y)$ = 0, so:
$$\text{Cov}(X, Y) = \text{E}(XY)$$
$= \text{P}(X = -1)(-1)(0) + \text{P}(X = 1, Y = 1)(1)(1) + \text{P}(X = 1, Y = -1)(-1)(1) = 0$

So, $\text{Cov}(X, Y) = 0$. But, X and Y are clearly dependent, and thus this is a valid counterexample.

5. Roll 2 dice. Let A be the number of 6's you get, and B be the number of 5's, find $\text{Cov}(A, B)$.

Solution: $\mathrm{E}(A) = \frac{1}{6}$ for one die, by linearity of expectation, two dice make $\frac{1}{3}$, same for $\mathrm{E}(B)$ $\mathrm{E}(A) = \frac{1}{3}$, $\mathrm{E}(B) = \frac{1}{3}$

AB can be either 0 (if no 5's or 6's show up) or 1 (get a 5 and a 6).

$$\begin{aligned}\mathrm{E}(AB) &= 1 \cdot \mathrm{P}[\text{get a 5 and a 6}] \\ &= \mathrm{P}[\text{first die = 5 and second die = 6}] + \mathrm{P}[\text{first die = 6 and second die = 5}] \\ &= \frac{1}{36} + \frac{1}{36}\end{aligned}$$

$$\begin{aligned}\mathrm{Cov}(AB) &= \mathrm{E}(AB) - \mathrm{E}(A) \cdot \mathrm{E}(B) \\ &= \frac{1}{18} - \frac{1}{9} \\ &= -\frac{1}{18}\end{aligned}$$