GRAPHS 2

COMPUTER SCIENCE MENTORS 70

September 10 to September 14, 2018

1 Graph Theory

1.1 Introduction

1. Let $G = (V, E)$ be an undirected graph. Match the term with the definition.

Path/Simple Path	Tour	Walk	Tournament	Cycle	Eulerian Tour

__________Sequence of edges.
__________Sequence of edges that does not repeat vertices.
__________Sequence of edges that starts and ends at the same vertex.
__________Sequence of edges that starts and ends on the same vertex and does not repeat any other vertices.
__________Sequence that uses each edge exactly once and starts and ends at the same vertex.
__________Directed graph in which every pair of distinct vertices is connected by a single directed edge.

Note: In CS 70, we typically assume paths are simple paths.

Additional Note: The questions below do not cover Eulerian tours, but they are an important topic included in the optional practice that you should review on your own.

Solution: walk: Sequences of edges.
path/simple path: Sequence of edges that does not repeat vertices.

tour: Sequence of edges that starts and ends at the same vertex.
cycle: Sequence of edges that starts and ends on the same vertex and does not repeat any other vertices.
eulerian tour: Sequence that uses each edge exactly once and starts and ends at the same vertex.
tournament: Directed graph in which every pair of distinct vertices is connected by a single directed edge.

1.2 Build-up Error

In this section we will work through an example of buildup error.
Faulty Claim: If a graph has average degree k, more than half the vertices must have degree at most k.
Proof: We use induction on the number of vertices n.
Base Case: A graph with just 1 vertex has average degree 0. 1 out of 1 vertices, or more than half of the vertices have degree 0.
Inductive Hypothesis: For a graph with n vertices that has average degree k, more than half of the vertices have degree at most k.
Inductive Step: Consider a graph of n vertices that has average degree k. By our inductive hypothesis, we claim that at least $\frac{n}{2}$ vertices have degree at most k. Add another vertex to this graph. In order for the graph to still have average degree k, we need to connect the new vertex to exactly $\frac{k}{2}$ vertices. Now we have an $n+1$ vertex graph with at least $\frac{n}{2}+1$ vertices with at most degree k. $\frac{n}{2}+1 \geq \frac{n+1}{2}$ as desired.

1. Give a counter-example to show the claim is false.

Solution: Consider K_8 where every node has degree 7, then remove 6 edges from 2 vertices. 6 vertices have degree at least 5 since in the form K_6, there are 16 edges, which corresponds to an average degree of $\frac{32}{8}$ of 4.

2. Since the claim is false, there must be an error in the proof. Explain the error.

Solution: One problem is that the proof doesn't work for odd numbers of vertices, as $\frac{n}{2}$ is not an integer then. The other, more fundamental problem, is that for $P(n+1)$ to be true, we must show that for every $(n+1)$-vertex graph with average degree k, more than half of the vertices must have degree at most k. Instead, the proof shows that every $(n+1)$-vertex graph with average degree k that can be constructed by adding a vertex of positive degree to an existing (n)-vertex graph with average degree k. Confirm that there is no way to build your counter-example

graph by the method in the proof.
More generally, this is an example of "build-up error". This error arises from a faulty assumption that every graph of size $n + 1$ with some property can be built by adding a vertex to an n vertex graph that also has that property. This assumption is correct in some cases, and incorrect in others.

1.3 Questions

1. Given a graph G with n vertices, where n is even, prove that if every vertex has degree $\frac{n}{2} + 1$, then G must contain a 3-cycle.

Solution: Let G be a graph with n vertices, where n is even, and every vertex has degree $\frac{n}{2} + 1$. Select any two vertices u and v, with an edge between them. There are $n - 2$ remaining vertices, and both u and v are connected to $\frac{n}{2}$ of these (because they have degree $\frac{n}{2} + 1$ and are connected to each other). Therefore, there must be some vertex w such that both u and v are connected to w (otherwise the set of $\frac{n}{2}$ vertices connected to u and the set of $\frac{n}{2}$ vertices connected to v would be disjoint, which contradicts the fact that there are only $n - 2$ of these vertices). Thus we have edges (u, v), (v, u) and (w, u), so the graph contains a 3-cycle.

2. Every tournament has a Hamiltonian path. (Recall that a Hamiltonian path is a path that visits each vertex exactly once.)

> **Solution:** *Base Case*: For $n = 1$ nodes, there is a trivial Hamiltonian path.
> *Inductive Hypothesis*: Assume that for a tournament with n nodes, there is a Hamiltonian path.
> *Inductive Step*: Consider a tournament T with $n + 1$ nodes. Take an arbitrary node x, and remove it along with its incident edges. The resulting subgraph T' is also a tournament (each node in T' still shares some edge with every other node in T'). By the Inductive Hypothesis, there is some Hamiltonian path in T'. Let this Hamiltonian Path be $v_1, v_2, v_3, \ldots, v_n$. Now we consider T. Note that since T is a tournament, x shares an edge with every other node in T. There are three possible cases:
>
> *Case 1*: Everybody beat x (there is no edge from x to any node in T'). Then there is an edge (v_n, x). Thus, there is a Hamiltonian Path in T, namely $v_1, v_2, v_3, \ldots, v_n, x$.
>
> *Case 2*: x beat everybody (there is no edge from any node in T' to x). Then there is an edge (x, v_1). Thus, there is a Hamiltonian Path in T, namely $x, v_1, v_2, v_3, \ldots, v_n$.
>
> *Case 3*: There is some v_i that is the last person who beat x, in the ordering $v_1, \ldots, v_n$. Note that v_i must exist because we are not in Case 2, and $i \neq n$ because we are not in Case 1. Then since v_i is the last person who beat x, there is an edge (v_i, x), and an edge (x, v_{i+1}). Thus, there is a Hamiltonian path in T, namely $v_1, v_2, v_3, \ldots, v_i, x, v_{i+1}, \ldots, v_n$. These are the only possible cases, so it must be that T has a Hamiltonian Path.
>
> Therefore by induction, any tournament has a Hamiltonian Path.

2 Trees

2.1 Introduction

> If complete graphs are maximally connected, then trees are the opposite: Removing just a single edge disconnects the graph! Formally, there are a number of equivalent definitions for identifying a graph $G = (V, E)$ as a tree.

Assume G is connected. There are 3 other properties we can use to define it as a tree.

1. G contains ________ cycles.

2. G has ________ edges.

3. Removing any additional edge will ______________

> **Solution:** no, $n-1$, disconnect G

One additional definition:

4. G is a tree if it has no cycles and ________________________

> **Solution:** adding any edge creates a cycle

> **Theorem**: G is connected and contains no cycles if and only if G is connected and has $n-1$ edges.

2.2 Questions

1. Now show that if a graph satisfies either of these two properties then it must be a tree:

   a If for every pair of vertices in a graph they are connected by exactly one simple path, then the graph must be a tree.

   Solution: Assume we have a graph with the property that there is a unique simple path between every pair of vertices. We will show that the graph is a tree, namely, it is connected and acyclic. First, the graph is connected because every pair of vertices is connected by a path. Moreover, the graph is acyclic because there is a unique path between every pair of vertices. More explicitly, if the graph has a cycle, then for any two vertices x, y in the cycle there are at least two simple paths between them (obtained by going from x to y through the right or left half of the cycle), contradicting the uniqueness of the path. Therefore, we conclude the graph is a tree.

   b If the graph has no simple cycles but has the property that the addition of any single edge (not already in the graph) will create a simple cycle, then the graph is a tree.

   Solution: Assume we have a graph with no simple cycles, but adding any edge will create a simple cycle. We will show that the graph is a tree. We know the graph is acyclic because it has no simple cycles. To show the graph is connected, we prove that any pair of vertices x, y are connected by a path. We consider two cases: If (x, y) is an edge, then clearly there is a path from x to y. Otherwise, if (x, y) is not an edge, then by assumption, adding the edge (x, y) will create a simple cycle. This means there is a simple path from x to y obtained by removing the edge (x, y) from this cycle. Therefore, we conclude the graph is a tree.

2. A **spanning tree** of a graph G is a subgraph of G that contains all the vertices of G and is a tree.
   Prove that a graph $G = (V, E)$ if connected if and only if it contains a spanning tree.

   Solution: First the if direction. If a graph contains a spanning tree, which is a connected graph that contains all the vertices, there is a path between any two vertices, so the graph is connected.

   Now the only if. Let G be a connected graph. Either G is already a tree, in which case it is its own spanning tree, or else there is an edge that can be removed from G while it remains connected. Because there are only a finite number of edges, we can continue this process until no more edges can be removed, at which point we will have found our spanning tree.

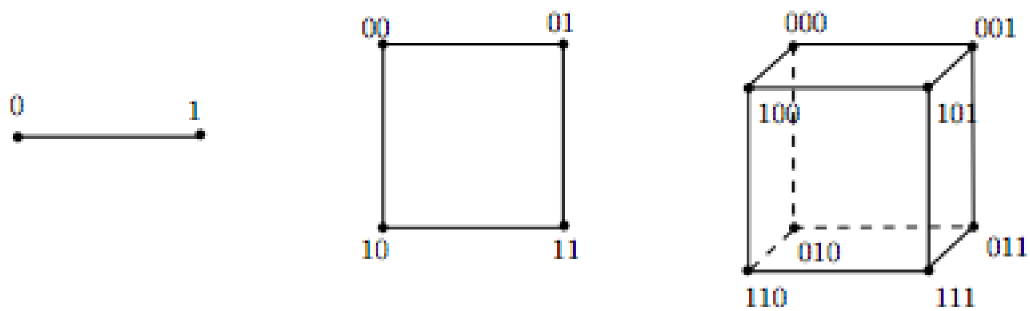
3 Hypercubes

3.1 Introduction

What is an n dimensional hypercube?

Bit definition: Two vertices x and y are adjacent and only if x and y differ in exactly one bit position.

Recursive definition: Define the 0-subcube as the $(n-1)$ dimensional hypercube with vertices labeled 0x (x is an element of $(0,1)^{n-1}$). Do the same for the 1-subcube with vertices labeled 1x. Then an n dimensional hypercube is created by placing an edge between 0x and 1x in the 0-subcube and 1-subcube respectively.



3.2 Questions

1. How many vertices and edges does an n dimensional hypercube have?

   Solution: 2^n vertices, $n * 2^{n-1}$ edges

2. How many edges do you need to cut from a hypercube to isolate one vertex in an n-dimensional hypercube?

   Solution: n because each node has n edges.

3. Prove that any cycle in an n-dimensional hypercube must have even length.

   Solution: Answer: Here are three ways to solve this problem: here we will argue via bit flips, but there also exist arguments using the parity of Hamming distance, or induction on n. Note that induction on n is more difficult and prone to build-up error.

Answer 1: Bit flips

Main idea: moving through an edge in a hypercube flips exactly one bit, and moreover each bit must be flipped an even number of times to end up at the starting vertex of the cycle.

Proof: Each edge of the hypercube flips exactly one bit position. Let E_i be the set of edges in the cycle that flip bit i. Then $|E_i|$ must be even. This is because bit i must be restored to its original value as we traverse the cycle, which means that bit i must be flipped an even number of times. Since each edge of the cycle must be in exactly one set E_j, the total number of edges in the cycle = $\sum_j |E_j|$ is a sum of even numbers and therefore even.