COMPUTABILITY, COUNTING, COMBINATORIAL PROOFS 6

COMPUTER SCIENCE MENTORS 70

October 8 to 12, 2018

1 Computability

1.1 Introduction

The Halting Problem: Does a given program ever halt when executed on a given input? This given input has to be general.

$$\texttt{TestHalt(P,x)} = \begin{cases} \text{"yes"}, & \text{if program } P \text{ halts on input } x \\ \text{"no"}, & \text{if program } P \text{ loops on input } x \end{cases}$$

How do we prove that `TestHalt` does not exist? Let's assume that it does, and hope we reach a contradiction.
Define another program:

```
Turing(P)
  if TestHalt(P,P) = "yes" then loop forever
  else halt
```

What happens when we call `Turing(Turing)`?

Case 1 : It halts. If `Turing(Turing)` halts then `TestHalt(Turing, Turing)` must have returned no. But `TestHalt(Turing Turing)` calls `Turing(Turing)` and calling `Turing(Turing)` must loop. But we assumed that `Turing(Turing)` halted. Contradiction.

Case 2 : It loops. This implies that `TestHalt(Turing, Turing)` returned yes, which by the way that `TestHalt` is defined implies that `Turing` halted. But we assumed that `Turing(Turing)` looped. Contradiction.

1.2 Questions

1. Determine the computability of the following tasks. If it's not computable, write a reduction or self-reference proof. If it is, write the program.

    1. You want to determine whether a program P on input x prints "Hello World!" Is there a computer program that can perform this task? Justify your answer?

    2. You want to determine whether a program P prints "Hello World!" before running the kth line of the program.

    3. You want to determine whether a program P prints "Hello World!" in the first k steps of its execution. Is there a computer program that can perform this task? Justify your answer?

2 Intro to Counting

Counting:
In this class, the basic premise of counting is determining the total number of possible ways something can be done. Reaching a particular outcome requires a number of specific choices to be made. To figure out the total number of possible outcomes, we multiply together the number of potential choices at each step.

1. You're getting ready in the morning, and you have to choose your outfit for the day.
   (a) You need to wear a necklace, a vest, and a sweater. Depending on the day, you decide whether it is worth wearing your watch. If you have 3 necklaces, 2 vests, and 4 sweaters, how many different combinations do you choose from each morning?

   (b) Now the order in which you put on your necklace, vest, and sweater matters. Specifically, your look after putting on necklace n first, vest v, and then sweater s is different than if you put on vest v first, necklace n, and then sweater s. When you put on your watch is irrelevant. Now how many options do you have?

Ordering and Combinations:
An important idea of counting is dealing with situations in which all of our choices must be drawn from the same set. Here is a chart which walks you through how to solve problems relating to this idea:

Order matters, with replacement Example: How many 3 letter "words" can we make with the letters a, b, c, and d assuming we can repeat letters? Answer: $4^3 = 64$ General problem: From a set of n items, how many ways can we choose k of them, assuming that we can choose the same item multiple times and the order in which we choose the items matters? General Form: n^k	**Order matters, without replacement** Example: How many 3 letter "words" can we make with the letters a, b, c, d, e, and f using each letter exactly once? Answer: $\frac{6!}{(6-3)!} = 120$ General problem: From a set of n items, how many ways can we choose k of them, assuming that we can choose a given item exactly once and the order in which we choose the items matters? Answer: $\mathrm{P}(n,k) = \frac{n!}{(n-k)!}$
Order doesn't matter, without replacement Example: How many ways can I pick a team of 3 from 7 possible people? Answer: $\frac{7!}{(7-3)!(7-4)!} = 35$ General problem: From a set of n items, how many ways can we choose k of them, assuming that we can choose a given item exactly once and the order that we choose the items doesn't matter? General Form: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$	**Special note: Sequencing** Example: How many different orderings are there of the letters in "CAT"? Answer: 3! How many different orderings are there of the letters in "BOOKKEEPER?"? Answer: $\frac{10!}{2!2!3!}$

1. How many ways are there to arrange the letters of the word "SUPERMAN"
   (a) On a straight line?
   (b) On a straight line, such that "SUPER" occurs as a substring?
   (c) On a circle?
   (d) On a circle, such that "SUPER" occurs as a substring?
2. (a) You have 15 chairs in a room and there are 9 people. How many different ways can everyone sit down?
   (b) How many ways are there to fill 9 of the 15 chairs? (We do not care who sits in them.)
3. The numbers 1447, 1005, and 1231 have something in common. Each of them is a four digit number that begins with 1 and has two identical digits. How many numbers like this are there?
4. How many ways can you deal 13 cards to each of 4 players so that each player gets one card of each of the 13 values (A, 2, 3. . .K)?
5. We grab a deck of cards and it is poker time. Remember, in poker, order does not matter.
   (a) How many ways can we have a hand with exactly one pair? This means a hand with ranks (a, a, b, c, d).
   (b) How many ways can we have a hand with four of a kind? This means a hand with ranks (a, a, a, a, b).
   (c) How many ways can we have a straight? A straight is 5 consecutive cards, that do not all necessarily have the same suit.
   (d) How many ways can we have a hand of all of the same suit?

(e) How many ways can we have a straight flush? This means we have a consecutive-rank hand of the same suit. For example, (2, 3, 4, 5, 6), all of spades, is a straight flush, while (2, 3, 5, 7, 8), all of spades, is NOT, as the ranks are not consecutive.

3 Counting

3.1 Introduction

Balls and Bins:

a. **Distributing n distinguishable balls amongst k distinguishable bins:** Each ball has k possible bins to go into, and there are n balls. Solution: k^n

b. **Distributing n indistinguishable balls amongst k distinguishable bins:**Solution: $\binom{n+k-1}{k-1}$
Note: Distributing balls among indistinguishable bins is not covered in CS 70!

The solution for case (b) initially seems somewhat unintuitive, but can be explained through an example.

How many ways can we distribute 7 dollar bills amongst 3 students?

Approaching this with the approaches we currently know fails: There are 7 possible options for the number of bills you give to the first student, but the number of bills you choose to give the first student has a *direct* effect on the numbers of bills you can give to the second student.

To solve this problem, we need to format it slightly differently: put the dollar bills on a line, and insert 2 dividers. Everything to the left of the first divider is given to the first student. Everything in between the dividers is given to the second student. Everything to the right of the second divider is given to the third student:

There are $\frac{9!}{7!2!} = \binom{9}{2} = 36$ ways to place 2 dividers among 9 positions such that the remaining positions are filled with dollar bills, and therefore 36 ways to distribute the money. This tactic of using dividers is commonly referred to *stars and bars* or *sticks and stones*. More generally, there are $\binom{n+k-1}{k-1}$ ways to distribute n indistinguishable items amongst k people.

3.2 Questions

1. How many ways are there to arrange the letters of the word "SUPERMAN"
   (a) On a straight line, such that "SUPER" occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

   (b) On a circle, such that "SUPER" occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

2. How many ways can you give 10 cookies to 4 friends?

3. How many solutions does $x+y+z = 10$ have, if all variables must be positive integers?

4. How many 5-digit sequences have the digits in non-decreasing order?

4 Combinatorial Proofs

4.1 Questions

1. Prove $k\binom{n}{k} = n\binom{n-1}{k-1}$ by a combinatorial proof.

2. Prove $a(n-a)\binom{n}{a} = n(n-1)\binom{n-2}{a-1}$ by a combinatorial proof.