Random Variables, Expectation, Variance

8

Computer Science Mentors 70

October 22 - 26, 2018

1 Expectation of Random Variables

1.1 Introduction

Random variable: a function $X : \omega \rightarrow R$ that assigns a real number to every outcome ω in the probability space.

Expectation: The expectation of a random variable X is defined as

$$\mathrm{E}(X) = \sum_{\alpha \in A} a * \mathrm{P}[X = a]$$

where the sum is over all possible values taken by the random variable. Expectation is usually denoted with the symbol μ.

Linearity of Expectation: For any random variables $X_1, X_2, ...X_n$, expectation is linear, i.e.:

$$E(X_1 + X_2 + ... + X_n) = E(X_1) + E(X_2) + ... + E(X_n)$$

This is true even when these random variables aren't independent.

1.2 Questions

1. You win the lottery with probability .01. If you win, you get \$10000. You lose with probability .99 and get no money. Define a random variable to represent this.

2. Given the random variable X defined as taking on the value 1 with probability 0.25, 2 with probability 0.5, and 20 with probability 0.25, what is the expectation of X?

3. Show that $E(aX + b) = aE[X] + b$ where X is any random variable.

4. Suppose X is a random variable. Does X always have to take on the value $E(X)$ at some point?

5. An urn contains n balls numbered $1, 2, \ldots, n$. We remove k balls at random (without replacement) and add up their numbers. Find the mean of the total.

6. A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence "book" appears?

7. In an arcade, you play game A 10 times and game B 20 times. Each time you play game A, you win with probability $\frac{1}{3}$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability $\frac{1}{5}$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

2 Variance

2.1 Introduction

Variance: The variance of a random variable X is defined as

$$\text{Var}(X) = \text{E}((X - \text{E}(X))^2) \ = \text{E}(X^2) - \text{E}(X)^2$$

The latter version of variance is the one we usually use in computations.

The square root of Var(X) is called the standard deviation of X. It is usually denoted with the variable σ.

Important property of variance: for some constant c,

$$\text{Var}(cX) = c^2 * \text{Var}(X)$$

.

2.2 Questions

1. **True or False?**
   Assume that X is a discrete random variable. If $Var(X) = 0$, then X is a constant.

2. Show that Var$(X) = \text{E}((X - \mu)^2) = \text{E}(X^2) - \mu^2$ where X is any random variable and $\mu = \text{E}[X]$.

3. Show that $Var(aX + b) = a^2 Var(X)$ where X is any random variable.

4. Let's consider the classic problems of flipping coins and rolling dice. Let X be a random variable for the number of coins that land on heads and Y be the value of the die roll.

   (a) What is the expected value of X after flipping 3 coins? What is the variance of X?

   (b) Let Y be the sum of rolling a dice 1 time. What is the expected value of Y?

(c) What is the variance of Y?

5. You are at a party with n people where you have prepared a red solo cup labeled with their name. Before handing red cups to your friends, you pick up each cup and put a sticker on it with probability $\frac{1}{2}$ (independently of the other cups). Then you hand back the cups according to a uniformly random permutation. Let X be the number of people who get their own cup back AND it has a sticker on it.

   (a) Compute the expectation $\mathrm{E}(X)$.

   (b) Compute the variance $\mathrm{Var}(X)$