Introduction to the Read Paper

Young Statisticians Section

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Riemann manifold Langevin and Hamiltonian Monte Carlo Methods, Girolami, M. & Calderhead, B., *J.R.Statist. Soc*. B (2011), **73**, *Part 2*

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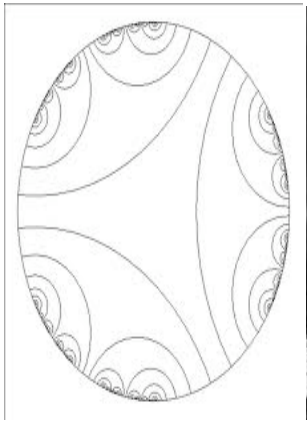


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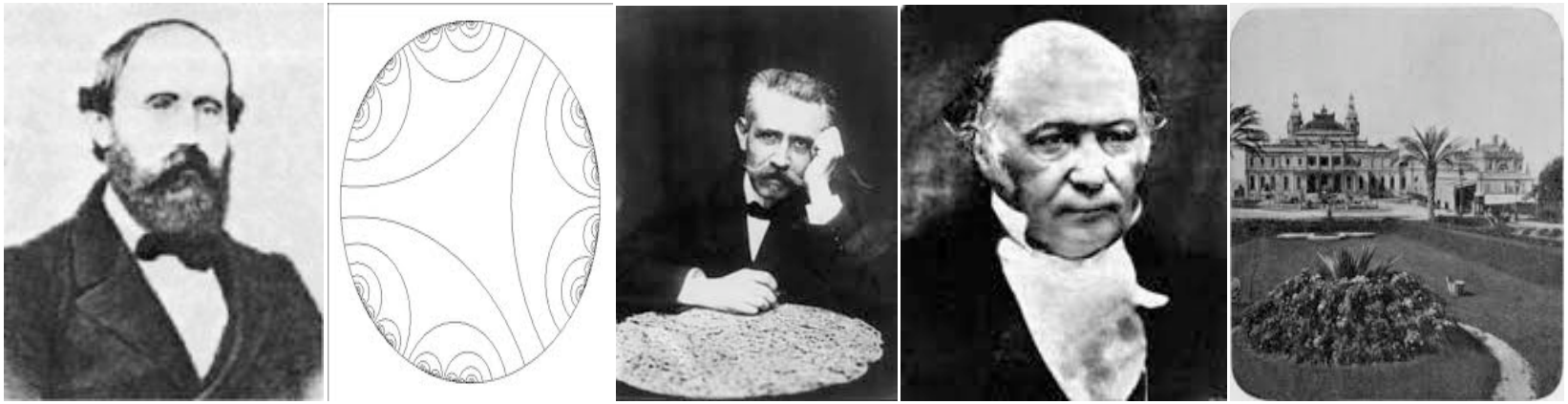

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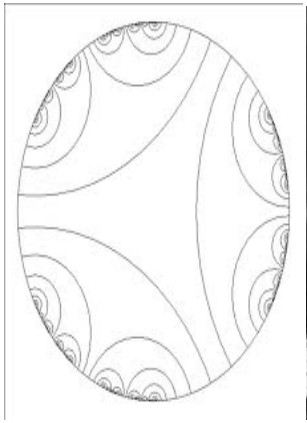


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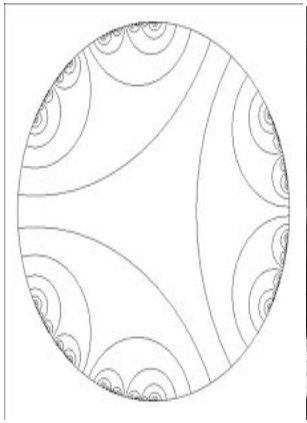


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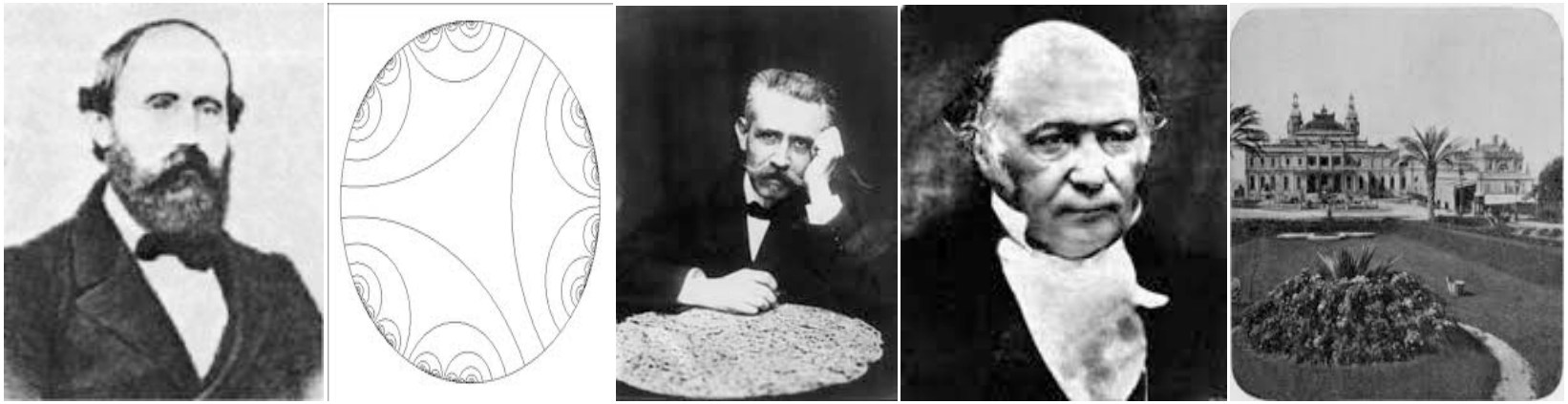

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- Focus on Hamiltonian Monte Carlo for next 27 minutes

Hamiltonian Monte Carlo for Computational Statistical Inference

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- Interpreted as separable Hamiltonian in position & momentum variables

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- Integrator provides proposals for $p(\theta|\mathbf{p})$ conditional

Illustrative Example - Bivariate Gaussian

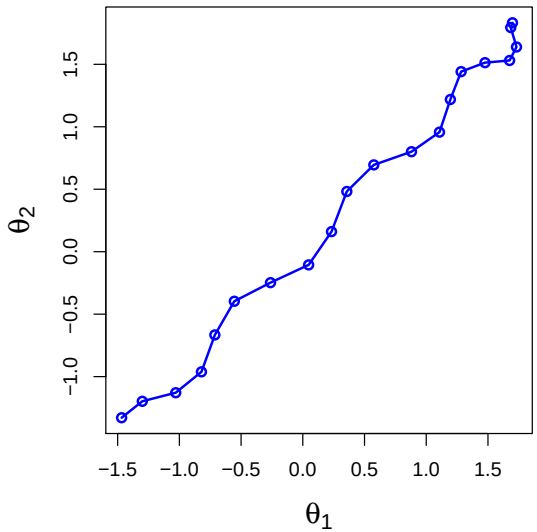
- Target density $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ where

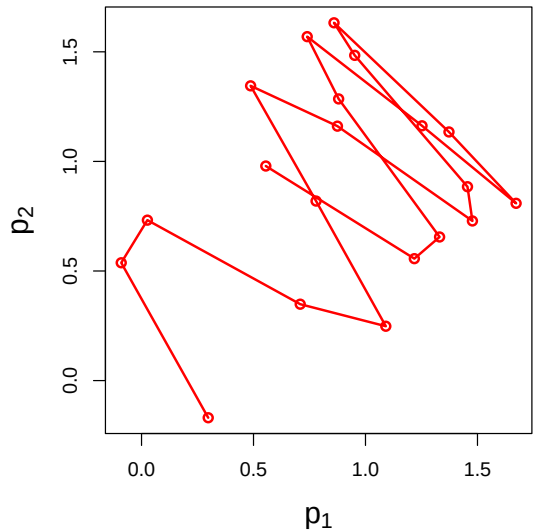
$$\mathbf{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

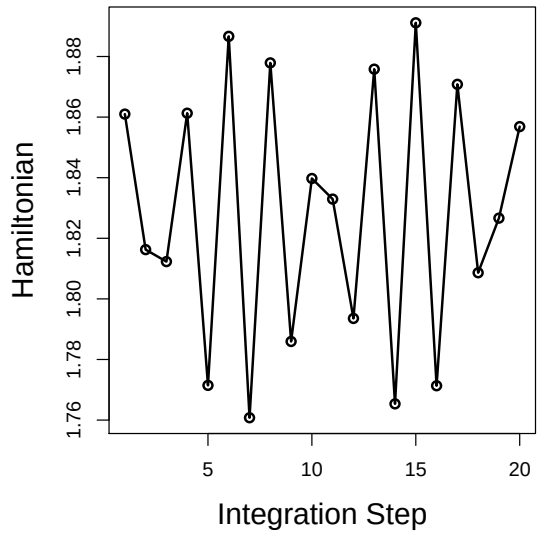
- For ρ large e.g. 0.98 sampling from this distribution is challenging
- Overall Hamiltonian

$$\frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{\Sigma}^{-1}\mathbf{x} + \frac{1}{2}\mathbf{p}^{\mathsf{T}}\mathbf{p}$$

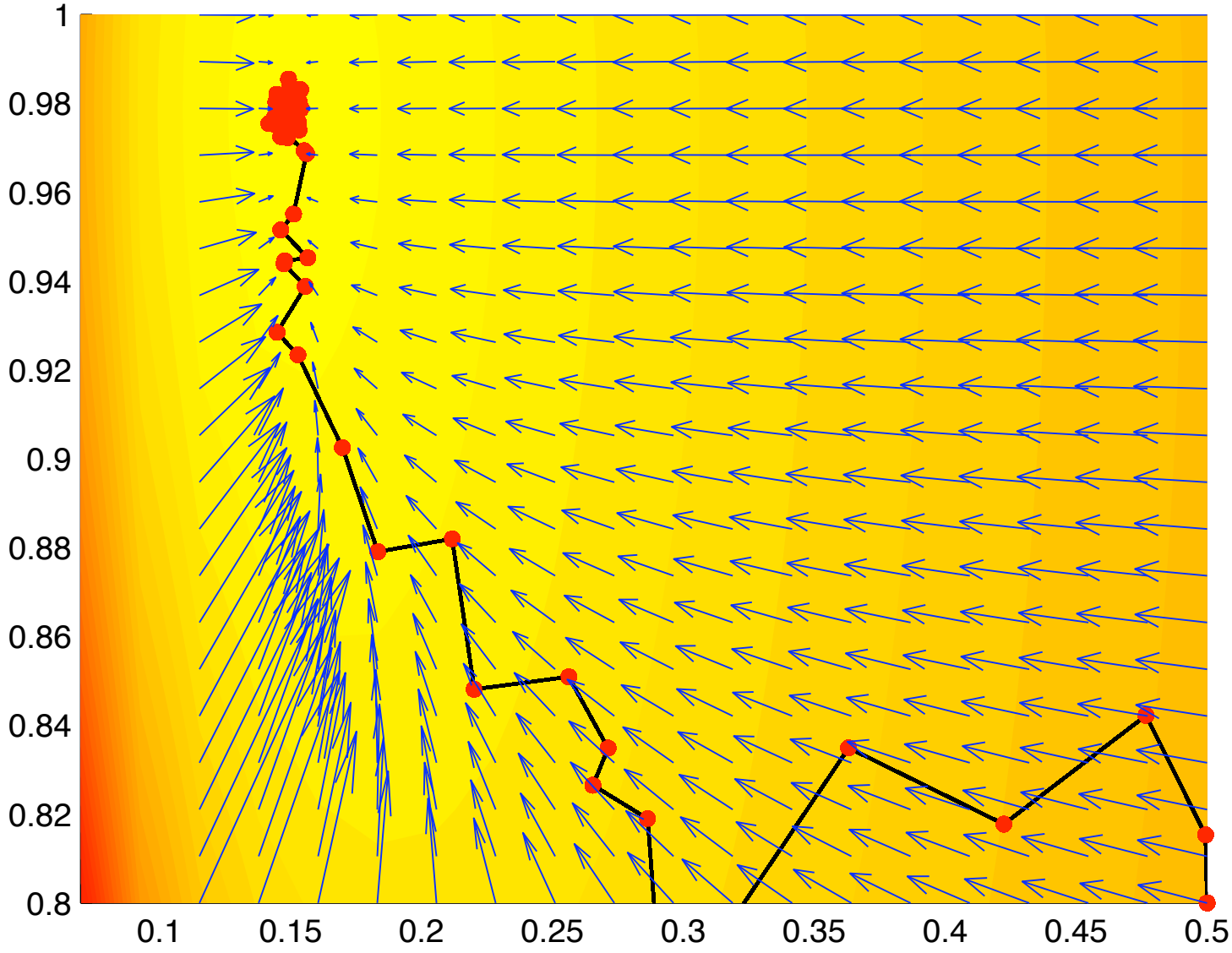
HMC Integration $\epsilon = 0.18$, $L = 20$, implicit identity matrix for metric







Metropolis Algorithm, Parameters of Stoch Vol Model, Acc Rate 25%



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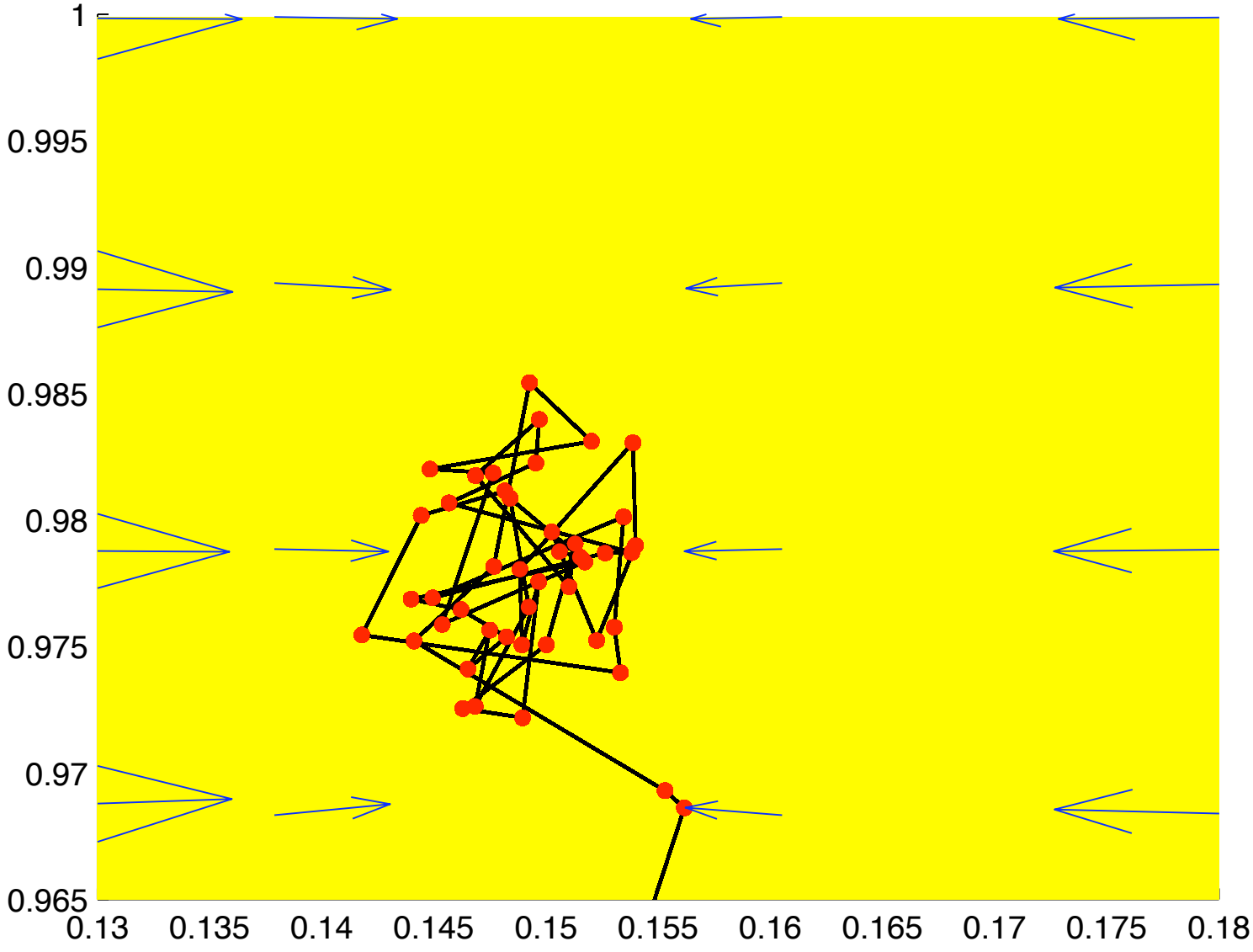

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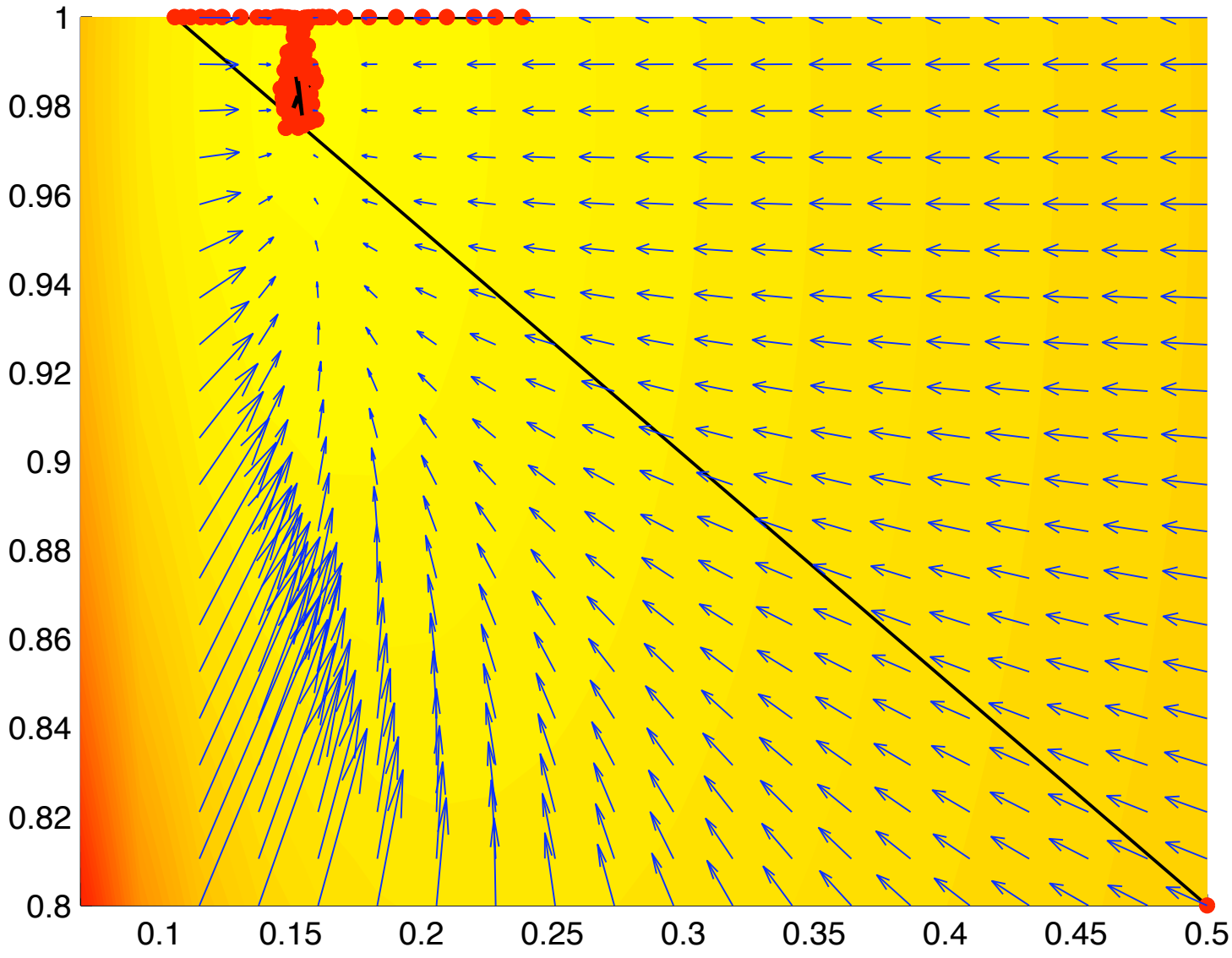

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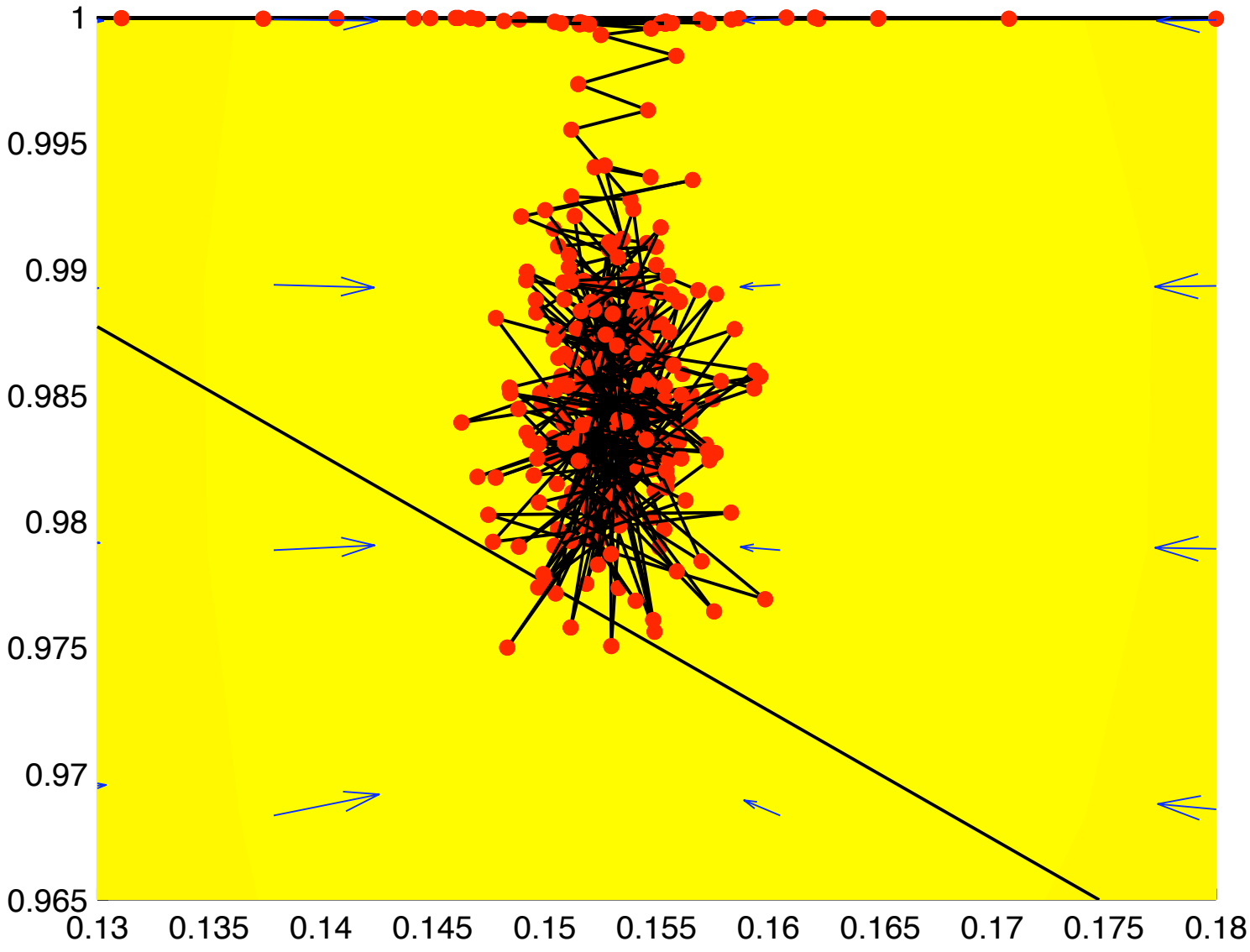

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- Can this weakness be resolved?

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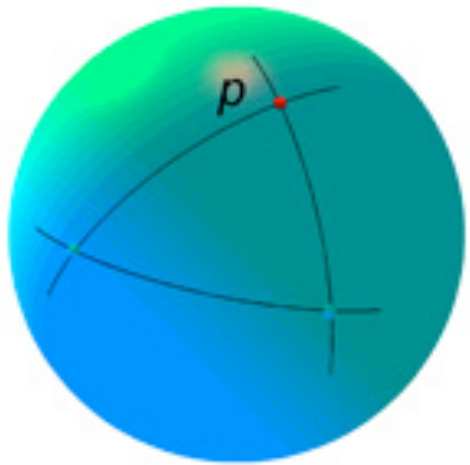

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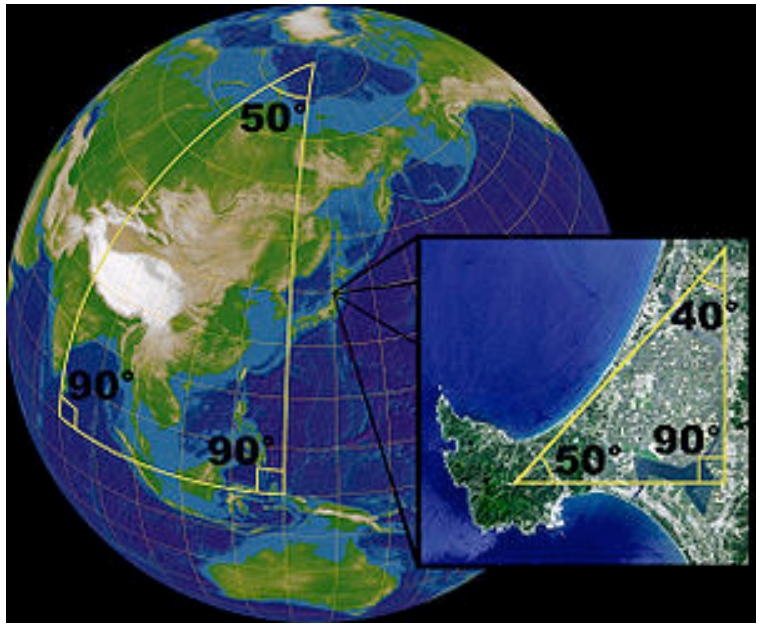
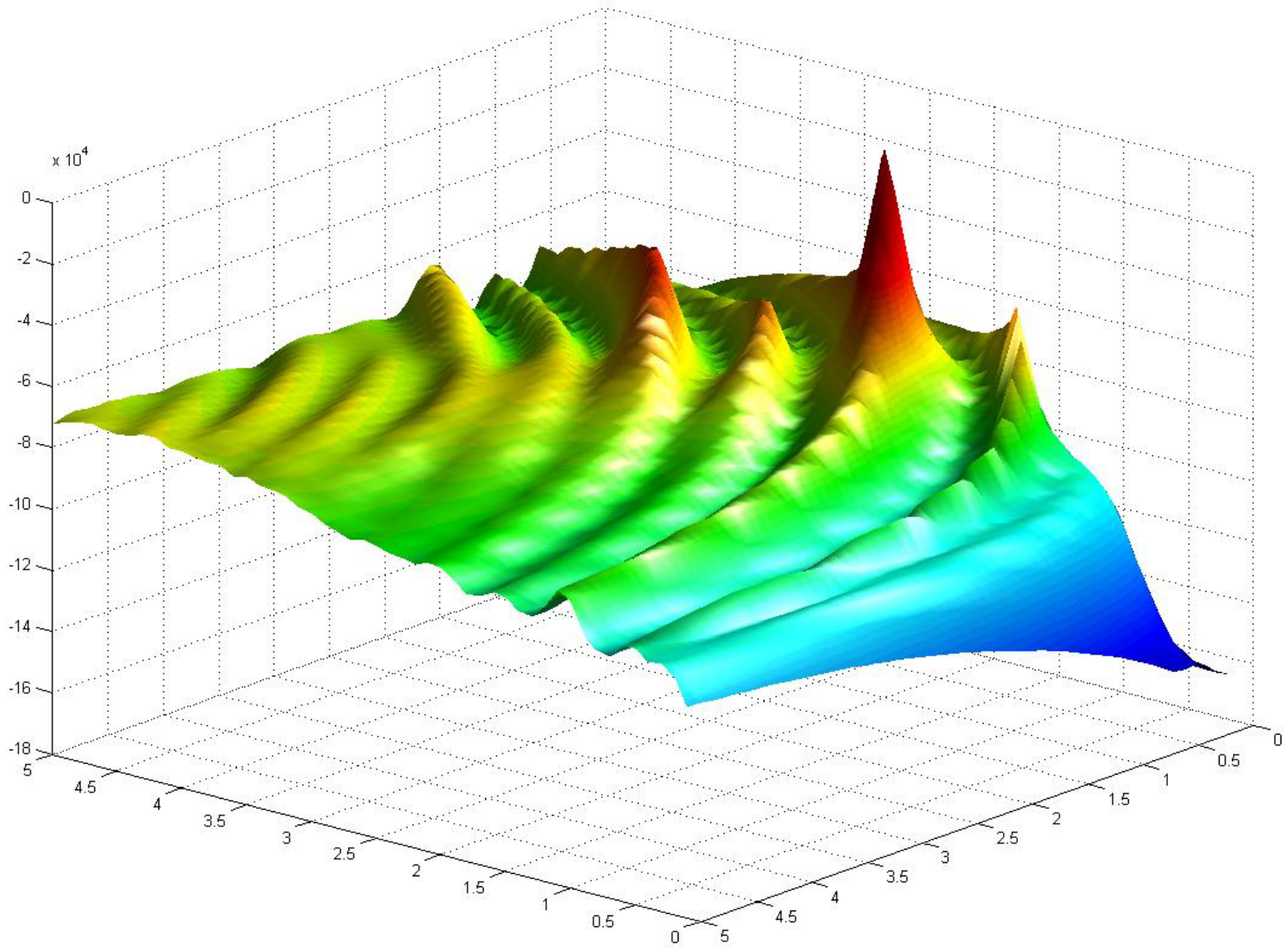


Manifold Structure in Density



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$$\begin{aligned} p(\mathbf{p}^{n+1}|\theta^{n}) &= \mathcal{N}(\mathbf{0}, \mathbf{G}(\theta^{n})) \\ \theta^{n+1}|\mathbf{p}^{n+1} &\sim p(\theta^{n+1}|\mathbf{p}^{n+1}) \end{aligned}$$

Riemannian Hamiltonian Monte Carlo

- The dynamics of the non-separable Hamiltonian follow as

$$\begin{aligned}
\frac{d\theta_i}{d\tau} &= \frac{\partial H}{\partial p_i} = \left(\mathbf{G}(\boldsymbol{\theta})^{-1}\mathbf{p}\right)_i \\
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- Require reversible, volume preserving integrator - 2'nd order semi-implicit integrator

$$\begin{aligned}
\mathbf{p}^{\tau+\frac{\epsilon}{2}} &= \mathbf{p}^{n} - \frac{\epsilon}{2}\nabla_{\boldsymbol{\theta}} H(\boldsymbol{\theta}^{\tau}, \mathbf{p}^{\tau+\frac{\epsilon}{2}}) && (1)\\
\boldsymbol{\theta}^{\tau+\epsilon} &= \boldsymbol{\theta}^{\tau} + \frac{\epsilon}{2}\left[\nabla_{\mathbf{p}} H(\boldsymbol{\theta}^{\tau}, \mathbf{p}^{\tau+\frac{\epsilon}{2}}) + \nabla_{\mathbf{p}} H(\boldsymbol{\theta}^{\tau+\epsilon}, \mathbf{p}^{\tau+\frac{\epsilon}{2}})\right] && (2)\\
\mathbf{p}^{\tau+\epsilon} &= \mathbf{p}^{\tau+\frac{\epsilon}{2}} - \frac{\epsilon}{2}\nabla_{\boldsymbol{\theta}} H(\boldsymbol{\theta}^{\tau+\epsilon}, \mathbf{p}^{\tau+\frac{\epsilon}{2}}) && (3)
\end{aligned}$$

Step (3) is explicit so require to solve (1) and (2) for $\mathbf{p}^{n+\frac{1}{2}}$ and $\boldsymbol{\theta}^{n+1}$ using e.g. fixed point iteration found to be very efficient in practice.

Illustrative Example - Bivariate Gaussian

- Target density $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ where

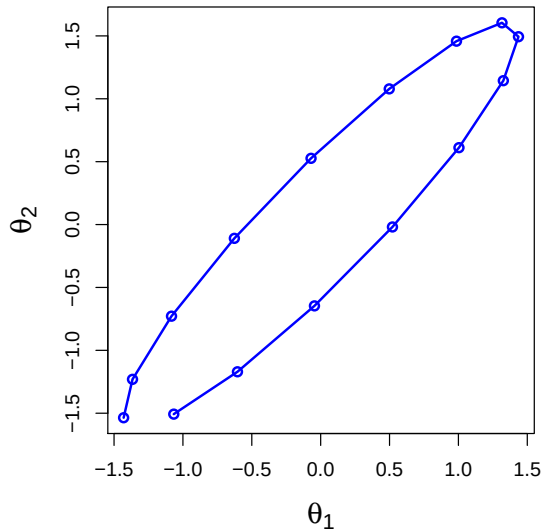
$$\mathbf{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

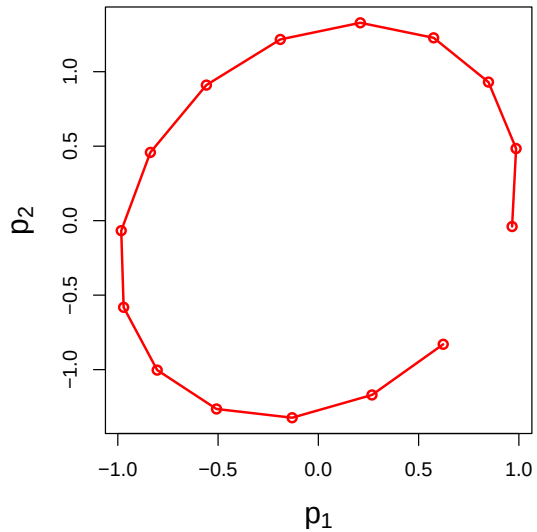
- For ρ large e.g. 0.98 sampling from this distribution is challenging
- Metric tensor defines flat manifold $\mathbf{\Sigma}^{-1}$
- Overall Hamiltonian

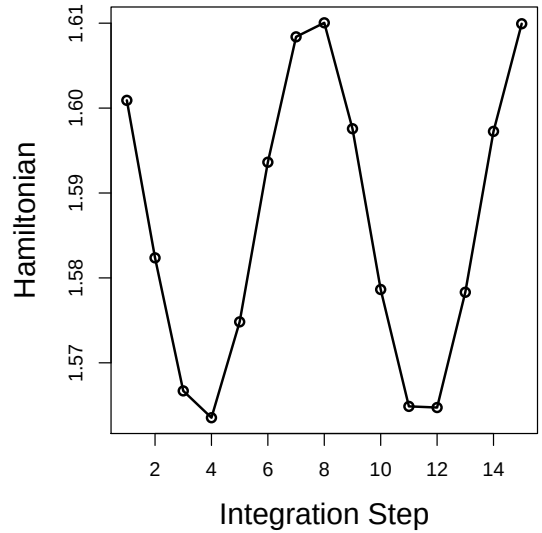
$$\frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{\Sigma}^{-1}\mathbf{x} + \frac{1}{2}\mathbf{p}^{\mathsf{T}}\mathbf{\Sigma}\mathbf{p}$$

- Stormer Verlet integrator is required

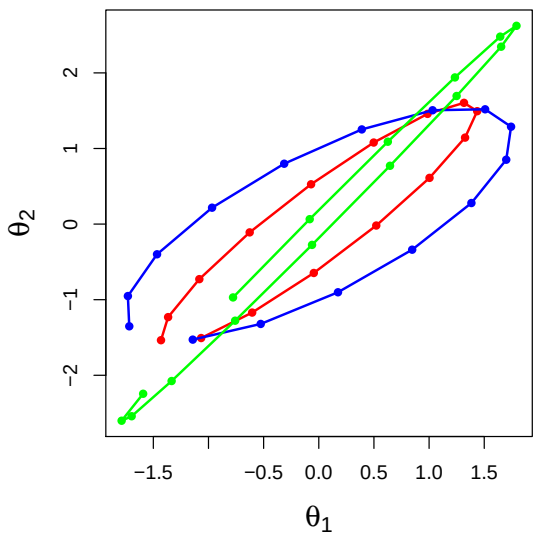
Illustrative Example - $\rho = 0.6$, RMHMC Integration $\epsilon = 0.8$, $L = 15$

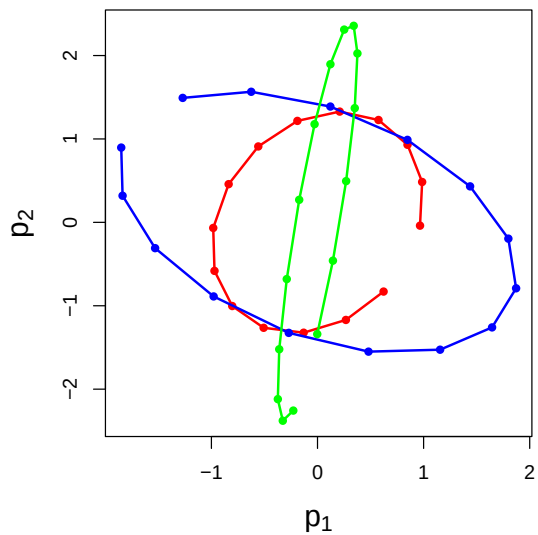


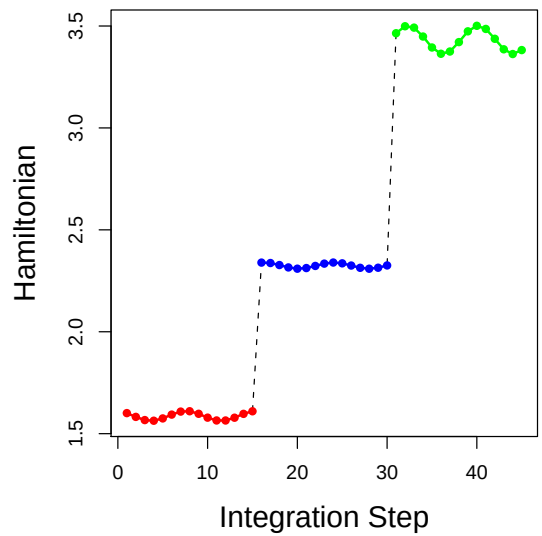




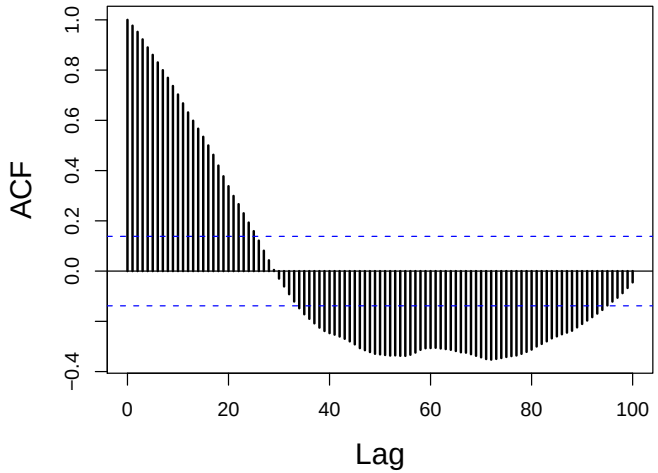
RMHMC Integration and 3 $\times$ Auxiliary Variable Sampling

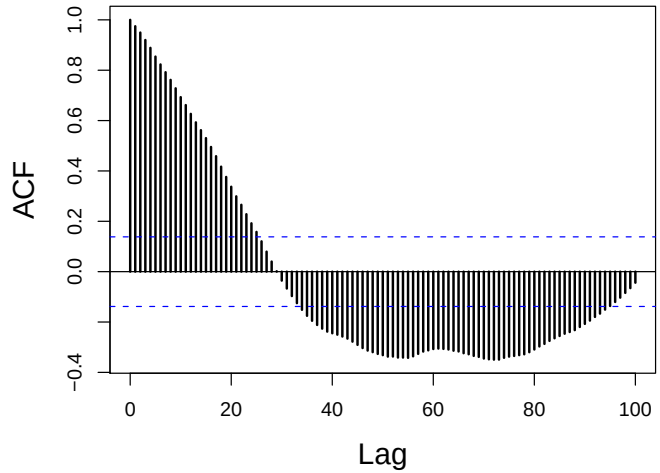


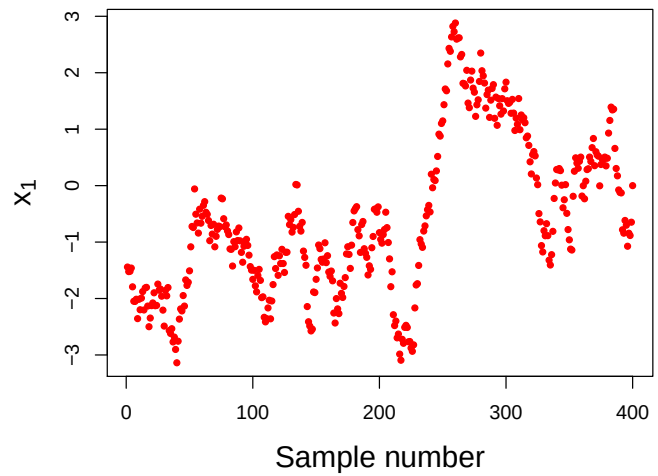


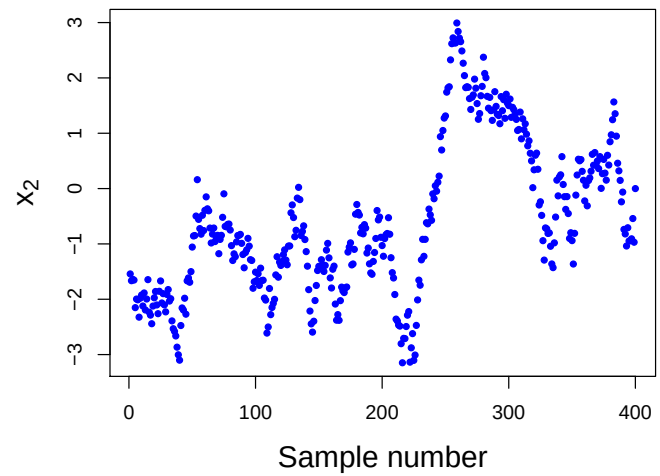


Gibbs Sampler Performance for $\rho = 0.98$

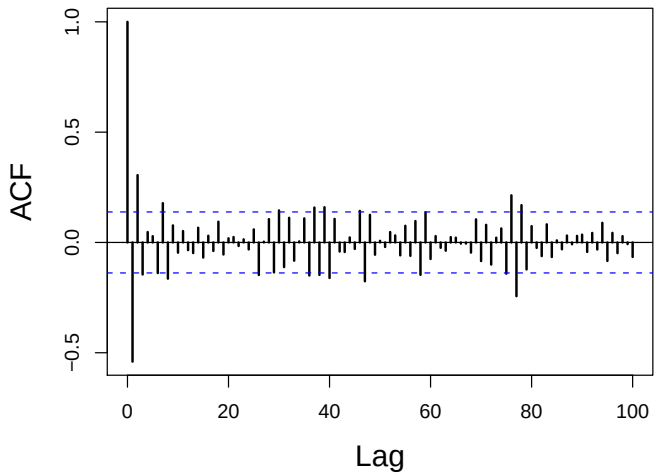


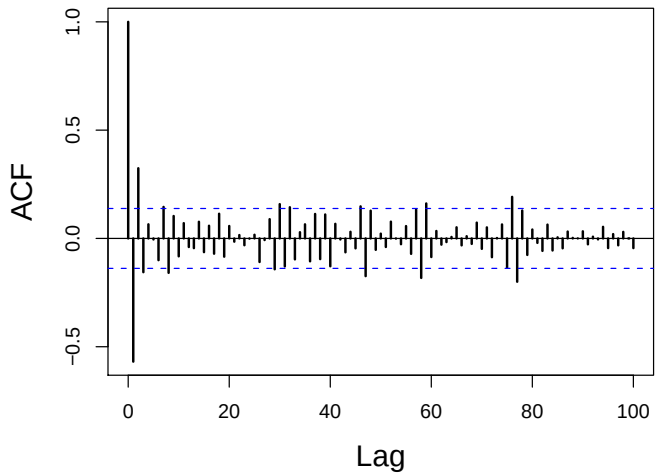


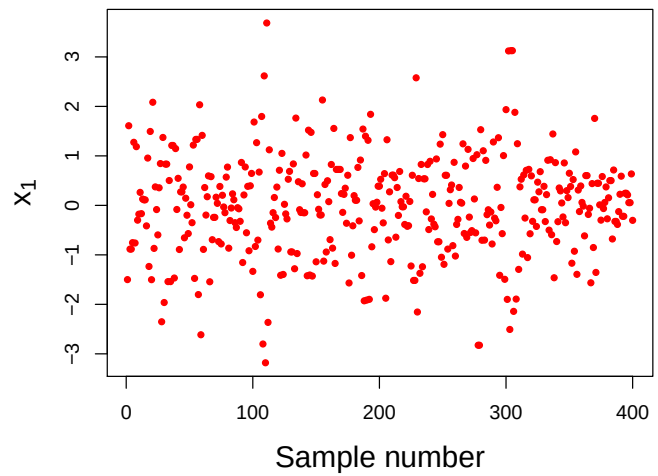


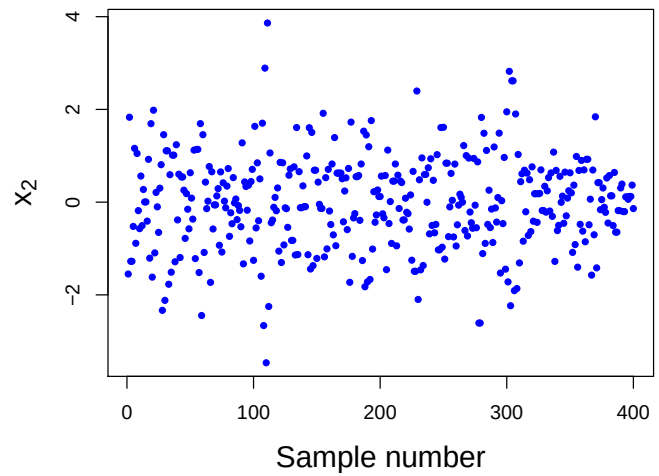


RMHMC Performance, $\rho = 0.98$, $\epsilon = 0.8$, $L = 15$

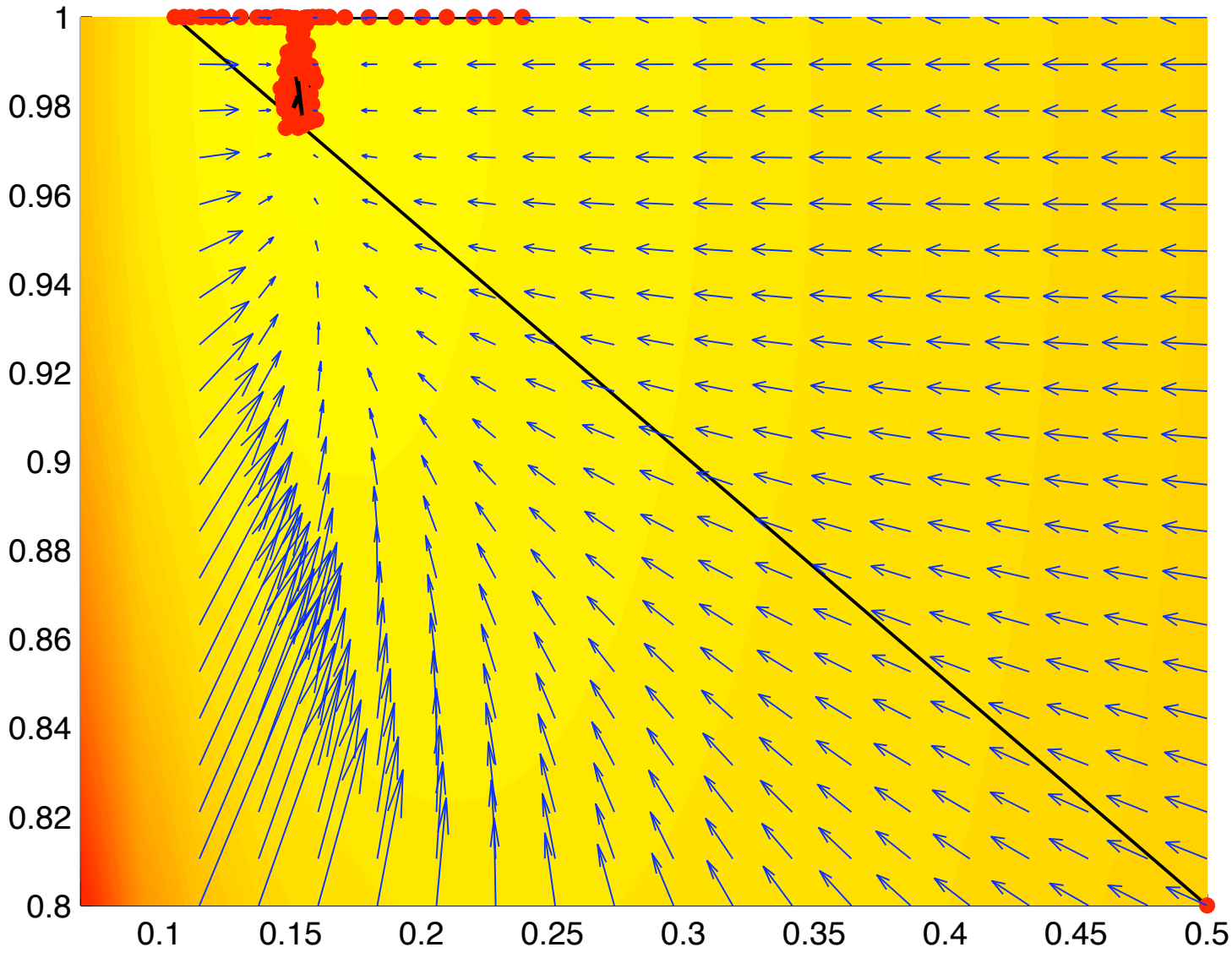








HMC Algorithm, Parameters of Stoch Vol Model, Acc Rate 95%



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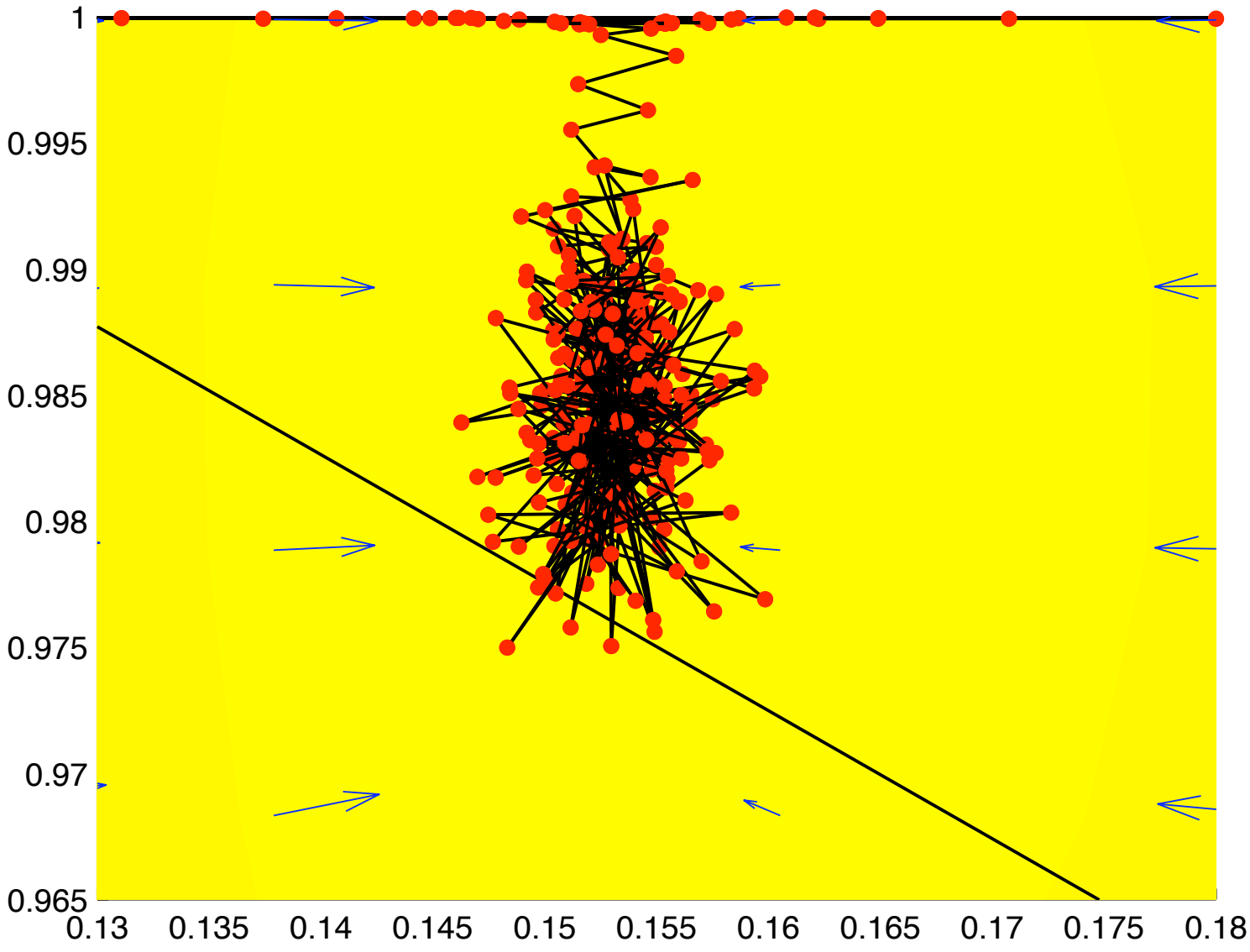

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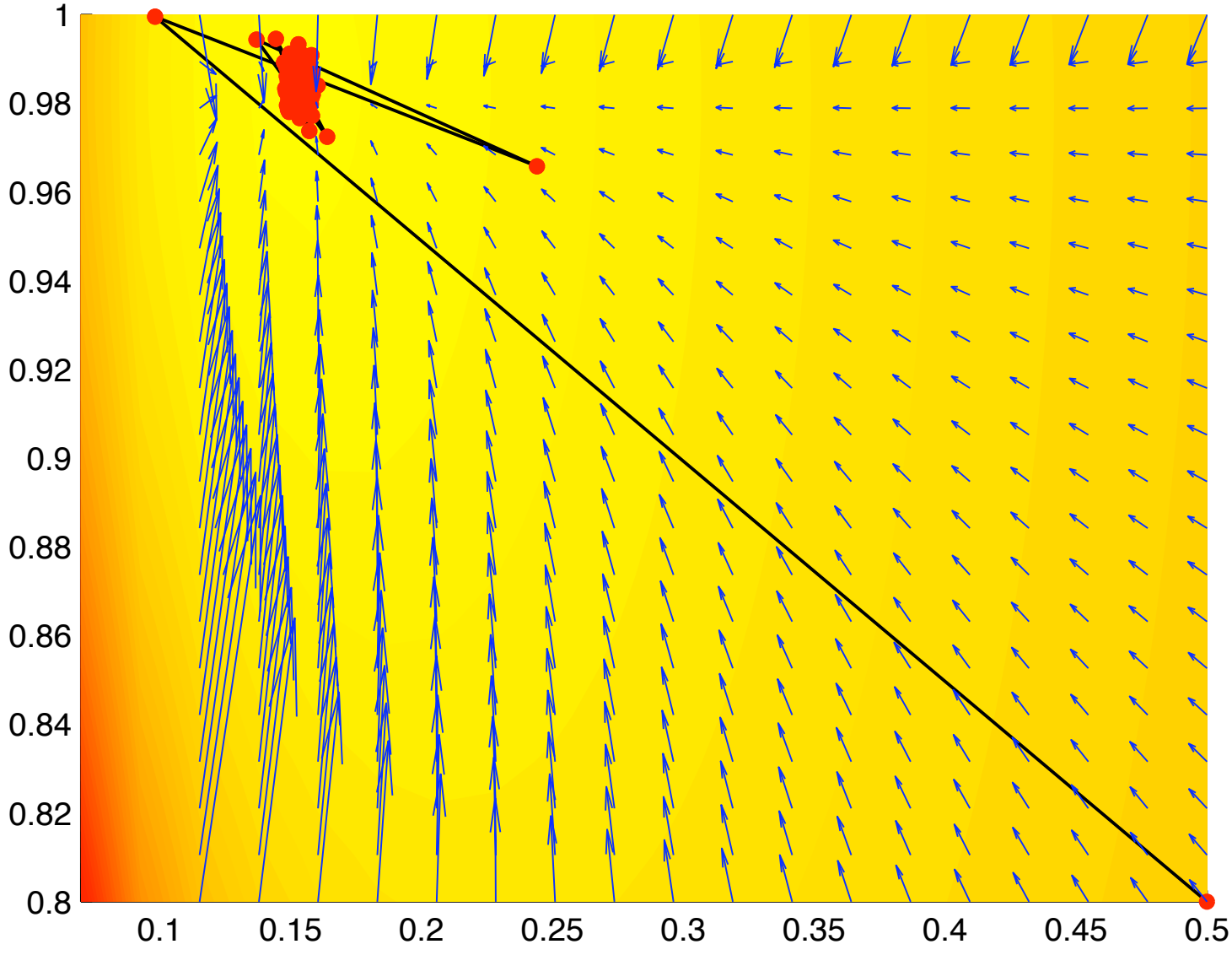

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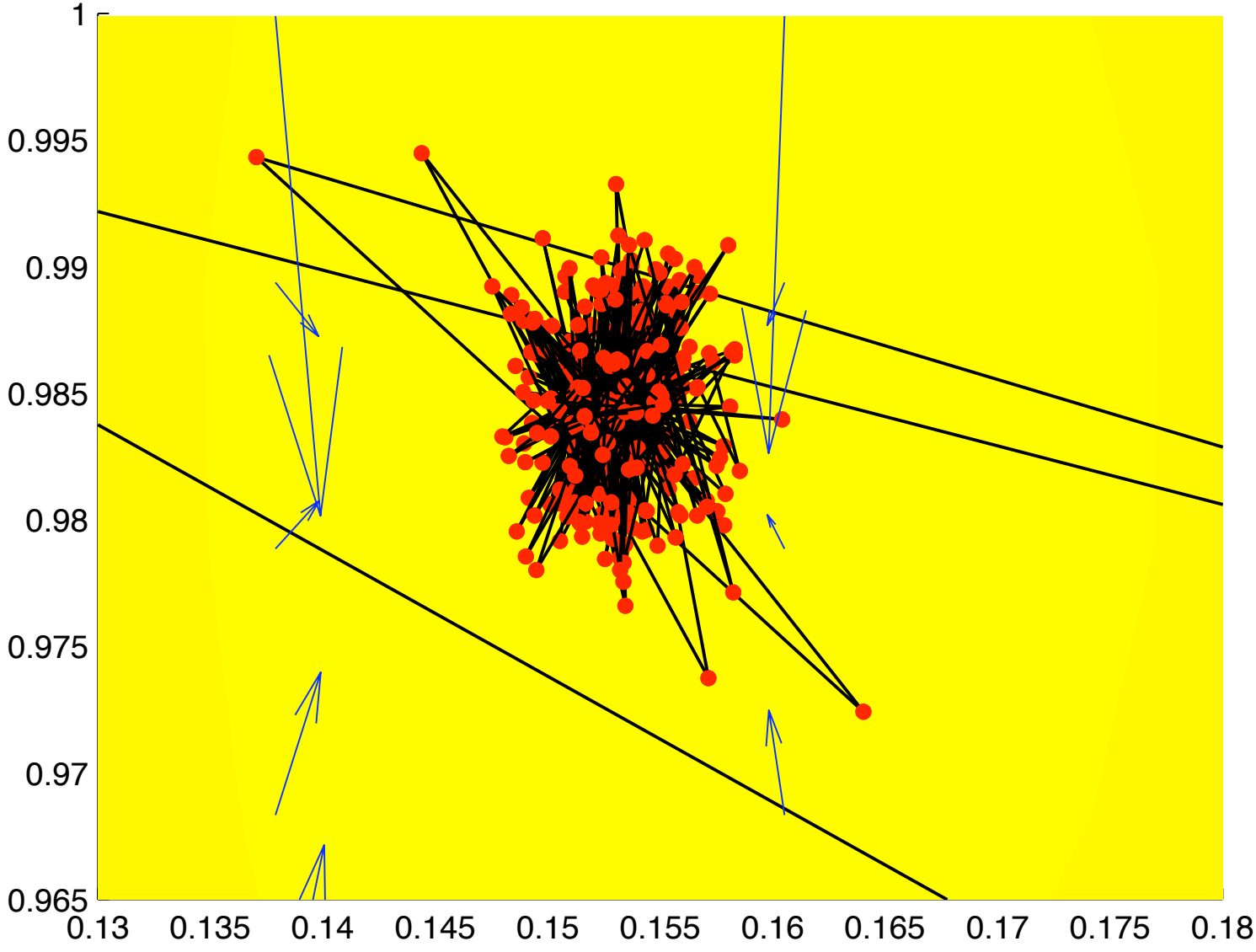

Log-Gaussian Cox Point Process with Latent Field

- Number of points in cells on 64×64 grid denoted by $\mathbf{Y} = \{Y_{i,j}\}$ conditionally independent and Poisson distributed with means $m\Lambda(i,j) = m\exp(X_{i,j})$

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- Latent field metric tensor defining flat manifold is 4096×4096, $\mathcal{O}(N^3)$ obtained from parameter conditional
- Metropolis and HMC fails.... completely. MALA requires transformation of latent field to sample - with separate tuning in transient and stationary phases - RMHMC requires no pilot tuning or additional transformations

RMHMC for Log-Gaussian Cox Point Processes

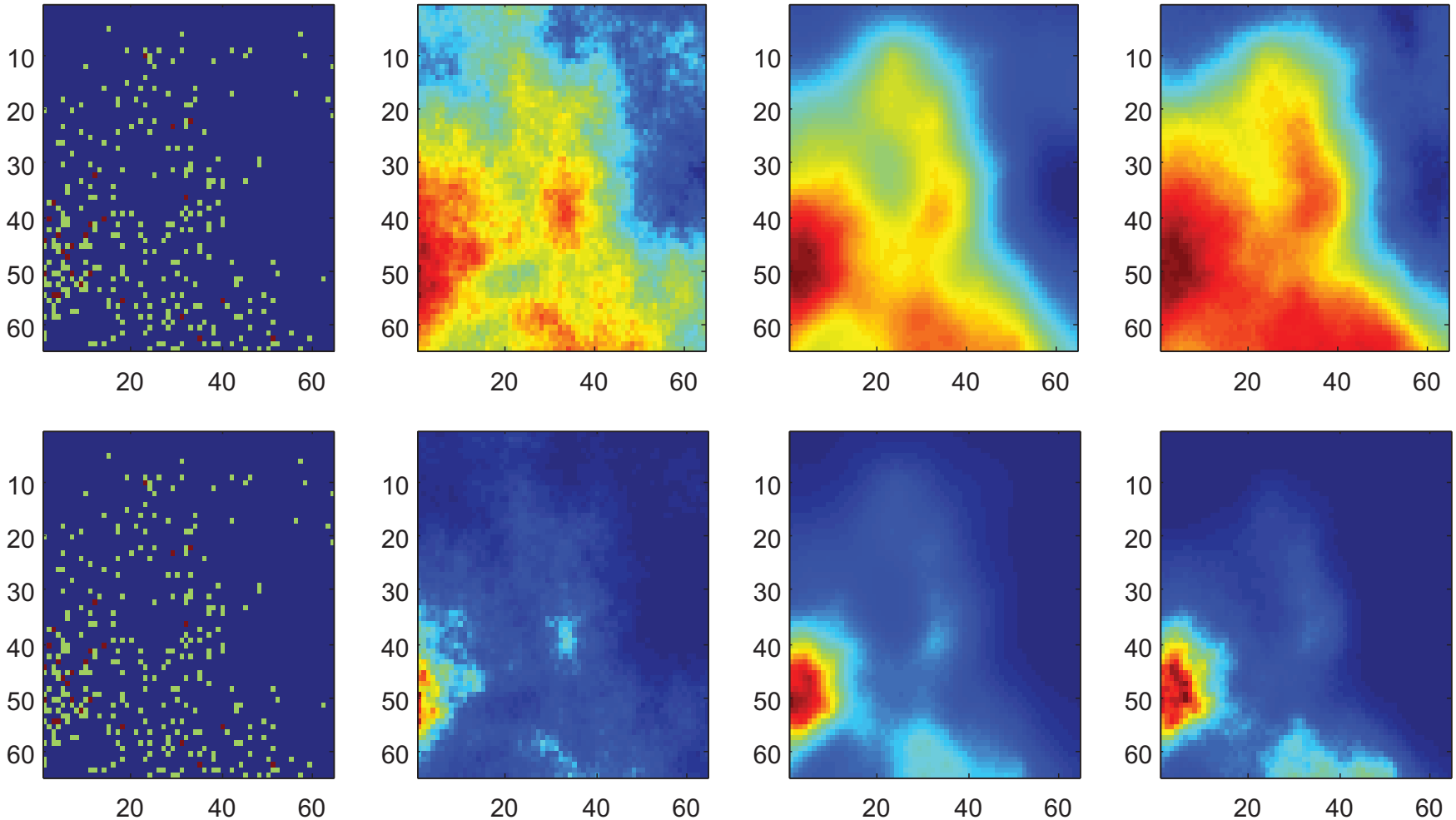

Figure: Data, Latent Field, Poisson Mean Field

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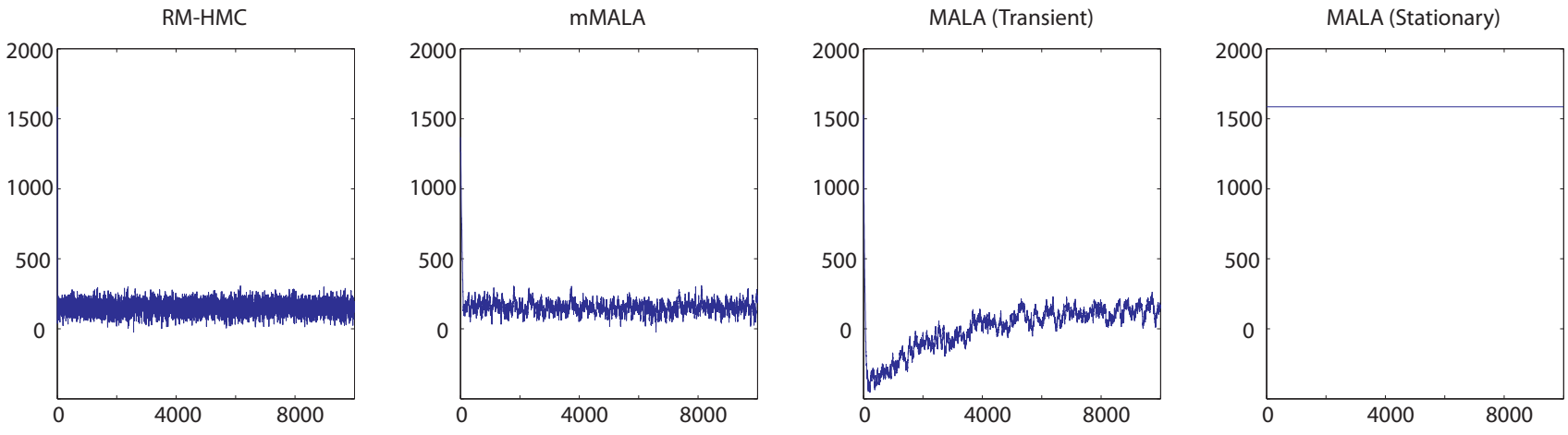

RMHMC for Log-Gaussian Cox Point Processes

Table: Sampling the latent variables of a Log-Gaussian Cox Process - Comparison of sampling methods

Method	Time	ESS (Min, Med, Max)	s/Min ESS	Rel. Speed
MALA (Transient)	31,577	(3, 8, 50)	10,605	$\times 1$
MALA (Stationary)	31,118	(4, 16, 80)	7836	$\times 1.35$
mMALA	634	(26, 84, 174)	24.1	$\times 440$
RMHMC	2936	(1951, 4545, 5000)	1.5	$\times 7070$

RMHMC for Log-Gaussian Cox Point Processes

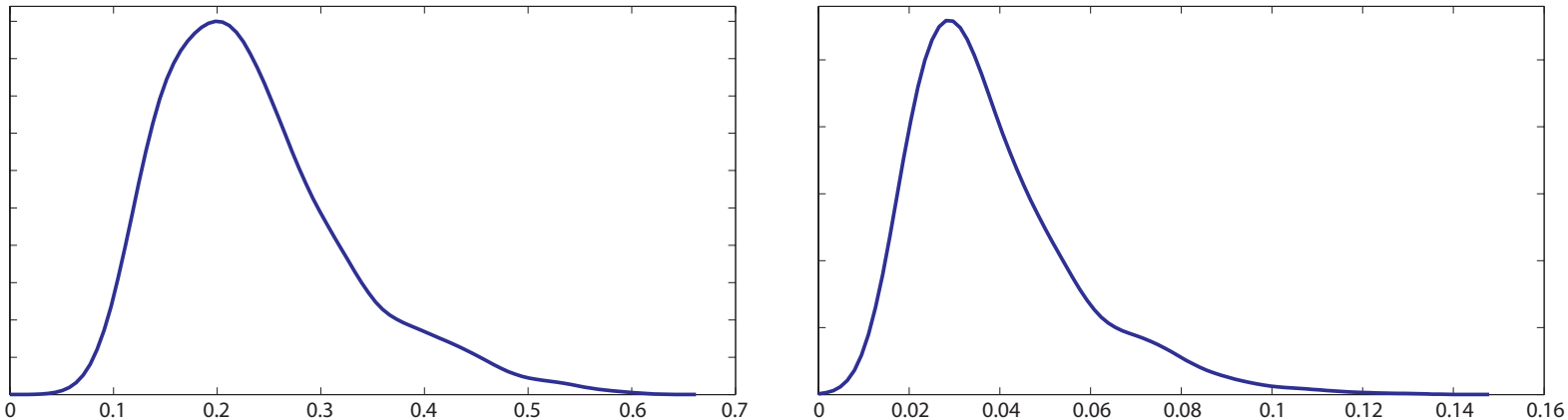

Figure: Kernel density estimates of the hyperparameter samples obtained from Gibbs style sampling from the Log-Gaussian Cox model. The true values are $\sigma = 0.19$ (left hand plot) and $\beta = 0.03$ (right hand plot).

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Codes to Replicate Results in Paper

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Funding and Research Group







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