

## Week 5, Lecture 9 - Dynamical models

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# Outline

- ▶ Administrative Issues
- ▶ Review ODE models
  - ▶ Some common constructions
- ▶ Classic analysis:
  - ▶ Stability analysis
  - ▶ Pseudo-steady-state
- ▶ Implementation:
  - ▶ Numerical integration
  - ▶ Stiff systems
  - ▶ Matrix exponentials

**Slides partly adapted from those by Bruce Tidor.**

# Ordinary Differential Equations

- ▶ ODE models are typically most useful when we already have an idea of the system components
  - ▶ As opposed to *data-driven* approaches when we don't know how to connect the data
  - ▶ Incredibly powerful for making specific predictions about how a system works
- ▶ Limits of these approaches:
  - ▶ Results can be extremely sensitive to missing components or model errors
  - ▶ Can quickly explode in complexity
  - ▶ May rely on variables that are impossible to measure

# Applications of ODE models: Molecular kinetics

Remember BE100!

Let's say we have two ligands that dimerize, then this dimer binds to a receptor as one unit:



If we want to know about how these species interact, we can model their behavior with the rate equations that describe this process.

# Applications of ODE models: Pharmacokinetics



# Applications of ODE models: Pharmacokinetics

- ▶ Central compartment corresponds to the plasma in the body.
  - ▶  $V_1$  is the distribution volume of plasma in the body.
  - ▶  $C_1$  is the concentration of drug in the plasma.
- ▶ Peripheral compartment represents a group of organs that significantly take up the particular drug.
  - ▶  $V_2$  is the volume of these group of organs.
  - ▶  $C_2$  is the concentration of drug in the group of organs.

# Applications of ODE models: Pharmacokinetics

- ▶  $k_e$  is the rate constant for clearance.
  - ▶  $k_e C_1 V_1$  is the mass of drug/time that's cleared.
- ▶  $k_1$  is the rate constant for mass transfer from the central to peripheral compartment.
  - ▶  $k_1 C_1 V_1$  is the mass of drug/time that transfers from the central to peripheral compartment.
- ▶  $k_2$  is the rate constant for mass transfer from the peripheral to central compartment.
  - ▶  $k_2 C_2 V_2$  is the mass of drug/time that transfers from the peripheral to the central compartment.

# Applications of ODE models: Pharmacokinetics

- ▶ Have a bolus i.v. injection
  - ▶ No drug in both compartments for  $t < 0$
  - ▶  $D \mu\text{g}$  of drug administered at once at  $t=0$
  - ▶ Drug distribution occurs instantaneously in the central compartment.
    - ▶ Also get well-mixed instantaneously.
    - ▶ Concentration in central compartment at  $t = 0$  is  $D \mu\text{g/mL}$
- ▶ No chemical reactions in the compartment



# Applications of ODE models: Population kinetics

## Lotka-Volterra Equations

## Note about difference from other models we've covered

- ▶ ODE models can be part of inference techniques just as elsewhere
  - ▶ If we have a symbolic integral, then fitting an ODE model to data is just non-linear least squares
- ▶ But we often don't have a symbolic expression of the answer
  - ▶ Have to simulate the model every time
  - ▶ Can only focus on the input-output we get from the black box
- ▶ In this respect, what we do with ODE models will be very similar to what you could do with any computational simulation

# Analytic vs Numerical Modeling

- ▶ Analytic
  - ▶ Wider range of parameters
  - ▶ Avoid numerical problems
  - ▶ Physical intuition more direct
  - ▶ Often must simplify model
- ▶ Numerical
  - ▶ Can handle complex models
  - ▶ Dependence on parameters & initial conditions
  - ▶ Physical insight may be difficult to extract
  - ▶ Convergence, numerical stability

Reality often requires handling in between:

- ▶ Use analytic treatment to study entire parameter space
- ▶ Use numerical treatment to study interesting regions
- ▶ Use both to handle complex behavior

# Stability Analysis

- ▶ Can solve for steady-states of a system

$$\frac{\delta F}{\delta t} = 0$$

- ▶ Results of this can be both stable or unstable points
  - ▶ With stable points, slope of  $\frac{\delta F}{\delta t}$  is negative
  - ▶ In multivariate case, this means eigenvalues of Jacobian are negative
- ▶ Steady-state points aren't necessarily realistic or feasible!
  - ▶ NNLSQ can solve for points
  - ▶ Only simulating system ensures they are accessible

# Generalization

- ▶ Linear models are easier to simulate and understand than non-linear
  - ▶ Linearity: If  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are both solutions, then  $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$  is also a solution
- ▶ Linear systems tend to be separable (effective decoupling)
- ▶ Non-linear systems exhibit interesting properties

# Linearity & Separability

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} & \kappa_{14} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} & \kappa_{24} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} & \kappa_{34} \\ \kappa_{41} & \kappa_{42} & \kappa_{43} & \kappa_{44} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} \lambda_{11} & 0 & 0 & 0 \\ 0 & \lambda_{22} & 0 & 0 \\ 0 & 0 & \lambda_{33} & 0 \\ 0 & 0 & 0 & \lambda_{44} \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$$



# Phase Portraits

$$\left. \begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \right\} \longrightarrow \dot{\vec{x}} = \vec{f}(\vec{x})$$



Non-linear systems

- ▶ No general analytic approach to finding trajectory
- ▶ So, goal is to understand qualitative trajectory behavior

## Features in Phase Portraits





# Solving a Set of Equations for Phase Portrait

- ▶ Numerical computation
  - ▶ i.e., Runge-Kutta integration
- ▶ Qualitative
  - ▶ Sufficient for some purposes
- ▶ Analytic
  - ▶ Elegant, though not always tractable

## Example – fixed points

$$\begin{aligned}\dot{x} &= x + e^{-y} \\ \dot{y} &= -y\end{aligned}$$

Nonlinear, because  
can't be represented as

$$\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned}$$

### Step 1: Find fixed points

Fixed points (also called stationary points) are those points where the time-derivative of each coordinate is zero.  $\dot{x} = 0$  and  $\dot{y} = 0$

$$\begin{aligned}0 &= x + e^{-y} &\Rightarrow & -x = e^{-y} = 1 \\ 0 &= -y &\Rightarrow & y = 0\end{aligned}$$

Thus, one fixed point at  $(x, y) = (-1, 0)$

# Example – stability

## Step 2: Determine stability of fixed points

- ▶ If the systems moves slightly away from each fixed point, will it return or will it move further away?
- ▶ Another way to ask the same question is to ask whether, as time approaches infinity, does the system tend toward or away from a given stable point.
- ▶ Note  $y$  solution must be of form:
  - ▶  $y = y_0 e^{-t}$  (because  $\dot{y} = \frac{dy}{dt} = -y$ )
  - ▶ So  $y \rightarrow 0$  for  $t \rightarrow \infty$
- ▶ Thus,  $\dot{x} = x + e^{-y}$  becomes  $\dot{x} \rightarrow x + 1$  for long times
  - ▶ This has exponentially growing solutions
  - ▶ Toward  $\infty$  for  $x > -1$  and  $-\infty$  for  $x < -1$

**Thus, overall solution grows exponentially in at least one dimension, and so is unstable.**

# Example – nullclines

## Step 3: Sketch nullclines

Nullclines are the sets of points for which  $\dot{x} = 0$  or  $\dot{y} = 0$ , so flow is either horizontal or vertical.

$\dot{y} = 0$  for  $0 = -y \rightarrow y = 0 \Rightarrow$  flow is horizontal at  $x$  - axis  
 $\dot{x} = x + 1$  here, so flow is to the right for  $x > -1$

Flow is vertical for  $\dot{x} = 0 \rightarrow 0 = x + e^{-y}$

The nullclines partition the space into classes of flow direction:

$$\dot{x}, \dot{y} \begin{cases} < \\ > \end{cases} 0$$



# Example – computed

## Step 4: Plot flow lines



# Existence & Uniqueness

Non-linear  $\dot{\mathbf{x}} = f(\mathbf{x})$  and given an initial condition.

- ▶ Existence and uniqueness of solution guaranteed if  $f$  is continuously differentiable
- ▶ Corollary: Trajectories **do not** intersect, because if they did, then there would be two solutions for the same initial condition at the crossing point

# Linearization About Fixed Points

Let  $\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}$  be a non - linear system with fixed point  $(x^*, y^*)$

$$0 = f(x^*, y^*) = g(x^*, y^*)$$

Let  $\begin{cases} u = x - x^* \\ v = y - y^* \end{cases}$  be deviations from fixed point

↓ Change of variable

$$\begin{aligned} \dot{u} &= \dot{x} \quad (x^* \text{ is constant}) \\ &= f(u + x^*, v + y^*) \\ &= f(x^*, y^*) + \underbrace{u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y}}_{\text{linear}} + O(u^2, v^2, uv) \quad \text{Taylor series expansion} \end{aligned}$$

Likewise,  $\dot{v} = u \frac{\partial g}{\partial x} + v \frac{\partial g}{\partial y} + O(u^2, v^2, uv)$

# Solving Linearized Systems

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\dot{\vec{x}} = \vec{A}\vec{x} \quad \vec{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{let } \vec{x}(t) = e^{\lambda t} \vec{v}$$

$$\lambda e^{\lambda t} \vec{v} = \vec{A} e^{\lambda t} \vec{v} \quad \lambda \vec{v} = \vec{A} \vec{v}$$

$$(\vec{A} - \lambda I) \vec{v} = 0$$

$$\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \vec{v} = 0$$

$$\det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

$$(a - \lambda)(d - \lambda) - bc = 0$$

$$\lambda^2 - \tau\lambda + \Delta = 0$$

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2} \quad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

$\tau = \text{trace}$

$\Delta = \text{determinant}$

If  $\lambda_1 \neq \lambda_2$ , then  $v_1$  &  $v_2$  are linearly independent and solutions of the following form are valid.

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$



## Example

$$\left. \begin{array}{l} \dot{x} = x + y \\ \dot{y} = 4x - 2y \\ (x, y)_{t=0} = (2, -3) \end{array} \right\} \begin{array}{l} \tau = -1 \\ \rightarrow \\ \Delta = -6 \end{array} \left\{ \begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = -3 \end{array} \right. \quad \begin{array}{l} \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \vec{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \end{array}$$

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} \quad \text{with } c_1 = c_2 = 1 \text{ from init. cond.}$$

Can draw phase portrait directly from eigenvalues & eigenvectors:



# More Examples

## Stable Node

(two real negative eigenvalues)



## Spiral

(two complex eigenvalues;  
Can be growing or shrinking)



## Unstable Node

(two real positive eigenvalues)



## Center

(purely imaginary eigenvalues)



Also, equal eigenvalues lead to stars & degenerate nodes

# Classification of Fixed Points



# Relevance for Nonlinear Dynamics

- ▶ So, we have said that we can find fixed points of nonlinear dynamics, linearize about each fixed point, and characterize the dynamics about each fixed point in the non-linear model by the corresponding linear model.
- ▶ Is this always true? Do the nonlinearities ever disturb this approach?
- ▶ A theorem can be proven which states
  - ▶ That all the regions on the previous slide are “robust” (nodes, spirals, saddles) and correspond between linear and nonlinear models.
  - ▶ But that all the lines on the previous slide are “delicate” (centers, stars, degenerate nodes, non-isolated fixed points) and can have different behaviors in linear and non-linear models.

# Bifurcations

- ▶ The phase portraits we have been looking at describe the trajectory of the system for a given set of initial conditions. However, for “fixed” parameters (rate constants in eqns, for instance).
- ▶ What we might like is a series of phase portraits corresponding to different sets of parameters.
- ▶ **Many** will be qualitatively similar. The interesting ones will be where a small change of parameters creates a qualitative change in the phase portrait (bifurcations).
- ▶ What we will find is that fixed points & closed orbits can be created/destroyed and stabilized/destabilized.

# Saddle-Node Bifurcation

$$\begin{aligned}\dot{x} &= \mu - x^2 \\ \dot{y} &= -y\end{aligned}$$



$\mu > 0$



$\mu = 0$



$\mu < 0$

# Genetic Control Network

Griffith (1971) model of genetic control:

- ▶  $x$  = protein concentration
- ▶  $y$  = mRNA concentration

$$\dot{x} = -ax + y$$

protein degrades and is synthesized from mRNA

$$\dot{y} = \frac{x^2}{1+x^2} - by$$

mRNA degrades and is stimulated by protein dimer



# Genetic Control Network

Biochemical version of a bistable switch:

1. Only stable points are no protein and mRNA or a fixed composition
2. If degradation rates too great, only stable point is origin





# Implementation - Testing

- ▶ Many properties one can test
  - ▶ Mass balance
  - ▶ Changes upon parameter adjustment
- ▶ Good to test these before and after integration

# Implementation

SciPy provides two interfaces for ODE solving:

- ▶ `scipy.integrate.ode`
- ▶ `scipy.integrate.odeint`

Notes:

- ▶ Both can solve stiff and non-stiff equations.
- ▶ `ode` has a number of different methods. Pay attention to the “`set_integrator`” option.

## Implementation - Example

The second order differential equation for the angle  $\theta$  of a pendulum acted on by gravity with friction can be written:

$$\theta''(t) + b * \theta'(t) + c * \sin(\theta(t)) = 0$$

where  $b$  and  $c$  are positive constants, and a prime (') denotes a derivative. To solve this equation with `odeint`, we must first convert it to a system of first order equations. By defining the angular velocity  $\omega(t) = \theta'(t)$ , we obtain the system:

$$\theta'(t) = \omega(t)$$

$$\omega'(t) = -b * \omega(t) - c * \sin(\theta(t))$$

## Implementation - Example

Let  $y$  be the vector  $[\theta, \omega]$ . We implement this system in python as:

```
def pend(y, t, b, c):  
    theta, omega = y  
    dydt = [omega, -b*omega - c*np.sin(theta)]  
    return dydt
```

We assume the constants are  $b = 0.25$  and  $c = 5.0$ :

```
b, c = 0.25, 5.0
```

## Implementation - Example

For initial conditions, we assume the pendulum is nearly vertical with  $\theta(0) = \pi - 0.1$ , and it initially at rest, so  $\omega(0) = 0$ . Then the vector of initial conditions is

```
y0 = [np.pi - 0.1, 0.0]
```

We generate a solution 101 evenly spaced samples in the interval  $0 \leq t \leq 10$ . So our array of times is:

```
t = np.linspace(0, 10, 101)
```

## Implementation - Example

Call `odeint` to generate the solution. To pass the parameters `b` and `c` to `pend`, we give them to `odeint` using the `args` argument.

```
from scipy.integrate import odeint  
sol = odeint(pend, y0, t, args=(b, c))
```

The solution is an array with shape `(101, 2)`. The first column is  $\theta(t)$ , and the second is  $\omega(t)$ . The following code plots both components.

## Implementation - Example

```
import matplotlib.pyplot as plt
plt.plot(t, sol[:, 0], 'b', label='theta(t)')
plt.plot(t, sol[:, 1], 'g', label='omega(t)')
plt.legend(loc='best')
plt.xlabel('t')
plt.grid()
plt.show()
```

# Implementation - Example





# Implementation - Stiff Systems

- ▶ Very roughly, most ODE solvers take steps inversely proportional to the rate at which the state is changing
- ▶ For systems where there are two processes operating on differing timescales, this can be problematic
  - ▶ If everything happens really fast, the system will come to equilibrium quickly
  - ▶ If everything is slow, you can take longer steps
- ▶ *Stiff* solvers additionally require the Jacobian matrix
  - ▶ This very roughly allows them to keep track of these differences in timescales
- ▶ `odeint` can automatically find this for you
  - ▶ Sometimes it's faster/better to provide this as parameter `Dfun`

# Implementation - Matrix Exponential

If  $J$  is the Jacobian matrix of an ODE model,  $y(t) = e^{Jt}y_0$ .

Matrix exponential is also implemented.

- ▶ `scipy.linalg.expm`
  - ▶ This method is numerically stable, but there are faster implementations elsewhere.
- ▶ A commonly used package is `expokit`

For linear systems, this can be  $>1000\times$  faster.

## Further Reading

- ▶ `scipy.linalg.expm`
- ▶ `scipy.integrate.odeint`
- ▶ Steven Strogatz, *Nonlinear Dynamics and Chaos*